

*Pacific
Journal of
Mathematics*

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IN A HYPERBOLIC SPACE $H^4(-1)$

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COMPLETE MINIMAL HYPERSURFACES IN A HYPERBOLIC SPACE $H^4(-1)$

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We study n -dimensional complete minimal hypersurfaces in the hyperbolic space $H^{n+1}(-1)$ of constant curvature -1 . We prove that a 3-dimensional complete minimal hypersurface with constant scalar curvature in $H^4(-1)$ satisfies $S \leq \frac{21}{29}$ by making use of the generalized maximum principle, where S denotes the squared norm of the second fundamental form of the hypersurface.

1. Introduction

Let M^n be an n -dimensional minimal hypersurface in the hyperbolic space $H^{n+1}(-1)$ of constant curvature -1 . A very important subject of study is the rigidity of complete minimal hypersurfaces in the hyperbolic space $H^{n+1}(-1)$. It is well known that there are many important results on the rigidity of compact minimal hypersurfaces in the unit sphere $S^{n+1}(1)$. For example, Simons [7], Chern, do Carmo and Kobayashi [3] and Lawson [4] prove that an n -dimensional compact minimal hypersurface in the unit sphere $S^{n+1}(1)$ is isometric to a totally geodesic sphere or a Clifford torus if the squared norm S of its second fundamental form satisfies $S \leq n$. In particular, for $n = 3$, it is known that a 3-dimensional compact minimal hypersurface in the unit sphere $S^4(1)$ with constant scalar curvature is isometric to a totally geodesic sphere or a Clifford torus or the Cartan minimal isoparametric hypersurface (see [1; 6]). On the other hand, Cheng and Wan [2] proved complete minimal hypersurfaces with constant scalar curvature in the Euclidean space \mathbb{R}^4 are isometric to the hyperplane \mathbb{R}^3 . But for complete minimal hypersurfaces in the hyperbolic space $H^{n+1}(-1)$, there are only few results on rigidity of complete minimal hypersurfaces. It is our main purpose to study the following conjecture:

Conjecture. A complete minimal hypersurface with constant scalar curvature in the hyperbolic space $H^4(-1)$ is isometric to the hyperbolic space $H^3(-1)$.

We will prove the following:

Peng is the corresponding author. This work was supported by JSPS Grant-in-Aid for Scientific Research No: 25K06992.

MSC2020: 53C40, 53C42.

Keywords: minimal hypersurfaces, hyperbolic space, constant scalar curvature, the generalized maximum principle.

Theorem 1.1. *A complete minimal hypersurface with constant scalar curvature in the hyperbolic space $H^4(-1)$ satisfies $S \leq \frac{21}{29}$, where S denotes the squared norm of the second fundamental form of the hypersurface.*

2. Basic formulas

Let M^n be an n -dimensional hypersurface in an $(n+1)$ -dimensional hyperbolic space $H^{n+1}(-1)$. At each point p in $H^{n+1}(-1)$, we choose a local orthonormal frame field $\{e_1, e_2, \dots, e_{n+1}\}$ and the dual coframe $\{\omega^1, \omega^2, \dots, \omega^{n+1}\}$ such that, restricted to M^n , $\{e_1, e_2, \dots, e_n\}$ is tangent to M^n . Structure equations of $H^{n+1}(-1)$ are given by

$$(2-1) \quad \begin{aligned} d\omega_A &= -\sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} + \sum_C \omega_{AC} \wedge \omega_{CB} &= \frac{1}{2} \sum_{C,D} K_{ABCD} \omega_C \wedge \omega_D, \end{aligned}$$

with

$$(2-2) \quad K_{ABCD} = -(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}).$$

If we restrict these forms to M^n , then $\omega^{n+1} = 0$. We have

$$(2-3) \quad \omega_{i,n+1} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}.$$

One calls

$$(2-4) \quad H = \frac{1}{n} \sum_i h_{ii}, \quad h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j$$

the mean curvature and the second fundamental form of M^n , respectively. If H is identically zero, M^n is called minimal. The structure equations of M^n are given by

$$(2-5) \quad \begin{aligned} d\omega_i &= -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} &= \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \end{aligned}$$

where

$$(2-6) \quad R_{ijkl} = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}).$$

For minimal hypersurfaces in $H^{n+1}(-1)$, we obtain

$$R = -n(n-1) - S,$$

where R and S denote the scalar curvature and the squared norm of the second fundamental form of M^n , respectively. From the structure equations of M^n , Codazzi equations and Ricci formulas are given by

$$h_{ijk} = h_{ikj}, \quad h_{ijkl} - h_{ijlk} = \sum_m h_{im} R_{mjkl} + \sum_m h_{mj} R_{mikl},$$

where $h_{ijk} = \nabla_k h_{ij}$ and $h_{ijkl} = \nabla_l \nabla_k h_{ij}$, respectively. Define f_3 and f_4 by

$$f_3 = \sum_{i,j,k=1}^n h_{ij} h_{jk} h_{ki} \quad \text{and} \quad f_4 = \sum_{i,j,k,l=1}^n h_{ij} h_{jk} h_{kl} h_{li},$$

respectively. We have, for minimal hypersurfaces,

$$(2-7) \quad \begin{aligned} \frac{1}{3} \Delta f_3 &= -(n + S) f_3 + 2C, \\ \frac{1}{4} \Delta f_4 &= -(n + S) f_4 + (2A + B), \end{aligned}$$

where

$$C = \sum_{i,j,k} \lambda_i h_{ijk}^2, \quad A = \sum_{i,j,k} \lambda_i^2 h_{ijk}^2, \quad B = \sum_{i,j,k} \lambda_i \lambda_j h_{ijk}^2$$

and λ_i 's are principal curvatures of M^n , that is,

$$\begin{aligned} \sum_i h_{ii} &= \sum_i \lambda_i = 0, \quad S = \sum_{i,j} h_{ij}^2 = \sum_i \lambda_i^2, \\ h_{ijij} - h_{jiji} &= (\lambda_i - \lambda_j)(-1 + \lambda_i \lambda_j). \end{aligned}$$

By a direct computation, we have

$$S = n(1 - n) - R, \quad \Delta h_{ij} = -(S + n)h_{ij}, \quad \frac{1}{2} \Delta S = -S(S + n) + \sum_{i,j,k} h_{ijk}^2.$$

If the squared norm S of the second fundamental form is constant, we have

$$\sum_{i,j,k} h_{ijk}^2 = S(S + n), \quad \sum_{i,j,k,l} h_{ijkl}^2 = S(S + n)(2n + 3 + S) + 3(A - 2B).$$

The following generalized maximum principle due to Omori [5] (see Yau [8]) will play an important role in this paper.

Theorem 2.1. *Let M^n be a complete Riemannian manifold with sectional curvature bounded from below. If a C^2 -function f is bounded from above in M^n , then there exists a sequence $\{p_k\}_{k=1}^\infty \subset M^n$ such that*

- (1) $\lim_{k \rightarrow \infty} f(p_k) = \sup_{M^n} f$,
- (2) $\lim_{k \rightarrow \infty} |\nabla f(p_k)| = 0$,
- (3) $\lim_{k \rightarrow \infty} \sup \nabla_l \nabla_l f(p_k) \leq 0$, for $l = 1, 2, \dots, n$.

3. Minimal hypersurfaces with two distinct principal curvatures

Theorem 3.1. *Let M^3 be a minimal hypersurface in $H^4(-1)$ with constant scalar curvature. If M^3 has two principal curvatures somewhere, we have $S \leq \frac{21}{29}$.*

Proof. We assume, at $p \in M^3$, that M^3 has two distinct principal curvatures. At p , we may choose an orthonormal frame e_1, e_2, e_3 such that $h_{ij} = \lambda_i \delta_{ij}$. We can assume

$$\lambda_1 = \lambda_2 = \lambda.$$

Since M^3 is minimal, we have

$$\lambda_3 = -2\lambda, \quad \lambda^2 = \frac{1}{6}S.$$

Because $\sum_i h_{ii} = 0$ and S is constant, we have

$$h_{11k} + h_{22k} + h_{33k} = 0, \quad h_{11k} + h_{22k} - 2h_{33k} = 0.$$

We obtain

$$h_{11k} + h_{22k} = 0, \quad h_{33k} = 0, \quad k = 1, 2, 3.$$

We can choose e_1, e_2 such that $h_{123}(p) = 0$ at p . In fact, if necessary, we make a rotation of e_1, e_2 with angle θ , which satisfies

$$\cos(-2\theta) = \frac{h_{223}(p)}{\sqrt{h_{223}^2(p) + h_{123}^2(p)}}, \quad \sin(-2\theta) = \frac{h_{123}(p)}{\sqrt{h_{223}^2(p) + h_{123}^2(p)}}.$$

Letting

$$a = h_{113}^2, \quad b = h_{111}^2 + h_{112}^2,$$

in view of

$$\begin{aligned} S(S+3) &= \sum_{i,j,k} h_{ijk}^2 = 3(h_{112}^2 + h_{113}^2 + h_{221}^2 + h_{223}^2) + (h_{111}^2 + h_{222}^2) \\ &= 6h_{113}^2 + 4(h_{111}^2 + h_{112}^2), \end{aligned}$$

we have

$$6a + 4b = S(S+3).$$

Since $n = 3$, we have

$$f_4 = \frac{1}{2}S^2, \quad 2A + B = \frac{1}{2}S^2(S+3).$$

Lemma 3.1. *h_{ijkl} are symmetric in i, j, k, l if i, j, k, l are not $\{1, 1, 3, 3\}, \{2, 2, 3, 3\}$ and*

$$\begin{aligned} h_{3311} = h_{3322} &= \frac{2}{3\lambda}(a+b), \quad h_{3333} = \frac{2a}{3\lambda}, \quad h_{3312} = 0, \quad h_{3313} = \frac{2}{3\lambda}h_{111}h_{113}, \\ h_{3323} &= \frac{2}{3\lambda}h_{112}h_{113}, \quad h_{1111} = h_{2222}, \quad h_{1133} = h_{2233} = -\frac{a}{3\lambda}. \end{aligned}$$

Proof. According to the Ricci formula,

$$\begin{aligned} h_{ijkl} - h_{ijlk} &= \sum_m h_{mj} R_{mikl} + \sum_m h_{im} R_{mjkl} \\ &= (\lambda_i - \lambda_j) R_{ijkl} \\ &= (\lambda_i - \lambda_j)(-1 + \lambda_i \lambda_j)(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}). \end{aligned}$$

Also $S = \sum_{i,j} h_{ij}^2$ is constant. We have

$$0 = \sum_{i,j} (h_{ij}^2)_{kl} = 2 \left(\sum_{i,j} h_{ijk} h_{ijl} + \sum_{i,j} h_{ijl} h_{ijk} \right) = 2 \left(\sum_{i,j} h_{ijk} h_{ijl} - 3\lambda h_{33kl} \right). \quad \square$$

Lemma 3.2. *We have*

$$(3-1) \quad x + 2y = \frac{26}{9}a^2 + \frac{7}{18}ab - b^2 + \frac{5}{4}Sb,$$

where

$$x = \lambda^2 [3(h_{1123}^2 + h_{2213}^2) + h_{1113}^2 + h_{2223}^2], \quad y = \lambda^2 (h_{1111}^2 + h_{1112}^2) + (a + b)\lambda h_{1111}.$$

Proof. We have

$$\begin{aligned} \sum_{i,j,k,l} h_{ijkl}^2 &= S(S+3)(S+9) + 3(A-2B) \\ &= S(S+3)(S+9) + 4(2A+B) - 5(A+2B) \\ &= S(S+3)(S+9) + 2S^2(S+3) - 5 \left(\sum_{i,j,k} h_{ijk}^2 \lambda_i^2 + 2 \sum_{i,j,k} h_{ijk}^2 \lambda_i \lambda_j \right) \\ &= 3S(S+3)^2 - \frac{5}{3} \sum_{i,j,k} h_{ijk}^2 (\lambda_i + \lambda_j + \lambda_k)^2, \end{aligned}$$

where

$$\sum_{i,j,k} h_{ijk}^2 (\lambda_i + \lambda_j + \lambda_k)^2 = 3 \sum_{i \neq k} h_{iik}^2 (2\lambda_i + \lambda_k)^2 + 9 \sum_i h_{iii}^2 \lambda_i^2 = 36\lambda^2 b.$$

We have

$$\begin{aligned} \sum_{i,j,k,l} h_{ijkl}^2 &= 3S(S+3)^2 - 60\lambda^2 b = 3S(S+3)^2 - 10Sb, \\ \sum_{i,j,k} h_{ijk1}^2 &= \sum_{i \neq j \neq k} h_{ijk1}^2 + 3 \sum_{i \neq k} h_{iik1}^2 + \sum_i h_{iii1}^2 \\ &= 6h_{1231}^2 + 3(h_{1121}^2 + h_{1131}^2 + h_{2211}^2 + h_{2231}^2 + h_{3311}^2) + (h_{1111}^2 + h_{2221}^2 + h_{3331}^2) \\ &= 3(2h_{1123}^2 + h_{1113}^2 + h_{2213}^2) + (h_{1111}^2 + 3h_{1112}^2) + h_{3331}^2 + (h_{2221}^2 + 3h_{2211}^2 + 3h_{3311}^2) \\ &= 3(2h_{1123}^2 + h_{1113}^2 + h_{2213}^2) + 4(h_{1111}^2 + h_{1112}^2) + h_{3331}^2 + 6(h_{1111} h_{3311} + h_{3311}^2). \end{aligned}$$

In the same way, we have

$$\begin{aligned} \sum_{i,j,k} h_{ijk2}^2 &= \sum_{i \neq j \neq k} h_{ijk2}^2 + 3 \sum_{i \neq k} h_{iik2}^2 + \sum_i h_{iii2}^2 \\ &= 3(2h_{2213}^2 + h_{1123}^2 + h_{2223}^2) + 4(h_{1111}^2 + h_{1112}^2) \\ &\quad + h_{3332}^2 + 3(2h_{1111}h_{3322} + h_{3311}^2 + h_{3322}^2), \\ \sum_{i,j,k} h_{ijk3}^2 &= \sum_{i \neq j \neq k} h_{ijk3}^2 + 3 \sum_{i \neq k} h_{iik3}^2 + \sum_i h_{iii3}^2 \\ &= 3(h_{1123}^2 + h_{2213}^2) + h_{1113}^2 + h_{2223}^2 \\ &\quad + 3(h_{1133}^2 + h_{2233}^2 + h_{3313}^2 + h_{3323}^2) + h_{3333}^2 \\ &= 3(h_{1123}^2 + h_{2213}^2) + h_{1113}^2 + h_{2223}^2 + \left(\frac{2a^2}{3\lambda^2} + \frac{4}{3\lambda^2}ab\right) + \frac{4a^2}{9\lambda^2} \\ &= 3(h_{1123}^2 + h_{2213}^2) + h_{1113}^2 + h_{2223}^2 + \frac{10a^2 + 12ab}{9\lambda^2}. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \sum_{i,j,k,l} h_{ijkl}^2 &= \sum_{i,j,k} h_{ijk1}^2 + \sum_{i,j,k} h_{ijk2}^2 + \sum_{i,j,k} h_{ijk3}^2 \\ &= 12(h_{1123}^2 + h_{2213}^2) + 4(h_{1113}^2 + h_{2223}^2) + 8(h_{1111}^2 + h_{1112}^2) \\ &\quad + 12h_{1111}h_{3311} + (h_{3331}^2 + h_{3332}^2) + 12h_{3311}^2 + \frac{10a^2 + 12ab}{9\lambda^2} \\ &= [12(h_{1123}^2 + h_{2213}^2) + 4(h_{1113}^2 + h_{2223}^2)] \\ &\quad + \left[8(h_{1111}^2 + h_{1112}^2) + \frac{8}{\lambda}h_{1111}(a + b)\right] \\ &\quad + \frac{4}{9\lambda^2}ab + 12\left[\frac{2}{3\lambda}(a + b)\right]^2 + \frac{10a^2 + 12ab}{9\lambda^2}. \end{aligned}$$

We infer, from the above formulas,

$$\frac{4}{\lambda^2}x + \frac{8}{\lambda^2}y + \frac{4ab + 48(a + b)^2 + 10a^2 + 12ab}{9\lambda^2} = 3S(S + 3)^2 - 10Sb,$$

that is,

$$\begin{aligned} x + 2y &= \frac{26}{9}a^2 + \frac{26}{9}ab + \frac{2}{3}b^2 - \frac{5}{12}S^2b \\ &= \frac{26}{9}a^2 + \frac{7}{18}ab - b^2 + \frac{5}{4}Sb. \end{aligned} \quad \square$$

Lemma 3.3. *We have*

$$(3-2) \quad x + 4a\lambda h_{1111} = -\frac{34}{9}a^2 - \frac{4}{3}ab + \frac{4}{3}b^2 + \lambda^2(72\lambda^2 + 18)a + \lambda^2(40\lambda^2 + 8)b.$$

Proof. Since $S = \sum_{i,j} h_{ij}^2$ is constant, we get, for any k, l, m ,

$$0 = \left(\sum_{i,j} h_{ij}^2 \right)_{klm} = 2 \sum_{i,j} (h_{ij}h_{ijklm} + h_{ijm}h_{ijkl} + h_{ijk}h_{ijlm} + h_{ijl}h_{ijkm}).$$

Since

$$\sum_{i,j} h_{ij}h_{ijklm} = -3\lambda h_{33klm},$$

we have

$$3\lambda h_{33klm} = \sum_{i,j} h_{ijm}h_{ijkl} + \sum_{i,j} h_{ijk}h_{ijlm} + \sum_{i,j} h_{ijl}h_{ijkm}.$$

Hence,

$$\sum_{k,l,m} h_{klm}h_{33klm} = \frac{1}{\lambda} \sum_{i,j,k,l,m} h_{ijk}h_{klm}h_{ijlm}.$$

On the other hand, we have

$$\begin{aligned} 0 &= \left(\sum_{i,j,k} h_{ijk}^2 \right)_{33} = 2 \sum_{i,j,k} (h_{ijk}h_{ijk33} + h_{ijk}^2). \\ \sum_{i,j,k} h_{ijk}(h_{33ijk} - h_{ijk33}) &= \sum_{i,j,k} h_{ijk}h_{33ijk} + \sum_{i,j,k} h_{ijk}^2 \\ &= \frac{1}{\lambda} \sum_{i,j,k,l,m} h_{ijk}h_{klm}h_{ijlm} + \frac{x}{\lambda^2} + \frac{10a^2 + 12ab}{9\lambda^2}. \end{aligned}$$

Since

$$\begin{aligned} &\sum_{i,j,k} h_{ijk}(h_{33ijk} - h_{ijk33}) \\ &= \sum_{i,j,k} h_{ijk}[h_{3i3jk} - h_{ijk33}] \\ &= \sum_{i,j,k} h_{ijk} \left[\left(h_{3ij3} + \sum_m h_{mi} R_{m33j} + \sum_m h_{3m} R_{mi3j} \right)_k - \left(h_{ij3k} + 2 \sum_m h_{mj} R_{mik3} \right)_3 \right] \\ &= \sum_{i,j,k} h_{ijk} \left[h_{3ij3k} - h_{ij3k3} + \sum_m h_{mik} R_{m33j} + \sum_m h_{3mk} R_{mi3j} - 2 \sum_m h_{mj3} R_{mik3} \right] \\ &\quad + \sum_{i,j,k,m} h_{ijk} h_{mi} (h_{m3} h_{3j} - h_{mj} h_{33})_k + \sum_{i,j,k,m} h_{ijk} h_{3m} (h_{m3} h_{ij} - h_{mj} h_{i3})_k \\ &\quad - 2 \sum_{i,j,k,m} h_{ijk} h_{mj} (h_{mk} h_{i3} - h_{m3} h_{ik})_3 \\ &= \sum_{i,j,k} h_{ijk} \left[2 \sum_m h_{mij} R_{m33k} + 5 \sum_m h_{3mj} R_{mi3k} \right] \\ &\quad + \sum_{i,j,k,m} h_{ijk} h_{mi} (h_{m3k} h_{3j} + h_{m3} h_{3jk} - h_{mj} h_{33}) \\ &\quad + \sum_{i,j,k,m} h_{ijk} h_{3m} (h_{m3} h_{ijk} - h_{mj} h_{i3} - h_{mj} h_{i3k} + h_{m3k} h_{ij}) \\ &\quad - 2 \sum_{i,j,k,m} h_{ijk} h_{mj} (h_{mk3} h_{i3} - h_{m3} h_{ik3}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i,j,k,m} \left[2h_{ijk}h_{mij}(-1+\lambda_k\lambda_3)(\delta_{k3}\delta_{3m}-\delta_{mk}\delta_{33}) \right. \\
&\quad \left. + 5h_{ijk}h_{3mj}(-1+\lambda_i\lambda_k)(\delta_{m3}\delta_{ik}-\delta_{mk}\delta_{i3}) \right] \\
&\quad + \sum_{i,k} \lambda_3\lambda_i h_{i3k}^2 - \sum_{i,j,k} \lambda_3\lambda_i h_{ijk}^2 \\
&\quad + \sum_{j,k} \lambda_3^2 h_{3jk}^2 + \sum_{i,j,k} h_{ijk}^2 \lambda_3^2 - \sum_{j,k} h_{3jk}^2 \lambda_3^2 \\
&\quad - \sum_{i,k} \lambda_3^2 h_{i3k}^2 - 2 \sum_{j,k} \lambda_3\lambda_j h_{3jk}^2 + 2 \sum_{i,k} \lambda_3^2 h_{i3k}^2 \\
&= \left[2 \sum_{i,j} h_{ij3}^2 (-1+\lambda_3^2) - 2 \sum_{i,j,k} h_{ijk}^2 (-1+\lambda_k\lambda_3) - 5 \sum_{j,k} h_{3jk}^2 (-1+\lambda_3\lambda_k) \right] \\
&\quad - \sum_{i,j} \lambda_3\lambda_i h_{ij3}^2 - \sum_{i,j,k} \lambda_3\lambda_i h_{ijk}^2 + \sum_{i,j,k} \lambda_3^2 h_{ijk}^2 + \sum_{i,j} \lambda_3^2 h_{ij3}^2 \\
&= \left[2(-1+4\lambda^2)(2a) - 2(- (6a+4b) - 2\lambda(4\lambda b)) + 5(1+2\lambda^2)(2a) \right] \\
&\quad + 4a\lambda^2 + 2\lambda(4\lambda b) + 4\lambda^2(6a+4b) + 8\lambda^2 a \\
&= (72\lambda^2 + 18)a + (40\lambda^2 + 8)b.
\end{aligned}$$

$$\begin{aligned}
&\sum_{i,j,k,l,m} h_{ijk}h_{klm}h_{ijlm} \\
&= \sum_{k,l,m} h_{klm}(h_{11k}h_{11lm} + h_{22k}h_{22lm} + 2h_{12k}h_{12lm} + 2h_{13k}h_{13lm} + 2h_{23k}h_{23lm}) \\
&= \sum_{k,l,m} h_{11k}h_{klm}(h_{11lm} - h_{22lm}) + \sum_{l,m} 2(h_{112}h_{1lm} - h_{111}h_{2lm})h_{12lm} \\
&\quad + \sum_{l,m} 2h_{113}h_{1lm}h_{13lm} - \sum_{l,m} 2h_{113}h_{2lm}h_{23lm} \\
&= \sum_k h_{11k} \left[h_{k11}(h_{1111} - h_{2211}) + h_{k22}(h_{1122} - h_{2222}) + 2h_{k13}(h_{1113} - h_{2213}) \right. \\
&\quad \left. + 2h_{k23}(h_{1123} - h_{2223}) + 2h_{k12}(h_{1112} - h_{2212}) \right] \\
&\quad + 2h_{112}(2h_{112}h_{1212} + 2h_{113}h_{1213} + h_{111}h_{1211} + h_{122}h_{1222}) \\
&\quad - 2h_{111}(2h_{212}h_{1212} + h_{211}h_{1211} + h_{222}h_{1222} + 2h_{223}h_{1223}) \\
&\quad + 2h_{113} \left[h_{111}h_{1311} + h_{113}(h_{1313} + h_{1331}) + 2h_{112}h_{1312} + h_{122}h_{1322} \right] \\
&\quad - 2h_{113} \left[h_{222}h_{2322} + 2h_{212}h_{2312} + h_{223}(h_{2323} + h_{2332}) + h_{211}h_{2311} \right] \\
&= (a+b)(h_{1111} - h_{2211}) + \sum_k h_{11k}^2 (h_{1111} - h_{2211}) \\
&\quad + 4bh_{1122} + 2a(h_{1133} + h_{3311} + h_{2233} + h_{3322}) \\
&\quad + 4h_{111}h_{113}h_{1113} - 4h_{112}h_{223}h_{2223} + 4h_{112}h_{113}h_{1123} - 4h_{113}h_{221}h_{2213} \\
&= 2(a+b)(h_{1111} - h_{2211}) + 4bh_{1122} \\
&\quad + 4h_{113}(h_{111}h_{1113} + h_{112}h_{2223} + h_{112}h_{1123} + h_{111}h_{2213}) \\
&\quad + 2a(h_{1133} + h_{3311} + h_{2233} + h_{3322})
\end{aligned}$$

$$\begin{aligned}
 &= 2(a+b)(h_{1111} - h_{2211}) + 4bh_{1122} \\
 &\quad - 4h_{113}(h_{111}h_{3313} + h_{112}h_{3323}) + 2a(h_{1133} + h_{3311} + h_{2233} + h_{3322}) \\
 &= 2(a+b)(2h_{1111} + h_{3311}) - 4b(h_{1111} + h_{3311}) \\
 &\quad - 4h_{113}(h_{111} \cdot \frac{2}{3\lambda}h_{111}h_{113} + h_{112} \cdot \frac{2}{3\lambda}h_{112}h_{113}) \\
 &\quad + 2a\left(-\frac{a}{3\lambda} + \frac{2}{3\lambda}(a+b) - \frac{a}{3\lambda} + \frac{2}{3\lambda}(a+b)\right) \\
 &= 4ah_{1111} + 2(a-b) \cdot \frac{2}{3\lambda}(a+b) - \frac{8}{3\lambda}a(h_{111}^2 + h_{112}^2) - \frac{4a^2}{3\lambda} + \frac{8a}{3\lambda}(a+b) \\
 &= 4ah_{1111} + \frac{8a^2 - 4b^2}{3\lambda}.
 \end{aligned}$$

Hence, we have

$$\begin{aligned}
 (72\lambda^2 + 18)a + (40\lambda^2 + 8)b &= \frac{4ah_{1111}}{\lambda} + \frac{8a^2 - 4b^2}{3\lambda^2} + \frac{x}{\lambda^2} + \frac{10a^2 + 12ab}{9\lambda^2}, \\
 x + 4a\lambda h_{1111} &= \lambda^2(72\lambda^2 + 18)a + \lambda^2(40\lambda^2 + 8)b - \frac{1}{9}(34a^2 + 12ab - 12b^2) \\
 &= -\frac{34}{9}a^2 - \frac{4}{3}ab + \frac{4}{3}b^2 + \lambda^2(72\lambda^2 + 18)a + \lambda^2(40\lambda^2 + 8)b. \quad \square
 \end{aligned}$$

In view of (3-1) and (3-2), we have from $6a + 4b = S(S + 3)$, $6\lambda^2 = S$,

$$\begin{aligned}
 (3-3) \quad &2\lambda^2(h_{1111}^2 + h_{1112}^2) - 2(a-b)\lambda h_{1111} \\
 &= \frac{60}{9}a^2 + \frac{31}{18}ab - \frac{7}{3}b^2 + \frac{5}{4}Sb - \lambda^2(72\lambda^2 + 18)a - \lambda^2(40\lambda^2 + 8)b \\
 &= \frac{20}{3}a^2 + \frac{31}{18}ab - \frac{7}{3}b^2 + \frac{5}{4}Sb - \frac{1}{6}S(12S + 18)a - \frac{1}{6}S\left(\frac{20}{3}S + 8\right)b \\
 &= \frac{20}{3}a^2 + \frac{31}{18}ab - \frac{7}{3}b^2 - 2S(S + 3)a + 3Sa - \frac{10}{9}S\left(S + \frac{3}{40}\right)b \\
 &= \frac{20}{3}a^2 + \frac{31}{18}ab - \frac{7}{3}b^2 - 2(6a + 4b)a + 3Sa - \frac{10}{9}S\left(S + 3 - \frac{117}{40}\right)b \\
 &= -\frac{16}{3}a^2 - \frac{233}{18}ab - \frac{61}{9}b^2 + S\left(3a + \frac{13}{4}b\right).
 \end{aligned}$$

According to

$$2\lambda^2(h_{1111}^2 + h_{1112}^2) - 2(a-b)\lambda h_{1111} \geq -\frac{1}{2}(a-b)^2,$$

we obtain

$$(3-4) \quad -\frac{29}{6}a^2 - \frac{251}{18}ab - \frac{113}{18}b^2 + S\left(3a + \frac{13}{4}b\right) \geq 0.$$

Since

$$\begin{aligned}
 &-\frac{29}{6}a^2 - \frac{58}{18}ab - \frac{13}{2}ab - \frac{13}{3}b^2 \\
 &= -\frac{29}{36}a(6a + 4b) - \frac{13}{12}(4b + 6a)b = -\frac{29}{36}S(S + 3)a - \frac{13}{12}S(S + 3)b,
 \end{aligned}$$

we have from (3-4)

$$\left(\frac{21}{36} - \frac{29}{36}S\right)Sa - \frac{76}{18}ab - \frac{35}{18}b^2 - \frac{13}{12}S^2b \geq 0.$$

Hence we have $S \leq \frac{21}{29}$. □

4. Proof of Theorem 1.1

In this section, we will give a proof of the Theorem 1.1.

Proof of Theorem 1.1. We choose a local frame field $\{e_1, e_2, e_3, e_4\}$ such that at any point p ,

$$h_{ij} = \lambda_i \delta_{ij}.$$

Since S is constant, we notice that the sectional curvature is bounded from below from Gauss equations. By making using of the generalized maximum principle due to Omori [5], there exists a sequence $\{p_k\}_{k=1}^\infty \subset M^3$ such that

$$\lim_{k \rightarrow \infty} f_3(p_k) = \sup_{M^3} f_3, \quad \lim_{k \rightarrow \infty} |\nabla f_3(p_k)| = 0, \quad \lim_{k \rightarrow \infty} \sup \nabla_l \nabla_l f_3(p_k) \leq 0 \text{ for } l = 1, 2, 3.$$

Since S is constant,

$$\sum_{i,j,k} h_{ijk}^2 = S(S+3), \quad \sum_{i,j,k,l} h_{ijkl}^2 = S(S+3)(S+9) + 3(A-2B),$$

we know that, for any i, j, k, l , $\{\lambda_i(p_k)\}$, $\{h_{ijk}(p_k)\}$ and $\{h_{ijkl}(p_k)\}$ are bounded sequences, respectively. Thus, we can assume, if necessary, by taking a subsequences of $\{p_m\}$,

$$\lim_{m \rightarrow \infty} \lambda_i(p_m) = \hat{\lambda}_i, \quad \lim_{m \rightarrow \infty} h_{ijk}(p_m) = \hat{h}_{ijk}, \quad \lim_{m \rightarrow \infty} h_{ijkl}(p_m) = \hat{h}_{ijkl} \quad \text{for all } i, j, k, l.$$

From now on, all the computations are considered for $\hat{\lambda}_i, \hat{h}_{ijk}$ and \hat{h}_{ijkl} . For simplicity, we omit $\hat{\cdot}$.

If the principal curvatures are the same, $S \equiv 0$ since M^3 is minimal. We only consider the following two cases.

Case 1. The number of distinct principal curvatures is two. By the same proof as in the Section 3, we get

$$S \leq \frac{21}{29}.$$

Case 2. All three principal curvatures are distinct. If f_3 is constant, M^3 is isoparametric and $S \equiv 0$. This is impossible. From now on, we suppose that f_3 is not constant. We will derive a contradiction. Without loss of the generality, we assume that $\lambda_1 < \lambda_2 < \lambda_3$. We also assume $\sup f_3 \neq 0$; otherwise we use $\inf f_3 \neq 0$.

Lemma 4.1. *We have*

$$h_{iik} = 0 \quad \text{for any } i, k \quad \text{and} \quad h_{123}^2 = \frac{1}{6}S(S+3).$$

Proof. Since $\sum_i h_{ii} = 0$ and $S = \sum_{i,j} h_{ij}^2$ is constant, we have

$$\sum_i h_{iik} = 0, \quad \sum_i h_{iik} \lambda_i = 0.$$

Since $\lim_{k \rightarrow \infty} |\nabla f_3(p_k)| = 0$, we have

$$\sum_i h_{iik} \lambda_i^2 = 0.$$

Since $\lambda_i \neq \lambda_j$ for $i \neq j$, we have $h_{iik} = 0$ for any i, k . From

$$S(S+3) = \sum_{i,j,k} h_{ijk}^2 = 6h_{123}^2,$$

we obtain

$$h_{123}^2 = \frac{1}{6}S(S+3). \quad \square$$

Lemma 4.2. *We have*

$$h_{iijk} = h_{iiik} = h_{kiii} = 0 \quad \text{for } i \neq j \neq k.$$

Proof. Since $\sum_{i,j,k} h_{ijk}^2 = S(S+3)$, we have $h_{123l} = 0$ for any l , i.e.,

$$(4-1) \quad h_{iijk} = 0 \quad \text{for } i \neq j \neq k.$$

Since $\sum_i h_{ii} = 0$ and $S = \sum_{i,j} h_{ij}^2$ is constant, we have

$$\sum_i h_{iijk} = 0, \quad \sum_i h_{iijk} \lambda_i = 0.$$

For $j \neq k$, using (4-1), we have

$$(4-2) \quad h_{jjjk} + h_{kkjk} = 0, \quad \sum_i h_{iijk} \lambda_i = 0 \quad \text{for } j \neq k.$$

From (4-2), we have $h_{jjjk} = h_{kkjk} = 0$ for $j \neq k$. □

Lemma 4.3. *We have*

$$\sum_{i,k} h_{iikk}^2 + 2 \sum_{i \neq k} h_{iikk}^2 = 3S(S+3)^2.$$

Proof. From

$$\begin{aligned} \sum_{i,j,k,l} h_{ijkl}^2 &= S(S+3)(S+9) + 3(A-2B), \\ 3(A-2B) &= 6h_{123}^2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\lambda_1\lambda_2 - 2\lambda_2\lambda_3 - 2\lambda_3\lambda_1) = 2S^2(S+3), \\ \sum_{i,j,k,l} h_{ijkl}^2 &= \sum_{i \neq j \neq k} h_{ijkl}^2 + 3 \sum_{i \neq k} h_{iikl}^2 + \sum_{i,l} h_{iili}^2 \\ &= 3 \sum_{i \neq k} h_{iikk}^2 + \sum_i h_{iiii}^2 = \sum_{i,k} h_{iikk}^2 + 2 \sum_{i \neq k} h_{iikk}^2, \end{aligned}$$

we have

$$\sum_{i,k} h_{iikk}^2 + 2 \sum_{i \neq k} h_{iikk}^2 = S(S+3)(S+9) + 2S^2(S+3) = 3S(S+3)^2. \quad \square$$

Lemma 4.4. *We have*

$$\sup f_3 > 0, \quad -\sqrt{\frac{1}{2}S} < \lambda_1 < -\sqrt{\frac{1}{6}S}, \quad -\sqrt{\frac{1}{6}S} < \lambda_2 < 0.$$

Proof. Since $\lim_{k \rightarrow \infty} \sup \Delta f_3(p_k) \leq 0$ and $\frac{1}{3} \Delta f_3 = -(S+3)f_3$, we have

$$\begin{aligned} 0 &\geq -(S+3) \lim_{k \rightarrow \infty} \sup f_3(p_k) \\ &= -(S+3) \sup_{M^3} f_3. \end{aligned}$$

We get $\sup f_3 > 0$. We also notice that $\lambda_1 < 0$ and $\lambda_3 > 0$. By a direct computation,

$$\sup f_3 = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = 3\lambda_1\lambda_2\lambda_3 = 3\lambda_i(\lambda_i^2 - \frac{1}{2}S) \quad \text{for all } i.$$

We obtain $\lambda_1^2 < \frac{1}{2}S$ and $\lambda_3^2 > \frac{1}{2}S$, $\lambda_2 < 0$ and

$$(4-3) \quad \lambda_1^2 + \lambda_2^2 < \frac{1}{2}S.$$

Because of $\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2 = \frac{1}{2}S$ and $\lambda_1 < \lambda_2 < 0$, we have

$$\lambda_1^2 > \frac{1}{6}S, \quad \lambda_2^2 < \frac{1}{6}S.$$

and

$$\frac{1}{6}S < \lambda_1^2 < \frac{1}{2}S, \quad 0 < \lambda_2^2 < \frac{1}{6}S, \quad \lambda_1 < \lambda_2 < 0. \quad \square$$

For simplicity, we use f_3 in place of $\sup f_3$ in the following.

Lemma 4.5. *We have*

$$h_{iikk} = -\frac{1}{3}(S+3)\lambda_i + g_i\lambda_k + wg_i g_k,$$

where

$$g_i = \lambda_i^2 - \frac{f_3}{S}\lambda_i - \frac{1}{3}S.$$

Proof. Taking derivatives of $\sum_i h_{ii} = 0$ and $\sum_{i,j} h_{ij}^2 = S$, we have

$$\sum_i h_{iikk} = 0, \quad \sum_i h_{iikk}\lambda_i = -\frac{1}{3}S(S+3).$$

We solve this rank-5 linear system of six equations with six unknowns h_{iikk} , $i \leq k$, with $h_{iijj} = h_{jjii} + (\lambda_i - \lambda_j)(-1 + \lambda_i\lambda_j)$. \square

Lemma 4.6. *We have*

$$\begin{aligned} f_5 &= \frac{5}{6}Sf_3, \quad f_6 = \frac{1}{3}f_3^2 + \frac{1}{4}S^3, \\ \sum_i g_i^2 &= \sum_i g_i\lambda_i^2 = \frac{1}{6}S^2 - \frac{f_3^2}{S}, \quad \sum_i g_i^4 = \frac{1}{2}\left(\frac{1}{6}S^2 - \frac{f_3^2}{S}\right)^2, \\ \sum_i g_i^2\lambda_i &= \frac{f_3^3}{S^2} - \frac{1}{6}Sf_3, \quad \sum_i g_i^2\lambda_i^2 = \frac{1}{36}S^3 - \frac{1}{6}f_3^2, \quad \sum_i g_i^3\lambda_i = 0. \end{aligned}$$

Proof. From $f_3 = 3\lambda_i(\lambda_i^2 - \frac{1}{2}S)$, for $i = 1, 2, 3$, we have

$$f_5 = \frac{5}{6}Sf_3, \quad f_6 = \frac{1}{3}f_3^2 + \frac{1}{4}S^3.$$

According to $g_i = \lambda_i^2 - \frac{f_3}{S}\lambda_i - \frac{1}{3}S$, we infer

$$\begin{aligned} \sum_i g_i^2 &= \frac{1}{6}S^2 - \frac{f_3^2}{S}, & \sum_i g_i^4 &= \frac{1}{2}\left(\frac{1}{6}S^2 - \frac{f_3^2}{S}\right)^2, \\ \sum_i g_i^2 \lambda_i &= \frac{f_3^3}{S^2} - \frac{1}{6}Sf_3, & \sum_i g_i^2 \lambda_i^2 &= \frac{1}{36}S^3 - \frac{1}{6}f_3^2, & \sum_i g_i \lambda_i^2 &= \frac{1}{6}S^2 - \frac{f_3^2}{S}, \end{aligned}$$

Because of $F_3 = 3g_i(g_i^2 - \frac{1}{2}F_2)$, for $i = 1, 2, 3$, we have

$$\sum_i g_i^3 \lambda_i = 0,$$

where $F_k = \sum_i g_i^k$. □

Lemma 4.7. *We have*

$$y = \left(\frac{1}{3} + \frac{1}{S}\right)f_3 \pm \left[\frac{f_3^2}{S^2}\left(\frac{19}{9}S^2 + \frac{8}{3}S + 1\right) + \frac{7}{9}S(S+6)\left(S + \frac{15}{7}\right)\right]^{\frac{1}{2}},$$

where

$$y = \left(\frac{1}{6}S^2 - \frac{f_3^2}{S}\right)w.$$

Proof. By using the Lemmas 4.5 and 4.6, we have

$$\begin{aligned} (4-4) \quad \sum_{i,k} h_{iikk}^2 &= \sum_{i,k} \left(-\frac{1}{3}(S+3)\lambda_i + g_i \lambda_k + w g_i g_k\right)^2 \\ &= \frac{1}{3}S(S+3)^2 + S\left(\frac{1}{6}S^2 - \frac{f_3^2}{S}\right) + w^2\left(\frac{1}{6}S^2 - \frac{f_3^2}{S}\right)^2, \end{aligned}$$

$$\begin{aligned} (4-5) \quad \sum_i h_{iiii}^2 &= \sum_i \left(-\frac{1}{3}(S+3)\lambda_i + g_i \lambda_i + w g_i^2\right)^2 \\ &= \frac{1}{9}(S+3)^2 S + \sum_i g_i^2 \lambda_i^2 + w^2 \sum_i g_i^4 + 2w \sum_i \lambda_i g_i^3 \\ &\quad - \frac{2}{3}(S+3) \sum_i g_i \lambda_i^2 - \frac{2}{3}(S+3)w \sum_i g_i^2 \lambda_i \\ &= \frac{1}{9}S(S+3)^2 + \frac{1}{6}S\left(\frac{1}{6}S^2 - \frac{f_3^2}{S}\right) + w^2 \frac{1}{2}\left(\frac{1}{6}S^2 - \frac{f_3^2}{S}\right)^2 \\ &\quad - \frac{2}{3}(S+3)\left(\frac{1}{6}S^2 - \frac{f_3^2}{S}\right) - \frac{2}{3}(S+3)w\left(\frac{f_3^3}{S^2} - \frac{1}{6}Sf_3\right), \end{aligned}$$

$$\begin{aligned} (4-6) \quad \sum_{i \neq k} h_{iikk}^2 &= \sum_{i,k} h_{iikk}^2 - \sum_i h_{iiii}^2 \\ &= \frac{2}{9}S(S+3)^2 + \left[\frac{5}{36}S^3 - \frac{5}{6}f_3^2\right] + \frac{2}{3}(S+3)\left(\frac{1}{6}S^2 - \frac{f_3^2}{S}\right) \\ &\quad + \frac{2}{3}w(S+3)\left(-\frac{1}{6}Sf_3 + \frac{f_3^3}{S^2}\right) + \frac{1}{2}w^2\left[\frac{1}{36}S^4 - \frac{1}{3}Sf_3^2 + \frac{f_3^4}{S^2}\right]. \end{aligned}$$

Substituting (4-4) and (4-6) into the Lemma 4.3 completes the proof. □

Lemma 4.8. *We have*

$$-\frac{3}{S}y(\lambda_l^2 - \frac{1}{6}S)(\lambda_l^2 - \frac{2}{3}S) \leq \lambda_l(\lambda_l^2 - \frac{1}{6}S)\left(\frac{9}{S}\lambda_l^4 - \frac{15}{2}\lambda_l^2 + 2S + 3\right).$$

Proof. Since

$$\frac{1}{3}(f_3)_{ll} = \sum_i h_{iill}\lambda_i^2 + 2\sum_{i,j} h_{ijl}^2\lambda_i,$$

we have

$$(4-7) \quad 0 \geq \frac{1}{3} \limsup_{k \rightarrow \infty} (f_3)_{ll} = \frac{1}{3} \lim_{k \rightarrow \infty} (f_3)_{ll} = \sum_i h_{iill}\lambda_i^2 + 2\sum_{i,j} h_{ijl}^2\lambda_i.$$

By a direct computation, we infer

$$(4-8) \quad \sum_i h_{iill}\lambda_i^2 = -\lambda_l(S+3)(\lambda_l^2 - \frac{1}{2}S) + \lambda_l\left(\frac{1}{6}S^2 - \frac{f_3^2}{S}\right) + \left(\lambda_l^2 - \frac{f_3}{S}\lambda_l - \frac{1}{3}S\right)y,$$

$$(4-9) \quad 2\sum_{i,j} h_{ijl}^2\lambda_i = -\frac{1}{3}S(S+3)\lambda_l.$$

By substituting (4-8) and (4-9) into (4-7), we have

$$\left(\lambda_l^2 - \frac{f_3}{S}\lambda_l - \frac{1}{3}S\right)y \leq \lambda_l\left[(S+3)(\lambda_l^2 - \frac{1}{2}S) - \left(\frac{1}{6}S^2 - \frac{f_3^2}{S}\right) + \frac{1}{3}S(S+3)\right]. \quad \square$$

If $y \geq 0$, by using the Lemma 4.6, we have

$$(4-10) \quad y = \left(\frac{1}{3} + \frac{1}{S}\right)f_3 + \left[\frac{f_3^2}{S^2}\left(\frac{19}{9}S^2 + \frac{8}{3}S + 1\right) + \frac{7}{9}S(S+6)\left(S + \frac{15}{7}\right)\right]^{\frac{1}{2}},$$

$$> \left(\frac{1}{3} + \frac{1}{S}\right)f_3 + \left[\frac{f_3^2}{S^2}\left(\frac{4}{3}S + 1\right)^2\right]^{\frac{1}{2}}$$

$$= \left(\frac{5}{3} + \frac{2}{S}\right)f_3.$$

By substituting (4-10) into the Lemma 4.8 with $l = 1$, we have

$$(4-11) \quad \left(\frac{24}{S} + \frac{18}{S^2}\right)\lambda_1^4 - \left(25 + \frac{21}{S}\right)\lambda_1^2 + 7S + 9 < 0.$$

We notice that the left-hand side of (4-11) is an increasing function of λ_1^2 for $\lambda_1^2 > \frac{1}{6}S$. Substituting $\lambda_1^2 = \frac{1}{6}S$ into (4-11), we have

$$S < -\frac{12}{7}.$$

It is a contradiction.

If $y < 0$, by taking $l = 2$ in the Lemma 4.8, we have

$$(4-12) \quad \frac{3}{S}y(\lambda_2^2 - \frac{2}{3}S) + \lambda_2\left(\frac{9}{S}\lambda_2^4 - \frac{15}{2}\lambda_2^2 + 2S + 3\right) \leq 0.$$

Because of

$$y = \left(\frac{1}{3} + \frac{1}{S}\right)f_3 - \left[\frac{f_3^2}{S^2}\left(\frac{19}{9}S^2 + \frac{8}{3}S + 1\right) + \frac{7}{9}S(S+6)\left(S + \frac{15}{7}\right)\right]^{\frac{1}{2}},$$

and since the left-hand side of (4-12) is an increasing function of λ_2 for $0 > \lambda_2 > -\sqrt{\frac{1}{6}S}$, substituting $\lambda_2 = -\sqrt{\frac{1}{6}S}$ into (4-12), we have

$$\text{LHS of (4-12)} > 0.$$

It is a contradiction. □

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Received August 4, 2024. Revised April 23, 2025.

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
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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

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PACIFIC JOURNAL OF MATHEMATICS

Volume 338 No. 2 October 2025

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