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We prove that Schubert and Richardson varieties in flag manifolds are uniquely determined by their equivariant cohomology classes, as well as a stronger result that replaces Schubert varieties with closures of Białynicki-Birula cells under suitable conditions. This is used to prove a conjecture of Buch, Chaput, and Perrin, stating that any two-pointed curve neighborhood representing a quantum cohomology product with a Seidel class is a Schubert variety. We pose a stronger conjecture which implies a Seidel multiplication formula in equivariant quantum K -theory, and prove this conjecture for cominuscle flag varieties.

1. Introduction

A Schubert variety Ω in a flag manifold $X = G/P$ is called *rigid* if it is uniquely determined by its class $[\Omega]$ in the cohomology ring $H^*(X)$. More precisely, if $Z \subset X$ is any irreducible closed subvariety such that $[Z]$ is a multiple of $[\Omega]$ in $H^*(X)$, then Z is a G -translate of Ω . This problem has been studied in numerous papers; see, e.g., [Hong 2005; 2007; Coskun 2011; 2014; 2018; Robles and The 2012; Coskun and Robles 2013; Hong and Mok 2020; Liu et al. 2024]. In this paper we show that all Schubert varieties and Richardson varieties are *equivariantly rigid*. In other words, if $T \subset G$ is a maximal torus, $\Omega \subset X$ is a T -stable Richardson variety, and $Z \subset X$ is a (nonempty) T -stable closed subvariety such that the T -equivariant class $[Z] \in H_T^*(X)$ is a multiple of $[\Omega]$, then $Z = \Omega$.

More generally, let T be an algebraic torus over an algebraically closed field, let X be a nonsingular projective T -variety, and let $\Omega \subset X$ be a T -stable closed subvariety. Let Ω^T denote the set of T -fixed points in Ω . We will say that Ω is T -convex if, for any T -stable closed subvariety $Z \subset X$ satisfying $Z^T \subset \Omega$, we have $Z \subset \Omega$. A fixed point $p \in X^T$ is called *fully definite* if all T -weights of the Zariski tangent space $T_p X$ belong to a strict half-space of the character lattice of T . We show that if

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all T -fixed points in X are fully definite, then any irreducible T -convex subvariety of X is also T -equivariantly rigid. Here Ω is called T -equivariantly rigid if Ω is determined by its class in the T -equivariant Chow cohomology ring of X .

Let $\mathbb{G}_m \subset T$ be a one-parameter subgroup such that $X^T = X^{\mathbb{G}_m}$, and assume that this fixed point set is finite. The associated Białyński-Birula decomposition of X is given by $X = \bigcup_{p \in X^T} X_p^+$, where $X_p^+ = \{x \in X : \lim_{t \rightarrow 0} t \cdot x = p\}$ is the Białyński-Birula cell of points attracted to p by the action of $t \in \mathbb{G}_m$. This decomposition is called a *stratification* if each cell closure $\overline{X_p^+} \subset X$ is a union of smaller cells. In this case we show that all cell closures are T -convex. We arrive at the following result combining [Theorem 4.3](#) and [Proposition 5.3](#).

Theorem. *Let X be a nonsingular projective T -variety with finitely many T -fixed points, and choose $\mathbb{G}_m \subset T$ such that $X^T = X^{\mathbb{G}_m}$.*

- (a) *Assume that the Białyński-Birula decomposition of X is a stratification. Then each cell closure $\overline{X_p^+}$ is T -convex.*
- (b) *Assume that all T -fixed points in X are fully definite. Then any irreducible T -convex subvariety of X is T -equivariantly rigid.*

This result applies to Schubert and Richardson varieties in flag varieties, as well as positroid varieties in Grassmannians, so these subvarieties are both T -convex and T -equivariantly rigid. However, projected Richardson varieties do not in general enjoy these properties; see [Remark 6.5](#). Our theorem also covers a class of horospherical varieties, which includes all nonsingular horospherical varieties of Picard rank 1 [[Pasquier 2009](#)].

Our theorem has additional applications in quantum Schubert calculus. Let $X = G/P$ be a complex flag manifold. A Schubert class $[X^w]$ is called a *Seidel class* if the Weyl group element w is the minimal representative of a point in some cominuscule flag variety G/Q . Multiplication by Seidel classes in the quantum cohomology ring $\text{QH}(X)$ is given by the identity $[X^w] \star [X^u] = q^{d(w,u)} [X^{wu}]$, where $d(w,u)$ is the unique minimal degree of a rational curve connecting the opposite Schubert varieties $X_{w_0 w}$ and X^u [[Seidel 1997](#); [Belkale 2004](#); [Chaput et al. 2009](#)]. This implies that $[X^{wu}]$ is equal to the class of the curve neighborhood $\Gamma_{d(w,u)}(X_{w_0 w}, X^u)$, defined as the union of all stable curves in X of degree $d(w,u)$ connecting $X_{w_0 w}$ to X^u . We conjectured in [[Buch et al. 2023](#)] that this curve neighborhood is in fact the translated Schubert variety

$$(1) \quad \Gamma_{d(w,u)}(X_{w_0 w}, X^u) = w^{-1} \cdot X^{wu}.$$

This has been proved in some cases when X is cominuscule, in all cases when X is a flag variety of type A [[Li et al. 2025](#); [Tarigradschi 2023](#)], and for $X = \text{SG}(2, 2n)$ [[Benedetti et al. 2024](#)]. Using that $\Gamma_{d(w,u)}(X_{w_0 w}, X^u)$ and $w^{-1} \cdot X^{wu}$ define the same class in $H_T^*(X)$ by an equivariant version of the Seidel multiplication formula

from [Chaput et al. 2009; Chaput and Perrin 2023], the identity (1) follows from our result that Schubert varieties are equivariantly rigid.

In this paper we conjecture the more general identity

$$(2) \quad \Gamma_{d(w,u)+e}(X_{w_0w}, X^u) = \Gamma_e(w^{-1}(X^{wu})),$$

where the right-hand side is the union of all stable curves of degree e that pass through $w^{-1} \cdot X^{wu}$. This union is a Schubert variety [Buch et al. 2013] whose Weyl group element was determined in [Buch and Mihalcea 2015]. Denote by $M_{d(w,u)+e}(X_{w_0w}, X^u)$ the moduli space of three-pointed stable maps to X of degree $d(w, u) + e$ and genus zero, which send the first two marked points to X_{w_0w} and X^u , respectively. We further conjecture that the evaluation map $\text{ev}_3 : M_{d(w,u)+e}(X_{w_0w}, X^u) \rightarrow \Gamma_e(w^{-1}(X^{wu}))$ is cohomologically trivial. This conjecture implies a Seidel multiplication formula in the equivariant quantum K -theory ring $\text{QK}_T(X)$. We prove this conjecture when X is a cominuscule flag variety, thereby obtaining an equivariant generalization of our Seidel multiplication formula from [Buch et al. 2023]. This generalized Seidel multiplication formula has also been obtained for Grassmannians of type A in [Gorbounov et al. 2025] using different methods. Based on suggestions from Mihail Tarigradschi, we finally apply the methods of [Tarigradschi 2023] to prove the identity (2) when $X = \text{GL}_n(\mathbb{C})/P$ is any flag manifold of Lie type A.

Our paper is organized as follows. In Section 2 we recall some basic facts and notation related to torus actions. In Section 3 we show that if all T -fixed points of X are fully definite, then the fixed point set Z^T of a T -stable subvariety $Z \subset X$ is determined by its equivariant class $[Z] \in H_T^*(X)$. This is used in Section 4 to prove part (b) of the above theorem. Section 5 proves part (a). Section 6 interprets our theorem for flag varieties, which is used in Section 7 to prove the conjecture about curve neighborhoods from [Buch et al. 2023]. Section 8 discusses the more general conjecture as well as its consequences in quantum K -theory. Finally, Section 9 interprets our theorem for certain horospherical varieties.

2. Torus actions

We work with varieties over a fixed algebraically closed field \mathbb{K} . Varieties are reduced but not necessarily irreducible. A point will always mean a closed point. The multiplicative group of \mathbb{K} is denoted by $\mathbb{G}_m = \mathbb{K} \setminus \{0\}$. An (algebraic) torus is a group variety isomorphic to $(\mathbb{G}_m)^r$ for some $r \in \mathbb{N}$.

Let $T = (\mathbb{G}_m)^r$ be an algebraic torus. Any rational representation V of T is a direct sum $V = \bigoplus_{\lambda} V_{\lambda}$ of weight spaces $V_{\lambda} = \{v \in V : t \cdot v = \lambda(t)v \ \forall t \in T\}$ defined by characters $\lambda : T \rightarrow \mathbb{G}_m$. The *weights* of V are the characters λ for which $V_{\lambda} \neq 0$. The group of all characters of T is called the *character lattice* and is isomorphic

to \mathbb{Z}^r . Given a T -variety X , we let $X^T \subset X$ denote the closed subvariety of T -fixed points. A subvariety $Z \subset X$ is called T -stable if $t \cdot z \in Z$ for all $t \in T$ and $z \in Z$. In this case Z is itself a T -variety.

Definition 2.1. The T -fixed point $p \in X$ is *nondegenerate* in X if T acts with nonzero weights on the Zariski tangent space $T_p X$. The point p is *fully definite* if all T -weights of $T_p X$ belong to a strict half-space of the character lattice of T .

Equivalently, $p \in X^T$ is fully definite in X if and only if there exists a cocharacter $\rho : \mathbb{G}_m \rightarrow T$ such that \mathbb{G}_m acts with strictly positive weights on $T_p X$ through ρ . For example, if $X = G/P$ is a flag variety and $T \subset G$ is a maximal torus, then all points of X^T are fully definite in X (see Section 6). Any nondegenerate T -fixed point must be isolated in X^T . Fully definite T -fixed points are called *attractive* in many sources, see, e.g., [Brion 1997]; here we follow the terminology from [Białynicki-Birula 1973].

Remark 2.2. If X is a normal quasiprojective T -variety, then $X^{\mathbb{G}_m} = X^T$ holds for all general cocharacters $\rho : \mathbb{G}_m \rightarrow T$. Here a cocharacter is called *general* if it avoids finitely many hyperplanes in the lattice of all cocharacters. This follows because X admits an equivariant embedding $X \subset \mathbb{P}(V)$, where V is a rational representation of T [Kambayashi 1966; Mumford 1965; Sumihiro 1974].

In the rest of this paper we let X be a nonsingular T -variety. The T -equivariant Chow cohomology ring of X will be denoted by $H_T^*(X)$; see [Fulton 1998; Anderson and Fulton 2024]. This is an algebra over the ring $H_T^*(\text{point})$, which may be identified with the symmetric algebra of the character lattice of T . Given a class $\sigma \in H_T^*(X)$ and a T -fixed point $p \in X^T$, we let $\sigma_p \in H_T^*(\text{point})$ denote the pullback of σ along the inclusion $\{p\} \rightarrow X$. When X is defined over $\mathbb{k} = \mathbb{C}$, Chow cohomology can be replaced with singular cohomology. In fact, our arguments will only depend on equivariant classes $[Z]_p \in H_T^*(\text{point})$ obtained by restricting the class of a T -stable closed subvariety $Z \subset X$ to a fixed point, and these restrictions are independent of the chosen cohomology theory. Similarly, we can use cohomology with coefficients in either \mathbb{Z} or \mathbb{Q} .

3. Equivariant local classes

Let Z be a T -variety, fix $p \in Z^T$, and let $\mathfrak{m} \subset \mathcal{O}_{Z,p}$ be the maximal ideal in the local ring of p . Then the tangent cone $C_p Z = \text{Spec}(\bigoplus \mathfrak{m}^i/\mathfrak{m}^{i+1})$ is a T -stable closed subscheme of the Zariski tangent space $T_p Z = (\mathfrak{m}/\mathfrak{m}^2)^\vee = \text{Spec}(\text{Sym}(\mathfrak{m}/\mathfrak{m}^2))$. The *local class* of Z at p is defined by (see [Anderson and Fulton 2024, §17.4])

$$(3) \quad \eta_p Z = [C_p Z] \in H_T^*(T_p Z) = H_T^*(\text{point}).$$

When p is a nonsingular point of Z , we have $\eta_p Z = 1$.

Proposition 3.1. *Let Z be a T -variety and let $p \in Z^T$ be fully definite in Z . Then $\eta_p Z \neq 0$ in $H_T^*(\text{point})$.*

Proof. We may assume that p is a singular point of Z , so that $C_p Z$ has positive dimension. Choose $\mathbb{G}_m \subset T$ such that \mathbb{G}_m acts with positive weights on $T_p Z$. It suffices to show that the class of $C_p Z$ is nonzero in $H_{\mathbb{G}_m}^*(T_p Z)$. Let $\{v_1, \dots, v_n\}$ be a basis of $T_p Z$ consisting of eigenvectors of \mathbb{G}_m . Then the action of \mathbb{G}_m is given by $t \cdot v_i = t^{a_i} v_i$ for positive integers $a_1, \dots, a_n > 0$. Set $A = \prod_{i=1}^n a_i$, and let \mathbb{G}_m act on $U = \mathbb{A}^n$ by $t \cdot u = t^A u$. Then the map $\phi : T_p Z \rightarrow U$ defined by

$$\phi(c_1 v_1 + \dots + c_n v_n) = (c_1^{A/a_1}, \dots, c_n^{A/a_n})$$

is a finite \mathbb{G}_m -equivariant morphism. By [Edidin and Graham 1998, Theorem 4] we obtain

$$H_{\mathbb{G}_m}^*(U \setminus \{0\}) \otimes \mathbb{Q} = H^*(\mathbb{P}U) \otimes \mathbb{Q},$$

where $\mathbb{P}U = (U \setminus \{0\})/\mathbb{G}_m \cong \mathbb{P}^{n-1}$ is the projective space of lines in U , and

$$\phi_*[C_p Z]|_{U \setminus \{0\}} = \deg(\phi)[\phi(C_p Z \setminus \{0\})/\mathbb{G}_m] \in H^*(\mathbb{P}U) \otimes \mathbb{Q}.$$

The result now follows from the fact that every nonempty closed subvariety of projective space defines a nonzero Chow class. \square

Corollary 3.2. *Let X be a nonsingular T -variety, $Z \subset X$ a T -stable closed subvariety, and $p \in Z^T$ a T -fixed point of Z . If p is nondegenerate in X and fully definite in Z , then $[Z]_p \neq 0 \in H_T^*(\text{point})$.*

Proof. By [Anderson and Fulton 2024, Proposition 17.4.1] we have

$$[Z]_p = c_m(T_p X/T_p Z) \cdot \eta_p Z,$$

where $m = \dim T_p X - \dim T_p Z$. The result therefore follows from Proposition 3.1, noting that T acts with nonzero weights on $T_p X/T_p Z$. \square

The following example rules out some potential generalizations of Corollary 3.2.

Example 3.3. Let \mathbb{G}_m act on \mathbb{A}^4 by

$$t \cdot (a, b, c, d) = (ta, tb, t^{-1}c, t^{-1}d).$$

Set $Z = V(ad - bc) \subset \mathbb{A}^4$, and let $p = (0, 0, 0, 0)$ be the origin in \mathbb{A}^4 . Then

$$T_p Z = T_p \mathbb{A}^4 = \mathbb{A}^4 \quad \text{and} \quad C_p Z = Z.$$

Since \mathbb{G}_m acts trivially on the equation $ad - bc$, we have $\eta_p Z = [Z] = 0$ in $H_{\mathbb{G}_m}^*(\mathbb{A}^4)$ (see [Anderson and Fulton 2024, §2.3]).

4. Rigidity of convex subvarieties

Let T be an algebraic torus and let X be a nonsingular T -variety. We will show in [Section 6](#) that Schubert varieties and Richardson varieties in a flag variety X satisfy the following two definitions.

Definition 4.1. A T -stable closed subvariety $\Omega \subset X$ is *T -equivariantly rigid* if it is uniquely determined by its T -equivariant cohomology class up to a constant. More precisely, if $Z \subset X$ is any T -stable closed subvariety such that $[Z] = c[\Omega]$ holds in $H_T^*(X)$ for some $0 \neq c \in \mathbb{Q}$, then $Z = \Omega$.

Definition 4.2. A T -stable closed subvariety $\Omega \subset X$ is *T -convex* if, for any T -stable closed subvariety $Z \subset X$ satisfying $Z^T \subset \Omega$, we have $Z \subset \Omega$.

When the action of T is clear from the context, we frequently drop T from the notation and write simply *equivariantly rigid* and *convex*. Both notions are properties of the T -equivariant embedding $\Omega \subset X$; for example, any T -variety is convex as a subvariety of itself. Intersections of T -convex subvarieties are again T -convex (with the reduced scheme structure). Most of this paper concerns applications of the following observation.

Theorem 4.3. *Let X be a nonsingular projective T -variety such that all fixed points $p \in X^T$ are fully definite in X . Then any irreducible T -convex subvariety of X is T -equivariantly rigid.*

Proof. Let $\Omega \subset X$ be irreducible and convex, and let $Z \subset X$ be any T -stable closed subvariety such that $[Z] = c[\Omega]$ holds in $H_T^*(X)$, with $0 \neq c \in \mathbb{Q}$. Then [Corollary 3.2](#) shows that $Z^T = \Omega^T = \{p \in X^T : [Z]_p \neq 0\}$. Since Ω is convex, we obtain $Z \subset \Omega$. Finally, the assumption $[Z] = c[\Omega]$ implies that Z and Ω have the same dimension, so we must have $Z = \Omega$. \square

Example 4.4. Let X be a nonsingular projective T -variety, let $H_T^*(X)$ be the T -equivariant Chow cohomology ring, and let \mathcal{L} be a T -equivariant line bundle. Given a section $f \in \Gamma(X, \mathcal{L})$, the associated divisor $D = Z(f)$ is T -stable if and only if f is semi-invariant, that is, $f \in \Gamma(X, \mathcal{L})_\lambda$ for some character λ . In this case f is an equivariant section of $\mathcal{L} \otimes \mathbb{K}_{-\lambda}$, and hence $[D] = c_1(\mathcal{L}) - c_1(\mathbb{K}_\lambda) \in H_T^*(X)$. Moreover, the T -stable effective Cartier divisors D' satisfying $[D'] = [D]$ are in bijective correspondence with $\mathbb{P}(\Gamma(X, \mathcal{L})_\lambda)$. It follows that if D is reduced and $\dim \Gamma(X, \mathcal{L}^{\otimes m})_{m\lambda} = 1$ for all $m \in \mathbb{N}$, then D is T -equivariantly rigid. This observation can be used to produce examples of equivariantly rigid subvarieties that are not convex. For example, if $T = (\mathbb{G}_m)^{n+1}$ acts on \mathbb{P}^n through the standard action on \mathbb{K}^{n+1} , then any reduced T -stable divisor $D \subset \mathbb{P}^n$ is equivariantly rigid, but D is convex only if it is irreducible; see [Theorem 6.3](#). We have not found an example of an irreducible T -stable subvariety that is equivariantly rigid but not convex.

5. Rigidity of Białyński-Birula cells

The multiplicative group \mathbb{G}_m is identified with the complement of the origin in \mathbb{A}^1 . Given a morphism of varieties $f : \mathbb{G}_m \rightarrow X$, we write $\lim_{t \rightarrow 0} f(t) = p$ if f can be extended to a morphism $\tilde{f} : \mathbb{A}^1 \rightarrow X$ such that $\tilde{f}(0) = p$. This limit is unique when it exists, and it always exists when X is complete.

Let X be a nonsingular projective \mathbb{G}_m -variety such that $X^{\mathbb{G}_m}$ is finite. Then each fixed point $p \in X^{\mathbb{G}_m}$ defines the (positive) Białyński-Birula cell

$$X_p^+ = \{x \in X : \lim_{t \rightarrow 0} t \cdot x = p\}.$$

A negative cell is similarly defined by $X_p^- = \{x \in X : \lim_{t \rightarrow 0} t^{-1} \cdot x = p\}$. By [Białyński-Birula 1973, Theorem 4.4], these cells form a locally closed decomposition of X ,

$$(4) \quad X = \bigcup_{p \in X^{\mathbb{G}_m}} X_p^+,$$

that is, a disjoint union of locally closed subsets. In addition, each cell X_p^+ is isomorphic to an affine space.

Lemma 5.1. *For any \mathbb{G}_m -stable closed subset $Z \subset X$, we have*

$$Z \subset \bigcup_{p \in Z^{\mathbb{G}_m}} X_p^+.$$

Proof. For any point $x \in Z$, we have $x \in X_p^+$, where $p = \lim_{t \rightarrow 0} t \cdot x \in Z^{\mathbb{G}_m}$. \square

Definition 5.2. A locally closed decomposition $X = \bigcup X_i$ is called a *stratification* if each subset X_i is nonsingular and its closure $\overline{X_i}$ is a union of subsets X_j of the decomposition.

The Białyński-Birula decomposition (4) typically fails to be a stratification, for example, when X is the blow-up of \mathbb{P}^2 at the point $[0, 1, 0]$, where \mathbb{G}_m acts on \mathbb{P}^2 by $t \cdot [x, y, z] = [x, ty, t^2z]$; see [Białyński-Birula 1976, Example 1]. Lemma 5.1 shows that the Białyński-Birula decomposition is a stratification if and only if $X_q^+ \subset \overline{X_p^+}$ holds for each fixed point $q \in (\overline{X_p^+})^{\mathbb{G}_m}$. It was proved in [Białyński-Birula 1976, Theorem 5] that the decomposition is a stratification when each positive cell X_p^+ meets each negative cell X_q^- transversally. In particular, this holds when $X = G/P$ is a flag variety and $\mathbb{G}_m \subset G$ is a general one-parameter subgroup; see [McGovern 2002, Example 4.2] or Lemma 6.1. When both the positive and negative Białyński-Birula decompositions are stratifications, all cells X_p^+ and X_q^- of complementary dimensions meet transversally, and hence the positive and negative cell closures form a pair of Poincaré dual bases of the cohomology ring $H^*(X)$; see [Benedetti and Perrin 2022, Lemma 3.11]. In this paper we utilize the following application, which is a consequence of Lemma 5.1.

Proposition 5.3. *Assume that the Białynicki-Birula decomposition of X is a stratification. Then each cell closure $\overline{X_p^+} \subset X$ is \mathbb{G}_m -convex.*

Corollary 5.4. *Let T be an algebraic torus and X a nonsingular projective T -variety such that all fixed points $p \in X^T$ are fully definite in X . Assume that $X^T = X^{\mathbb{G}_m}$ for some one-parameter subgroup $\mathbb{G}_m \subset T$, such that the associated Białynicki-Birula decomposition of X is a stratification. Then each cell closure $\overline{X_p^+}$ is T -convex and T -equivariantly rigid.*

Proof. The cell X_p^+ is T -stable because T is commutative and $p \in X^T$. The result now follows from [Theorem 4.3](#) and [Proposition 5.3](#). \square

Question 5.5. We do not know whether [Proposition 5.3](#) and [Corollary 5.4](#) are true without the assumption that the Białynicki-Birula decomposition of X is a stratification. It would be very interesting to settle this question.

Example 5.6. Let X be a nonsingular projective toric variety, with torus $T \subset X$, and choose $\mathbb{G}_m \subset T$ such that $X^T = X^{\mathbb{G}_m}$. We show that the conclusion of [Corollary 5.4](#) holds, even though the Białynicki-Birula decomposition is rarely a stratification. All fixed points $p \in X^T$ are fully definite in X , as the weights of $T_p X$ form a basis of the character lattice of T . The T -orbits $O_\tau \subset X$ correspond to the cones τ of the fan defining X , and we have $O_\sigma \subset \overline{O_\tau}$ if and only if τ is a face of σ ; see [[Fulton 1993](#), §3.1]. In particular, the T -fixed points in X correspond to the maximal cones σ . Since X is complete, each cone τ is the intersection of the maximal cones σ corresponding to the T -fixed points in $\overline{O_\tau}$. Since all cell closures $\overline{X_p^+}$ are T -orbit closures, it suffices to show that each orbit closure $\overline{O_\tau}$ is T -convex. Let $Z \subset X$ be a T -stable closed subvariety such that $Z^T \subset \overline{O_\tau}$. We may assume that Z is irreducible, in which case $Z = \overline{O_\kappa}$ is also a T -orbit closure. Since κ is the intersection of the maximal cones given by the fixed points in Z^T , we obtain $\tau \subset \kappa$ and $\overline{O_\tau} \subset \overline{O_\kappa}$, as required. Now assume that X has dimension two. By [[Białynicki-Birula 1973](#), Corollary 1 of Theorem 4.5], there is a unique repulsive fixed point $b \in X^{\mathbb{G}_m}$ with $X_b^+ = \{b\}$, and a unique attractive fixed point $a \in X^{\mathbb{G}_m}$ such that X_a^+ is a dense open subset of X . For all other fixed points $p \in X^{\mathbb{G}_m} \setminus \{a, b\}$, the cell $X_p^+ \cong \mathbb{A}^1$ is a line. If the Białynicki-Birula decomposition of X is a stratification, then $b \in X_p^+$ for all $p \in X^{\mathbb{G}_m}$. The T -fixed point b corresponds to a maximal cone σ , and b is connected to exactly two T -stable lines corresponding to the rays forming the boundary of this cone. We deduce that X contains at most four T -fixed points. Higher-dimensional toric varieties for which the Białynicki-Birula decomposition is not a stratification can be constructed by taking products. We do not know if the cell closures $\overline{X_p^+}$ are \mathbb{G}_m -convex when X is a toric variety.¹

¹Teddy Gonzales and Chayim Lowen [[≥ 2025](#)] have recently produced several examples showing that X_p^+ may not be \mathbb{G}_m -convex when X is a nonsingular projective toric variety.

6. Rigidity of Richardson varieties

Let $X = G/P = \{g \cdot P : g \in G\}$ be a flag variety defined by a connected reductive linear algebraic group G and a parabolic subgroup P . Fix a maximal torus T and a Borel subgroup B such that $T \subset B \subset P \subset G$. The opposite Borel subgroup $B^- \subset G$ is defined by $B^- \cap B = T$. Let Φ be the root system of nonzero weights of $T_1 G$, the tangent space of G at the identity element. The positive roots Φ^+ are the nonzero weights of $T_1 B$. Let $W = N_G(T)/T$ be the Weyl group of G , $W_P = N_P(T)/T$ the Weyl group of P , and let $W^P \subset W$ be the subset of minimal representatives of the cosets in W/W_P . The set of T -fixed points in X is given by $X^T = \{w \cdot P : w \in W\}$, where each point $w \cdot P$ depends only on the coset wW_P in W/W_P . Each fixed point $w \cdot P$ defines the *Schubert varieties* $X_w = \overline{Bw \cdot P}$ and $X^w = \overline{B^-w \cdot P}$. For $w \in W^P$ we have $\dim(X_w) = \text{codim}(X^w, X) = \ell(w)$. The Bruhat order \leq on W^P is defined by

$$u \leq w \iff X_u \subset X_w \iff X^u \supset X^w \iff X^u \cap X_w \neq \emptyset.$$

A *Richardson variety* is any nonempty intersection $X_w^u = X_w \cap X^u$ of opposite Schubert varieties in X . More generally, any G -translate of X_w^u will be called a Richardson variety. Any Richardson variety is reduced, irreducible, and rational; see [Deodhar 1977; Brion and Kumar 2005, §2].

Recall that a cocharacter $\rho : \mathbb{G}_m \rightarrow T$ is *strongly dominant* if $\langle \alpha, \rho \rangle > 0$ for all positive roots $\alpha \in \Phi^+$, where $\langle \alpha, \rho \rangle \in \mathbb{Z}$ is defined by $\alpha(\rho(t)) = t^{\langle \alpha, \rho \rangle}$ for $t \in \mathbb{G}_m$. The following lemma is well known; see, e.g., [McGovern 2002, Example 4.2] or [Benedetti and Perrin 2022, Corollary 3.14].

Lemma 6.1. *Let $\rho : \mathbb{G}_m \rightarrow T$ be a strongly dominant cocharacter. Then the associated Białynicki-Birula cells of X are given by $X_p^+ = B \cdot p$, for $p \in X^T$.*

Proof. Let \mathbb{G}_m act on G by conjugation through ρ . The fixed point set for this action is [Springer 1998, (7.1.2), (7.6.4)]

$$T = \{g \in G : tgt^{-1} = g \ \forall t \in \mathbb{G}_m\},$$

and the corresponding Białynicki-Birula cell is [Springer 1998, (8.2.1)]

$$B = \{g \in G : \lim_{t \rightarrow 0} tgt^{-1} \in T\}.$$

This implies $B \cdot p \subset X_p^+$ for any fixed point $p \in X^{\mathbb{G}_m}$. We deduce from (4) that the positive Białynicki-Birula cells in X are the B -orbits. \square

Lemma 6.2. *Let Y be any G -variety, and $\Omega \subset Y$ a T -stable closed subvariety. Any T -stable G -translate of Ω has the form $w \cdot \Omega$, with $w \in N_G(T)$.*

Proof. Let $\Omega' = g \cdot \Omega$ be a T -stable translate, and let $H \subset G$ be the stabilizer of Ω' . Since T and gTg^{-1} are maximal tori in H , we can choose $h \in H$ such that $T = hgTg^{-1}h^{-1}$. We obtain $hg \in N_G(T)$ and $\Omega' = h \cdot \Omega' = hg \cdot \Omega$, as required. \square

Theorem 6.3. *Any T -stable Richardson variety in the flag variety $X = G/P$ is T -convex and T -equivariantly rigid.*

Proof. It follows from [Proposition 5.3](#) and [Lemma 6.1](#) that all Schubert varieties X_w and X^u are convex. This implies that every Richardson variety $X_w^u = X_w \cap X^u$ is convex; hence all T -stable Richardson varieties in X are convex by [Lemma 6.2](#). The B -fixed point $p = 1 \cdot P$ is fully definite in X because the weights of $T_p X$ are a subset of the negative roots of G . Since W acts transitively on X^T , this implies that all T -fixed points in X are fully definite. The result therefore follows from [Theorem 4.3](#). \square

Let $E = G/B$ denote the variety of complete flags, and let $\pi : E \rightarrow X$ be the natural projection. A *projected Richardson variety* in X is the image $\Pi_w^u(X) = \pi(E_w^u)$ of a Richardson variety in E . Projected Richardson varieties in the Grassmannian $X = \text{Gr}(m, n)$ of type A , obtained as images of Richardson varieties in $\text{Fl}(n)$, are also called *positroid varieties*.

Corollary 6.4. *Let $X = \text{Gr}(m, n)$ be a Grassmannian of type A , and let $T = (\mathbb{G}_m)^n$ act on X through the diagonal action on \mathbb{K}^n . Then all positroid varieties in X are T -convex and T -equivariantly rigid.*

Proof. It was proved in [\[Knutson et al. 2013\]](#) that any positroid variety Ω is defined by Plucker equations. Equivalently, Ω is an intersection of T -stable Schubert divisors, so Ω is convex by [Theorem 6.3](#) and equivariantly rigid by [Theorem 4.3](#). \square

Remark 6.5. [Corollary 6.4](#) does not hold for projected Richardson varieties in arbitrary flag varieties $X = G/P$. Each simple root β defines a projected Richardson divisor $D_\beta = \Pi_{w_0^\beta}^{\beta}(X)$, where w_0^β denotes the longest element in W^P . It frequently happens that two distinct divisors $D_{\beta'}$ and $D_{\beta''}$ have the same T -equivariant cohomology and K -theory classes, which implies that these divisors are not equivariantly rigid. For example, this is the case for the quadric hypersurfaces of dimensions seven and eight, of Lie types B_4 and D_5 , and the two-step flag variety $\text{Fl}(1, 4; 5)$ of type A_4 . For other flag varieties X , all projected Richardson varieties have distinct equivariant classes, but some projected Richardson divisor D_β contains all T -fixed points in X , which rules out that D_β is convex. For example, this is the case for the Lagrangian Grassmannian $\text{LG}(2, 4)$ of type C_2 and the maximal orthogonal Grassmannian $\text{OG}(4, 8)$ of type D_4 . This is a special case of [\[Benedetti and Perrin 2022, Lemma 3.1\]](#), which can be used to produce many more examples.

Any element $u \in W$ has a unique factorization $u = u^P u_P$ for which $u^P \in W^P$ and $u_P \in W_P$, called the *parabolic factorization* with respect to P . This factorization is *reduced* in the sense that $\ell(u) = \ell(u^P) + \ell(u_P)$. The parabolic factorization of

the longest element $w_0 \in W$ is $w_0 = w_0^P w_{0,P}$, where w_0^P and $w_{0,P}$ are the longest elements in W^P and W_P , respectively. Since w_0 and $w_{0,P}$ are self-inverse, we have $w_{0,P} = w_0 w_0^P$. As preparation for the next section, we prove the following identity of Schubert varieties.

Lemma 6.6. *Let $Q \subset G$ be a parabolic subgroup containing B and set $w = w_0^Q$. Then $w^{-1} \cdot X^w = X_{w_0 w}$.*

Proof. Since $X_{w_0, Q}$ is a Q -stable Schubert variety, we have $X_{w_0, Q} = w_{0, Q} \cdot X_{w_0, Q}$. By translating both sides by $w = w_0^Q$, we obtain $w \cdot X_{w_0 w} = w_0 \cdot X_{w_0 w} = X^w$. \square

7. Seidel neighborhoods

In this section we prove a conjecture about curve neighborhoods from [Buch et al. 2023]. Since this conjecture and its proof relies on the moduli space of stable maps, we will restrict our attention to varieties defined over the field $\mathbb{k} = \mathbb{C}$ of complex numbers. As in Section 6, we let $X = G/P$ denote a flag variety.

For any effective degree $d \in H_2(X, \mathbb{Z})$, we let $M_d = \overline{\mathcal{M}}_{0,3}(X, d)$ denote the Kontsevich moduli space of three-pointed stable maps to X of degree d and genus zero; see [Fulton and Pandharipande 1997]. The evaluation map $\text{ev}_i : M_d \rightarrow X$, defined for $1 \leq i \leq 3$, sends a stable map to the image of the i -th marked point in its domain. Given two opposite Schubert varieties X_v and X^u , the *Gromov–Witten variety* $M_d(X_v, X^u)$ is the variety of stable maps that send the first two marked points to X_v and X^u :

$$M_d(X_v, X^u) = \text{ev}_1^{-1}(X_v) \cap X_2^{-1}(X^u) \subset M_d.$$

The *curve neighborhood* $\Gamma_d(X_v, X^u)$ is the union of all stable curves of degree d in X connecting X_v and X^u :

$$\Gamma_d(X_v, X^u) = \text{ev}_3(M_d(X_v, X^u)) \subset X.$$

Let $\mathbb{Z}[q] = \text{Span}_{\mathbb{Z}}\{q^d : d \in H_2(X, \mathbb{Z}) \text{ effective}\}$ be the semigroup ring defined by the effective curve classes on X . The equivariant quantum cohomology ring of X is an algebra over $H_T^*(\text{point}) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$, which is defined by $\text{QH}_T(X) = H_T^*(X) \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ as a module. The *quantum product* of two opposite Schubert classes is given by

$$[X_v] \star [X^u] = \sum_{d \geq 0} q^d \text{ev}_{3,*}[M_d(X_v, X^u)],$$

where the sum is over all effective degrees $d \in H_2(X; \mathbb{Z})$.

A simple root $\gamma \in \Phi^+$ is called *cominuscul* if, when the highest root is written in the basis of simple roots, the coefficient of γ is one. The flag variety G/Q is *cominuscul* if Q is a maximal parabolic subgroup corresponding to a *cominuscul* simple root γ , that is, s_γ is the unique simple reflection in W^Q . Let $W^{\text{comin}} \subset W$

be the subset of point representatives of cominuscule flag varieties of G , together with the identity element:

$$W^{\text{comin}} = \{w_0^Q : G/Q \text{ is cominuscule}\} \cup \{1\}.$$

This is a subgroup of W , which is isomorphic to the quotient of the coweight lattice of Φ modulo the coroot lattice [Bourbaki 1981, Proposition VI.2.6]. The isomorphism sends w_0^Q to the class of the fundamental coweight ω_γ^\vee corresponding to Q . Notice that γ is the unique simple root for which $w_0^Q \cdot \gamma < 0$. In the following we set $d(w_0^Q, u) = \omega_\gamma^\vee - u^{-1} \cdot \omega_\gamma^\vee \in H_2(X; \mathbb{Z})$ for any $u \in W$. Here we identify the group $H_2(X, \mathbb{Z})$ with a quotient of the coroot lattice, by mapping each simple coroot β^\vee to the curve class $[X_{s_\beta}]$ if $s_\beta \in W^P$, and to zero otherwise.

The Seidel representation of W^{comin} on $\text{QH}(X)/\langle q-1 \rangle$ is defined by $w \cdot [X^u] = [X^w] \star [X^u]$ for $w \in W^{\text{comin}}$ and $u \in W$. In fact, we have [Seidel 1997; Belkale 2004; Chaput et al. 2009]

$$(5) \quad [X^w] \star [X^u] = q^{d(w,u)} [X^{wu}]$$

in the (nonequivariant) quantum ring $\text{QH}(X)$. This implies that $d(w, u)$ is the unique minimal degree d for which $\Gamma_d(X_{w_0w}, X^u)$ is not empty [Fulton and Woodward 2004; Buch et al. 2020]. More generally, it was proved in [Chaput et al. 2009; Chaput and Perrin 2023] that the identity

$$(6) \quad [X^w] \star [w \cdot X^u] = q^{d(w,u)} [X^{wu}]$$

holds in the equivariant quantum cohomology ring $\text{QH}_T(X)$. We will discuss generalizations to quantum K -theory in Section 8.

It follows from (5) and the definition of the quantum product in $\text{QH}(X)$ that $[\Gamma_{d(w,u)}(X_{w_0w}, X^u)] = [X^{wu}]$ holds in $H^*(X)$. Conjecture 3.11 from [Buch et al. 2023] asserts that $\Gamma_{d(w,u)}(X_{w_0w}, X^u)$ is in fact equal to the translated Schubert variety $w^{-1} \cdot X^{wu}$. This is proved below as a consequence of Theorem 6.3 and (6). This result was known when $X = G/P$ is cominuscule and $w = w_0^P$ [Buch et al. 2023], when X is a Grassmannian of type A and $[X^w]$ is a special Seidel class [Li et al. 2025, Corollary 4.6], when X is any flag variety of type A [Tarigradschi 2023], and when X is the symplectic Grassmannian $\text{SG}(2, 2n)$ [Benedetti et al. 2024, Theorem 8.1].

Theorem 7.1. *Let $X = G/P$ be a complex flag variety. For $w \in W^{\text{comin}}$ and $u \in W$ we have $\Gamma_{d(w,u)}(X_{w_0w}, X^u) = w^{-1} \cdot X^{wu}$.*

Proof. By applying w^{-1} to both sides of (6) and using Lemma 6.6, we obtain

$$[X_{w_0w}] \star [X^u] = q^{d(w,u)} [w^{-1} \cdot X^{wu}]$$

in $\mathrm{QH}_T(X)$. By definition of the quantum product, this implies that

$$[w^{-1} \cdot X^{wu}] = \mathrm{ev}_{3,*}[M_{d(w,u)}(X_{w_0w}, X^u)] = c [\Gamma_{d(w,u)}(X_{w_0w}, X^u)]$$

holds in $H_T^*(X)$, where c is the degree of the map

$$\mathrm{ev}_3 : M_{d(w,u)}(X_{w_0w}, X^u) \rightarrow \Gamma_{d(w,u)}(X_{w_0w}, X^u).$$

The result therefore follows from [Theorem 6.3](#). \square

8. Seidel products in quantum K -theory

In this section we discuss a generalization of the Seidel multiplication formula to quantum K -theory. We start by briefly recalling the definition of quantum K -theory. A more detailed discussion can be found in [\[Buch et al. 2018a, §2\]](#).

Let $X = G/P$ be a flag variety defined over $\mathbb{k} = \mathbb{C}$. The equivariant K -theory ring $K^T(X)$ is an algebra over the representation ring $\Gamma = K^T(\mathrm{point})$. The equivariant quantum K -theory ring $\mathrm{QK}_T(X)$ was originally constructed by Givental [\[2000\]](#) and Lee [\[2004\]](#). This ring is an algebra over the formal power series ring $\Gamma[[q]] = \Gamma[[q_\beta : s_\beta \in W^P]]$, which has one variable q_β for each simple reflection s_β in W^P . As a module over $\Gamma[[q]]$ we have $\mathrm{QK}_T(X) = K^T(X) \otimes_\Gamma \Gamma[[q]]$. The *undeformed product* of two opposite Schubert classes in $\mathrm{QK}_T(X)$ is defined by

$$[\mathcal{O}_{X_v}] \odot [\mathcal{O}_{X^u}] = \sum_{d \geq 0} q^d \mathrm{ev}_{3,*}[\mathcal{O}_{M_d(X_v, X^u)}].$$

Let $\Psi : \mathrm{QK}_T(X) \rightarrow \mathrm{QK}_T(X)$ be the $\Gamma[[q]]$ -linear map defined by

$$\Psi([\mathcal{O}_{X^w}]) = \sum_{d \geq 0} q^d [\mathcal{O}_{\Gamma_d(X^w)}],$$

where the curve neighborhood $\Gamma_d(X^w) = \mathrm{ev}_2(\mathrm{ev}_1^{-1}(X^w))$ is defined using the evaluation maps from M_d . This curve neighborhood is a Schubert variety in X by [\[Buch et al. 2013, Proposition 3.2\(b\)\]](#), whose Weyl group element was determined in [\[Buch and Mihalcea 2015\]](#). By [\[Buch et al. 2018a, Proposition 2.3\]](#), Givental's *quantum K -theory product* \star is given by

$$(7) \quad [\mathcal{O}_{X_v}] \star [\mathcal{O}_{X^u}] = \Psi^{-1}([\mathcal{O}_{X_v}] \odot [\mathcal{O}_{X^u}]).$$

The following conjecture is the K -theoretic analogue of the Seidel multiplication formula (6) in $\mathrm{QH}_T(X)$ proved in [\[Chaput et al. 2009; Chaput and Perrin 2023\]](#).

Conjecture 8.1. *For $w \in W^{\mathrm{comin}}$ and $u \in W$ we have*

$$[\mathcal{O}_{X_{w_0w}}] \star [\mathcal{O}_{X^u}] = q^{d(w,u)} [\mathcal{O}_{w^{-1} \cdot X^{wu}}] \quad \text{and} \quad [\mathcal{O}_{X^w}] \star [\mathcal{O}_{w \cdot X^u}] = q^{d(w,u)} [\mathcal{O}_{X^{wu}}]$$

in $\mathrm{QK}_T(X)$.

The two identities in [Conjecture 8.1](#) are equivalent by [Lemma 6.6](#). The nonequivariant case of this conjecture was proved in [[Buch et al. 2023](#), Corollary 3.7] when X is a cominuscule flag variety. When X is a Grassmannian of type A, the conjecture is equivalent to [[Gorbounov et al. 2025](#), Corollary 10.4]. We will prove [Conjecture 8.1](#) for cominuscule flag varieties below, based on the following conjectural generalization of [Theorem 7.1](#). Recall that a morphism $\pi : Z \rightarrow Y$ is called *cohomologically trivial* if $\pi_*\mathcal{O}_Z = \mathcal{O}_Y$ and $R^j\pi_*\mathcal{O}_Z = 0$ for $j \geq 1$.

Conjecture 8.2. *Let $w \in W^{\text{comin}}$, $u \in W$, and let $e \in H_2(X, \mathbb{Z})$ be effective.*

- (a) *We have $\Gamma_{d(w,u)+e}(X_{w_0w}, X^u) = \Gamma_e(w^{-1} \cdot X^{wu})$.*
- (b) *The evaluation map $\text{ev}_3 : M_{d(w,u)+e}(X_{w_0w}, X^u) \rightarrow \Gamma_{d(w,u)+e}(X_{w_0w}, X^u)$ is cohomologically trivial.*

[Conjecture 8.2](#) is a variant of the quantum-equals-classical theorem for Gromov–Witten invariants as stated in [[Buch et al. 2018b](#), Theorem 4.1]; see also [[Xu 2024](#), Theorem 1.2]. The conjecture is true for $e = 0$; part (a) is equivalent to [Theorem 7.1](#), and part (b) holds because the map $\text{ev}_3 : M_{d(w,u)}(X_{w_0w}, X^u) \rightarrow \Gamma_{d(w,u)}(X_{w_0w}, X^u)$ is birational by [[Belkale 2004](#); [Chaput et al. 2009](#)], and $M_{d(w,u)}(X_{w_0w}, X^u)$ has rational singularities by [[Buch et al. 2013](#), Corollary 3.1]. For $e \geq 0$, [Theorem 7.1](#) implies that

$$(8) \quad \Gamma_e(w^{-1} \cdot X^{wu}) = \Gamma_e(\Gamma_{d(w,u)}(X_{w_0w}, X^u)) \subset \Gamma_{d(w,u)+e}(X_{w_0w}, X^u),$$

and $\Gamma_{d(w,u)+e}(X_{w_0w}, X^u)$ is irreducible by Corollary 3.8 in [[Buch et al. 2013](#)]. [Conjecture 8.2\(a\)](#) is therefore true if and only if $\Gamma_{d(w,u)+e}(X_{w_0w}, X^u)$ and $\Gamma_e(X^{wu})$ have the same dimension. We prove below that [Conjecture 8.2\(a\)](#) is true when $X = \text{GL}(n)/P$ is any flag variety of Lie type A. [Conjecture 8.1](#) follows from [Conjecture 8.2](#) by the following observation.

Lemma 8.3. *Given $w \in W^{\text{comin}}$ and $u \in W$, the identity*

$$[\mathcal{O}_{X_{w_0w}}] \star [\mathcal{O}_{X^u}] = q^{d(w,u)} [\mathcal{O}_{w^{-1} \cdot X^{wu}}]$$

holds in $\text{QK}_T(X)$ if and only if

$$(9) \quad \text{ev}_{3,*} [\mathcal{O}_{M_{d(w,u)+e}(X_{w_0w}, X^u)}] = [\mathcal{O}_{\Gamma_e(w^{-1} \cdot X^{wu})}]$$

holds in $K_T(X)$ for all effective degrees $e \in H_2(X, \mathbb{Z})$.

Proof. Both assertions are equivalent to the identity

$$[\mathcal{O}_{X_{w_0w}}] \odot [\mathcal{O}_{X^u}] = \sum_{e \geq 0} q^{d(w,u)+e} [\mathcal{O}_{\Gamma_e(w^{-1} \cdot X^{wu})}]$$

by the definition (7) of the quantum product in $\text{QK}_T(X)$. □

Theorem 8.4. *Conjectures 8.1 and 8.2 hold when X is a cominuscule flag variety.*

Proof. Assume that X is cominuscule. Then [Conjecture 8.2\(b\)](#) is a special case of [\[Buch et al. 2018b, Theorem 4.1\]](#), and [Conjecture 8.2\(a\)](#) follows from [Theorem 7.1](#) and [\[Buch et al. 2022, Corollary 8.24\]](#), noting that $q^{d(w,u)}$ is the maximal power of q occurring in the quantum cohomology product $[X_{w_0w}] \star [X^u]$ by [\[Belkale 2004; Chaput et al. 2009\]](#). This proves [Conjecture 8.2](#), which implies [Conjecture 8.1](#) by [Lemma 8.3](#). \square

We finish this section by proving that [Conjecture 8.2\(a\)](#) can be reduced to the case where X is a flag variety of Picard rank 1. In particular, [Conjecture 8.2\(a\)](#) follows from [Theorem 8.4](#) in type A. These results were proved for $e = 0$ in [\[Tarigradschi 2023\]](#). We thank Mihail Tarigradschi for suggesting that his methods might apply to the general case of our conjecture.

Recall that $X = G/P$. Let $Q_1, Q_2 \subset G$ be parabolic subgroups such that $P = Q_1 \cap Q_2$. Set $Y_i = G/Q_i$ and let $\pi_i : X \rightarrow Y_i$ be the projection, for $i \in \{1, 2\}$. Given a degree $d \in H_2(X, \mathbb{Z})$, we also let d denote the image $\pi_{i,*}(d)$ of this degree in $H_2(Y_i, \mathbb{Z})$. Let $\Gamma_d(Y_{i,v}, Y_i^u) \subset Y_i$ be the union of all stable curves of degree d in Y_i that connect the Schubert varieties $Y_{i,v} = \pi_i(X_v)$ and $Y_i^u = \pi_i(X^u)$, for $u, v \in W$. The next result generalizes [\[Björner and Brenti 2005, Theorem 2.6.1; Tarigradschi 2023, Lemma 4\]](#).

Lemma 8.5. *We have $\Gamma_d(X^u) = \pi_1^{-1}(\Gamma_d(Y_1^u)) \cap \pi_2^{-1}(\Gamma_d(Y_2^u))$.*

Proof. Let $\text{dist}_X(X_v, X^u)$ denote the unique minimal degree of a rational curve in X connecting X_v and X^u . It follows from [\[Buch et al. 2020, Theorem 5\]](#) that this degree is uniquely determined by $\pi_{i,*}(\text{dist}_X(X_v, X^u)) = \text{dist}_{Y_i}(Y_{i,v}, Y_i^u)$ for $i \in \{1, 2\}$. Using that $v \cdot P \in \Gamma_d(X^u)$ holds if and only if $d \geq \text{dist}_X(X_v, X^u)$, we deduce that $\Gamma_d(X^u)$ and $\pi_1^{-1}(\Gamma_d(Y_1^u)) \cap \pi_2^{-1}(\Gamma_d(Y_2^u))$ contain the same T -fixed points. The lemma follows from this, as both sets are B^- -stable subvarieties of X . \square

The following result implies that [Conjecture 8.2\(a\)](#) follows from the case where X has Picard rank 1. It was proved for $e = 0$ in [\[Tarigradschi 2023, Theorem 3\]](#).

Theorem 8.6. *Let $X = G/P$, $Y_1 = G/Q_1$, and $Y_2 = G/Q_2$ be flag varieties such that $P = Q_1 \cap Q_2$. Let $w \in W^{\text{comin}}$, $u \in W$, and let $e \in H_2(X, \mathbb{Z})$ be any effective degree. If $\Gamma_{d(w,u)+e}(Y_{i,w_0w}, Y_i^u) = \Gamma_e(w^{-1} \cdot Y_i^{wu})$ holds for $i \in \{1, 2\}$, then $\Gamma_{d(w,u)+e}(X_{w_0w}, X^u) = \Gamma_e(w^{-1} \cdot X^{wu})$.*

Proof. The assumptions and [Lemma 8.5](#) imply that

$$\begin{aligned} \Gamma_{d(w,u)+e}(X_{w_0w}, X^u) &\subset \pi_1^{-1}(\Gamma_{d(w,u)+e}(Y_{1,w_0w}, Y_1^u)) \cap \pi_2^{-1}(\Gamma_{d(w,u)+e}(Y_{2,w_0w}, Y_2^u)) \\ &= \pi_1^{-1}(\Gamma_e(w^{-1} \cdot Y_1^{wu})) \cap \pi_2^{-1}(\Gamma_e(w^{-1} \cdot Y_2^{wu})) = \Gamma_e(w^{-1} \cdot X^{wu}), \end{aligned}$$

and the opposite inclusion holds by [\(8\)](#). \square

Corollary 8.7. *[Conjecture 8.2\(a\)](#) is true when $X = \text{GL}(n)/P$ has Lie type A.*

Proof. This follows from [Theorems 8.4](#) and [8.6](#), noting that all flag varieties of type A with Picard rank 1 are Grassmannians, and therefore cominuscule. \square

9. Horospherical varieties of Picard rank 1

In this section we interpret [Theorem 4.3](#) and [Proposition 5.3](#) for a class of horospherical varieties that includes all nonsingular projective horospherical varieties of Picard rank 1 (except flag varieties) by Pasquier’s classification [\[2009\]](#). Let G be a connected reductive linear algebraic group, $B \subset G$ a Borel subgroup, and $T \subset B$ a maximal torus. Let V_1 and V_2 be irreducible rational representations of G , and let $v_i \in V_i$ be a highest-weight vector of weight λ_i , for $i \in \{1, 2\}$. We assume that $\lambda_1 \neq \lambda_2$. Define

$$X = \overline{G \cdot [v_1 + v_2]} \subset \mathbb{P}(V_1 \oplus V_2).$$

If X is normal, then X is a horospherical variety of rank 1; see [\[Timashev 2011, Chapter 7\]](#). We will assume that X is nonsingular and $\mathbb{K} = \mathbb{C}$, even though many claims hold more generally; this implies that X is fibered over a flag variety G/P_{12} with nonsingular horospherical fibers of Picard rank 1; see [Remark 9.5](#). Any G -translate of a B -orbit closure in X will be called a *Schubert variety*. Our next result uses the action of $T \times \mathbb{G}_m$ on X defined by $(t, z) \cdot [u_1 + u_2] = t \cdot [u_1 + zu_2]$, for $u_i \in V_i$. We have $X^{T \times \mathbb{G}_m} = X^T$, and a Schubert variety is T -stable if and only if it is $T \times \mathbb{G}_m$ -stable.

Theorem 9.1. *Any T -stable Schubert variety in X is $T \times \mathbb{G}_m$ -convex and $T \times \mathbb{G}_m$ -equivariantly rigid.*

Before proving [Theorem 9.1](#), we sketch elementary proofs of some basic facts about X , which are also consequences of general results about spherical varieties; see [\[Timashev 2011; Perrin 2014; Pasquier 2009\]](#).

Given an element $[u_1 + u_2] \in \mathbb{P}(V_1 \oplus V_2)$, we will always assume $u_i \in V_i$, and i will always mean an element from $\{1, 2\}$. We consider $\mathbb{P}(V_i)$ as a subvariety of $\mathbb{P}(V_1 \oplus V_2)$. Let $\pi_i : \mathbb{P}(V_1 \oplus V_2) \setminus \mathbb{P}(V_{3-i}) \rightarrow \mathbb{P}(V_i)$ denote the projection from V_{3-i} , defined by $\pi_i([u_1 + u_2]) = [u_i]$. Set $X_0 = G \cdot [v_1 + v_2] \subset \mathbb{P}(V_1 \oplus V_2)$, $X_i = G \cdot [v_i] \subset \mathbb{P}(V_i)$, and $X_{12} = G \cdot ([v_1], [v_2]) \subset \mathbb{P}(V_1) \times \mathbb{P}(V_2)$. Since v_i is a highest-weight vector, the stabilizer $P_i = G_{[v_i]}$ is a parabolic subgroup containing B . It follows that $X_i \cong G/P_i$ and $X_{12} \cong G/(P_1 \cap P_2)$ are flag varieties. In particular, X_i is closed in $\mathbb{P}(V_i)$, and X_{12} is closed in $\mathbb{P}(V_1) \times \mathbb{P}(V_2)$. Notice also that $X_0 \cong G/H$, where $H \subset P_1 \cap P_2$ is the kernel of the character $\lambda_1 - \lambda_2 : P_1 \cap P_2 \rightarrow \mathbb{G}_m$. This shows that X_0 is a \mathbb{G}_m -bundle over X_{12} , so X is a nonsingular projective horospherical variety of rank 1 (but not necessarily of Picard rank 1; see [Remark 9.5](#)).

Let W be the Weyl group of G , and recall the notation from [Section 6](#).

Lemma 9.2. *We have $X = X_0 \cup X_1 \cup X_2$. The B -orbit closures in X are*

$$\overline{Bw \cdot [v_i]} = \bigcup_{w' \leq w} Bw' \cdot [v_i] \quad \text{for } w \in W^{P_i} \text{ and } i \in \{1, 2\},$$

$$\overline{Bw \cdot [v_1 + v_2]} = \bigcup_{w' \leq w} (Bw' \cdot [v_1 + v_2] \cup Bw' \cdot [v_1] \cup Bw' \cdot [v_2]) \quad \text{for } w \in W^{P_1 \cap P_2}.$$

Proof. Set $\mathbb{P}_0 = \mathbb{P}(V_1 \oplus V_2) \setminus (\mathbb{P}(V_1) \cup \mathbb{P}(V_2))$. Since $\lambda_1 \neq \lambda_2$, it follows that $\overline{T \cdot [v_1 + v_2]}$ is the line through $[v_1]$ and $[v_2]$ in $\mathbb{P}(V_1 \oplus V_2)$. This implies $X_0 = (\pi_1 \times \pi_2)^{-1}(X_{12})$; hence X_0 is closed in \mathbb{P}_0 and $X_0 = X \cap \mathbb{P}_0$. We also have $X_i \subset X \cap \mathbb{P}(V_i) \subset \pi_i^{-1}(X_i) \cap \mathbb{P}(V_i) = X_i$, which proves the first claim. To finish the proof, it suffices to show $w' \cdot [v_i] \in \overline{Bw \cdot [v_1 + v_2]}$ if and only if $w' \leq w$ (when $w' \in W^{P_i}$). The implication ‘if’ holds because $w' \cdot [v_i] \in \overline{T w' \cdot [v_1 + v_2]}$, and ‘only if’ holds because $\pi_i(\overline{Bw \cdot [v_1 + v_2]} \setminus X_{3-i}) \subset \overline{Bw \cdot [v_i]}$. \square

Define an alternative action of P_i on V_{3-i} by $p \bullet u = \lambda_i(p)^{-1} p \cdot u$, and use this action to form the space

$$G \times^{P_i} V_{3-i} = \{[g, u] : g \in G, u \in V_{3-i}\} / \{[gp, u] = [g, p \bullet u] : p \in P_i\}.$$

Define a morphism of varieties $\phi_i : G \times^{P_i} V_{3-i} \rightarrow \mathbb{P}(V_1 \oplus V_2)$ by

$$\phi_i([g, u]) = g \cdot [v_i + u].$$

This is well defined since $p \cdot (v_i + u) = \lambda_i(p)(v_i + p \bullet u)$ holds for $p \in P_i$ and $u \in V_{3-i}$. Set $E_i = (P_i \bullet v_{3-i}) \cup \{0\} \subset V_{3-i}$. Noting that E_i is the cone over $P_i \cdot [v_{3-i}] \cong P_i / (P_1 \cap P_2)$, it follows that E_i is closed in V_{3-i} .

Lemma 9.3. *The restricted map $\phi_i : G \times^{P_i} E_i \rightarrow X_0 \cup X_i$ is an isomorphism of varieties. In particular, $E_i \subset V_{3-i}$ is a linear subspace.*

Proof. Assume $\phi_i([g, u]) = \phi_i([g', u'])$, and set $p = g^{-1}g'$. We obtain $p \in P_i$ and $[v_i + u] = p \cdot [v_i + u'] = [v_i + p \bullet u']$ in $\mathbb{P}(V_1 \oplus V_2)$; hence

$$[g, u] = [g, p \bullet u'] = [gp, u'] = [g', u']$$

in $G \times^{P_i} V_{3-i}$. We deduce that $\phi_i : G \times^{P_i} E_i \rightarrow X_0 \cup X_i$ is bijective, so the lemma follows from Zariski’s main theorem, using that $X_0 \cup X_i$ is nonsingular. \square

Fix a strongly dominant cocharacter $\rho : \mathbb{G}_m \rightarrow T$. For $a \in \mathbb{Z}$, define the map $\rho_a : \mathbb{G}_m \rightarrow T \times \mathbb{G}_m$ by $\rho_a(z) = (\rho(z), z^a)$. The resulting action of \mathbb{G}_m on X is given by $\rho_a(z) \cdot [u_1 + u_2] = \rho(z) \cdot [u_1 + z^a u_2]$.

Lemma 9.4. *All T -fixed points in X are fully definite for the action of $T \times \mathbb{G}_m$.*

Proof. Lemma 9.3 shows that $[v_1]$ has a $T \times \mathbb{G}_m$ -stable open neighborhood in X isomorphic to $B^- \cdot [v_1] \times E_1$, where the action is given by $(t, z) \cdot (x, u) = (t \cdot x, t \bullet z u)$. If a is sufficiently negative, then \mathbb{G}_m acts through ρ_a on $T_{[v_1]}X = T_{[v_1]}X_1 \oplus E_1$ with strictly negative weights; hence $[v_1]$ is fully definite in X for the action of $T \times \mathbb{G}_m$. A symmetric argument shows that $[v_2]$ is fully definite. The result follows from this, since all T -fixed points in X are obtained from $[v_1]$ or $[v_2]$ by the action of the Weyl group W . \square

Proof of Theorem 9.1. For a sufficiently negative, it follows from Lemma 6.1 that the Białynicki-Birula cells of X defined by ρ_a are

$$X_{w \cdot [v_1]}^+ = Bw \cdot [v_1] \quad \text{and} \quad X_{w \cdot [v_2]}^+ = Bw \cdot [v_1 + v_2] \cup Bw \cdot [v_2].$$

These cells form a stratification of X by Lemma 9.2, so Proposition 5.3 implies that $\overline{Bw \cdot [v_1]}$ and $\overline{Bw \cdot [v_1 + v_2]}$ are $T \times \mathbb{G}_m$ -convex for $w \in W$. A symmetric argument applies to $\overline{Bw \cdot [v_2]}$; hence all T -stable Schubert varieties in X are $T \times \mathbb{G}_m$ -convex by Lemma 6.2. The result now follows from Theorem 4.3 and Lemma 9.4. \square

Remark 9.5. The exact sequence of [Perrin 2014, Theorem 3.2.4] implies that $\text{Pic}(X)$ is a free abelian group of rank equal to the rank of X (which is one) plus the number of B -stable prime divisors in X that do not contain a G -orbit. Any B -stable prime divisor meeting X_0 has the form $D = \overline{Bw_0 s_\beta \cdot [v_1 + v_2]}$, where β is a simple root, and Lemma 9.2 shows that D contains X_i if and only if β is a root of P_i . Let $P_{12} \subset G$ be the parabolic subgroup generated by P_1 and P_2 . We obtain $\text{Pic}(X) \cong \mathbb{Z} \oplus \text{Pic}(G/P_{12})$. Let $\pi : X \rightarrow G/P_{12}$ be the map defined by $\pi(g \cdot [v_1 + v_2]) = \pi(g \cdot [v_i]) = g \cdot P_{12}$. This is a G -equivariant morphism of varieties, as its restriction to $X_0 \cup X_i$ is the composition of $\pi_i : X_0 \cup X_i \rightarrow G/P_i$ with the projection $G/P_i \rightarrow G/P_{12}$. The fibers of π are translates of $\pi^{-1}(1 \cdot P_{12}) = \overline{L \cdot [v_1 + v_2]} \subset \mathbb{P}(V_1 \oplus V_2)$, where L is the Levi subgroup of P_{12} containing T . Moreover, $\pi^{-1}(1 \cdot P_{12})$ is a nonsingular projective horospherical variety of Picard rank 1, so it is either a flag variety or one of the nonhomogeneous spaces from Pasquier's classification [2009].

Question 9.6. Let X be any projective G -horospherical variety fibered over a flag variety G/P with nonsingular horospherical fibers of Picard rank 1. Is it true that X is isomorphic to an orbit closure $\overline{G \cdot [v_1 + v_2]} \subset \mathbb{P}(V)$, where V is a rational representation of G , and $v_1, v_2 \in V$ are highest-weight vectors?

Example 9.7. Let X be the blow-up of \mathbb{P}^2 at a point p , let $\pi : X \rightarrow \mathbb{P}^1$ be the morphism defined by projection from p , and set $G = \text{SL}(2, \mathbb{C})$. Then X is G -horospherical and fibered over \mathbb{P}^1 with fiber \mathbb{P}^1 . This variety X is isomorphic to $\overline{G \cdot [v_1 + v_2]} \subset \mathbb{P}(V_1 \oplus V_2)$, where v_1 is a highest-weight vector in $V_1 = \mathbb{C}^2$, and v_2 is a highest-weight vector in $V_2 = \text{Sym}^2(\mathbb{C}^2)$.

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References

- [Anderson and Fulton 2024] D. Anderson and W. Fulton, *Equivariant cohomology in algebraic geometry*, Cambridge Studies in Advanced Mathematics **210**, Cambridge Univ. Press, 2024. [MR](#)
- [Belkale 2004] P. Belkale, “Transformation formulas in quantum cohomology”, *Compos. Math.* **140**:3 (2004), 778–792. [MR](#)
- [Benedetti and Perrin 2022] V. Benedetti and N. Perrin, “Cohomology of hyperplane sections of (co)adjoint varieties”, preprint, 2022. [arXiv 2207.02089](#)
- [Benedetti et al. 2024] V. Benedetti, N. Perrin, and W. Xu, “Quantum K -theory of $IG(2, 2n)$ ”, *Int. Math. Res. Not.* **2024**:22 (2024), 14061–14093. [MR](#)
- [Białynicki-Birula 1973] A. Białynicki-Birula, “Some theorems on actions of algebraic groups”, *Ann. of Math. (2)* **98** (1973), 480–497. [MR](#)
- [Białynicki-Birula 1976] A. Białynicki-Birula, “Some properties of the decompositions of algebraic varieties determined by actions of a torus”, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **24**:9 (1976), 667–674. [MR](#)
- [Björner and Brenti 2005] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Graduate Texts in Mathematics **231**, Springer, 2005. [MR](#)
- [Bourbaki 1981] N. Bourbaki, *Éléments de mathématique: groupes et algèbres de Lie. Chapitres 4, 5 et 6*, Masson, Paris, 1981. Reprinted by Springer, Berlin, 2007. [MR](#)
- [Brion 1997] M. Brion, “Equivariant Chow groups for torus actions”, *Transform. Groups* **2**:3 (1997), 225–267. [MR](#)
- [Brion and Kumar 2005] M. Brion and S. Kumar, *Frobenius splitting methods in geometry and representation theory*, Progress in Mathematics **231**, Birkhäuser, Boston, MA, 2005. [MR](#)
- [Buch and Mihalcea 2015] A. S. Buch and L. C. Mihalcea, “Curve neighborhoods of Schubert varieties”, *J. Differential Geom.* **99**:2 (2015), 255–283. [MR](#)
- [Buch et al. 2013] A. S. Buch, P.-E. Chaput, L. C. Mihalcea, and N. Perrin, “Finiteness of cominuscule quantum K -theory”, *Ann. Sci. Éc. Norm. Supér. (4)* **46**:3 (2013), 477–494. [MR](#)
- [Buch et al. 2018a] A. S. Buch, P.-E. Chaput, L. C. Mihalcea, and N. Perrin, “A Chevalley formula for the equivariant quantum K -theory of cominuscule varieties”, *Algebr. Geom.* **5**:5 (2018), 568–595. [MR](#)
- [Buch et al. 2018b] A. S. Buch, P.-E. Chaput, L. C. Mihalcea, and N. Perrin, “Projected Gromov–Witten varieties in cominuscule spaces”, *Proc. Amer. Math. Soc.* **146**:9 (2018), 3647–3660. [MR](#)
- [Buch et al. 2020] A. S. Buch, S. Chung, C. Li, and L. C. Mihalcea, “Euler characteristics in the quantum K -theory of flag varieties”, *Selecta Math. (N.S.)* **26**:2 (2020), art. id. 29. [MR](#)
- [Buch et al. 2022] A. S. Buch, P.-E. Chaput, L. C. Mihalcea, and N. Perrin, “Positivity of minuscule quantum K -theory”, preprint, 2022. [arXiv 2205.08630](#)
- [Buch et al. 2023] A. S. Buch, P.-E. Chaput, and N. Perrin, “Seidel and Pieri products in cominuscule quantum K -theory”, preprint, 2023. [arXiv 2308.05307](#)
- [Chaput and Perrin 2023] P.-E. Chaput and N. Perrin, “Affine symmetries in quantum cohomology: corrections and new results”, *Math. Res. Lett.* **30**:2 (2023), 341–374. [MR](#)
- [Chaput et al. 2009] P.-E. Chaput, L. Manivel, and N. Perrin, “Affine symmetries of the equivariant quantum cohomology ring of rational homogeneous spaces”, *Math. Res. Lett.* **16**:1 (2009), 7–21. [MR](#)
- [Coskun 2011] I. Coskun, “Rigid and non-smoothable Schubert classes”, *J. Differential Geom.* **87**:3 (2011), 493–514. [MR](#)

- [Coskun 2014] I. Coskun, “Rigidity of Schubert classes in orthogonal Grassmannians”, *Israel J. Math.* **200**:1 (2014), 85–126. [MR](#)
- [Coskun 2018] I. Coskun, “Restriction varieties and the rigidity problem”, pp. 49–95 in *Schubert varieties, equivariant cohomology and characteristic classes — IMPANGA 15*, edited by J. Buczyński et al., Eur. Math. Soc., Zürich, 2018. [MR](#)
- [Coskun and Robles 2013] I. Coskun and C. Robles, “Flexibility of Schubert classes”, *Differential Geom. Appl.* **31**:6 (2013), 759–774. [MR](#)
- [Deodhar 1977] V. V. Deodhar, “Some characterizations of Bruhat ordering on a Coxeter group and determination of the relative Möbius function”, *Invent. Math.* **39**:2 (1977), 187–198. [MR](#)
- [Edidin and Graham 1998] D. Edidin and W. Graham, “Equivariant intersection theory”, *Invent. Math.* **131**:3 (1998), 595–634. [MR](#)
- [Fulton 1993] W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies **131**, Princeton Univ. Press, 1993. [MR](#)
- [Fulton 1998] W. Fulton, *Intersection theory*, 2nd ed., Ergebnisse der Math. (3) **2**, Springer, 1998. [MR](#)
- [Fulton and Pandharipande 1997] W. Fulton and R. Pandharipande, “Notes on stable maps and quantum cohomology”, pp. 45–96 in *Algebraic geometry, II* (Santa Cruz, CA, 1995), edited by J. Kollár et al., Proc. Sympos. Pure Math. **62**, Amer. Math. Soc., Providence, RI, 1997. [MR](#)
- [Fulton and Woodward 2004] W. Fulton and C. Woodward, “On the quantum product of Schubert classes”, *J. Algebraic Geom.* **13**:4 (2004), 641–661. [MR](#)
- [Givental 2000] A. Givental, “On the WDVV equation in quantum K -theory”, *Michigan Math. J.* **48** (2000), 295–304. [MR](#)
- [Gonzales and Lowen \geq 2025] T. Gonzales and C. Lowen, “Counterexamples to \mathbb{G}_m -convexity for toric varieties”, In preparation.
- [Gorbounov et al. 2025] V. Gorbounov, C. Korff, and L. C. Mihalea, “Quantum K -theory of Grassmannians from a Yang–Baxter algebra”, preprint, 2025. [arXiv 2503.08602](#)
- [Hong 2005] J. Hong, “Rigidity of singular Schubert varieties in $\text{Gr}(m, n)$ ”, *J. Differential Geom.* **71**:1 (2005), 1–22. [MR](#)
- [Hong 2007] J. Hong, “Rigidity of smooth Schubert varieties in Hermitian symmetric spaces”, *Trans. Amer. Math. Soc.* **359**:5 (2007), 2361–2381. [MR](#)
- [Hong and Mok 2020] J. Hong and N. Mok, “Schur rigidity of Schubert varieties in rational homogeneous manifolds of Picard number one”, *Selecta Math. (N.S.)* **26**:3 (2020), art. id. 41. [MR](#)
- [Kambayashi 1966] T. Kambayashi, “Projective representation of algebraic linear groups of transformations”, *Amer. J. Math.* **88** (1966), 199–205. [MR](#)
- [Knutson et al. 2013] A. Knutson, T. Lam, and D. E. Speyer, “Positroid varieties: juggling and geometry”, *Compos. Math.* **149**:10 (2013), 1710–1752. [MR](#)
- [Lee 2004] Y.-P. Lee, “Quantum K -theory, I: Foundations”, *Duke Math. J.* **121**:3 (2004), 389–424. [MR](#)
- [Li et al. 2025] C. Li, Z. Liu, J. Song, and M. Yang, “On Seidel representation in quantum K -theory of Grassmannians”, *Sci. China Math.* **68**:7 (2025), 1523–1548. [MR](#)
- [Liu et al. 2024] Y. Liu, A. Sheshmani, and S.-T. Yau, “Multi-rigidity of Schubert classes in partial flag varieties”, preprint, 2024. [arXiv 2410.21726](#)
- [McGovern 2002] W. M. McGovern, “The adjoint representation and the adjoint action”, pp. 159–238 in *Algebraic quotients, torus actions and cohomology, the adjoint representation and the adjoint action*, Encyclopaedia Math. Sci. **131**, Springer, 2002. [MR](#)

- [Mumford 1965] D. Mumford, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, (N.F.) **34**, Springer, 1965. [MR](#)
- [Pasquier 2009] B. Pasquier, “On some smooth projective two-orbit varieties with Picard number 1”, *Math. Ann.* **344**:4 (2009), 963–987. [MR](#)
- [Perrin 2014] N. Perrin, “On the geometry of spherical varieties”, *Transform. Groups* **19**:1 (2014), 171–223. [MR](#)
- [Robles and The 2012] C. Robles and D. The, “Rigid Schubert varieties in compact Hermitian symmetric spaces”, *Selecta Math. (N.S.)* **18**:3 (2012), 717–777. [MR](#)
- [Seidel 1997] P. Seidel, “ π_1 of symplectic automorphism groups and invertibles in quantum homology rings”, *Geom. Funct. Anal.* **7**:6 (1997), 1046–1095. [MR](#)
- [Springer 1998] T. A. Springer, *Linear algebraic groups*, 2nd ed., Progress in Mathematics **9**, Birkhäuser, Boston, MA, 1998. [MR](#)
- [Sumihiro 1974] H. Sumihiro, “Equivariant completion”, *J. Math. Kyoto Univ.* **14** (1974), 1–28. [MR](#)
- [Tarigradschi 2023] M. Tarigradschi, “Curve neighborhoods of Seidel products in quantum cohomology”, preprint, 2023. [arXiv 2309.05985](#)
- [Timashev 2011] D. A. Timashev, *Homogeneous spaces and equivariant embeddings*, Encyclopaedia of Mathematical Sciences **138**, Springer, 2011. [MR](#)
- [Xu 2024] W. Xu, “Quantum K -theory of incidence varieties”, *Eur. J. Math.* **10**:2 (2024), art. id. 22. [MR](#)

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Equivariant rigidity of Richardson varieties	209
ANDERS S. BUCH, PIERRE-EMMANUEL CHAPUT and NICOLAS PERRIN	
Genus three Goeritz groups of connected sums of two lens spaces	231
HAO CHEN and YANQING ZOU	
Complete minimal hypersurfaces in a hyperbolic space $H^4(-1)$	251
QING-MING CHENG and YEJUAN PENG	
The reciprocal complement of a polynomial ring in several variables over a field	267
NEIL EPSTEIN, LORENZO GUERRIERI and K. ALAN LOPER	
On A-packets containing unitary lowest-weight representations of $U(p, q)$	295
SHUJI HORINAGA	
An evolution of matrix-valued orthogonal polynomials	325
ERIK KOELINK, PABLO ROMÁN and WADIM ZUDILIN	
Defect relation of $n + 1$ components through the GCD method	349
MIN RU and JULIE TZU-YUEH WANG	
The tangent spaces of Teichmüller space from an energy-conscious perspective	373
DIVYA SHARMA and MICHAEL S. WEISS	