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The *reciprocal complement* $R(D)$ of an integral domain D is the subring of its fraction field generated by the reciprocals of its nonzero elements. Many properties of $R(D)$ are determined when D is a polynomial ring in $n \geq 2$ variables over a field. In particular, $R(D)$ is an n-dimensional, local, non-Noetherian, non-integrally closed, non-factorial, atomic G-domain, with infinitely many prime ideals at each height other than 0 and n.

1. Introduction

Let D be an integral domain with fraction field F. What can we say about the subring $R(D)$ of F generated by the reciprocals of all the nonzero elements of D (called the *reciprocal complement*, or *ring of reciprocals*, of D)?

Simple as the above question is, it appears to be a new one, and as we will see in this paper, the answer can be both surprising and satisfying. The question arises naturally in the study of *Egyptian domains*, which extends the notion of Egyptian fractions from the integers to arbitrary integral domains. This study was initiated in [Guerrieri et al. 2024] and continued in [Epstein 2024a]. Recall [Guerrieri et al. 2024] that an integral domain D is *Egyptian* if every element of its fraction field F can be written as a sum of (resp., of distinct) reciprocals of elements of D (in general, such an element of F is called *Egyptian* or *D-Egyptian*). From the viewpoint of the above question then, D is Egyptian if and only if $F = R(D)$. So the distinction between $R(D)$ and F can be seen as a measure of how far an integral domain is from being Egyptian.

The idea of Egyptian domains ultimately comes from the way the ancient Egyptians represented fractions. Namely, they represented an element of $\mathbb{Q} \cap (0, 1)$ as a sum of reciprocals of (distinct) positive integers (so-called *unit fractions*). More than eight centuries ago, Fibonacci [Dunton and Grimm 1966] showed that this is always

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possible. However, it is far from unique. There are always infinitely many ways to represent a positive rational number as a sum of distinct unit fractions. For the past century or so, number theorists have taken up questions of the diversity of ways to represent fractions as unit fractions. Indeed, such questions can always be rephrased as diophantine equations. Most prominently, the Erdős–Straus conjecture posits that for any $n \geq 5$, the number $4/n$ can be written as a sum of at most three unit fractions.

In addition to its connection with Egyptian fractions, the setting of Egyptian domains and reciprocal complements also isolates natural properties of affine semigroups, rings, and even of algebraic varieties. For instance, it can distinguish whether a subsemigroup Λ of $\mathbb{Q}^n$ is a group, in the following sense. Let D be an Egyptian domain (e.g., $\mathbb{Z}$, or any field). Then $D[\Lambda]$ is Egyptian if and only if Λ is a group (see [Guerrieri et al. 2024, Proposition 3] for "if" and [Epstein 2024a, Theorem 2.6] for "only if"). On the other hand, any *local* domain is Egyptian [Guerrieri et al. 2024, Example 3] and in an affine-local sense, any domain that is finitely generated over a field is *locally* Egyptian [Epstein 2024a, Corollary 3.11], even though $k[x]$ is not Egyptian. Thus, the Egyptian property is an essentially global property that cannot be checked locally, unlike many ring-theoretic properties. Passing to algebraic varieties, Dario Spirito [2025, Theorem 2.1] has shown that when k is an algebraically closed field, and D is a one-dimensional finitely generated k-algebra that is a domain, then D is Egyptian if and only if there is some realization $X \subseteq \mathbb{A}^n_k$ of D that is regular at ∞ such that $|\overline{X} \setminus X| \neq 1$, where $\overline{X}$ is the projective closure of X in $\mathbb{P}^n_k$. Otherwise, if $\{p\} = \overline{X} \setminus X$, he shows that the reciprocal complement of D is isomorphic to $\mathcal{O}_{\overline{X},\{p\}}$.

The first-named author called an integral domain *Bonaccian* if for any nonzero $f \in F$, either f or $1/f$ can be written as a sum of reciprocals from D. Equivalently, $R(D)$ is a valuation domain. He then showed that a Euclidean domain is always Bonaccian [Epstein 2024c], and indeed $R(D)$ is either a DVR or a field. In particular, he [Epstein 2024b] showed that the reciprocal complement of $K[X]$ (K a field, X an indeterminate) is $K[T]_{(T)}$, where $T = 1/X$.

In the current paper, we show that no such thing is true for the reciprocal complement of a polynomial ring in two or more variables over a field. Indeed, let $D = K[X_1, \dots, X_n]$ and $R = R(D)$. Then R has many properties like those of D, but also many interesting, even exotic features. Our main results include the following:

- Any prime ideal of R is generated by elements of the form $1/f$, where $f \in D \setminus K$. This is a special case of a result that holds in any reciprocal complement (see Proposition 2.8).
- R is a local ring whose unique maximal ideal is generated by all elements of the form $1/f$ for $f \in D \setminus K$ (see Theorem 3.3). This is a special case of a result that holds in any reciprocal complement (see Theorem 2.4).

- For every $1 \leq i \leq n-1$, R has infinitely many primes of height i (see Theorem 6.6).
- $\dim R = n$ (see Theorem 4.4).
- R is atomic (see Theorem 3.11).
- For every $j \leq n$, there is a prime ideal $\mathfrak{p} \in \operatorname{Spec} R$ such that $R_{\mathfrak{p}}$ is isomorphic to the reciprocal complement of a polynomial ring in j variables over a field (see Proposition 4.2).
- R is a *G-domain*. In fact, $R\big[\prod_{i=1}^{n} X_i\big] = \operatorname{Frac} R$ (see Proposition 2.9).
- For certain height-one primes $\mathfrak{p}$, we have that $R_{\mathfrak{p}}$ is a DVR (see Lemma 3.6).
- If $n \geq 2$, R is not Noetherian. In fact it is not even coherent (see Corollary 5.7).
- If $n \geq 2$, R is not integrally closed (see Theorem 5.8).
- When $n \leq 2$ and $\operatorname{ht} \mathfrak{p} = 1$, $R_{\mathfrak{p}}$ is always Noetherian (see Theorem 7.3).
- When $n = 2$, any finitely generated ideal is contained in all but finitely many prime ideals (see Theorem 7.5). Since all but two prime ideals of R have height one, this behavior can be seen as an extreme version of Krull's principal ideal theorem.

A key to our results has been a change in perspective, wherein one uses the K-automorphism σ of the field $K(X_1, \dots, X_n)$ that sends $X_i \mapsto 1/X_i$. Then $R^* := \sigma(R(D))$ contains D as a subring, even though R^* and R are isomorphic as rings. The effect of this on reciprocals of explicit polynomials is captured in Lemma 3.5. We will sometimes use the R^* point of view to analyze R, and sometimes the R point of view.

We also use valuations in a variety of ways to control the behavior of prime ideals in R.

Prior to this paper, there were four standard constructions that generally lead to non-Noetherian rings in ways that are essentially different from one another:

- polynomial rings in infinitely many variables over a field or $\mathbb{Z}$ (and their quotients),
- putting a valuation on a field with value group not isomorphic to $\mathbb{Z}$ and extracting the valuation ring,
- pullbacks, and
- rings of integer-valued polynomials.

Now we know there is a fifth such construction: the reciprocal complement. The fact that the reciprocal complement of such a well-behaved ring as $K[X_1, \dots, X_d]$ for $d > 1$ is non-Noetherian, not integrally closed, and so forth (see above) indicates that these properties probably also fail for reciprocal complements of many other otherwise well-behaved integral domains. This provides a fertile ground for and source of problems for factorization theory and other investigations in non-Noetherian commutative algebra, being so different (as seen in the list of properties above) from valuation rings, pullbacks, integer-valued polynomial rings, and infinite-dimensional polynomial rings.

2. General properties

In this section, we determine some properties of the reciprocal complement $R = R(T)$ of any integral domain T. In particular, we show that R is always local (see Theorem 2.4), that the prime ideals of R are generated by reciprocals of elements of T (see Proposition 2.8), and that if T is finitely generated over a field, then the fraction field of R is a finitely generated R-algebra (see Proposition 2.9).

Definition 2.1. For any integral domain T, we let $R(T)$ be the *reciprocal complement* of T. That is, if $F(T)$ is the fraction field of T, then $R(T)$ is the subring of $F(T)$ generated by all terms of the form $1/f$, where $f \in T \setminus \{0\}$. Equivalently, $R(T)$ is the set of all finite sums $1/f_1 + \cdots + 1/f_t$, where $t \geq 0$ and each f_i is an element of $T \setminus \{0\}$.

Let T be an integral domain. Let E be the set of Egyptian elements of T, and set $G := E \cup \{0\}$. Recall [Guerrieri et al. 2024, Proposition 8(3), where G is called E_3] that G is a subring of T. Since it must then be an integral domain, $E = G \setminus \{0\}$ is a multiplicatively closed subset of T.

Proposition 2.2. *Let T, E, G be as above. Then*

(1) $R(E^{-1}T) = R(T)$, *and*

(2) *the set of Egyptian elements of $E^{-1}T$, along with* 0, *coincides with the fraction field of G.*

Proof. To prove (1), note that taking the reciprocal complement preserves inclusion; hence, $R(E^{-1}T) \supseteq R(T)$. Conversely, let $y \in R(E^{-1}T)$. By definition there exists $d_1, \dots, d_n \in T$ and $e_1, \dots, e_n \in E$ such that

$$y = \frac{e_1}{d_1} + \cdots + \frac{e_n}{d_n}.$$

But each e_i is an Egyptian element of T, and thus we can write $e_i = \sum_{j=1}^{n_i} (1/d_{i_j})$. It follows that

$$y = \sum_{i=1}^{n} \frac{1}{d_i} \left(\sum_{j=1}^{n_i} \frac{1}{d_{i_j}} \right) \in R(T).$$

To prove (2), let K the fraction field of G. Pick an $(E^{-1}T)$-Egyptian element x of $E^{-1}T$. Hence, $x \in R(E^{-1}T) = R(T)$. We can write $x = d/e$ with $d \in T$ and

$e \in E = G \setminus \{0\}$. But $e \in R(T)$ and therefore $d = ex \in R(T)$. It follows that $d \in R(T) \cap T = G$. Thus $x = d/e \in K$.

Conversely, let $0 \neq x \in K$. Then $x = g/h$, where $g, h \in E$. Write $g = \sum_{i=1}^{s}(1/d_i)$, $d_i \in T$. Then $x = g/h = \sum_{i=1}^{s} 1/(d_i/g)$. Since each d_i/g is an element of $E^{-1}T$, we have $x \in R(E^{-1}T)$. Since $g \in T$ and $h \in E$, we have $x \in E^{-1}T$. Hence x is an Egyptian element of $E^{-1}T$. $\square$

Lemma 2.3. *Let K be a field. Let T be a K-algebra all of whose Egyptian elements are in K. Let $x_1, \ldots, x_n \in T \setminus K$ and $u \in K \setminus \{0\}$. Then*

$$y = u + \frac{1}{x_1} + \cdots + \frac{1}{x_n}$$

is a unit in $R := R(T)$.

Proof. We can reduce to the case $u = 1$ by dividing all the x_i's by u.

If $n = 0$, the statement is vacuously true. So we may assume $n \geq 1$ and work by induction on n. Set $\alpha_i := x_i^{-1}$ for each $1 \leq i \leq n$.

If $y = 0$, then $\alpha_n = -\left(1 + \sum_{i=1}^{n-1} \alpha_i\right) \in U(R)$ by the inductive hypothesis. Thus, $x_n = \alpha_n^{-1} \in R$, so x_n is an Egyptian element of T, whence $x_n \in K$ by the assumptions on T. But that contradicts the assumption on x_n that $x_n \notin K$. Hence, $y \neq 0$.

Next, notice that

$$H_n := \frac{\prod_{i=1}^{n} \alpha_i}{1 + \alpha_1 + \cdots + \alpha_n} = \frac{1}{\prod_{i=1}^{n} x_i + \sum_{i=1}^{n}\left(\prod_{j \neq i} x_i\right)} \in R(T).$$

Starting from this fact we prove by reverse induction that

$$H_k := \frac{\prod_{i=1}^{k} \alpha_i}{1 + \alpha_1 + \cdots + \alpha_n} \in R(T)$$

also for every $0 \leq k \leq n$. Suppose that $H_{k+1} \in R(T)$. Then notice that

$$\prod_{i=1}^{k} \alpha_i - H_{k+1} = \frac{\left(\prod_{i=1}^{k} \alpha_i\right)\left(1 + \sum_{i \neq k+1} \alpha_i\right)}{1 + \alpha_1 + \cdots + \alpha_n} = \left(1 + \sum_{i \neq k+1} \alpha_i\right) H_k.$$

Since by the original inductive hypothesis $1 + \sum_{i \neq k+1} \alpha_i$ is a unit in $R(T)$, we get $H_k \in R(T)$. In particular, $H_0 \in R(T)$, as was to be shown. $\square$

Theorem 2.4. *Let T be an integral domain. Then $R(T)$ is a local ring, with maximal ideal generated by all elements of the form $1/x$, with $0 \neq x \in T$ not an Egyptian element.*

Proof. First assume that all the Egyptian elements of T are in a subring K of T that is a field. Let $\mathfrak{m}$ be the set of finite sums of elements of the form $1/x$, where $x \in T \setminus K$. Since any multiple of a nonunit of T is a nonunit of T, and all units of T are in K, it follows that $\mathfrak{m}$ is an ideal of $R(T)$. Moreover, any element of $R(T) \setminus \mathfrak{m}$ is a unit of $R(T)$ by Lemma 2.3. Thus it suffices to show that $\mathfrak{m}$ does not contain a unit of $R(T)$.

Let $\alpha \in \mathfrak{m}$. Write $\alpha = 1/x_1 + \cdots + 1/x_n$, with each x_i in $T \setminus K$. We proceed by induction on n to show that α is not a unit. If $n = 0$ (so $\alpha = 0$) the claim is vacuously true. If $n > 0$ and α is a unit of $R(T)$, then $\alpha^{-1} \in R(T)$. We have

$$x_1 = (1/x_1)^{-1} = \frac{1}{\alpha - \sum_{i=2}^{n}(1/x_i)} = \frac{\alpha^{-1}}{1 - \alpha^{-1}\sum_{i=2}^{n}(1/x_i)}. \tag{1}$$

But since $\alpha^{-1} \in R(T)$, $\sum_{i=2}^{n}(1/x_i) \in \mathfrak{m}$ by the inductive hypothesis, and $\mathfrak{m}$ is an ideal, we have $-\alpha^{-1}\sum_{i=2}^{n}(1/x_i) \in \mathfrak{m}$. By Lemma 2.3, it follows that the denominator of (1) is a unit. Hence, $x_1 \in R(T)$, so that $1/x_1 \in U(R(T)) \subseteq K$, contradicting the fact that $x_1 \notin K$.

Finally, we drop the assumption on T. Let E be the set of Egyptian elements of T. Then by Proposition 2.2, $R(T) = R(E^{-1}T)$, and $E^{-1}T$ has all its Egyptian elements in the subfield $E^{-1}G$, where G is the subring $E \cup \{0\}$ of T. Then by the first part of the proof, $R(T)$ is a local ring whose maximal ideal $\mathfrak{m}$ is generated by all elements of the form $1/(x/e)$, where $x \in T \setminus \{0\}$, $e \in E$, and $x/e \notin E^{-1}G$. First note that since every $e \in E$ is a unit of $R(E)$, it follows that $\mathfrak{m}$ is generated by those elements $1/x$ where $x \in T \setminus \{0\}$ and $x \notin E^{-1}G$. But since $E^{-1}G \cap T = G$, the result follows. □

We will see shortly that every prime ideal of $R(T)$ shares the property with $\mathfrak{m}$ that it is generated by reciprocals of elements of T. First, we need the following notion of length.

Definition 2.5. For $\alpha \in R(T)$, the *T-length* of α, denoted by $\ell_T(\alpha)$ (or the *length of α*, denoted by $\ell(\alpha)$, if the ring is understood) is the minimum number t such that there exist $f_1, \ldots, f_t \in T$ such that $\alpha = 1/f_1 + \cdots + 1/f_t$. For α in the fraction field of T but not in $R(T)$, we write $\ell_T(\alpha) = \infty$.

Lemma 2.6. *Let T be an integral domain and $0 \neq \alpha \in R(T)$. Write $\alpha = \sum_{i=1}^{t}(1/f_i)$, where $t = \ell_T(\alpha)$ and each f_i is an element of $T \setminus \{0\}$. Then, in $R(T)$, α is a factor of the product of all the elements $1/f_i$.*

Proof. Set $F := \prod_{i=1}^{t} f_i$. Since $F/f_i \in T$ for each i, we have $F\alpha \in T$, and since both F and α are nonzero elements of the fraction field of T, we have $F\alpha \neq 0$. Hence $1/(F\alpha) \in R(T)$. Then the equation $1/F = 1/(F\alpha) \cdot \alpha$ finishes the proof. □

Lemma 2.7. *Let T be an integral domain and $0 \neq \alpha \in R(T)$. Let $t = \ell_T(\alpha)$ and write $\alpha = \sum_{i=1}^{t} 1/f_i$ with $f_i \in T \setminus \{0\}$. Then $\ell_T(\alpha - (1/f_t)) = t - 1$.*

Proof. Write $\beta = \alpha - 1/f_t$. Since $\beta = \sum_{i=1}^{t-1}(1/f_i)$, we have $\ell_T(\beta) \leq t - 1$. Write $\beta = \sum_{j=1}^{s}(1/g_j)$, where $s = \ell_T(\beta)$ and each g_j is an element of $T \setminus \{0\}$. Then $\alpha = (1/f_t) + \sum_{j=1}^{s}(1/g_j)$. Thus,

$$t = \ell_T(\alpha) \leq s + 1 \leq (t-1) + 1 = t.$$

Hence, $s + 1 = t$, as was to be shown. □

As a consequence of the above two lemmas, we obtain the following result about the generators of any prime ideal of $R(T)$, which recapitulates the fact about the maximal ideal of $R(T)$ given in Theorem 2.4.

Proposition 2.8. *Any prime ideal of* $R(T)$ *is generated by elements of the form* $1/f$, *where* $f \in T$.

Proof. Let $0 \neq \alpha \in \mathfrak{p}$ and $t = \ell_T(\alpha)$. We proceed by induction on t to show that α is a sum of elements of $\mathfrak{p}$ of the form $1/f$, where $f \in T$.

When $t = 1$, it is clear. Suppose $t > 1$. Write $\alpha = \sum_{i=1}^{t}(1/f_i)$, where $f_i \in T$. By Lemma 2.6, $\prod_{i=1}^{t}(1/f_i)$ is a multiple of $\alpha \in R(D)$. Hence, $\prod_{i=1}^{t}(1/f_i) \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, it follows that $1/f_i \in \mathfrak{p}$ for some $1 \leq i \leq t$. Let $\beta = \alpha - 1/f_i$. Clearly $\beta \in \mathfrak{p}$, and by Lemma 2.7, $\ell_T(\beta) = t - 1$, so by the inductive hypothesis, β is a sum of elements of the form $1/g \in \mathfrak{p}$, with $g \in T$. Thus, $\alpha = \beta + 1/f_i$ is also such a sum. □

We culminate this section with a result on reciprocal complements of finitely generated K-algebras.

Proposition 2.9. *Let* L/K *be a field extension, let* $f_1, \dots, f_n \in L$, *and let* $T = K[f_1, \dots, f_n]$. *Then* $R(T)\big[\prod_{i=1}^{n} f_i\big] = \operatorname{Frac} T$. *Hence* $1/\prod_{i=1}^{n} f_i \in \mathfrak{p}$ *for every nonzero prime ideal* $\mathfrak{p}$ *of* $R(T)$.

Proof. Write $g = \prod_{i=1}^{n} f_i$. First, note that since $g \in T$, we have $1/g \in R(T)$.

We have

$$f_1 = \frac{g}{\prod_{i=2}^{n} f_i} = g \cdot \frac{1}{\prod_{i=2}^{n} f_i} \in R(T)[g],$$

since $1/\big(\prod_{i=2}^{n} f_i\big) \in R(T)$. By symmetry, we have $f_1, \dots, f_n \in R(T)[g]$. Obviously $K \subseteq R(T)$ as well, so $T = K[f_1, \dots, f_n] \subseteq R(T)[g]$. Let $\alpha \in \operatorname{Frac} T$. We may write $\alpha = u/v$ with $u, v \in T$ and $v \neq 0$. Then, since $u \in T \subseteq R(T)[g]$ and $1/v \in R(T) \subseteq R(T)[g]$, we have $\alpha = u \cdot (1/v) \in R(T)[g]$. Thus, $\operatorname{Frac} T \subseteq R(T)[g]$, but the reverse containment is obvious, so $R(T)[g] = \operatorname{Frac} T$.

The final statement follows from [Kaplansky 1970, Theorem 19]. □

Remark 2.10. Recall that a *G-domain* is an integral domain whose fraction field is a finitely generated algebra over it [Kaplansky 1970, Definition following Theorem 18]. Hence, Proposition 2.9 implies that for any integral domain T that is finitely generated over a field, $R(T)$ is a G-domain.

3. Properties and bounds on the ring of polynomial reciprocals

In this section, we give bounds on the reciprocal complement of a polynomial ring in n variables. That is, we exhibit rings that it is contained in and rings that it contains. We also show it is atomic, but fails unique factorization. A main tool is the map σ, an involution on $K(X_1, \dots, X_n)$, which makes our ring isomorphic to an overring of $K[X_1, \dots, X_n]$.

Notation 3.1. Let $D = D_n = K[X_1, \dots, X_n]$, the polynomial ring in n variables over a field K, where $n \geq 1$. Let $F = F_n$ the fraction field of D_n. That is, $F_n = K(X_1, \dots, X_n)$. We set $R := R_n = R(D_n)$.

We let $\sigma = \sigma_n : F_n \to F_n$ be the unique K-algebra homomorphism that sends $X_i \mapsto 1/X_i$ for $1 \leq i \leq n$. Note that $\sigma \circ \sigma = 1_F$; hence σ is an *involution*, whence a K-automorphism of F. For any subring T of F, we set $T^* := \sigma(T)$.

We define $2n$ functions $t_i, a_i : D \setminus \{0\} \to \mathbb{N}_0$ for $1 \leq i \leq n$ as follows. We set $t_i(f) = c$ if $X_i^c \mid f$ but $X_i^{c+1} \nmid f$, and we let $a_i(f) = \deg_{X_i}(f) - t_i(f)$, where $\deg_{X_i}(f)$ is equal to the degree of f as a polynomial in X_i with coefficients in $K[X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n]$. Then, for any $f \in D \setminus \{0\}$, we write $f = \left(\prod_{i=1}^n X_i^{t_i(f)}\right) f_0$, where $f_0 \in D \setminus \bigcup_{i=1}^n X_i D$.

For any n-tuple $(u_1, \dots, u_n) \in \mathbb{Z}^n$, write $\boldsymbol{u} := (u_1, \dots, u_n)$ and $\boldsymbol{X}^{\boldsymbol{u}} := \prod_{i=1}^n X_i^{u_i}$.

Our first goal will be to prove that the maximal ideal is generated by the reciprocals of the nonconstant polynomials.

Lemma 3.2. *Let T be an $\mathbb{N}$-graded integral domain. Then all the Egyptian elements of T are in T_0, its 0-th graded component.*

Proof. For $f \in T$, we let its *degree* be the degree of its largest nonzero graded component. Note that degree is then additive in T; that is, if $f, g \in T \setminus \{0\}$, then $\deg(fg) = \deg(f) + \deg(g)$. Also, if $f + g \neq 0$, then $\deg(f + g) \leq \max\{\deg f, \deg g\}$.

With this in mind, let $f \in T$ be Egyptian. Write

$$f = \frac{1}{f_1} + \cdots + \frac{1}{f_s}$$

with $f_j \in T$ for each j. Clearing denominators by multiplying through by $\prod_{i=1}^s f_i$, we have

$$f f_1 \cdots f_s = \sum_{i=1}^s \prod_{j \neq i} f_j.$$

By equating degrees, it follows that

$$\deg(f) + \sum_{i=1}^s \deg(f_i) \leq \max_i \sum_{j \neq i} \deg(f_j).$$

Since all degrees are nonnegative, it follows that $\deg f = 0$, so that $f \in T_0$. □

Theorem 3.3. R (as in Notation 3.1) *is a local ring, with maximal ideal generated by all elements of the form $1/f$, with f a nonconstant polynomial.*

Proof. By Theorem 2.4, $R(D)$ is a local ring with maximal ideal $\mathfrak{m}$ generated by all elements of the form $1/f$ with $0 \neq f \in D$ non-Egyptian. However, by Lemma 3.2 (and using the standard grading on the polynomial ring), no nonconstant polynomial can be Egyptian. Since the nonzero constant polynomials are units and hence Egyptian, the result follows. □

Proposition 3.4. *Let $0 \le j < n$ be integers. Then $R_n \cap F_j = R_j$ and $R_n^* \cap F_j = R_j^*$.*

Proof. We need only prove the first statement, since it then follows that

$$R_n^* \cap F_j = \sigma(R_n) \cap F_j = \sigma(R_n \cap F_j) = \sigma(R_j) = R_j^*.$$

Moreover, by an easy induction, we may assume $j = n - 1$.

It is clear that $R_{n-1} \subseteq R_n \cap F_{n-1}$. So let $\alpha \in R_n \cap F_{n-1}$. Then $\alpha = \sum_{i=1}^t (1/f_i)$, where each f_i is an element of $D_n \setminus \{0\}$. Reorder the f_i such that $f_1, \dots, f_s \in D_{n-1}$ and $f_{s+1}, \dots, f_t \in D_n \setminus D_{n-1}$. Then for $1 \le i \le s$, we have $1/f_i \in R_{n-1} \subseteq R_n \cap F_{n-1}$. Let $\beta = \alpha - \sum_{i=1}^s (1/f_i)$; then we have

$$\beta = \sum_{i=s+1}^{t} \frac{1}{f_i}. \tag{2}$$

Assume $\beta \ne 0$. Then multiplying (2) by $\prod_{i=s+1}^t f_i$, we have

$$\beta f_{s+1} \cdots f_t = \sum_{i=s+1}^{t} \prod_{\substack{j \ne i \\ j > s}} f_j.$$

With respect to the polynomial ring $F_{n-1}[X_n]$, note that $\beta \in F_{n-1}$ and each f_i for $i > s$ is a nonconstant polynomial. Say $\deg f_i = d_i > 0$ for each $i > s$. Then the left hand side above has degree $\sum_{i=s+1}^t d_i$, whereas the right hand side has degree $\le \max_i \left\{ \sum_{j \ne i,\ j > s} d_j \right\} < \sum_{s=i+1}^t d_i$, a contradiction. Hence $\beta = 0$. That is, $\alpha = \sum_{i=1}^t (1/f_i) \in R_{n-1}$, since each f_i is an element of D_{n-1}. □

Lemma 3.5. *Let $f \in D \setminus \{0\}$. Then*

$$\sigma\left(\frac{1}{f}\right) = \frac{\boldsymbol{X}^{\boldsymbol{a}(f) + \boldsymbol{t}(f)}}{f^*},$$

where $f^* \in D \setminus \bigcup_{i=1}^n X_i D$. Moreover, $\boldsymbol{a}(f) = \boldsymbol{a}(f^*)$ and $f = \boldsymbol{X}^{\boldsymbol{t}(f)} f^{**}$.

Proof. First suppose $f \in D \setminus \bigcup_{i=1}^n X_i D$, so that $\boldsymbol{t}(f) = \boldsymbol{0}$. Write $f = \sum_{\boldsymbol{j} \in \mathbb{N}_0^n} u_{\boldsymbol{j}} \boldsymbol{X}^{\boldsymbol{j}}$, where each $u_{\boldsymbol{j}}$ is in K and $u_{\boldsymbol{j}} = 0$ for all but finitely many n-tuples $\boldsymbol{j}$. Then $a_i(f) = \deg_{X_i}(f) = \max\{c \mid \exists\, \boldsymbol{j} \text{ with } j_i = c \text{ and } u_{\boldsymbol{j}} \ne 0\}$ for each $1 \le i \le n$. We have

$$\sigma\left(\frac{1}{f}\right) = \frac{1}{\sum_{\boldsymbol{j}} u_{\boldsymbol{j}} / \boldsymbol{X}^{\boldsymbol{j}}} = \frac{\boldsymbol{X}^{\boldsymbol{a}(f)}}{\sum_{\boldsymbol{j}} u_{\boldsymbol{j}} \boldsymbol{X}^{\boldsymbol{a}(f) - \boldsymbol{j}}}.$$

Let f^* denote the expression in the denominator above. Note that $\boldsymbol{a}(f) - \boldsymbol{j} \in \mathbb{N}_0^n$ whenever $u_{\boldsymbol{j}} \ne 0$, since for each such $\boldsymbol{j}$ we have $j_i \le a_i(f)$ for all $1 \le i \le n$. Hence f^* is a true polynomial. Moreover, for each i, since $X_i \nmid f$, there is some $\boldsymbol{j}$ with $u_{\boldsymbol{j}} \ne 0$ and $j_i = 0$, and hence $a_i(f) - j_i = a_i(f)$. Thus, $a_i(f^*) = a_i(f)$. Finally, for each i, there is some $\boldsymbol{j}$ with $u_{\boldsymbol{j}} \ne 0$ and $j_i = a_i(f)$. Hence $a_i(f) - j_i = 0$, so $X_i \nmid f^*$, whence $f_i^* \in D \setminus \bigcup_{i=1}^n X_i D$.

For the final claim, we have

$$\sigma\left(\frac{1}{f^*}\right) = \frac{\boldsymbol{X}^{\boldsymbol{a}(f^*)}}{\sum_j u_j \boldsymbol{X}^{\boldsymbol{a}(f^*)-(\boldsymbol{a}(f)-\boldsymbol{j})}} = \frac{\boldsymbol{X}^{\boldsymbol{a}(f^*)}}{\sum_j u_j \boldsymbol{X}^{\boldsymbol{j}}} = \frac{\boldsymbol{X}^{\boldsymbol{a}(f^*)}}{f}.$$

Now we go to the general case, where $\boldsymbol{t}(f)$ is not necessarily the zero vector. We have $f = \boldsymbol{X}^{\boldsymbol{t}(f)} f_0$, where $f_0 \in D \setminus \bigcup_{i=1}^n X_i D$. Then

$$\sigma\left(\frac{1}{f}\right) = \left(\prod_{i=1}^n \sigma\left(\frac{1}{X_i}\right)^{t_i(f)}\right)\sigma\left(\frac{1}{f_0}\right) = \boldsymbol{X}^{\boldsymbol{t}(f)}\sigma\left(\frac{1}{f_0}\right) = \frac{\boldsymbol{X}^{\boldsymbol{a}(f_0)+\boldsymbol{t}(f)}}{f_0^*}.$$

Moreover, $a_i(f) = \deg_{X_i}(f) - t_i(f) = \deg_{X_i}(f_0) = a_i(f_0)$ for each $1 \le i \le n$, so $\boldsymbol{a}(f) = \boldsymbol{a}(f_0)$. Setting $f^* = f_0^*$, we have $f^{**} = f_0^{**} = f_0$, so that $f = \boldsymbol{X}^{\boldsymbol{t}(f)} f_0 = \boldsymbol{X}^{\boldsymbol{t}(f)} f_0^{**}$, completing the proof. □

Lemma 3.6. *We have $R^* \subseteq D_{(X_i)}$ for each $1 \le i \le n$. In particular there are n distinct height-one prime ideals $\mathfrak{p}_i$ of R^* obtained as centers of the X_i-adic valuations of D, and $R^*_{\mathfrak{p}_i} = D_{(X_i)}$.*

Proof. Choose i with $1 \le i \le n$. Let $f \in D$. Let v_i be the X_i-adic valuation function. Then by Lemma 3.5, $v_i(\sigma(1/f)) = t_i(f) + a_i(f) - v_i(f^*) \ge 0$, as $a_i(f) = \deg_{X_i}(f^*)$, and $v_i(f^*)$ cannot exceed the X_i-degree of f^*. Since every nonzero element $\alpha \in R^*$ is a sum of terms of the form $\sigma(1/f)$, it follows that $v_i(R^*) \ge 0$.

Now let $1 \le i < j \le n$, and let $\mathfrak{p}_i$ and $\mathfrak{p}_j$ be the centers of the X_i- and X_j-adic valuations on D in R^*, respectively. Since $v_i(X_i) = 1$ but $v_j(X_i) = 0$, we have $X_i \in \mathfrak{p}_i \setminus \mathfrak{p}_j$. Similarly, $v_i(X_j) = 0$ and $v_j(X_j) = 1$, so $X_j \in \mathfrak{p}_j \setminus \mathfrak{p}_i$.

For the final claim, let $\alpha \in D_{(X_i)}$. Then $\alpha = f/g$ for some $f \in D$ and $g \in D \setminus X_i D$. If $g \in \mathfrak{p}_i$, then $g \in X_i D_{(X_i)} \cap D = X_i D$, which is a contradiction. Hence, $f \in R^*$ and $g \in R^* \setminus \mathfrak{p}_i$, so $f/g \in R^*_{\mathfrak{p}_i}$. Thus, $R^*_{\mathfrak{p}_i} = D_{(X_i)}$, whence $\operatorname{ht} \mathfrak{p}_i = 1$. □

Lemma 3.7. *Let $\mathfrak{p}$ be a nonzero prime ideal of R. Then $1/X_i \in \mathfrak{p}$ for some $1 \le i \le n$.*

Proof. By Proposition 2.9, $1/\left(\prod_{i=1}^n X_i\right) = \prod_{i=1}^n (1/X_i) \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, some $1/X_i$ is in $\mathfrak{p}$. □

Lemma 3.8. *Let $f \in D$. If $f(\mathbf{0}) \neq 0$ then f is a unit in R^*. In particular,*

$$K[X_1, \dots, X_n]_{(X_1,\dots,X_n)} \subseteq R^*.$$

Proof. We prove this by induction on n. If $n = 0$, then the result is vacuous. Thus, let $n \ge 1$ and assume the result true for smaller n.

By way of contradiction suppose that $f \in \mathfrak{p}$ for a prime ideal $\mathfrak{p}$ of R^*. By Lemma 3.7, some X_i is in $\mathfrak{p}$; without loss of generality assume $i = n$, so that $X_n \in \mathfrak{p}$. Then $f = X_n g + h$ for some $g \in D_n$ and $h \in D_{n-1} \setminus (X_1, \dots, X_{n-1})D_{n-1}$. Then $h \in \mathfrak{p} \cap D_{n-1} \setminus (X_1, \dots, X_{n-1})D_{n-1}$, so that, by the inductive hypothesis,

$1/h \in K[X_1, \ldots, X_{n-1}]_{(X_1,\ldots,X_{n-1})} \subseteq R^*_{n-1}$. Hence also $1/h \in R^*$. But then $1 = (1/h)(f - X_n g) \in \mathfrak{p}$, a contradiction. □

Lemma 3.9. *Let w be the* ***order valuation*** *of $D_{(X_1,\ldots,X_n)}$ on F, i.e., the unique valuation on F such that for any nonzero $g \in D$, $w(g) = \max\{j \mid g \in (X_1, \ldots, X_n)^j\}$. Let $(W, \mathfrak{m}_W)$ be the corresponding DVR. Then $R^* \subseteq W$, and $\mathfrak{m}_W \cap R^*$ is the maximal ideal $\mathfrak{m}$ of R^*.*

Proof. By construction $D \subseteq W$ and $X_i \in \mathfrak{m}_W$ for all i, so that $w(X_i) > 0$. We first show that for any nonconstant polynomial f not divisible by any of the variables, $w(\sigma(1/f)) \geq 1$. Under these assumptions, $\sigma(1/f) = \boldsymbol{X}^{\boldsymbol{a}(f)}/f^*$ by Lemma 3.5. Note that there is some i and some monomial m in f^* such that $X_i^{a_i(f)} \nmid m$. Thus, $w(f^*) \leq w(m) \leq (a_i(f) - 1) + \sum_{j \neq i} a_j(f) < \sum_j a_j(f)$. Thus, $w(\sigma(1/f)) = \sum_j a_j(f) - w(f^*) > 0$.

For a general nonconstant polynomial f, we have $f = \boldsymbol{X}^{\boldsymbol{t}(f)} f_0$, where f_0 is not a multiple of any of the X_i, and if f_0 is constant then some $t_i(f) > 0$. Hence, $w(\sigma(1/f)) = \sum_{i=1}^n t_i(f) w(X_i) + w(\sigma(1/f_0)) > 0$.

Now let $\alpha \in R^*$. Write $\alpha = u + \sum_{j=1}^t \sigma(1/f_j)$, where $u \in K$ and each f_j is in $D \setminus K$. Since $K \subseteq D \subseteq W$, by the above we have $\alpha \in W$, whence $R^* \subseteq W$. If $u = 0$ then $w(\alpha) \geq \min\{w(\sigma(1/f_i) \mid 1 \leq i \leq t\} > 0$, so that $\alpha \in \mathfrak{m}_W$. Thus by Theorem 3.3, $\mathfrak{m} \subseteq \mathfrak{m}_W \cap R^*$, but then since $\mathfrak{m}$ is maximal, the result follows. □

The following result must be well known but we provide a proof for the convenience of the reader.

Lemma 3.10. *Let $(T, \mathfrak{m})$ be a local integral domain with fraction field F, and let $(V, \mathfrak{n})$ be a discrete rank-one valuation ring such that $T \subseteq V \subseteq F$ and $\mathfrak{n} \cap T = \mathfrak{m}$. Then T is atomic, and any $x \in \mathfrak{m} \setminus \mathfrak{n}^2$ is an irreducible element of T.*

Proof. We begin with the second statement. Let $x \in \mathfrak{m} \setminus \mathfrak{n}^2$. Write $x = st$ with $s, t \in T$ and s not a unit. Then if v is the valuation function of v, we have $v(s) \geq 1$, so that $1 = v(x) = v(st) = v(s) + v(t) \geq 1 + v(t)$, whence $v(t) = 0$, so that $t \in T \setminus \mathfrak{n} = T \setminus \mathfrak{m}$ and is thus a unit. Thus, x is irreducible.

For the first statement, let $A = \{x \in \mathfrak{m} \mid x$ cannot be written as a product of irreducible elements$\}$. If $A \neq \varnothing$, choose $a \in A$ such that $v(a) \leq v(b)$ for all $b \in A$. Since a is not irreducible, we may write $a = bc$ for some nonunits $b, c \in T$; hence $b, c \in \mathfrak{m}$. But then $v(c) \geq 1$, so $v(b) = v(a) - v(c) < v(a)$. Thus, $b \notin A$, so b can be written as a product of irreducible elements. By the same argument, the same holds for c. Hence, $a = bc$ is also a product of irreducible elements, which is a contradiction. Thus, $A = \varnothing$, so every element of T is either a unit or a product of irreducibles. That is, T is atomic. □

Theorem 3.11. *The ring R is atomic. That is, every nonzero nonunit element factors into a product of irreducible elements.*

Proof. Since $R \cong R^*$, we may work with R^*. By Lemma 3.9, the maximal ideal of R^* is the center of a rank-one discrete valuation. The result then follows from Lemma 3.10. □

Remark 3.12. However, R is not a UFD provided $n \geq 2$. To see this (and working in R^*), use the labels $X := X_1$ and $Y := X_2$, and first note that $s := \sigma(1/(X+Y)) = XY/(X+Y) \in R^*$, and also that $t := X^2/(X+Y) = X - s$ and $u := Y^2/(X+Y) = Y - s \in R^*$. But each of s, t, u has value 1 in the order valuation w from Lemma 3.9, hence must be irreducible elements of R^* by Lemma 3.10.

Then we have $X^2Y^2/(X+Y)^2 = s^2 = tu$, so if R^* were a UFD, s would be associate to either t or u. But if s is an associate of t, then $s \in tR^*$, which implies that $Y/X \in R^*$. And if s is an associate of u, then $u \in sR^*$, which again implies that $Y/X \in R^*$. So in either case we obtain $X/Y = \sigma(Y/X) \in R = R(D) \subseteq R(K(X_3, \dots, X_n)[X, Y])$, contradicting [Epstein 2024c, Example 2.9].

4. Dimension

In this section, we show that R has the same Krull dimension as D (see Theorem 4.4).

Lemma 4.1. *For any* $1 \leq i \leq n$, *we have*

$$R[X_i] = R(K(X_i)[X_1, \dots, \widehat{X}_i, \dots, X_n]).$$

Hence, $R^*[X_i^{-1}]$ *is isomorphic to the reciprocal complement of the polynomial ring in* $n-1$ *variables over a field.*

Proof. Without loss of generality set $i = n$. Set $S := R(K(X_n)[X_1, \dots, X_{n-1}])$.

For the forward containment, first note that $X_n = 1/(1/X_n) \in S$ since $1/X_n \in K(X_n)[X_1, \dots, X_{n-1}]$. Moreover, since $D_n \subseteq K(X_n)[X_1, \dots, X_{n-1}]$ and $R(-)$ preserves containment, we have $R \subseteq S$. Thus, $R[X_n] \subseteq S$.

For the reverse, let $0 \neq f \in K(X_n)[X_1, \dots, X_{n-1}]$. By finding a common denominator to the $K(X_n)$-coefficients of the monomials in $X_1, \dots, X_{n-1}$, we may write $f = g/h$, where $g \in D_n$ and $h \in K[X_n]$. Write $h = \sum_{i=0}^{t} c_i X_n^i$ with all $c_i \in K$. Then

$$\frac{1}{f} = \frac{h}{g} = \sum_{i=0}^{t} \frac{c_i}{g} \cdot X_n^i.$$

But for each i, if $c_i = 0$, then $c_i/g = 0 \in R_n$; otherwise $c_i/g = (1/g/c_i) \in R$. Thus, $1/f = \sum_{i=0}^{t} (c_i/g) \cdot X_n^i \in R[X_n]$. Hence, $S \subseteq R[X_n]$, completing the proof. □

Proposition 4.2. *Let j and n be integers with $0 \leq j \leq n$. Then there is a unique prime ideal P of R such that R_P is the reciprocal complement of $L[X_1, \dots, X_j]$, where $L = K(X_{j+1}, \dots, X_n)$.*

Proof. By applying induction to Lemma 4.1, we have

$$R[X_{j+1}, \dots, X_n] = R(L[X_1, \dots, X_j]) =: S.$$

But, since S is a local ring by Theorem 2.4, and since each X_i^{-1} is an element of R, there is a unique prime ideal P of R maximal with respect to avoiding all of X_i^{-1} for $j+1 \leq i \leq n$ such that $R_P = S$. □

Lemma 4.3. *For each i, there is a unique prime ideal $\mathfrak{q}_i$ of R maximal with respect to not containing $1/X_i$. We have $R_{\mathfrak{q}_i} = R[X_i]$.*

Proof. We may assume $i = n \geq 1$. By Lemma 4.1, $R[X_n]$ is the reciprocal complement of the polynomial ring in $n-1$ variables over a field, which by Theorem 2.4 is local. Hence by elementary localization theory, there is a unique prime ideal $\mathfrak{q}_n$ of R maximal with respect to avoiding X_n^{-1}, and $R_{\mathfrak{q}_n} = R[X_n] = R(K(X_n)[X_1, \ldots, X_{n-1}])$. □

Theorem 4.4. $\dim R = n$.

Proof. We proceed by induction on n. Of course $R_0 = K$, which has dimension 0, so we may assume $n > 0$. By Theorem 3.3, R is local, and its maximal ideal $\mathfrak{m}$ contains the reciprocals of all the variables. Let $\mathfrak{q} = \mathfrak{q}_n$ be as in Lemma 4.3, so that $1/X_n \notin \mathfrak{q}_n$ and $R_\mathfrak{q}$ is isomorphic to the reciprocal complement of a polynomial ring in $n-1$ variables over a field by Lemma 4.1. Then by the inductive hypothesis, $\operatorname{ht}\mathfrak{q} = \dim R_\mathfrak{q} = n-1$. Since $1/X_n \in \mathfrak{m} \setminus \mathfrak{q}$, we have $\operatorname{ht}\mathfrak{m} > \operatorname{ht}\mathfrak{q} = n-1$, whence $\dim R = \operatorname{ht}\mathfrak{m} \geq n$.

On the other hand, since R^* is an overring of the n-dimensional Noetherian domain D, we have $\dim R = \dim R^* \leq n$; see [Anderson et al. 1988]. Hence $\dim R = n$. □

5. Exotic properties of *R*

For most of this section, we work in two variables, so that $D = K[X, Y]$, where $X = X_1$ and $Y = X_2$ for short. Then the notation D_n, R_n, etc. when $n \neq 2$ will stand for the corresponding rings of other dimensions. We show that R_n^* is not integrally closed when $n \geq 2$. We also show that it is not a finite conductor domain, hence not coherent, and is thus also non-Noetherian.

Before we begin, recall the following presumably well-known result:

Lemma 5.1. *Let $A \subseteq B$ be integral domains such that B is free as an A-module. Then $B \cap \operatorname{Frac}(A) = A$.*

Proof. Let $b \in B \cap \operatorname{Frac} A$. Write $b = x/y$ with $x, y \in A$ and $y \neq 0$. Then $x = yb \in yB \cap A = (yA)B \cap A = yA$, where the latter equation holds by freeness. Thus, $x = ya$ for some $a \in A$, whence $yb = ya$, so by cancellation, $b = a \in A$. □

Our methodology here is to construct a family of valuation rings that contain R^*, which serve as a tool to analyze the elements and prime ideals of our ring. We must start with notation that will be useful:

Notation 5.2. Choose two relatively prime positive integers p and q with $p < q$, such that neither p nor q is a multiple of char K. Let K' be the smallest field extension of K that contains all the primitive p-th and q-th roots of 1. Let $L/K'(X, Y)$ be generated by elements s and t such that $s^p = X$ and $t^q = Y$. Note that $K'[s, t]$ is free as a $K'[X, Y]$-module on the basis $\{s^i t^j \mid 0 \le i < p,\ 0 \le j < q\}$.

Lemma 5.3. *Let $g \in K[X, Y]$. Then $s - t \mid g$ in $K'[s, t]$ if and only if $X^q - Y^p \mid g$ in $K[X, Y]$.*

Proof. Since $K'[X, Y] \cap K(X, Y) = K[X, Y]$ by Lemma 5.1, we may assume $K = K'$. Thus, we may let ξ_p (resp. ξ_q) be a primitive p-th (resp. q-th) root of unity in K. Then

$$\prod_{i=0}^{p-1}\prod_{i=0}^{q-1}(\xi_p^i s - \xi_q^j t) = \prod_{i=0}^{p-1}((\xi_p^i s)^q - t^q) = \prod_{k=0}^{p-1}(\xi_p^k s^q - t^q) = (-1)^p \prod_{k=0}^{p-1}(t^q - \xi_p^k s^q)$$
$$= (-1)^p (t^{pq} - s^{pq}) = (-1)^p (Y^p - X^q).$$

The second equality above holds because p and q are relatively prime, so that the order of $q + p\mathbb{Z}$ in $\mathbb{Z}/p\mathbb{Z}$ must be p.

Also note that for each pair (i, j) of integers with $0 \le i < p$ and $0 \le j < q$, there is a unique $\tau_{ij} \in \operatorname{Aut}_{K(X,Y)} L$ such that $\tau_{ij}(s) = \xi_p^i s$ and $\tau_{ij}(t) = \xi_q^j t$. Thus, if $s - t \mid g$ in $K[s, t]$, then for each i and j, we have $\tau_{ij}(s - t) = \xi_p^i s - \xi_q^j t \mid \tau_{ij}(g) = g$. Since the $\xi_p^i s - \xi_q^j t$ are mutually nonassociate irreducible elements of $K[s, t]$, a UFD, it follows that $X^q - Y^p = \pm \prod_{i,j}(\xi_p^i s - \xi_q^j t) \mid g$ in $K[s, t]$. Hence,

$$\frac{g}{X^q - Y^p} \in K(X, Y) \cap K[s, t] = K[X, Y],$$

again by Lemma 5.1, which implies that $X^q - Y^p \mid g$ in $K[X, Y]$.

For the converse, simply note that if $X^q - Y^p \mid g$ in $K[X, Y]$, then as $s - t \mid X^q - Y^p$ in $K[s, t]$ and $K[X, Y] \subset K[s, t]$, it follows by transitivity of divisibility that $s - t \mid g$ in $K[s, t]$. □

Notation 5.4. Let $u := s - t$. Then s and u are algebraically independent over K', and $K'[s, t] = K'[s, u]$. We define a valuation $w := w_h := w_{p,q,h}$ on $K'[s, u]$ by setting $w(s) = 1$ and $w(u) = h$ for some integer $h \ge 1$, and for any nonzero $f = \sum_{i,j} c_{ij} s^i u^j$ in $K'[s, t]$, where $c_{ij} \in K'$, we set

$$w(f) = \min\{w(s^i u^j) \mid c_{ij} \ne 0\} = \min\{i + hj \mid c_{ij} \ne 0\}.$$

Then we let $W := W_h := W_{p,q,h}$ be the corresponding valuation ring in the field L. Clearly $K'[s, u] \subseteq W$. Set $V := W \cap F$ (denoted by V_h or $V_{p,q,h}$ if needed) and let $v = v_h = v_{p,q,h}$ be the corresponding valuation on F.

Lemma 5.5. *The valuation ring V is an overring of $R^* := R_2^*$ if and only if $h \le pq + 1$. If $h < pq + 1$, then $\sigma(1/f) \in \mathfrak{m}_V$ for all $f \in D \setminus K$.*

Suppose on the other hand that $h = pq + 1$. *Then for an irreducible polynomial* $f \in K[X, Y]$, *with* $\alpha = \sigma(1/f^*)$ *and* $\theta = \sigma(1/(X^q - Y^p)) = X^q Y^p/(Y^p - X^q)$, *we have* $v(\alpha) = 0$ *if and only if* $\alpha = \theta^m \delta^{-1}$ *for some* $m \geq 1$ *and some element* $\delta \in K[\theta] \setminus (\theta)K[\theta]$. *Otherwise* $v(\alpha) \geq p$.

Proof. Since $X = X^*$ and $Y = Y^*$, we have $v(\sigma(1/X^*)) = v(X) = w(s^p) = pw(s) = p$, and $w(t) = w(s-u) = 1$, so $v(\sigma(1/Y^*)) = v(Y) = w(t^q) = qw(t) = q$. Hence, V is an overring of $K[X, Y]$, with $(X, Y) \subseteq \mathfrak{m}_V$.

Now suppose (i, j) is a pair of integers with $0 \leq i < p$, $0 \leq j < q$, and $(i, j) \neq (0, 0)$. Then $\xi_p^i s - \xi_q^j t = (\xi_p^i - \xi_q^j)s + \xi_p^j u$, so since $\xi_p^i - \xi_q^j \in K \setminus \{0\}$, we have $w(\xi_p^i s - \xi_q^j t) = 1$. Therefore,

$$\begin{aligned} v(X^q - Y^p) &= w\left(\pm \prod_{i=0}^{p-1} \prod_{i=0}^{q-1} (\xi_p^i s - \xi_q^j t)\right) = w(s-t) + \sum_{i,j}^{(i,j)\neq(0,0)} w(\xi_p^i s - \xi_q^j t) \\ &= h + pq - 1. \end{aligned}$$

It follows that

$$v(\theta) = v\left(\frac{X^q Y^p}{X^q - Y^p}\right) = qp + pq - (h + pq - 1) = pq + 1 - h.$$

Thus, if $h > pq + 1$, we have $v(\theta) < 0$, so that $\theta \notin V$ and $R^* \nsubseteq V$. But as long as $h \leq pq + 1$ we have $v(\theta) \geq 0$, with $v(\theta) > 0 \iff h < pq + 1$. From now on we assume $h \leq pq + 1$.

Now let $f \in K[X, Y]$ be nonconstant, irreducible, and not associate to any of X, Y, $X^q - Y^p$. Then for some $c_{ij} \in K$, we have

$$\begin{aligned} f &= \sum_{i,j} c_{ij} X^i Y^j = \sum_{ij} c_{ij} s^{pi} (s-u)^{qj} \\ &= \sum_{i,j} c_{ij} s^{pi} \sum_{k=0}^{qj} (-1)^k \binom{qj}{k} s^{qj-k} u^k \\ &= \left(\sum_{i,j} c_{ij} s^{pi+qj}\right) + u \cdot \sum_{i,j} c_{ij} s^{pi} \sum_{k=1}^{qj} (-1)^k \binom{qj}{k} s^{qj-k} u^{k-1}. \end{aligned} \tag{3}$$

Then since f is not associate to (hence not divisible by) $X^q - Y^p$ in $K[X, Y]$, it follows from Lemma 5.3 that $u \nmid f$ in $K'[s, u]$. Therefore, $f = f_1 + uf_2$ with $0 \neq f_1 \in K'[s]$ and $f_2 \in K'[s, u]$. in particular, $f_1 = \sum_{i,j} c_{ij} s^{pi+qj}$. Then $v(f) = w(f) \leq w(f_1) = \min\{pi + qj \mid c_{ij} \neq 0\}$. As usual, recalling the notation of Lemma 3.5, write $\alpha = \sigma(1/f^*) = X^a Y^b/f$, where $(a, b) = (a_1(f), a_2(f))$, $a \geq 1$, and $b \geq 1$. Thus in the sums in (3) above, we have $i \leq a$ and $j \leq b$ for all pairs (i, j) such that $c_{ij} \neq 0$. Hence, $w(f_1)$ takes the form $pi + qj$ for some $i \leq a$ and $j \leq b$.

Therefore $v(\alpha) = pa + qb - v(f) \geq p(a-i) + q(b-j) \geq 0$, and if it is nonzero it must be at least p. Since R^* is generated as a K-algebra by all such terms $\sigma(1/f^*)$, it follows that $R^* \subseteq V$.

Now, suppose $\delta \in K[\theta] \setminus (\theta)K[\theta]$. Then by Theorem 3.3, δ is a unit of R^*, hence also in V, so $v(\delta) = 0$. Hence for any nonnegative integer m, we have $v(\theta^m/\delta) = m \cdot (pq + 1 - h)$. Thus, it has value 0 if and only if $h = pq + 1$, and is otherwise positive.

It remains to show that if $h = pq + 1$ and $v(\alpha) = 0$, then there exist some $m \geq 0$ and some $\delta \in K[\theta] \setminus (\theta)K[\theta]$ with $\alpha = \theta^m/\delta$, whereas if $h < pq + 1$ then $v(\alpha) > 0$. To prove this, let f, α, f_1, f_2 be as above. We proceed by induction on the number $\ell = a + b = a_1(f) + a_2(f)$, noting that the statement is vacuously true for $\ell = 0, 1$.

We first dispense with the case that some monomial $c_{ij}s^{pi+qj}$ appearing in f_1 satisfies either $i < a$ or $j < b$. Then $v(f) \leq w(f_1) \leq pi + qj \leq pa + qb - p$, so that $v(\alpha) = pa + qb - v(f) \geq p$.

Thus, we may assume that $f_1 = c_{ab}s^{pa+qb}$, so that $c_{ab} \neq 0$. Set $g := f - c_{ab}X^aY^b$. Then rewriting g as an element of $K'[s, u]$, we have $g = g_1 + ug_2$, where $g_1 = f_1 - c_{ab}s^{pa+qb} = 0$. Hence $u \mid g$ in $K'[s, t]$, whence $X^q - Y^p \mid g$ in $K[X, Y]$ by Lemma 5.3. That is, we have $f = c_{ab}X^aY^b + (X^q - Y^p)^m H$, where $m \geq 1$ and $H \in K[X, Y]$ is relatively prime to each of X, Y, and $X^q - Y^p$. Thus $a \geq p$ and $b \geq q$. Also note that $a_1(H) \leq a - qm$ and $a_2(H) \leq b - pm$.

Set $\alpha' := X^aY^b/((Y^p - X^q)^m H)$. Then there are nonnegative integers e_1, e_2 with $\alpha' = X^{e_1}Y^{e_2}\theta^m\sigma(1/H^*)$. In particular, $e_1 = a - mp - a_1(H)$ and $e_2 = b - mq - a_2(H)$. Thus, $c_{ab}\alpha' + 1$ is a unit of R^*, so since $\alpha = \alpha'/(c_{ab}\alpha' + 1)$, we have $v(\alpha') = v(\alpha)$, which we assume to be 0. But $v(\alpha') = e_1p + e_2q + m(pq + 1 - h) + v(\sigma(1/H^*))$, whence, since $v(\alpha') = 0$, we have $e_1 = e_2 = 0$, and every irreducible factor τ of H satisfies $v(\sigma(1/\tau^*)) = 0$. Moreover if $h < pq + 1$ it further follows that $m = 0$, so that $f = c_{ab}X^aY^b$, contradicting the fact that f is relatively prime to X and Y, finishing this case.

Then in the remaining case (where $h = pq + 1$), by the inductive hypothesis each such τ satisfies $\sigma(1/\tau^*) = \theta^{m(\tau)}/\delta(\tau)$. As these terms are multiplicative, there is some $k \in \mathbb{N}$ and $\epsilon \in K[\theta] \setminus (\theta)K[\theta]$ with $\sigma(1/H^*) = \theta^k/\epsilon$ Thus, we have

$$\alpha = \frac{\alpha'}{c_{ab}\alpha' + 1} = \frac{\theta^{m+k}/\epsilon}{c_{ab}(\theta^{m+k}/\epsilon) + 1} = \frac{\theta^{m+k}}{c_{ab}\theta^{m+k} + \epsilon}.$$

Since $c_{ab}\theta^{m+k} + \epsilon \in K[\theta] \setminus (\theta)K[\theta]$, we are done. □

Recall (see [Zafrullah 1978]) that an integral domain is a *finite conductor domain* if the intersection of any pair of principal ideals is finitely generated.

Theorem 5.6. *For any $n \geq 2$, the ideal $(1/X_1)R_n \cap (1/X_2)R_n$ is not finitely generated. Hence R_n is not a finite conductor domain.*

Proof. Let $n \geq 3$ and suppose $(1/X_1)R_n \cap (1/X_2)R_n = (\alpha_1, \ldots, \alpha_t)R_n$ for some $\alpha_1, \ldots, \alpha_t \in R_n$. Let $S = R_n[X_3, \ldots, X_n]$. Let $L = K(X_3, \ldots, X_n)$. By Lemma 4.1, we have $S = R(L[X_1, X_2])$. Let $(-)'$ denote the image of an element of R_n in S. Then $\alpha_j' \in (1/X_1)S \cap (1/X_2)S$ for all j, so $(\alpha_1', \ldots, \alpha_t') \subseteq (1/X_1)S \cap (1/X_2)S$. Conversely let $u \in (1/X_1)S \cap (1/X_2)S$. Then by clearing denominators, there is some positive integer d such that $(X_3 \cdots X_n)^{-d}u \in (1/X_1)R_n \cap (1/X_2)R_n = (\alpha_1, \ldots, \alpha_t)R_n$. Since $S = R_n[X_3, \ldots, X_n]$, it follows that $u \in (\alpha_1', \ldots, \alpha_t')S$. Thus, $(1/X_1)S \cap (1/X_2)S$ is a finitely generated ideal, and we have reduced to the 2-dimensional case. So from now on we assume $n = 2$ and we rewrite $X = X_1$ and $Y = X_2$. For the rest of the proof, we pass to the R^* notation.

Suppose $XR^* \cap YR^* = (\alpha_1, \ldots, \alpha_t)$ for some finite list of nonzero $\alpha_i \in R^*$; a contradiction will complete the proof. Then there exist $\beta_i, \gamma_i \in R^*$ with $\alpha_i = X\beta_i = Y\gamma_i$ for all i. Write $\gamma_i = c_i + \sum_{j=1}^{m_i} \sigma(1/f_{ij})$, where $c_i \in K$, $m_i \geq 0$, and each f_{ij} is in $D \setminus K$. If some $c_i \neq 0$, then γ_i is a unit by Lemma 2.3, so $Y/X = \gamma_i^{-1}\beta_i \in R^*$, which is false by [Epstein 2024c, Example 2.9]. Hence $m_i \geq 1$ and $\gamma_i = \sum_{j=1}^{m_i} \sigma(1/f_{ij})$. Choose some positive integer q that is not a multiple of $\operatorname{char} K$ and such that $q > \max\{\deg_X f_{ij} \mid 1 \leq i \leq t,\ 1 \leq j \leq m_i\}$. Set $v := v_{1,q,q+1}$ and $\theta = \theta_{1,q} = X^q Y/(Y - X^q)$ as in Lemma 5.5.

Then $Y\theta = X^q \cdot (\theta + Y) \in XR^*$, and therefore $Y\theta \in XR^* \cap YR^*$. It follows that $\theta \in (\gamma_1, \ldots, \gamma_t)$. Since $v(\theta) = 0$, it follows that for some pair (i, j), we have $v(\sigma(1/f_{ij})) = 0$. By Lemma 5.5, there exists some positive integer m and some element $\delta \in K[\theta] \setminus (\theta)K[\theta]$ such that $\sigma(1/f_{ij}) = \theta^m \delta^{-1}$. Write $f = f_{ij}$.

Let $d = \deg_X(f)$ and $e = \deg_Y(f)$. Then by Lemma 3.5, we have $\sigma(1/f) = X^d Y^e/f^*$, where $\deg_X(f^*) \leq d$ and $\deg_Y(f^*) \leq e$.

Write $\delta = c_0 + \sum_{i=1}^{s} c_i \theta^i$, where each c_i is in K and $c_0 \neq 0$. Then

$$\frac{X^d Y^e}{f^*} = \sigma(1/f) = \theta^m \delta^{-1} = \frac{(X^q Y)^m/(Y - X^q)^m}{c_0 + \sum_{i=1}^{s} c_i (X^q Y)^i/(Y - X^q)^i}.$$

If $m \geq s$, then the latter equation simplifies to an equation where both the numerator and denominator of each fraction is a polynomial, as follows:

$$\frac{X^d Y^e}{f^*} = \frac{X^{qm} Y^m}{c_0(Y - X^q)^m + \sum_{i=1}^{s} c_i (X^q Y)^i (Y - X^q)^{m-i}}.$$

Since $q > d$ by the choice of q, we have $qm > d$. It follows that $X \mid c_0(Y - X^q)^m$, which contradicts the fact that $c_0 \in K^\times$.

On the other hand if $m < s$, then the equation simplifies with numerators and denominators being polynomials, as follows:

$$\frac{X^d Y^e}{f^*} = \frac{X^{qm} Y^m (Y - X^q)^s}{c_0(Y - X^q)^s + \sum_{i=1}^{s} c_i (X^q Y)^i (Y - X^q)^{s-i}}.$$

Cross-multiplying, we have

$$X^{qm}Y^m(Y-X^q)^s f^* = X^dY^e \cdot \left(c_0(Y-X^q)^s + \sum_{i=1}^{s} c_i(X^qY)^i(Y-X^q)^{s-i}\right).$$

Since $qm > d$, it follows that $X \mid c_0(Y-X^q)^s$, which again contradicts the fact that $c_0 \in K^\times$. □

Recall that a ring is *coherent* if every finitely generated ideal is finitely presented. The coherent rings include the Noetherian rings and also all valuation domains (see [Bourbaki 1972, Chapter I, §2, Exercise 12 and Chapter VI, §1, Exercise 3]).

Corollary 5.7. *For any $n \geq 2$, the ring R_n is not coherent. Hence it is non-Noetherian.*

Proof. This follows from Theorem 5.6 and [Chase 1960, Theorem 2.2]. □

The next result is notably unlike the behavior of localized polynomial rings.

Theorem 5.8. *For any $n \geq 2$, R_n is not integrally closed.*

Proof. We first consider the 2-dimensional case. Let p and q be relatively prime integers with $1 < p < q$, such that neither p nor q is a multiple of char K. By elementary number theory, there is a unique pair of integers c and d with $qd - pc = 1$, $0 < c < q$, and $0 < d < p$. Consider the element $\beta := \beta_{p,q} := (X^{2q-c}Y^d)/(X^q - Y^p) \in F$. We claim that β is integral over R^*—in fact, $\beta^p \in R^*$—but $\beta \notin R^*$.

To see that $\beta^p \in R^*$, simply note the following:

$$\beta^p = \frac{X^{(2q-c)p}Y^{dp}}{(X^q-Y^p)^p} = \left(\frac{X^qY^p}{X^q-Y^p}\right)^d \cdot \left(\frac{X^qY^p}{X^q-Y^p} + X^q\right)^{p-d} \cdot X,$$

which is in R^* since $X \in R^*$ and $\sigma(1/(Y^p - X^q)) = X^qY^p/(X^q - Y^p) \in R^*$.

On the other hand, let $v = v_{p,q,pq+1}$. Then

$$v(\beta) = v\left(\frac{X^{2q-c}Y^d}{X^q-Y^p}\right) = (2q-c)p + qd - 2pq = qd - pc = 1.$$

Suppose $\beta \in R^*$. Since $v(\beta) > 0$, it follows that $\beta \in \mathfrak{m}_V \cap R^* \subseteq \mathfrak{m}$, the maximal ideal of R^*. So by Theorem 3.3, we have $\beta = \sum_{i=1}^{t} \sigma(1/f_i^*)$ for nonconstant polynomials $f_i \in K[X, Y]$. By reordering, let $f_1, \dots, f_s$ be the polynomials whose only irreducible factor is $X^q - Y^p$ up to associate and multiplicity, whereas each of $f_{s+1}, \dots, f_t$ has an irreducible factor not associate to $X^q - Y^p$. Set $\gamma := \sum_{i=1}^{s} \sigma(1/f_i^*)$ and $\delta := \sum_{i=s+1}^{t} \sigma(1/f_i^*)$, so that $\beta = \gamma + \delta$. By Lemma 5.5, we have $v(\delta) \geq p$, so that since $v(\beta) = 1 < p$, we have $v(\gamma) = 1$.

On the other hand, for $1 \leq i \leq s$, there exist $\lambda_i \in K$ and $\ell_i \in \mathbb{N}_0$ with $f_i = \lambda_i(X^q - Y^p)^{\ell_i}$. Thus, $\sigma(1/f^*) = \lambda_i\theta^{\ell_i}$, so that by Lemma 5.5 we have $v(\sigma(1/f_i^*)) =$

$\ell_i v(\theta) = 0$. Thus, either $\gamma = 0$ or $v(\gamma) = 0$, either of which is a contradiction. Hence, $\beta \notin R^*$.

Finally, we pass to the n-dimensional case. We have $\beta^p \in R_2^* \subseteq R_n^*$. Since $R_2^* = R_n^* \cap K(X, Y)$ by Proposition 3.4, and $\beta \in K(X, Y) \setminus R_2^*$, it follows that $\beta \notin R_n^*$. □

6. The abundance of prime ideals in *R*

In this section, we show that R_n, which as we have seen is far from Noetherian when $n > 1$ (see Corollary 5.7), does have infinitely many prime ideals of each height other than 0 and n, a property enjoyed by any n-dimensional Noetherian ring, but not by some non-Noetherian rings (e.g., any valuation domain of dimension at least 2). We start with the following result to bootstrap our efforts.

Proposition 6.1. *For any $n \geq 2$, R_n has infinitely many height-one prime ideals.*

Proof. In this proof, we use R^* notation.

First suppose $n = 2$. For any relatively prime pair (p, q) of positive integers with $p < q$, let V, v, θ, and h be as in Lemma 5.5, with $h = pq + 1$. Let $\mathfrak{p} = \mathfrak{p}_{p,q}$ be the contraction of $\mathfrak{m}_V$ to R^*. Then since $v(\theta) = 0$, we have $\theta \notin \mathfrak{p}$. Since $\mathfrak{p}$ is a nonzero prime but not the maximal ideal of R^* (as $\theta \in \mathfrak{m}$), it follows that $\mathfrak{p}$ is a height-one prime.

On the other hand, let (r, s) be a different pair of relatively prime positive integers with $r < s$. We claim that $v(X^s - Y^r) = \min\{v(X^s), v(Y^r)\} = \min\{ps, qr\}$. Otherwise we would have $v(X^s) = v(Y^r)$, whence $ps = qr$. But then by assumption of relatively prime pairs, we would have $p = r$ and $q = s$, contradicting the assumption of distinctness. Therefore, $v(\theta_{r,s}) = v(X^s Y^r) - v(X^s - Y^r) = \max\{qr, ps\}$. Thus, $\theta_{r,s} \in \mathfrak{p}_{p,q}$. But by the proof of Lemma 5.5, $\theta_{r,s} \notin \mathfrak{p}_{r,s}$. Hence, $\mathfrak{p}_{p,q} \neq \mathfrak{p}_{r,s}$. Since there are infinitely many such pairs of integers, it follows that R^* has infinitely many height-one primes.

Finally, we drop the assumption that $n = 2$. By Proposition 4.2, there is a prime ideal Q of R^* such that R_Q^* is isomorphic to the reciprocal complement of $L[X, Y]$ for some field L. But then by the dimension 2 part of the proof above, R_Q^* has infinitely many height-one primes. Thus, there are infinitely many height-one primes of R^* that are contained in Q. □

Notation 6.2. Recall that given a valuation ring with fraction field K and an indeterminate t over K, the ring $V(t)$ is a valuation ring of $K(t)$ called the trivial extension of V. Given $\varphi = \sum_{j=0}^{e} f_j t^j$ with $f_j \in K$, then the value of φ with respect to $V(t)$ is $\min_j\{v(f_j)\}$ (see [Gilmer 1972, p. 218]).

By Lemma 4.3, there is a prime ideal $Q \in \operatorname{Spec} R_n$ such that

$$(R_n)_Q = R(K(X_n)[X_1, \ldots, X_{n-1}]).$$

Fix this prime for the next two lemmas.

Lemma 6.3. *Let V be a valuation overring of R_{n-1}; then the trivial extension $V(X_n)$ is an overring of $(R_n)_Q$, where Q is as in Notation 6.2.*

Proof. By the comment before the Lemma, it suffices to show that $1/\varphi \in V(X_n)$ for every $\varphi \in K(X_n)[X_1, \dots, X_{n-1}]$. Since $K(X_n) \subseteq V(X_n)$, we may assume $\varphi \in K[X_1, \dots, X_n]$. Let v^* be the valuation for $V(X_n)$; write $\varphi = \sum_{j=0}^{e} f_j X_n^e$ with $f_j \in K[X_1, \dots, X_{n-1}]$. Since V is an overring of R_{n-1}, we have that $v(f_j) \leq 0$ whenever $f_j \neq 0$. Hence, $v^*(1/\varphi) = -\min\{v(f_j) \mid 0 \leq j \leq e \text{ and } f_j \neq 0\} \geq 0$. □

Lemma 6.4. *Let $\mathfrak{p} \in \operatorname{Spec} R_{n-1}$, and let V be a valuation overring of R_{n-1} centered on $\mathfrak{p}$. Let $\mathfrak{p}'$ be the center of $V(X_n)$ in R_n. Then $\operatorname{ht} \mathfrak{p}' \geq \operatorname{ht} \mathfrak{p}$, with equality if $\operatorname{ht} \mathfrak{p} \in \{0, n-2, n-1\}$.*

Proof. Let $i = \operatorname{ht} \mathfrak{p}$. If $i = 0$, then $\mathfrak{p} = (0)$, so that $V = \operatorname{Frac} R_{n-1} = K(X_1, \dots, X_{n-1})$, whence $\mathfrak{p}' = (0)$. Assume by induction that $i \geq 1$ and the inequality holds for all primes with smaller height. Note that $\mathfrak{p}' \cap R_{n-1} = \mathfrak{p}$.

Let $\mathfrak{q} \subsetneq \mathfrak{p}$ with $\mathfrak{q} \in \operatorname{Spec} R_{n-1}$ and $\operatorname{ht} \mathfrak{q} = i-1$. Let W be a valuation overring of R_{n-1} centered on $\mathfrak{q}$; let $\mathfrak{q}'$ be the center of $W(X_n)$ in R_n. To show that $\mathfrak{q}' \subseteq \mathfrak{p}'$, it suffices by Proposition 2.8 to show that for any $\varphi \in K[X_1, \dots, X_n]$ with $1/\varphi \in \mathfrak{q}'$, we have $1/\varphi \in \mathfrak{p}'$. Write $\varphi = \sum_{j=0}^{e} f_j X_n^j$, where $f_j \in K[X_1, \dots, X_{n-1}$. Since $1/\varphi \in \mathfrak{q}'$, there is some $0 \leq k \leq e$ with $1/f_k \in \mathfrak{q}$, by the way the valuation on $W(X_n)$ is defined. Thus, $1/f_k \in \mathfrak{p}$, so $v^*(1/\varphi) = -\min\{v(f_j) \mid 0 \leq j \leq e\} \geq -v(f_k) > 0$. Hence $\mathfrak{q}' \subseteq \mathfrak{p}'$. On the other hand $\mathfrak{q}' \neq \mathfrak{p}'$, since for any $\alpha \in \mathfrak{p} \setminus \mathfrak{q}$, we have $\alpha \in \mathfrak{p}' \setminus \mathfrak{q}'$. Thus, $\mathfrak{q}' \subsetneq \mathfrak{p}'$, so that

$$\operatorname{ht} \mathfrak{p}' \geq 1 + \operatorname{ht} \mathfrak{q}' \geq 1 + (i-1) = i,$$

with the second inequality by the inductive hypothesis.

Suppose $i = n-1$. Since $X_n \notin \mathfrak{p}'$, we have that $\mathfrak{p}'$ is not the maximal ideal of R_n, so that $\operatorname{ht} \mathfrak{p}' \leq n-1$. But also $\operatorname{ht} \mathfrak{p}' \geq \operatorname{ht} \mathfrak{p} = n-1$, so that $\operatorname{ht} \mathfrak{p}' = n-1$.

Finally, suppose $i = n-2$. Let $\mathfrak{m}$ be the maximal ideal of R_{n-1}. Since $\mathfrak{m}$ contains all nonunits of R_{n-1} and $\operatorname{ht} \mathfrak{m} = n-1$, there is some $\alpha \in \mathfrak{m} \setminus \mathfrak{p}$. Thus $\alpha \notin \mathfrak{p}'$. But since $V(X_n) \supseteq (R_n)_Q$ by Lemma 6.3, we have $\mathfrak{p}' \subseteq Q$. Moreover, the containment must be strict, since $\alpha \in \mathfrak{m} \subseteq Q$ but $\alpha \notin \mathfrak{p}'$. Thus, $\operatorname{ht} \mathfrak{p}' \leq n-2$, so $\operatorname{ht} \mathfrak{p}' = \operatorname{ht} \mathfrak{p} = n-2$. □

Lemma 6.5. *Let $\mathfrak{p}$ be a prime ideal of R_{n-1} of height $n-2$. Let V a valuation ring of R_{n-1} centered on $\mathfrak{p}$. Let κ_V be the residue field of V and let $\pi : V(X_n) \twoheadrightarrow \kappa_V(X_n)$ be the canonical surjection. Let $W := \pi^{-1}(\kappa_V[X_n^{-1}]_{(X_n^{-1})})$. Then W is a valuation overring of R_n centered on a prime ideal $\mathfrak{a}$ with $\operatorname{ht} \mathfrak{a} = n-1$, such that $\mathfrak{a} \cap R_{n-1} = \mathfrak{p}$.*

Proof. We have that W is a valuation ring with quotient field $K(X_1, \dots, X_n)$ by [Bastida and Gilmer 1973, Theorem 2.1(h)]. If G is the value group of V, then $G \oplus \mathbb{Z}$, ordered lexicographically, is the value group of W. In particular, given $\varphi =$

$\sum_{j=0}^{e} f_j X_n^j \in K[X_1, \dots, X_n]$, with each f_j in $K[X_1, \dots, X_{n-1}]$, the valuation w is given by $w(\varphi) = (v(f_k), -k)$, where k is the largest index i with $0 \le i \le e$ such that $v(f_i) \le v(f_j)$ for all $0 \le j \le e$. Then $w(1/\varphi) = (-v(f_k), k) \ge (0, 0)$ since $1/f_k \in R_{n-1} \subseteq V$, whence $v(f_k) \le 0$.

Set $\mathfrak{a} := \mathfrak{m}_W \cap R_n$ and $\mathfrak{p}' := \mathfrak{m}_{V(X_n)} \cap R_n$. By standard pullback results, the maximal ideal of $V(X_n)$ is a nonmaximal prime of W; thus $\mathfrak{p}' \subseteq \mathfrak{a}$. On the other hand, $X_n^{-1} \in \mathfrak{a} \setminus \mathfrak{p}'$, so that $\mathfrak{p}' \subsetneq \mathfrak{a}$. Since $\mathfrak{p}$ is a nonmaximal ideal of R_{n-1}, there is some nonunit α of R_{n-1} (hence also of R_n) that avoids $\mathfrak{p}$. We have $w(\alpha) = (v(\alpha), 0) = (0, 0)$, so that $\alpha \notin \mathfrak{a}$. Thus, $\mathfrak{p}' \subsetneq \mathfrak{a} \subsetneq \mathfrak{m}_{R_n}$, so that since $\operatorname{ht} \mathfrak{p}' = n - 2$ by Lemma 6.4, we have $\operatorname{ht} \mathfrak{a} = n - 1$.

Now, $\mathfrak{p} = \mathfrak{p}' \cap R_{n-1} \subseteq \mathfrak{a} \cap R_{n-1}$. Hence, $\operatorname{ht}(\mathfrak{a} \cap R_{n-1}) \ge \operatorname{ht} \mathfrak{p} = n - 2$. But $\alpha \in \mathfrak{m}_{R_{n-1}} \setminus \mathfrak{a}$, so $\operatorname{ht}(\mathfrak{a} \cap R_{n-1}) = n - 2$, whence $\mathfrak{a} \cap R_{n-1} = \mathfrak{p}$. □

Theorem 6.6. *For every* $1 \le i \le n - 1$, *there exist infinitely many primes of* R_n *of height* i.

Proof. When $n = 0, 1$, the statement is vacuous. Moreover, since when $n \ge 2$ we know that R_n has infinitely many height-one primes by Proposition 6.1, the result holds for $n = 2$. Thus, we assume inductively that $n > 2$ and the result holds for smaller n. Since $(R_n)_Q = R(K(X_n)[X_1, \dots, X_{n-1}])$ (see Notation 6.2), it has infinitely many primes of height i for $1 \le i \le n - 2$, which then restrict to distinct primes of these heights in R_n via the localization map. So we need only show that R_n has infinitely many primes of height $n - 1$.

Let $\mathfrak{p}$ and $\mathfrak{q}$ be distinct prime ideals of height $n-2$ in R_{n-1}. By Lemma 6.5, there are valuation overrings W_1 and W_2 of R_n whose centers in R_n are height $n-1$ primes $\mathfrak{p}'$ and $\mathfrak{q}'$ such that $\mathfrak{p}' \cap R_{n-1} = \mathfrak{p}$ and $\mathfrak{q}' \cap R_{n-1} = \mathfrak{q}$. Since $\mathfrak{p} \ne \mathfrak{q}$, it follows that $\mathfrak{p}' \ne \mathfrak{q}'$. Since there are infinitely many primes of height $n - 2$ in R_{n-1} by the inductive hypothesis, it follows that there are infinitely many primes of height $n - 1$ in R_n. □

7. The dimension 2 case

In this section, we work in two variables, so that $D = K[X, Y]$ where $X = X_1$, $Y = X_2$ for short, $R = R(K[X, Y])$, $F = K(X, Y)$, etc. We have done likewise in many results earlier in the paper in service of extending the results to higher dimensions. However, for each of the results in this section, either we do not know how to extend it into higher dimension, or else we know it to be false in higher dimension. In the dimension 2 case, we will show that R has an integral overring that is not finitely generated, that localizing R at height-one primes always yields Noetherian domains, and that any finitely generated proper ideal lives in almost all height-one primes.

We start by expanding Theorem 5.8 to show that the integral closure if R^* is quite a bit larger than R^* itself:

Proposition 7.1. *There is an overring S of $R = R_2$, that is integral over R but not finitely generated over it.*

Proof. As usual, we will work with R^* instead of R.

Let $\Sigma := \{\beta_{p,q}\}$ as in the proof of Theorem 5.8, where the pairs (p, q) range over all relatively prime pairs of integers $1 < p < q$ such that neither p nor q is a multiple of $\operatorname{char} K$. Let $S := R^*[\Sigma]$. Then, as seen in the proof of Theorem 5.8, each $\beta_{p,q}$ is in the integral closure of R^*. Hence, S is integral over R^*.

Suppose that S is finitely generated as an R^*-algebra. Then there is a finite list of such pairs $\{(p_i, q_i)\}_{1\le i\le s}$ such that $S = R^*[\beta_{p_1,q_1}, \dots, \beta_{p_s,q_s}]$. Choose $r > \max\{p_i \mid 1 \le i \le s\}$ such that $r > 1$. Then $(r, r+1)$ is such a pair, so $\beta_{r,r+1} \in S$. Let $v = v_{r,r+1,r^2+r+1}$.

Then for any relatively prime pair (p, q) with $p < r$, we claim that $v(X^q - Y^p) = \min\{v(X^q), v(Y^p)\}$. If this were not the case, we would have $rq = v(X^q) = v(Y^p) = (r+1)p$, so that $(q-p)r = p$, contradicting the facts that $q - p \ge 1$ and $r > p$. Thus, $v(X^q - Y^p) = \min\{rq, (r+1)p\}$.

Now choose any $(p, q) = (p_i, q_i)$ with $1 \le i \le s$. Let (c, d) be the unique pair of integers with $0 < c < q$, $0 < d < p$, and $qd = pc + 1$. Then

$$\begin{aligned} v(\beta_{p,q}) = v\left(\frac{X^{2q-c}Y^d}{X^q - Y^p}\right) &= (2q-c)r + d(r+1) - \min\{rq, (r+1)p\} \\ &\ge (2q-c)r + d(r+1) - rq = (q-c)r + (r+1)d \ge 2r+1. \end{aligned}$$

On the other hand, by the proof of Theorem 5.8, we have $v(\beta_{r,r+1}) = 1$.

Let $Z_1, \dots, Z_s$ be algebraically independent indeterminates over R^*. It follows that if $g \in R^*[Z_1, \dots, Z_s]$ such that $\beta_{r,r+1} = g(\beta_{p_1,q_1}, \dots, \beta_{p_s,q_s})$, then g has a nonzero constant term c, and $1 = v(\beta_{r,r+1}) = v(c)$. But this contradicts the fact (see Lemma 5.5) that every element of R^* has value either 0 or $\ge r$ under v. Thus, S is not finitely generated as an R^*-algebra. □

Our main result for dimension 2 shows that the localizations at height-one primes are surprisingly well behaved. First, though, we need the following lemma.

Lemma 7.2. *Let $\mathfrak{p} \in \operatorname{Spec} R_n$ and $1 \le i \le n$ such that $\mathfrak{p} \not\subseteq \mathfrak{q}_i$, where $\mathfrak{q}_i$ is as in Lemma 4.3. Then $1/X_i \in \mathfrak{p}$.*

The $\mathfrak{q}_i$ are mutually incomparable, and for each i, $1/X_j \in \mathfrak{q}_i$ for each $j \neq i$.

If $n = 2$, then any nonzero prime distinct from $\mathfrak{q}_1$ and $\mathfrak{q}_2$ contains $(1/X_1, 1/X_2)R$. Hence with the notation of Lemma 3.6, $\mathfrak{p}_1 = \mathfrak{q}_2$ and $\mathfrak{p}_2 = \mathfrak{q}_1$.

Proof. By Lemma 4.3, any prime ideal avoiding X_i^{-1} must be contained in $\mathfrak{q}_i$. Thus $1/X_i \in \mathfrak{p}$.

Now let i and j be distinct integers between 1 and n. Combining Lemmas 4.1 and 4.3 and Theorem 4.4 yields $\operatorname{ht} \mathfrak{q}_i = \operatorname{ht} \mathfrak{q}_j = n - 1$. Thus, $\mathfrak{q}_i$ and $\mathfrak{q}_j$ must be incomparable. Since $\mathfrak{q}_i \not\subseteq \mathfrak{q}_j$, we have $X_j \in \mathfrak{q}_i$ by the first paragraph.

In the $n = 2$ case, by Theorem 4.4 we have that any nonmaximal nonzero prime has height one. In particular, if $\mathfrak{p}$ is distinct from $\mathfrak{q}_1$ and $\mathfrak{q}_2$, then since $\operatorname{ht}\mathfrak{p} = 1 = \operatorname{ht}\mathfrak{q}_1 = \operatorname{ht}\mathfrak{q}_2$, we have $\mathfrak{p} \not\subseteq \mathfrak{q}_i$ for $i = 1, 2$. Since $1/X_i \in \mathfrak{p}_i \setminus \mathfrak{q}_i$ for $i = 1, 2$, the final claim follows. □

Theorem 7.3. *Let $\mathfrak{p}$ be a height-one prime ideal of $R = R_2$. Then $R_{\mathfrak{p}}$ is a Noetherian one-dimensional local domain.*

Proof. By Lemma 3.6, we may assume $\mathfrak{p} \neq \mathfrak{p}_1, \mathfrak{p}_2$. We work in R^* and use the notation $D = K[X, Y]$ and $R^* = \sigma(D)$. Since $\mathfrak{p}$ is not maximal there must exist $f \in (X, Y)D$ irreducible and not associate to either X or Y in D, such that $\alpha := \sigma(1/f^*) \notin \mathfrak{p}$. As neither α nor α^{-1} are in $S = D_{(X,Y)}$ it follows from [Seidenberg 1953, Theorem 7] that $S[\alpha]$ is a two-dimensional Noetherian ring having a height-one prime ideal $\mathfrak{q}$ generated by X and Y. Hence $S[\alpha]_{\mathfrak{q}} = D[\alpha]_{(X,Y)}$ is a one-dimensional local Noetherian domain.

Moreover, since $S \subseteq R^*$, we get $S[\alpha] \subseteq R^*$. Let us show that $\mathfrak{p} \cap S[\alpha] = \mathfrak{q}$. By Lemma 7.2, $X, Y \in \mathfrak{p}$, so that $\mathfrak{q} \subseteq \mathfrak{p} \cap S[\alpha]$. For the reverse containment, it will be enough to show that $\mathfrak{p} \cap S[\alpha]$ is a height-one prime, for which it will suffice to show it is not a maximal ideal.

Every maximal ideal of $S[\alpha]$ that contains $\mathfrak{q}$ is of the form $(\mathfrak{q}, h(\alpha))$ where $h \in K[T]$ is an irreducible monic polynomial. This is because by [Seidenberg 1953, Theorem 7], we have $S[\alpha]/\mathfrak{q} \cong K[T]$, where T is an indeterminate over K and $\overline{2}2\alpha \mapsto T$ in the isomorphism. We know that $\alpha \notin \mathfrak{p}$. If $h(T) \neq T$, then it has a nonzero constant term, so by Theorem 3.3, $h(\alpha)$ is a unit in R^*, whence $h(\alpha) \notin \mathfrak{p}$. On the other hand, $\alpha \notin \mathfrak{p}$ by assumption.

Thus $\mathfrak{p} \cap S[\alpha] = \mathfrak{q}$, so that $S[\alpha]_{\mathfrak{q}} \subseteq R^*_{\mathfrak{p}}$. By the Krull–Akizuki theorem [Matsumura 1986, Theorem 11.7], $R^*_{\mathfrak{p}}$ is Noetherian, finishing the proof. □

Remark 7.4. Suppose, in the setting of Theorem 7.3, that there exists $f \in (X, Y)D \setminus (X, Y)^2 D$ such that $1/f^* \notin \mathfrak{p}$. Then $R_{\mathfrak{p}}$ is not merely one-dimensional and Noetherian, but a DVR.

To see this, and continuing the notation in the proof above, let Z be an indeterminate over D and let $\pi : K[X, Y, Z] \to D[\alpha]$ be the unique K-algebra homomorphism that fixes D and sends $Z \mapsto \alpha$. Note that π is surjective. Since $D[\alpha]$ has dimension 2, the kernel of π is a height-one prime of $K[X, Y, Z]$. It is clear that the polynomial $h = fZ - X^{a_1(f)} Y^{a_2(f)}$ is irreducible in $K[X, Y, Z]$ and contained in the kernel of π. It follows that

$$D[\alpha] \cong \frac{K[X, Y, Z]}{(h)}.$$

The ring

$$S[\alpha]_{\mathfrak{q}} \cong \left(\frac{K[X, Y, Z]}{(h)}\right)_{(X,Y)}$$

is a DVR if and only if h is a regular parameter in $K[X, Y, Z]_{(X,Y)}$. This happens if and only if $f \in (X, Y)D \setminus (X, Y)^2 D$. In case $S[\alpha]_{\mathfrak{q}}$ is a DVR, we clearly have $S[\alpha]_{\mathfrak{q}} = R^*_{\mathfrak{p}}$ since a DVR has no proper overring other than its fraction field.

Next we show a "cofinite character" result that is somewhat dual to the finite character property of Krull domains.

Theorem 7.5. *Every finitely generated proper ideal of $R = R_2$ is contained in all but finitely many prime ideals.*

Proof. Every nonunit element $\varphi \in R^*$ can be written as a finite sum $\varphi = \varphi_1 + \cdots + \varphi_t$ such that any φ_j is a finite product of elements of the form $\sigma(1/f)$ with f irreducible in $K[X, Y]$. Thus it is sufficient to prove that any $\sigma(1/f)$ with f irreducible is contained in all but finitely many primes of R^*. Since we already know this fact for X and Y by Lemma 3.7, we can assume f is not an associate of X nor Y. Set $\alpha = \sigma(1/f)$. We know that α is in the maximal ideal of R^*. By the proof of Theorem 7.3, we get that if a prime $\mathfrak{p}$ of R^* does not contain α, then $R^*_{\mathfrak{p}}$ contains the one-dimensional Noetherian local domain $D[\alpha]_{(X,Y)}$. Suppose there exist two distinct nonzero prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of R^* with $\alpha \notin \mathfrak{p} \cup \mathfrak{q}$. Necessarily $\mathfrak{p}$ and $\mathfrak{q}$ have height one. We show that $R^*_{\mathfrak{p}}$ and $R^*_{\mathfrak{q}}$ cannot be contained in a common valuation overring. Suppose by way of contradiction that $R^*_{\mathfrak{p}} \cup R^*_{\mathfrak{q}} \subseteq V$ for some valuation ring V contained in $K(X, Y)$. Then V is an overring of $D[\alpha]_{(X,Y)}$, hence a DVR. Since $R^*_{\mathfrak{p}}$ and $R^*_{\mathfrak{q}}$ are one-dimensional, the maximal ideal $\mathfrak{m}_V$ of V contains both the maximal ideals of $R^*_{\mathfrak{p}}$ and $R^*_{\mathfrak{q}}$. Therefore the intersection $R^*_{\mathfrak{p}} \cap R^*_{\mathfrak{q}}$ is local. For this observe that given two nonunits $\beta, \theta \in R^*_{\mathfrak{p}} \cap R^*_{\mathfrak{q}}$, we have $\beta, \theta \in \mathfrak{p}R^*_{\mathfrak{p}} \cup \mathfrak{q}R^*_{\mathfrak{q}} \subseteq \mathfrak{m}_V$ and hence their sum $\beta + \theta \in \mathfrak{m}_V \cap R^*_{\mathfrak{p}} \cap R^*_{\mathfrak{q}} \subseteq \mathfrak{p}R^*_{\mathfrak{p}} \cap \mathfrak{q}R^*_{\mathfrak{q}}$ is a nonunit in $R^*_{\mathfrak{p}} \cap R^*_{\mathfrak{q}}$. Moreover, we have $R^* \subseteq R^*_{\mathfrak{p}} \cap R^*_{\mathfrak{q}} \subseteq R^*_{\mathfrak{p}}$. Because $\alpha^{-1} \in (R^*_{\mathfrak{p}} \cap R^*_{\mathfrak{q}}) \setminus R^*$, the maximal ideal of $R^*_{\mathfrak{p}} \cap R^*_{\mathfrak{q}}$ must contract to a nonzero prime ideal of R^* not containing α. But any such ideal has height one, so that $R^*_{\mathfrak{p}} \cap R^*_{\mathfrak{q}}$ contains the localization of R^* at some height-one prime. This is a contradiction since two localizations at distinct height-one primes cannot be comparable with respect to inclusion. Hence, $R^*_{\mathfrak{p}}$ and $R^*_{\mathfrak{q}}$ cannot have a common valuation overring.

Suppose there are infinitely many distinct primes of R^* not containing α. Then the above implies that $D[\alpha]_{(X,Y)}$ has infinitely many valuation overrings, but it is clearly a contradiction since $D[\alpha]_{(X,Y)}$ is local, Noetherian and one-dimensional. □

Recall that in a domain S, with fraction field F, for any S-submodule I of F we can write $I^{-1} := \{x \in F \mid xI \subseteq S\}$. Then for any ideal I of S, we set $I_v := (I^{-1})^{-1}$ and $I_t := \bigcup\{J_v \mid J \subseteq I \text{ and } J \text{ is finitely generated}\}$. A *t-ideal* is then an ideal I such that $I = I_t$. If there is a unique maximal element among the proper t-ideals of S, we say S is *t-local*. See [Fontana and Zafrullah 2019].

Corollary 7.6. *The ring $R = R_2$ is t-local.*

Proof. In fact the unique maximal ideal $\mathfrak{m}$ of R is a t-ideal. To see this, let I be a finitely generated proper ideal of R. Then by Theorem 7.5, I is contained in some height-one prime ideal $\mathfrak{p}$. But $\mathfrak{p}$ is a t-ideal by [Elliott 2019, top of p. 23]. Hence, $I_t \subseteq \mathfrak{p}_t = \mathfrak{p} \subset \mathfrak{m}$. Thus, $\mathfrak{m}_t = \mathfrak{m}$. □

8. Questions

The study of reciprocal complements is an entirely new field of inquiry. There are many interesting questions one could ask about this particular R, or about reciprocal complements in general. The following are just some questions that occurred to these authors, but such questions are easy to generate. As seen below, some of these questions have had progress on them since they were first proposed in an earlier draft of this paper.

Question 1. What can be said about the integral closure of R, where R is the reciprocal complement of a polynomial ring in two or more variables over a field? In particular:

(a) Is the integral closure of R infinitely generated over R? (We can't conclude this from Proposition 7.1 since R is not a Noetherian ring, so finitely generated R-modules can have infinitely generated submodules.)

(b) Is the integral closure of R local? If not, is it at least semilocal?

(c) Is the integral closure of R completely integrally closed?

Question 2. Let R be the reciprocal complement of a polynomial ring in finitely many variables over a field. Is R a *strong Bézout intersection domain* (SBID) (see [Guerrieri and Loper 2021])? That is, is it true that every finite intersection of non-comparable principal ideals fails to be finitely generated?

Some positive evidence is given by Theorem 5.6, and also by Corollary 7.6, since any SBID is t-local.

Question 3. Let D be a Noetherian domain. Is $R(D)$ a G-domain?

For some evidence of this, see Proposition 2.9. More generally, by [Guerrieri 2025, Theorem 2.12], the above holds whenever $\dim R(D) < \infty$.

Question 4. Let D be an integral domain of dimension ≥ 2. Assume that any nonzero Egyptian element of D is a unit. Must $R(D)$ be non-Noetherian?

We have seen in Corollary 5.7 that $D = D_n$ is an example of the above phenomenon when $n \geq 2$. More generally, by [Guerrieri 2025, Corollary 2.9], the answer is yes whenever $\dim R(D) \geq 2$.

Question 5. For any integral domain D, must we have $\dim R(D) \leq \dim(D)$?

Note that there is no hope for *equality* in the above, as we have for $D = D_n$ by Theorem 4.4. Indeed, the quantity $\varphi(D) := \dim(D) - \dim R(D)$ can be any nonnegative value ω, by letting A be a Jaffard (e.g., Noetherian) Egyptian domain of dimension ω and $D = A[X_1, \ldots, X_d]$; then by Proposition 2.2 and Theorem 4.4, $\dim R(D) = d$, but $\dim D = d + \dim A$, so $\varphi(D) = \dim A = \omega$. Moreover, one can make A have any dimension ω by letting $G = \mathbb{Z}^{\oplus \omega}$ and $A = K[G]$ for any field K, which is Egyptian by [Guerrieri et al. 2024, Proposition 3].

By [Guerrieri 2025, Theorem 5.5], for any nonnegative integer c, one can construct integral domains D all of whose Egyptian elements are units, such that $\varphi(D) = c$.

The above question has a positive answer when D is finitely generated over a field or falls into certain classes of semigroup algebras [Guerrieri 2025, Theorem 3.2, Remark 4.11].

Question 6. Let D be a Noetherian integral domain with $\dim D = n \geq 2$. Are there infinitely many prime ideals of $R(D)$ of height i for each $1 \leq i \leq n - 1$?

We see an example of this phenomenon when $D = D_n$ by Theorem 6.6. If D is not restricted to be Noetherian, however, there are counterexamples [Guerrieri 2025, Theorem 4.2].

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