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GENUS THREE GOERITZ GROUPS OF CONNECTED SUMS OF TWO LENS SPACES

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We prove that the mapping class groups of the genus three Heegaard splittings of the connected sums of two lens spaces are finitely generated, and the corresponding reducing sphere complexes are all connected.

1. Introduction

It is well known that every closed orientable 3-manifold N admits a *Heegaard splitting* $V \cup_\Sigma W$, which is a decomposition of N into two handlebodies V and W of the same genus. Their common boundary surface Σ is called *Heegaard surface*, and the genus of Σ is called the genus of the Heegaard splitting. Two Heegaard splittings of N are said to be isotopic if their corresponding Heegaard surfaces are ambient isotopic. The *Goeritz group* $\mathcal{G}(N, \Sigma)$, first introduced by Goeritz [1933], is the group of isotopy classes of orientation-preserving diffeomorphisms of N that preserve these two handlebodies of the splitting setwise. As a subgroup of the mapping class group of Σ, there is an open question about it in [Gordon 2007].

Question. Is the Goeritz group finite or finitely generated?

By the works of Johnson [2010], Namazi [2007], as well as Zou and Qiu [2020], the Goeritz groups of almost all Heegaard splittings of distance at least 2 are finite. However, if $V \cup_\Sigma W$ is weakly reducible, or equivalently, of distance at most 1, $\mathcal{G}(N, \Sigma)$ is infinite as shown in Namazi's construction. We focus on studying the finite generation problem for the Goeritz group of reducible Heegaard splittings.

A Heegaard splitting $V \cup_\Sigma W$ is reducible if there is a 2-sphere $S \subset N$ that intersects Σ transversely in one essential simple closed curve. Such a 2-sphere S is called a reducing sphere for Σ. When $V \cup_\Sigma W$ is reducible, we can decompose it as a connected sum of two Heegaard splittings of smaller genus, denoted by $V \cup_\Sigma W = N_1 \sharp N_2$, where $N_1 = V_1 \cup_{\Sigma_1} W_1$ and $N_2 = V_2 \cup_{\Sigma_2} W_2$. A natural question arises: If both $\mathcal{G}(N_1, \Sigma_1)$ and $\mathcal{G}(N_2, \Sigma_2)$ are finitely generated, is $\mathcal{G}(N, \Sigma)$ also finitely generated?

MSC2020: 57K30, 57K31.
Keywords: reducing sphere complex, Goeritz group, Heegaard splitting.

If $g(\Sigma_1)+g(\Sigma_2)=2$, Cho and Koda [2019] proved that $\mathcal{G}(N,\Sigma)$ is finitely presented. Here we study the case $g(\Sigma_1)+g(\Sigma_2)=3$ and give an answer to the Question as follows.

Theorem 1.1. *If $N=V\cup_\Sigma W$ is a genus three Heegaard splitting*[1] *for a connected sum of two lens spaces ($\neq S^3$, $S^1\times S^2$), then $\mathcal{G}(N,\Sigma)$ is finitely generated.*

Furthermore, we study the reducing sphere complex $\mathscr{R}$ for $V\cup_\Sigma W$, which is a subcomplex of the curve complex spanned by those curves that bound disks in both handlebodies. As a corollary, we have the following.

Corollary 1.2. *Under the same condition as in Theorem 1.1, $\mathscr{R}$ is connected.*

For any reducible Heegaard splitting $V\cup_\Sigma W=(V_1\cup_{\Sigma_1}W_1)\,\sharp\,(V_2\cup_{\Sigma_2}W_2)$, let $\mu=S\cap\Sigma$ be the intersection of a reducing sphere S and the Heegaard surface Σ. Although the method in the proof of Theorem 1.1 does not apply in general, it provides insight into the widely studied subgroup $G_\mu\leqslant\mathcal{G}(N,\Sigma)$, the stabilizer of μ, which is a key subgroup of $\mathcal{G}(N,\Sigma)$. By its definition, it is not hard to see that there is a natural homomorphism from G_μ to $\mathcal{G}(N_i,\Sigma_i)$ for each i. Thus, it is of interest to determine whether G_μ is finitely generated (or finitely presented) when both of those two Goeritz groups are finitely generated (or finitely presented). Using standard combinatorial techniques, we obtain the following result.

Theorem 1.3. *If $\mathcal{G}(N_1,\Sigma_1)$ and $\mathcal{G}(N_2,\Sigma_2)$ are both finitely generated* (*or finitely presented*), *then so is G_μ.*[2]

Overview of the proof. We first show that the Goeritz group under consideration can be generated by three stabilizers of reducing curves, as shown in Theorem 4.8. As a corollary, the corresponding reducing sphere complex is connected. Next, we carefully study the stabilizer of a reducing sphere and give a proof of Theorem 1.3. Finally, by the previous work [Cho and Koda 2019] on genus two reducible Heegaard splittings, we arrive at the finite generation of each stabilizer.

This paper is organized as follows. We introduce some notations in Section 2 and study two classes of automorphisms, *eyeglass twist* and *visional bubble move*, in Section 3. Next, we carefully study the properties of three stabilizers in Section 4. After all preparations have been done, we complete the proof of Theorem 1.1 and Theorem 1.3 in Section 5.

Notations. We respectively denote the isotopy class of a curve μ in a surface and of a diffeomorphism h of a manifold by $\bar{\mu}$ and $\bar{h}$. When μ is endowed with an orientation, we denote it by $\vec{\mu}$. For an oriented curve $\vec{\mu}$, we denote its isotopy by $[\vec{\mu}]$.

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[1] By Haken's lemma, the Heegaard splitting is reducible.

[2] Here, N_i $(i=1,2)$ is not necessarily a lens space.

2. Preliminaries

Throughout the paper, we respectively denote the isotopy class of a curve μ and of a diffeomorphism f by $\bar{\mu}$ and $\bar{f}$. From now on, we assume that N is the connected sum of two lens spaces unless otherwise specified, and $V \cup_\Sigma W$ is a genus three Heegaard splitting of N. Let $\mathrm{Diff}^+(N, \Sigma)$ be a subgroup of $\mathrm{Diff}^+(N)$ defined as

$$\mathrm{Diff}^+(N, \Sigma) \overset{\text{def}}{=} \{ f \in \mathrm{Diff}^+(N) : f(\Sigma) = \Sigma \text{ and } f \text{ preserves the orientation of } \Sigma \}.$$

It is clear that if an orientation-preserving diffeomorphism of N preserves the Heegaard splitting of N, it must preserve the orientation of Σ. Hence, the natural homomorphism $\rho_1 : \mathrm{Diff}^+(N, \Sigma) \to \mathcal{G}(N, \Sigma)$ is an epimorphism.

Definition 2.1. Two reducing spheres S_1, S_2 (for Σ) are isotopic if there is an isotopy

$$H_t : (N, \Sigma) \to (N, \Sigma), \quad 0 \le t \le 1,$$

such that $H_0 = \mathrm{id}$ and $H_1(S_1) = S_2$.

Definition 2.2. A triplet $\mathcal{T} = (S_1, S_2, S_3)$ of pairwise nonisotopic reducing spheres for Σ is called a *sphere triplet* (for Σ), and spheres S_i $(i = 1, 2, 3)$ are called the components of $\mathcal{T}$. We say the triplet is *complete* if the reducing spheres are pairwise disjoint (i.e., its three components span a 2-simplex in the corresponding reducing sphere complex).

Definition 2.3. Two sphere triplets $\mathcal{T}_1, \mathcal{T}_2$ are isotopic if there is an isotopy

$$H_t : (N, \Sigma) \to (N, \Sigma), \quad 0 \le t \le 1,$$

such that $H_0 = \mathrm{id}$ and $H_1(\mathcal{T}_1) = \mathcal{T}_2$.

Note 2.4. We usually make no notational distinction between triplets and their isotopy classes when the context is clear.

Definition 2.5. Two sphere triplets $\mathcal{T}_1, \mathcal{T}_2$ are *congruent* if they differ by a permutation. For instance, (S_1, S_2, S_3) is congruent with (S_3, S_1, S_2).

We designate a complete sphere triplet $\mathcal{T} = (S_1, S_2, S_3)$ for Σ, as depicted in Figure 1, such that (1) S_i $(i = 1, 2)$ cuts off a genus one Heegaard splitting of $M_i \setminus B^3$; (2) S_3 cuts off a genus one Heegaard splitting of a 3-ball. Clearly, S_1 and S_2 are two reducible 2-spheres and cobound $S^2 \times I$ in N. We also write $\mu_i = S_i \cap \Sigma$, for $i = 1, 2, 3$. Throughout the remainder of this paper, we fix the notations S_i and μ_i for these designated reducing spheres in Figure 1 and corresponding reducing curves respectively.

Lemma 2.6. *Up to congruences, $\mathcal{G}(N, \Sigma)$ acts transitively on the set of isotopy classes of complete sphere triplets for Σ.*

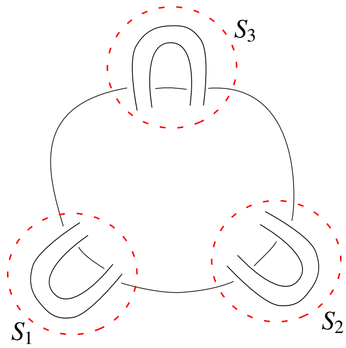

Figure 1. Heegaard surface Σ and triplet $\mathcal{T} = (S_1, S_2, S_3)$.

Proof. It suffices to show that for any given triplet $\mathcal{T}' = (S_1', S_2', S_3')$, there exists a diffeomorphism $h \in \mathrm{Diff}^+(N, \Sigma)$ such that $h(\mathcal{T})$ is congruent to $\mathcal{T}'$. Firstly, each reducing sphere S_i' cuts off a genus one summand of Σ. Then, by uniqueness of the prime decomposition of 3-manifolds, one of the three reducing spheres bounds a 3-ball $\mathcal{B} \subset N$. By assumption, $N \setminus \mathrm{int}(\mathcal{B})$ does not admit a genus one Heegaard splitting. This implies that $\Sigma \cap \mathcal{B}$ is a torus with a open disk removed. Thus, the other two reducing spheres are isotopic in N and each cuts off a genus one Heegaard splitting of a once-punctured lens space. It follows that $\bigcup_{i=1}^3 S_i'$ divides N into four parts, a 3-ball, a thrice-punctured 3-sphere, and two once-punctured lens spaces. Since $\bigcup_{i=1}^3 S_i'$ divides N into four parts of the same diffeomorphism type as those divided by $\bigcup_{i=1}^3 S_i$, we glue all diffeomorphisms of these four parts along spheres to obtain the desired h. □

Using similar arguments, we can prove the following lemma.

Lemma 2.7. *If S is a common component of these two complete sphere triplets $\mathcal{T}$ and $\mathcal{T}'$, then there exists a diffeomorphism $h \in \mathrm{Diff}^+(N, \Sigma)$ such that $h(\mathcal{T}) = \mathcal{T}'$ and $h(S) = S$.*

3. Eyeglass twist and visional bubble move

A Heegaard splitting $N = A \cup_\Sigma B$ is weakly reducible if there are two properly embedded disjoint essential disks, $a \subset A$ and $b \subset B$. We call (a, b) a weakly reducing pair for Σ. An eyeglass is a triple (a, b, λ), where (a, b) is a weakly reducing pair for Σ and $\lambda \subset \Sigma$ is an arc connecting a and b with its interior disjoint from them. For an eyeglass $\eta = (a, b, \lambda)$, we refer to (a, b) as the lenses of η and λ as the bridge of η. Given a normal direction $\vec{n}$ pointing toward the interior of B, we can push the 1-handle $a \times I$ around the circumference of the disk b in a counterclockwise direction as in Figure 2 (left). In fact, it is exactly an excursion

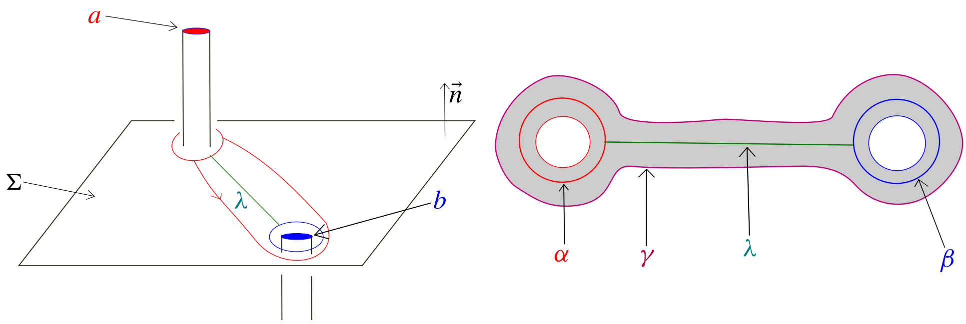

Figure 2. Left: eyeglass twist. Right: regular neighborhood Δ.

of the handlebody A that ends at the initial position. More formally, an eyeglass η defines a natural automorphism $T_\eta : (N, \Sigma) \to (N, \Sigma)$, known as the *positive eyeglass twist*. The inverse of this operation, which involves a clockwise excursion of A, is called *negative eyeglass twist* and denoted by $T_{\bar{\eta}}$. For an eyeglass twist T_η, the eyeglass η is referred to as its base eyeglass. It is not hard to see that an eyeglass twist preserves the isotopy classes of its lenses.

Note 3.1. The above definition does not depend on the order of the two lenses of the eyeglass. In other words, if $\eta = (a, b, \lambda)$ and $\eta' = (b, a, \lambda)$, then we have $T_\eta = T_{\eta'}$.

Let $\eta = (a, b, \lambda)$ be an eyeglass, where $\alpha \overset{\text{def}}{=} \partial a$, $\beta \overset{\text{def}}{=} \partial b$, and let Δ be a regular neighborhood of $\alpha \cup \lambda \cup \beta$ in surface Σ. Denote by γ the component (as in Figure 2 (right)) of $\partial\Delta$, which is isotopic to neither α nor β. We can now describe the above situation as

$$T_\eta = \tau_\alpha \cdot \tau_\beta \cdot \tau_\gamma^{-1},$$

where $\tau_{[\,\cdot\,]}$ denotes the *left-handed Dehn twist*. See more details in [Zupan 2020, Lemma 2.5].

Remark. Although different choices of regular neighborhoods of the eyeglass η yield different eyeglass twists, they are all equivalent up to isotopy. Therefore, for the eyeglass η, we obtain two eyeglass twists $T_\eta, T_{\bar{\eta}} \in \mathcal{G}(N, \Sigma)$.

Definition 3.2. Suppose η_1 and η_2 are two eyeglasses in N. They are isotopic if there is an isotopy $H_t : (N, \Sigma) \to (N, \Sigma)$, $0 \le t \le 1$, such that $H_0 = \text{id}$ and $H_1(\eta_1) = \eta_2$. Furthermore, the isotopy class of an eyeglass η is denoted by $[\eta]$.

It is not hard to see that the eyeglass twist T_η depends only on the isotopy class of η. In analogy to the case for Dehn twists, we have the following lemma.

Lemma 3.3. *Given any* $\varphi \in \mathcal{G}(N, \Sigma)$, *we have* $T_{\varphi(\eta)} = \varphi \cdot T_\eta \cdot \varphi^{-1}$.

When a lens of an eyeglass η is decomposed into two disks, the corresponding eyeglass twist T_η can be expressed as the composition of two new eyeglass twists. We write it as the following lemma.

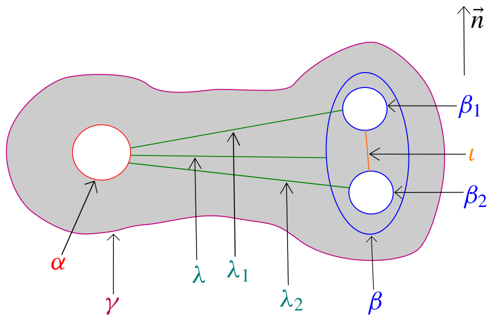

Figure 3. Composition of eyeglass twists.

Lemma 3.4 [Freedman and Scharlemann 2018, Figure 8]. *Let $\eta = (a, b, \lambda)$ be an eyeglass in N. If b is the band sum of two disks b_1 and b_2 along arc ι such that $\beta_i = \partial b_i$ is disjoint from both the bridge λ and the disk a, then for $i = 1, 2$, we can choose a proper arc λ_i which connects α and β_i, in the planar surface bounded by α, β_1, β_2 and γ. Given such a choice, we can obtain two new eyeglasses, η_1 $(= (a, b_1, \lambda_1))$ and η_2 $(= (a, b_2, \lambda_2))$. Moreover, there exists a suitable choice $\{\lambda_i\}$ such that T_η is a composition of T_{η_1} and T_{η_2}.*

Proof. Let $P \subset \Sigma$ be the pair of pants bounded by β_1, β_2, and β, and let $p = \lambda \cap \beta$. Let $\ell \subset P$ be an embedded arc that connects p and another point in β. Clearly, ℓ divides P into two annuli A_1 and A_2 that contain β_1 and β_2, respectively. Next, we choose for each $i \in \{1, 2\}$ an embedded arc $\lambda_i' \subset A_i$ that connects p and β_i. Finally, let $\lambda_i = \lambda \cup \lambda_i'$. We can verify that $\{\lambda_i\}$ is a desired choice. We also provide a specific choice in Figure 3, in which $T_\eta = T_{\eta_1} \cdot T_{\eta_2}$. □

Definition 3.5. Suppose η is an eyeglass in N, and S a reducing sphere for Σ. We say S separates η if these two lenses of η are disjoint from S and lie in different components of $N \setminus S$.

Definition 3.6. Suppose S is a reducing sphere for Σ, $\mu_s = S \cap \Sigma$, and $\eta \subset N$ (with $\partial\eta = \alpha \cup \beta \cup \lambda)$) an eyeglass with lenses disjoint from S. The *geometric intersection number* between S and η is defined as $I(\eta, S) = \tilde{I}(\lambda, \mu_s)$, where $\tilde{I}(\cdot, \cdot)$ is the geometric intersection number up to isotopies (of Σ) that leave α and β invariant.

Definition 3.7. For any separating reducing sphere S for Σ, we associate it with a subgroup $\mathcal{E}_k(S)$ of $\mathcal{G}(N, \Sigma)$ for each $k \in \mathbb{N}_+$ defined as

$$\mathcal{E}_k(S) = \langle E_k(S) \rangle,$$

where

$$E_k(S) = \big\{ T_\eta \in \mathcal{G}(N, \Sigma) : S \text{ separates } \eta \text{ and } I(\eta, S) \le k \big\}.$$

By definition, we have $E_k(S) \subset E_{k+1}(S)$. Then we have the ascending sequence $\mathcal{E}_1(S) \le \mathcal{E}_2(S) \le \mathcal{E}_3(S) \le \cdots$.

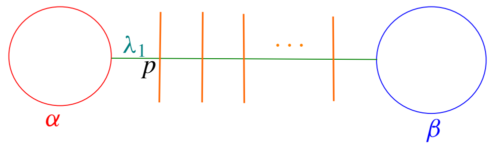

Figure 4. Intersection pattern.

Let us return to the setting from the previous section, in which $N = V \cup_\Sigma W$ is a genus three Heegaard splitting for a connected sum of two lens spaces and each S_i ($i = 1, 2, 3$) is a fixed separating reducing sphere for Σ, as in the previous section. Then we have the following lemma.

Lemma 3.8. *For $i = 1, 2, 3$ and $k \in \mathbb{N}_+$, we have $\mathcal{E}_{k+1}(S_i) \leqslant \mathcal{E}_k(S_i)$.*

Proof. It is sufficient to prove that $E_{k+1}(S_i) \subset \mathcal{E}_k(S_i)$. Consider an eyeglass twist T_η (suppose $\eta = (a, b, \lambda)$ and $\partial\eta = \alpha \cup \beta \cup \lambda$) representing an element of $E_{k+1}(S_i)$. We aim to show that $T_\eta \in E_k(S_i)$. Without loss of generality, we assume that β lies in the genus one component of $\Sigma \setminus S_i$, and the bridge λ intersects $\mu_i (= S_i \cap \Sigma)$ at $k+1$ points. Let one of these points, say p, be closest to α, as depicted in Figure 4. The point p divides λ into two segments, λ_1 and λ_2, where λ_1 denotes the one that is disjoint from μ_i.

Next, we choose an arc λ_3 in the pair of pants $\Sigma \setminus (S_i \cup \beta)$ that connects the point p and β, and whose interior does not intersect β (see Figure 5 (left)). Then we obtain a new eyeglass $\eta' = (a, b, \lambda_1 \cup \lambda_3)$, with the corresponding two eyeglass twists φ_1 and φ_2 ($\varphi_1 = T_{\eta'}$, $\varphi_2 = T_{\bar{\eta}'}$). Let $\gamma \subset \Sigma$ be a curve such that α, β, and γ cobound a pair of pants $\Delta \subset \Sigma$ containing η', as shown in Figure 6. Notice that $I(\eta', S_i) = 1$, which implies that $\varphi_1, \varphi_2 \in \mathcal{E}_k(S_i)$.

Pushing the 1-handle $a \times I$ along the path ι, as illustrated by the green line in Figure 5 (left), produces an eyeglass twist. This twist is exactly φ_2 ($= \tau_\alpha^{-1} \cdot \tau_\beta^{-1} \cdot \tau_\gamma$). After the excursion of the 1-handle $a \times I$ along path ι, the intersection points p and p' are eliminated, as shown in Figure 5 (right). To see this, we assume that φ_2 is supported in Δ. Thus, we only need to observe where the arcs $\lambda \cap \Delta$ are sent by φ_2. The precise picture is illustrated in Figure 5 (middle). Let $\ell \subset \lambda \cap \Delta$ be the component that is closest to α (i.e., $\{p, p'\} \subset \ell$). After having removed all bigons, we can see that $\varphi_2(\ell)$ is disjoint from S_i (See details in Figure 5 (right).) On the other hand, for any other component $\ell' \subset \lambda \cap \Delta$, $|\varphi_2(\ell') \cap S_i| = |\ell' \cap S_i|$. This implies that $I(\varphi_2(\eta), S_i) < k+1$. By Lemma 3.3, we then have $T_\eta = \varphi_2^{-1} \cdot T_{\varphi_2(\eta)} \cdot \varphi_2$. Since both $T_{\varphi_2(\eta)}$ and φ_2 belong to $\mathcal{E}_k(S_i)$, it follows that $T_\eta \in \mathcal{E}_k(S_i)$. □

In summary, we have the equation

$$\mathcal{E}_1(S_i) = \mathcal{E}_2(S_i) = \mathcal{E}_3(S_i) = \cdots$$

for $i = 1, 2, 3$.

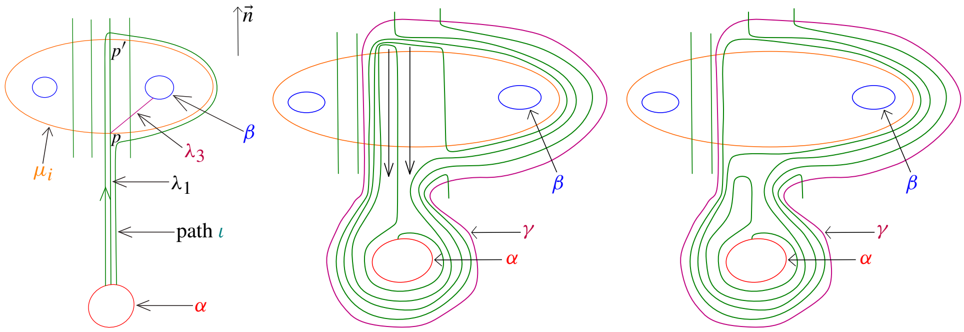

Figure 5. Left: new eyeglass. Middle: new intersection pattern. Right: new pattern with bigons removed.

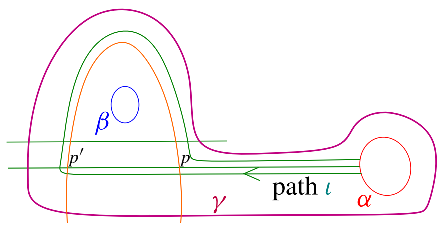

Figure 6. Specific neighborhood Δ of η.

Scharlemann [2022] defines a class of automorphisms of a Heegaard splitting, called bubble moves, which generalize one of the five automorphisms proposed by Powell [1980]. In this work, we extend the concept to a more general setting.

Definition 3.9. For a 3-manifold N with a Heegaard splitting $N = A \cup_\Sigma B$, a bubble is a 3-submanifold $\mathcal{B}$ of N, whose boundary is a 2-sphere. If the boundary $\partial\mathcal{B}$ is a reducing sphere for Σ, we call $\mathcal{B}$ a *bubble* for Σ. The genus of $\mathcal{B} \cap \Sigma$ is referred to as the *genus* of $\mathcal{B}$. In addition, a bubble is called *trivial* if it is a 3-ball, otherwise it is called *essential*.

Definition 3.10 [Scharlemann 2022, Section 2]. Let $N = A \cup_\Sigma B$ be a Heegaard splitting, and $\mathcal{B}$ a trivial bubble for Σ. A *bubble move* is an isotopy of $\mathcal{B}$ along a closed path in $\Sigma \setminus \text{int}(\mathcal{B})$ that starts and ends at $\mathcal{B}$, returning $(\mathcal{B}, \mathcal{B} \cap \Sigma)$ to itself. See Figure 7 (left).

For essential bubbles, we introduce a new class of automorphisms defined as follows.

Definition 3.11 (visional bubble move). Let $N = A \cup_\Sigma B$ be a Heegaard splitting for a closed 3-manifold N, and $\mathcal{B}$ a bubble for Σ with S as its boundary. The submanifold $N \setminus \text{int}(\mathcal{B})$ is also a bubble for Σ, which we refer to as the *dual bubble* of $\mathcal{B}$ and denote by $\mathcal{B}'$. By capping off the sphere boundary of $\mathcal{B}'$ with a 3-ball, we obtain a new manifold $N(\mathcal{B})$. The manifold $N(\mathcal{B})$ inherits a Heegaard splitting,

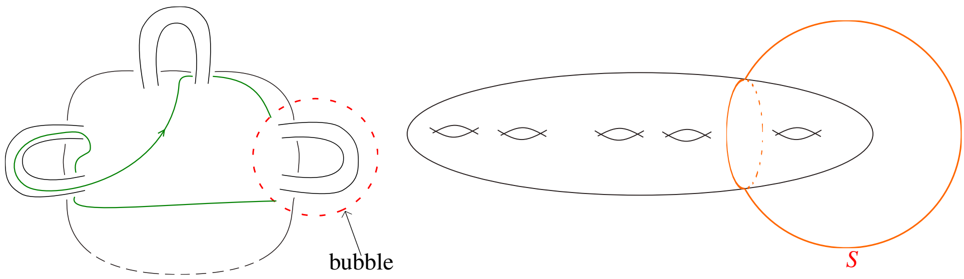

Figure 7. Left: bubble move. Right: Heegaard surface Σ'.

with its Heegaard surface Σ' being the boundary sum of $\Sigma \setminus \mathrm{int}(\mathcal{B})$ and a bordered torus (torus with an open disk removed), as illustrated in Figure 7 (right). Clearly, $N(\mathcal{B}) \setminus \mathrm{int}(\mathcal{B}')$ is a trivial bubble in $N(\mathcal{B})$, bounded by the sphere S, denoted by $\mathcal{B}^3$. As in the case of trivial bubbles, a bubble ($\mathcal{B}^3$) move induces a diffeomorphism $h : N(\mathcal{B}) \to N(\mathcal{B})$ such that $h|_{\mathcal{B}^3} = \mathrm{id}$. We then glue the two diffeomorphisms $h|_{\mathcal{B}'} : \mathcal{B}' \to \mathcal{B}'$ and $\mathrm{id} : \mathcal{B} \to \mathcal{B}$ along the sphere S to obtain a diffeomorphism $\tilde{h} : (N, \Sigma) \to (N, \Sigma)$, which we refer to as a *visional bubble* ($\mathcal{B}$) *move*.

In our setting, each sphere S_i ($i \in \{1, 2, 3\}$) bounds a genus one bubble, denoted by $\mathcal{B}_i$. It is easy to see that a visional $\mathcal{B}_i$ move fixes the sphere S_i. In other words, any visional $\mathcal{B}_i$ move lies in the stabilizer $\mathcal{H}_i \leqslant \mathcal{G}(N, \Sigma)$ of the isotopy class of the curve $\mu_i = S_i \cap \Sigma$. Furthermore, let $\mathcal{H}$ be the subgroup of $\mathcal{G}(N, \Sigma)$ generated by $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$; i.e., $\mathcal{H} = \langle \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 \rangle$.

4. Stabilizers of reducing 2-spheres

To prove the main theorem, we first show that $\mathcal{G}(N, \Sigma) = \mathcal{H}$ in Theorem 4.8. Then we will prove that $\mathcal{H}$ is finitely generated in the next section. As a corollary, the reducing sphere complex $\mathscr{R}$ is connected. Before proving Theorem 4.8, we introduce the following two lemmas.

Lemma 4.1. *$\mathcal{E}_1(S_i) < \mathcal{H}$ for $i = 1, 2, 3$.*

Proof. It is sufficient to prove that all generators of $\mathcal{E}_1(S_i)$ belong to the subgroup $\mathcal{H}$; i.e., $E_1(S_i) \subseteq \mathcal{H}$ for $i = 1, 2, 3$. Without loss of generality, we assume that $i = 1$. For any element $T_\eta \in E_1(S_1)$, $\eta = (a, b, \lambda)$ is an eyeglass such that S_1 separates η and the bridge λ intersects S_1 transversely at one point p_1. In this case, both $\alpha = \partial a$ and $\beta = \partial b$ are disjoint from $S_1 \cap \Sigma$. We assume the following conditions:

(1) a and b are, respectively, a disk in V and W.

(2) α and β lie in the genus two and the genus one component of $\Sigma \setminus S_1$ respectively.

We prove by induction on the geometric intersection number $I(\alpha, \mu_3)$ that $T_\eta \in \mathcal{H}$. The base case $\alpha \cap \mu_3 = \varnothing$ is divided into the following Case 1 and Case 2.

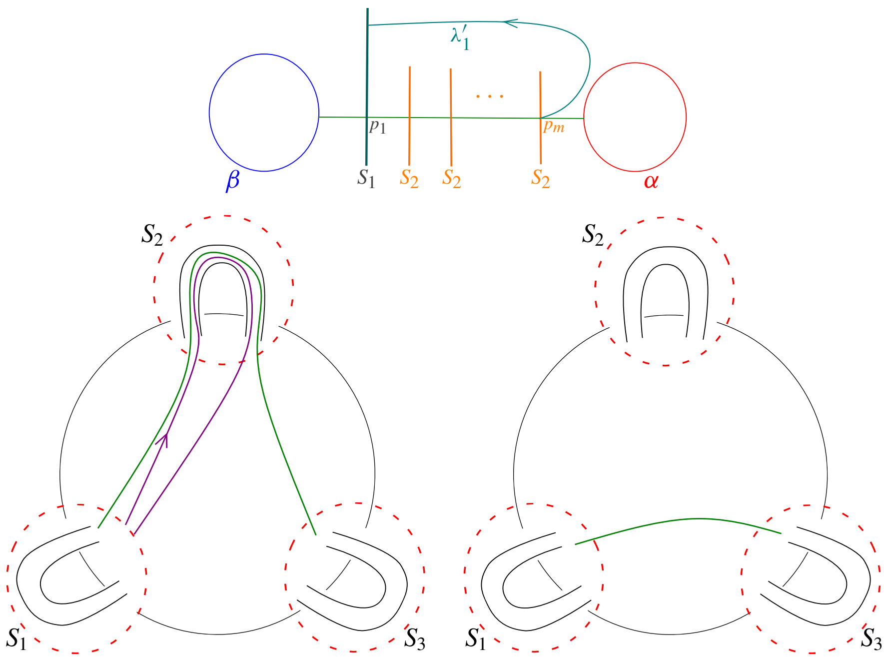

Figure 8. Top: finding a new arc λ_1'. Bottom left: a visional $\mathcal{B}_1$ move along the pink path. Bottom right: the reduction.

Case 1. $\alpha \cap \mu_3 = \varnothing$ and α lies in the genus one component of $\Sigma \setminus S_3$. Let $p_m \in \lambda \cap S_2$ be the point that is closest to α and $\lambda_1 \subset \lambda$ be the subarc bounded by p_0 and p_m, as shown in Figure 8 (top row). Then we choose an arc $\lambda_1' \subset \Sigma$ connecting p_m and a point of S_1, with its interior disjoint from S_1, S_2, and α. Let $\lambda' = \lambda_1 \cup \lambda_1'$. We endow λ' with an orientation as illustrated in Figure 8 (top row) and write $\vec{\lambda}'$ for the resulting oriented arc. Notice that the visional $\mathcal{B}_1$ move along $\vec{\lambda}'$, denoted by ψ_1, can reduce the intersection number $I(\eta, S_2)$. To be precise, $\psi_1 \in \mathcal{H}_1$ satisfies

$$I(\psi_1(\eta), S_2) = 0, \quad \psi_1(\alpha) = \alpha, \quad \psi_1(\beta) = \beta.$$

Figure 8 (bottom row) illustrates how a visional bubble move reduces the intersection.

Note that $\psi_1(\eta)$ is disjoint from S_2. It means that $T_{\psi_1(\eta)} \in \mathcal{H}_2$. Then we have

$$T_\eta = \psi_1^{-1} \cdot T_{\psi_1(\eta)} \cdot \psi_1 \in \mathcal{H}.$$

Case 2. $\alpha \cap \mu_3 = \varnothing$ and α lies in the genus two component of $\Sigma \setminus S_3$. Similarly, we can find a visional $\mathcal{B}_1$ move $\psi_2 \in \mathcal{H}_1$ such that

$$I(\psi_2(\eta), S_3) = 0, \quad \psi_2(\alpha) = \alpha, \quad \psi_2(\beta) = \beta.$$

It follows that $T_{\psi_2(\eta)} \in \mathcal{H}_3$ and $T_\eta = \psi_2^{-1} \cdot T_{\psi_2(\eta)} \cdot \psi_2 \in \mathcal{H}$.

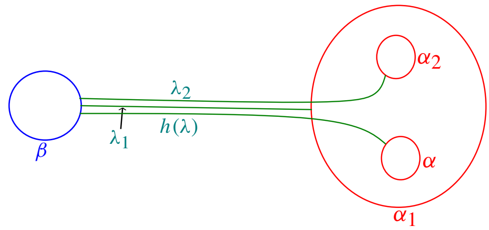

Figure 9. New eyeglasses.

Assume the statement is true for $I(\alpha, \mu_3) \le 2n$. We consider the case that $I(\alpha, \mu_3) = 2n+2$. We first isotope η so that all the endpoints of the bridge λ lie in the genus two component of $\Sigma \setminus S_3$, while keeping the curves α and β invariant during the isotopy. Next, we apply a visional $\mathcal{B}_1$ move $\bar{h} \in \mathcal{H}_1$ such that $I(h(\alpha), \mu_3) \leqslant I(\alpha, \mu_3)$ and $h(\lambda)$ is disjoint from μ_3.

Let D and D' be, respectively, the disks bounded by $h(\alpha)$ and μ_3 in V. Without loss of generality, we assume that $|D \cap D'|$ is minimal. Let $D'' \subset D'$ be an outermost subdisk cut off by D. Doing a compression on D along D'' results in two essential disks D_1 and D_2, with boundaries α_1 and α_2. Since $h(\lambda) \cap D'' \subset h(\lambda) \cap \mu_3 = \varnothing$, $\alpha_1 \cup \alpha_2$ intersects $h(\lambda)$ at most once. Thus, we consider the following two subcases:

Case 3.1. $(\alpha_1 \cup \alpha_2) \cap h(\lambda) = \varnothing$. By Lemma 3.4, $T_{h(\eta)}$ is a composition of two eyeglass twists whose bases have a smaller intersection with μ_3. Since the intersection number is smaller, the induction hypothesis applies. Therefore, by the induction assumption, we have $T_{h(\eta)} \in \mathcal{H}$.

Case 3.2. $|(\alpha_1 \cup \alpha_2) \cap h(\lambda)| = 1$. Without loss of generality, we assume that α_1 intersects $h(\lambda)$ at one point. We now construct two new eyeglasses η_1 $(=(D_1, b, \lambda_1))$ and η_2 $(= (D_2, b, \lambda_2))$, as depicted in Figure 9. By Lemma 3.4, we know that $T_{\eta_1} = T_{\eta_2} \cdot T_{h(\eta)}$. From the induction assumption, it follows that $T_{h(\eta)} \in \mathcal{H}$. □

Lemma 4.2. *For any eyeglass* $\eta = (a, b, \lambda)$ *(with* $\partial\eta = (\alpha, \beta, \lambda)$*) satisfying that both* α *and* β *lie in the genus two component of* $\Sigma \setminus S_i$ $(i = 1$ *or* $2)$*, we have* $T_\eta \in \mathcal{H}$.

Proof. Without loss of generality, we assume that both α and β lie in the genus two component of $\Sigma \setminus S_1$. The other case is similar. So we omit it. If λ is disjoint from $\mu_1 = S_1 \cap \Sigma$, then T_η fixes μ_1. This means that $T_\eta \in \mathcal{H}$. So we assume that neither α nor β is isotopic to μ_1, and $\lambda \cap \mu_1 \neq \varnothing$. After performing three compressions on Σ along a, b, and the disk $D \subset V$ bounded by μ_1, we obtain a 2-sphere S.

Claim 4.3. *The sphere* S *contains a scar*[3] *of* D.

[3]A disk compression will produce two copies of the surgery disk in the resulting surface, which we refer to as scars of the surgery disk.

Proof. Suppose that the conclusion is false. In this case, one of α and β is separating in Σ. Suppose it is α. Then β lies in the genus one component of $\Sigma \setminus \alpha$. Since α is disjoint from μ_1, it means that λ is disjoint from μ_1, a contradiction. □

Hence, S contains a scar of D and the scars of a and b. Then we choose two disjoint simple closed curves $\ell_1, \ell_2 \subset S$ such that ℓ_1 cuts off a disk that contains only the scars of a, while ℓ_2 cuts off a disk containing only the scars of b. Since ℓ_1 (resp. ℓ_2) is a band sum of two scars of a (resp. b), ℓ_1 (resp. ℓ_2) bounds an essential disk in V (resp. W). Moreover, ℓ_1, ℓ_2, and the reducing curve μ_1 $(= \partial D)$ cobound a pair of pants in Σ. Then ℓ_1 is the band sum of μ_1 and ℓ_2. Hence ℓ_1 bounds a disk in W. So ℓ_1 is a reducing curve. Similarly, ℓ_2 is also a reducing curve. Then, there are two disjoint reducing spheres S_{ℓ_1} and S_{ℓ_2} such that $S_{\ell_i} \cap \Sigma = \ell_i$. By definition, these three spheres (S_{ℓ_1}, S_{ℓ_2} and S_1) constitute a complete sphere triplet, denoted by $\mathcal{T}'$. Note that S_1 is a common 2-sphere of $\mathcal{T}$ $(= (S_1, S_2, S_3))$ and $\mathcal{T}'$. By Lemma 2.7, there exists an element $\phi \in \mathcal{H}_i$ such that $\phi(\mathcal{T}') = \mathcal{T}$. To prove $T_\eta \in \mathcal{H}$, it suffices to prove $T_{\phi(\eta)} \in \mathcal{H}$.

If α is isotopic to ℓ_1, then $\phi(\alpha)$ is isotopic to one of μ_1, μ_2, and μ_3. So $T_{\phi(\eta)} \in \mathcal{H}$. Otherwise, S_{ℓ_1} separates η. Then $\phi(S_{\ell_1})$ separates $\phi(\eta)$. By Lemma 3.8, we know that $\mathcal{E}_{I(\phi(\eta),\phi(S_{\ell_1}))}(\phi(S_{\ell_1})) = \mathcal{E}_1(\phi(S_{\ell_1}))$. On the other hand, $\phi(\mathcal{T}') = \mathcal{T}$ implies that $\phi(S_{\ell_1})$ is exactly one component of $\mathcal{T}$. Further, by Lemma 4.1, it follows that $\mathcal{E}_1(\phi(S_{\ell_1})) < \mathcal{H}$. In summary, we have

$$T_{\phi(\eta)} \in E_{I(\phi(\eta),\phi(S_{\ell_1}))}(\phi(S_{\ell_1})) \subseteq \mathcal{E}_{I(\phi(\eta),\phi(S_{\ell_1}))}(\phi(S_{\ell_1})) = \mathcal{E}_1(\phi(S_{\ell_1})) < \mathcal{H}. \quad \square$$

Recently, the classical "Haken's lemma" has been strengthened into the "strong Haken's lemma" by several authors in various ways [Scharlemann 2024; Hensel and Schultens 2024; Taylor 2025], which says that a sphere set in a 3-manifold can be isotoped to be aligned with the given Heegaard surface (See the following definition for "aligned".)

Definition 4.4 [Freedman and Scharlemann 2024, Section 1]. A sphere set $E \subset N$ is a compact properly embedded surface in N such that each component of E is a sphere. Then, a Heegaard surface Σ and a sphere set E in N $(= A \cup_\Sigma B)$ are *aligned* if they are transverse, and each component of E intersects Σ in at most one circle. In addition, each disk component of $E \setminus \Sigma$ is essential in either A or B.

Definition 4.5 (bubble-sum). Suppose $R_1 \subset N$ is a reducing sphere for Σ and $\mathcal{B} \subset N$ a trivial bubble disjoint from R_1. We will use the notation $R_2 = \partial\mathcal{B}$, $\gamma_1 = R_1 \cap \Sigma$, and $\gamma_2 = R_2 \cap \Sigma$. Let $\lambda \subset \Sigma$ be an embedded arc that connects γ_1 and γ_2, with its interior disjoint from γ_1 and γ_2. Then for a good closed neighborhood $N(R_1 \cup R_2 \cup \lambda)$, $\partial N(R_1 \cup R_2 \cup \lambda)$ consists of copies of $R_1 \cup R_2$ and a new reducing sphere R_3. The reducing sphere R_3 is called a bubble-sum of R_1 and R_2 along λ.

Definition 4.6 [Freedman and Scharlemann 2024, Definition 1.4]. Let E_0 and E_1 be two sphere sets aligned with Σ in N. E_0 and E_1 are *equivalent* if there is an isotopy $H : N \times I \to N$ with $H_s : \Sigma \times \{s\} \to \Sigma$ for $0 \leq s \leq 1$ such that H_1 maps E_0 to E_1.

Freedman and Scharlemann [2024] prove that any two alignments of a sphere set are related by a sequence of *bubble-sums*, which are called "bubble moves" in their paper, and *eyeglass twists*.

Theorem 4.7 [Freedman and Scharlemann 2024, Theorem 1.6]. *If E_0 and E_1 are two sphere sets that are properly isotopic in N and each aligns with Σ, then up to equivalence, E_1 can be obtained from E_0 by a sequence of bubble-sums and eyeglass twists.*[4]

There is a bijection between the isotopy classes of reducing curves and the isotopy classes of reducing spheres. We identify a reducing sphere with its corresponding reducing curve. Therefore, the reducing sphere complex can be treated as a subcomplex of the curve complex of Σ. In subsequent arguments, we will use the symbol $\mathscr{R}$ to represent the reducing sphere complex for the Heegaard splitting $N = V \cup_\Sigma W$.

Theorem 4.8. $\mathcal{G}(N, \Sigma) = \mathcal{H}$.

Proof. We divide the proof into two cases: (a) $N_1 \neq N_2$; (b) $N_1 = N_2$.

Case (a). Let $\mathscr{R}^0$ be the 0-skeleton of $\mathscr{R}$. As both $\mathcal{G}(N, \Sigma)$ and $\mathcal{H}$ naturally act on $\mathscr{R}^0$, we denote the corresponding orbit containing the isotopy class $\bar{\mu}_i$ by $\mathscr{O}_i$ and $\mathscr{O}_i'$ respectively. Since we have assumed that $N_1 \neq N_2$, we know that $\mathscr{O}_1 \cap \mathscr{O}_2 = \varnothing$. To prove the theorem, it suffices[5] to show that $\mathscr{O}_1 = \mathscr{O}_1'$.

Given any reducing sphere S for Σ with the intersection curve μ $(= S \cap \Sigma)$ representing an isotopy class $\bar{\mu}$ of $\mathscr{O}_1$, we will prove that $\bar{\mu} \in \mathscr{O}_1'$. If it does, we immediately obtain the desired result $\mathscr{O}_1 = \mathscr{O}_1'$.

By an innermost curve argument, these two essential spheres S and S_1 are isotopic. Both S and S_1 are aligned with Σ. By Theorem 4.7, S is related to S_1 by a sequence of bubble-sums and eyeglass twists. Thus, there is a sequence of reducing curves λ_i in Σ such that λ_{i+1} can be obtained from λ_i by a bubble-sum or an eyeglass twist

$$\mu_1 = \Sigma \cap S_1 = \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n = \Sigma \cap S = \mu. \tag{1}$$

We first prove by induction that $\bar{\lambda}_i \in \mathscr{O}_1' \cup \mathscr{O}_2'$. The base case $\bar{\lambda}_1 \in \mathscr{O}_1' \cup \mathscr{O}_2'$ clearly holds. We assume that $\bar{\lambda}_i \in \mathscr{O}_1' \cup \mathscr{O}_2'$. Then we prove that $\bar{\lambda}_{i+1} \in \mathscr{O}_1' \cup \mathscr{O}_2'$. Without loss of generality, we assume that $\bar{\lambda}_i \in \mathscr{O}_1'$. It means that there is an element $\bar{g} \in \mathcal{H}$ such that $\bar{g}(\bar{\lambda}_i) = \bar{\mu}_1$. Let $g \in \mathrm{Diff}^+(N, \Sigma)$ be a representative of $\bar{g}$ such that $g(\lambda_i) = \mu_1$. Then there are two cases as follows.

[4]Each involved eyeglass twist must have its base lenses disjoint from the sphere set on which it acts. See details in [Freedman and Scharlemann 2024, Section 1].

[5]Assume that $\mathscr{O}_1 = \mathscr{O}_1'$. Then, for any $\phi \in \mathcal{G}(N, \Sigma)$, we can find an element $\varphi \in \mathcal{H}$ such that $\phi(\bar{\mu}_1) = \varphi(\bar{\mu}_1)$. So $\varphi^{-1} \cdot \phi \in \mathcal{H}_1 \leqslant \mathcal{H}$. It follows that $\phi \in \mathcal{H}$.

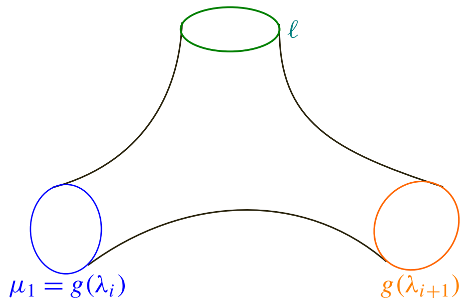

Figure 10. A pair of pants bounded by $g(\lambda_{i+1})$, ℓ, and μ_1.

Case 1. λ_{i+1} is obtained from λ_i by a bubble-sum. It implies that $g(\lambda_{i+1})$ can be obtained from $g(\lambda_i)$ by a bubble-sum. In this bubble-sum, the involved trivial bubble has sphere boundary S_ℓ, where $S_\ell \cap \Sigma = \ell$. By Definition 4.5, the three curves $g(\lambda_{i+1})$, ℓ, and μ_1 $(= g(\lambda_i))$ cobound a pair of pants illustrated as in Figure 10. Denote by $S_{g(\lambda_{i+1})}$ the reducing 2-sphere intersecting transversely with Σ in $g(\lambda_{i+1})$. Note that $S_1 \cup S_\ell \cup S_{g(\lambda_{i+1})}$ divides N into four submanifolds of the same diffeomorphism type as those divided by $S_1 \cup S_3 \cup S_2$. It follows that there is a diffeomorphism $h \in \mathrm{Diff}^+(N, \Sigma)$ such that $h(S_1, S_\ell, S_{g(\lambda_{i+1})}) = (S_1, S_3, S_2)$. Thus, we have $h \cdot g(\lambda_{i+1}) = \mu_2$. Furthermore, $\bar{h} \cdot \bar{g}(\bar{\lambda}_{i+1}) = \bar{\mu}_2$, where $\bar{g} \in \mathcal{H}$ and $\bar{h} \in \mathcal{H}_1$. So $\bar{\lambda}_{i+1} \in \mathscr{O}_2'$.

Case 2. λ_{i+1} is obtained from λ_i by an eyeglass twist $T_{\eta_i'}$. It means that $g(\lambda_{i+1})$ can be obtained from $g(\lambda_i)$ $(= \mu_1)$ by the eyeglass twist $T_{g(\eta_i')}$. Write $\eta_i = g(\eta_i')$. From the hypotheses of Theorem 4.7, we know that the lenses of η_i' are disjoint from λ_i. This implies the lenses of η_i are also disjoint from μ_1. Accordingly, there are two subcases as follows.

Subcase 2.1. S_1 separates η_i. By Lemma 3.8 and Lemma 4.1, we have $T_{\eta_i} \in \mathcal{H}$. Since $T_{\eta_i}^{-1} \cdot g(\lambda_{i+1}) = \mu_1$, we have that $\bar{\lambda}_{i+1} \in \mathscr{O}_1'$.

Subcase 2.2. S_1 does not separate η_i. Then these two lenses of η_i both lie in the component of $N \setminus S_1$ which contains the genus two component of $\Sigma \setminus S_1$. By Lemma 4.2, $T_{\eta_i} \in \mathcal{H}$. So we have $\bar{\lambda}_{i+1} \in \mathscr{O}_1'$.

The above argument completes the proof for the statement that $\bar{\lambda}_i \in \mathscr{O}_1' \cup \mathscr{O}_2'$ for all $i \leqslant n$. In particular, $\bar{\mu} = \bar{\lambda}_n \in \mathscr{O}_1' \cup \mathscr{O}_2'$. However, $\bar{\mu} \in \mathscr{O}_1$, $\mathscr{O}_1 \cap \mathscr{O}_2 = \varnothing$, and $\mathscr{O}_i' \subset \mathscr{O}_i$. This implies that $\bar{\mu} \in \mathscr{O}_1'$. This completes the proof of the case (a).

Case (b). Since any lens space has a unique genus one Heegaard splitting up to diffeomorphism, we can construct a diffeomorphism $f \in \mathrm{Diff}^+(N, \Sigma)$ such that $f(S_1, S_2, S_3) = (S_2, S_1, S_3)$, as in the proof of Lemma 2.6. This means that $\mathscr{O}_1' = \mathscr{O}_2'$. Then we can prove by induction on the sequence (1), as in Case (a), that $\bar{\lambda}_i \in \mathscr{O}_1'$. It follows that $\mathscr{O}_1' = \mathscr{O}_1$. Overall, $\mathcal{G}(N, \Sigma) = \mathcal{H}$. □

We use the above results to prove the connectedness of the reducing sphere complexes $\mathscr{R}$.

Proof of Corollary 1.2. These three reducing 2-spheres S_1, S_2 and S_3 are contained in a same component of $\mathscr{R}$, say $\mathscr{R}'$. To prove the connectedness of $\mathscr{R}$, it suffices to show that $\mathscr{R} = \mathscr{R}'$.

Given any reducing sphere $S \in \mathscr{R}$, there are two other reducing spheres S' and S'' in $\mathscr{R}$ such that the sphere triplet $\mathcal{T}' = (S, S', S'')$ is complete. By Lemma 2.6, there is an element $\nu \in \mathcal{G}(N, \Sigma)$ such that $\nu(\mathcal{T})$ is congruent to $\mathcal{T}'$. By Theorem 4.8,

$$\nu = \theta_n \cdot \theta_{n-1} \cdots \theta_2 \cdot \theta_1,$$

where $\theta_i \in \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$.

Let $\nu_i = \theta_i \cdot \theta_{i-1} \cdots \theta_2 \cdot \theta_1$ $(0 \leqslant i \leqslant n)$ and $\mathcal{T}_i = \nu_i(\mathcal{T})$. Then we obtain a sequence of triplets

$$\mathcal{T} = \mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{n-1}, \mathcal{T}_n = \mathcal{T}'.$$

We identify a complete sphere triplet with a 2-simplex of $\mathscr{R}$. Then we prove by induction that $\mathcal{T}_i \subset \mathscr{R}'$. The base case $\mathcal{T}_0 \subset \mathscr{R}'$ clearly holds. We assume that $\mathcal{T}_k \subset \mathscr{R}'$. Without loss of generality, we assume that $\theta_{k+1} \in \mathcal{H}_1$. By the induction assumption, $\mathcal{T}_k$ and S_1 are contained in the same component $\mathscr{R}'$. So $\theta_{k+1}(\mathcal{T}_k)$ and $\theta_{k+1}(S_1)$ are also in the same component of $\mathscr{R}$. Since $\theta_{k+1}(S_1) = S_1 \in \mathscr{R}'$, we have that $\mathcal{T}_{k+1} = \theta_{k+1}(\mathcal{T}_k) \subset \mathscr{R}'$. Therefore, $S \in \mathcal{T}' = \mathcal{T}_n \subset \mathscr{R}'$. □

5. Finitely many generators

In this section, we prove that each $\mathcal{H}_i$ is finitely generated. Then, by Theorem 4.8, $\mathcal{G}(N, \Sigma)$ $(= \langle \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 \rangle)$ is also finitely generated. Before that, we first introduce the following definition.

Definition 5.1. Let $N = A \cup_\Sigma B$ be a Heegaard splitting and D a union of finitely many disjoint *marked*[6] disks in Σ. In this case, we define the Goeritz group $\mathcal{G}(N, \Sigma, D)$ to be the group of diffeomorphisms of N that preserve the Heegaard splitting and fix D pointwise, modulo isotopies[7] that leave Σ and D invariant.

Note 5.2. Unless otherwise specified, all manifolds considered in this section are not assumed to be the connected sum of lens spaces.

Suppose $\mathcal{B}$ is a bubble for Σ, with the boundary sphere $S = \partial\mathcal{B}$ intersecting Σ in an essential simple closed curve μ. Let D_a be the disk $S \cap A$ and $\Sigma_{\mathcal{B}}$ be the bordered surface $\Sigma \backslash \mathrm{int}(\mathcal{B})$. Denote by $\Sigma(\mathcal{B})$ the closed surface $\Sigma_{\mathcal{B}} \cup D_a$. Since S is separating

[6]In classical surface mapping class group theory, the surfaces with marked points are frequently considered. Here, we simply follow this tradition by discussing mapping class groups of surfaces with marked disks.

[7]Precisely, such an isotopy $H_t : N \to N,\ 0 \leq t \leq 1$, is required to satisfy that $H_t(\Sigma, D) = (\Sigma, D)$ and $H_0|_D = H_1|_D = \mathrm{id}$.

in N, we cap off the sphere boundary S of the dual bubble $\mathcal{B}'$ $(= N \setminus \mathrm{int}(\mathcal{B}))$ with a 3-ball to obtain a new closed orientable 3-manifold $N(\mathcal{B})$. It is not hard to see that $\Sigma(\mathcal{B})$ is a Heegaard surface for $N(\mathcal{B})$. Putting an orientation for the curve μ, we get the oriented curve $\vec{\mu}$ and its isotopy class $[\vec{\mu}]$. Denote by $G_{\vec{\mu}}$ the stabilizer of $[\vec{\mu}]$:

$$G_{\vec{\mu}} \overset{\text{def}}{=} \{\phi \in \mathcal{G}(N, \Sigma) : \phi([\vec{\mu}]) = [\vec{\mu}]\}.$$

We define a subgroup of $\mathrm{Diff}^+(N, \Sigma)$ by

$$\mathrm{Diff}^+(N, \Sigma, S) \overset{\text{def}}{=} \{f \in \mathrm{Diff}^+(N, \Sigma) : f|_S = \mathrm{id}\}.$$

By the definition, if $f \in \mathrm{Diff}^+(N, \Sigma, S)$, then $f(\mathcal{B}') = \mathcal{B}'$ $(\mathcal{B}' = \overline{N \setminus \mathcal{B}})$. We associate it with an element of the Goeritz group $\mathcal{G}(N(\mathcal{B}), \Sigma(\mathcal{B}), D_a)$ as follows. Since $N(\mathcal{B})$ is the union of B' and a 3-ball, $f|_{\mathcal{B}'}$ can be naturally extended into a diffeomorphism $\hat{f} : (N(\mathcal{B}), \Sigma(\mathcal{B}), D_a) \to (N(\mathcal{B}), \Sigma(\mathcal{B}), D_a)$. All different extensions of f are pairwise isotopic. Subsequently, we obtain a map $\rho_2 : \mathrm{Diff}^+(N, \Sigma, S) \to \mathcal{G}(N(\mathcal{B}), \Sigma(\mathcal{B}), D_a)$. It is not hard to verify that ρ_2 is also an epimorphism.

Restricting the natural homomorphism $\rho_1 : \mathrm{Diff}^+(N, \Sigma) \to \mathcal{G}(N, \Sigma)$ to the subgroup $\mathrm{Diff}^+(N, \Sigma, S)$ results in a restriction map, which we still denote by ρ_1. It is easy to see that $\rho_1(\mathrm{Diff}^+(N, \Sigma, S)) = G_{\vec{\mu}}$.

We want to define a homomorphism $\rho : G_{\vec{\mu}} \to \mathcal{G}(N(\mathcal{B}), \Sigma(\mathcal{B}), D_a)$ so that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Diff}^+(N, \Sigma, S) & \xrightarrow{\ \rho_1\ } & G_{\vec{\mu}} \\ & \searrow^{\rho_2} & \downarrow \rho \\ & & \mathcal{G}(N(\mathcal{B}), \Sigma(\mathcal{B}), D_a) \end{array}$$

If such a ρ exists, it is uniquely determined by the requirement. Its existence is guaranteed by the following lemma.

Lemma 5.3. *For any diffeomorphism* $f \in \mathrm{Diff}^+(N, \Sigma, S)$, *if* $\rho_1(f) = \mathrm{id}$, *then* $\rho_2(f) = \mathrm{id}$.

Proof. Assume that $\vec{\alpha}$ is an oriented essential simple closed curve in $\Sigma_{\mathcal{B}}$. Since $\rho_1(f) = \mathrm{id}$, $f(\vec{\alpha})$ is isotopic to $\vec{\alpha}$ in Σ. By [Farb and Margalit 2012, Lemma 3.16], we know that $f(\vec{\alpha})$ is also isotopic to $\vec{\alpha}$ in $\Sigma_{\mathcal{B}}$. In other words, f preserves the isotopy classes of all oriented essential simple closed curves in $\Sigma_{\mathcal{B}}$. We can prove by the *Alexander method*[8] that such a diffeomorphism is isotopic to a power of the Dehn twist τ_μ. This means that $\rho_2(f) = \mathrm{id}$. □

[8] Here, we first choose a collection $\{\gamma_i\}$ of essential simple closed curves in $\Sigma_{\mathcal{B}}$, as shown in Figure 11. By the Alexander method [Farb and Margalit 2012, Proposition 2.8], we know that $f\big(\bigcup \gamma_i\big)$ is isotopic to $\bigcup \gamma_i$ relative to μ $(= \partial \Sigma_{\mathcal{B}})$. So we assume that $f\big(\bigcup \gamma_i\big) = \bigcup \gamma_i$. In addition, f also

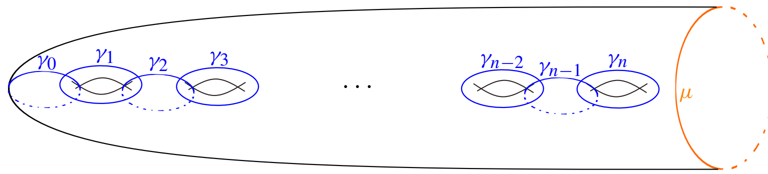

Figure 11. The surface $\Sigma_{\mathcal{B}}$.

It follows that such ρ exists and is surjective, and the kernel of ρ is denoted by $\mathcal{I}(\rho)$. Then we have the exact sequence

$$1 \to \mathcal{I}(\rho) \xrightarrow{i} G_{\vec{\mu}} \xrightarrow{\rho} \mathcal{G}(N(\mathcal{B}), \Sigma(\mathcal{B}), D_a) \to 1. \tag{2}$$

Similarly, for the dual bubble $\mathcal{B}'$, we also have a dual exact sequence

$$1 \to \mathcal{I}(\rho') \xrightarrow{i'} G_{\vec{\mu}} \xrightarrow{\rho'} \mathcal{G}(N(\mathcal{B}'), \Sigma(\mathcal{B}'), D_a) \to 1. \tag{3}$$

Lemma 5.4. *The composite homomorphism $\rho' \cdot i$ is an epimorphism.*

Proof. By a similar argument for the dual bubble $\mathcal{B}'$, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Diff}^+(N, \Sigma, S) & \xrightarrow{\rho_1} & G_{\vec{\mu}} \\ & \searrow^{\rho_2'} & \downarrow \rho' \\ & & \mathcal{G}(N(\mathcal{B}'), \Sigma(\mathcal{B}'), D_a) \end{array}$$

Since ρ_2 is surjective, for any element $\phi \in \mathcal{G}(N(\mathcal{B}'), \Sigma(\mathcal{B}'), D_a)$, we can find a diffeomorphism $f \in \mathrm{Diff}^+(N, \Sigma, S)$ such that $\rho_2'(f) = \phi$. We extend $f|_{\mathcal{B}}$ by the identity to obtain a diffeomorphism $\hat{f} \in \mathrm{Diff}^+(N, \Sigma, S)$. It is not hard to see that $\rho' \cdot \rho_1(\hat{f}) = \rho_2'(\hat{f}) = \rho_2'(f) = \phi$ and $\rho_1(\hat{f}) \in \mathcal{I}(\rho)$. The lemma follows immediately. □

Subsequently, we have the exact sequence

$$1 \to \mathcal{I}(\rho') \cap \mathcal{I}(\rho) \xrightarrow{i'''} \mathcal{I}(\rho) \xrightarrow{\rho' \cdot i} \mathcal{G}(N(\mathcal{B}'), \Sigma(\mathcal{B}'), D_a) \to 1. \tag{4}$$

Since the marked disk D_a can be treated as a marked point in $\Sigma(\mathcal{B})$, we apply the description of the kernel of the *capping homomorphism*[9] [Farb and Margalit 2012, Proposition 3.19] to obtain $\mathcal{I}(\rho') \cap \mathcal{I}(\rho) = \langle \tilde{\tau}_\mu \rangle$, where $\tilde{\tau}_\mu$ is the extension of the Dehn twist τ_μ to the whole manifold N.

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preserves the orientation of γ_i. This implies that f fixes each vertex and each edge of $\bigcup \gamma_i$ (regard $\bigcup \gamma_i$ as a graph). Without loss of generality, we may further assume that $f|_{\bigcup \gamma_i} = \mathrm{id}$. On the other hand, $\Sigma_{\mathcal{B}} \setminus (\bigcup \gamma_i)$ is an annulus. It follows that f is isotopic to a power of the *Dehn twist* τ_μ.

[9]Let $f \in \mathrm{Diff}^+(N, \Sigma, S)$ be a diffeomorphism that represents an element of $\mathcal{I}(\rho') \cap \mathcal{I}(\rho)$. Then, with D_a identified with a marked point in $\Sigma(\mathcal{B})$, $f|_{\Sigma_{\mathcal{B}}}$ represents an element of the kernel of the capping homomorphism $\mathrm{Cap} : \mathrm{Mod}(\Sigma_{\mathcal{B}}, \partial \Sigma_{\mathcal{B}}) \to \mathrm{Mod}(\Sigma(\mathcal{B}), D_a)$. By [Farb and Margalit 2012, Proposition 3.19], $f|_{\Sigma_{\mathcal{B}}}$ is isotopic to a power of the Dehn twist along μ $(= \partial \Sigma_{\mathcal{B}})$, and so is $f|_{\Sigma_{\mathcal{B}'}}$. This implies that f is isotopic to a power of the Dehn twist along μ.

Lemma 5.5. *If both* $\mathcal{G}(N(\mathcal{B}), \Sigma(\mathcal{B}))$ *and* $\mathcal{G}(N(\mathcal{B}'), \Sigma(\mathcal{B}'))$ *are finitely generated* (*or finitely presented*), *then* $G_{\vec{\mu}}$ *is finitely generated* (*or finitely presented*) *as well.*

Proof. We first prove that $\mathcal{G}(N(\mathcal{B}), \Sigma(\mathcal{B}), D_a)$ is finitely generated (or finitely presented). If the genus $g(\Sigma(\mathcal{B}))$ is 1, we have $\mathcal{G}(N(\mathcal{B}), \Sigma(\mathcal{B}), D_a) = \mathcal{G}(N(\mathcal{B}), \Sigma(\mathcal{B}))$. Then there is nothing to prove. If $g(\Sigma(\mathcal{B})) \geqslant 2$, we apply the *Birman exact sequence* [Birman 1969] for the pair $(\Sigma(\mathcal{B}), D_a)$ to obtain the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \xrightarrow{\text{push}} & \mathcal{G}(N(\mathcal{B}), \Sigma(\mathcal{B}), D_a) & \xrightarrow{\text{forget}} & \mathcal{G}(N(\mathcal{B}), \Sigma(\mathcal{B})) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i} & & \\ 1 & \longrightarrow & \pi_1(\Sigma(\mathcal{B})) & \xrightarrow{\text{push}} & \mathrm{Mod}(\Sigma(\mathcal{B}), D_a) & \xrightarrow{\text{forget}} & \mathrm{Mod}(\Sigma(\mathcal{B})) & \longrightarrow & 1 \end{array}$$

where i denotes the inclusion map. The Birman exact sequence provides a description for the kernel of the *forget* map, which asserts that the kernel is generated by the isotopies (of $\Sigma(\mathcal{B})$) that push D_a along a closed path (that begins and ends at D_a) in $\Sigma(\mathcal{B})$. Note that all such isotopies can be extended to the whole manifold $N(\mathcal{B})$. It follows that $K = \pi_1(\Sigma(\mathcal{B}))$. Since both $\pi_1(\Sigma(\mathcal{B}))$ and $\mathcal{G}(N(\mathcal{B}), \Sigma(\mathcal{B}))$ are finitely generated (or finitely presented), so is $\mathcal{G}(N(\mathcal{B}), \Sigma(\mathcal{B}), D_a)$. Similarly, $\mathcal{G}(N(\mathcal{B}'), \Sigma(\mathcal{B}'), D_a)$ is also finitely generated (or finitely presented).

By the exact sequence (4), $\mathcal{I}(\rho)$ is finitely generated (or finitely presented). Then by the exact sequence (2), $G_{\vec{\mu}}$ is also finitely generated (or finitely presented). □

With the above preparations completed, we can present the proof of Theorem 1.3.

Proof of Theorem 1.3. By Lemma 5.5, we know that $G_{\vec{\mu}}$ is finitely generated (or finitely presented). Since $G_{\vec{\mu}}$ is a subgroup of G_μ with index at most two, G_μ is also finitely generated (or finitely presented). □

The genus at most two Goeritz groups for lens spaces or their connected sum have been shown to be finitely generated in [Cho 2013; Cho and Koda 2016; 2019]. Then by Theorem 1.3, the stabilizer G_{μ_i}, which is exactly the group $\mathcal{H}_i$, is finitely generated. By Theorem 4.8, it follows that $\mathcal{G}(N, \Sigma)$ is finitely generated. So we complete the proof of Theorem 1.1.

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