Pacific Journal of Mathematics

TAMAGAWA NUMBERS OF QUASISPLIT GROUPS OVER FUNCTION FIELDS

RALF KÖHL, M. M. RADHIKA AND ANKIT RAI

Volume 338 No. 1 September 2025

PACIFIC JOURNAL OF MATHEMATICS
Vol. 338, No. 1, 2025
https://doi.org/10.2140/pjm.2025.338.139

TAMAGAWA NUMBERS OF QUASISPLIT GROUPS OVER FUNCTION FIELDS

RALF KÖHL, M. M. RADHIKA AND ANKIT RAI

To Günter Harder, with the deepest admiration

We use Morris' theory of Eisenstein series for reductive groups over global function fields in order to extend Harder's computation of Tamagawa numbers to quasisplit groups.

1. Introduction

Let F be a global field and $\mathbb{A}$ be the adèles over F. For an algebraic group G defined over F, an invariant $\tau(G) \in \mathbb{R}$ called the Tamagawa number can be associated to G. If we let $\mathbb{G}$ denote the group $G(\mathbb{A})$, this is the volume of a subspace (denoted by $G(F)\backslash\mathbb{G}_1$ in Section 2.3) of the space $G(F)\backslash\mathbb{G}$ with respect to a certain left $G(F)$-invariant Haar measure on $\mathbb{G}$ called the Tamagawa measure. It was conjectured by Weil that for an absolutely simple simply connected algebraic group G over a global field, the Tamagawa number $\tau(G)$ equals 1. This was first proved for split groups over number fields by Langlands [1966] and over function fields by Harder [1974]. The proof given by Langlands was rewritten in the adelic language for quasisplit groups by Rapoport [1976] and Lai [1980], thus giving a unified proof for the split and quasisplit groups over a number field.

Using Arthur's trace formula, Kottwitz [1988] proved Weil's conjecture over number fields. The proof of Weil's conjecture over function fields for any semisimple group G was given by Gaitsgory and Lurie [2019] by a method different than the one used in the earlier works of Langlands, Lai, Rapoport and Kottwitz. In another direction the theory of Eisenstein series was developed for general reductive groups over function fields in the works of Morris [1982a; 1982b]. Now that this theory is well developed, it is natural to proceed as in the works of Harder and Lai to directly prove Weil's conjecture for quasisplit groups over function fields. The present article should be considered as a contribution towards confirming Weil's conjecture for function fields via the strategy used for number fields. The main theorem of this article is as stated.

MSC2020: primary 20G35; secondary 11M38, 11R56, 11R58, 22E55.
Keywords: Tamagawa number, Weil conjecture, quasisplit groups, Eisenstein series.

Theorem 1.1. *Let F be a function field of a smooth projective curve over $\mathbb{F}_q$ where $q \neq 2$ and G be a quasisplit semisimple simply connected group over F. Then*

$$\tau(G) = 1.$$

For nonquasisplit groups either the methods of Kottwitz will have to be used or, alternatively, some other way to establish that the Tamagawa number does not change when passing to inner forms. However, given the unsatisfactory state of the trace formula over function fields, at the moment one cannot proceed further with the methods of Kottwitz. Nevertheless, some progress towards Arthur's trace formula over function fields has been made in [Ngô Dac 2009].

Tamagawa [1966] originally observed that the group $\mathrm{SO}_q(\mathbb{A})$ can be endowed with a natural measure such that the Minkowski–Siegel formula is equivalent to the assertion that the Tamagawa number (i.e., the volume with respect to this natural measure) be 2. Weil [1982] subsequently observed that for simply connected groups one should expect the value 1, which as — outlined above — has been confirmed by Kottwitz [1988] for number fields and by Gaitsgory and Lurie [2019] for function fields.

The organization of the article is as follows. Section 2.1 recalls the basics on reductive groups over global and local fields, root systems and sets up the notation for the subsequent sections. In Section 2.3 the Tamagawa measure for semisimple groups, and more generally for reductive groups is defined following the work of Oesterlé [1984]. Section 2.4 deals with quasicharacters on tori.

The aim of Section 3 is to prove Theorem 1.1. Section 3.1 contains generalities on Eisenstein series. In Section 3.2 we follow the methods of Lai [1980] and Rapoport [1976] for computing certain intertwining operators for groups over functions fields and thus, obtain precise information about their poles and zeros (see Theorem 3.3). Sections 3.3 and 3.4 are devoted to proving the main theorem.

Appendices A–C comprise the proofs of a few technical lemmas used in the main content of this article. These results are well known and have been added here with the intention of improving the exposition of this article.

After preparing the initial version of the present paper we learned from G. Prasad that our results have also been achieved by E. Kushnirsky in the unpublished part of his Ph.D. thesis [Kushnirsky 1999] following the methods of Lai and Rapoport. Although the basic approach is the same, there are a few noteworthy differences. In addition to providing enough background to make the article more accessible and self-contained, we also provide more details for our proofs, especially for Lemma 2.1 and Proposition 3.4. As will be evident from the strategy of the proof (or, in fact, rather the proof itself), Theorems 3.3 and 3.12 form the technical heart of our approach: the former proves an assertion about the poles of the intertwining operator for the longest element of the Weyl group, the latter provides a description of the projection operator onto the constant functions. Although the strategy of the proof of Theorem 3.3 has become standard and the proof of Theorem 3.12 follows Harder's approach, it is important — also for the clarity of the exposition — to include a complete chain of arguments.

1.1. ***Strategy of the proof.*** Recall that the Tamagawa number of a semisimple algebraic group defined over a global field F is the volume of the quotient space $G(F)\backslash\mathbb{G}$ with respect to the measure induced by the Tamagawa measure on $\mathbb{G}$ (see Section 2.3). Langlands implemented the idea of computing the constant terms of pseudo-Eisenstein series in two different ways and comparing the results to deduce Weil's conjecture for split groups over number fields. He further speculated that his proof may admit generalizations. A comparison of these two different ways of computing the constant terms expresses the Tamagawa number of G in terms of the Tamagawa number of a maximal split torus of G.

We now explain our strategy for the proof of Weil's conjecture for quasisplit groups over function fields followed in this article. Let θ be a pseudo-Eisenstein series whose constant term does not vanish; there certainly exists many such. We refer the reader to Section 3.1 for more details. We know $\theta \in L^2(G(F)\backslash\mathbb{G})$. The constant term of θ is $(\theta, 1)$, where $(\, , \,)$ denotes the L^2-inner product, and is given by an easy computation (see Lemma 3.2). Another interpretation of $(\theta, 1)$ is as the projection of θ onto the subspace of constant functions. We exploit this interpretation and following [Harder 1974], show that the projection

$$P : L^2(G(F)\backslash\mathbb{G}) \to \mathbb{C}$$

onto the constants, restricted to the closed subspace $\mathcal{E}^\vee \subset L^2(G(F)\backslash\mathbb{G})$ generated by the pseudo-Eisenstein series, admits an explicit description. This description is stated in Theorem 3.12, where the constant c is computed in Lemma 2.1 and $c' = q^{\dim(N)(1-g)}$.

Finally, the comparison of the two ways of computing the constant term is completed in Section 3.4. The equations in Section 3.4, are simplified using the explicit computation of local intertwining operators from Sections 3.2.1.1 and 3.2.1.2. Also used is the fact that the intertwining operator $M(w_0, s\rho)$ has a pole of order $r = F\text{-rk}(G)$ (see Theorem 3.3). This in turn is same as the order of pole at $s = 1$ of the Artin L-function $L(s, X^*(A))$ attached to the $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-representation $X^*(A \times_F F^{\mathrm{sep}})$. The latter statement is true because A is a quasisplit torus. That the groups under consideration be quasisplit is crucial for the proof of Theorem 3.3. The importance of this assumption is emphasized in the reduction to the F-rank 1 quasisplit groups using the Bhanu Murthy–Gindikin–Karpelevitch formula. In the rank 1 case the theorem is proved by explicit computation in each case (see Section 3.2.2.1). With the final expression of the projection $P\theta$ in hand, the rest of the proof is explained in Section 3.4.

2. Basics and notation

2.1. ***Generalities about quasisplit reductive groups.*** Let F be a function field of a smooth projective curve or genus g defined over $\mathbb{F}_q$ for $q \neq 2$. Let F^{sep} be a separable closure of F and $\bar{F}$ be the algebraic closure. For any place v of F, let F_v denote the corresponding local field, $k(v)$ be the residue field at v, $\mathcal{O}_v$ be the ring of integers in F_v, and π_v be a uniformizer of F_v. Let G be a quasisplit reductive group defined over F, and $B \subset G$ be an F-Borel subgroup fixed throughout this article. Let $B = A \cdot N$ be a Levi decomposition, where N is the unipotent radical and $A \subset B$ is a maximal torus defined over F chosen such that the maximal split subtorus $A_d \subset A$ is a maximal split torus of G. Let N^- be the opposite unipotent radical.

Given a reductive algebraic group G defined over F that is unramified at a place v of F, it is known that $G \times_F F_v$ admits a smooth reductive model $\mathcal{G}$ over $\mathrm{Spec}(\mathcal{O}_v)$. Then $K_v = \mathcal{G}(\mathcal{O}_v)$ is a hyperspecial maximal compact subgroup. If G is ramified at v, let K_v denote a special maximal compact subgroup of $G(F_v)$ which always exists by Bruhat–Tits theory. The choice of K_v is hereby fixed for the rest of this article.

Let $\mathbb{G}$, $\mathbb{B}$, $\mathbb{N}$ and $\mathbb{K}$ respectively denote the groups $G(\mathbb{A})$, $B(\mathbb{A})$, $N(\mathbb{A})$, and $\prod_v K_v$. We have the Iwasawa decomposition

$$\mathbb{G} = \mathbb{K}\mathbb{B}.$$

Recall that a quasicharacter is a continuous homomorphism from $A(F)\backslash A(\mathbb{A})$ to $\mathbb{C}^\times$. A character $\lambda : A \to \mathbb{G}_m$, defined over F, gives a quasicharacter $\lambda : A(F)\backslash A(\mathbb{A}) \to q^{\mathbb{Z}}$ defined to be the composite map

$$A(F)\backslash A(\mathbb{A}) \to F^\times\backslash \mathbb{A}^\times \to q^{\mathbb{Z}}.$$

Denote with $X^*(A)$ (resp. $X^*(A_d)$) the group of characters of the torus A (resp. A_d) defined over F and with $\Lambda(A)$ the set of quasicharacters of A.

2.1.1. *Root systems.* Let $G \supset B \supset A$ be as before. Let $\Pi_F \subset X^*(A_d)$ be the subset of nontrivial weights of A_d on $\mathrm{Lie}(G)$. Let $X_*(A_d)$ be the set of cocharacters of A_d and $\Pi_F^\vee \subset X_*(A_d)$ be the set of coroots. The root data $(\Pi_F, X^*(A_d))$ can be enhanced to the tuple $(X^*(A_d), \Pi_F, X_*(A_d), \Pi_F^\vee)$ called the relative root datum. Denote the absolute root datum by $(X^*(A \times_F F^{\mathrm{sep}}), \Pi, X_*(A \times_F F^{\mathrm{sep}}), \Pi^\vee)$. Let $X^*_+(A \times_F F^{\mathrm{sep}})$ and $X^*_-(A \times_F F^{\mathrm{sep}})$, respectively, denote the weight lattice of the universal cover and the root lattice of $G \times_F F^{\mathrm{sep}}$. In the rest of this paper we assume that G is simply connected unless otherwise stated. Note that this assumption implies $X^*_-(A \times_F F^{\mathrm{sep}}) = X^*_+(A \times_F F^{\mathrm{sep}})$. Let Π^+ and Π_F^+ respectively denote the set of positive absolute roots and the set of positive relative roots of G with respect to B. Let ρ be the half sum of positive relative roots counted with multiplicity. We can also define ρ to be the element of $X^*(A_d)$ or $X^*(A)$ given by $a \mapsto \det(\mathrm{Ad}(a|_{\mathrm{Lie}(N)}))^{1/2}$.

Let $W_F := N_G(A)(F)/Z_G(A)(F)$ and $W = N_G(A)(F^{\mathrm{sep}})/Z_G(A)(F^{\mathrm{sep}})$ be the relative Weyl group and the absolute Weyl group respectively. We have an embedding $W_F \hookrightarrow W$. Recall that there is a W_F-equivariant positive definite bilinear form $\langle\cdot,\cdot\rangle : X^*(A_d)_{\mathbb{R}} \times X^*(A_d)_{\mathbb{R}} \to \mathbb{R}$ such that, the coroot $a^\vee$ corresponding to the root $a \in \Pi_F$ is the element $2a/\langle a, a\rangle$ under the isomorphism $X^*(A_d) \simeq X_*(A_d)$ given by $\langle\cdot,\cdot\rangle$. The set $(\mathbb{Z}\Pi_F^\vee)^* \subset X^*(A_d)_{\mathbb{Q}}$, defined under the pairing $X^*(A_d)_{\mathbb{Q}} \times X_*(A_d)_{\mathbb{Q}} \to \mathbb{Q}$, is called the relative weight lattice of G.

2.1.2. *Groups over local fields.* Given a place v of F, the group $G \times_F F_v$ is quasisplit as G was assumed to be quasisplit. Furthermore, if $G \times_F F_v$ splits over an unramified extension E of F_v, then G admits a canonical (up to isomorphism) smooth reductive model over $\mathcal{O}_v$ and thus, a canonical (up to the action of the adjoint group) choice of maximal hyperspecial compact subgroup $K_v = G(\mathcal{O}_v)$. If $G \times_F F_v$ does not split over an unramified extension then it is possible to construct a *parahoric*[1] group scheme over $\mathrm{Spec}(\mathcal{O}_v)$ and we define $K_v := G(\mathrm{Spec}(\mathcal{O}_v))$. This is again a maximal compact subgroup of $G(F_v)$. We assume these choices have been made and fixed for the rest of the article.

In the later sections we will need a classification of quasisplit groups over function fields of characteristic $\neq 2$. Thắng [2022] gives a complete classification of these groups which was started in the seminal work of Bruhat and Tits [1972]. According to the table in [Thắng 2022], up to central isogeny there are two quasisplit absolutely simple algebraic groups of relative rank 1. They are

(1) SL_2,

(2) $\mathrm{SU}(3, E_v/F_v)$, where E_v/F_v is a quadratic extension.

2.2. ***Dual groups.*** We will recall the definition of the dual groups and setup some more notation here. Let G be a quasisplit group over any field F, A be a maximal torus in G defined over F, and let E/F be a separable extension such that $G\times_F E$ is a split reductive group. Let $\Psi(G) := (X^*(A\times_F E), \Pi_E, X_*(A\times_F E), \Pi_E^\vee)$ be the root datum of the split reductive group $G \times_F E$. Consider the dual root datum $\Psi(G)^\vee := (X_*(A\times_F E), \Pi_E^\vee, X^*(A\times_F E), \Pi_E)$. By the existence theorem there exists a reductive algebraic group $\hat{G}$ and a maximal torus $\hat{A} \subset \hat{G}$ such that the root datum of $(\hat{G}, \hat{A})$ is isomorphic to the root datum $\Psi(G)^\vee := (X_*(A \times_F E), \Pi_E^\vee, X^*(A \times_F E), \Pi_E)$. Observe that $\mathrm{Gal}(E/F)$ acts on the root datum $\Psi(G)$ and consequently, we get a Galois action on the dual root datum $\Psi(G)^\vee$. This will induce an action of $\mathrm{Gal}(E/F)$ on the associated dual group $\hat{G}$ as explained below.

Let $\hat{G}$ and $\hat{A}$ be as in the previous paragraph. Then the construction of the Langlands dual group gives a canonical identification $\eta : \hat{A}(\mathbb{C}) \to (X^*(A\times_F E)\otimes\mathbb{C})^\times$. Let $\Delta \subset \Pi_E$ be the set of simple roots. For $\alpha_i \in \Delta$, choose the vectors $X_{\alpha_i^\vee} \in \mathrm{Lie}(\hat{G})$

[1]Not a reductive group scheme.

such that $\sigma(X_{\alpha_i^\vee}) = X_{\sigma\alpha_i^\vee}$ for every $\sigma \in \mathrm{Gal}(E/F)$. This gives us a pinning $(X_*(A\times_F E), \Pi_E^\vee, X^*(A\times_F E), \Pi_E, \{X_{\alpha^\vee}\}_{\alpha^\vee\in\Delta^\vee})$ of $\hat{G}$ equipped with a $\mathrm{Gal}(E/F)$ action. Since $\mathrm{Gal}(E/F)$ acts on the dual root datum, this action can be lifted to an action on the group $\hat{G}$ using the splitting of the short exact sequence

$$1 \to \mathrm{Inn}(\hat{G}) \to \mathrm{Aut}(\hat{G}) \to \mathrm{Aut}(\Psi(G)^\vee) \to 1$$

provided by the pinning.

2.3. ***Haar measures.*** Let $X_1, \dots, X_{\dim(G)}$ be a basis for the vector space $\mathrm{Lie}(G)$ such that $X_1, \dots, X_{\dim(N)}$ is a basis for $\mathrm{Lie}(N)$, $X_{\dim(N)+1}, \dots, X_{\dim(N)+\dim(A)}$ is a basis of $\mathrm{Lie}(A)$, and finally the remaining vectors form a basis for $\mathrm{Lie}(N^-)$, where G, A, N and N^- are as in Section 2.1. Let $\{X_i^*\}_i$ be the dual basis. Let ω be the unique left G-invariant form which at identity is given by the vector $X_1^* \wedge \cdots \wedge X_{\dim(G)}^*$. We may similarly define ω_N, ω_A, and ω_{N^-} for the respective groups N, A, and N^-. This induces a form ω_v on $G \times_F F_v$. Let $\underline{G}_v$ denote the smooth reductive model of $G \times_F F_v$ over $\mathrm{Spec}(\mathcal{O}_v)$. Denote by $\mathrm{ord}_e(\omega_v)$ the number n such that $(\omega_v)_e(\wedge^{\dim(G)} \mathrm{Lie}(\underline{G}_v)) = \pi_v^n$. The form ω_v defines a left $G(F_v)$-invariant measure on $G(F_v)$ denoted by $\bar{\mu}_{v,\omega}$. For all places $v \notin S$, normalize $\bar{\mu}_{v,\omega}$ as

$$\bar{\mu}_{v,\omega}(G(\mathcal{O}_v)) = \frac{\sharp G(k(v))}{(\sharp k(v))^{\dim(G)+\mathrm{ord}_e(\omega_v)}}$$

(see [Oesterlé 1984, Section 2.5]). For $v \in S$, we refer the reader to [Bourbaki 1971, Section 10.1.6] for the construction of the Haar measure $\bar{\mu}_{v,\omega}$ (denoted by $\mathrm{mod}(\omega_v)$ in Bourbaki) on $G(F_v)$.

We need more preliminaries before defining the Tamagawa measure on $\mathbb{G}$. For $v \notin S$ denote by $L_v(s, X^*(G))$ the local Artin L-function associated to the $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-representation $X^*(G \times_F F^{\mathrm{sep}}) \otimes \mathbb{C}$, where $X^*(G \times_F F^{\mathrm{sep}})$ denotes the group of characters of G defined over F^{sep}. Renormalize the measure $\bar{\mu}_{v,\omega}$ on $G(F_v)$ to $L_v(1, X^*(G))\bar{\mu}_{v,\omega}$, and denote the renormalized measure by $\mu_{v,\omega}$. The unnormalized Tamagawa measure on $\mathbb{G}$ is then defined as the measure $\bar{\mu} := \prod_v \mu_{v,\omega}$. Similarly, on $N(F_v)$, $A(F_v)$ and $N^-(F_v)$, we can define the measures dn_{v,ω_N}, da_{v,ω_A} and $dn^-_{v,\omega_{N^-}}$, respectively, giving the unnormalized Tamagawa measures $\overline{dn}$, $\overline{da}$ and $\overline{dn^-}$, respectively, on $\mathbb{N}$, $A(\mathbb{A})$ and $\mathbb{N}^-$. Since the forms ω and, therefore, ω_N, ω_A, and ω_{N^-} are fixed, we shall omit them without confusion in the discussion henceforth. Thus, for example, dn_{v,ω_N} will be denoted by dn_v.

Let $L^S(s, X^*(G))$ denote the product of local L-functions $L_v(s, X^*(G))$ for $v \notin S$. We normalize $\bar{\mu}$ to

$$\mu := q^{\dim(G)(1-g)} \frac{\bar{\mu}}{\lim_{s\to 1}(1-s)^{\mathrm{rk}\, X^*(G)} L^S(s, X^*(G))},$$

and call it the Tamagawa measure of $\mathbb{G}$. Since μ is a product of measures with a normalization factor, it is true, but not immediately obvious, that this measure is well defined (see [Oesterlé 1984, Definition 4.7]). When G is semisimple, $\mathbb{G}$ is a unimodular group and hence, the measure μ descends to $G(F)\backslash\mathbb{G}$. The Tamagawa number is then defined by

$$\tau(G) := \mathrm{vol}_{\mu}(G(F)\backslash\mathbb{G}).$$

In the sequel, we require the definition of the Tamagawa number of a torus, so we define the Tamagawa number for a general reductive group G here. Consider the kernel $\mathbb{G}_1$ of the homomorphism $\mathbb{G} \xrightarrow{\mathfrak{I}} \hom_{\mathbb{Z}}(X^*(G), q^{\mathbb{Z}})$ defined by the map $g \mapsto (\chi \mapsto \|\chi(g)\|)$ where $g := (g_v) \in \mathbb{G}$. The image of $\mathbb{G}$ under $\mathfrak{I}$ is of finite index (see [Oesterlé 1984, Proposition 5.6]), and the Tamagawa number of G is defined by

$$\tau(G) := \frac{\mathrm{vol}_{\mu}(G(F)\backslash\mathbb{G}_1)}{(\log q)^{\mathrm{rk}\, X^*(G)}[\hom_{\mathbb{Z}}(X^*(G), q^{\mathbb{Z}}) : \mathfrak{I}(\mathbb{G})]}.$$

Henceforth the notation G will denote a semisimple group. Choose a Haar measure on F_v such that $\mathrm{vol}(\mathcal{O}_v) = 1$. Recall that $\overline{da}$ and $\overline{dn}$ are the unnormalized Tamagawa measures on $A(\mathbb{A})$ and $\mathbb{N}$ respectively. Let dk be the unique left invariant (and hence right invariant) Haar measure on $\mathbb{K}$ such that $\mathrm{vol}_{dk}(\mathbb{K}) = 1$. Using the Iwasawa decomposition $\mathbb{G} = \mathbb{N}A(\mathbb{A})\mathbb{K}$, $\rho^{-2}(a)\,\overline{dn}\,\overline{da}\,dk$ is a left invariant Haar measure on $\mathbb{G}$. Thus, there exists a positive constant κ such that

$$\bar{\mu} = \kappa\rho^{-2}(a)\,\overline{dn}\,\overline{da}\,dk.$$

Let w_0 be the longest element of the Weyl group that sends all the positive roots to the negative roots and $\dot{w}_0$ be a representative in $N_G(A)(F)$ such that $\dot{w}_{0v}$ belongs to K_v for all $v \notin S$. Then $N(F_v)A(F_v)\,\dot{w}_0 N(F_v)$ is a dense open subset of $G(F_v)$ and has full measure. Thus, comparing the measures[2] $\mu_{v,\omega}$ and $\rho^{-2}(a)\,dn_v\,da_v\,dn'_v$ we get

$$\mu_{v,\omega} = c_v\rho^{-2}(a)\,dn_v\,da_v\,dn'_v,$$

where $c_v = L_v(1, X^*(G))/L_v(1, X^*(A))$ when $v \notin S$, and $c_v = 1$ otherwise.

2.4. ***Quasicharacters on tori.*** Let A be a quasisplit torus defined over F and $r = F\text{-}\mathrm{rk}(A)$. The results obtained in this section will be applied to the torus $A \subset B \subset G$ chosen in Section 2.1. Recall that such a torus is necessarily quasisplit.

The map

$$\mathfrak{I} : A(\mathbb{A}) \to \hom(X^*(A), q^{\mathbb{Z}}), \quad a \mapsto (\chi \mapsto \|\chi(a)\|)$$

defined in [Oesterlé 1984] induces a map

$$\mathfrak{I}^*_{\mathbb{C}} : X^*(A) \otimes \mathbb{C} \to \hom(A(\mathbb{A})/A(\mathbb{A})_1, \mathbb{C}^\times), \quad \sum_i c_i\chi_i \mapsto \Big(a \mapsto \prod_i \|\chi_i(a)\|^{c_i}\Big).$$

[2] dn'_v is the measure on $N(F_v)$. The prime is meant to differentiate the two appearances of N.

The map $\mathfrak{I}_{\mathbb{C}}^*$ is surjective and $X^*(A) \otimes \frac{2\pi\iota}{\log q}\mathbb{Z}$ is a finite index subgroup of $\ker(\mathfrak{I}_{\mathbb{C}}^*)$. Both these assertions follow from the existence of the commutative diagram

$$\begin{array}{ccc} X^*(A) \otimes \mathbb{C} & \xrightarrow{\mathfrak{I}_{\mathbb{C}}^*} & \hom(A(\mathbb{A})/A(\mathbb{A})_1, \mathbb{C}^\times) \\ \downarrow \wr & & \wr \downarrow \\ X^*(A_d) \otimes \mathbb{C} & \xrightarrow[\mathfrak{I}_{\mathbb{C}}^*]{} & \hom(A_d(\mathbb{A})/A_d(\mathbb{A})_1, \mathbb{C}^\times) \end{array} \tag{1}$$

where the vertical arrows are induced by the inclusion $A_d \subset A$. The right vertical arrow is an isomorphism since the obvious inclusion $A_d(\mathbb{A})/A_d(\mathbb{A})_1 \hookrightarrow A(\mathbb{A})/A(\mathbb{A})_1$ is an isomorphism. This follows from the fact that the anisotropic part of the torus is contained in $A(\mathbb{A})_1$. The left vertical arrow is an isomorphism since the torus A is quasisplit, which implies that the map $X^*(A) \to X^*(A_d)$ is injective and the image is of finite index. Because the kernel of the bottom arrow in (1) is known to be $X^*(A_d) \otimes \frac{2\pi\iota}{\log q}\mathbb{Z}$ (see [Harder 1974]), $X^*(A) \otimes \frac{2\pi\iota}{\log q}\mathbb{Z}$ is a finite index subgroup of the kernel of the top arrow. The induced map on the quotient is again denoted by $\mathfrak{I}_{\mathbb{C}}^*$:

$$X^*(A) \otimes \mathbb{C} \Big/ \left(X^*(A) \otimes \frac{2\pi\iota}{\log q}\mathbb{Z} \right) \xrightarrow{\mathfrak{I}_{\mathbb{C}}^*} \hom(A(\mathbb{A})/A(\mathbb{A})_1, \mathbb{C}^\times). \tag{2}$$

Fix a coordinate system on $X^*(A) \otimes \mathbb{C}/\big(X^*(A) \otimes \frac{2\pi\iota}{\log q}\mathbb{Z}\big)$ as follows. Let $\{\varpi_i\}$ be the fundamental weights of the group G. Denote by $[\varpi_i]$ the sum over $\mathrm{Gal}(E/F)$-orbit of ϖ_i. Since G is assumed to be simply connected we have the equality $X^*(A \times_F F^{\mathrm{sep}}) = \bigoplus_i \mathbb{Z}\varpi_i$. Moreover, G is quasisplit and hence by Lemma B.1 A is a quasisplit torus. Now, using [Oesterlé 1984, Theorem 2.4] we get that $[\varpi_i]$ is a $\mathbb{Z}$-basis of $X^*(A)$. The above choice of coordinate system induces the isomorphism

$$\mathbb{Z}^r \xrightarrow{\xi} X^*(A). \tag{3}$$

A small computation shows that $\xi(1, 1, \ldots, 1) = \rho$. To a quasicharacter given by $\lambda \in X^*(A) \otimes \mathbb{C}$, we fix the notation $(\lambda_1, \ldots, \lambda_r) = \xi^{-1}(\lambda)$.

For a quasicharacter $\lambda \in \Lambda(A)$ define $\Re\lambda(t) := |\lambda(t)| \in \mathbb{R}$ and

$$\Lambda_\sigma(A) := \{\lambda \in \Lambda(A) \mid \Re(\lambda) = \sigma\}.$$

The latter is a translate of $\Lambda_0(A)$ which is the Pontryagin dual of $A(F)\backslash A(\mathbb{A})$. Equip $\Lambda_0(A)$ with the Haar measure $d\lambda$ that is dual to the measure on $A(F)\backslash A(\mathbb{A})$ induced by $\overline{da}$. The measure on $\Lambda_\sigma(A)$ is then the unique left $\Lambda_0(A)$-invariant measure such that the volume remains the same. We fix this measure for the future computations.

2.4.1. *Comparison of measures on quasicharacters.* The short exact sequence

$$1 \to A(F)\backslash A(\mathbb{A})_1 \to A(F)\backslash A(\mathbb{A}) \to A(\mathbb{A})/A(\mathbb{A})_1 \to 1$$

of locally compact abelian groups gives the exact sequence

$$1 \to \hom(A(\mathbb{A})/A(\mathbb{A})_1, S^1) \to \Lambda_0(A) \to \hom(A(F)\backslash A(\mathbb{A})_1, S^1) \to 1.$$

Since the last term is discrete we get that $\hom(A(\mathbb{A})/A(\mathbb{A})_1, S^1) = \Lambda_0(A)^\circ$. The pullback of the measure $d\lambda|_{\Lambda_0(A)^\circ}$ along the map $\mathfrak{I}^*_{\mathbb{C}}$, denoted by $d\lambda|_V$, where

$$V = \frac{X^*(A) \otimes \mathbb{R}}{2\pi/\log(q) X^*(A)},$$

can be compared with the dual measure on $X^*(A) \otimes \mathbb{R}$. Arguing as in [Lai 1980, Lemma 6.7] we get the following:

Lemma 2.1.

$$d\lambda|_V = \frac{[\hom(X^*(A), q^{\mathbb{Z}}), \operatorname{im} \mathfrak{I}]}{\operatorname{vol}_{\overline{da}}(A(F)\backslash A(\mathbb{A})_1)} \left(\frac{\log q}{2\pi}\right)^r dz_1 \wedge \cdots \wedge dz_r.$$

Proof. Recall the map $\mathbb{C}^r \xrightarrow{\xi} X^*(A) \otimes \mathbb{C}$ in (3) giving the isomorphism $\mathbb{C}^r / \frac{2\pi}{\log q}\mathbb{Z}^r \simeq X^*(A) \otimes \mathbb{C} / \frac{2\pi}{\log q} X^*(A)$. Equip the latter space with the measure that assigns mass 1 to the fundamental domain $X^*(A) \otimes \mathbb{R} / \frac{2\pi}{\log q} X^*(A)$, which under the above isomorphism equals $\left(\frac{\log q}{2\pi}\right)^r dz_1 \wedge \cdots \wedge dz_r$. Denote by $\mathfrak{I}^\vee : (\hom(X^*(A), q^{\mathbb{Z}}))^\vee \to \hom(A(\mathbb{A})/A(\mathbb{A})_1, S^1)$ the map induced by $\mathfrak{I}$ on the Pontryagin dual. We get the short exact sequence

$$1 \to (\hom(X^*(A), q^{\mathbb{Z}})/\operatorname{im} \mathfrak{I})^\vee \to (\hom(X^*(A), q^{\mathbb{Z}}))^\vee \xrightarrow{\mathfrak{I}^\vee} \hom(A(\mathbb{A})/A(\mathbb{A})_1, S^1) \to 1.$$

The term in the middle is isomorphic to $X^*(A) \otimes \mathbb{R} / \frac{2\pi}{\log q} X^*(A)$ and the first term is abstractly isomorphic to $\hom(X^*(A), q^{\mathbb{Z}})/\operatorname{im} \mathfrak{I}$ since it is finite. Note that the quotient measure on $A(\mathbb{A})/A(\mathbb{A})_1$ is $\operatorname{vol}_{\overline{da}}(A(F)\backslash A(\mathbb{A})_1)$ times the counting measure and hence, the dual measure $d\lambda$ assigns the mass $1/\operatorname{vol}_{\overline{da}}(A(F)\backslash A(\mathbb{A})_1)$ to $\hom(A(\mathbb{A})/A(\mathbb{A})_1, S^1)$. The pullback of this measure along $\mathfrak{I}^*_{\mathbb{C}}$ is a Haar measure which assigns mass $\frac{|\hom(X^*(A), q^{\mathbb{Z}})/\operatorname{im} \mathfrak{I}|}{\operatorname{vol}_{\overline{da}}(A(F)\backslash A(\mathbb{A})_1)}$ to $(\hom(X^*(A), q^{\mathbb{Z}}))^\vee$, whereas the Haar measure $\left(\frac{\log q}{2\pi}\right)^r dz_1 \wedge \cdots \wedge dz_r$ assigns it mass 1. Hence the claim. □

3. Determining the Tamagawa numbers

3.1. ***Eisenstein series.*** Let $\varphi : \mathbb{N}B(F)\backslash \mathbb{G}/\mathbb{K} \to \mathbb{C}$ be a compactly supported measurable function. Let φ_v denote the local components of φ such that $\varphi = \prod_v \varphi_v$. For any $\lambda \in \Lambda(A)$ the Fourier transform is defined by

$$\hat{\varphi}(\lambda)(g) := \int_{A(F)\backslash A(\mathbb{A})} \varphi(ag)\lambda^{-1}(a)\rho^{-1}(a)\, \overline{da}.$$

Let $\hat{\varphi}(\lambda)_v$ denote the restriction of $\hat{\varphi}(\lambda)$ to $G(F_v)$. Then for $g = (g_v)_v$ we have $\hat{\varphi}(\lambda)(g) = \prod_v \hat{\varphi}(\lambda)_v(g_v)$. Note that $\hat{\varphi}(\lambda)(g)$ is determined by its value at 1 and we denote this value simply by $\hat{\varphi}(\lambda)$. On applying Fourier inversion

$$\varphi(g) = \int_{\Re(\lambda)=\lambda_0} \hat{\varphi}(\lambda)(g)\, d\lambda,$$

with λ_0 such that the composite $F^*\backslash\mathbb{A}^* \xrightarrow{\alpha^\vee} A_d(F)\backslash A_d(\mathbb{A}) \xrightarrow{\lambda_0} \mathbb{R}^\times$ given by $|\cdot|^{s_\alpha}$ satisfies $s_\alpha > 1$ (see footnote [3]) for any coroot $\alpha^\vee$. For φ as above define the theta series

$$\theta_\varphi(g) := \sum_{\gamma \in B(F)\backslash G(F)} \varphi(\gamma g).$$

The above series converges uniformly on compact subsets of $G(F)\backslash\mathbb{G}$ (see [Morris 1982a, Section 2.3]). In fact, the support of θ_φ is compact and hence, it is in $L^2(G(F)\backslash\mathbb{G})$. Define

$$E(g, \hat{\varphi}(\lambda)) := \sum_{\gamma \in B(F)\backslash G(F)} \hat{\varphi}(\lambda)(\gamma g).$$

We have

$$\begin{aligned}\theta_\varphi(g) = \sum_{\gamma \in B(F)\backslash G(F)} \varphi(\gamma g) &= \sum_{\gamma \in B(F)\backslash G(F)} \int_{\Re(\lambda)=\lambda_0} \hat{\varphi}(\lambda)(\gamma g)\, d\lambda \\ &\overset{*}{=} \int_{\Re(\lambda)=\lambda_0} \sum_{\gamma \in B(F)\backslash G(F)} \hat{\varphi}(\lambda)(\gamma g)\, d\lambda \\ &= \int_{\Re(\lambda)=\lambda_0} E(g, \hat{\varphi}(\lambda))\, d\lambda.\end{aligned}$$

The assumption on λ_0 is used in the equality marked with $*$ above. The Eisenstein series $E(g, \hat{\varphi}(\lambda))$ is a priori defined on the domain $\lambda \in X^*(A)\otimes\mathbb{C}/\big(X^*(A)\otimes \frac{2\pi\iota}{\log q}\mathbb{Z}\big)$ and $\Re(\lambda) - \rho \in C$,[4] but can be continued meromorphically to all of $X^*(A) \otimes \mathbb{C}/\big(X^*(A) \otimes \frac{2\pi\iota}{\log q}\mathbb{Z}\big)$.[5]

For $w \in W_F$, let $\dot{w}$ denote a lift to $N_G(A)(F)$. Set

$${}^{\dot{w}}N = \dot{w}N\dot{w}^{-1} \quad \text{and} \quad N^{\dot{w}} = \dot{w}N^-\dot{w}^{-1} \cap N.$$

Recall the definition of the local and global intertwining operators,

$$(M_v(w,\lambda)\,\hat{\varphi}(\lambda)_v)(g_v) = \int_{N^{\dot{w}}(F_v)} \hat{\varphi}(\lambda)_v(\dot{w}n_v g_v)\, dn_v \quad \text{for any } g_v \in G(F_v), \tag{4}$$

$$\begin{aligned}(M(w,\lambda)\,\hat{\varphi}(\lambda))(g) &= \int_{{}^{\dot{w}}N(F)\cap N(F)\backslash\mathbb{N}} \hat{\varphi}(\lambda)(\dot{w}ng)\, \overline{dn} \\ &= \mathrm{vol}({}^{\dot{w}}N(F)\cap N(F)\backslash({}^{\dot{w}}\mathbb{N}\cap\mathbb{N})) \int_{\dot{w}\mathbb{N}\dot{w}\cap\mathbb{N}\backslash\mathbb{N}} \hat{\varphi}(\lambda)(\dot{w}ng)\, \overline{dn} \\ &= \mathrm{vol}({}^{\dot{w}}N(F)\cap N(F)\backslash({}^{\dot{w}}\mathbb{N}\cap\mathbb{N})) \int_{\mathbb{N}^{\dot{w}}} \hat{\varphi}(\lambda)(\dot{w}ng)\, \overline{dn}.\end{aligned} \tag{5}$$

[3] Note that $|\lambda| : A_d(F)\backslash A_d(\mathbb{A}) \to \mathbb{R}^{>0}$ factors through the quotient $A_d(\mathbb{A})/A_d(F)A_d(\mathbb{A})_1$ since $A_d(F)\backslash A(\mathbb{A})_1$ is a compact group. Therefore λ_0 is defined by an element of $X^*(A_d)\otimes\mathbb{R}$ and hence the corresponding real numbers s_α defined above determines λ_0.

[4] C is the positive Weyl chamber; or in other words $(\Re(\lambda), \alpha) > (\alpha, \rho)$.

[5] The Eisenstein series can be extended to other connected components as well.

The last equality in (5) follows from

$$^{\dot{w}}N(F) \cap N(F)\backslash \mathbb{N} = {}^{\dot{w}}N(F) \cap N(F)\backslash ({}^{\dot{w}}\mathbb{N} \cap \mathbb{N})\mathbb{N}^{\dot{w}},$$

and the left $\mathbb{N}$-invariance of $\hat{\varphi}(\lambda)$. Observe that for $w = w_0$ the group ${}^{\dot{w}}\mathbb{N} \cap \mathbb{N}$ is trivial and hence combining (4) and (5) we get

$$M(w_0, \lambda) = \prod_v{}' M_v(w_0, \lambda).$$

Note that $M(w, \lambda)(\hat{\varphi}(\lambda))$ is right $\mathbb{K}$-invariant, left $\mathbb{N}$-invariant and transforms via λ on $A(\mathbb{A})$. Hence $M(w, \lambda)(\hat{\varphi}(\lambda))$ is a scalar multiple of $\hat{\varphi}(\lambda)$. The operator $M(w, \lambda)$ can thus be confused with the scalar. We record here the following proposition and a lemma to be used in the later sections.

Proposition 3.1 [Morris 1982a, Section 3.1, Lemma]. *The constant term of the Eisenstein series is given by the following formula*:

$$E^B(g) := \int_{N(F)\backslash \mathbb{N}} E(ng, \hat{\varphi}(\lambda))\, \overline{dn} = \sum_{w \in W_F} (M(w, \lambda)\, \hat{\varphi}(\lambda))(g).$$

Lemma 3.2.

$$(\theta_\varphi, 1) = \int_{B(F)\backslash \mathbb{G}} \varphi(g)\, \overline{d\mu} = \kappa \int_{N(F)\backslash \mathbb{N}} \int_{A(F)\backslash A(\mathbb{A})} \int_K \varphi(nak)\, \rho^{-2}(a)\, \overline{dn}\, \overline{da}\, dk$$
$$= \kappa q^{-\dim(N)(1-g)} \hat{\varphi}(\rho).$$

3.2. ***Intertwining operators.*** The aim of this subsection is to prove the following:

Theorem 3.3. *The intertwining operator $M(w_0, \lambda)$ has a simple pole along each of the hyperplanes $\lambda_i = 1$ in the region $1 - \epsilon < (\Re\lambda)_i < 1 + \epsilon$ for some $\epsilon > 0$. In particular*, *$M(w_0, s\rho)$ has a pole of order F-rk(G) at $s = 1$.*

We carry forward the strategy of the proof used in [Lai 1980; Rapoport 1976] for our case. Below, we rework all the details for the case over function fields. The strategy is to reduce the calculation of the integrals defining certain intertwining operators to the case of quasisplit semisimple simply connected rank 1 groups. This is accomplished via the Bhanu Murthy–Gindikin–Karpelevitch formula. The theorem in the case of quasisplit semisimple simply connected F-rank 1 groups is then proved by explicit computations. This result is summarized in Section 3.2.2.1 and follows from the computations of the local intertwining operators in Section 3.2.1.3. Theorem 3.3 is crucial for the proof of the main theorem as explained in Section 1.1. We remark here that, unlike the strategy followed in this article, Harder [1974, Section 2.3] explicitly computes an expression for the Eisenstein series and concludes Theorem 3.3 as a corollary of his results.

3.2.1. *Local intertwining operators.* We will require the computation of the local intertwining operators $M_v(w_0, \rho)$ for the ramified and unramified places of F.

3.2.1.1. *$M_v(w_0,\rho)$ for ramified places.* Let G, F and S be as in Section 2. Thus, for any $v\notin S$ the group $G\times_F F_v$ splits over an unramified extension of F_v. Let $\dot w_0$ denote a representative in $N_G(A)(F)$ of the longest Weyl group element $w_0\in W_F$ as in Section 2.3. Let $\mathbb{A}_S$ denote the ring of adèles over F with trivial component outside S. For $n\in N(\mathbb{A}_S)$ we write the Iwasawa decomposition of $\dot w_0 n$ as

$$\dot w_0 n = n_1(n)\,a(n)\,k(n)\in N(\mathbb{A})A(\mathbb{A})\mathbb{K}.$$

Proposition 3.4 ([Lai 1980, Section 5.2] for number fields). *For any finite set S' containing the set of ramified places, let $M_{S'}(w_0,\rho)=\prod_{v\in S'}M_v(w_0,\rho)$. Then*

$$M_{S'}(w_0,\rho)=\int_{(\dot w_0 N)(\mathbb{A}_{S'})}|\rho|^2(a(n))\,\overline{dn}=\kappa\Big(\prod_{v\notin S'}\mathrm{vol}(K_v)\Big)^{-1}\Big(\prod_{v\in S'}c_v\Big).$$

Proof. Let f be a right $\mathbb{K}$-invariant function on $\mathbb{G}$ defined for any $g=nak\in\mathbb{G}$ as

$$f(g)=\begin{cases}0 & \text{if } g_v\notin K_v \text{ for some } v\notin S',\\ h(n_{S'},a_{S'}) & \text{otherwise, where } n_{S'}=(n_v)_{v\in S'},\ a_{S'}=(a_v)_{v\in S'},\\ & \text{and } h: N(\mathbb{A}_{S'})\times A(\mathbb{A}_{S'})\to\mathbb{R} \text{ is any integrable function.}\end{cases}$$

Using the equality $\bar\mu=\kappa|\rho|^{-2}(a)\,\overline{dn}\,\overline{da}\,dk$ we get

$$\begin{aligned}\int_{\mathbb{G}}f(g)\bar\mu&=\kappa\int_{\mathbb{N}A(\mathbb{A})\mathbb{K}}f(nak)|\rho|^{-2}(a)\,\overline{dn}\,\overline{da}\,dk\\&=\kappa\int_{N(\mathbb{A}_{S'})A(\mathbb{A}_{S'})}h(n_{S'},a_{S'})|\rho|^{-2}(a)\,\overline{dn}\,\overline{da}.\end{aligned}$$

The largest Bruhat cell $B\dot w_0 N$ has full measure with respect to $\bar\mu$ and hence the left-hand side of the above integral equals the integral on this cell. Using the Iwasawa decomposition of $\dot w_0 n'\in\dot w_0 N$ in the Bruhat decomposition of g we get $na\dot w_0 n'=nan_1(n')\,a(n')\,k(n')$. In the following we omit the subscript S' in the integrand for convenience. We further let

$$\kappa':=\Big(\prod_{v\notin S'}\mathrm{vol}(K_v)\Big)\Big(\prod_{v\in S'}c_v\Big).$$

Then

$$\begin{aligned}(6)\quad &\int_{\mathbb{B}\dot w_0\mathbb{N}}f(g)\bar\mu\\&=\kappa'\int_{B(\mathbb{A}_{S'})\,\dot w_0\,N(\mathbb{A}_{S'})}f\big(nan_1(n')\,a^{-1}aa(n')\,k(n')\big)|\rho|^{-2}(a)\,\overline{dn}\,\overline{da}\,\overline{dn'}\\&=\kappa'\int_{\dot w_0 N(\mathbb{A}_{S'})}\int_{A(\mathbb{A}_{S'})}\int_{N(\mathbb{A}_{S'})}h\big(n(an_1(n')\,a^{-1}),aa(n')\big)|\rho|^{-2}(a)\,\overline{dn}\,\overline{da}\,\overline{dn'}\\&=\kappa'\int_{\dot w_0 N(\mathbb{A}_{S'})}\int_{A(\mathbb{A}_{S'})}\int_{N(\mathbb{A}_{S'})}h(n,aa(n'))|\rho|^{-2}(a)\,\overline{dn}\,\overline{da}\,\overline{dn'}.\end{aligned}$$

The last equality follows from the right invariance of the Haar measure $\overline{dn}$ on $\mathbb{N}$. Now using the right invariance of $\overline{da}$ we obtain

$$\begin{aligned}(6) &= \kappa' \int_{\dot{w}_0 N(\mathbb{A}_{S'})} \int_{A(\mathbb{A}_{S'})} \int_{N(\mathbb{A}_{S'})} h(n,a)|\rho|^{-2}(a)|\rho|^2(a(n'))\,\overline{dn}\,\overline{da}\,\overline{dn'} \\ &= \kappa' \int_{\dot{w}_0 N(\mathbb{A}_{S'})} |\rho|^2(a(n'))\,\overline{dn'} \int_{A(\mathbb{A}_{S'})} \int_{N(\mathbb{A}_{S'})} h(n,a)|\rho|^{-2}(a)\,\overline{dn}\,\overline{da}.\end{aligned}$$

Since h can be chosen such that the integral is nonzero, comparing the right-hand side of the two equations implies

$$\kappa' \int_{\dot{w}_0 N(\mathbb{A}_{S'})} |\rho|^2(a(n'))\,\overline{dn'} = \kappa.$$

Thus, we get the desired result

$$\int_{\dot{w}_0 N(\mathbb{A}_{S'})} |\rho|^2(a(n'))\,\overline{dn'} = \kappa \Big(\prod_{v\notin S'} \operatorname{vol}(K_v)\Big)^{-1} \Big(\prod_{v\in S'} c_v\Big). \qquad \square$$

3.2.1.2. *$M_v(w_0, \rho)$ for unramified places.*

Corollary 3.5. *Suppose $v \notin S$. Then*

$$M_v(w_0,\rho) = \int_{\dot{w}_{0v} N_v} |\rho|^2 a(n_v)\,dn_v = \operatorname{vol}(K_v)\frac{L_v(1, X^*(A))}{L_v(1, X^*(G))}.$$

The corollary simply follows from the proposition above by expanding the set S to $S' = S \cup \{v\}$. We record here a lemma regarding the holomorphicity of the local intertwining operators and refer the reader to [Lai 1980, Lemma 6.9] for a proof.

Lemma 3.6. *The intertwining operators $M_v(w_0, \lambda)$ are holomorphic for $\lambda \in \Lambda_0(A)$ and $\Re(\lambda) \geq 1$.*

3.2.1.3. *Local intertwining operators for rank 1 groups.* We continue to denote a nonarchimedean local field by F_v. Let G be a semisimple group defined over F_v with F_v-rank 1 which splits over an unramified extension E_v of F_v. Let $\dot{w}_{0v} \in K_v$ be a representative of the longest Weyl group element $w_0 \in W_F$ as in Section 2.3. Using the right K_v-invariance of $\hat{\varphi}(\lambda)$, the integral in (4) is equal to

$$\int_{N(F_v)} \hat{\varphi}(\lambda)(\dot{w}_{0v}\, n_v\, \dot{w}_{0v}^{-1})\,dn_v = \int_{N^-(F_v)} \hat{\varphi}(\lambda)(n_v)\,dn_v^-.$$

This integral depends only on the choice of the Haar measure on the group $N^-(F_v)$, which is taken to be the Tamagawa measure on $N^-(F_v)$. Theorem 3.10 will allow us to reduce the computation of the local intertwining operators to the case of F-rank 1 groups. Here we will give an explicit computation of $M_v(w_0, \lambda)$ for F-rank 1 groups.

Fix $\lambda \in X^*(A)\otimes\mathbb{C}$ and suppose that $\hat{t} \in \hat{A}$ is such that for any $\mu \in X_*(A\times_{F_v} E_v)$ the equality $\hat{t}(\mu) = |\pi_v|^{\langle\lambda,\mu\rangle}$ holds. Let $\hat{\mathfrak{u}}$ be the Lie subalgebra of $\mathrm{Lie}(\hat{G})$ corresponding to the unipotent radical N.

Theorem 3.7 ([Rapoport 1976, Theorem 4.2.1] for local fields of characteristic 0). *Suppose E_v/F_v is the unramified extension that splits G and let $\sigma \in \mathrm{Gal}(E_v/F_v)$ be the Frobenius element. Then*

$$M_v(w_0, \lambda) = \frac{\det(I - |\pi_v|\,\mathrm{Ad}(\sigma\hat{t})|_{\hat{\mathfrak{u}}})}{\det(I - \mathrm{Ad}(\sigma\hat{t})|_{\hat{\mathfrak{u}}})}. \tag{7}$$

Proof. The proof will be done in three steps by verifying the above formula first for semisimple rank 1 groups and then for higher-rank groups.

Step 1. The theorem is true in the case of absolutely simple simply connected groups of semisimple F_v-rank 1. We quote the results of Rapoport and Lai below.

Proposition 3.8 [Rapoport 1976, Section 4.4(a)]. *For the group SL_2 the intertwining operator $M_v(w_0, s\rho)$ is given by*

$$M_v(w_0, s\rho) = \frac{(1-q^{-s-1})}{(1-q^{-s})}.$$

Proposition 3.9 [Lai 1980, Proposition 3.4]. *Let E_v/F_v be a quadratic unramified extension of F_v and $\mathrm{SU}(3, E_v/F_v)$ be the quasisplit group defined over F_v. Suppose 2 is invertible in F_v. Then*

$$M_v(w_0, s\rho) = \frac{(1-q^{-2s-2})(1+q^{-2s-1})}{(1-q^{-2s})(1+q^{-2s})}.$$

Step 2. If the theorem is true for G then it is true for any central isogeny $\widetilde{G} \to G$.

Let $\widetilde{G} \to G$ be a central isogeny. The notation $\sim$ will denote the corresponding objects for $\widetilde{G}$. It is clear that the right-hand side of (7) is the same for $\widetilde{G}$ and G. Further, the isogeny induces the isomorphisms

$$\widetilde{W}_{F_v} \xrightarrow{\sim} W_{F_v} \quad\text{and}\quad X^*(A)\otimes\mathbb{C} \xrightarrow{\sim} X^*(\tilde{A})\otimes\mathbb{C},$$

where the image of ρ under the latter isomorphism is $\tilde{\rho}$. Also, the images of $\widetilde{N}$, $\tilde{A}$ and $\widetilde{K}$ are N, A and K respectively; and $\widetilde{N}^- \xrightarrow{\sim} N^-$. Thus the image of $\tilde{n}\tilde{a}\tilde{k}$ maps to nak which is the Iwasawa decomposition.

Step 3. Let $G = \mathrm{Res}_{E_v'/F_v} G'$ for a quasisplit simply connected semisimple group G' defined over $E_v' \subset E_v$ which splits over E_v and let the degree of the unramified extension E_v'/F_v be n. If the theorem is true for G' then it is true for G.

Alphabets with superscript $'$ will denote the corresponding objects for the group G'. The Weyl groups $W'_{E_v'}$ and W_{F_v} are the same. Also, if $A = R_{E_v'/F_v}(A')$ we can identify $\hat{A}$ with $\prod_{\mathrm{Gal}(E_v'/F_v)} \widehat{A'}$. We have $\hat{\mathfrak{u}} = \prod_{\mathrm{Gal}(E_v'/F_v)} \widehat{\mathfrak{u}'}$. Since $\lambda \in X^*(A)\otimes\mathbb{C}$

we get that $\hat{t} \in \hat{A}$ is mapped to a diagonal element $\operatorname{diag}(\widehat{t'}, \widehat{t'}, \ldots, \widehat{t'}) \in \prod_{\mathrm{Gal}(E'_v/F_v)} \widehat{A'}$ under the identification above:

$$I - \mathrm{Ad}(\sigma\hat{t}) = \begin{pmatrix} I & & & & -\mathrm{Ad}(\sigma'\widehat{t'}) \\ -\mathrm{Ad}(\sigma'\widehat{t'}) & I & & & \\ & -\mathrm{Ad}(\sigma'\widehat{t'}) & I & & \\ & & \ddots & \ddots & \\ & & & -\mathrm{Ad}(\sigma'\widehat{t'}) & I \end{pmatrix}.$$

Note that $\mathrm{Ad}(\hat{t})$ is a diagonal matrix. Then

$$\begin{aligned} \det(I - \mathrm{Ad}(\sigma\hat{t})) &= \det(I - (\mathrm{Ad}(\sigma'\widehat{t'}))^n|_{\widehat{\mathfrak{u}'}}) = \det(I - (\mathrm{Ad}(\sigma'\widehat{t'})^n)|_{\widehat{\mathfrak{u}'}}), \\ \det(I - |\pi_v|\,\mathrm{Ad}(\sigma\hat{t})|_{\hat{\mathfrak{u}}}) &= \det(I - |\pi'_v|'\,\mathrm{Ad}(\sigma'\widehat{t'^n})|_{\widehat{\mathfrak{u}'}}), \end{aligned} \tag{8}$$

where $|\cdot|'$ is the absolute value on E'_v. The left-hand side of (7) for G is equal to

$$\begin{aligned} &\int_{(N^-)^{w_0}(F_v)} |\lambda|(a(n^-)) \cdot |\rho|(a(n^-))\, dn^- \\ &\qquad = \int_{(N^{-'})^{w_0}(E'_v)} |N_{E'_v/F_v}\,\lambda'| \cdot |N_{E'_v/F_v}\,\rho'|(a'(n^{-'}))\, dn^{-'} \\ &\qquad = \int_{(N^{-'})^{w_0}(E'_v)} |\lambda'|'\,|\rho'|'(a'(n^{-'}))\, dn^{-'} = M'_v(w_0, \lambda'), \end{aligned}$$

where $\lambda' \in X^*(A') \otimes \mathbb{C}$ corresponds to $\lambda \in X^*(A) \otimes \mathbb{C}$ under the identification $X^*(A') \simeq X^*(A)$ via the norm map. Finally, the equality $M_v(w_0, \lambda) = M_{v'}(w_0, \lambda')$ along with (8) and Lemma A.1 proves that (7) holds for G. □

3.2.1.4. *Formula of Bhanu Murthy, Gindikin and Karpelevitch.* Let F_v be a nonarchimedean local field, G a semisimple linear algebraic group defined over F_v, P_0 (can take it to be B in our case) a minimal parabolic over F_v, and N_0 be its unipotent radical. Let A_0 be the maximal torus contained in P_0, defined over F_v, and let K be a special maximal compact subgroup of G which satisfies $P_0(F_v)K = G(K_v)$. Denote by $X^*(A_0)$ the set of characters of A_0 defined over F_v. Let $G(\alpha) \subset G$ be the semisimple group of rank 1 whose Lie algebra is generated by the radical space corresponding to the roots that are multiples of α (here α is indivisible). Note that this is the same as the derived subgroup of the centralizer of $\ker(\alpha)^0_{\mathrm{red}}$. It is known that this will be a connected simply connected smooth group scheme of rank 1 [Conrad 2015, Corollary 9.5.11]. Define $P_0(\alpha) := P_0 \cap G(\alpha)$, then $N_0(\alpha)$, $A_0(\alpha)$ and $K(\alpha)$ can be defined similarly sharing the same properties as N_0, A_0 and K, but with respect to the group $G(\alpha)$. There is a unique opposite parabolic subgroup $P_0(-\alpha)$ which we denote by $\bar{P}_0(\alpha)$. Let $N^-(\alpha)$ be the unipotent radical of the opposite parabolic.

Let P be a parabolic defined over F_v such that $P \supset P_0$, and let N be the unipotent radical of P. Let $\Pi'_+(P)$ be the set of indivisible roots α such that $N(\alpha) \subset N$. We denote by N^- the opposite unipotent group to N. For $\lambda \in X^*(A_0) \otimes \mathbb{C}$, denote by Φ^λ the function over $G(F_v)$ that associates to the element $g_v = n_v.a_v.k_v \in G(F_v)$ the complex number $\Phi^\lambda(g_v) = |\lambda|(a_v)|\rho|(a_v)$. Let $\lambda(\alpha)$ denote the projection of $\lambda \in X^*(A_0) \otimes \mathbb{C}$ under the map $X^*(A_0) \otimes \mathbb{C} \to X^*(A_0(\alpha)) \otimes \mathbb{C}$ induced by the natural inclusion $A_0(\alpha) \subset A_0$. We state the following:

Theorem 3.10. *The integral*

$$\int_{N^-(F_v)} \Phi^\lambda(n_v^-)\, dn_v^-$$

converges for any $\lambda \in X^*(A_0) \otimes \mathbb{C}$ *with* $\mathrm{Re}(\langle \lambda, \alpha^\vee \rangle) > 0$ *for all* $\alpha \in \Pi'_+(P)$. *There exists a constant depending on the choice of Haar measure and up to this constant the value is*

$$\prod_{\alpha \in \Pi'_+(P)} \int_{N^-(\alpha)(F_v)} \Phi^{\lambda(\alpha)}(n_v^-)\, dn_v^-.$$

If the semisimple group G is the local place of a semisimple group defined over a global field and if the Haar measure is deduced from the Tamagawa measure then this constant is 1 *for almost all places* v.

As an application of the above we have a straightforward generalization of Proposition 3.9. We refer the interested readers to [Lai 1980].

3.2.2. *Proof of Theorem 3.3.* The proof of theorem will be completed in two steps. The first step is via explicit computations for F-rank 1 groups, and the second step is using the method of Bhanu Murthy and Gindikin–Karpelevitch for reduction of higher-rank case to that of rank 1 groups.

3.2.2.1. *Case of rank 1 groups.* The four cases of quasisplit simply connected groups of F-rank 1 are as described below. The local intertwining operators at the unramified places are known from Theorem 3.7. Since the group is ramified at finitely many places and the local intertwining operators are holomorphic in the region $1-\epsilon < s < 1+\epsilon$ (Lemma 3.6), we may put them together as a single function which do not contribute to the poles. A calculation using the explicit expressions of the intertwining operators now enables us to identify the global operator with a product of zeta functions and certain meromorphic functions $f_1(s)$, $f_2(s)$, $f_3(s)$, and $f_4(s)$ of $s \in \mathbb{C}$, which are holomorphic near $s = 1$. This is summarized below.

(1) For $G = \mathrm{SL}_2$,

$$M(w_0, s\rho) = \frac{\zeta_F(s)}{\zeta_F(s+1)} = \zeta_F(s) f_1(s).$$

(2) For $G = \mathrm{SU}(3, E/F)$, where E is a quadratic extension of F,

$$M(w_0, s\rho) = \zeta_E(s) f_2(s).$$

(3) For $G = \mathrm{Res}_{E'/F}(\mathrm{SL}_2)$, by the proof of Step 3 in Theorem 3.7,

$$M(w_0, s\rho) = \frac{\zeta_E(s)}{\zeta_E(s+1)} = \zeta_E(s) f_3(s).$$

(4) For $G = \mathrm{Res}_{E'/F}\,\mathrm{SU}(3, E/F)$, where E is a quadratic extension of E',

$$M(w_0, s\rho) = \zeta_{E'}(s) f_4(s).$$

It is clear from the above list that the theorem holds for the F-rank 1 groups.

3.2.2.2. *Case of higher-rank groups.* We denote by $M^{G(\alpha)}(w_0, \lambda)$ the intertwining operator for the F-rank 1 semisimple simply connected group $G(\alpha) \subset G$ where w_0 is the largest element in the Weyl group of $G(\alpha)$. Writing $\lambda = (s_1, s_2, \ldots, s_r)$ in the coordinate system given by ξ (see (3)), Theorem 3.10 implies the following equality up to a scalar:

$$M(w_0, \lambda) = \prod_{\substack{\alpha_i \text{ positive} \\ \text{and simple}}} M^{G(\alpha)}(w_0, \lambda|_{G(\alpha)}) \prod_{\substack{\alpha \text{ positive, indivisible} \\ \text{and nonsimple}}} M^{G(\alpha)}(w_0, \lambda|_{G(\alpha)}).$$

For s_i in the region $1 - \epsilon < s_i < 1 + \epsilon$, the poles of $M(w_0, (s_i))$ are determined by the poles of the operators on the right-hand side. In the case when $\alpha = \alpha_i$ is a positive simple root then $\lambda|_{G(\alpha_i)} = s_i$. If α is not a simple root then $\Re(\lambda|_{G(\alpha)})$ lies outside the domain $(1 - \epsilon, 1 + \epsilon)$.

Note that $G(\alpha)$ is isomorphic to one of the four cases discussed above up to central isogeny. Reading the poles of the intertwining operators on the right-hand side from the list for rank 1 groups, we can see that $M(w_0, \lambda)$ has simple poles along the hyperplanes $s_i = 1$ when $(\Re(s_1), \Re(s_2), \ldots, \Re(s_r)) \in (1 - \epsilon, 1 + \epsilon)^r$.

The second part follows by restricting to the case of $\lambda = s\rho = (s, s, \ldots, s)$.

3.3. ***Prerequisites for the computation.*** For any $h \in \mathcal{H}(G)$, the Hecke algebra, we define

$$T_h(\theta_\varphi)(g) := \int_{A(F)\backslash A(\mathbb{A})} h(a^{-1})\theta_\varphi(ag)\,\overline{da}.$$

The operator T_h enjoys the following property as can be seen from the integral representation above.

Proposition 3.11. *The operator T_h defines a self-adjoint bounded operator on the closed subspace of $L^2(G(F)\backslash\mathbb{G}, \bar{\mu})$ generated by the function θ_φ, such that if $\hat{\psi}(\lambda) = \hat{h}(\lambda)\,\hat{\varphi}(\lambda)$, then $\theta_\psi = T_h(\theta_\varphi)$. The norm of T_h is bounded above by $\hat{h}(\rho)$.*

Proof. The existence of T_h follows from [Morris 1982a, Lemma, p. 136]. □

Let $\mathcal{E}^\vee$ be the closure of the subspace of $L^2(G(F)\backslash\mathbb{G})$ generated by the pseudo-Eisenstein series θ_φ where φ is a compactly supported function on $A(\mathbb{A})/A(F)$. Then the constant function belongs to $\mathcal{E}^\vee$ [Mœglin and Waldspurger 1995, Chapter II, Section 1.12]. The main theorem of this section is the computation of the projection of the pseudo-Eisenstein series θ_φ onto the constant function.

Choose $h \in \mathcal{H}(G)$ as below and consider the positive normal operator

$$T := T_h \circ (T_h)^*/(\hat{h}(\rho))^2.$$

(1) Choose a place $v_0 \notin S$: via Satake isomorphism there exists $h_{v_0} \in \mathcal{H}(G \times_F F_{v_0})$ a bi-K_{v_0}-invariant function such that its Fourier transform satisfies $\hat{h}_{v_0}(s_{v_0}) = \sum_{w \in W_F} (\sharp k(v_0))^{-\langle \rho, w s_{v_0} \rangle}$.

(2) At places $v \neq v_0$ define h_v to be the characteristic function of K_v.

Following Harder, we prove:

Theorem 3.12. *The sequence of positive normal operators $T^n : \mathcal{E}^\vee \to \mathcal{E}^\vee$ converges to the operator $P : \mathcal{E}^\vee \to \mathcal{E}^\vee$ which is the projection onto the constant functions. Explicitly*

$$P(\theta_\varphi) = c \log(q)^r \operatorname{res}_{s=1} E(g, s\rho)\, \hat{\varphi}(s\rho) = cc' \log(q)^r \lim_{s \to 1} (s-1)^r M(w_0, s\rho)\, \hat{\varphi}(s\rho),$$

where c and c' are the constants satisfying

$$d\lambda|_L = c \left(\frac{\log q}{2\pi} \right)^r dz_1 \wedge \cdots \wedge dz_r, \qquad L = \frac{X^*(A) \otimes \mathbb{R}}{2\pi/\log(q) X^*(A)},$$
$$\operatorname{res}_{\lambda=\rho}(E(x, \lambda)\, \hat{\varphi}(\lambda)) = c' \lim_{s \to 1} (s-1)^r M(w_0, s\rho)\, \hat{\varphi}(s\rho).$$

Proof. The proof is analogous to that of [Harder 1974, pp. 301–303]. We summarize the key steps below. Recall that

$$\theta_\varphi(g) = \int_{\Re(\lambda)=\lambda_0} E(g, \hat{\varphi}(\lambda))\, d\lambda. \tag{9}$$

Let λ_i denote the i-th coordinate of λ under the parametrization given by ξ. It is well known that the poles of the Eisenstein series are the same as those of the constant term of the series. Using Theorem 3.3 we get that the Eisenstein series $E(g, \lambda)$ has simple poles along the hyperplanes $\lambda_i = 1$ in the region $1 - \epsilon < (\Re\lambda)_i < 1 + \epsilon$ for small ϵ. The residue theorem gives us the equality

$$\int_{\Re\lambda=\sigma} E(g, \lambda) = \int_{\Re\lambda=\sigma'} E(g, s) + \int_{\Re\mu=\sigma''} E^{\{\alpha_1\}}(g, \mu),$$

where $1 - \epsilon < \sigma'_1 < 1$, $\sigma''_1 = 1$, $\mu_1 = 1$, and $E^{\{\alpha_1\}}(g, \mu) = \lim_{\lambda_1 \to 1} (\lambda_1 - 1) E(g, \lambda)$. Using the residue theorem iteratively, and applying T^n, we get

$$T^n(\theta_\varphi)(g) = c \log(q)^r \operatorname{res}_{\lambda=\rho}(E(g, \lambda)\, \hat{\varphi}(\lambda)) + R.$$

In the above equation R is the sum of expressions of the form

$$T^n\left(\int_{\Lambda_{\sigma'}(A)} \widetilde{E}(g,\lambda)\,\hat{\varphi}(\lambda)\,d\lambda\right) = \int_{\Lambda_{\sigma'}(A)} \widetilde{E}(g,\lambda)\,\hat{\varphi}(\lambda)\left(\frac{\hat{h}(\lambda)}{\hat{h}(\rho)}\right)^{2n} d\lambda,$$

where $1-\epsilon < \sigma_i' \leq 1$ for all $i = 1, 2, \ldots, n$, $\sigma_j' = 1$ for $j \in S \subsetneq \{1, 2, \ldots, n\}$, S nonempty, and $\widetilde{E}(g,\lambda) = \lim_{\lambda_j \to 1} \prod_{j\in S}(\lambda_j - 1) E(g,\lambda)$. Note that for $\lambda \in \Lambda_{\sigma'}(A)$, the inequality $\hat{h}(\lambda) < \hat{h}(\rho)$ holds (see Lemma C.1). Hence we get

$$\lim_{n\to\infty} T^n(\theta_\varphi) = c \log(q)^r \operatorname{res}_{\lambda=\rho}(E(x,\lambda)\,\hat{\varphi}(\lambda)). \tag{10}$$

The above limit and the equality is to be understood as pointwise convergence. Proposition 3.11 implies that the spectrum of the self-adjoint positive operator T is concentrated on [0, 1] and hence $T^n \to P$ where P is the projection onto the subspace

$$\{e \in \mathcal{E}^\vee \mid Te = e\}.$$

This observation of Harder coupled with the pointwise convergence result from (10) implies that the equality (10) in fact holds in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$. This finishes the proof of the first equality. Following the arguments in [Harder 1974, pp. 289–290] we get that $\operatorname{res}_{s=1} E(g, s\rho)\,\hat{\varphi}(s\rho) = q^{\dim(N)(1-g)} \operatorname{res}_{s=1} E^B(g, s\rho)\,\hat{\varphi}(s\rho)$. Now using the formula for the constant term from Proposition 3.1 and observing that the intertwining operators $M(w, s\rho)$ has poles of order $< r$ for $w \neq w_0$ we get the second equality with $c' = q^{\dim(N)(1-g)}$. □

3.4. ***A final computation.*** We will complete the proof of the Weil conjecture in the case of quasisplit group over function fields in this section.

We begin with the equality

$$\begin{aligned}
P\theta_\varphi &= cc' \log(q)^r \lim_{t\to 1}(t-1)^r M(w_0, t\rho)\,\hat{\varphi}(t\rho)\\
&= cc' \log(q)^r \operatorname{res}_{s=1}\big(L^S(s, X^*(A))\big) \lim_{t\to 1} \frac{M(w_0, t\rho)\,\hat{\varphi}(t\rho)}{L(t, A)}\\
&= cc' \log(q)^r \operatorname{res}_{s=1}\big(L^S(s, X^*(A))\big) \lim_{t\to 1} \frac{M^S(w_0, t\rho)\hat{\varphi}^S(t\rho)}{L^S(t, X^*(A))} \prod_{v\in S} M_v(w_0, \rho)\hat{\varphi}_v(t\rho)\\
&= cc' \log(q)^r \operatorname{res}_{s=1}\big(L^S(s, X^*(A))\big) \lim_{t\to 1} \frac{M^S(w_0, t\rho)\hat{\varphi}^S(t\rho)}{L^S(t, X^*(A))} \prod_{v\in S} M_v(w_0, \rho)\hat{\varphi}_v(t\rho)\\
&= cc' \log(q)^r \operatorname{res}_{s=1}\big(L^S(s, X^*(A))\big)\\
&\qquad \times \Big(\prod_{v\notin S} \operatorname{vol}(K_v)\,\hat{\varphi}^S(t\rho)\Big)\Big(\kappa\Big(\prod_{v\notin S} \operatorname{vol}(K_v)\Big)^{-1} \prod_{v\in S} \hat{\varphi}_v(t\rho)\Big)\\
&= cc' \log(q)^r \operatorname{res}_{s=1}\big(L^S(s, X^*(A))\big)\,\kappa\hat{\varphi}(\rho).
\end{aligned}$$

Since P is the projection operator onto the constants we have $(\theta_\varphi, 1) = (P\theta_\varphi, 1)$. The right-hand side equals

$$q^{-\dim(G)(1-g)}\tau(G)\, cc' \log(q)^r \operatorname{res}_{s=1}\big(L(s, X^*(A))\big)\kappa\hat{\varphi}(\rho).$$

Since we can surely have functions φ with $\hat{\varphi}(\rho) \neq 0$, we get the equality

$$\tau(G) = \frac{q^{(\dim(G)-\dim(N))(1-g)}}{cc' \log(q)^r \operatorname{res}_{s=1}\big(L(s, X^*(A))\big)} = \tau(A).$$

The last equality follows from the explicit value of c obtained in Section 2.4.1 and of c' obtained in the proof of Theorem 3.12. We know from [Oesterlé 1984, Chapter II, Theorem 1.3(d)] that $\tau(\operatorname{Res}_{E/F}(\mathbb{G}_m)) = \tau(\mathbb{G}_m)$. Using Lemma B.1 and the fact that the Tamagawa number of a split torus is 1, we get

$$\tau(G) = \tau(A) = 1.$$

Appendix A. Dual groups and restriction of scalars

Let $E \supset E' \supset F$ be a tower of unramified extensions of local fields. Let A' be a torus defined over E' which splits over E and consider $A = \operatorname{Res}_{E'/F} A'$. We have the $\operatorname{Gal}(E/E')$-equivariant isomorphism

$$\hat{A}^{\operatorname{Gal}(E/E')} \cong \prod_{\operatorname{Gal}(E'/F)} \widehat{A'}^{\operatorname{Gal}(E/E')} \tag{11}$$

and the action of $\operatorname{Gal}(E'/F)$ is given by permuting the indices. Hence

$$\hat{A}^{\operatorname{Gal}(E/F)} \cong \widehat{A'}^{\operatorname{Gal}(E/E')}.$$

The inclusion $\widehat{A'}^{\operatorname{Gal}(E/E')} \hookrightarrow \hat{A}^{\operatorname{Gal}(E/E')}$ can be identified under the isomorphism (11) with the diagonal embedding $\widehat{A'}^{\operatorname{Gal}(E/E')} \hookrightarrow \prod_{\operatorname{Gal}(E'/F)} \widehat{A'}^{\operatorname{Gal}(E/E')}$. Define the map $\eta : \hat{A}(\mathbb{C}) \to X^*(A \times_F \bar{F}) \otimes \mathbb{C}$ by the condition that $\mu(\hat{t}) = |\pi_F|^{\langle \eta(\hat{t}), \mu\rangle}$ for all $\mu \in X_*(A \times_F \bar{F})$. Similarly, we may define $\eta' : \widehat{A'}(\mathbb{C}) \to X^*(A' \times_F \bar{F}) \otimes \mathbb{C}$.

Lemma A.1. *Let $\hat{t} \in \hat{A}^{\operatorname{Gal}(E'/F)}$ and $\widehat{t'} \in \widehat{A'}^{\operatorname{Gal}(E/E')}$ be such that under the isomorphism (11) we have $\hat{t} = (\widehat{t'}, \widehat{t'}, \dots, \widehat{t'})$. Further assume that $\lambda = \eta(\hat{t})$ and $\operatorname{Nm}_{E'/F}(\lambda') = \lambda$, then $\eta'(\widehat{t'}^{\,n}) = \lambda'$.*

Proof. Note that there is the following commutative diagram:

$$\begin{array}{ccc} X_*(A') \times X^*(A') & \longrightarrow & \mathbb{Z} \\ \uparrow \qquad\qquad \downarrow{\scriptstyle \operatorname{Nm}_{E'/F}} & & \downarrow{\scriptstyle \times[E':F]} \\ X_*(A) \times X^*(A) & \longrightarrow & \mathbb{Z}, \end{array} \tag{12}$$

where the leftmost vertical arrow is an isomorphism given by the adjunction of restriction and extension of scalars. For $\hat{t}$, λ, λ' as in the statement of the lemma, and $\mu \in X_*(A)$,

$$\mu(\hat{t}) = |\pi_F|^{\langle \eta(\hat{t}),\mu\rangle} = |\pi_F|^{\langle \mathrm{Nm}_{E'/F}(\lambda'),\mu\rangle} = |\pi_F|^{n\langle \lambda',\mu\rangle} = |\pi_{E'}|^{\langle \lambda',\mu\rangle}.$$

Recall that $\hat{t} = (\widehat{t'}, \widehat{t'}, \ldots, \widehat{t'})$, hence $\mu(\hat{t}) = \mu(\widehat{t'})^n$ for any $\mu \in X_*(A) = X_*(A')$. Thus, we get $|\pi_{E'}|^{\langle \lambda',\mu\rangle} = \mu(\widehat{t'}^n)$. Hence by definition of η' we get $\eta'(\widehat{t'}^n) = \lambda'$. □

Appendix B. Quasisplit tori in simply connected groups

We state the following lemma from [Rapoport 1976, Lemma 6.1.2] for the sake of completeness.

Lemma B.1. *Suppose G is a simply connected quasisplit group over a field F. Let A be a maximal torus defined over F which is contained in a Borel subgroup defined over F. Then A is a product of tori of the form $\mathrm{Res}_{E_i/F}\,\mathbb{G}_m$, where E_i/F are finite separable extension of F.*

Proof. Let $X^*(A \times_F F^{\mathrm{sep}})$ be the set of characters of A defined over F^{sep}. Then the Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ acts on the group $X^*(A \times_F F^{\mathrm{sep}})$. When G is quasisplit the restriction map $\Pi_{F^{\mathrm{sep}}} \to \Pi_F$ is surjective and the fibers are exactly the $\mathrm{Gal}(F^{\mathrm{sep}}/F)$-orbits. This implies that the set of absolute simple roots restricting to a given relative simple root is permuted by the Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$. We may use [Springer 2009, Exercise 13.1.5(4)] to conclude the lemma. □

Appendix C. A lemma

This is [Harder 1974, Lemma 3.2.3(a)] for quasisplit groups.

Lemma C.1. *The inequality $\hat{h}(\lambda) < \hat{h}(\rho)$ holds.*

Proof. The proof follows as in [Harder 1974, Lemma 3.2.3] which in turn depends on [Harder 1974, Lemma 3.2.1(ii)]. We need only prove an analogous result to the latter lemma for the quasisplit case. That is, to show that

$$\{\sigma = (\sigma_i) \mid 1 - \epsilon < \Re(\sigma_i) \le 1 \text{ for all } i\} \subset \{\sigma \mid \Re(\sigma) \in \mathrm{ConvHull}(W_F \cdot \rho)\}.$$

Note that the restriction map $X^*(A \times \bar{F}) \twoheadrightarrow X^*(A)$ in our chosen coordinate system (3) can be identified with the map "average over the Galois orbits". This is a convex map and hence preserves convex domains. Since the lemma is known for the convex hull of the Weyl conjugate of ρ in $X^*(A \times \bar{F})$, the lemma follows in the quasisplit case as well. □

Acknowledgements

The authors thank a referee for very valuable comments and suggestions that helped to considerably improve the article.

Köhl was supported by public funding of the state of Schleswig-Holstein, Germany, and partially supported by Deutsche Forschungsgemeinschaft (DFG) via grant agreement KO 4323/15 "Lattices of minimal covolume in simple Lie groups and applications". Radhika is supported by a postdoctoral fellowship from the NBHM grant 02018/2/2023-R&D-II/7839 under Kerala School of Mathematics. Rai is partially supported by postdoctoral fellowship at Indian Institute of Technology Bombay and at Chennai Mathematical Institute.

References

[Bourbaki 1971] N. Bourbaki, *Variétés différentielles et analytiques*: *fascicule de résultats*, 8–15, Actualités Scientifiques et Industrielles **1347**, Hermann, Paris, 1971. MR

[Bruhat and Tits 1972] F. Bruhat and J. Tits, "Groupes réductifs sur un corps local", *Inst. Hautes Études Sci. Publ. Math.* **41** (1972), 5–251. MR

[Conrad 2015] B. Conrad, "Reductive groups over fields", lecture notes, Stanford, 2015, see https://tinyurl.com/ConradHandout.

[Gaitsgory and Lurie 2019] D. Gaitsgory and J. Lurie, *Weil's conjecture for function fields*, *I*, Annals of Mathematics Studies **199**, Princeton Univ. Press, 2019. MR

[Harder 1974] G. Harder, "Chevalley groups over function fields and automorphic forms", *Ann. Math.* (2) **100** (1974), 249–306.

[Kottwitz 1988] R. E. Kottwitz, "Tamagawa numbers", *Ann. Math.* (2) **127**:3 (1988), 629–646.

[Kushnirsky 1999] E. Kushnirsky, *Some arithmetical applications of integration on reductive groups*, Ph.D. thesis, University of Michigan, 1999, see https://www.proquest.com/docview/304518405. MR

[Lai 1980] K. F. Lai, "Tamagawa number of reductive algebraic groups", *Compositio Math.* **41**:2 (1980), 153–188. MR

[Langlands 1966] R. P. Langlands, "The volume of the fundamental domain for some arithmetical subgroups of Chevalley groups", pp. 143–148 in *Algebraic Groups and Discontinuous Subgroups* (Boulder, Colo., 1965), edited by A. Borel and G. D. Mostow, Amer. Math. Soc., Providence, RI, 1966. MR

[Mœglin and Waldspurger 1995] C. Mœglin and J.-L. Waldspurger, *Spectral decomposition and Eisenstein series: a paraphrase of scripture*, Cambridge Tracts in Mathematics **113**, Cambridge Univ. Press, 1995. MR

[Morris 1982a] L. E. Morris, "Eisenstein series for reductive groups over global function fields, I: The cusp form case", *Canadian J. Math.* **34**:1 (1982), 91–168. MR

[Morris 1982b] L. E. Morris, "Eisenstein series for reductive groups over global function fields, II: The general case", *Canadian J. Math.* **34**:5 (1982), 1112–1182. MR

[Ngô Dac 2009] T. Ngô Dac, "Sur le développement spectral de la formule des traces d'Arthur–Selberg sur les corps de fonctions", *Bull. Soc. Math. France* **137**:4 (2009), 545–586. MR

[Oesterlé 1984] J. Oesterlé, "Nombres de Tamagawa et groupes unipotents en caractéristique p", *Invent. Math.* **78** (1984), 13–88.

[Rapoport 1976] M. Rapoport, "Détermination du nombre de tamagawa des groupes semi-simples quasi-déployés sur des corps de nombres", preprint, 1976, see https://www.math.uni-bonn.de/people/rapoport/myalggeom/preprints/Determinationdunombredetamagawa.pdf.

[Springer 2009] T. A. Springer, *Linear algebraic groups*, 2nd ed., Birkhäuser, Boston, MA, 2009. MR

[Tamagawa 1966] T. Tamagawa, "Adèles", pp. 113–121 in *Algebraic Groups and Discontinuous Subgroups* (Boulder, Colo., 1965), edited by A. Borel and G. D. Mostow, Amer. Math. Soc., Providence, RI, 1966. MR

[Thắng 2022] N. Q. Thắng, "On the Tits indices of absolutely almost simple algebraic groups over local and global fields", *J. Pure Appl. Algebra* **226**:9 (2022), art. id. 107031. MR

[Weil 1982] A. Weil, *Adeles and algebraic groups*, Progress in Mathematics **23**, Birkhäuser, Boston, MA, 1982. MR

Received September 29, 2023. Revised June 17, 2025.

RALF KÖHL
DEPARTMENT OF MATHEMATICS
KIEL UNIVERSITY
KIEL
GERMANY
koehl@math.uni-kiel.de

M. M. RADHIKA
KERALA SCHOOL OF MATHEMATICS
KOZHIKODE
KERALA
INDIA
mmr@ksom.res.in

ANKIT RAI
CHENNAI MATHEMATICAL INSTITUTE
CHENNAI
TAMIL NADU
INDIA
ankitr@cmi.ac.in

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2025 is US $677/year for the electronic version, and $917/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 338 No. 1 September 2025
