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COMPLETE MINIMAL HYPERSURFACES IN A HYPERBOLIC SPACE $H^4(-1)$

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We study n-dimensional complete minimal hypersurfaces in the hyperbolic space $H^{n+1}(-1)$ of constant curvature -1. We prove that a 3-dimensional complete minimal hypersurface with constant scalar curvature in $H^4(-1)$ satisfies $S \leq \frac{21}{29}$ by making use of the generalized maximum principle, where S denotes the squared norm of the second fundamental form of the hypersurface.

1. Introduction

Let M^n be an n-dimensional minimal hypersurface in the hyperbolic space $H^{n+1}(-1)$ of constant curvature -1. A very important subject of study is the rigidity of complete minimal hypersurfaces in the hyperbolic space $H^{n+1}(-1)$. It is well known that there are many important results on the rigidity of compact minimal hypersurfaces in the unit sphere $S^{n+1}(1)$. For example, Simons [7], Chern, do Carmo and Kobayashi [3] and Lawson [4] prove that an n-dimensional compact minimal hypersurface in the unit sphere $S^{n+1}(1)$ is isometric to a totally geodesic sphere or a Clifford torus if the squared norm S of its second fundamental form satisfies $S \leq n$. In particular, for $n = 3$, it is known that a 3-dimensional compact minimal hypersurface in the unit sphere $S^4(1)$ with constant scalar curvature is isometric to a totally geodesic sphere or a Clifford torus or the Cartan minimal isoparametric hypersurface (see [1; 6]). On the other hand, Cheng and Wan [2] proved complete minimal hypersurfaces with constant scalar curvature in the Euclidean space $\mathbb{R}^4$ are isometric to the hyperplane $\mathbb{R}^3$. But for complete minimal hypersurfaces in the hyperbolic space $H^{n+1}(-1)$, there are only few results on rigidity of complete minimal hypersurfaces. It is our main purpose to study the following conjecture:

Conjecture. A complete minimal hypersurface with constant scalar curvature in the hyperbolic space $H^4(-1)$ is isometric to the hyperbolic space $H^3(-1)$.

We will prove the following:

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Theorem 1.1. *A complete minimal hypersurface with constant scalar curvature in the hyperbolic space $H^4(-1)$ satisfies $S \leq \frac{21}{29}$, where S denotes the squared norm of the second fundamental form of the hypersurface.*

2. Basic formulas

Let M^n be an n-dimensional hypersurface in an $(n+1)$-dimensional hyperbolic space $H^{n+1}(-1)$. At each point p in $H^{n+1}(-1)$, we choose a local orthonormal frame field $\{e_1, e_2, \dots, e_{n+1}\}$ and the dual coframe $\{\omega^1, \omega^2, \dots, \omega^{n+1}\}$ such that, restricted to M^n, $\{e_1, e_2, \dots, e_n\}$ is tangent to M^n. Structure equations of $H^{n+1}(-1)$ are given by

$$\begin{aligned} d\omega_A &= -\sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \\ d\omega_{AB} + \sum_C \omega_{AC} \wedge \omega_{CB} &= \tfrac{1}{2} \sum_{C,D} K_{ABCD}\, \omega_C \wedge \omega_D, \end{aligned} \tag{2-1}$$

with

$$K_{ABCD} = -(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{2-2}$$

If we restrict these forms to M^n, then $\omega^{n+1} = 0$. We have

$$\omega_{i,n+1} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2-3}$$

One calls

$$H = \frac{1}{n}\sum_i h_{ii}, \quad h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j \tag{2-4}$$

the mean curvature and the second fundamental form of M^n, respectively. If H is identically zero, M^n is called minimal. The structure equations of M^n are given by

$$\begin{aligned} d\omega_i &= -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} &= \tfrac{1}{2} \sum_{k,l} R_{ijkl}\, \omega_k \wedge \omega_l, \end{aligned} \tag{2-5}$$

where

$$R_{ijkl} = -(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}). \tag{2-6}$$

For minimal hypersurfaces in $H^{n+1}(-1)$, we obtain

$$R = -n(n-1) - S,$$

where R and S denote the scalar curvature and the squared norm of the second fundamental form of M^n, respectively. From the structure equations of M^n, Codazzi equations and Ricci formulas are given by

$$h_{ijk} = h_{ikj}, \quad h_{ijkl} - h_{ijlk} = \sum_m h_{im} R_{mjkl} + \sum_m h_{mj} R_{mikl},$$

where $h_{ijk} = \nabla_k h_{ij}$ and $h_{ijkl} = \nabla_l \nabla_k h_{ij}$, respectively. Define f_3 and f_4 by

$$f_3 = \sum_{i,j,k=1}^{n} h_{ij} h_{jk} h_{ki} \quad \text{and} \quad f_4 = \sum_{i,j,k,l=1}^{n} h_{ij} h_{jk} h_{kl} h_{li},$$

respectively. We have, for minimal hypersurfaces,

$$\begin{aligned} \tfrac{1}{3}\Delta f_3 &= -(n+S) f_3 + 2C, \\ \tfrac{1}{4}\Delta f_4 &= -(n+S) f_4 + (2A+B), \end{aligned} \tag{2-7}$$

where

$$C = \sum_{i,j,k} \lambda_i h_{ijk}^2, \quad A = \sum_{i,j,k} \lambda_i^2 h_{ijk}^2, \quad B = \sum_{i,j,k} \lambda_i \lambda_j h_{ijk}^2$$

and λ_i's are principal curvatures of M^n, that is,

$$\sum_i h_{ii} = \sum_i \lambda_i = 0, \quad S = \sum_{i,j} h_{ij}^2 = \sum_i \lambda_i^2,$$

$$h_{ijij} - h_{jiji} = (\lambda_i - \lambda_j)(-1 + \lambda_i \lambda_j).$$

By a direct computation, we have

$$S = n(1-n) - R, \quad \Delta h_{ij} = -(S+n) h_{ij}, \quad \tfrac{1}{2}\Delta S = -S(S+n) + \sum_{i,j,k} h_{ijk}^2.$$

If the squared norm S of the second fundamental form is constant, we have

$$\sum_{i,j,k} h_{ijk}^2 = S(S+n), \quad \sum_{i,j,k,l} h_{ijkl}^2 = S(S+n)(2n+3+S) + 3(A-2B).$$

The following generalized maximum principle due to Omori [5] (see Yau [8]) will play an important role in this paper.

Theorem 2.1. *Let M^n be a complete Riemannian manifold with sectional curvature bounded from below. If a C^2-function f is bounded from above in M^n, then there exists a sequence $\{p_k\}_{k=1}^{\infty} \subset M^n$ such that*

(1) $\lim_{k\to\infty} f(p_k) = \sup_{M^n} f$,

(2) $\lim_{k\to\infty} |\nabla f(p_k)| = 0$,

(3) $\lim_{k\to\infty} \sup \nabla_l \nabla_l f(p_k) \le 0$, *for* $l = 1, 2, \ldots, n$.

3. Minimal hypersurfaces with two distinct principal curvatures

Theorem 3.1. *Let M^3 be a minimal hypersurface in $H^4(-1)$ with constant scalar curvature. If M^3 has two principal curvatures somewhere, we have $S \leq \frac{21}{29}$.*

Proof. We assume, at $p \in M^3$, that M^3 has two distinct principal curvatures. At p, we may choose an orthonormal frame e_1, e_2, e_3 such that $h_{ij} = \lambda_i \delta_{ij}$. We can assume

$$\lambda_1 = \lambda_2 = \lambda.$$

Since M^3 is minimal, we have

$$\lambda_3 = -2\lambda, \quad \lambda^2 = \tfrac{1}{6}S.$$

Because $\sum_i h_{ii} = 0$ and S is constant, we have

$$h_{11k} + h_{22k} + h_{33k} = 0, \quad h_{11k} + h_{22k} - 2h_{33k} = 0.$$

We obtain

$$h_{11k} + h_{22k} = 0, \quad h_{33k} = 0, \qquad k = 1, 2, 3.$$

We can choose e_1, e_2 such that $h_{123}(p) = 0$ at p. In fact, if necessary, we make a rotation of e_1, e_2 with angle θ, which satisfies

$$\cos(-2\theta) = \frac{h_{223}(p)}{\sqrt{h_{223}^2(p) + h_{123}^2(p)}}, \quad \sin(-2\theta) = \frac{h_{123}(p)}{\sqrt{h_{223}^2(p) + h_{123}^2(p)}}.$$

Letting

$$a = h_{113}^2, \quad b = h_{111}^2 + h_{112}^2,$$

in view of

$$\begin{aligned} S(S+3) &= \sum_{i,j,k} h_{ijk}^2 = 3(h_{112}^2 + h_{113}^2 + h_{221}^2 + h_{223}^2) + (h_{111}^2 + h_{222}^2) \\ &= 6h_{113}^2 + 4(h_{111}^2 + h_{112}^2), \end{aligned}$$

we have

$$6a + 4b = S(S+3).$$

Since $n = 3$, we have

$$f_4 = \tfrac{1}{2}S^2, \quad 2A + B = \tfrac{1}{2}S^2(S+3).$$

Lemma 3.1. *h_{ijkl} are symmetric in i, j, k, l if i, j, k, l are not $\{1, 1, 3, 3\}$, $\{2, 2, 3, 3\}$ and*

$$h_{3311} = h_{3322} = \frac{2}{3\lambda}(a+b), \quad h_{3333} = \frac{2a}{3\lambda}, \quad h_{3312} = 0, \quad h_{3313} = \frac{2}{3\lambda}h_{111}h_{113},$$
$$h_{3323} = \frac{2}{3\lambda}h_{112}h_{113}, \quad h_{1111} = h_{2222}, \quad h_{1133} = h_{2233} = -\frac{a}{3\lambda}.$$

Proof. According to the Ricci formula,

$$\begin{aligned} h_{ijkl} - h_{ijlk} &= \sum_m h_{mj} R_{mikl} + \sum_m h_{im} R_{mjkl} \\ &= (\lambda_i - \lambda_j) R_{ijkl} \\ &= (\lambda_i - \lambda_j)(-1 + \lambda_i \lambda_j)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \end{aligned}$$

Also $S = \sum_{i,j} h_{ij}^2$ is constant. We have

$$0 = \sum_{i,j} (h_{ij}^2)_{kl} = 2\Big(\sum_{i,j} h_{ijk} h_{ijl} + \sum_{i,j} h_{ij} h_{ijkl} \Big) = 2\Big(\sum_{i,j} h_{ijk} h_{ijl} - 3\lambda h_{33kl} \Big). \qquad \square$$

Lemma 3.2. *We have*

$$x + 2y = \tfrac{26}{9} a^2 + \tfrac{7}{18} ab - b^2 + \tfrac{5}{4} Sb, \tag{3-1}$$

where

$$x = \lambda^2 [3(h_{1123}^2 + h_{2213}^2) + h_{1113}^2 + h_{2223}^2], \quad y = \lambda^2 (h_{1111}^2 + h_{1112}^2) + (a+b)\lambda h_{1111}.$$

Proof. We have

$$\begin{aligned} \sum_{i,j,k,l} h_{ijkl}^2 &= S(S+3)(S+9) + 3(A - 2B) \\ &= S(S+3)(S+9) + 4(2A + B) - 5(A + 2B) \\ &= S(S+3)(S+9) + 2S^2(S+3) - 5\Big(\sum_{i,j,k} h_{ijk}^2 \lambda_i^2 + 2 \sum_{i,j,k} h_{ijk}^2 \lambda_i \lambda_j \Big) \\ &= 3S(S+3)^2 - \tfrac{5}{3} \sum_{i,j,k} h_{ijk}^2 (\lambda_i + \lambda_j + \lambda_k)^2, \end{aligned}$$

where

$$\sum_{i,j,k} h_{ijk}^2 (\lambda_i + \lambda_j + \lambda_k)^2 = 3 \sum_{i \neq k} h_{iik}^2 (2\lambda_i + \lambda_k)^2 + 9 \sum_i h_{iii}^2 \lambda_i^2 = 36 \lambda^2 b.$$

We have

$$\sum_{i,j,k,l} h_{ijkl}^2 = 3S(S+3)^2 - 60\lambda^2 b = 3S(S+3)^2 - 10Sb,$$

$$\begin{aligned} \sum_{i,j,k} h_{ijk1}^2 &= \sum_{i \neq j \neq k} h_{ijk1}^2 + 3 \sum_{i \neq k} h_{iik1}^2 + \sum_i h_{iii1}^2 \\ &= 6h_{1231}^2 + 3(h_{1121}^2 + h_{1131}^2 + h_{2211}^2 + h_{2231}^2 + h_{3311}^2) + (h_{1111}^2 + h_{2221}^2 + h_{3331}^2) \\ &= 3(2h_{1123}^2 + h_{1113}^2 + h_{2213}^2) + (h_{1111}^2 + 3h_{1112}^2) + h_{3331}^2 + (h_{2221}^2 + 3h_{2211}^2 + 3h_{3311}^2) \\ &= 3(2h_{1123}^2 + h_{1113}^2 + h_{2213}^2) + 4(h_{1111}^2 + h_{1112}^2) + h_{3331}^2 + 6(h_{1111} h_{3311} + h_{3311}^2). \end{aligned}$$

In the same way, we have

$$\begin{aligned}\sum_{i,j,k} h_{ijk2}^2 &= \sum_{i\neq j\neq k} h_{ijk2}^2 + 3\sum_{i\neq k} h_{iik2}^2 + \sum_i h_{iii2}^2 \\ &= 3(2h_{2213}^2 + h_{1123}^2 + h_{2223}^2) + 4(h_{1111}^2 + h_{1112}^2) \\ &\qquad + h_{3332}^2 + 3(2h_{1111}h_{3322} + h_{3311}^2 + h_{3322}^2),\end{aligned}$$

$$\begin{aligned}\sum_{i,j,k} h_{ijk3}^2 &= \sum_{i\neq j\neq k} h_{ijk3}^2 + 3\sum_{i\neq k} h_{iik3}^2 + \sum_i h_{iii3}^2 \\ &= 3(h_{1123}^2 + h_{2213}^2) + h_{1113}^2 + h_{2223}^2 \\ &\qquad + 3(h_{1133}^2 + h_{2233}^2 + h_{3313}^2 + h_{3323}^2) + h_{3333}^2 \\ &= 3(h_{1123}^2 + h_{2213}^2) + h_{1113}^2 + h_{2223}^2 + \left(\frac{2a^2}{3\lambda^2} + \frac{4}{3\lambda^2}ab\right) + \frac{4a^2}{9\lambda^2} \\ &= 3(h_{1123}^2 + h_{2213}^2) + h_{1113}^2 + h_{2223}^2 + \frac{10a^2 + 12ab}{9\lambda^2}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}\sum_{i,j,k,l} h_{ijkl}^2 &= \sum_{i,j,k} h_{ijk1}^2 + \sum_{i,j,k} h_{ijk2}^2 + \sum_{i,j,k} h_{ijk3}^2 \\ &= 12(h_{1123}^2 + h_{2213}^2) + 4(h_{1113}^2 + h_{2223}^2) + 8(h_{1111}^2 + h_{1112}^2) \\ &\qquad + 12h_{1111}h_{3311} + (h_{3331}^2 + h_{3332}^2) + 12h_{3311}^2 + \frac{10a^2 + 12ab}{9\lambda^2} \\ &= [12(h_{1123}^2 + h_{2213}^2) + 4(h_{1113}^2 + h_{2223}^2)] \\ &\qquad + \left[8(h_{1111}^2 + h_{1112}^2) + \frac{8}{\lambda}h_{1111}(a+b)\right] \\ &\qquad + \frac{4}{9\lambda^2}ab + 12\left[\frac{2}{3\lambda}(a+b)\right]^2 + \frac{10a^2 + 12ab}{9\lambda^2}.\end{aligned}$$

We infer, from the above formulas,

$$\frac{4}{\lambda^2}x + \frac{8}{\lambda^2}y + \frac{4ab + 48(a+b)^2 + 10a^2 + 12ab}{9\lambda^2} = 3S(S+3)^2 - 10Sb,$$

that is,

$$\begin{aligned}x + 2y &= \tfrac{26}{9}a^2 + \tfrac{26}{9}ab + \tfrac{2}{3}b^2 - \tfrac{5}{12}S^2 b \\ &= \tfrac{26}{9}a^2 + \tfrac{7}{18}ab - b^2 + \tfrac{5}{4}Sb.\end{aligned}$$

□

Lemma 3.3. *We have*

$$x + 4a\lambda h_{1111} = -\tfrac{34}{9}a^2 - \tfrac{4}{3}ab + \tfrac{4}{3}b^2 + \lambda^2(72\lambda^2 + 18)a + \lambda^2(40\lambda^2 + 8)b. \tag{3-2}$$

Proof. Since $S=\sum_{i,j} h_{ij}^2$ is constant, we get, for any k, l, m,

$$0=\Big(\sum_{i,j} h_{ij}^2\Big)_{klm}=2\sum_{i,j}(h_{ij}h_{ijklm}+h_{ijm}h_{ijkl}+h_{ijk}h_{ijlm}+h_{ijl}h_{ijkm}).$$

Since

$$\sum_{i,j} h_{ij}h_{ijklm}=-3\lambda h_{33klm},$$

we have

$$3\lambda h_{33klm}=\sum_{i,j} h_{ijm}h_{ijkl}+\sum_{i,j} h_{ijk}h_{ijlm}+\sum_{i,j} h_{ijl}h_{ijkm}.$$

Hence,

$$\sum_{k,l,m} h_{klm}h_{33klm}=\frac{1}{\lambda}\sum_{i,j,k,l,m} h_{ijk}h_{klm}h_{ijlm}.$$

On the other hand, we have

$$0=\Big(\sum_{i,j,k} h_{ijk}^2\Big)_{33}=2\sum_{i,j,k}(h_{ijk}h_{ijk33}+h_{ijk3}^2).$$

$$\begin{aligned}\sum_{i,j,k} h_{ijk}(h_{33ijk}-h_{ijk33})&=\sum_{i,j,k} h_{ijk}h_{33ijk}+\sum_{i,j,k} h_{ijk3}^2\\&=\frac{1}{\lambda}\sum_{i,j,k,l,m} h_{ijk}h_{klm}h_{ijlm}+\frac{x}{\lambda^2}+\frac{10a^2+12ab}{9\lambda^2}.\end{aligned}$$

Since

$$\begin{aligned}&\sum_{i,j,k} h_{ijk}(h_{33ijk}-h_{ijk33})\\
&=\sum_{i,j,k} h_{ijk}[h_{3i3jk}-h_{ijk33}]\\
&=\sum_{i,j,k} h_{ijk}\Big[\Big(h_{3ij3}+\sum_m h_{mi}R_{m33j}+\sum_m h_{3m}R_{mi3j}\Big)_k-\Big(h_{ij3k}+2\sum_m h_{mj}R_{mik3}\Big)_3\Big]\\
&=\sum_{i,j,k} h_{ijk}\Big[h_{3ij3k}-h_{ij3k3}+\sum_m h_{mik}R_{m33j}+\sum_m h_{3mk}R_{mi3j}-2\sum_m h_{mj3}R_{mik3}\Big]\\
&\qquad+\sum_{i,j,k,m} h_{ijk}h_{mi}(h_{m3}h_{3j}-h_{mj}h_{33})_k+\sum_{i,j,k,m} h_{ijk}h_{3m}(h_{m3}h_{ij}-h_{mj}h_{i3})_k\\
&\qquad-2\sum_{i,j,k,m} h_{ijk}h_{mj}(h_{mk}h_{i3}-h_{m3}h_{ik})_3\\
&=\sum_{i,j,k} h_{ijk}\Big[2\sum_m h_{mij}R_{m33k}+5\sum_m h_{3mj}R_{mi3k}\Big]\\
&\qquad+\sum_{i,j,k,m} h_{ijk}h_{mi}(h_{m3k}h_{3j}+h_{m3}h_{3jk}-h_{mjk}h_{33})\\
&\qquad+\sum_{i,j,k,m} h_{ijk}h_{3m}(h_{m3}h_{ijk}-h_{mjk}h_{i3}-h_{mj}h_{i3k}+h_{m3k}h_{ij})\\
&\qquad-2\sum_{i,j,k,m} h_{ijk}h_{mj}(h_{mk3}h_{i3}-h_{m3}h_{ik3})\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i,j,k,m} \big[2h_{ijk}h_{mij}(-1+\lambda_k\lambda_3)(\delta_{k3}\delta_{3m}-\delta_{mk}\delta_{33}) \\
&\qquad\qquad\qquad +5h_{ijk}h_{3mj}(-1+\lambda_i\lambda_k)(\delta_{m3}\delta_{ik}-\delta_{mk}\delta_{i3})\big] \\
&\qquad +\sum_{i,k}\lambda_3\lambda_i h_{i3k}^2-\sum_{i,j,k}\lambda_3\lambda_i h_{ijk}^2 \\
&\qquad +\sum_{j,k}\lambda_3^2 h_{3jk}^2+\sum_{i,j,k}h_{ijk}^2\lambda_3^2-\sum_{j,k}h_{3jk}^2\lambda_3^2 \\
&\qquad -\sum_{i,k}\lambda_3^2 h_{i3k}^2-2\sum_{j,k}\lambda_3\lambda_j h_{3jk}^2+2\sum_{i,k}\lambda_3^2 h_{i3k}^2 \\
&= \Big[2\sum_{i,j}h_{ij3}^2(-1+\lambda_3^2)-2\sum_{i,j,k}h_{ijk}^2(-1+\lambda_k\lambda_3)-5\sum_{j,k}h_{3jk}^2(-1+\lambda_3\lambda_k)\Big] \\
&\qquad\qquad -\sum_{i,j}\lambda_3\lambda_i h_{ij3}^2-\sum_{i,j,k}\lambda_3\lambda_i h_{ijk}^2+\sum_{i,j,k}\lambda_3^2 h_{ijk}^2+\sum_{i,j}\lambda_3^2 h_{ij3}^2 \\
&= \big[2(-1+4\lambda^2)(2a)-2(-(6a+4b)-2\lambda(4\lambda b))+5(1+2\lambda^2)(2a)\big] \\
&\qquad\qquad +4a\lambda^2+2\lambda(4\lambda b)+4\lambda^2(6a+4b)+8\lambda^2 a \\
&= (72\lambda^2+18)a+(40\lambda^2+8)b.
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i,j,k,l,m} h_{ijk}h_{klm}h_{ijlm} \\
&= \sum_{k,l,m} h_{klm}(h_{11k}h_{11lm}+h_{22k}h_{22lm}+2h_{12k}h_{12lm}+2h_{13k}h_{13lm}+2h_{23k}h_{23lm}) \\
&= \sum_{k,l,m} h_{11k}h_{klm}(h_{11lm}-h_{22lm})+\sum_{l,m}2(h_{112}h_{1lm}-h_{111}h_{2lm})h_{12lm} \\
&\qquad\qquad +\sum_{l,m}2h_{113}h_{1lm}h_{13lm}-\sum_{l,m}2h_{113}h_{2lm}h_{23lm} \\
&= \sum_k h_{11k}\big[h_{k11}(h_{1111}-h_{2211})+h_{k22}(h_{1122}-h_{2222})+2h_{k13}(h_{1113}-h_{2213}) \\
&\qquad +2h_{k23}(h_{1123}-h_{2223})+2h_{k12}(h_{1112}-h_{2212})\big] \\
&\qquad +2h_{112}(2h_{112}h_{1212}+2h_{113}h_{1213}+h_{111}h_{1211}+h_{122}h_{1222}) \\
&\qquad -2h_{111}(2h_{212}h_{1212}+h_{211}h_{1211}+h_{222}h_{1222}+2h_{223}h_{1223}) \\
&\qquad +2h_{113}\big[h_{111}h_{1311}+h_{113}(h_{1313}+h_{1331})+2h_{112}h_{1312}+h_{122}h_{1322}\big] \\
&\qquad -2h_{113}\big[h_{222}h_{2322}+2h_{212}h_{2312}+h_{223}(h_{2323}+h_{2332})+h_{211}h_{2311}\big] \\
&= (a+b)(h_{1111}-h_{2211})+\sum_k h_{11k}^2(h_{1111}-h_{2211}) \\
&\qquad +4bh_{1122}+2a(h_{1133}+h_{3311}+h_{2233}+h_{3322}) \\
&\qquad +4h_{111}h_{113}h_{1113}-4h_{112}h_{223}h_{2223}+4h_{112}h_{113}h_{1123}-4h_{113}h_{221}h_{2213} \\
&= 2(a+b)(h_{1111}-h_{2211})+4bh_{1122} \\
&\qquad +4h_{113}(h_{111}h_{1113}+h_{112}h_{2223}+h_{112}h_{1123}+h_{111}h_{2213}) \\
&\qquad +2a(h_{1133}+h_{3311}+h_{2233}+h_{3322})
\end{aligned}
$$

$$
\begin{aligned}
&= 2(a+b)(h_{1111}-h_{2211})+4bh_{1122}\\
&\qquad -4h_{113}(h_{111}h_{3313}+h_{112}h_{3323})+2a(h_{1133}+h_{3311}+h_{2233}+h_{3322})\\
&= 2(a+b)(2h_{1111}+h_{3311})-4b(h_{1111}+h_{3311})\\
&\qquad -4h_{113}\Big(h_{111}\cdot\frac{2}{3\lambda}h_{111}h_{113}+h_{112}\cdot\frac{2}{3\lambda}h_{112}h_{113}\Big)\\
&\qquad +2a\Big(-\frac{a}{3\lambda}+\frac{2}{3\lambda}(a+b)-\frac{a}{3\lambda}+\frac{2}{3\lambda}(a+b)\Big)\\
&= 4ah_{1111}+2(a-b)\cdot\frac{2}{3\lambda}(a+b)-\frac{8}{3\lambda}a(h_{111}^2+h_{112}^2)-\frac{4a^2}{3\lambda}+\frac{8a}{3\lambda}(a+b)\\
&= 4ah_{1111}+\frac{8a^2-4b^2}{3\lambda}.
\end{aligned}
$$

Hence, we have

$$(72\lambda^2+18)a+(40\lambda^2+8)b=\frac{4ah_{1111}}{\lambda}+\frac{8a^2-4b^2}{3\lambda^2}+\frac{x}{\lambda^2}+\frac{10a^2+12ab}{9\lambda^2},$$

$$
\begin{aligned}
x+4a\lambda h_{1111} &= \lambda^2(72\lambda^2+18)a+\lambda^2(40\lambda^2+8)b-\tfrac{1}{9}(34a^2+12ab-12b^2)\\
&= -\tfrac{34}{9}a^2-\tfrac{4}{3}ab+\tfrac{4}{3}b^2+\lambda^2(72\lambda^2+18)a+\lambda^2(40\lambda^2+8)b. \qquad \square
\end{aligned}
$$

In view of (3-1) and (3-2), we have from $6a+4b=S(S+3)$, $6\lambda^2=S$,

(3-3)
$$
\begin{aligned}
&2\lambda^2(h_{1111}^2+h_{1112}^2)-2(a-b)\lambda h_{1111}\\
&= \tfrac{60}{9}a^2+\tfrac{31}{18}ab-\tfrac{7}{3}b^2+\tfrac{5}{4}Sb-\lambda^2(72\lambda^2+18)a-\lambda^2(40\lambda^2+8)b\\
&= \tfrac{20}{3}a^2+\tfrac{31}{18}ab-\tfrac{7}{3}b^2+\tfrac{5}{4}Sb-\tfrac{1}{6}S(12S+18)a-\tfrac{1}{6}S\big(\tfrac{20}{3}S+8\big)b\\
&= \tfrac{20}{3}a^2+\tfrac{31}{18}ab-\tfrac{7}{3}b^2-2S(S+3)a+3Sa-\tfrac{10}{9}S\big(S+\tfrac{3}{40}\big)b\\
&= \tfrac{20}{3}a^2+\tfrac{31}{18}ab-\tfrac{7}{3}b^2-2(6a+4b)a+3Sa-\tfrac{10}{9}S\big(S+3-\tfrac{117}{40}\big)b\\
&= -\tfrac{16}{3}a^2-\tfrac{233}{18}ab-\tfrac{61}{9}b^2+S\big(3a+\tfrac{13}{4}b\big).
\end{aligned}
$$

According to

$$2\lambda^2(h_{1111}^2+h_{1112}^2)-2(a-b)\lambda h_{1111}\ge -\tfrac{1}{2}(a-b)^2,$$

we obtain

$$-\tfrac{29}{6}a^2-\tfrac{251}{18}ab-\tfrac{113}{18}b^2+S\big(3a+\tfrac{13}{4}b\big)\ge 0. \tag{3-4}$$

Since

$$
\begin{aligned}
&-\tfrac{29}{6}a^2-\tfrac{58}{18}ab-\tfrac{13}{2}ab-\tfrac{13}{3}b^2\\
&\qquad = -\tfrac{29}{36}a(6a+4b)-\tfrac{13}{12}(4b+6a)b=-\tfrac{29}{36}S(S+3)a-\tfrac{13}{12}S(S+3)b,
\end{aligned}
$$

we have from (3-4)

$$\big(\tfrac{21}{36}-\tfrac{29}{36}S\big)Sa-\tfrac{76}{18}ab-\tfrac{35}{18}b^2-\tfrac{13}{12}S^2b\ge 0.$$

Hence we have $S\le\frac{21}{29}$. $\square$

4. Proof of Theorem 1.1

In this section, we will give a proof of the Theorem 1.1.

Proof of Theorem 1.1. We choose a local frame field $\{e_1, e_2, e_3, e_4\}$ such that at any point p,

$$h_{ij} = \lambda_i \delta_{ij}.$$

Since S is constant, we notice that the sectional curvature is bounded from below from Gauss equations. By making using of the generalized maximum principle due to Omori [5], there exists a sequence $\{p_k\}_{k=1}^{\infty} \subset M^3$ such that

$$\lim_{k\to\infty} f_3(p_k) = \sup_{M^3} f_3, \quad \lim_{k\to\infty} |\nabla f_3(p_k)| = 0, \quad \lim_{k\to\infty} \sup \nabla_l \nabla_l f_3(p_k) \le 0 \text{ for } l = 1, 2, 3.$$

Since S is constant,

$$\sum_{i,j,k} h_{ijk}^2 = S(S+3), \qquad \sum_{i,j,k,l} h_{ijkl}^2 = S(S+3)(S+9) + 3(A-2B),$$

we know that, for any i, j, k, l, $\{\lambda_i(p_k)\}$, $\{h_{ijk}(p_k)\}$ and $\{h_{ijkl}(p_k)\}$ are bounded sequences, respectively. Thus, we can assume, if necessary, by taking a subsequences of $\{p_m\}$,

$$\lim_{m\to\infty} \lambda_i(p_m) = \hat{\lambda}_i, \quad \lim_{m\to\infty} h_{ijk}(p_m) = \hat{h}_{ijk}, \quad \lim_{m\to\infty} h_{ijkl}(p_m) = \hat{h}_{ijkl} \quad \text{for all } i, j, k, l.$$

From now on, all the computations are considered for $\hat{\lambda}_i, \hat{h}_{ijk}$ and $\hat{h}_{ijkl}$. For simplicity, we omit $\hat{}$.

If the principal curvatures are the same, $S \equiv 0$ since M^3 is minimal. We only consider the following two cases.

Case 1. The number of distinct principal curvatures is two. By the same proof as in the Section 3, we get

$$S \le \tfrac{21}{29}.$$

Case 2. All three principal curvatures are distinct. If f_3 is constant, M^3 is isoparametric and $S \equiv 0$. This is impossible. From now on, we suppose that f_3 is not constant. We will derive a contradiction. Without loss of the generality, we assume that $\lambda_1 < \lambda_2 < \lambda_3$. We also assume $\sup f_3 \neq 0$; otherwise we use $\inf f_3 \neq 0$.

Lemma 4.1. *We have*

$$h_{iik} = 0 \quad \textit{for any } i, k \qquad \textit{and} \qquad h_{123}^2 = \tfrac{1}{6} S(S+3).$$

Proof. Since $\sum_i h_{ii} = 0$ and $S = \sum_{i,j} h_{ij}^2$ is constant, we have

$$\sum_i h_{iik} = 0, \quad \sum_i h_{iik}\lambda_i = 0.$$

Since $\lim_{k\to\infty} |\nabla f_3(p_k)| = 0$, we have

$$\sum_i h_{iik}\lambda_i^2 = 0.$$

Since $\lambda_i \neq \lambda_j$ for $i \neq j$, we have $h_{iik} = 0$ for any i, k. From

$$S(S+3) = \sum_{i,j,k} h_{ijk}^2 = 6h_{123}^2,$$

we obtain

$$h_{123}^2 = \tfrac{1}{6} S(S+3). \qquad \square$$

Lemma 4.2. *We have*

$$h_{iijk} = h_{iiik} = h_{kiii} = 0 \quad \textit{for } i \neq j \neq k.$$

Proof. Since $\sum_{i,j,k} h_{ijk}^2 = S(S+3)$, we have $h_{123l} = 0$ for any l, i.e.,

$$h_{iijk} = 0 \quad \text{for } i \neq j \neq k. \tag{4-1}$$

Since $\sum_i h_{ii} = 0$ and $S = \sum_{i,j} h_{ij}^2$ is constant, we have

$$\sum_i h_{iijk} = 0, \qquad \sum_i h_{iijk}\lambda_i = 0.$$

For $j \neq k$, using (4-1), we have

$$h_{jjjk} + h_{kkjk} = 0, \quad \sum_i h_{iijk}\lambda_i = 0 \qquad \text{for } j \neq k. \tag{4-2}$$

From (4-2), we have $h_{jjjk} = h_{kkjk} = 0$ for $j \neq k$. $\square$

Lemma 4.3. *We have*

$$\sum_{i,k} h_{iikk}^2 + 2\sum_{i\neq k} h_{iikk}^2 = 3S(S+3)^2.$$

Proof. From

$$\begin{aligned}
&\sum_{i,j,k,l} h_{ijkl}^2 = S(S+3)(S+9) + 3(A-2B),\\
&3(A-2B) = 6h_{123}^2(\lambda_1^2+\lambda_2^2+\lambda_3^2-2\lambda_1\lambda_2-2\lambda_2\lambda_3-2\lambda_3\lambda_1) = 2S^2(S+3),\\
&\sum_{i,j,k,l} h_{ijkl}^2 = \sum_{\substack{i\neq j\neq k\\ l}} h_{ijkl}^2 + 3\sum_{\substack{i\neq k\\ l}} h_{iikl}^2 + \sum_{i,l} h_{iiil}^2\\
&\qquad = 3\sum_{i\neq k} h_{iikk}^2 + \sum_i h_{iiii}^2 = \sum_{i,k} h_{iikk}^2 + 2\sum_{i\neq k} h_{iikk}^2,
\end{aligned}$$

we have

$$\sum_{i,k} h_{iikk}^2 + 2\sum_{i\neq k} h_{iikk}^2 = S(S+3)(S+9) + 2S^2(S+3) = 3S(S+3)^2. \qquad \square$$

Lemma 4.4. *We have*

$$\sup f_3 > 0, \quad -\sqrt{\tfrac{1}{2}S} < \lambda_1 < -\sqrt{\tfrac{1}{6}S}, \quad -\sqrt{\tfrac{1}{6}S} < \lambda_2 < 0.$$

Proof. Since $\lim_{k\to\infty}\sup \Delta f_3(p_k) \le 0$ and $\frac{1}{3}\Delta f_3 = -(S+3)f_3$, we have

$$\begin{aligned} 0 &\ge -(S+3)\lim_{k\to\infty}\sup f_3(p_k) \\ &= -(S+3)\sup_{M^3} f_3. \end{aligned}$$

We get $\sup f_3 > 0$. We also notice that $\lambda_1 < 0$ and $\lambda_3 > 0$. By a direct computation,

$$\sup f_3 = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 = 3\lambda_1\lambda_2\lambda_3 = 3\lambda_i\big(\lambda_i^2 - \tfrac{1}{2}S\big) \quad \text{for all } i.$$

We obtain $\lambda_1^2 < \frac{1}{2}S$ and $\lambda_3^2 > \frac{1}{2}S$, $\lambda_2 < 0$ and

$$\lambda_1^2 + \lambda_2^2 < \tfrac{1}{2}S. \tag{4-3}$$

Because of $\lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2 = \frac{1}{2}S$ and $\lambda_1 < \lambda_2 < 0$, we have

$$\lambda_1^2 > \tfrac{1}{6}S, \quad \lambda_2^2 < \tfrac{1}{6}S.$$

and

$$\tfrac{1}{6}S < \lambda_1^2 < \tfrac{1}{2}S, \quad 0 < \lambda_2^2 < \tfrac{1}{6}S, \quad \lambda_1 < \lambda_2 < 0. \qquad \square$$

For simplicity, we use f_3 in place of $\sup f_3$ in the following.

Lemma 4.5. *We have*

$$h_{iikk} = -\tfrac{1}{3}(S+3)\lambda_i + g_i\lambda_k + w g_i g_k,$$

where

$$g_i = \lambda_i^2 - \frac{f_3}{S}\lambda_i - \tfrac{1}{3}S.$$

Proof. Taking derivatives of $\sum_i h_{ii} = 0$ and $\sum_{i,j} h_{ij}^2 = S$, we have

$$\sum_i h_{iikk} = 0, \quad \sum_i h_{iikk}\lambda_i = -\tfrac{1}{3}S(S+3).$$

We solve this rank-5 linear system of six equations with six unknowns h_{iikk}, $i \le k$, with $h_{iijj} = h_{jjii} + (\lambda_i - \lambda_j)(-1 + \lambda_i\lambda_j)$. $\square$

Lemma 4.6. *We have*

$$f_5 = \tfrac{5}{6}Sf_3, \quad f_6 = \tfrac{1}{3}f_3^2 + \tfrac{1}{4}S^3,$$

$$\sum_i g_i^2 = \sum_i g_i\lambda_i^2 = \tfrac{1}{6}S^2 - \frac{f_3^2}{S}, \quad \sum_i g_i^4 = \tfrac{1}{2}\Big(\tfrac{1}{6}S^2 - \frac{f_3^2}{S}\Big)^2,$$

$$\sum_i g_i^2\lambda_i = \frac{f_3^3}{S^2} - \tfrac{1}{6}Sf_3, \quad \sum_i g_i^2\lambda_i^2 = \tfrac{1}{36}S^3 - \tfrac{1}{6}f_3^2, \quad \sum_i g_i^3\lambda_i = 0.$$

Proof. From $f_3 = 3\lambda_i\big(\lambda_i^2 - \frac{1}{2}S\big)$, for $i = 1, 2, 3$, we have

$$f_5 = \tfrac{5}{6}Sf_3, \quad f_6 = \tfrac{1}{3}f_3^2 + \tfrac{1}{4}S^3.$$

According to $g_i = \lambda_i^2 - \frac{f_3}{S}\lambda_i - \frac{1}{3}S$, we infer

$$\sum_i g_i^2 = \tfrac{1}{6}S^2 - \frac{f_3^2}{S}, \quad \sum_i g_i^4 = \tfrac{1}{2}\Big(\tfrac{1}{6}S^2 - \frac{f_3^2}{S}\Big)^2,$$

$$\sum_i g_i^2\lambda_i = \frac{f_3^3}{S^2} - \tfrac{1}{6}Sf_3, \quad \sum_i g_i^2\lambda_i^2 = \tfrac{1}{36}S^3 - \tfrac{1}{6}f_3^2, \quad \sum_i g_i\lambda_i^2 = \tfrac{1}{6}S^2 - \frac{f_3^2}{S},$$

Because of $F_3 = 3g_i\big(g_i^2 - \frac{1}{2}F_2\big)$, for $i = 1, 2, 3$, we have

$$\sum_i g_i^3\lambda_i = 0,$$

where $F_k = \sum_i g_i^k$. □

Lemma 4.7. *We have*

$$y = \Big(\tfrac{1}{3} + \frac{1}{S}\Big)f_3 \pm \Big[\frac{f_3^2}{S^2}\big(\tfrac{19}{9}S^2 + \tfrac{8}{3}S + 1\big) + \tfrac{7}{9}S(S+6)\big(S + \tfrac{15}{7}\big)\Big]^{\frac{1}{2}},$$

where

$$y = \Big(\tfrac{1}{6}S^2 - \frac{f_3^2}{S}\Big)w.$$

Proof. By using the Lemmas 4.5 and 4.6, we have

$$\sum_{i,k} h_{iikk}^2 = \sum_{i,k}\big(-\tfrac{1}{3}(S+3)\lambda_i + g_i\lambda_k + wg_ig_k\big)^2 = \tfrac{1}{3}S(S+3)^2 + S\Big(\tfrac{1}{6}S^2 - \frac{f_3^2}{S}\Big) + w^2\Big(\tfrac{1}{6}S^2 - \frac{f_3^2}{S}\Big)^2, \tag{4-4}$$

$$\begin{aligned}\sum_i h_{iiii}^2 &= \sum_i\big(-\tfrac{1}{3}(S+3)\lambda_i + g_i\lambda_i + wg_i^2\big)^2 \\ &= \tfrac{1}{9}(S+3)^2S + \sum_i g_i^2\lambda_i^2 + w^2\sum_i g_i^4 + 2w\sum_i\lambda_ig_i^3 \\ &\qquad - \tfrac{2}{3}(S+3)\sum_i g_i\lambda_i^2 - \tfrac{2}{3}(S+3)w\sum_i g_i^2\lambda_i \\ &= \tfrac{1}{9}S(S+3)^2 + \tfrac{1}{6}S\Big(\tfrac{1}{6}S^2 - \frac{f_3^2}{S}\Big) + w^2\tfrac{1}{2}\Big(\tfrac{1}{6}S^2 - \frac{f^2}{S}\Big)^2 \\ &\qquad - \tfrac{2}{3}(S+3)\Big(\tfrac{1}{6}S^2 - \frac{f_3^2}{S}\Big) - \tfrac{2}{3}(S+3)w\Big(\frac{f_3^3}{S^2} - \tfrac{1}{6}Sf_3\Big),\end{aligned} \tag{4-5}$$

$$\begin{aligned}\sum_{i\neq k} h_{iikk}^2 &= \sum_{i,k} h_{iikk}^2 - \sum_i h_{iiii}^2 \\ &= \tfrac{2}{9}S(S+3)^2 + \big[\tfrac{5}{36}S^3 - \tfrac{5}{6}f_3^2\big] + \tfrac{2}{3}(S+3)\Big(\tfrac{1}{6}S^2 - \frac{f_3^2}{S}\Big) \\ &\qquad + \tfrac{2}{3}w(S+3)\Big(-\tfrac{1}{6}Sf_3 + \frac{f_3^3}{S^2}\Big) + \tfrac{1}{2}w^2\Big[\tfrac{1}{36}S^4 - \tfrac{1}{3}Sf_3^2 + \frac{f_3^4}{S^2}\Big].\end{aligned} \tag{4-6}$$

Substituting (4-4) and (4-6) into the Lemma 4.3 completes the proof. □

Lemma 4.8. *We have*

$$-\frac{3}{S}y\big(\lambda_l^2-\tfrac{1}{6}S\big)\big(\lambda_l^2-\tfrac{2}{3}S\big)\le\lambda_l\big(\lambda_l^2-\tfrac{1}{6}S\big)\Big(\frac{9}{S}\lambda_l^4-\tfrac{15}{2}\lambda_l^2+2S+3\Big).$$

Proof. Since

$$\tfrac{1}{3}(f_3)_{ll}=\sum_i h_{iill}\lambda_i^2+2\sum_{i,j}h_{ijl}^2\lambda_i,$$

we have

$$0\ge\tfrac{1}{3}\lim_{k\to\infty}\sup(f_3)_{ll}=\tfrac{1}{3}\lim_{k\to\infty}(f_3)_{ll}=\sum_i h_{iill}\lambda_i^2+2\sum_{i,j}h_{ijl}^2\lambda_i. \tag{4-7}$$

By a direct computation, we infer

$$\sum_i h_{iill}\lambda_i^2=-\lambda_l(S+3)\big(\lambda_l^2-\tfrac{1}{2}S\big)+\lambda_l\Big(\tfrac{1}{6}S^2-\frac{f_3^2}{S}\Big)+\Big(\lambda_l^2-\frac{f_3}{S}\lambda_l-\tfrac{1}{3}S\Big)y, \tag{4-8}$$

$$2\sum_{i,j}h_{ijl}^2\lambda_i=-\tfrac{1}{3}S(S+3)\lambda_l. \tag{4-9}$$

By substituting (4-8) and (4-9) into (4-7), we have

$$\Big(\lambda_l^2-\frac{f_3}{S}\lambda_l-\tfrac{1}{3}S\Big)y\le\lambda_l\Big[(S+3)\big(\lambda_l^2-\tfrac{1}{2}S\big)-\Big(\tfrac{1}{6}S^2-\frac{f_3^2}{S}\Big)+\tfrac{1}{3}S(S+3)\Big].\qquad\square$$

If $y\ge 0$, by using the Lemma 4.6, we have

$$\begin{aligned} y&=\Big(\tfrac{1}{3}+\frac{1}{S}\Big)f_3+\Big[\frac{f_3^2}{S^2}\big(\tfrac{19}{9}S^2+\tfrac{8}{3}S+1\big)+\tfrac{7}{9}S(S+6)\big(S+\tfrac{15}{7}\big)\Big]^{\frac{1}{2}},\\ &>\Big(\tfrac{1}{3}+\frac{1}{S}\Big)f_3+\Big[\frac{f_3^2}{S^2}\big(\tfrac{4}{3}S+1\big)^2\Big]^{\frac{1}{2}}\\ &=\Big(\tfrac{5}{3}+\frac{2}{S}\Big)f_3. \end{aligned} \tag{4-10}$$

By substituting (4-10) into the Lemma 4.8 with $l=1$, we have

$$\Big(\frac{24}{S}+\frac{18}{S^2}\Big)\lambda_1^4-\Big(25+\frac{21}{S}\Big)\lambda_1^2+7S+9<0. \tag{4-11}$$

We notice that the left-hand side of (4-11) is an increasing function of λ_1^2 for $\lambda_1^2>\frac{1}{6}S$. Substituting $\lambda_1^2=\frac{1}{6}S$ into (4-11), we have

$$S<-\tfrac{12}{7}.$$

It is a contradiction.

If $y<0$, by taking $l=2$ in the Lemma 4.8, we have

$$\frac{3}{S}y\big(\lambda_2^2-\tfrac{2}{3}S\big)+\lambda_2\Big(\frac{9}{S}\lambda_2^4-\tfrac{15}{2}\lambda_2^2+2S+3\Big)\le 0. \tag{4-12}$$

Because of

$$y=\Big(\tfrac{1}{3}+\frac{1}{S}\Big)f_3-\Big[\frac{f_3^2}{S^2}\big(\tfrac{19}{9}S^2+\tfrac{8}{3}S+1\big)+\tfrac{7}{9}S(S+6)\big(S+\tfrac{15}{7}\big)\Big]^{\frac{1}{2}},$$

and since the left-hand side of (4-12) is an increasing function of λ_2 for $0 > \lambda_2 > -\sqrt{\frac{1}{6}S}$, substituting $\lambda_2 = -\sqrt{\frac{1}{6}S}$ into (4-12), we have

$$\text{LHS of (4-12)} > 0.$$

It is a contradiction. □

References

[1] S. Chang, "On minimal hypersurfaces with constant scalar curvatures in S^4", *J. Differential Geom.* **37**:3 (1993), 523–534. MR

[2] Q. M. Cheng and Q. R. Wan, "Complete hypersurfaces of $\mathbb{R}^4$ with constant mean curvature", *Monatsh. Math.* **118**:3-4 (1994), 171–204. MR

[3] S. S. Chern, M. do Carmo, and S. Kobayashi, "Minimal submanifolds of a sphere with second fundamental form of constant length", pp. 59–75 in *Functional Analysis and Related Fields* (Chicago, IL, 1968), edited by F. E. Browder and M. Stone, Springer, 1970. MR

[4] H. B. Lawson, Jr., "Local rigidity theorems for minimal hypersurfaces", *Ann. of Math.* (2) **89** (1969), 187–197. MR

[5] H. Omori, "Isometric immersions of Riemannian manifolds", *J. Math. Soc. Japan* **19** (1967), 205–214. MR

[6] C.-K. Peng and C.-L. Terng, "Minimal hypersurfaces of spheres with constant scalar curvature", pp. 177–198 in *Seminar on minimal submanifolds*, edited by E. Bombieri, Ann. of Math. Stud. **103**, Princeton Univ. Press, 1983. MR

[7] J. Simons, "Minimal varieties in riemannian manifolds", *Ann. of Math.* (2) **88** (1968), 62–105. MR

[8] S. T. Yau, "Harmonic functions on complete Riemannian manifolds", *Comm. Pure Appl. Math.* **28** (1975), 201–228. MR

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