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# EQUIVARIANT RIGIDITY OF RICHARDSON VARIETIES

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**We prove that Schubert and Richardson varieties in flag manifolds are uniquely determined by their equivariant cohomology classes, as well as a stronger result that replaces Schubert varieties with closures of Białynicki-Birula cells under suitable conditions. This is used to prove a conjecture of Buch, Chaput, and Perrin, stating that any two-pointed curve neighborhood representing a quantum cohomology product with a Seidel class is a Schubert variety. We pose a stronger conjecture which implies a Seidel multiplication formula in equivariant quantum $K$-theory, and prove this conjecture for cominuscule flag varieties.**

## 1. Introduction

A Schubert variety $\Omega$ in a flag manifold $X = G/P$ is called *rigid* if it is uniquely determined by its class $[\Omega]$ in the cohomology ring $H^*(X)$. More precisely, if $Z \subset X$ is any irreducible closed subvariety such that $[Z]$ is a multiple of $[\Omega]$ in $H^*(X)$, then $Z$ is a $G$-translate of $\Omega$. This problem has been studied in numerous papers; see, e.g., [Hong 2005; 2007; Coskun 2011; 2014; 2018; Robles and The 2012; Coskun and Robles 2013; Hong and Mok 2020; Liu et al. 2024]. In this paper we show that all Schubert varieties and Richardson varieties are *equivariantly rigid*. In other words, if $T \subset G$ is a maximal torus, $\Omega \subset X$ is a $T$-stable Richardson variety, and $Z \subset X$ is a (nonempty) $T$-stable closed subvariety such that the $T$-equivariant class $[Z] \in H_T^*(X)$ is a multiple of $[\Omega]$, then $Z = \Omega$.

More generally, let $T$ be an algebraic torus over an algebraically closed field, let $X$ be a nonsingular projective $T$-variety, and let $\Omega \subset X$ be a $T$-stable closed subvariety. Let $\Omega^T$ denote the set of $T$-fixed points in $\Omega$. We will say that $\Omega$ is *$T$-convex* if, for any $T$-stable closed subvariety $Z \subset X$ satisfying $Z^T \subset \Omega$, we have $Z \subset \Omega$. A fixed point $p \in X^T$ is called *fully definite* if all $T$-weights of the Zariski tangent space $T_p X$ belong to a strict half-space of the character lattice of $T$. We show that if

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all $T$-fixed points in $X$ are fully definite, then any irreducible $T$-convex subvariety of $X$ is also $T$-equivariantly rigid. Here $\Omega$ is called $T$-equivariantly rigid if $\Omega$ is determined by its class in the $T$-equivariant Chow cohomology ring of $X$.

Let $\mathbb{G}_m \subset T$ be a one-parameter subgroup such that $X^T = X^{\mathbb{G}_m}$, and assume that this fixed point set is finite. The associated Białynicki-Birula decomposition of $X$ is given by $X = \bigcup_{p \in X^T} X_p^+$, where $X_p^+ = \{x \in X : \lim_{t \to 0} t \cdot x = p\}$ is the Białynicki-Birula cell of points attracted to $p$ by the action of $t \in \mathbb{G}_m$. This decomposition is called a *stratification* if each cell closure $\overline{X_p^+} \subset X$ is a union of smaller cells. In this case we show that all cell closures are $T$-convex. We arrive at the following result combining Theorem 4.3 and Proposition 5.3.

**Theorem.** *Let $X$ be a nonsingular projective $T$-variety with finitely many $T$-fixed points, and choose $\mathbb{G}_m \subset T$ such that $X^T = X^{\mathbb{G}_m}$.*

(a) *Assume that the Białynicki-Birula decomposition of $X$ is a stratification. Then each cell closure $\overline{X_p^+}$ is $T$-convex.*

(b) *Assume that all $T$-fixed points in $X$ are fully definite. Then any irreducible $T$-convex subvariety of $X$ is $T$-equivariantly rigid.*

This result applies to Schubert and Richardson varieties in flag varieties, as well as positroid varieties in Grassmannians, so these subvarieties are both $T$-convex and $T$-equivariantly rigid. However, projected Richardson varieties do not in general enjoy these properties; see Remark 6.5. Our theorem also covers a class of horospherical varieties, which includes all nonsingular horospherical varieties of Picard rank 1 [Pasquier 2009].

Our theorem has additional applications in quantum Schubert calculus. Let $X = G/P$ be a complex flag manifold. A Schubert class $[X^w]$ is called a *Seidel class* if the Weyl group element $w$ is the minimal representative of a point in some cominuscule flag variety $G/Q$. Multiplication by Seidel classes in the quantum cohomology ring $\mathrm{QH}(X)$ is given by the identity $[X^w] \star [X^u] = q^{d(w,u)}[X^{wu}]$, where $d(w,u)$ is the unique minimal degree of a rational curve connecting the opposite Schubert varieties $X_{w_0 w}$ and $X^u$ [Seidel 1997; Belkale 2004; Chaput et al. 2009]. This implies that $[X^{wu}]$ is equal to the class of the curve neighborhood $\Gamma_{d(w,u)}(X_{w_0 w}, X^u)$, defined as the union of all stable curves in $X$ of degree $d(w,u)$ connecting $X_{w_0 w}$ to $X^u$. We conjectured in [Buch et al. 2023] that this curve neighborhood is in fact the translated Schubert variety

$$\Gamma_{d(w,u)}(X_{w_0 w}, X^u) = w^{-1} \cdot X^{wu}. \tag{1}$$

This has been proved in some cases when $X$ is cominuscule, in all cases when $X$ is a flag variety of type A [Li et al. 2025; Tarigradschi 2023], and for $X = \mathrm{SG}(2, 2n)$ [Benedetti et al. 2024]. Using that $\Gamma_{d(w,u)}(X_{w_0 w}, X^u)$ and $w^{-1} \cdot X^{wu}$ define the same class in $H_T^*(X)$ by an equivariant version of the Seidel multiplication formula from [Chaput et al. 2009; Chaput and Perrin 2023], the identity (1) follows from our result that Schubert varieties are equivariantly rigid.

In this paper we conjecture the more general identity

$$\Gamma_{d(w,u)+e}(X_{w_0w}, X^u) = \Gamma_e(w^{-1}(X^{wu})), \tag{2}$$

where the right-hand side is the union of all stable curves of degree $e$ that pass through $w^{-1} \cdot X^{wu}$. This union is a Schubert variety [Buch et al. 2013] whose Weyl group element was determined in [Buch and Mihalcea 2015]. Denote by $M_{d(w,u)+e}(X_{w_0w}, X^u)$ the moduli space of three-pointed stable maps to $X$ of degree $d(w,u)+e$ and genus zero, which send the first two marked points to $X_{w_0w}$ and $X^u$, respectively. We further conjecture that the evaluation map $\mathrm{ev}_3 : M_{d(w,u)+e}(X_{w_0w}, X^u) \to \Gamma_e(w^{-1}(X^{wu}))$ is cohomologically trivial. This conjecture implies a Seidel multiplication formula in the equivariant quantum $K$-theory ring $\mathrm{QK}_T(X)$. We prove this conjecture when $X$ is a cominuscule flag variety, thereby obtaining an equivariant generalization of our Seidel multiplication formula from [Buch et al. 2023]. This generalized Seidel multiplication formula has also been obtained for Grassmannians of type A in [Gorbounov et al. 2025] using different methods. Based on suggestions from Mihail Tarigradschi, we finally apply the methods of [Tarigradschi 2023] to prove the identity (2) when $X = \mathrm{GL}_n(\mathbb{C})/P$ is any flag manifold of Lie type A.

Our paper is organized as follows. In Section 2 we recall some basic facts and notation related to torus actions. In Section 3 we show that if all $T$-fixed points of $X$ are fully definite, then the fixed point set $Z^T$ of a $T$-stable subvariety $Z \subset X$ is determined by its equivariant class $[Z] \in H_T^*(X)$. This is used in Section 4 to prove part (b) of the above theorem. Section 5 proves part (a). Section 6 interprets our theorem for flag varieties, which is used in Section 7 to prove the conjecture about curve neighborhoods from [Buch et al. 2023]. Section 8 discusses the more general conjecture as well as its consequences in quantum $K$-theory. Finally, Section 9 interprets our theorem for certain horospherical varieties.

## 2. Torus actions

We work with varieties over a fixed algebraically closed field $\mathbb{K}$. Varieties are reduced but not necessarily irreducible. A point will always mean a closed point. The multiplicative group of $\mathbb{K}$ is denoted by $\mathbb{G}_m = \mathbb{K} \smallsetminus \{0\}$. An (algebraic) torus is a group variety isomorphic to $(\mathbb{G}_m)^r$ for some $r \in \mathbb{N}$.

Let $T = (\mathbb{G}_m)^r$ be an algebraic torus. Any rational representation $V$ of $T$ is a direct sum $V = \bigoplus_\lambda V_\lambda$ of weight spaces $V_\lambda = \{v \in V : t \cdot v = \lambda(t)v \ \forall t \in T\}$ defined by characters $\lambda : T \to \mathbb{G}_m$. The *weights* of $V$ are the characters $\lambda$ for which $V_\lambda \neq 0$. The group of all characters of $T$ is called the *character lattice* and is isomorphic to $\mathbb{Z}^r$. Given a $T$-variety $X$, we let $X^T \subset X$ denote the closed subvariety of $T$-fixed points. A subvariety $Z \subset X$ is called *$T$-stable* if $t \cdot z \in Z$ for all $t \in T$ and $z \in Z$. In this case $Z$ is itself a $T$-variety.

**Definition 2.1.** The $T$-fixed point $p \in X$ is *nondegenerate* in $X$ if $T$ acts with nonzero weights on the Zariski tangent space $T_pX$. The point $p$ is *fully definite* if all $T$-weights of $T_pX$ belong to a strict half-space of the character lattice of $T$.

Equivalently, $p \in X^T$ is fully definite in $X$ if and only if there exists a cocharacter $\rho : \mathbb{G}_m \to T$ such that $\mathbb{G}_m$ acts with strictly positive weights on $T_pX$ though $\rho$. For example, if $X = G/P$ is a flag variety and $T \subset G$ is a maximal torus, then all points of $X^T$ are fully definite in $X$ (see Section 6). Any nondegenerate $T$-fixed point must be isolated in $X^T$. Fully definite $T$-fixed points are called *attractive* in many sources, see, e.g., [Brion 1997]; here we follow the terminology from [Białynicki-Birula 1973].

**Remark 2.2.** If $X$ is a normal quasiprojective $T$-variety, then $X^{\mathbb{G}_m} = X^T$ holds for all general cocharacters $\rho : \mathbb{G}_m \to T$. Here a cocharacter is called *general* if it avoids finitely many hyperplanes in the lattice of all cocharacters. This follows because $X$ admits an equivariant embedding $X \subset \mathbb{P}(V)$, where $V$ is a rational representation of $T$ [Kambayashi 1966; Mumford 1965; Sumihiro 1974].

In the rest of this paper we let $X$ be a nonsingular $T$-variety. The $T$-equivariant Chow cohomology ring of $X$ will be denoted by $H_T^*(X)$; see [Fulton 1998; Anderson and Fulton 2024]. This is an algebra over the ring $H_T^*(\text{point})$, which may be identified with the symmetric algebra of the character lattice of $T$. Given a class $\sigma \in H_T^*(X)$ and a $T$-fixed point $p \in X^T$, we let $\sigma_p \in H_T^*(\text{point})$ denote the pullback of $\sigma$ along the inclusion $\{p\} \to X$. When $X$ is defined over $\mathbb{K} = \mathbb{C}$, Chow cohomology can be replaced with singular cohomology. In fact, our arguments will only depend on equivariant classes $[Z]_p \in H_T^*(\text{point})$ obtained by restricting the class of a $T$-stable closed subvariety $Z \subset X$ to a fixed point, and these restrictions are independent of the chosen cohomology theory. Similarly, we can use cohomology with coefficients in either $\mathbb{Z}$ or $\mathbb{Q}$.

## 3. Equivariant local classes

Let $Z$ be a $T$-variety, fix $p \in Z^T$, and let $\mathfrak{m} \subset \mathcal{O}_{Z,p}$ be the maximal ideal in the local ring of $p$. Then the tangent cone $C_pZ = \operatorname{Spec}\big(\bigoplus \mathfrak{m}^i/\mathfrak{m}^{i+1}\big)$ is a $T$-stable closed subscheme of the Zariski tangent space $T_pZ = (\mathfrak{m}/\mathfrak{m}^2)^\vee = \operatorname{Spec}(\operatorname{Sym}(\mathfrak{m}/\mathfrak{m}^2))$. The *local class* of $Z$ at $p$ is defined by (see [Anderson and Fulton 2024, §17.4])

$$\eta_pZ = [C_pZ] \in H_T^*(T_pZ) = H_T^*(\text{point}). \tag{3}$$

When $p$ is a nonsingular point of $Z$, we have $\eta_pZ = 1$.

**Proposition 3.1.** *Let $Z$ be a $T$-variety and let $p \in Z^T$ be fully definite in $Z$. Then $\eta_p Z \neq 0$ in $H_T^*(\text{point})$.*

*Proof.* We may assume that $p$ is a singular point of $Z$, so that $C_p Z$ has positive dimension. Choose $\mathbb{G}_m \subset T$ such that $\mathbb{G}_m$ acts with positive weights on $T_p Z$. It suffices to show that the class of $C_p Z$ is nonzero in $H^*_{\mathbb{G}_m}(T_p Z)$. Let $\{v_1, \dots, v_n\}$ be a basis of $T_p Z$ consisting of eigenvectors of $\mathbb{G}_m$. Then the action of $\mathbb{G}_m$ is given by $t \cdot v_i = t^{a_i} v_i$ for positive integers $a_1, \dots, a_n > 0$. Set $A = \prod_{i=1}^n a_i$, and let $\mathbb{G}_m$ act on $U = \mathbb{K}^n$ by $t \cdot u = t^A u$. Then the map $\phi : T_p Z \to U$ defined by

$$\phi(c_1 v_1 + \cdots + c_n v_n) = (c_1^{A/a_1}, \dots, c_n^{A/a_n})$$

is a finite $\mathbb{G}_m$-equivariant morphism. By [Edidin and Graham 1998, Theorem 4] we obtain

$$H^*_{\mathbb{G}_m}(U \smallsetminus \{0\}) \otimes \mathbb{Q} = H^*(\mathbb{P}U) \otimes \mathbb{Q},$$

where $\mathbb{P}U = (U \smallsetminus \{0\})/\mathbb{G}_m \cong \mathbb{P}^{n-1}$ is the projective space of lines in $U$, and

$$\phi_*[C_p Z]|_{U \smallsetminus \{0\}} = \deg(\phi)[\phi(C_p Z \smallsetminus \{0\})/\mathbb{G}_m] \in H^*(\mathbb{P}U) \otimes \mathbb{Q}.$$

The result now follows from the fact that every nonempty closed subvariety of projective space defines a nonzero Chow class. □

**Corollary 3.2.** *Let $X$ be a nonsingular $T$-variety, $Z \subset X$ a $T$-stable closed subvariety, and $p \in Z^T$ a $T$-fixed point of $Z$. If $p$ is nondegenerate in $X$ and fully definite in $Z$, then $[Z]_p \neq 0 \in H_T^*(\text{point})$.*

*Proof.* By [Anderson and Fulton 2024, Proposition 17.4.1] we have

$$[Z]_p = c_m(T_p X / T_p Z) \cdot \eta_p Z,$$

where $m = \dim T_p X - \dim T_p Z$. The result therefore follows from Proposition 3.1, noting that $T$ acts with nonzero weights on $T_p X / T_p Z$. □

The following example rules out some potential generalizations of Corollary 3.2.

**Example 3.3.** Let $\mathbb{G}_m$ act on $\mathbb{A}^4$ by

$$t.(a, b, c, d) = (ta,\ tb,\ t^{-1}c,\ t^{-1}d)\,.$$

Set $Z = V(ad - bc) \subset \mathbb{A}^4$, and let $p = (0, 0, 0, 0)$ be the origin in $\mathbb{A}^4$. Then

$$T_p Z = T_p \mathbb{A}^4 = \mathbb{A}^4 \quad \text{and} \quad C_p Z = Z.$$

Since $\mathbb{G}_m$ acts trivially on the equation $ad - bc$, we have $\eta_p Z = [Z] = 0$ in $H^*_{\mathbb{G}_m}(\mathbb{A}^4)$ (see [Anderson and Fulton 2024, §2.3]).

## 4. Rigidity of convex subvarieties

Let $T$ be an algebraic torus and let $X$ be a nonsingular $T$-variety. We will show in Section 6 that Schubert varieties and Richardson varieties in a flag variety $X$ satisfy the following two definitions.

**Definition 4.1.** A $T$-stable closed subvariety $\Omega \subset X$ is *$T$-equivariantly rigid* if it is uniquely determined by its $T$-equivariant cohomology class up to a constant. More precisely, if $Z \subset X$ is any $T$-stable closed subvariety such that $[Z] = c\,[\Omega]$ holds in $H_T^*(X)$ for some $0 \neq c \in \mathbb{Q}$, then $Z = \Omega$.

**Definition 4.2.** A $T$-stable closed subvariety $\Omega \subset X$ is *$T$-convex* if, for any $T$-stable closed subvariety $Z \subset X$ satisfying $Z^T \subset \Omega$, we have $Z \subset \Omega$.

When the action of $T$ is clear from the context, we frequently drop $T$ from the notation and write simply *equivariantly rigid* and *convex*. Both notions are properties of the $T$-equivariant embedding $\Omega \subset X$; for example, any $T$-variety is convex as a subvariety of itself. Intersections of $T$-convex subvarieties are again $T$-convex (with the reduced scheme structure). Most of this paper concerns applications of the following observation.

**Theorem 4.3.** *Let $X$ be a nonsingular projective $T$-variety such that all fixed points $p \in X^T$ are fully definite in $X$. Then any irreducible $T$-convex subvariety of $X$ is $T$-equivariantly rigid.*

*Proof.* Let $\Omega \subset X$ be irreducible and convex, and let $Z \subset X$ be any $T$-stable closed subvariety such that $[Z] = c\,[\Omega]$ holds in $H_T^*(X)$, with $0 \neq c \in \mathbb{Q}$. Then Corollary 3.2 shows that $Z^T = \Omega^T = \{p \in X^T : [Z]_p \neq 0\}$. Since $\Omega$ is convex, we obtain $Z \subset \Omega$. Finally, the assumption $[Z] = c\,[\Omega]$ implies that $Z$ and $\Omega$ have the same dimension, so we must have $Z = \Omega$. $\square$

**Example 4.4.** Let $X$ be a nonsingular projective $T$-variety, let $H_T^*(X)$ be the $T$-equivariant Chow cohomology ring, and let $\mathcal{L}$ be a $T$-equivariant line bundle. Given a section $f \in \Gamma(X, \mathcal{L})$, the associated divisor $D = Z(f)$ is $T$-stable if and only if $f$ is semi-invariant, that is, $f \in \Gamma(X, \mathcal{L})_\lambda$ for some character $\lambda$. In this case $f$ is an equivariant section of $\mathcal{L} \otimes \mathbb{K}_{-\lambda}$, and hence $[D] = c_1(\mathcal{L}) - c_1(\mathbb{K}_\lambda) \in H_T^*(X)$. Moreover, the $T$-stable effective Cartier divisors $D'$ satisfying $[D'] = [D]$ are in bijective correspondence with $\mathbb{P}(\Gamma(X, \mathcal{L})_\lambda)$. It follows that if $D$ is reduced and $\dim \Gamma(X, \mathcal{L}^{\otimes m})_{m\lambda} = 1$ for all $m \in \mathbb{N}$, then $D$ is $T$-equivariantly rigid. This observation can be used to produce examples of equivariantly rigid subvarieties that are not convex. For example, if $T = (\mathbb{G}_m)^{n+1}$ acts on $\mathbb{P}^n$ through the standard action on $\mathbb{K}^{n+1}$, then any reduced $T$-stable divisor $D \subset \mathbb{P}^n$ is equivariantly rigid, but $D$ is convex only if it is irreducible; see Theorem 6.3. We have not found an example of an irreducible $T$-stable subvariety that is equivariantly rigid but not convex.

## 5. Rigidity of Białynicki-Birula cells

The multiplicative group $\mathbb{G}_m$ is identified with the complement of the origin in $\mathbb{A}^1$. Given a morphism of varieties $f : \mathbb{G}_m \to X$, we write $\lim_{t\to 0} f(t) = p$ if $f$ can be extended to a morphism $\bar{f} : \mathbb{A}^1 \to X$ such that $\bar{f}(0) = p$. This limit is unique when it exists, and it always exists when $X$ is complete.

Let $X$ be a nonsingular projective $\mathbb{G}_m$-variety such that $X^{\mathbb{G}_m}$ is finite. Then each fixed point $p \in X^{\mathbb{G}_m}$ defines the (positive) Białynicki-Birula cell

$$X_p^+ = \{x \in X : \lim_{t\to 0} t \cdot x = p\}\,.$$

A negative cell is similarly defined by $X_p^- = \{x \in X : \lim_{t\to 0} t^{-1} \cdot x = p\}$. By [Białynicki-Birula 1973, Theorem 4.4], these cells form a locally closed decomposition of $X$,

$$X = \bigcup_{p\in X^{\mathbb{G}_m}} X_p^+\,, \tag{4}$$

that is, a disjoint union of locally closed subsets. In addition, each cell $X_p^+$ is isomorphic to an affine space.

**Lemma 5.1.** *For any $\mathbb{G}_m$-stable closed subset $Z \subset X$, we have*

$$Z \subset \bigcup_{p\in Z^{\mathbb{G}_m}} X_p^+.$$

*Proof.* For any point $x \in Z$, we have $x \in X_p^+$, where $p = \lim_{t\to 0} t \cdot x \in Z^{\mathbb{G}_m}$. □

**Definition 5.2.** A locally closed decomposition $X = \bigcup X_i$ is called a *stratification* if each subset $X_i$ is nonsingular and its closure $\overline{X_i}$ is a union of subsets $X_j$ of the decomposition.

The Białynicki-Birula decomposition (4) typically fails to be a stratification, for example, when $X$ is the blow-up of $\mathbb{P}^2$ at the point [0, 1, 0], where $\mathbb{G}_m$ acts on $\mathbb{P}^2$ by $t \cdot [x, y, z] = [x, ty, t^2 z]$; see [Białynicki-Birula 1976, Example 1]. Lemma 5.1 shows that the Białynicki-Birula decomposition is a stratification if and only if $X_q^+ \subset \overline{X_p^+}$ holds for each fixed point $q \in (\overline{X_p^+})^{\mathbb{G}_m}$. It was proved in [Białynicki-Birula 1976, Theorem 5] that the decomposition is a stratification when each positive cell $X_p^+$ meets each negative cell $X_q^-$ transversally. In particular, this holds when $X = G/P$ is a flag variety and $\mathbb{G}_m \subset G$ is a general one-parameter subgroup; see [McGovern 2002, Example 4.2] or Lemma 6.1. When both the positive and negative Białynicki-Birula decompositions are stratifications, all cells $X_p^+$ and $X_q^-$ of complementary dimensions meet transversally, and hence the positive and negative cell closures form a pair of Poincaré dual bases of the cohomology ring $H^*(X)$; see [Benedetti and Perrin 2022, Lemma 3.11]. In this paper we utilize the following application, which is a consequence of Lemma 5.1.

**Proposition 5.3.** *Assume that the Białynicki-Birula decomposition of $X$ is a stratification. Then each cell closure $\overline{X_p^+} \subset X$ is $\mathbb{G}_m$-convex.*

**Corollary 5.4.** *Let $T$ be an algebraic torus and $X$ a nonsingular projective $T$-variety such that all fixed points $p \in X^T$ are fully definite in $X$. Assume that $X^T = X^{\mathbb{G}_m}$ for some one-parameter subgroup $\mathbb{G}_m \subset T$, such that the associated Białynicki-Birula decomposition of $X$ is a stratification. Then each cell closure $\overline{X_p^+}$ is $T$-convex and $T$-equivariantly rigid.*

*Proof.* The cell $X_p^+$ is $T$-stable because $T$ is commutative and $p \in X^T$. The result now follows from Theorem 4.3 and Proposition 5.3. □

**Question 5.5.** We do not know whether Proposition 5.3 and Corollary 5.4 are true without the assumption that the Białynicki-Birula decomposition of $X$ is a stratification. It would be very interesting to settle this question.

**Example 5.6.** Let $X$ be a nonsingular projective toric variety, with torus $T \subset X$, and choose $\mathbb{G}_m \subset T$ such that $X^T = X^{\mathbb{G}_m}$. We show that the conclusion of Corollary 5.4 holds, even though the Białynicki-Birula decomposition is rarely a stratification. All fixed points $p \in X^T$ are fully definite in $X$, as the weights of $T_pX$ form a basis of the character lattice of $T$. The $T$-orbits $O_\tau \subset X$ correspond to the cones $\tau$ of the fan defining $X$, and we have $O_\sigma \subset \overline{O_\tau}$ if and only if $\tau$ is a face of $\sigma$; see [Fulton 1993, §3.1]. In particular, the $T$-fixed points in $X$ correspond to the maximal cones $\sigma$. Since $X$ is complete, each cone $\tau$ is the intersection of the maximal cones $\sigma$ corresponding to the $T$-fixed points in $\overline{O_\tau}$. Since all cell closures $\overline{X_p^+}$ are $T$-orbit closures, it suffices to show that each orbit closure $\overline{O_\tau}$ is $T$-convex. Let $Z \subset X$ be a $T$-stable closed subvariety such that $Z^T \subset \overline{O_\tau}$. We may assume that $Z$ is irreducible, in which case $Z = \overline{O_\kappa}$ is also a $T$-orbit closure. Since $\kappa$ is the intersection of the maximal cones given by the fixed points in $Z^T$, we obtain $\tau \subset \kappa$ and $\overline{O_\kappa} \subset \overline{O_\tau}$, as required. Now assume that $X$ has dimension two. By [Białynicki-Birula 1973, Corollary 1 of Theorem 4.5], there is a unique repulsive fixed point $b \in X^{\mathbb{G}_m}$ with $X_b^+ = \{b\}$, and a unique attractive fixed point $a \in X^{\mathbb{G}_m}$ such that $X_a^+$ is a dense open subset of $X$. For all other fixed points $p \in X^{\mathbb{G}_m} \smallsetminus \{a, b\}$, the cell $X_p^+ \cong \mathbb{A}^1$ is a line. If the Białynicki-Birula decomposition of $X$ is a stratification, then $b \in \overline{X_p^+}$ for all $p \in X^{\mathbb{G}_m}$. The $T$-fixed point $b$ corresponds to a maximal cone $\sigma$, and $b$ is connected to exactly two $T$-stable lines corresponding to the rays forming the boundary of this cone. We deduce that $X$ contains at most four $T$-fixed points. Higher-dimensional toric varieties for which the Białynicki-Birula decomposition is not a stratification can be constructed by taking products. We do not know if the cell closures $\overline{X_p^+}$ are $\mathbb{G}_m$-convex when $X$ is a toric variety.[1]

[1] Teddy Gonzales and Chayim Lowen [≥ 2025] have recently produced several examples showing that $\overline{X_p^+}$ may not be $\mathbb{G}_m$-convex when $X$ is a nonsingular projective toric variety.

## 6. Rigidity of Richardson varieties

Let $X = G/P = \{g.P : g \in G\}$ be a flag variety defined by a connected reductive linear algebraic group $G$ and a parabolic subgroup $P$. Fix a maximal torus $T$ and a Borel subgroup $B$ such that $T \subset B \subset P \subset G$. The opposite Borel subgroup $B^- \subset G$ is defined by $B^- \cap B = T$. Let $\Phi$ be the root system of nonzero weights of $T_1 G$, the tangent space of $G$ at the identity element. The positive roots $\Phi^+$ are the nonzero weights of $T_1 B$. Let $W = N_G(T)/T$ be the Weyl group of $G$, $W_P = N_P(T)/T$ the Weyl group of $P$, and let $W^P \subset W$ be the subset of minimal representatives of the cosets in $W/W_P$. The set of $T$-fixed points in $X$ is given by $X^T = \{w \cdot P : w \in W\}$, where each point $w \cdot P$ depends only on the coset $wW_P$ in $W/W_P$. Each fixed point $w \cdot P$ defines the *Schubert varieties* $X_w = \overline{Bw \cdot P}$ and $X^w = \overline{B^- w \cdot P}$. For $w \in W^P$ we have $\dim(X_w) = \operatorname{codim}(X^w, X) = \ell(w)$. The Bruhat order $\leq$ on $W^P$ is defined by

$$u \leq w \iff X_u \subset X_w \iff X^u \supset X^w \iff X^u \cap X_w \neq \varnothing.$$

A *Richardson variety* is any nonempty intersection $X_w^u = X_w \cap X^u$ of opposite Schubert varieties in $X$. More generally, any $G$-translate of $X_w^u$ will be called a Richardson variety. Any Richardson variety is reduced, irreducible, and rational; see [Deodhar 1977; Brion and Kumar 2005, §2].

Recall that a cocharacter $\rho : \mathbb{G}_m \to T$ is *strongly dominant* if $\langle \alpha, \rho \rangle > 0$ for all positive roots $\alpha \in \Phi^+$, where $\langle \alpha, \rho \rangle \in \mathbb{Z}$ is defined by $\alpha(\rho(t)) = t^{\langle \alpha, \rho \rangle}$ for $t \in \mathbb{G}_m$. The following lemma is well known; see, e.g., [McGovern 2002, Example 4.2] or [Benedetti and Perrin 2022, Corollary 3.14].

**Lemma 6.1.** *Let $\rho : \mathbb{G}_m \to T$ be a strongly dominant cocharacter. Then the associated Białynicki-Birula cells of $X$ are given by $X_p^+ = B \cdot p$, for $p \in X^T$.*

*Proof.* Let $\mathbb{G}_m$ act on $G$ by conjugation through $\rho$. The fixed point set for this action is [Springer 1998, (7.1.2), (7.6.4)]

$$T = \{g \in G : tgt^{-1} = g \ \forall t \in \mathbb{G}_m\},$$

and the corresponding Białynicki-Birula cell is [Springer 1998, (8.2.1)]

$$B = \{g \in G : \lim_{t \to 0} tgt^{-1} \in T\}.$$

This implies $B \cdot p \subset X_p^+$ for any fixed point $p \in X^{\mathbb{G}_m}$. We deduce from (4) that the positive Białynicki-Birula cells in $X$ are the $B$-orbits. □

**Lemma 6.2.** *Let $Y$ be any $G$-variety, and $\Omega \subset Y$ a $T$-stable closed subvariety. Any $T$-stable $G$-translate of $\Omega$ has the form $w \cdot \Omega$, with $w \in N_G(T)$.*

*Proof.* Let $\Omega' = g \cdot \Omega$ be a $T$-stable translate, and let $H \subset G$ be the stabilizer of $\Omega'$. Since $T$ and $gTg^{-1}$ are maximal tori in $H$, we can choose $h \in H$ such that $T = hgTg^{-1}h^{-1}$. We obtain $hg \in N_G(T)$ and $\Omega' = h \cdot \Omega' = hg \cdot \Omega$, as required. □

**Theorem 6.3.** *Any $T$-stable Richardson variety in the flag variety $X = G/P$ is $T$-convex and $T$-equivariantly rigid.*

*Proof.* It follows from Proposition 5.3 and Lemma 6.1 that all Schubert varieties $X_w$ and $X^u$ are convex. This implies that every Richardson variety $X_w^u = X_w \cap X^u$ is convex; hence all $T$-stable Richardson varieties in $X$ are convex by Lemma 6.2. The $B$-fixed point $p = 1 \cdot P$ is fully definite in $X$ because the weights of $T_pX$ are a subset of the negative roots of $G$. Since $W$ acts transitively on $X^T$, this implies that all $T$-fixed points in $X$ are fully definite. The result therefore follows from Theorem 4.3. □

Let $E = G/B$ denote the variety of complete flags, and let $\pi : E \to X$ be the natural projection. A *projected Richardson variety* in $X$ is the image $\Pi_w^u(X) = \pi(E_w^u)$ of a Richardson variety in $E$. Projected Richardson varieties in the Grassmannian $X = \mathrm{Gr}(m, n)$ of type A, obtained as images of Richardson varieties in $\mathrm{Fl}(n)$, are also called *positroid varieties*.

**Corollary 6.4.** *Let $X = \mathrm{Gr}(m, n)$ be a Grassmannian of type A, and let $T = (\mathbb{G}_m)^n$ act on $X$ through the diagonal action on $\mathbb{K}^n$. Then all positroid varieties in $X$ are $T$-convex and $T$-equivariantly rigid.*

*Proof.* It was proved in [Knutson et al. 2013] that any positroid variety $\Omega$ is defined by Plucker equations. Equivalently, $\Omega$ is an intersection of $T$-stable Schubert divisors, so $\Omega$ is convex by Theorem 6.3 and equivariantly rigid by Theorem 4.3. □

**Remark 6.5.** Corollary 6.4 does not hold for projected Richardson varieties in arbitrary flag varieties $X = G/P$. Each simple root $\beta$ defines a projected Richardson divisor $D_\beta = \Pi^{s_\beta}_{w_0^P}(X)$, where $w_0^P$ denotes the longest element in $W^P$. It frequently happens that two distinct divisors $D_{\beta'}$ and $D_{\beta''}$ have the same $T$-equivariant cohomology and $K$-theory classes, which implies that these divisors are not equivariantly rigid. For example, this is the case for the quadric hypersurfaces of dimensions seven and eight, of Lie types $B_4$ and $D_5$, and the two-step flag variety $\mathrm{Fl}(1, 4; 5)$ of type $A_4$. For other flag varieties $X$, all projected Richardson varieties have distinct equivariant classes, but some projected Richardson divisor $D_\beta$ contains all $T$-fixed points in $X$, which rules out that $D_\beta$ is convex. For example, this is the case for the Lagrangian Grassmannian $\mathrm{LG}(2, 4)$ of type $C_2$ and the maximal orthogonal Grassmannian $\mathrm{OG}(4, 8)$ of type $D_4$. This is a special case of [Benedetti and Perrin 2022, Lemma 3.1], which can be used to produce many more examples.

Any element $u \in W$ has a unique factorization $u = u^P u_P$ for which $u^P \in W^P$ and $u_P \in W_P$, called the *parabolic factorization* with respect to $P$. This factorization is *reduced* in the sense that $\ell(u) = \ell(u^P) + \ell(u_P)$. The parabolic factorization of the longest element $w_0 \in W$ is $w_0 = w_0^P w_{0,P}$, where $w_0^P$ and $w_{0,P}$ are the longest elements in $W^P$ and $W_P$, respectively. Since $w_0$ and $w_{0,P}$ are self-inverse, we have $w_{0,P} = w_0 w_0^P$. As preparation for the next section, we prove the following identity of Schubert varieties.

**Lemma 6.6.** *Let $Q \subset G$ be a parabolic subgroup containing $B$ and set $w = w_0^Q$. Then $w^{-1} \cdot X^w = X_{w_0 w}$.*

*Proof.* Since $X_{w_{0,Q}}$ is a $Q$-stable Schubert variety, we have $X_{w_{0,Q}} = w_{0,Q} \cdot X_{w_{0,Q}}$. By translating both sides by $w = w_0^Q$, we obtain $w \cdot X_{w_0 w} = w_0 \cdot X_{w_0 w} = X^w$. □

## 7. Seidel neighborhoods

In this section we prove a conjecture about curve neighborhoods from [Buch et al. 2023]. Since this conjecture and its proof relies on the moduli space of stable maps, we will restrict our attention to varieties defined over the field $\mathbb{K} = \mathbb{C}$ of complex numbers. As in Section 6, we let $X = G/P$ denote a flag variety.

For any effective degree $d \in H_2(X, \mathbb{Z})$, we let $M_d = \overline{\mathcal{M}}_{0,3}(X, d)$ denote the Kontsevich moduli space of three-pointed stable maps to $X$ of degree $d$ and genus zero; see [Fulton and Pandharipande 1997]. The evaluation map $\mathrm{ev}_i : M_d \to X$, defined for $1 \leq i \leq 3$, sends a stable map to the image of the $i$-th marked point in its domain. Given two opposite Schubert varieties $X_v$ and $X^u$, the *Gromov–Witten variety* $M_d(X_v, X^u)$ is the variety of stable maps that send the first two marked points to $X_v$ and $X^u$:

$$M_d(X_v, X^u) = \mathrm{ev}_1^{-1}(X_v) \cap X_2^{-1}(X^u) \subset M_d.$$

The *curve neighborhood* $\Gamma_d(X_v, X^u)$ is the union of all stable curves of degree $d$ in $X$ connecting $X_v$ and $X^u$:

$$\Gamma_d(X_v, X^u) = \mathrm{ev}_3(M_d(X_v, X^u)) \subset X.$$

Let $\mathbb{Z}[q] = \mathrm{Span}_{\mathbb{Z}}\{q^d : d \in H_2(X, \mathbb{Z}) \text{ effective}\}$ be the semigroup ring defined by the effective curve classes on $X$. The equivariant quantum cohomology ring of $X$ is an algebra over $H_T^*(\text{point}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$, which is defined by $\mathrm{QH}_T(X) = H_T^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a module. The *quantum product* of two opposite Schubert classes is given by

$$[X_v] \star [X^u] = \sum_{d \geq 0} q^d \, \mathrm{ev}_{3,*}[M_d(X_v, X^u)],$$

where the sum is over all effective degrees $d \in H_2(X; \mathbb{Z})$.

A simple root $\gamma \in \Phi^+$ is called *cominuscule* if, when the highest root is written in the basis of simple roots, the coefficient of $\gamma$ is one. The flag variety $G/Q$ is cominuscule if $Q$ is a maximal parabolic subgroup corresponding to a cominuscule simple root $\gamma$, that is, $s_\gamma$ is the unique simple reflection in $W^Q$. Let $W^{\mathrm{comin}} \subset W$

be the subset of point representatives of cominuscule flag varieties of $G$, together with the identity element:

$$W^{\text{comin}} = \{w_0^Q : G/Q \text{ is cominuscule}\} \cup \{1\}.$$

This is a subgroup of $W$, which is isomorphic to the quotient of the coweight lattice of $\Phi$ modulo the coroot lattice [Bourbaki 1981, Proposition VI.2.6]. The isomorphism sends $w_0^Q$ to the class of the fundamental coweight $\omega_\gamma^\vee$ corresponding to $Q$. Notice that $\gamma$ is the unique simple root for which $w_0^Q \cdot \gamma < 0$. In the following we set $d(w_0^Q, u) = \omega_\gamma^\vee - u^{-1} \cdot \omega_\gamma^\vee \in H_2(X;\mathbb{Z})$ for any $u \in W$. Here we identify the group $H_2(X,\mathbb{Z})$ with a quotient of the coroot lattice, by mapping each simple coroot $\beta^\vee$ to the curve class $[X_{s_\beta}]$ if $s_\beta \in W^P$, and to zero otherwise.

The *Seidel representation* of $W^{\text{comin}}$ on $\mathrm{QH}(X)/\langle q-1\rangle$ is defined by $w \cdot [X^u] = [X^w] \star [X^u]$ for $w \in W^{\text{comin}}$ and $u \in W$. In fact, we have [Seidel 1997; Belkale 2004; Chaput et al. 2009]

$$[X^w] \star [X^u] = q^{d(w,u)}[X^{wu}] \tag{5}$$

in the (nonequivariant) quantum ring $\mathrm{QH}(X)$. This implies that $d(w,u)$ is the unique minimal degree $d$ for which $\Gamma_d(X_{w_0 w}, X^u)$ is not empty [Fulton and Woodward 2004; Buch et al. 2020]. More generally, it was proved in [Chaput et al. 2009; Chaput and Perrin 2023] that the identity

$$[X^w] \star [w \cdot X^u] = q^{d(w,u)}[X^{wu}] \tag{6}$$

holds in the equivariant quantum cohomology ring $\mathrm{QH}_T(X)$. We will discuss generalizations to quantum $K$-theory in Section 8.

It follows from (5) and the definition of the quantum product in $\mathrm{QH}(X)$ that $[\Gamma_{d(w,u)}(X_{w_0 w}, X^u)] = [X^{wu}]$ holds in $H^*(X)$. Conjecture 3.11 from [Buch et al. 2023] asserts that $\Gamma_{d(w,u)}(X_{w_0 w}, X^u)$ is in fact equal to the translated Schubert variety $w^{-1} \cdot X^{wu}$. This is proved below as a consequence of Theorem 6.3 and (6). This result was known when $X = G/P$ is cominuscule and $w = w_0^P$ [Buch et al. 2023], when $X$ is a Grassmannian of type A and $[X^w]$ is a special Seidel class [Li et al. 2025, Corollary 4.6], when $X$ is any flag variety of type A [Tarigradschi 2023], and when $X$ is the symplectic Grassmannian $\mathrm{SG}(2, 2n)$ [Benedetti et al. 2024, Theorem 8.1].

**Theorem 7.1.** *Let $X = G/P$ be a complex flag variety. For $w \in W^{\text{comin}}$ and $u \in W$ we have $\Gamma_{d(w,u)}(X_{w_0 w}, X^u) = w^{-1} \cdot X^{wu}$.*

*Proof.* By applying $w^{-1}$ to both sides of (6) and using Lemma 6.6, we obtain

$$[X_{w_0 w}] \star [X^u] = q^{d(w,u)}[w^{-1} \cdot X^{wu}]$$

in $\mathrm{QH}_T(X)$. By definition of the quantum product, this implies that

$$[w^{-1} \cdot X^{wu}] = \mathrm{ev}_{3,*}[M_{d(w,u)}(X_{w_0 w}, X^u)] = c\,[\Gamma_{d(w,u)}(X_{w_0 w}, X^u)]$$

holds in $H_T^*(X)$, where $c$ is the degree of the map

$$\mathrm{ev}_3 : M_{d(w,u)}(X_{w_0 w}, X^u) \to \Gamma_{d(w,u)}(X_{w_0 w}, X^u).$$

The result therefore follows from Theorem 6.3. □

## 8. Seidel products in quantum $K$-theory

In this section we discuss a generalization of the Seidel multiplication formula to quantum $K$-theory. We start by briefly recalling the definition of quantum $K$-theory. A more detailed discussion can be found in [Buch et al. 2018a, §2].

Let $X = G/P$ be a flag variety defined over $\mathbb{K} = \mathbb{C}$. The equivariant $K$-theory ring $K^T(X)$ is an algebra over the representation ring $\Gamma = K^T(\text{point})$. The equivariant quantum $K$-theory ring $\mathrm{QK}_T(X)$ was originally constructed by Givental [2000] and Lee [2004]. This ring is an algebra over the formal power series ring $\Gamma[\![q]\!] = \Gamma[\![q_\beta : s_\beta \in W^P]\!]$, which has one variable $q_\beta$ for each simple reflection $s_\beta$ in $W^P$. As a module over $\Gamma[\![q]\!]$ we have $\mathrm{QK}_T(X) = K^T(X) \otimes_\Gamma \Gamma[\![q]\!]$. The *undeformed product* of two opposite Schubert classes in $\mathrm{QK}_T(X)$ is defined by

$$[\mathcal{O}_{X_v}] \odot [\mathcal{O}_{X^u}] = \sum_{d \geq 0} q^d \,\mathrm{ev}_{3,*}[\mathcal{O}_{M_d(X_v, X^u)}].$$

Let $\Psi : \mathrm{QK}_T(X) \to \mathrm{QK}_T(X)$ be the $\Gamma[\![q]\!]$-linear map defined by

$$\Psi([\mathcal{O}_{X^w}]) = \sum_{d \geq 0} q^d [\mathcal{O}_{\Gamma_d(X^w)}],$$

where the curve neighborhood $\Gamma_d(X^w) = \mathrm{ev}_2(\mathrm{ev}_1^{-1}(X^w))$ is defined using the evaluation maps from $M_d$. This curve neighborhood is a Schubert variety in $X$ by [Buch et al. 2013, Proposition 3.2(b)], whose Weyl group element was determined in [Buch and Mihalcea 2015]. By [Buch et al. 2018a, Proposition 2.3], Givental's *quantum $K$-theory product* $\star$ is given by

$$[\mathcal{O}_{X_v}] \star [\mathcal{O}_{X^u}] = \Psi^{-1}([\mathcal{O}_{X_v}] \odot [\mathcal{O}_{X^u}]). \tag{7}$$

The following conjecture is the $K$-theoretic analogue of the Seidel multiplication formula (6) in $\mathrm{QH}_T(X)$ proved in [Chaput et al. 2009; Chaput and Perrin 2023].

**Conjecture 8.1.** *For $w \in W^{\mathrm{comin}}$ and $u \in W$ we have*

$$[\mathcal{O}_{X_{w_0 w}}] \star [\mathcal{O}_{X^u}] = q^{d(w,u)} [\mathcal{O}_{w^{-1}.X^{wu}}] \quad \text{and} \quad [\mathcal{O}_{X^w}] \star [\mathcal{O}_{w.X^u}] = q^{d(w,u)} [\mathcal{O}_{X^{wu}}]$$

*in* $\mathrm{QK}_T(X)$.

The two identities in Conjecture 8.1 are equivalent by Lemma 6.6. The nonequivariant case of this conjecture was proved in [Buch et al. 2023, Corollary 3.7] when $X$ is a cominuscule flag variety. When $X$ is a Grassmannian of type A, the conjecture is equivalent to [Gorbounov et al. 2025, Corollary 10.4]. We will prove Conjecture 8.1 for cominuscule flag varieties below, based on the following conjectural generalization of Theorem 7.1. Recall that a morphism $\pi : Z \to Y$ is called *cohomologically trivial* if $\pi_* \mathcal{O}_Z = \mathcal{O}_Y$ and $R^j \pi_* \mathcal{O}_Z = 0$ for $j \geq 1$.

**Conjecture 8.2.** *Let $w \in W^{\text{comin}}$, $u \in W$, and let $e \in H_2(X, \mathbb{Z})$ be effective.*

(a) *We have $\Gamma_{d(w,u)+e}(X_{w_0 w}, X^u) = \Gamma_e(w^{-1} \cdot X^{wu})$.*

(b) *The evaluation map $\text{ev}_3 : M_{d(w,u)+e}(X_{w_0 w}, X^u) \to \Gamma_{d(w,u)+e}(X_{w_0 w}, X^u)$ is cohomologically trivial.*

Conjecture 8.2 is a variant of the quantum-equals-classical theorem for Gromov–Witten invariants as stated in [Buch et al. 2018b, Theorem 4.1]; see also [Xu 2024, Theorem 1.2]. The conjecture is true for $e = 0$; part (a) is equivalent to Theorem 7.1, and part (b) holds because the map $\text{ev}_3 : M_{d(w,u)}(X_{w_0 w}, X^u) \to \Gamma_{d(w,u)}(X_{w_0 w}, X^u)$ is birational by [Belkale 2004; Chaput et al. 2009], and $M_{d(w,u)}(X_{w_0 w}, X^u)$ has rational singularities by [Buch et al. 2013, Corollary 3.1]. For $e \geq 0$, Theorem 7.1 implies that

$$\Gamma_e(w^{-1}.X^{wu}) = \Gamma_e(\Gamma_{d(w,u)}(X_{w_0 w}, X^u)) \subset \Gamma_{d(w,u)+e}(X_{w_0 w}, X^u), \tag{8}$$

and $\Gamma_{d(w,u)+e}(X_{w_0 w}, X^u)$ is irreducible by Corollary 3.8 in [Buch et al. 2013]. Conjecture 8.2(a) is therefore true if and only if $\Gamma_{d(w,u)+e}(X_{w_0 w}, X^u)$ and $\Gamma_e(X^{wu})$ have the same dimension. We prove below that Conjecture 8.2(a) is true when $X = \text{GL}(n)/P$ is any flag variety of Lie type A. Conjecture 8.1 follows from Conjecture 8.2 by the following observation.

**Lemma 8.3.** *Given $w \in W^{\text{comin}}$ and $u \in W$, the identity*

$$[\mathcal{O}_{X_{w_0 w}}] \star [\mathcal{O}_{X^u}] = q^{d(w,u)} [\mathcal{O}_{w^{-1}.X^{wu}}]$$

*holds in $\text{QK}_T(X)$ if and only if*

$$\text{ev}_{3,*}[\mathcal{O}_{M_{d(w,u)+e}(X_{w_0 w}, X^u)}] = [\mathcal{O}_{\Gamma_e(w^{-1}.X^{wu})}] \tag{9}$$

*holds in $K_T(X)$ for all effective degrees $e \in H_2(X, \mathbb{Z})$.*

*Proof.* Both assertions are equivalent to the identity

$$[\mathcal{O}_{X_{w_0 w}}] \odot [\mathcal{O}_{X^u}] = \sum_{e \geq 0} q^{d(w,u)+e} [\mathcal{O}_{\Gamma_e(w^{-1}.X^{wu})}]$$

by the definition (7) of the quantum product in $\text{QK}_T(X)$. □

**Theorem 8.4.** *Conjectures 8.1 and 8.2 hold when $X$ is a cominuscule flag variety.*

*Proof.* Assume that $X$ is cominuscule. Then Conjecture 8.2(b) is a special case of [Buch et al. 2018b, Theorem 4.1], and Conjecture 8.2(a) follows from Theorem 7.1 and [Buch et al. 2022, Corollary 8.24], noting that $q^{d(w,u)}$ is the maximal power of $q$ occurring in the quantum cohomology product $[X_{w_0w}]\star[X^u]$ by [Belkale 2004; Chaput et al. 2009]. This proves Conjecture 8.2, which implies Conjecture 8.1 by Lemma 8.3. □

We finish this section by proving that Conjecture 8.2(a) can be reduced to the case where $X$ is a flag variety of Picard rank 1. In particular, Conjecture 8.2(a) follows from Theorem 8.4 in type A. These results were proved for $e=0$ in [Tarigradschi 2023]. We thank Mihail Tarigradschi for suggesting that his methods might apply to the general case of our conjecture.

Recall that $X=G/P$. Let $Q_1$, $Q_2\subset G$ be parabolic subgroups such that $P=Q_1\cap Q_2$. Set $Y_i=G/Q_i$ and let $\pi_i : X\to Y_i$ be the projection, for $i\in\{1,2\}$. Given a degree $d\in H_2(X,\mathbb{Z})$, we also let $d$ denote the image $\pi_{i,*}(d)$ of this degree in $H_2(Y_i,\mathbb{Z})$. Let $\Gamma_d(Y_{i,v}, Y_i^u)\subset Y_i$ be the union of all stable curves of degree $d$ in $Y_i$ that connect the Schubert varieties $Y_{i,v}=\pi_i(X_v)$ and $Y_i^u=\pi_i(X^u)$, for $u, v\in W$. The next result generalizes [Björner and Brenti 2005, Theorem 2.6.1; Tarigradschi 2023, Lemma 4].

**Lemma 8.5.** *We have* $\Gamma_d(X^u)=\pi_1^{-1}(\Gamma_d(Y_1^u))\cap\pi_2^{-1}(\Gamma_d(Y_2^u))$.

*Proof.* Let $\operatorname{dist}_X(X_v, X^u)$ denote the unique minimal degree of a rational curve in $X$ connecting $X_v$ and $X^u$. It follows from [Buch et al. 2020, Theorem 5] that this degree is uniquely determined by $\pi_{i,*}(\operatorname{dist}_X(X_v, X^u))=\operatorname{dist}_{Y_i}(Y_{i,v}, Y_i^u)$ for $i\in\{1,2\}$. Using that $v\cdot P\in\Gamma_d(X^u)$ holds if and only if $d\geq\operatorname{dist}_X(X_v, X^u)$, we deduce that $\Gamma_d(X^u)$ and $\pi_1^{-1}(\Gamma_d(Y_1^u))\cap\pi_2^{-1}(\Gamma_d(Y_2^u))$ contain the same $T$-fixed points. The lemma follows from this, as both sets are $B^-$-stable subvarieties of $X$. □

The following result implies that Conjecture 8.2(a) follows from the case where $X$ has Picard rank 1. It was proved for $e=0$ in [Tarigradschi 2023, Theorem 3].

**Theorem 8.6.** *Let $X=G/P$, $Y_1=G/Q_1$, and $Y_2=G/Q_2$ be flag varieties such that $P=Q_1\cap Q_2$. Let $w\in W^{\mathrm{comin}}$, $u\in W$, and let $e\in H_2(X,\mathbb{Z})$ be any effective degree. If $\Gamma_{d(w,u)+e}(Y_{i,w_0w}, Y_i^u)=\Gamma_e(w^{-1}\cdot Y_i^{wu})$ holds for $i\in\{1,2\}$, then $\Gamma_{d(w,u)+e}(X_{w_0w}, X^u)=\Gamma_e(w^{-1}\cdot X^{wu})$.*

*Proof.* The assumptions and Lemma 8.5 imply that

$$\begin{aligned}\Gamma_{d(w,u)+e}(X_{w_0w}, X^u)\subset\pi_1^{-1}(\Gamma_{d(w,u)+e}(Y_{1,w_0w}, Y_1^u))\cap\pi_2^{-1}(\Gamma_{d(w,u)+e}(Y_{2,w_0w}, Y_2^u))\\ =\pi_1^{-1}(\Gamma_e(w^{-1}\cdot Y_1^{wu}))\cap\pi_2^{-1}(\Gamma_e(w^{-1}\cdot Y_2^{wu}))=\Gamma_e(w^{-1}\cdot X^{wu}),\end{aligned}$$

and the opposite inclusion holds by (8). □

**Corollary 8.7.** *Conjecture 8.2(a) is true when $X=\mathrm{GL}(n)/P$ has Lie type A.*

*Proof.* This follows from Theorems 8.4 and 8.6, noting that all flag varieties of type A with Picard rank 1 are Grassmannians, and therefore cominuscule. □

## 9. Horospherical varieties of Picard rank 1

In this section we interpret Theorem 4.3 and Proposition 5.3 for a class of horospherical varieties that includes all nonsingular projective horospherical varieties of Picard rank 1 (except flag varieties) by Pasquier's classification [2009]. Let $G$ be a connected reductive linear algebraic group, $B \subset G$ a Borel subgroup, and $T \subset B$ a maximal torus. Let $V_1$ and $V_2$ be irreducible rational representations of $G$, and let $v_i \in V_i$ be a highest-weight vector of weight $\lambda_i$, for $i \in \{1, 2\}$. We assume that $\lambda_1 \neq \lambda_2$. Define

$$X = \overline{G \cdot [v_1 + v_2]} \subset \mathbb{P}(V_1 \oplus V_2).$$

If $X$ is normal, then $X$ is a horospherical variety of rank 1; see [Timashev 2011, Chapter 7]. We will assume that $X$ is nonsingular and $\mathbb{K} = \mathbb{C}$, even though many claims hold more generally; this implies that $X$ is fibered over a flag variety $G/P_{12}$ with nonsingular horospherical fibers of Picard rank 1; see Remark 9.5. Any $G$-translate of a $B$-orbit closure in $X$ will be called a *Schubert variety*. Our next result uses the action of $T \times \mathbb{G}_m$ on $X$ defined by $(t, z)\cdot[u_1+u_2] = t\cdot[u_1+zu_2]$, for $u_i \in V_i$. We have $X^{T\times\mathbb{G}_m} = X^T$, and a Schubert variety is $T$-stable if and only if it is $T \times \mathbb{G}_m$-stable.

**Theorem 9.1.** *Any $T$-stable Schubert variety in $X$ is $T \times \mathbb{G}_m$-convex and $T \times \mathbb{G}_m$-equivariantly rigid.*

Before proving Theorem 9.1, we sketch elementary proofs of some basic facts about $X$, which are also consequences of general results about spherical varieties; see [Timashev 2011; Perrin 2014; Pasquier 2009].

Given an element $[u_1 + u_2] \in \mathbb{P}(V_1 \oplus V_2)$, we will always assume $u_i \in V_i$, and $i$ will always mean an element from $\{1, 2\}$. We consider $\mathbb{P}(V_i)$ as a subvariety of $\mathbb{P}(V_1 \oplus V_2)$. Let $\pi_i : \mathbb{P}(V_1 \oplus V_2) \smallsetminus \mathbb{P}(V_{3-i}) \to \mathbb{P}(V_i)$ denote the projection from $V_{3-i}$, defined by $\pi_i([u_1 + u_2]) = [u_i]$. Set $X_0 = G \cdot [v_1 + v_2] \subset \mathbb{P}(V_1 \oplus V_2)$, $X_i = G \cdot [v_i] \subset \mathbb{P}(V_i)$, and $X_{12} = G \cdot ([v_1], [v_2]) \subset \mathbb{P}(V_1) \times \mathbb{P}(V_2)$. Since $v_i$ is a highest-weight vector, the stabilizer $P_i = G_{[v_i]}$ is a parabolic subgroup containing $B$. It follows that $X_i \cong G/P_i$ and $X_{12} \cong G/(P_1 \cap P_2)$ are flag varieties. In particular, $X_i$ is closed in $\mathbb{P}(V_i)$, and $X_{12}$ is closed in $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$. Notice also that $X_0 \cong G/H$, where $H \subset P_1 \cap P_2$ is the kernel of the character $\lambda_1 - \lambda_2 : P_1 \cap P_2 \to \mathbb{G}_m$. This shows that $X_0$ is a $\mathbb{G}_m$-bundle over $X_{12}$, so $X$ is a nonsingular projective horospherical variety of rank 1 (but not necessarily of Picard rank 1; see Remark 9.5).

Let $W$ be the Weyl group of $G$, and recall the notation from Section 6.

**Lemma 9.2.** *We have $X = X_0 \cup X_1 \cup X_2$. The $B$-orbit closures in $X$ are*

$$\overline{Bw \cdot [v_i]} = \bigcup_{w' \leq w} Bw' \cdot [v_i] \quad \textit{for } w \in W^{P_i} \textit{ and } i \in \{1, 2\},$$

$$\overline{Bw \cdot [v_1 + v_2]} = \bigcup_{w' \leq w} (Bw' \cdot [v_1 + v_2] \cup Bw' \cdot [v_1] \cup Bw' \cdot [v_2]) \quad \textit{for } w \in W^{P_1 \cap P_2}.$$

*Proof.* Set $\mathbb{P}_0 = \mathbb{P}(V_1 \oplus V_2) \smallsetminus (\mathbb{P}(V_1) \cup \mathbb{P}(V_2))$. Since $\lambda_1 \neq \lambda_2$, it follows that $\overline{T \cdot [v_1 + v_2]}$ is the line through $[v_1]$ and $[v_2]$ in $\mathbb{P}(V_1 \oplus V_2)$. This implies $X_0 = (\pi_1 \times \pi_2)^{-1}(X_{12})$; hence $X_0$ is closed in $\mathbb{P}_0$ and $X_0 = X \cap \mathbb{P}_0$. We also have $X_i \subset X \cap \mathbb{P}(V_i) \subset \pi_i^{-1}(X_i) \cap \mathbb{P}(V_i) = X_i$, which proves the first claim. To finish the proof, it suffices to show $w' \cdot [v_i] \in \overline{Bw \cdot [v_1 + v_2]}$ if and only if $w' \leq w$ (when $w' \in W^{P_i}$). The implication 'if' holds because $w' \cdot [v_i] \in \overline{Tw' \cdot [v_1 + v_2]}$, and 'only if' holds because $\pi_i(\overline{Bw \cdot [v_1 + v_2]} \smallsetminus X_{3-i}) \subset \overline{Bw \cdot [v_i]}$. □

Define an alternative action of $P_i$ on $V_{3-i}$ by $p \bullet u = \lambda_i(p)^{-1} p \cdot u$, and use this action to form the space

$$G \times^{P_i} V_{3-i} = \{[g, u] : g \in G,\ u \in V_{3-i}\}/\{[gp, u] = [g, p \bullet u] : p \in P_i\}.$$

Define a morphism of varieties $\phi_i : G \times^{P_i} V_{3-i} \to \mathbb{P}(V_1 \oplus V_2)$ by

$$\phi_i([g, u]) = g \cdot [v_i + u].$$

This is well defined since $p \cdot (v_i + u) = \lambda_i(p)(v_i + p \bullet u)$ holds for $p \in P_i$ and $u \in V_{3-i}$. Set $E_i = (P_i \bullet v_{3-i}) \cup \{0\} \subset V_{3-i}$. Noting that $E_i$ is the cone over $P_i \cdot [v_{3-i}] \cong P_i/(P_1 \cap P_2)$, it follows that $E_i$ is closed in $V_{3-i}$.

**Lemma 9.3.** *The restricted map $\phi_i : G \times^{P_i} E_i \to X_0 \cup X_i$ is an isomorphism of varieties. In particular, $E_i \subset V_{3-i}$ is a linear subspace.*

*Proof.* Assume $\phi_i([g, u]) = \phi_i([g', u'])$, and set $p = g^{-1}g'$. We obtain $p \in P_i$ and $[v_i + u] = p \cdot [v_i + u'] = [v_i + p \bullet u']$ in $\mathbb{P}(V_1 \oplus V_2)$; hence

$$[g, u] = [g, p \bullet u'] = [gp, u'] = [g', u']$$

in $G \times^{P_i} V_{3-i}$. We deduce that $\phi_i : G \times^{P_i} E_i \to X_0 \cup X_i$ is bijective, so the lemma follows from Zariski's main theorem, using that $X_0 \cup X_i$ is nonsingular. □

Fix a strongly dominant cocharacter $\rho : \mathbb{G}_m \to T$. For $a \in \mathbb{Z}$, define the map $\rho_a : \mathbb{G}_m \to T \times \mathbb{G}_m$ by $\rho_a(z) = (\rho(z), z^a)$. The resulting action of $\mathbb{G}_m$ on $X$ is given by $\rho_a(z) \cdot [u_1 + u_2] = \rho(z) \cdot [u_1 + z^a u_2]$.

**Lemma 9.4.** *All $T$-fixed points in $X$ are fully definite for the action of $T \times \mathbb{G}_m$.*

*Proof.* Lemma 9.3 shows that $[v_1]$ has a $T \times \mathbb{G}_m$-stable open neighborhood in $X$ isomorphic to $B^- \cdot [v_1] \times E_1$, where the action is given by $(t, z) \cdot (x, u) = (t \cdot x, t \bullet zu)$. If $a$ is sufficiently negative, then $\mathbb{G}_m$ acts through $\rho_a$ on $T_{[v_1]}X = T_{[v_1]}X_1 \oplus E_1$ with strictly negative weights; hence $[v_1]$ is fully definite in $X$ for the action of $T \times \mathbb{G}_m$. A symmetric argument shows that $[v_2]$ is fully definite. The result follows from this, since all $T$-fixed points in $X$ are obtained from $[v_1]$ or $[v_2]$ by the action of the Weyl group $W$. □

*Proof of Theorem 9.1.* For $a$ sufficiently negative, it follows from Lemma 6.1 that the Białynicki-Birula cells of $X$ defined by $\rho_a$ are

$$X^+_{w\cdot[v_1]} = Bw\cdot[v_1] \quad \text{and} \quad X^+_{w\cdot[v_2]} = Bw\cdot[v_1+v_2]\cup Bw\cdot[v_2].$$

These cells form a stratification of $X$ by Lemma 9.2, so Proposition 5.3 implies that $\overline{Bw\cdot[v_1]}$ and $\overline{Bw\cdot[v_1+v_2]}$ are $T\times\mathbb{G}_m$-convex for $w\in W$. A symmetric argument applies to $\overline{Bw\cdot[v_2]}$; hence all $T$-stable Schubert varieties in $X$ are $T\times\mathbb{G}_m$-convex by Lemma 6.2. The result now follows from Theorem 4.3 and Lemma 9.4. □

**Remark 9.5.** The exact sequence of [Perrin 2014, Theorem 3.2.4] implies that $\mathrm{Pic}(X)$ is a free abelian group of rank equal to the rank of $X$ (which is one) plus the number of $B$-stable prime divisors in $X$ that do not contain a $G$-orbit. Any $B$-stable prime divisor meeting $X_0$ has the form $D=\overline{Bw_0s_\beta\cdot[v_1+v_2]}$, where $\beta$ is a simple root, and Lemma 9.2 shows that $D$ contains $X_i$ if and only if $\beta$ is a root of $P_i$. Let $P_{12}\subset G$ be the parabolic subgroup generated by $P_1$ and $P_2$. We obtain $\mathrm{Pic}(X)\cong\mathbb{Z}\oplus\mathrm{Pic}(G/P_{12})$. Let $\pi:X\to G/P_{12}$ be the map defined by $\pi(g\cdot[v_1+v_2])=\pi(g\cdot[v_i])=g\cdot P_{12}$. This is a $G$-equivariant morphism of varieties, as its restriction to $X_0\cup X_i$ is the composition of $\pi_i:X_0\cup X_i\to G/P_i$ with the projection $G/P_i\to G/P_{12}$. The fibers of $\pi$ are translates of $\pi^{-1}(1\cdot P_{12})=\overline{L\cdot[v_1+v_2]}\subset\mathbb{P}(V_1\oplus V_2)$, where $L$ is the Levi subgroup of $P_{12}$ containing $T$. Moreover, $\pi^{-1}(1\cdot P_{12})$ is a nonsingular projective horospherical variety of Picard rank 1, so it is either a flag variety or one of the nonhomogeneous spaces from Pasquier's classification [2009].

**Question 9.6.** Let $X$ be any projective $G$-horospherical variety fibered over a flag variety $G/P$ with nonsingular horospherical fibers of Picard rank 1. Is it true that $X$ is isomorphic to an orbit closure $\overline{G\cdot[v_1+v_2]}\subset\mathbb{P}(V)$, where $V$ is a rational representation of $G$, and $v_1,v_2\in V$ are highest-weight vectors?

**Example 9.7.** Let $X$ be the blow-up of $\mathbb{P}^2$ at a point $p$, let $\pi:X\to\mathbb{P}^1$ be the morphism defined by projection from $p$, and set $G=\mathrm{SL}(2,\mathbb{C})$. Then $X$ is $G$-horospherical and fibered over $\mathbb{P}^1$ with fiber $\mathbb{P}^1$. This variety $X$ is isomorphic to $\overline{G\cdot[v_1+v_2]}\subset\mathbb{P}(V_1\oplus V_2)$, where $v_1$ is a highest-weight vector in $V_1=\mathbb{C}^2$, and $v_2$ is a highest-weight vector in $V_2=\mathrm{Sym}^2(\mathbb{C}^2)$.

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