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# ON $A$-PACKETS CONTAINING UNITARY LOWEST-WEIGHT REPRESENTATIONS OF $U(p,q)$

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**We determine all the Arthur packets containing an irreducible unitary lowest-weight representation $\pi$ of a real unitary group $G = U(p,q)$, including nonscalar cases. Our methods are the Barbasch–Vogan parametrization of representations of $G$ and Trapa's algorithm to calculate the cohomological inductions. In particular, we show that an Arthur packet has at most one irreducible unitary lowest-weight representation of $G$. As a consequence, if an irreducible unitary lowest-weight representation $\pi$ exists in the Arthur packet of $\psi$, we give an explicit formula for the lowest $K$-type of $\pi$.**

## 1. Introduction

One of the fundamental problems in number theory is to investigate arithmetic properties of holomorphic cusp forms on Hermitian symmetric spaces and the geometry of Shimura varieties. For instance, in the case of Siegel modular forms, the detailed analysis of Fourier coefficients and zeta integrals leads to arithmeticity of standard $L$-values and the cohomology of Siegel modular varieties. In contrast, holomorphic cusp forms associated with unitary groups remain less understood despite their importance.

Recently, there has been significant progress in Arthur's endoscopic classification of automorphic representations on classical groups, and now it is possible to study the automorphic forms systematically. When we try to apply these advances to the study of holomorphic cusp forms on Hermitian symmetric spaces, we face problems in the local representation theory. A key local question is how to classify the local Arthur packets containing a given unitary lowest-weight representation. In this paper, we completely determine the local Arthur packets containing a given unitary lowest-weight representation for unitary groups using the Barbasch–Vogan parametrization of irreducible admissible representations.

To state the main theorem, we recall lowest-weight representations and Mœglin and Renard's description of Arthur packets. Let $G = U(p,q)$ with $N = p+q$ and $K$ be the maximal compact subgroup of $G$. With the usual choice of positive roots, the

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highest weights of irreducible representations of $K$ are $(\lambda_1,\dots,\lambda_p,\lambda_{p+1},\dots,\lambda_N)\in\mathbb{Z}^N$ satisfying

$$\lambda_1\geq\cdots\geq\lambda_p,\quad \lambda_{p+1}\geq\cdots\geq\lambda_N.$$

We denote by $\lambda$ the irreducible representation of $K$ with highest-weight $\lambda$ for short. For each irreducible representation $\lambda$ of $K$, there exists a unique irreducible lowest-weight representation $\pi_\lambda$ with the lowest $K$-type $\lambda$. For an irreducible representation $\pi$, let $\chi_\pi$ be the infinitesimal character of $\pi$.

Let us recall the Arthur classification for $G$. The Arthur classification associates the $A$-parameters $\psi$ with a finite set $\Pi(\psi)$, called the Arthur packet (or $A$-packet) for $\psi$, consisting of unitary representations of finite length of $G$. The $A$-parameters are equivalence classes of $N$-dimensional representations of $\mathbb{C}^\times\times\mathrm{SL}_2(\mathbb{C})$ with suitable properties. The $A$-packet satisfies several properties that can be easily investigated by $\psi$. For example, all the representations in $\Pi(\psi)$ have the same infinitesimal characters $\chi_\psi$, and their Harish-Chandra parameter is given by the exponents in the representation $\mathbb{C}^\times\to\mathrm{GL}_N(\mathbb{C})$ defined by $z\mapsto\psi\bigl(z,\mathrm{diag}((z/\bar{z})^{1/2},(z/\bar{z})^{-1/2})\bigr)$. When the corresponding Harish-Chandra parameter for $\psi$ is integral, the parameter $\psi$ is called good or good parity. The good $A$-parameters $\psi$ of $G$ can be viewed as a formal sum

$$\psi=\bigoplus_{i=1}^{r}\chi_{t_i}\otimes S_{a_i}$$

such that $\sum_i a_i=N$, $S_m$ is the $m$-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ and $t_i+a_i+N\in2\mathbb{Z}$, where $\chi_t$ is the character of $\mathbb{C}^\times$ defined by $\chi_t(z)=z^{t/2}\bar{z}^{-t/2}$. Suppose $t_i\geq t_{i+1}$ and $a_i\geq a_{i+1}$ if $t_i=t_{i+1}$. Put

$$\mathcal{D}(\psi)=\Bigl\{(p_i,q_i)\in(\mathbb{Z}_{\geq0}\times\mathbb{Z}_{\geq0})^r\mid p_i+q_i=a_i,\ \sum_{i=1}^{r}p_i=p,\ \sum_{i=1}^{r}q_i=q\Bigr\}.$$

For each $\underline{d}\in\mathcal{D}(\psi)$, we will attach a cohomological induction $\mathscr{A}_{\underline{d}}(\psi)=A_{\mathfrak{q}(x_{\underline{d}})}(\mu_{\underline{d}})$; see (4-3). By [Mœglin and Renard 2019, Théorème 1.1], the $A$-packet $\Pi(\psi)$ is equal to the set

$$\Pi(\psi)=\{\mathscr{A}_{\underline{d}}(\psi)\mid\underline{d}\in\mathcal{D}(\psi)\}.$$

For a good $A$-parameter $\psi=\bigoplus_{i=1}^{r}\chi_{t_i}\otimes S_{a_i}$, let $j=j(\psi)$ be the minimal number $i$ so that $\sum_{\ell=1}^{i}a_\ell\geq p$. Put $a_{<i}=\sum_{\ell<i}a_\ell$ and $a_{>i}=\sum_{\ell>i}a_\ell$. Define $\underline{d}_0\in\mathcal{D}(\psi)$ by

$$\underline{d}_0=\underline{d}_0(\psi)=\{(a_1,0),\dots,(a_{j-1},0),(p_j,q_j),(0,a_{j+1}),\dots,(0,a_r)\}$$

where $p_j=p-a_{<j}$, $q_j=q-a_{>j}$. Let $\nu_i$ be the segment $\bigl[\frac{1}{2}(t_i-a_i+1),\frac{1}{2}(t_i+a_i-1)\bigr]$ and

$$\nu_{<i}=\bigsqcup_{k<i}\nu_k,\quad \nu_{>i}=\bigsqcup_{i<k}\nu_k.$$

Here, we consider the union as multisets. The multisets $\nu_{\leq i}$ and $\nu_{\geq i}$ are defined similarly.

For an irreducible representation $\lambda = (\lambda_1, \dots, \lambda_N)$ of $K$, put

$$p' = p'(\lambda) = \#\{i \mid \lambda_p = \lambda_i,\ 1 \leq i \leq p\}, \quad q' = q'(\lambda) = \#\{i \mid \lambda_i = \lambda_{p+1},\ p+1 \leq i \leq N\}.$$

Set

$$P = P(\lambda) = \left\{\lambda_p - \tfrac{1}{2}(N-1), \lambda_{p-1} - \tfrac{1}{2}(N-1) + 1 \dots, \lambda_1 + \tfrac{1}{2}(p-q-1)\right\},$$
$$Q = Q(\lambda) = \left\{\lambda_N + \tfrac{1}{2}(p-q+1), \lambda_{N-1} + \tfrac{1}{2}(p-q+3), \dots, \lambda_{p+1} + \tfrac{1}{2}(N-1)\right\}.$$

The multiset $P \sqcup Q$ can be identified with the infinitesimal character of the lowest-weight representation $\pi_\lambda$. We define the segments $P'$ and $Q'$ by

$$P' = \left[\lambda_p - \tfrac{1}{2}(N-1), \lambda_p - \tfrac{1}{2}(N+1) + p'\right],$$
$$Q' = \left[\lambda_{p+1} + \tfrac{1}{2}(N+1) - q', \lambda_{p+1} + \tfrac{1}{2}(N-1)\right].$$

Put $I = P' \cap Q'$.

**Lemma 1.1** (Lemma 4.6). *Let $\psi = \bigoplus_{i=1}^{r} \chi_{t_i,s} \otimes S_{a_i}$ be an A-parameter. If $\Pi(\psi)$ contains an irreducible lowest-weight representation $\pi$, the parameter $\psi$ is good and $\chi_\psi = \chi_\pi$. Moreover, if $\mathscr{A}_{\underline{d}}(\psi) \in \Pi(\psi)$ is nonzero and lowest weight, there exists $j$ such that $q_i = 0$ for any $i < j$ and $p_\ell = 0$ for any $\ell > j$, i.e., $\underline{d} = \underline{d}_0$.*

As a consequence of this lemma, there exists at most one unitary lowest-weight representation in $\Pi(\psi)$. We now state the main theorem of the present paper.

**Theorem 1.2** (Theorem 4.7). *Let $\lambda = (\lambda_1, \dots, \lambda_N)$ be an irreducible representation of $K$ and $\pi_\lambda$ be the irreducible lowest-weight representation with lowest $K$-type $\lambda$. Suppose that $\psi$ is a good A-parameter such that $\chi_\psi = \chi_{\pi_\lambda}$ and $\mathscr{A}_{\underline{d}_0}(\psi)$ is nonzero.*

(1) *If $N - p' \leq \lambda_p - \lambda_{p+1} < N - q'$, the packet $\Pi(\psi)$ contains $\pi_\lambda$ if and only if $\left[\lambda_p - \tfrac{1}{2}(N-1), \lambda_{p+1} + \tfrac{1}{2}(N-1)\right] \subset \nu_j \subset P'$.*

(2) *If $N - q' \leq \lambda_p - \lambda_{p+1} < N - p'$, the packet $\Pi(\psi)$ contains $\pi_\lambda$ if and only if either*

- *$\nu_{\leq j} = P$, or*
- *$\left[\lambda_p - \tfrac{1}{2}(N-1), \lambda_{p+1} + \tfrac{1}{2}(N-1)\right] \subset \nu_j \subset Q'$.*

(3) *If $N - p', N - q' \leq \lambda_p - \lambda_{p+1}$, the packet $\Pi(\psi)$ contains $\pi_\lambda$ if and only if either*

- *$P \subset \nu_{\leq j} \subset P \sqcup I$, or*
- *$I \subset \nu_j \subset Q'$.*

(4) *If $\lambda_p - \lambda_{p+1} < N - p', N - q'$, the packet $\Pi(\psi)$ contains $\pi_\lambda$ if and only if $\left[\lambda_p - \tfrac{1}{2}(N-1), \lambda_{p+1} + \tfrac{1}{2}(N-1)\right] = \nu_j$.*

Conversely, we have the following:

**Theorem 1.3** (Corollary 4.8). *Let $\psi = \bigoplus_{i=1}^{r} \chi_{t_i} \otimes S_{a_i}$ be a good A-parameter. The packet $\Pi(\psi)$ contains a nonzero unitary lowest-weight representation if and only if both*

- *$\nu_{<j}$ and $\nu_{>j}$ are multiplicity free, and*
- *$\#(\nu_j \cap \nu_{>j}) \leq p_j$ and $\#(\nu_j \cap \nu_{<j}) \leq q_j$.*

*When $\Pi(\psi)$ contains a nonzero unitary lowest-weight representation $\pi$ in $\Pi(\psi)$, the lowest K-type $\lambda$ of $\pi$ is given as follows*:

(1) *When $q_j = 0$, the lowest K-type $\lambda$ of $\pi$ satisfies $P(\lambda) = \nu_{\leq j}$ and $Q(\lambda) = \nu_{>j}$.*

(2) *When $p_j = \#(\nu_j \cap \nu_{>j})$ and $q_j \neq 0$, the lowest K-type $\lambda$ of $\pi$ satisfies $P(\lambda) = \nu_{<j} \sqcup (\nu_j \cap \nu_{>j})$ and $Q(\lambda) = \nu_{\geq j} \setminus (\nu_j \cap \nu_{>j})$.*

(3) *When $q_j = \#(\nu_j \cap \nu_{<j}) \neq 0$, the lowest K-type $\lambda$ of $\pi$ satisfies*

$$P(\lambda) = \nu_{\leq j} \setminus (\nu_{<j} \cap \nu_j) \quad \text{and} \quad Q(\lambda) = (\nu_j \cap \nu_{<j}) \sqcup \nu_{>j}.$$

(4) *When $p_j \neq \#(\nu_j \cap \nu_{>j})$ and $q_j \neq \#(\nu_j \cap \nu_{<j})$, put $\nu_{<j} \sqcup \nu_{>j} = \{\sigma_1, \dots, \sigma_{N-\#(\nu_j)}\}$. Let $i_0$ be the minimal integer such that*

$$1 \leq i_0 \leq N - \#(\nu_j) \quad \text{and} \quad \#(\nu_j) - i_0 + 1 + \#\{x \in \nu_{<j} \sqcup \nu_{>j} \mid x > \nu_{j,i_0}\} = p.$$

*Then, the lowest K-type $\lambda = (\lambda_1, \dots, \lambda_N)$ of $\pi$ is given by*

$$\lambda_i = \begin{cases} \sigma_i - \frac{1}{2}(p-q+1) + i & \text{if } i < p - \#(\nu_j) + i_0, \\ \nu_{j,1} + \frac{1}{2}(N+1) - \#(\nu_j) & \text{if } p - \#(\nu_j) + i_0 \leq i \leq p, \\ \nu_{j,1} - \frac{1}{2}(N-1) & \text{if } p+1 \leq i \leq p + i_0 - 1, \\ \sigma_{i-\#(\nu_j)} - \frac{1}{2}(N+1) - p + i & \text{if } p + i_0 \leq i. \end{cases}$$

To conclude the introduction, we give some remarks and one application of the present paper. In the proof of the main theorem, we do not calculate the $K$-types of cohomological inductions except for special cases. Our proof is based on the Barbasch–Vogan parametrization of representations of $G$. This parametrization says that for any irreducible representation $\pi$, the map $\pi \mapsto (\mathrm{Ann}(\pi), \mathrm{AS}(\pi))$ is injective, where $\mathrm{Ann}(\pi)$ is the annihilator and $\mathrm{AS}(\pi)$ is the asymptotic support. The invariants $\mathrm{Ann}(\pi)$ and $\mathrm{AS}(\pi)$ can be described as certain tableaux in our case. Trapa [2001] gave an algorithm to compute such invariants for cohomological inductions $A_{\mathfrak{q}}(\mu)$. We calculate the tableaux and investigate the conditions where the cohomological inductions $A_{\mathfrak{q}}(\mu)$ are isomorphic to a given $\pi_\lambda$. When $G$ is not a unitary group, the problem becomes complicated, and there are at least two difficulties that do not occur in the unitary group case: one, the description of unipotent $A$-parameters and two, the reducibility of $A_{\mathfrak{q}}(\mu)$ in the weakly fair range.

Finally, the results of this paper have been applied to the birational geometry of Shimura varieties of $U(1, n)$ in [Horinaga et al. 2025]. In the study of the Kodaira dimension of Shimura varieties, it is crucial to show that the existence of low-weight cusp forms, i.e., cusp forms whose weights are less than that of the discrete series. The method is based on Arthur's multiplicity formula, and the result in the present paper plays a vital role.

## 2. Unitary groups and representations

Here, we review definitions of unitary groups and representations. We also recall the Barbasch–Vogan parametrization of the representations, which is key to our study.

**2.1.** ***Tableau notation.*** By a segment, we mean a set of the form $\{a, a+1, \dots, a+n\}$, say $[a, a+n]$, for a real number $a$ and $n \in \mathbb{Z}_{\geq 0}$. In this paper, the segments are always regarded as multisets, that is, sets with multiplicities. For segments $\nu_1 = [a, b]$ and $\nu_2 = [c, d]$, we say that $\nu_1$ and $\nu_2$ are linked (resp. $\nu_1 \leq \nu_2$) if either $c = b+1$ or $a = d+1$ (resp. $a \leq c$ and $b \leq d$) holds.

For a partition $n = n_1 + \cdots + n_\ell$ with $n_1 \geq \cdots \geq n_\ell$, we have a diagram with $n_i$ boxes in the $i$-th row. This diagram is called a Young diagram of size $n = n_1 + \cdots + n_\ell$. If $\nu = (\nu_1, \dots, \nu_n)$ is an $n$-tuple of real numbers, a $\nu$-quasitableau is defined as a tableau such that the shape is a Young diagram of size $n$ and the entries are an arrangement of $\nu_1, \dots, \nu_n$. For a $\nu$-quasitableau $T$, we say that $T$ is $\nu$-antitableau if entries strictly decrease down along each column and weakly decrease along each row. This definition is the same as the definition of the semistandard tableau by replacing "decreasing" with "increasing."

We define a $(p, q)$-signed tableau as an equivalence class of Young diagrams whose boxes are $p$ plus boxes and $q$ minus boxes so that the signs alternate across the row. Here, we say that two signed tableaux are equivalent if the signatures $(p, q)$ are the same and coincide by interchanging rows of the same length. For a tableau $T$ with entries, we say that the $(a, b)$-th entry of $T$ is the entry in the $a$-th row and the $b$-th column.

The definitions above are those in the previous works [Huang 2025; Chengyu 2025; Trapa 2001] on the nonvanishing of cohomological inductions. See these references for examples of tableaux.

**2.2.** ***Unitary groups.*** For a Lie group $H$, we denote by $\mathfrak{h}$ (resp. $\mathfrak{h}_{\mathbb{C}}$) the Lie algebra of $H$ (resp. complexification of $\mathfrak{h}$) and by $\mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ the universal enveloping algebra of $\mathfrak{h}_{\mathbb{C}}$. Fix a positive integer $N$ with a partition $N = p + q$ and $p, q \geq 0$. We define the unitary group $G = U(p, q)$ by

$$G = U(p, q) = \{g \in \mathrm{GL}_N(\mathbb{C}) \mid {}^t\bar{g} I_{p,q} g = I_{p,q}\}, \quad I_{p,q} = \begin{pmatrix} \mathbb{1}_p & \\ & -\mathbb{1}_q \end{pmatrix}.$$

Here, $\bar{g}$ is the complex conjugate of $g$. For a Cartan involution $\theta : g \mapsto \mathrm{Ad}(I_{p,q})({}^t\bar{g}^{-1})$, let $K$ be the group of the fixed points of $\theta$. Then, $K$ is a maximal compact subgroup of $G$, which is isomorphic to $U(p) \times U(q)$. The Cartan involution $\theta$ induces an involution $\theta$ on $\mathfrak{g}$ and a decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-.$$

Here, $\mathfrak{p}$ is the $(-1)$-eigenspace of $\theta$ on $\mathfrak{g}$, and $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) corresponds to the holomorphic (resp. antiholomorphic) tangent space of a Hermitian symmetric space $G/K$. Let $T$ be the diagonal subgroup of $G$. Then, $T$ is a Cartan subgroup of $G$ and $K$. Define $e_i \in \mathfrak{t}_{\mathbb{C}}^*$ by $e_i(\mathrm{diag}(t_1, \dots, t_N)) = t_i$. We regard $\mathfrak{t}_{\mathbb{C}}^*$ as $\mathbb{C}^N$ by the basis $\{e_1, \dots, e_N\}$. Then, the root system $\triangle$ of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$ is equal to

$$\triangle = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq N\}.$$

We choose a positive system $\triangle^+$ as

$$\triangle^+ = \{e_i - e_j \mid 1 \leq i < j \leq N\}.$$

Let $\triangle_c$ (resp. $\triangle_n$) be the compact (resp. noncompact) root system with the positive system $\triangle_c^+ = \triangle_c \cap \triangle^+$ (resp. $\triangle_n^+ = \triangle_n \cap \triangle^+$), explicitly,

$$\triangle_c = \{\pm(e_i - e_j) \mid 1 \leq i < j \leq p \text{ or } p+1 \leq i < j \leq N\}$$

and

$$\triangle_n = \{\pm(e_i - e_j) \mid 1 \leq i \leq p < j \leq N\}.$$

Then, the root system of $\mathfrak{p}_+$ associated with $\mathfrak{t}_{\mathbb{C}}$ is $\triangle_n^+$. Let $\mathfrak{b}$ be the Borel subalgebra of $\mathfrak{g}_{\mathbb{C}}$ associated with $\triangle^+$ and $\mathfrak{b}^-$ be the opposite of $\mathfrak{b}$. For a Lie subalgebra $\mathfrak{u}$ of $\mathfrak{g}_{\mathbb{C}}$ stable under the adjoint action of $\mathfrak{t}_{\mathbb{C}}$, let $\rho(\mathfrak{u})$ be half the sum of roots in $\mathfrak{u}$. Put $\rho = \rho(\mathfrak{b})$.

For $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}^N$ with $\lambda_1 \geq \cdots \geq \lambda_p$ and $\lambda_{p+1} \geq \cdots \geq \lambda_N$, let $F(\lambda)$ be an irreducible representation of $K$ with the highest-weight $\lambda$. We often write $F(\lambda)$ as $\lambda$, for short. The restriction of an admissible $(\mathfrak{g}, K)$-module $\pi$ to $K$ is decomposed as a direct sum

$$\pi|_K = \bigoplus_{\lambda} F(\lambda)^{\oplus m_\pi(\lambda)}, \quad m_\pi(\lambda) \in \mathbb{Z}_{\geq 0},$$

where $\lambda$ runs over all $\triangle_c^+$-dominant integral weights. The nonnegative integer $m_\pi(\lambda)$ is the multiplicity of $F(\lambda)$ in $\pi$. By a $K$-type of $\pi$, we mean $F(\lambda)$ with $m_\pi(\lambda) \neq 0$.

The restriction of $\pi$ to $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ defines a character $\chi_\pi$ of $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$. The character $\chi_\pi$ is called the infinitesimal character of $\pi$. By the Harish-Chandra isomorphism, infinitesimal characters are parametrized by $\mathfrak{t}_{\mathbb{C}}^*/W$, where $W = W(G; T)$ is the Weyl group of $G$ for $T$. For $\lambda \in \mathfrak{t}_{\mathbb{C}}^*/W$, let $\chi_\lambda$ (or $\lambda$ for short) be the corresponding character of $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$. We say that an infinitesimal character $\nu$ is integral if $\nu$ is in the image of $\mathbb{Z}^N + \rho$. In this paper, we regard integral infinitesimal characters as multisets with $N$-elements or an element $(x_1, \dots, x_N)$ in $\mathbb{Z}^N + \frac{1}{2}(N-1)$ with $x_1 \geq \cdots \geq x_N$.

**2.3.** ***Unitary lowest-weight representations.*** For a $(\mathfrak{g}, K)$-module $\pi$, we say that $\pi$ is lowest-weight if there exists $v \in \pi$ such that $v$ generates $\pi$ and $v$ is annihilated by $\mathfrak{b}^-$. Let $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}^N \subset \mathfrak{t}_{\mathbb{C}}^*$ be a $\Delta_c^+$-dominant integral weight. We regard the irreducible representation $F(\lambda)$ of $K$ as an irreducible $\mathfrak{p}_- \oplus \mathfrak{k}_{\mathbb{C}}$ module by letting $\mathfrak{p}_-$ act trivially. Set

$$N(\lambda) = \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathcal{U}(\mathfrak{p}_- \oplus \mathfrak{k}_{\mathbb{C}})} F(\lambda).$$

The module $N(\lambda)$ is called the parabolic Verma module and has the unique irreducible quotient $L(\lambda)$ by [Humphreys 2008, §9.4]. By [Enright et al. 1983, Theorem 2.4] the module $L(\lambda)$ is unitarizable if $\lambda_p - \lambda_{p+1} \geq N - p' - q'$ where $p' = p'(\lambda) = \#\{i \mid \lambda_i = \lambda_p,\ i \leq p\}$ and $q' = q'(\lambda) = \#\{i \mid \lambda_i = \lambda_{p+1},\ p+1 \leq i \leq N\}$. In particular, $L(\lambda)$ is a discrete series representation if $\lambda_p - \lambda_{p+1} > N - 1$, and is limits of discrete series if $\lambda_p - \lambda_{p+1} = N - 1$. The infinitesimal character of $\pi_\lambda$ equals

$$\left(\lambda_1 + \tfrac{1}{2}(p-q-1), \dots, \lambda_p - \tfrac{1}{2}(N-1), \lambda_{p+1} + \tfrac{1}{2}(N-1), \dots, \lambda_N + \tfrac{1}{2}(p-q+1)\right).$$

Note that the infinitesimal character $\chi_{\pi_\lambda}$ is integral by $\lambda \in \mathbb{Z}^N$.

**2.4.** ***Barbasch–Vogan parametrization of representations of G.*** For the details of this subsection, we refer to [Trapa 2001, §4,5; Barbasch and Vogan 1983]. We introduce two invariants $\mathrm{Ann}(\pi)$ and $\mathrm{AS}(\pi)$ associated to $(\mathfrak{g}, K)$-modules $\pi$. For a $(\mathfrak{g}, K)$-module $\pi$, let $\mathrm{Ann}(\pi)$ be the annihilator of $\pi$ in $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$. When $\pi$ is irreducible, the ideal $\mathrm{Ann}(\pi)$ is called a primitive ideal. Primitive ideals for an irreducible representation with infinitesimal character $\nu$ are parametrized by $\nu$-antitableau. The asymptotic support $\mathrm{AS}(\pi)$ is defined via the local behavior of the character of $\pi$. By definition, $\mathrm{AS}(\pi)$ is a union of nilpotent orbits of $\mathfrak{g}_{\mathbb{C}}$.

**Theorem 2.1** [Barbasch and Vogan 1983, Theorem 4.2; Trapa 2001, Theorem 6.1]. *For irreducible $(\mathfrak{g}, K)$-modules $\pi$ and $\pi'$ with integral infinitesimal characters, the representation $\pi$ is isomorphic to $\pi'$ if and only if*

$$(\mathrm{Ann}(\pi), \mathrm{AS}(\pi)) = (\mathrm{Ann}(\pi'), \mathrm{AS}(\pi')).$$

Trapa and Vogan [Trapa 2001, Conjecture 1.1] conjectured that the cohomological inductions in the weakly fair range exhaust the unitary $(\mathfrak{g}, K)$-modules with integral infinitesimal characters. As far as the author knows, the conjecture has been proven only for specific cases such as $U(n, 1)$ and $U(n, 2)$ (see [Wong and Zhang 2024] for details).

## 3. Cohomological inductions

In this section, we introduce the cohomological inductions $A_{\mathfrak{q}}(\mu)$, recall their basic properties, review Trapa's algorithm to determine the tableaux for $A_{\mathfrak{q}}(\mu)$, and state a nonvanishing criterion for certain $A_{\mathfrak{q}}(\mu)$.

**3.1.** ***θ-stable parabolic subalgebras and cohomological induction.*** Take $x \in \sqrt{-1}\mathfrak{k}$. Since the action $\mathrm{ad}(x)$ on $\mathfrak{g}_{\mathbb{C}}$ is diagonalizable with real eigenvalues, we define the subalgebras of $\mathfrak{g}_{\mathbb{C}}$ by

$$\mathfrak{q} = \mathfrak{q}(x) = \text{sum of root vectors with nonnegative eigenvalues,}$$
$$\mathfrak{u} = \mathfrak{u}(x) = \text{sum of root vectors with positive eigenvalues,}$$
$$\mathfrak{l} = \mathfrak{l}(x) = \text{sum of root vectors with zero eigenvalues.}$$

We call parabolic subalgebras $\mathfrak{q}$ of $\mathfrak{g}_{\mathbb{C}}$ obtained in the above way $\theta$-stable parabolic subalgebras of $\mathfrak{g}_{\mathbb{C}}$. By a conjugate of an element in $K$, we may assume $x \in \sqrt{-1}\,\mathfrak{t}$. For $\mu \in \mathfrak{t}_{\mathbb{C}}^*$, let $\mathbb{C}_\mu$ denote the character of $\mathfrak{l}$ with $\mathbb{C}_\mu|_{\mathfrak{t}_{\mathbb{C}}} = \mu$, if it exists. We then obtain the cohomological induction $A_{\mathfrak{q}}(\mu)$ by the induction of $\mathbb{C}_\mu$ as in [Knapp and Vogan 1995, (5.6)].

For $G = U(p, q)$, $\theta$-stable parabolic subalgebras $\mathfrak{q}(x)$ arise from $\underline{d} = \{(p_i, q_i)_{1 \le i \le r}\}$ and an element $x_{\underline{d}}$ in $\sqrt{-1}\,\mathfrak{t} \cong \mathbb{R}^N$ of the form

$$x_{\underline{d}} = (\underbrace{x_1, \dots, x_1}_{p_1}, \dots, \underbrace{x_r, \dots, x_r}_{p_r}, \underbrace{x_1, \dots, x_1}_{q_1}, \dots, \underbrace{x_r, \dots, x_r}_{q_r}), \quad x_1 > \dots > x_r, \tag{3-1}$$

such that $(p_i, q_i) \in (\mathbb{Z}_{\ge 0})^2$ with $(p_i, q_i) \ne (0, 0)$ for any $i$ and $p_1 + \dots + p_r = p$, $q_1 + \dots + q_r = q$. Set $\mathfrak{q}_{\underline{d}} = \mathfrak{q}(x_{\underline{d}})$. The centralizer $C_G(\mathfrak{q}_{\underline{d}})$ is a connected reductive group $L_{\underline{d}}$ isomorphic to

$$L_{\underline{d}} \cong U(p_1, q_1) \times \dots \times U(p_r, q_r)$$

such that $\mathfrak{l}_{\underline{d}} = \mathfrak{l}(x_{\underline{d}})_{\mathbb{C}} = \mathrm{Lie}(L_{\underline{d}}) \otimes \mathbb{C}$. Note that our choice of $\mathfrak{q}_{\underline{d}}$ or the hermitian form $I_{p,q}$ is different from that of [Mœglin and Renard 2019; Trapa 2001; Vogan 1997], but the same as [Ichino 2022]. We say that a $\theta$-stable parabolic subalgebra $\mathfrak{q}$ is holomorphic if there exists $j$ such that $q_i = 0$ for any $i < j$ and $p_\ell = 0$ for any $\ell > j$, in other words, $\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}} \subset \mathfrak{p}_+$. The choices for a holomorphic $\theta$-stable parabolic subalgebra are the same as [Trapa 2001; Vogan 1997].

**3.2.** ***Properties of $A_{\mathfrak{q}}(\mu)$.*** Here, we review the basic properties of $A_{\mathfrak{q}}(\mu)$. Let $\underline{d} = \{(p_i, q_i)_{1 \le i \le r}\}$ be a set of pairs of nonnegative integers with $(p_i, q_i) \ne 0$, $\sum_i p_i = p$, and $\sum_i q_i = q$. Take $\mu \in \mathbb{Z}^N$ such that

$$\mu = (\underbrace{\mu_1, \dots, \mu_1}_{a_1}, \dots, \underbrace{\mu_r, \dots, \mu_r}_{a_r}), \quad a_i = p_i + q_i.$$

We define the segments $\nu_i$ associated with $\underline{d}$ and $\mu$ by

$$\nu_i = \nu_i(\underline{d}, \mu) = \left[\mu_i + \tfrac{1}{2}(N+1) - a_{\le i},\ \mu_i + \tfrac{1}{2}(N-1) - a_{<i}\right]. \tag{3-2}$$

Here, $a_{<i} = \sum_{\ell < i} a_\ell$ and $a_{\le i} = \sum_{\ell \le i} a_\ell$. We denote the cohomological induction $A_{\mathfrak{q}_{\underline{d}}}(\mu)$ by $A(\mathfrak{q}_{\underline{d}}, \nu_1, \dots, \nu_r) = A(\underline{d}, \nu_1, \dots, \nu_r)$ for short. We say that the cohomological induction $A_{\mathfrak{q}_{\underline{d}}}(\mu)$ is in the weakly fair range (resp. mediocre range) if $\mu_i-\mu_{i+1}\geq -\frac{1}{2}(a_i+a_{i+1})$ for any $i$ (resp. $\mu_i-\mu_j\geq -\max\{a_i,a_j\}-\sum_{i<k<j}a_k$ for any $i<j$). We also say that $\nu=\mu+\rho=\bigsqcup_i \nu_i$ is in the weakly fair range (resp. mediocre range) if $A_{\mathfrak{q}_{\underline{d}}}(\mu)$ is so.

The cohomological induction $A_{\mathfrak{q}_{\underline{d}}}(\mu)$ is in the weakly fair range (resp. mediocre range) if and only if the mean value in $\nu_i$ is greater than or equal to the mean value in $\nu_{i+1}$ (resp. $\nu_i\not\succeq\nu_j$ for any $i<j$) by the explicit calculation (see [Chengyu 2025, Lemma 2.4]).

The following statements are well known (for example, see [Knapp and Vogan 1995; Adams 1987, Lemma 4.2; Huang and Pandžić 2006, Theorem 6.4.4; Trapa 2001, Theorem 3.1 (iv)]):

- In the weakly fair range, $A_{\mathfrak{q}}(\mu)$ is unitarizable. Moreover, it is zero or irreducible for $G=U(p,q)$.
- For $A_{\mathfrak{q}}(\mu)$ in the mediocre range,

$$\dim \operatorname{Hom}_K(F(\lambda),A_{\mathfrak{q}}(\mu))=\sum_{w\in W^1}\operatorname{sgn}(w)P_{\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}}\big(w(\lambda+\rho_c)-(\mu+2\rho(\mathfrak{u}\cap\mathfrak{p})+\rho_c)\big)$$

holds. Here, $W^1$ is the subgroup of the Weyl group of $K$ with respect to $T$ consisting of $w$ for which $\alpha\in\triangle_c^+$, $w^{-1}\alpha<0$ implies $\alpha\in\triangle(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})$, and $P_{\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}}$ is the partition function with respect to $\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}$, i.e., $P_{\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}}(x)$ is the multiplicity of weight $x$ in the symmetric algebra $S(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})^{\mathfrak{l}\cap\mathfrak{k}\cap\mathfrak{n}}$.

- If $\mu+2\rho(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})$ is $\triangle_c^+$-dominant, $\mu+2\rho(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})$ occurs in $A_{\mathfrak{q}}(\mu)$ and $K$-types in $A_{\mathfrak{q}}(\mu)$ are of the form

$$\mu+2\rho(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})+\sum_{\alpha\in\triangle(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})}n_\alpha\alpha,\quad n_\alpha\geq 0. \tag{3-3}$$

Calculating the $K$-type formula directly seems complicated, even when $\mathfrak{q}$ is a holomorphic $\theta$-stable parabolic subalgebra. To avoid such complexity, we will calculate the tableaux described below instead.

**3.3. *Tableaux associated with $A_{\mathfrak{q}}(\mu)$.*** Let $\pi$ be a cohomological induction in the weakly fair range. We associate two invariants $\operatorname{Ann}(\pi)$ and $\operatorname{AS}(\pi)$ to $\pi$ in Section 2.4. The annihilator $\operatorname{Ann}(\pi)$ can be regarded as a $\nu$-antitableau, but $\operatorname{AS}(\pi)$ is a union of unipotent orbits. If $\pi$ is isomorphic to a cohomological induction $A_{\mathfrak{q}}(\mu)$ in the weakly fair range, the asymptotic support $\operatorname{AS}(\pi)$ is a single unipotent orbit by [Trapa 2001, Proposition 5.4]. Indeed, for $A_{\mathfrak{q}}(\mu)$ in the good range, its asymptotic support is a single unipotent orbit. Since we can obtain $\pi$ as a translation of cohomological inductions in the good range, one can show that the asymptotic supports coincide. Hence, $\operatorname{AS}(A_{\mathfrak{q}}(\mu))$ is a single unipotent orbit. Thus, we may associate a $\nu$-antitableau and a $(p,q)$-signed tableau for a cohomological induction in the weakly fair range by [Collingwood and McGovern 1993, Theorem 9.3.3]. The examples of tableaux associated with $A_{\mathfrak{q}}(\mu)$ are available in [Huang 2025; Chengyu 2025; Trapa 2001].

In the following, we associate tableaux for $A_{\mathfrak{q}}(\mu)$. Let $\mathfrak{q}$ be the $\theta$-stable parabolic subalgebra corresponding to $\{(p_i, q_i)_{1\le i\le r}\}$ and $\mu$ be a $\Delta_c^+$-dominant integral weight such that $A_{\mathfrak{q}}(\mu)$ is in the mediocre range. We first construct the $(p, q)$-signed tableau inductively. Let $S_1$ be the Young diagram of size $1 + \cdots + 1$ with $p_1 + q_1$ boxes filled with $p_1$ plus boxes and $q_1$ minus boxes. Suppose that the $\left(\sum_{i<k} p_i, \sum_{i<k} q_i\right)$-signed tableau $\bigsqcup_{i<k} S_i$ for $k \le r$ is defined. We then construct the signed tableau $\bigsqcup_{i\le k} S_i$ by adding $p_k$ boxes filled with $+$ and $q_k$ boxes filled with $-$, from top to bottom, to each row-ends of $\bigsqcup_{i<k} S_i$ such that

- at most one box is added in each row-end, and
- the signs in $\bigsqcup_{i\le k} S_i$ are alternating across the row.

Then, $\bigsqcup_{i\le k} S_i$ is defined as a Young diagram with decreasing rows by rearranging the rows. The resulting tableau $S = \bigsqcup_{i\le r} S_i$ is the asymptotic support of $A_{\mathfrak{q}}(\mu)$.

We next construct the $\nu$-antitableau. For $A_{\mathfrak{q}}(\mu)$, the shape of $(p, q)$-signed tableau and the $\nu$-antitableau are the same. The shape $S$ of the $(p, q)$-signed tableau is partitioned into $\bigsqcup_{1\le i\le r} S_i$, which is the same as in the definition of the $(p, q)$-signed tableau. For each $S_i$, we fill $\nu_{i,1}, \ldots, \nu_{i,a_i}$ from top to bottom, where $\nu_i = \nu_i(\underline{d}, \mu) = \{\nu_{i,1}, \ldots, \nu_{i,a_i}\}$ with $\nu_{i,1} > \cdots > \nu_{i,a_i}$. Then, $S$ is a $\nu$-quasitableau. When the $S$ is a $\nu$-antitableau, let $\operatorname{Ann}(A_{\mathfrak{q}}(\mu)) = S$, which is possibly equivalent to the formal zero tableau explained below. We introduce the two invariants $\operatorname{overlap}(S_i, S_{i+1})$ and $\operatorname{sing}(S_i, S_{i+1})$ associated with cohomological inductions. Set

$$\operatorname{sing}(S_i, S_{i+1}) = \#(\nu_i \cap \nu_{i+1}).$$

For $S_i$ and $S_{i+1}$ as in the definition of $(p, q)$-signed tableau $S = \bigsqcup_i S_i$, let $m$ be the largest integer such that for any $i$ with $i \le m$, the $(a_i - m + i)$-th box of $S_i$ is strictly left to the $i$-th box of $S_{i+1}$. If such an integer $m$ does not exist, set $m = 0$. We then define $\operatorname{overlap}(S_i, S_{i+1})$ by the nonnegative integer $m$. In the following, we give an algorithm [Trapa 2001, Procedure 7.5] to obtain a $\nu$-antitableau or the formal zero tableau from the tableau $S$. By [Trapa 2001, Theorem 7.9], the cohomological induction $A_{\mathfrak{q}}(\mu)$ is zero if and only if the tableau $S$ is equivalent to the formal zero tableau. More precisely, let $S' = \bigsqcup_i S'_i$ be the tableau after Trapa's algorithm. Let $\nu'_i$ be the segment consisting of the entries of $S'_i$. The cohomological induction $A_{\mathfrak{q}}(\mu)$ is nonzero if and only if the resulting tableau $S' = \bigsqcup_i S'_i$ satisfies

- $\nu_i \ge \nu_{i+1}$, and
- $\operatorname{overlap}(S'_i, S'_{i+1}) \ge \operatorname{sing}(S'_i, S'_{i+1})$ for any $i$.

We give Trapa's algorithm to transform the tableau to $\nu$-antitableau or the formal zero tableau. For $S = \bigsqcup_i S_i$, the algorithm is generated by replacing adjacent skew columns $S_i \sqcup S_{i+1}$ with $S'_i \sqcup S'_{i+1}$. Set $R = S_i \sqcup S_{i+1}$ and $R' = S'_i \sqcup S'_{i+1}$. When $R'$ is the formal zero tableau, we understand that the tableau $S$ is equivalent to the formal zero tableau. We review the construction of $R'$, which is an arrangement of $\nu_i \sqcup \nu_{i+1}$ with the shape $R$.

(1) If $\mathrm{overlap}(S_i, S_{i+1}) = \mathrm{sing}(S_i, S_{i+1}) < \min\{a_i, a_{i+1}\}$ or $\mathrm{overlap}(S_i, S_{i+1}) > \mathrm{sing}(S_i, S_{i+1})$, set $R' = R$. Here, $a_i = \#(\nu_i)$.

(2) If $\mathrm{overlap}(S_i, S_{i+1}) < \mathrm{sing}(S_i, S_{i+1})$, then $R'$ is the formal zero tableau.

(3) Assume $\mathrm{overlap}(S_i, S_{i+1}) = \mathrm{sing}(S_i, S_{i+1}) = a_{i+1}$. In this case, $\nu_{i+1} \subset \nu_i$. We define $R'$ by induction on $m = \nu_{i+1,a_{i+1}} - \nu_{i,a_i}$. When $m = 0$, set $R' = R$. When $m > 0$, set $\nu_{i+1}(-) = \{\nu_{i+1,1} - 1, \ldots, \nu_{i+1,a_{i+1}} - 1\}$. We define $S_{i+1}(-)$ by the tableau with the shape $S_{i+1}$ filled with $\nu_{i+1}(-)$. Set $R(-) = S_i \sqcup S_{i+1}(-)$. By the induction hypothesis, $R(-)'$ is defined. Then, in $R(-)'$, there exists at most one box $B$ filled with $\nu_{i+1,1} - 1$ and strictly to the right of the unique box filled with $\nu_{i+1,1}$ in $R(-)'$. If the box $B$ exists, add one to the entry in $B$. If no such box exists, add one to the entry in the left-most box filled with $\nu_{i+1,1} - 1$ in $R(-)'$. We denote the resulting tableau by $R(-)'_1$. Now construct $R(-)'_2$ by the same procedure applied to $R(-)'_1$, but instead considering the entries $\nu_{i+1,2}$ and $\nu_{i+1,2} - 2$. By the same procedure, we get $R(-)'_{a_{i+1}}$. Set $R' = R(-)'_{a_{i+1}}$.

(4) Assume $\mathrm{overlap}(S_i, S_{i+1}) = \mathrm{sing}(S_i, S_{i+1}) = a_i$. In this case $\nu_i \subset \nu_{i+1}$. We define $R'$ by induction on $m = \nu_{i+1,1} - \nu_{i,1}$. When $m = 0$, set $R' = R$. When $m > 0$, set $\nu_i(+) = \{\nu_{i,1} + 1, \ldots, \nu_{i,a_i} + 1\}$. We define the tableau $S_i(+)$ by the tableau with the shape $S_i$ filled with $\nu_i(+)$. Set $R(+) = S_i(+) \sqcup S_{i+1}$. By the induction hypothesis, $R(+)'$ is defined. Then, in $R(+)'$, there exists at most one box $B$ filled with $\nu_{i,a_i} + 1$ and strictly to the left of the unique box filled with $\nu_{i,a_i}$ in $R(+)'$. If the box $B$ exists, subtract one from the entry in the box $B$. If no such box exists, subtract one from the entry in the right-most box filled with $\nu_{i,a_i} + 1$ in $R(+)'$. We denote the resulting tableau by $R(+)'_1$. Now construct $R(+)'_2$ by the same procedure applied to $R(+)'_1$, by the same procedure again, we get $R(+)'_{a_i}$. Set $R' = R(+)'_{a_i}$.

We still need to consider a partition of the resulting tableau $R'$ into $R' = S'_i \sqcup S'_{i+1}$. The last box in $R'$ is the right-most box filled with $\nu'_{i+1,a_{i+1}} = \min\{\nu_{i,a_i}, \nu_{i+1,a_{i+1}}\}$. The box next to the last box is the right-most box filled with $\nu_{i+1,a_{i+1}} + 1$. This procedure stops when the entry of the box reaches $\min\{\nu_{k,1}, \nu_{k+1,1}\}$. Let $S'_i$ be the tableau of the remaining boxes. We then obtain the partition $R' = S'_i \sqcup S'_{i+1}$.

If interested, the author recommends to calculate examples and check the well-definedness of the above definition. The explicit formula for the overlap is investigated in [Chengyu 2025, Theorem 3.6; Huang 2025, Lemma 5.4]. For the explicit description of entries in the tableau $R'$, see [Huang 2025, §4.5].

**3.4.** ***Unitary lowest-weight representations as cohomological inductions.*** In this subsection, we determine a cohomological induction that is isomorphic to a given lowest-weight representation $\pi_\lambda$. The following is one of the easiest cases in which to calculate the $K$-types.

**Lemma 3.1.** *Let $\mathfrak{q}_{\underline{d}}$ be a holomorphic $\theta$-stable parabolic subalgebra corresponding to $\underline{d} = \{(p_i, q_i)_i\}$ such that $p_i q_i = 0$ for any $i$ and $\pi = A_{\mathfrak{q}_{\underline{d}}}(\mu)$ be a cohomological induction in the mediocre range. Put $j = \max\{i \mid q_i = 0\}$. Then, $\pi$ is nonzero if and only if the multisets $\nu_{\leq j}$ and $\nu_{>j}$ are multiplicity free. If $\pi$ is nonzero, then $\pi$ is a unitary lowest-weight representation with the lowest $K$-type*

$$\mu + 2\rho(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}) = \mu + (\underbrace{q, \ldots, q}_{p}, \underbrace{-p, \ldots, -p}_{q}).$$

*Moreover, let*

$$\sigma_i = \begin{cases} \mu_i + \frac{1}{2}(p - q + 1) - i & \text{if } 1 \leq i \leq p, \\ \mu_i + \frac{1}{2}(N+1) - (i - p) & \text{if } p + 1 \leq i \leq N. \end{cases}$$

*We denote by $i_0$ the maximal positive integer such that $\sigma_{p+\min\{p,q\}+1-i_0} \geq \sigma_{p+1-i_0}$, if it exists. If there is no such integer, put $i_0 = 0$. Then, the first column of the tableau* $\mathrm{Ann}(\pi)$ *consists of*

$$\sigma_1, \sigma_2, \ldots, \sigma_{p-i_0}, \underbrace{\sigma_{p+\min\{p,q\}+1-i_0}, \ldots, \sigma_{p+\min\{p,q\}}}_{i_0}, \sigma_{p+\min\{p,q\}+1}, \ldots, \sigma_N$$

*and the second column consists of*

$$\sigma_{p+1}, \ldots, \sigma_{p+\min\{p,q\}-i_0}, \underbrace{\sigma_{p+1-i_0}, \ldots, \sigma_p}_{i_0}.$$

*Proof.* Before applying the algorithm, the first column of the tableau $\mathrm{Ann}(\pi)$ consists of

$$\underbrace{\sigma_1, \ldots, \sigma_p}_{p}, \underbrace{\sigma_{p+\min\{p,q\}+1}, \ldots, \sigma_N}_{q-\min\{p,q\}}$$

and the second column consists of

$$\sigma_{p+1}, \ldots, \sigma_{p+\min\{p,q\}}$$

from top to bottom. To apply Trapa's algorithm, we use the partition $\mathrm{AS}(\pi) = \bigsqcup_i S_i$. Let $\nu_i$ be the segment defined in (3-2). Then, $\nu_i = [\sigma_{a_{\leq i}}, \sigma_{a_{<i}+1}]$ and the tableaux $S_i$ are filled with $\nu_i$. By the algorithm for the partition $\bigsqcup_{i \leq j} S_i$ and $\bigsqcup_{i>j} S_i$, the representation $\pi$ is zero if $\nu_{\leq j}$ or $\nu_{>j}$ is not multiplicity free.

Suppose that $\nu_{\leq j}$ and $\nu_{>j}$ are multiplicity free. By the algorithm, the diagram $\bigsqcup_i S_i$ is invariant under the algorithm if $\nu_j \not\subset \nu_{j+1}$ and $\nu_j \not\supset \nu_{j+1}$. Consider the partition $S_j \sqcup S_{j+1}$. When $\nu_j \supset \nu_{j+1}$, the tableau $S_j$ consists only of the first column with the entries $\sigma_{p+1-a_j}, \ldots, \sigma_p$ and the tableau $S_{j+1}$ consists only of the second column with the entries $\sigma_{p+1}, \ldots, \sigma_{p+a_{j+1}}$. Hence, the tableau $S_j \sqcup S_{j+1}$ is invariant under the algorithm, but the partition $S'_j \sqcup S'_{j+1}$ is different. The tableau $S'_j$ consists only of the first column with entries $\sigma_{p+1-a_j}, \ldots, \sigma_{p+a_{j+1}}$ and the tableau $S'_{j+1}$ consists of the remaining boxes. Then, if $S'_{j+1}$ has a box in the first column, the maximal entry is $\sigma_{p+a_{j+1}} - 1$. The tableau $\left(\bigsqcup_{i \leq j-1} S_i\right) \sqcup S'_j$ is invariant under the algorithm by definition. Since the entry $\sigma_{p+a_{j+1}} - 1$ is greater than or equal to the maximal entry in $S_{j+2}$, the tableau $S'_{j+1} \sqcup S_{j+2}$ is invariant under the algorithm. Continuing the same procedure, one obtains the resulting tableau $\mathrm{Ann}(\pi)$. This shows that $i_0 = 0$ and the representation $\pi$ is nonzero. The lowest $K$-type of $\pi$ is given as (3-3), since $\mu + 2\rho(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}})$ is $\Delta_c^+$-dominant.

When $\nu_j \subset \nu_{j+1}$, we first assume that $a_{j+1} > \min\{p, q\}$. By the same procedure as above, if $\sigma_p > \sigma_{p+\min\{p,q\}+1}$, the tableau $S'_j \sqcup S'_{j+1}$ is the same as $S_j \sqcup S_{j+1}$ and $i_0 = 0$. The first column of the tableau $S'_j$ consists of $\sigma_{p+1-a_j}, \ldots, \sigma_p$ and the second column consists of $\sigma_{p+1}, \ldots, \sigma_{p+1-a_j} + 1$. Since $\nu_{\leq j}$ is multiplicity free, we have $\sigma_{p-a_j} \geq \sigma_{p+1-a_j}$. The tableau $S_{j-1} \sqcup S'_j$ is stable under the algorithm. Similarly, $S'_{j+1} \sqcup S_{j+2}$ is stable. Hence, the tableau $\bigsqcup_i S_i$ is stable under the algorithm. If $\sigma_p \leq \sigma_{p+\min\{p,q\}+1}$, the first column of the tableau $S'_j \sqcup S'_{j+1}$ consists of

$$\underbrace{\sigma_{p+\min\{p,q\}-a_j+1}, \ldots, \sigma_{p+\min\{p,q\}}}_{a_j}, \underbrace{\sigma_{p+\min\{p,q\}+1}, \ldots, \sigma_{p+a_{j+1}}}_{a_{j+1}-\min\{p,q\}}$$

and the second column consists of

$$\underbrace{\sigma_{p+1}, \ldots, \sigma_{p+\min\{p,q\}-a_j}}_{\min\{p,q\}-a_j}, \underbrace{\sigma_{p+1-a_j}, \ldots, \sigma_p}_{a_j}.$$

The first column of $S'_j$ consists of

$$\sigma_{p+\min\{p,q\}+1-a_j}, \ldots, \sigma_p$$

and the second column consists of

$$\sigma_{p+1}, \ldots, \sigma_{p+\min\{p,q\}-a_j}.$$

By the same procedure to the end, the statement follows.

We suppose $\nu_j \subset \nu_{j+1}$ and $a_{j+1} \leq \min\{p, q\}$. Then, $i_0 = 0$. Since $\nu_{>j}$ is multiplicity free, $\sigma_p$ is greater than or equal to $\sigma_{p+\min\{p,q\}}$. In this case, the tableau $S'_j \sqcup S'_{j+1}$ is the same as $S_j \sqcup S'_{j+1}$, but the tableau $S'_{j+1}$ consists only of the second column with entries $\sigma_p + 1, \ldots, \sigma_{p+\min\{p,q\}}$. By the routine discussion, the tableau $\bigsqcup_i S_i$ is invariant under the algorithm. □

We explicitly describe $\pi_\lambda$ in terms of cohomological induction as follows:

**Lemma 3.2.** *Let $\pi_\lambda$ be a unitary lowest-weight representation with lowest $K$-type $\lambda = (\lambda_1, \dots \lambda_N)$.*

(1) *When $\lambda_p - \lambda_{p+1} < N - p', N - q'$, let $\mathfrak{q}$ be the $\theta$-stable parabolic subalgebra corresponding to*

$$\{\underbrace{(1,0), \dots, (1,0)}_{p-p'}, (p', N - (\lambda_p - \lambda_{p+1}) - p'), \underbrace{(0,1), \dots, (0,1)}_{\lambda_p - \lambda_{p+1} - p + p'}\}$$

*and*

$$\mu = (\mu_1, \dots, \mu_N)$$

*be an element in $\mathbb{Z}^N$ defined by*

$$\mu_i = \begin{cases} \lambda_i - q & \text{if } i \le p - p', \\ \lambda_{p+1} + p - p' & \text{if } p - p' < i \le N - (\lambda_p - \lambda_{p+1}) - p', \\ \lambda_i + p & \text{if } N - (\lambda_p - \lambda_{p+1}) - p' < i. \end{cases}$$

*Then, $A_{\mathfrak{q}}(\mu) \cong \pi_\lambda$.*

(2) *When $\min\{N - p', N - q'\} \le \lambda_p - \lambda_{p+1}$, let $\mathfrak{q}$ be the $\theta$-stable parabolic subalgebra corresponding to*

$$\{\underbrace{(1,0), \dots, (1,0)}_{p-p'}, (p', 0), (0, q'), \underbrace{(0,1), \dots, (0,1)}_{q-q'}\}$$

*and $\mu$ be an element in $\mathbb{Z}^N$ defined by*

$$\mu = (\underbrace{\lambda_1 - q, \dots, \lambda_p - q}_{p}, \underbrace{\lambda_{p+1} + p, \dots, \lambda_N + p}_{q}).$$

*Then, $A_{\mathfrak{q}}(\mu) \cong \pi_\lambda$.*

(3) *Let $\mathfrak{q}$ be the $\theta$-stable parabolic subalgebra corresponding to*

$$\{\underbrace{(1,0), \dots, (1,0)}_{p-p'}, (p', 0), (0, q'), \underbrace{(0,1), \dots, (0,1)}_{q-q'}\}$$

*and $\mu$ be an element in $\mathbb{Z}^N$ defined by $\mu = (\mu_1, \dots, \mu_N)$ such that $\mu_1 \ge \dots \ge \mu_p$, $\mu_{p+1} \ge \dots \ge \mu_N$, $\mu_p = \mu_{p-1} = \dots = \mu_{p-p'+1}$, and $\mu_{p+1} = \mu_{p+2} = \dots = \mu_{p+q'}$. Suppose that $A_{\mathfrak{q}}(\mu)$ is in the mediocre range. Then, $A_{\mathfrak{q}}(\mu)$ is a nonzero lowest-weight representation and the lowest $K$-type $\lambda = (\lambda_1, \dots, \lambda_N)$ of $A_{\mathfrak{q}}(\mu)$ satisfies*

$$\min\{N - p'(\lambda), N - q'(\lambda)\} \le \lambda_p - \lambda_{p+1},$$

*where $p'(\lambda) = \#\{i \mid 1 \le i \le p,\ \lambda_i = \lambda_p\}$ and $q'(\lambda) = \#\{i \mid p+1 \le i \le N,\ \lambda_i = \lambda_{p+1}\}$.*

*Proof.* We calculate the $K$-types of $A_{\mathfrak{q}}(\mu)$. For (1), we have

$$2\rho(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}}) = (\underbrace{q,\dots,q}_{p-p'}, \underbrace{\lambda_p-\lambda_{p+1}-p+p',\dots,\lambda_p-\lambda_{p+1}-p+p'}_{p'}, \underbrace{-p+p',\dots,-p+p'}_{N-(\lambda_p-\lambda_{p+1})-p')}, \underbrace{-p,\dots,-p}_{\lambda_p-\lambda_{p+1}-p+p'})$$

and

$$\mu+2\rho(\mathfrak{u}\cap\mathfrak{p}) = (\lambda_1,\dots,\lambda_p,\lambda_{p+1},\dots,\lambda_N).$$

Since $\mu+2\rho(\mathfrak{u}\cap\mathfrak{p}_{\mathbb{C}})$ is $\triangle_c^+$-dominant, the representation $A_{\mathfrak{q}}(\mu)$ is isomorphic to $\pi_\lambda$ by (3-3). The statement (2) is a restatement of Lemma 3.1. We prove (3). Since $\mathfrak{q}$ is holomorphic, $A_{\mathfrak{q}}(\mu)$ is a lowest-weight representation by Lemma 3.1. Suppose $A_{\mathfrak{q}}(\mu)\cong\pi_\lambda$. In this case, we have

$$\nu_{p-p'+1} = \bigl[\mu_p-\tfrac{1}{2}(p-q-1),\, \mu_p-\tfrac{1}{2}(p-q+1)+p'\bigr]$$

and

$$\nu_{p-p'+2} = \bigl[\mu_{p+1}-\tfrac{1}{2}(p-q-1)-q',\, \mu_{p+1}-\tfrac{1}{2}(p-q+1)\bigr].$$

Here, $\nu_i$ is the segment defined in (3-2). Since it is in the mediocre range, either

$$\mu_p-\tfrac{1}{2}(p-q+1)+p' \ge \mu_{p+1}-\tfrac{1}{2}(p-q+1)$$

or

$$\mu_p-\tfrac{1}{2}(p-q-1) \ge \mu_{p+1}-\tfrac{1}{2}(p-q-1)-q'$$

holds. This is equivalent to

$$\mu_p-\mu_{p+1}\ge -p' \quad\text{or}\quad \mu_p-\mu_{p+1}\ge -q'.$$

By (2), we have

$$\mu_p=\lambda_p-q \quad\text{and}\quad \mu_{p+1}=\lambda_{p+1}+p.$$

Then, the statement (3) follows from $p'(\lambda)\ge p'$ and $q'(\lambda)\ge q'$. □

The signed tableaux for unitary lowest-weight representations are as follows:

**Corollary 3.3.** *Let $\pi$ be a unitary lowest-weight representation. Then the signed tableau* AS$(\pi)$ *has at most two columns. If the tableau* AS$(\pi)$ *has only one column, $\pi$ is a character. If* AS$(\pi)$ *has two columns, the signs are arranged in the order of $+$ and $-$ in rows with two boxes. In particular, the tableau* AS$(\pi)$ *is uniquely determined by its shape.*

*Proof.* This follows from Lemma 3.2 and the definition of $(p,q)$-signed tableau for $A_{\mathfrak{q}}(\mu)$. □

**3.5.** ***Nonvanishing criterion for $A_{\mathfrak{q}}(\mu)$.*** For a $\theta$-stable maximal parabolic subalgebra $\mathfrak{q}$, Trapa's algorithm gives the following nonvanishing criterion for $A_{\mathfrak{q}}(\mu)$.

**Lemma 3.4** [Huang 2025, Lemma 2.6]. *Let $\mathfrak{q}$ be a $\theta$-stable parabolic subalgebra corresponding to $\underline{d} = \{(p_1, q_1), (p_2, q_2)\}$ and $\mu$ be a $\triangle_c^+$-dominant integral weight such that $A_{\mathfrak{q}}(\mu)$ is in the mediocre range. Then, $A_{\mathfrak{q}}(\mu)$ is nonzero if and only if*

$$\min\{p_1, q_2\} + \min\{q_1, p_2\} \geq \#(\nu_1 \cap \nu_2).$$

Take a holomorphic $\theta$-stable parabolic subalgebra associated to $\underline{d} = \{(p_i, q_i)_{1\leq i\leq r}\}$. Let $j = j(\underline{d})$ be the minimal integer $i$ such that $a_{\leq i} \geq p$. The cohomological inductions $A_{\mathfrak{q}}(\mu)$ for holomorphic $\theta$-stable parabolic subalgebras $\mathfrak{q}$ are lowest weight by [Adams 1987, Lemma 1.7]. We will show that the converse holds in Lemma 4.6. We can calculate the nonvanishing conditions and tableaux of such cohomological induction as follows:

**Lemma 3.5.** *Let $\mathfrak{q}_{\underline{d}}$ be a holomorphic $\theta$-stable parabolic subalgebra corresponding to $\underline{d} = \{(p_i, q_i)_{1\leq i\leq r}\}$ with $a_i = p_i + q_i$ and $\pi$ be a cohomological induction $A_{\mathfrak{q}_{\underline{d}}}(\mu)$ in the mediocre range. Then, $\pi$ is nonzero if and only if*

- *$\nu_{<j}$ and $\nu_{>j}$ are multiplicity free, and*
- *$\#(\nu_j \cap \nu_{<j}) \leq q_j$ and $\#(\nu_j \cap \nu_{>j}) \leq p_j$.*

*When $\pi$ is nonzero and $q_j \neq 0$, we have $\nu_{<j} \cap \nu_{>j} = \varnothing$. Write*

$$\nu_j = \{\nu_{j,1}, \dots, \nu_{j,a_j}\} \quad \textit{and} \quad \nu_{<j} \sqcup \nu_{>j} = \{\sigma_1, \dots, \sigma_t\}$$

*such that $\nu_{j,1} > \cdots > \nu_{j,a_j}$ and $\sigma_1 > \cdots > \sigma_t$. Put $\nu_j \cap (\nu_{<j} \sqcup \nu_{>j}) = \{\sigma_{f+1}, \dots, \sigma_g\}$. Let $m = \min\{f, q_j - \#(\nu_j \cap \nu_{<j})\}$ and let $i_0$ be the maximal integer such that $1 \leq i_0 \leq g - f$ and $\sigma_{f+i_0} \geq \nu_{j,m+i_0}$. The first column of the tableau $\operatorname{Ann}(\pi)$ consists of*

$$\sigma_1, \sigma_2, \dots, \sigma_f, \underbrace{\sigma_{f+1}, \sigma_{f+2}, \dots, \sigma_{f+i_0}}_{i_0},$$
$$\nu_{j,m+i_0+1}, \nu_{j,m+i_0+2}, \dots, \nu_{j,g}, \nu_{j,g+1}, \dots, \nu_{j,a_j}, \sigma_{f+p-m+1}, \dots, \sigma_t$$

*and the second column consists of*

$$\underbrace{\nu_{j,1}, \nu_{j,2} \dots, \nu_{j,m}}_{m}, \underbrace{\nu_{j,m+1}, \nu_{j,m+2}, \dots, \nu_{j,m+i_0}}_{i_0},$$
$$\underbrace{\sigma_{f+i_0+1}, \sigma_{f+i_0+2}, \dots, \sigma_g}_{g-i_0}, \sigma_{g+1}, \dots, \sigma_{\min\{t, f+p-m\}}$$

*from top to bottom. Here, we understand that there is no box next to the box filled with $\nu_{j,a_j}$ if $t \leq f + p - m$ in the first column. In particular, if $m \neq 0$, then the* $(1, 2)$*-th entry of* $\operatorname{Ann}(\pi)$ *is $\nu_{j,1}$.*

*Proof.* The proof is the same as Lemma 3.1. The case $q_j = 0$ follows from Lemma 3.1. When $q_j \neq 0$, we compute the tableau $\mathrm{Ann}(\pi)$. If $\nu_{<j} \cap \nu_{>j} \neq \varnothing$, take $x \in \nu_{<j} \cap \nu_{>j}$. Since any element in $\nu$ has multiplicity at most two, any element in $\nu_j$ is greater than or less than $x$. If $x$ is less than any element in $\nu$, the segment $\nu_j$ is contained in $\nu_{j-1}$, since $\pi$ is in the mediocre range. Applying the algorithm to the tableau $S_{j-1} \sqcup S_j$, the representation $A_{\mathfrak{q}_{\underline{d}}}(\mu)$ is zero by $p_j \neq 0$. If $x$ is greater than any element in $\nu_j$, the representation is zero by the same method as $q_j \neq 0$. Hence, $\nu_{<j} \cap \nu_{>j} = \varnothing$.

If $\nu_{<j}$ or $\nu_{>j}$ is not multiplicity one, we have $\pi = 0$. Hence, we may assume that $\nu_{<j}$ and $\nu_{>j}$ are multiplicity free. Let $h$ be the integer such that $\nu_{<j} = \{\sigma_1, \ldots, \sigma_h\}$ with $h = p - p_j = f + \#(\nu_j \cap \nu_{<j})$. Set $m' = \min\{h, q_j\}$. Before Trapa's algorithm, the first column of the tableau $\mathrm{Ann}(\pi)$ consists of

$$\sigma_1, \ldots, \sigma_h, \nu_{j,m'+1}, \ldots, \nu_{j,a_j}, \ldots$$

and the second column consists of

$$\nu_{j,1}, \nu_{j,2}, \ldots, \nu_{j,m'}, \sigma_{h+1}, \ldots$$

from top to bottom. Let $\nu_{j-1} = \{\sigma_k, \ldots, \sigma_h\}$ and $S_i$ be the tableau consisting of entries with elements in $\nu_i$ defined in Trapa's algorithm. Consider $R = S_{j-1} \sqcup S_j$. By Trapa's algorithm, we have $R = R'$ if $\sigma_h \geq \nu_{j,m'}$ or $\nu_{j-1} \not\subset \nu_j$. In this case, $R$ is equivalent to zero when $q_j < \#(\nu_{j-1} \cap \nu_j)$. If not, the first column of $R'$ consists of

$$\nu_{j,m'-h+k}, \ldots, \nu_{j,m'}, \nu_{j,m'+1}, \ldots$$

and the second column consists of

$$\nu_{j,1}, \ldots, \nu_{j,m'-h+k-1}, \sigma_k, \ldots, \sigma_h, \sigma_{h+1}, \ldots$$

from top to bottom. If $\sigma_h < \nu_{j,m'}$, then $\sigma_k < \nu_{j,m'-h+k}$. Trapa's algorithm for $R = S'_{j-2} \sqcup S'_{j-1}$ is similar and $R$ is equivalent to zero if $q_j - \#(\nu_{j-1}) < \#(\nu_{j-2} \cap \nu'_{j-1})$. Also, the algorithm for $S'_j \sqcup S_{j+1}$ is similar. Continuing this procedure to the end, one obtains the lemma. Note that $f - m = h - m'$ by elementary computations. □

The following statements, which are helpful to compute $A_{\mathfrak{q}}(\mu)$, show that when the associated segments are linked, we may replace the linked segments or, conversely, partition them into smaller segments.

**Corollary 3.6.** *Under the same notation as in Lemma 3.5, suppose $A_{\mathfrak{q}}(\mu)$ is nonzero. Let $\nu'_1 \sqcup \cdots \sqcup \nu'_k = \nu_{<j}$ be a partition of $\nu_{<j}$ into segments with $\nu'_1 > \cdots > \nu'_k$ and $\nu''_1 \sqcup \cdots \sqcup \nu''_\ell = \nu_{>j}$ be a partition of $\nu_{>j}$ into segments with $\nu''_1 > \cdots > \nu''_\ell$. Let $\mathfrak{q}'$ be the $\theta$-stable parabolic subalgebra associated with*

$$\big\{(\#(\nu'_1), 0), \ldots, (\#(\nu'_k), 0), (p - \#(\nu_{<j}), q - \#(\nu_{>j})), (0, \#(\nu''_1)), \ldots, (0, \#(\nu''_\ell))\big\}.$$

*Put $\pi = A(\mathfrak{q}', \nu'_1 \sqcup \cdots \sqcup \nu'_k \sqcup \nu_j \sqcup \nu''_1 \sqcup \cdots \sqcup \nu''_\ell)$. Then, if $\pi$ is in the mediocre range, we have $A_{\mathfrak{q}}(\mu) \cong \pi$.*

*Proof.* The statement follows from Lemma 3.5, since the tableau $A_{\mathfrak{q}}(\mu)$ does not depend on a partition of $\nu_{<j}$ and $\nu_{>j}$. □

**Corollary 3.7.** *Under the same notation as in Lemma 3.5, suppose $A_{\mathfrak{q}}(\mu)$ is nonzero. When $\nu_{j-1} \subset \nu_j$ (resp. $\nu_{j+1} \subset \nu_j$), let $\mathfrak{q}'$ be the $\theta$-stable parabolic subalgebra corresponding to*

$$\big\{(p_1, q_1), \dots, (p_{j-2}, q_{j-2}), (p_j + p_{j-1}, q_j - p_{j-1}), \\ (0, p_{j-1}), (p_{j+1}, q_{j+1}), \dots, (p_r, q_r)\big\}$$

$$\big(\textit{resp. } \big\{(p_1, q_1), \dots, (p_{j-1}, q_{j-1}), (q_{j+1}, 0), \\ (p_j - q_{j+1}, q_j + q_{j-1}), (p_{j+2}, q_{j+2}), \dots, (p_r, q_r)\big\}\big)$$

*and $\pi = A(\mathfrak{q}', \nu_1, \dots, \nu_j, \nu_{j-1}, \dots, \nu_r)$ (resp. $\pi = A(\mathfrak{q}', \nu_1, \dots, \nu_{j+1}, \nu_j, \dots, \nu_r)$). If $\pi$ is in the mediocre range, one has $A_{\mathfrak{q}}(\mu) \cong \pi$.*

*Proof.* The statement follows from Lemmas 3.1 and 3.5 by the explicit computation of tableaux. □

**Remark 3.8.** One can prove Corollaries 3.6 and 3.7 by the induction in stages of $A_{\mathfrak{q}}(\mu)$ (see [Huang 2025, §5.6; Trapa 2001, Lemma 3.9]). Note that Corollary 3.7 is a special case of [Huang 2025, Proposition 4.9; Trapa 2001, Lemma 9.3].

## 4. *A*-parameters and main theorem

We recall Mœglin and Renard's description of $A$-parameters in terms of cohomological inductions, and state the main theorem of this paper.

**4.1. *A-parameters.*** The $A$-parameters $\psi$ are defined by a formal sum

$$\psi = \bigoplus_{i=1}^{r} \chi_{t_i, s_i} \otimes S_{a_i}, \tag{4-1}$$

where $\chi_{t,s}$ is the character of $\mathbb{C}^\times$ defined by $z \mapsto (z/\bar{z})^{t/2}(z\bar{z})^{s/2}$, $S_m$ is the irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ with dimension $m$, and the triplets $(t, s, a)$ run over multisets on $(t, s, a) \in \mathbb{Z} \times \sqrt{-1}\,\mathbb{R} \times \mathbb{Z}_{>0}$. When $s = 0$, we write $\chi_t = \chi_{t,s}$. This definition of $\chi_t$ differs slightly from that of [Ichino 2022] but is the same as [Mœglin and Renard 2019]. For an $A$-parameter $\psi = \bigoplus_i \chi_{t_i,s_i} \otimes S_{a_i}$, we say that $\psi$ is good if $\frac{1}{2}(t_i + a_i + N) \in \mathbb{Z}$ and $s_i = 0$ for any $i$. Associated with an $A$-parameter $\psi$, we obtain the component group $\mathfrak{S}_\psi$. It is isomorphic to a free $\mathbb{Z}/2\mathbb{Z}$-module

$$\mathfrak{S}_\psi = (\mathbb{Z}/2\mathbb{Z})e_1 \oplus \cdots \oplus (\mathbb{Z}/2\mathbb{Z})e_r.$$

Let $\Pi(\psi)$ be the $A$-packet of $\psi$, that is, the set of semisimple representations of $G$ satisfying the standard and twisted endoscopic character relations (see [Atobe et al. 2024, §1.6; Kaletha et al. 2014, (1.6.1)]).

**Remark 4.1.** In the usual definition, an $A$-parameter is a homomorphism $\psi_{\mathbb{R}}$ from $W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C})$ to the $L$-group of the unitary group. Here, $W_{\mathbb{R}}$ is the real Weil group. By the base change $\psi_{\mathbb{R}}|_{\mathbb{C}^\times \times \mathrm{SL}_2(\mathbb{C})}$, the parameter $\psi_{\mathbb{R}}$ can be identified with the formal sum (4-1). For details, see [Gan et al. 2012, Theorem 8.1].

**4.2. *Mœglin–Renard's construction of A-packets.*** Take a good $A$-parameter

$$\psi = \bigoplus_{i=1}^{r} \chi_{t_i} \otimes S_{a_i}.$$

We define the infinitesimal character $\chi_\psi$ of $\psi$ by the multiset

$$\bigsqcup_{i=1}^{r} \left\{ \tfrac{1}{2}(t_i + a_i - 1), \tfrac{1}{2}(t_i + a_i - 3), \dots, \tfrac{1}{2}(t_i - a_i + 1) \right\} = \bigsqcup_{i=1}^{r} \left[ \tfrac{1}{2}(t_i - a_i + 1), \tfrac{1}{2}(t_i + a_i - 1) \right]. \tag{4-2}$$

Then, all the representations in $\Pi(\psi)$ have the same infinitesimal character with the Harish-Chandra parameter (4-2). Following Théorème 1.1 in [Mœglin and Renard 2019], we describe the representations in $\Pi(\psi)$. Put

$$\mathcal{D}(\psi) = \left\{ (p_i, q_i)_{i=1,\dots,r} \in (\mathbb{Z}_{\geq 0})^2 \mid p_i + q_i = a_i \text{ for any } i \text{ and } \sum_{i=1}^{r} p_i = p, \sum_{i=1}^{r} q_i = q \right\}.$$

For $\underline{d} \in \mathcal{D}(\psi)$, set

$$\mu_{\underline{d}} = (\underbrace{\mu_1, \dots, \mu_1}_{p_1}, \dots, \underbrace{\mu_r, \dots, \mu_r}_{p_r}, \underbrace{\mu_1, \dots, \mu_1}_{q_1}, \dots, \underbrace{\mu_r, \dots, \mu_r}_{q_r}),$$

where $\mu_i = \frac{1}{2}(t_i + a_i - N) + a_{<i}$ and $a_{<i} = \sum_{j<i} a_j$. Note that there is a typo in [Mœglin and Renard 2019, (4.2)]: $\frac{1}{2}(t_i + a_i - N) - a_{<i}$ should be $\frac{1}{2}(t_i + a_i - N) + a_{<i}$. For $x_{\underline{d}}$ as in (3-1), we define the cohomological induction by

$$\mathscr{A}_{\underline{d}}(\psi) = A_{\mathfrak{q}(x_{\underline{d}})}(\mu_{\underline{d}}) \tag{4-3}$$

and a character $\varepsilon_{\underline{d}}$ on $\mathfrak{S}_\psi$ by

$$\varepsilon_{\underline{d}}(e_i) = (-1)^{p_i a_{<i} + q_i(a_{<i}+1) + a_i(a_i-1)/2}$$

for any $e_i \in \mathfrak{S}_\psi$. The following is proved in [Mœglin and Renard 2019, Théorème 1.1]. We choose the same Whittaker datum $\mathfrak{w}$ as [Mœglin and Renard 2019] (see [Atobe 2020, Appendix A]).

**Theorem 4.2.** *Let $\psi$ be a good $A$-parameter with $\psi = \bigoplus_{i=1}^{r} \chi_{t_i} \otimes S_{a_i}$. Suppose that $t_1 \geq \cdots \geq t_r$ and $a_i \geq a_{i+1}$ if $t_i = t_{i+1}$. We then have*

$$\Pi(\psi) = \{ \mathscr{A}_{\underline{d}}(\psi) \mid \underline{d} \in \mathcal{D}(\psi) \}.$$

*The character of $\mathfrak{S}_\psi$ associated with $\mathscr{A}_{\underline{d}}(\psi)$ is equal to $\varepsilon_{\underline{d}}$. Moreover, the multiplicity one holds in $\Pi(\psi)$.*

**Remark 4.3.** The correspondence $\mathscr{A}_{\underline{d}}(\psi)$ to $\varepsilon_{\underline{d}}$ depends on the choice of Whittaker datum $\mathfrak{w}$. We may explicitly calculate the dependence. See [Kaletha et al. 2014, Theorem 1.6.1] for details.

**Remark 4.4.** The statement does not imply that the cohomological induction $\mathscr{A}_{\underline{d}}(\psi)$ is nonzero. Trapa [2001] gives an algorithm to determine whether cohomological inductions of $G$ are zero. Recently, Huang [2025] and Chengyu [2025] independently considered the nonvanishing of cohomological inductions of real unitary groups with Chengyu treating the "nice" case and Huang the general case.

**Remark 4.5.** For general $\psi$, representations in the $A$-packet $\Pi(\psi)$ consist of the parabolic induction from the representations in the packet $\Pi(\psi_0)$ for certain good $A$-parameter $\psi_0 \subset \psi$. Here, $\psi_0$ is a good $A$-parameter of a unitary group that is a subgroup of $G$.

**4.3.** ***Main theorem.*** In this subsection, we state the main theorem following the notation as in the introduction. The lemma below plays a crucial role in stating and proving the main theorem.

**Lemma 4.6.** *Let $\psi = \bigoplus_{i=1}^{r} \chi_{t_i,s} \otimes S_{a_i}$ be an $A$-parameter. If $\Pi(\psi)$ contains an irreducible lowest-weight representation $\pi$, the parameter $\psi$ is good and $\chi_\psi = \chi_\pi$. Moreover, if $\mathscr{A}_{\underline{d}}(\psi) \in \Pi(\psi)$ is nonzero and lowest weight, there exists $j$ such that $q_i = 0$ for any $i < j$ and $p_\ell = 0$ for any $\ell > j$, i.e., $\underline{d} = \underline{d}_0$ and $\mathfrak{q}_{\underline{d}}$ is holomorphic.*

*Proof.* The goodness of $\psi$ follows from the construction of general $\psi$ as in [Mœglin and Renard 2019, Proposition 5.2]. The condition $\chi_\psi = \chi_\pi$ is obvious since representations in $\Pi(\psi)$ have the same infinitesimal character $\chi_\psi$. For the second statement, consider the signed tableau $\mathrm{Ann}(\mathscr{A}_{\underline{d}}(\psi))$. If there exist different integers $k$ and $\ell$ with $p_k q_k p_\ell q_\ell \neq 0$, the signed tableau $\mathrm{Ann}(\mathscr{A}_{\underline{d}}(\psi))$ satisfies either

- there exists a row with three or more boxes, or
- there exists a row with two boxes arranged in the order of $-$ and $+$.

Then $\mathscr{A}_{\underline{d}}(\psi)$ is not of a unitary lowest-weight representation by Corollary 3.3. □

The following is the main theorem of this paper. Note that the nonvanishing condition $\mathscr{A}_{\underline{d}_0}(\psi)$ is determined in Lemma 3.5 (see also Corollary 4.8).

**Theorem 4.7.** *Let $\lambda = (\lambda_1, \ldots, \lambda_N)$ be a $\Delta_c^+$-dominant integral weight and $\pi_\lambda$ be the irreducible lowest-weight representation of lowest $K$-type $\lambda$. Let $\psi$ be a good $A$-parameter with $\chi_\psi = \chi_{\pi_\lambda}$ such that $\mathscr{A}_{\underline{d}_0}(\psi)$ is nonzero.*

(1) *If $N - p' \leq \lambda_p - \lambda_{p+1} < N - q'$, the packet $\Pi(\psi)$ contains $\pi_\lambda$ if and only if $\left[\lambda_p - \frac{1}{2}(N-1), \lambda_{p+1} + \frac{1}{2}(N-1)\right] \subset \nu_j \subset P'$.*

(2) *If $N - q' \leq \lambda_p - \lambda_{p+1} < N - p'$, the packet $\Pi(\psi)$ contains $\pi_\lambda$ if and only if either*

- $\nu_{\leq j} = P$, *or*
- $\left[\lambda_p - \frac{1}{2}(N-1), \lambda_{p+1} + \frac{1}{2}(N-1)\right] \subset \nu_j \subset Q'$.

(3) *If* $N-p', N-q' \leq \lambda_p - \lambda_{p+1}$, *the packet* $\Pi(\psi)$ *contains* $\pi_\lambda$ *if and only if either*

- $P \subset \nu_{\leq j} \subset P \sqcup I$, *or*
- $I \subset \nu_j \subset Q'$.

(4) *If* $\lambda_p - \lambda_{p+1} < N-p', N-q'$, *the packet* $\Pi(\psi)$ *contains* $\pi_\lambda$ *if and only if* $\left[\lambda_p - \frac{1}{2}(N-1), \lambda_{p+1} + \frac{1}{2}(N-1)\right] = \nu_j$.

Our proof, provided in Section 5, is based on the explicit computation of $K$-types and the associated tableaux of $\mathscr{A}_{\underline{d}}(\psi)$. More precisely, the if part follows from Lemma 3.2 and Corollaries 3.6–3.7. For the only if part, we divide the cases into $q_j = 0$ or $q_j \neq 0$. When $q_j = 0$, the statement follows from Lemma 3.1. When $q_j \neq 0$, we calculate the associated tableau $\mathrm{Ann}(\mathscr{A}_{\underline{d}}(\psi))$ in Lemmas 5.2, 5.3, and 5.4. Then, the theorem follows.

As a consequence of Theorem 4.7, we can determine the lowest $K$-type of the lowest-weight representation in $\Pi(\psi)$.

**Corollary 4.8.** *Let* $\psi = \bigoplus_{i=1}^{r} \chi_{t_i} \otimes S_{a_i}$ *be a good A-parameter. The packet* $\Pi(\psi)$ *contains a nonzero unitary lowest-weight representation if and only if both*

- $\nu_{<j}$ *and* $\nu_{>j}$ *are multiplicity free*, *and*
- $\#(\nu_j \cap \nu_{>j}) \leq p_j$ *and* $\#(\nu_j \cap \nu_{<j}) \leq q_j$.

*When* $\Pi(\psi)$ *contains a nonzero unitary lowest-weight representation* $\pi$ *in* $\Pi(\psi)$, *the lowest* $K$*-type* $\lambda$ *of* $\pi$ *is given as follows*:

(1) *When* $q_j = 0$, *the lowest* $K$*-type* $\lambda$ *of* $\pi$ *satisfies* $P(\lambda) = \nu_{\leq j}$ *and* $Q(\lambda) = \nu_{>j}$.

(2) *When* $p_j = \#(\nu_j \cap \nu_{>j})$ *and* $q_j \neq 0$, *the lowest* $K$*-type* $\lambda$ *of* $\pi$ *satisfies* $P(\lambda) = \nu_{<j} \sqcup (\nu_j \cap \nu_{>j})$ *and* $Q(\lambda) = \nu_{\geq j} \setminus (\nu_j \cap \nu_{>j})$.

(3) *When* $q_j = \#(\nu_j \cap \nu_{<j}) \neq 0$, *the lowest* $K$*-type* $\lambda$ *of* $\pi$ *satisfies*

$$P(\lambda) = \nu_{\leq j} \setminus (\nu_{<j} \cap \nu_j) = \{\sigma_1, \ldots, \sigma_p\} \quad \textit{and} \quad Q(\lambda) = (\nu_j \cap \nu_{<j}) \sqcup \nu_{>j}.$$

(4) *When* $p_j \neq \#(\nu_j \cap \nu_{>j})$ *and* $q_j \neq \#(\nu_j \cap \nu_{<j})$, *set* $\nu_{<j} \sqcup \nu_{>j} = \{\sigma_1, \ldots, \sigma_{N-\#(\nu_j)}\}$. *Let* $i_0$ *be the minimal integer such that*

$$1 \leq i_0 \leq \#(\nu_j) \quad \textit{and} \quad \#(\nu_j) - i_0 + 1 + \#\{x \in \nu_{<j} \sqcup \nu_{>j} \mid x > \nu_{j,i_0}\} = p.$$

*Then, the lowest* $K$*-type* $\lambda = (\lambda_1, \ldots, \lambda_N)$ *of* $\pi$ *is given by*

$$\lambda_i = \begin{cases} \sigma_i - \frac{1}{2}(p-q+1) + i & \textit{if } i < p - \#(\nu_j) + i_0, \\ \nu_{j,1} + \frac{1}{2}(N+1) - \#(\nu_j) & \textit{if } p - \#(\nu_j) + i_0 \leq i \leq p, \\ \nu_{j,1} - \frac{1}{2}(N-1) & \textit{if } p+1 \leq i \leq p + i_0 - 1, \\ \sigma_{i-\#(\nu_j)} - \frac{1}{2}(N+1) - p + i & \textit{if } p + i_0 \leq i. \end{cases}$$

*Proof.* To show the nonvanishing condition, it suffices to consider the case $\underline{d} = \underline{d}_0$ by Lemma 4.6. The nonvanishing condition for $\mathscr{A}_{\underline{d}_0}(\psi)$ is given in Lemma 3.5. Suppose that $\mathscr{A}_{\underline{d}_0}(\psi)$ is nonzero. Then, the statement (1) follows from Lemma 3.1. For (2) and (3), the statements follow from Lemma 3.2 and Corollaries 3.6–3.7. For (4), we have $\lambda_p - \lambda_{p+1} < N - p'(\lambda), N - q'(\lambda)$ by Theorem 4.7. Now, $i_0$ as in (4) exists. We then have $\nu_j = \left[\lambda_p - \frac{1}{2}(N-1), \lambda_{p+1} + \frac{1}{2}(N-1)\right]$ and $P(\lambda) = \{\nu_{j,i_0}, \nu_{j,i_0+1} \dots, \nu_{j,\#(\nu_j)}\} \sqcup \{x \in \nu \setminus \nu_j \mid x > \nu_{j,i_0}\}$. The statement follows from Lemma 3.2(1) and Corollary 3.7. □

## 5. Proof of main theorem

We prove the main theorem in Sections 5.2, 5.3, 5.4, and 5.5.

**5.1.** ***Cohomological induction for holomorphic θ-stable parabolic subalgebras.*** To show the main theorem, we will need Lemma 5.1 below.

By Lemma 4.6, it suffices to consider $\underline{d} = \underline{d}_0$. We already described the nonvanishing conditions of such cohomological inductions in Lemma 3.5. In the following lemma, we will investigate the necessary condition that $\mathscr{A}_{\underline{d}_0}(\psi)$ is isomorphic to $\pi_\lambda$ with a given $\lambda$. A key point here is that the cohomological induction $\mathscr{A}_{\underline{d}}(\psi)$ is in the weakly fair range.

**Lemma 5.1.** *Take an irreducible unitary lowest-weight representation $\pi_\lambda$ with lowest $K$-type $\lambda$. Let $\psi$ be an $A$-parameter with $\pi_\lambda \in \Pi(\psi)$. Let $\nu_i$ be the segments associated with $\psi$ and $\underline{d}_0$.*

(1) *As multisets, $\nu_{<j}$ and $\nu_{>j}$ are multiplicity free.*

(2) *If $\nu_j \subset \nu_{>j}$, then $q_j = 0$.*

(3) $\nu_j \not\subset \nu_{<j}$.

(4) $I \subset \nu_{<j} \sqcup \nu_{>j}$.

(5) *If $\nu_{<j} \cap \nu_{>j} \neq \varnothing$, then $\nu_j \subset \nu_{>j}$ and $q_j = 0$.*

(6) *If $\nu_{<j} \cap \nu_{>j} \cap I \neq \varnothing$ and $I \cap \nu_j \neq \varnothing$, then $\nu_j \subset I \subset \nu_{>j}$ and $q_j = 0$.*

(7) *If $\nu_{<j} \cap \nu_{>j} \cap I = \varnothing$, then $I \subset \nu_j$.*

(8) *If $I \neq \varnothing$, then $I \cap \nu_j \neq \varnothing$.*

*Proof.* The statements (1), (2), (3), and (5) follow immediately from Lemma 3.5. Note that $p_j \neq 0$ by definition of $j$. For (4), consider the multiplicities of each element in $\nu$. The multiplicities of elements in $I$ are two in $\nu$. Thus, $\nu_{<j} \sqcup \nu_{>j}$ contains $I$, since $\nu_j$ is a set.

Set $I = [x, y]$ and $\nu_j = [\alpha, \beta]$. By $\pi_\lambda \in \Pi(\psi)$, one has $P \sqcup Q = \nu$. For (6), assume $\nu_{<j} \cap \nu_{>j} \cap I \neq \varnothing$ and $\nu_j \cap I \neq \varnothing$. Since any element in $I$ has multiplicity two in $\nu$, one has $\nu_{<j} \cap \nu_{>j} \cap I = I \setminus (I \cap \nu_j)$ by (1). Moreover, $I \cap \nu_j$ is a segment since $I$

and $\nu_i$ are segments. Hence, $\nu_{<j} \cap \nu_{>j} \cap I$ contains $x$ or $y$. When $y \in \nu_{<j} \cap \nu_{>j}$, we denote by $z$ the minimal member in $\nu_{<j} \cap \nu_{>j} \cap I$. The maximal member in $\nu_j \cap I$ is $z-1$. By the weakly fair property, the set $\nu_{<j}$ does not contain $z-1$. We thus have $z-1 \in \nu_{>j}$. Then, the set $\nu_{>j}$ contains $I$ and in particular, $\nu_j \subset I \subset \nu_{>j}$. Here, we use the fact that the real numbers $x-1$ and $y+1$ have multiplicity at most one in $\nu$. When $x \in \nu_{<j} \cap \nu_{>j}$, one has $\nu_j \subset I \subset \nu_{<j}$ by the same discussion. This case does not happen by (3). Hence, we have $\nu_j \subset I \subset \nu_{>j}$ and then $q_j = 0$.

The statement (7) follows immediately from the fact that the multiplicities of elements in $I$ are two.

For (8), suppose $I \neq \varnothing$ and $I \cap \nu_j = \varnothing$. By the proof of Lemma 3.5, we have $\nu_j \subset \nu_{>j}$ and $\alpha \leq \beta < x$. Then, $q_j = 0$, and there exists an element $t$ in $P' \sqcup Q'$ with multiplicity two such that $t < x$. The existence of $t$ implies $\lambda_p - \lambda_{p+1} < N - q'$. By $q_j = 0$ and Lemma 3.2(3), we may assume $N - p' \leq \lambda_p - \lambda_{p+1}$. Then, the set $I$ is equal to $Q'$. In this case, $x - 1 \notin Q$, but $x - 1 \in P \cap \nu_{>j}$ by definition. Moreover, the segment $\nu_{>j}$ contains the set $Q \setminus Q'$ and $I = Q'$, and then $Q \subset \nu_{>j}$. Hence,

$$\#\nu_{>j} \geq \#\{x-1\} + \#Q \geq q + 1.$$

This contradicts the definition of $j$. Hence $I \cap \nu_j \neq \varnothing$. □

In the following sections, we complete the proof of Theorem 4.7.

**5.2.** ***Proof of main theorem: the case $N - p' \leq \lambda_p - \lambda_{p+1} < N - q'$.*** We first show the only if part. By the assumption, one has $I = Q'$. Put $\nu_j = [\alpha, \beta]$.

When $q_j = 0$, by Lemma 3.1, we have $\nu_{\leq j} = P$ and $I = Q' \subset \nu_{>j} = Q$ if $\mathscr{A}_{\underline{d}_0}(\psi) \cong \pi_\lambda$. In this case, one has $\alpha = \lambda_p - \frac{1}{2}(N-1)$. Note that by the weakly fair property, we have $\beta \geq \lambda_{p+1} + \frac{1}{2}(N-1)$. Hence, the segment $\nu_j$ contains $\left[\lambda_p - \frac{1}{2}(N-1), \lambda_{p+1} + \frac{1}{2}(N-1)\right]$.

When $q_j \neq 0$, the segment $\nu_j$ is not contained in $\nu_{>j}$ by Lemma 5.1 (2). Also, $\nu_{<j} \cap \nu_{>j} = \varnothing$ and $\nu_j$ contains $I$ by Lemma 5.1. Set

$$\nu \setminus \nu_j = \{\sigma_1, \dots, \sigma_{N-\#(\nu_j)}\}, \quad \nu_j \cap (\nu_{<j} \sqcup \nu_{>j}) = \{\sigma_{f+1}, \dots, \sigma_g\}$$

with $\sigma_1 > \cdots > \sigma_{N-\#(\nu_j)}$. Define $\underline{d} = \{(p_i, q_i)\}$ and $\nu'_i$ by

$$(p_i, q_i) = \begin{cases} (1,0) & \text{if } i < p - p_j, \\ (p_j, q_j) & \text{if } i = p_j, \\ (0,1) & \text{if } i > p_j, \end{cases} \qquad \nu'_i = \begin{cases} \{\sigma_i\} & \text{if } i < p - p_j, \\ \nu_j & \text{if } i = p - p_j, \\ \{\sigma_{i-1}\} & \text{if } i > p - p_j, \end{cases}$$

and

$$\pi(\psi) = A(\mathfrak{q}_{\underline{d}}, \nu'_1, \dots, \nu'_{N-\#(\nu_j)+1}). \tag{5-1}$$

By Corollary 3.6, $\pi(\psi)$ is in the mediocre range and isomorphic to $\mathscr{A}_{\underline{d}_0}(\psi)$.

**Lemma 5.2.** *If* $\pi(\psi)\cong\pi_\lambda$, *then* $\left[\lambda_p-\frac{1}{2}(N-1),\lambda_{p+1}+\frac{1}{2}(N-1)\right]\subset\nu_j\subset P'$.

*Proof.* Since the segment $\nu_j$ contains $I$, it suffices to show $\alpha=\lambda_p-\frac{1}{2}(N-1)$. Suppose $\alpha\geq\lambda_p-\frac{1}{2}(N-1)$. By Lemma 3.5, one has $\#(\nu_{<j}\cap\nu_j)\leq q_j=q-\#\nu_{>j}$. Since the set $(\nu_{<j}\cap\nu_j)\sqcup\nu_{>j}$ contains $Q$ in this case, one has $\#\nu_{>j}+\#(\nu_{<j}\cap\nu_j)\geq q$. Hence, we have $\nu_{>j}\sqcup(\nu_{<j}\cap\nu_j)=Q$ and in particular, $\nu_{>j}\subset Q$. By $I\subset\nu_j$, the multiset $\nu_j$ is not multiplicity free or $\nu_{>j}\not\subset Q$ if $\alpha>\lambda_p-\frac{1}{2}(N-1)$. This shows $\alpha\leq\lambda_p-\frac{1}{2}(N-1)$.

It remains to show $\alpha\geq\lambda_p-\frac{1}{2}(N-1)$. Suppose $\alpha<\lambda_p-\frac{1}{2}(N-1)$. To show $\pi(\psi)\ncong\pi_\lambda$ under this assumption, recall the tableau $\mathrm{Ann}(\pi_\lambda)$. By Lemma 3.2(2), the first column of the tableau $\mathrm{Ann}(\pi_\lambda)$ consists of entries

$$\lambda_1+\tfrac{1}{2}(p-q-1),\lambda_2+\tfrac{1}{2}(p-q-3),\dots,\lambda_p-\tfrac{1}{2}(N-1),\dots$$

and the second column consists of entries

$$\lambda_{p+1}+\tfrac{1}{2}(N-1),\lambda_{p+2}+\tfrac{1}{2}(N-3),\dots,\lambda_{p+\min\{p,q\}}+\tfrac{1}{2}(N+1)-\min\{p,q\}$$

from top to bottom. This is invariant under Trapa's algorithm. We describe the tableau $\mathrm{Ann}(\pi(\psi))$ as follows. By $\#(\nu_{<j})+p_j+q_j-\#(\nu_{<j}\cap\nu_j)>p$ and $\#(\nu_{<j})+p_j=p$, we have $q_j-\#(\nu_{<j}\cap\nu_j)>0$. When there is no $x\in\nu$ with $\beta<x$, the first column of $\pi(\psi)$ consists of

$$\beta=\lambda_1+\tfrac{1}{2}(p-q-1),\beta-1,\dots,\alpha,\dots$$

from top to bottom. In particular, the entry next to $\lambda_p-\frac{1}{2}(N-1)$ is $\lambda_p-\frac{1}{2}(N+1)$. However, the entry next to $\lambda_p-\frac{1}{2}(N-1)$ in the first column of $\mathrm{Ann}(\pi_\lambda)$ does not equal $\lambda_p-\frac{1}{2}(N+1)$. Indeed, if $q\leq p$, there is no such box. If $q>p$, the entry is $\lambda_{2p+1}+\frac{1}{2}(N+1)-(p+1)=\lambda_{2p+1}-\frac{1}{2}(p-q+1)$. We then have

$$\begin{aligned}\lambda_p-\tfrac{1}{2}(N+1)-\bigl(\lambda_{2p+1}-\tfrac{1}{2}(p-q+1)\bigr)&=\lambda_p-\lambda_{2p+1}-q\\&=\lambda_p-\lambda_{p+1}-(N-p)+(\lambda_{p+1}-\lambda_{2p+1})\\&\geq\lambda_{p+1}-\lambda_{2p+1}.\end{aligned}$$

For the last line, we use $P=P'$ and $N-p'\leq\lambda_p-\lambda_{p+1}$. By $N-p=N-p'<N-q'$, one has $q'<p$ and then $\lambda_{p+1}-\lambda_{2p+1}>0$. Hence, the tableaux $\mathrm{Ann}(\pi_\lambda)$ and $\mathrm{Ann}(\pi(\psi))$ are different and in particular, the representations are different. We may assume that there exists $x\in\nu$ such that $x>\beta$. Put $f=\#\{x\in\nu\mid x>\beta\}$. Let $m=\min\{f,q_j-\#(\nu_j\cap\nu_{<j})\}$ and $i_0$ be the maximal integer such that $1\leq i_0\leq g-f$ and $\sigma_{f+i_0}\geq\nu_{j,m+i_0}$. Here, $\nu_j=\{\nu_{j,1},\dots,\nu_{j,a_j}\}$ with $\nu_{j,1}>\cdots>\nu_{j,a_j}$. By assumption, $m$ is positive. By Lemma 3.5, the $(1,2)$-th entry in $\mathrm{Ann}(\pi(\psi))$ is $\beta$. Hence, we have $\beta=\lambda_{p+1}+\frac{1}{2}(N-1)$. This shows $i_0\geq\#I$. The second column of $\mathrm{Ann}(\pi(\psi))$ consists of

$$\nu_{j,1},\nu_{j,2},\dots,\nu_{j,m+i_0},\dots$$

from top to bottom. In particular, the entry next to $\nu_{j,\#(I)} = \lambda_{p+1} + \frac{1}{2}(N+1) - q'$ is $\nu_{j,\#(I)+1} = \nu_{j,\#(I)} - 1$. Note that $\lambda_{p+1} + \frac{1}{2}(N-1) - q'$ is in $\nu$ by $\lambda_p - \lambda_{p+1} < N - q'$. However, in the second column of $\mathrm{Ann}(\pi_\lambda)$, the entry next to $\lambda_{p+1} + \frac{1}{2}(N+1) - q'$ is $\lambda_{p+q'+1} + \frac{1}{2}(N-1) - q' < \nu_{j,\#(I)+1}$, if it exists. Hence, the representation $\lambda_{p+1} + \frac{1}{2}(N+1) - q'$ is not isomorphic to $\pi_\lambda$ since the associated $\nu$-antitableau tableaux are different. □

We show the converse. Suppose that $\mathscr{A}_{\underline{d}_0}(\psi)$ satisfies

$$\left[\lambda_p - \tfrac{1}{2}(N-1), \lambda_{p+1} + \tfrac{1}{2}(N-1)\right] \subset \nu_j \subset P'.$$

The nonvanishing of $\mathscr{A}_{\underline{d}_0}(\psi)$ follows from Lemma 3.5. Since the multiset $\nu \setminus \nu_j = \nu_{<j} \sqcup \nu_{>j}$ is multiplicity free, the representation $\mathscr{A}_{\underline{d}_0}(\psi)$ is isomorphic to $\pi_\lambda$ by Lemma 3.2 and Corollaries 3.6–3.7.

**5.3.** ***Proof of main theorem: the case $N - q' \le \lambda_p - \lambda_{p+1} < N - p'$.*** We first show the only if part. Let $P'' = (P \cap Q') \setminus I$ and $p'' = \#(P'')$. One has $I = P'$ and $I \cap \nu_j \neq \varnothing$ by Lemma 5.1(8). When $q_j = 0$, by Lemma 3.1, the representation $\mathscr{A}_{\underline{d}_0}(\psi)$ is isomorphic to $\pi_\lambda$ if and only if $\nu_{\le j} = P$.

We consider the case where $q_j \neq 0$. Then, the multiset $\nu_{<j} \sqcup \nu_{>j}$ is multiplicity free and $\nu_j$ is not contained in $\nu_{>j}$. Let $\pi(\psi)$ be the cohomological induction defined in the same way in (5-1). The statement follows from this lemma:

**Lemma 5.3.** *If $\pi(\psi) \cong \pi_\lambda$, we then have*

$$\left[\lambda_p - \tfrac{1}{2}(N-1), \lambda_{p+1} + \tfrac{1}{2}(N-1)\right] \subset \nu_j \subset Q'.$$

*Proof.* Put $\nu_j = [\alpha, \beta]$. Since $\nu_j$ contains $I$, one has $\alpha \le \lambda_p - \frac{1}{2}(N-1)$. It remains to show $\beta = \lambda_{p+1} + \frac{1}{2}(N-1)$. Suppose that $\beta < \lambda_{p+1} + \frac{1}{2}(N-1)$. We then have $p_j \ge \#(\nu_j \cap \nu_{>j})$ and the multiset $\nu_{\le j}$ contains a set $P$ properly, since $\nu_{<j}$ contains $\{x \in \nu \mid x > \beta\}$. We show that $\#(\nu_j \cap \nu_{>j}) - p_j$ is positive. By $\#(\nu_j \cap \nu_{>j}) = \#I + \#P'' - \#(\nu_{<j} \cap \nu_j)$, one has

$$\begin{aligned}\#(\nu_j \cap \nu_{>j}) - p_j &= p' + p'' - \#(\nu_j \cap \nu_{<j}) - p_j \\ &= p' + p'' - \#(\nu_j \cap \nu_{<j}) - (p - \#(\nu_{<j})) \\ &= \#(\nu_{<j}) - \#(\nu_j \cap \nu_{<j}) - (p - p' - p'') > 0.\end{aligned}$$

The last inequality follows from $\beta < \lambda_{p+1} + \frac{1}{2}(N-1)$. Hence $\pi(\psi)$ is zero. This is a contradiction. Therefore we have $\beta \ge \lambda_{p+1} + \frac{1}{2}(N-1)$.

It remains to show $\beta \le \lambda_{p+1} + \frac{1}{2}(N-1)$. Put $f = \#\left\{x \in \nu \mid x > \lambda_{p+1} + \frac{1}{2}(N-1)\right\}$. Suppose $\beta > \lambda_{p+1} + \frac{1}{2}(N-1)$. By assumption, we have $q_j > \#(\nu_j \cap \nu_{<j})$ and $f \neq 0$. We recall the tableau $\mathrm{Ann}(\pi_\lambda)$. By Lemmas 3.2 and 3.5, the second column of the

tableau $\mathrm{Ann}(\pi_\lambda)$ consists of

$$\lambda_{p+1}+\tfrac{1}{2}(N-1), \lambda_{p+2}+\tfrac{1}{2}(N-3), \dots, \lambda_{p+\min\{p,q\}+1-i_0}+\tfrac{1}{2}(N-1)-\min\{p,q\}+i_0, \underbrace{\lambda_{p+1-i_0}-\tfrac{1}{2}(N+1)+i_0, \dots, \lambda_p-\tfrac{1}{2}(N-1)}_{i_0}, \dots.$$

Here, $i_0$ is the maximal positive integer such that

$$\lambda_{p+\min\{p,q\}+1-i_0}+\tfrac{1}{2}(N-1)-\min\{p,q\}+i_0 \geq \lambda_{p+1-i_0}-\tfrac{1}{2}(N+1)+i_0$$

if it exists. If no such $i_0$ exists, $i_0$ is defined as 0. In particular, the $(1,2)$-th entry is $\lambda_{p+1}+\frac{1}{2}(N-1)$. In this case, the number of boxes in the second column from the top to the box filled with $\lambda_p-\frac{1}{2}(N-1)$ is greater than $p'+p''$. If there exists $x \in \nu$ with $x > \beta$, the $(1,2)$-th entry of $\mathrm{Ann}(\pi(\psi))$ is $\beta$ by $q_j > \#(\nu_j \cap \nu_{>j})$. Thus, the tableaux $\mathrm{Ann}(\pi_\lambda)$ and $\mathrm{Ann}(\pi(\psi))$ are different by the assumption $\beta > \lambda_{p+1}+\frac{1}{2}(N-1)$. In other words, the representations $\pi_\lambda$ and $\pi(\psi)$ are different. The remaining case is that there is no $x \in \nu$ with $x > \beta$. Then, the second column of $\mathrm{Ann}(\mathscr{A}_{\underline{d}_0}(\psi))$ consists of entries

$$\lambda_{f+1}+\tfrac{1}{2}(p-q-1)-f, \lambda_{f+2}+\tfrac{1}{2}(N-3)-f, \dots, \lambda_p-\tfrac{1}{2}(N-1), \dots$$

from top to bottom. The number of boxes from the top to the box filled with $\lambda_p-\frac{1}{2}(N-1)$ is $p'+p''$, different from that of $\mathrm{Ann}(\pi_\lambda)$. Therefore, the representation $\mathscr{A}_{\underline{d}_0}(\psi)$ is not isomorphic to $\pi_\lambda$. We conclude that $\beta = \lambda_{p+1}+\frac{1}{2}(N-1)$. □

It remains to show the converse. Suppose that $\mathscr{A}_{\underline{d}_0}(\psi)$ satisfies $\nu_{\leq j} = P$. Then, $\mathscr{A}_{\underline{d}_0}(\psi) \cong \pi_\lambda$ by Lemma 3.1. Suppose next that $\mathscr{A}_{\underline{d}_0}(\psi)$ satisfies

$$\left[\lambda_p-\tfrac{1}{2}(N-1), \lambda_{p+1}+\tfrac{1}{2}(N-1)\right] \subset \nu_j \subset Q'.$$

In this case, by the explicit computation of $p_j$ and $q_j$, the representation $\mathscr{A}_{\underline{d}_0}(\psi)$ is nonzero and isomorphic to $\pi_\lambda$ by Lemma 3.2 and Corollaries 3.6–3.7.

**5.4.** ***Proof of main theorem: the case $N-p', N-q' \leq \lambda_p-\lambda_{p+1}$.*** We first show the only if part. When $q_j = 0$, by Lemma 3.1, we have $\nu_{\leq j} = P$ and $\nu_{>j} = Q$. In particular, $P \subset \nu_{\leq j} \subset P \sqcup I$. When $q_j \neq 0$, the segment $\nu_j$ contains $I$. When $\nu_j \cap (Q' \setminus I) = \varnothing$, we have $\nu_{\leq j} \subset P \sqcup I$ and $P \subset \nu_{\leq j}$ since $\nu_{>j}$ is multiplicity free by Lemma 3.5. We may assume that $\nu_j \cap (Q' \setminus I) \neq \varnothing$ and $\lambda_1+\frac{1}{2}(p-q-1) > \lambda_{p+1}+\frac{1}{2}(N-1)$. In fact, if $\lambda_1+\frac{1}{2}(p-q-1) = \lambda_{p+1}+\frac{1}{2}(N-1)$, the segment $\nu_j$ is automatically contained in $Q'$. Consider the tableaux for $\mathrm{Ann}(\mathscr{A}_{\underline{d}_0}(\psi))$ and $\mathrm{Ann}(\pi_\lambda)$. We show that $\nu_j$ is contained in $Q'$ if $\mathscr{A}_{\underline{d}_0}(\psi) \cong \pi_\lambda$. Note that we have $q_j - \#(\nu_j \cap \nu_{<j}) > 0$. Indeed, by assumption, the multiset $\nu_{\leq j}$ contains $P$ properly and then $P \sqcup (\nu_j \cap \nu_{<j}) \subsetneq \nu_{\leq j}$. We then have $p + \#(\nu_j \cap \nu_{<j}) < \#\nu_{<j} + p_j + q_j$

and, in particular, $0 < q_j - \#(\nu_j \cap \nu_{<j})$ by $p_j + \#(\nu_{<j}) = p$. By Lemma 3.2, the first column of $\mathrm{Ann}(\pi_\lambda)$ is

$$\lambda_1 + \tfrac{1}{2}(p-q-1), \lambda_1 + \tfrac{1}{2}(p-q-3), \ldots, \lambda_p - \tfrac{1}{2}(N-1), \ldots$$

and the second column is

$$\lambda_{p+1} + \tfrac{1}{2}(N-1), \lambda_{p+2} + \tfrac{1}{2}(N-3), \ldots, \lambda_{p+\min\{p,q\}} + \tfrac{1}{2}(N+1) - \min\{p,q\}.$$

Put $\nu_j = [\alpha, \beta]$. Suppose $\beta = \lambda_1 + \frac{1}{2}(p-q-1)$. Then

$$P = P' \quad \text{and} \quad \beta > \lambda_{p+1} + \tfrac{1}{2}(N-1).$$

The first column of $\mathrm{Ann}(\mathscr{A}_{\underline{d}_0}(\psi))$ consists of

$$\beta, \beta-1, \ldots, \alpha, \ldots$$

and the second column consists of

$$\lambda_{p+1} + \tfrac{1}{2}(N-1), \lambda_{p+2} + \tfrac{1}{2}(N-3), \ldots, \lambda_{p+\ell} + \tfrac{1}{2}(N+1) - \ell, \alpha - 1, \ldots$$

from top to bottom. Here, $\ell$ is the unique positive integer such that

$$\lambda_{p+\ell} + \tfrac{1}{2}(N+1) - \ell = \lambda_p - \tfrac{1}{2}(N-1).$$

Note that in the second column of $\mathrm{Ann}(\mathscr{A}_{\underline{d}_0}(\psi))$, the box next to the box filled with $\lambda_{p+\ell} + \frac{1}{2}(N+1) - \ell$ exists if and only if $\beta > \lambda_{p+1} + \frac{1}{2}(N-1)$ and $\nu_j \sqcup I \neq \nu$. Contrary to this, in the second column of $\mathrm{Ann}(\pi_\lambda)$, the box next to the box filled with $\lambda_{p+\ell} + \frac{1}{2}(N+1) - \ell$ exists if and only if $\lambda_1 + \frac{1}{2}(p-q-1) > \lambda_{p+1} + \frac{1}{2}(N-1)$ and $I \neq Q$. Hence, under our assumption, there exists a box next to the box filled with $\lambda_{p+\ell} + \frac{1}{2}(N+1) - \ell$ in the second column of $\mathrm{Ann}(\pi_\lambda)$ and it is equal to $\lambda_{p+\ell} + \frac{1}{2}(N-1) - \ell$. We may additionally assume $\nu \neq \nu_j \sqcup I$. The entry next to $\lambda_{p+\ell} + \frac{1}{2}(N+1) - \ell$ in the second column of $\mathrm{Ann}(\mathscr{A}_{\underline{d}_0}(\psi))$ is strictly less than $\lambda_{p+\ell} + \frac{1}{2}(N+1) - \ell - 1$. This shows that the tableaux $\mathrm{Ann}(\mathscr{A}_{\underline{d}_0}(\psi))$ and $\mathrm{Ann}(\pi_\lambda)$ are different. Suppose $\beta \neq \lambda_1 + \frac{1}{2}(p-q-1)$. Then, the $(1,2)$-th entry in $\mathrm{Ann}(\mathscr{A}_{\underline{d}_0}(\psi))$ is $\beta$ by $q_j > \#(\nu_j \cap \nu_{<j})$. Hence, we have $\beta = \lambda_{p+1} + \frac{1}{2}(N-1)$ if $\mathscr{A}_{\underline{d}_0}(\psi) \cong \pi_\lambda$. In other words, one has $I \subset \nu_j \subset Q'$ if $\mathscr{A}_{\underline{d}_0}(\psi) \cong \pi_\lambda$.

The converse follows from Lemmas 3.2–3.5 and Corollaries 3.6–3.7. This completes the proof.

**5.5.** ***Proof of main theorem: the case $\lambda_p - \lambda_{p+1} < N - p', N - q'$.*** We first show the only if part. Put $\nu_j = [\alpha, \beta]$. In this case, one has $q_j > 0$ by Lemma 3.2(3). Then, $\nu_j$ contains $I$ and $\nu_{<j} \sqcup \nu_{>j}$ is multiplicity free. Let $\pi(\psi)$ be the cohomological induction defined in the same way in (5-1). The statement follows from this lemma:

**Lemma 5.4.** *If $\pi(\psi) \cong \pi_\lambda$, one has* $\nu_j = \left[\lambda_p - \frac{1}{2}(N-1), \lambda_{p+1} + \frac{1}{2}(N-1)\right]$.

*Proof.* The statement follows from the explicit calculation of the associated tableaux. Recall the tableau $\mathrm{Ann}(\pi_\lambda)$. Put $f = \#\{x \in \nu \mid x > \lambda_{p+1} + \frac{1}{2}(N-1)\}$ and $m = \min\{f, N-(\lambda_p - \lambda_{p+1}) - p' - p''\}$, where $p'' = \#((P \cap Q') \setminus I)$. When $f \neq 0$, the $(1,2)$-th entry in $\mathrm{Ann}(\pi_\lambda)$ is $\lambda_{p+1} + \frac{1}{2}(N-1)$ by Lemmas 3.2(1) and 3.5. Let $i_0$ be the maximal positive integer such that $\lambda_{f+i_0} + \frac{1}{2}(p-q+1) - (f+i_0) \geq \lambda_{p+1} + \frac{1}{2}(N+1) - m - i_0$, if it exists. If there is no such $i_0$, set $i_0 = 0$. Then, the first column of $\mathrm{Ann}(\pi_\lambda)$ consists of

$$\underbrace{\lambda_1 + \tfrac{1}{2}(p-q-1), \lambda_2 + \tfrac{1}{2}(p-q-3), \dots, \lambda_f + \tfrac{1}{2}(p-q+1) - f,}_{f}$$

$$\left.\begin{aligned}&\lambda_{f+1} + \tfrac{1}{2}(p-q+1) - (f+1), \lambda_{f+2} + \tfrac{1}{2}(p-q+1) - (f+2),\\ &\qquad\qquad \dots, \lambda_{f+i_0} + \tfrac{1}{2}(p-q+1) - (f+i_0)\end{aligned}\right\} i_0$$

$$\begin{aligned}&\lambda_{p+m+i_0+1} + \tfrac{1}{2}(N-1) - (m+i_0), \lambda_{p+m+i_0+2} + \tfrac{1}{2}(N-1) - (m+i_0+1),\\ &\qquad\qquad \dots, \lambda_p - \tfrac{1}{2}(N-1), \dots\end{aligned}$$

from top to bottom. The entry adjacent to $\lambda_p - \frac{1}{2}(N-1)$ is strictly less than $\lambda_p - \frac{1}{2}(N-1) - 1$ since $p - f$ is strictly greater than the number of elements in $\nu$ with multiplicity two. When $f = 0$, the $(1,2)$-th entry in $\mathrm{Ann}(\pi_\lambda)$ is the maximal member in $\nu$ with multiplicity two. This is greater than or equal to $\lambda_p - \frac{1}{2}(N-1) + p'$. Then the first column consists of

$$\lambda_{p+1} + \tfrac{1}{2}(N-1), \lambda_{p+1} + \tfrac{1}{2}(N-3), \dots, \lambda_p - \tfrac{1}{2}(N-3), \lambda_p - \tfrac{1}{2}(N-1), \dots$$

from top to bottom.

We claim $\beta = \lambda_{p+1} + \frac{1}{2}(N-1)$. Note that $q_j > \#(\nu_j \cap \nu_{<j})$ and $p_j > \#(\nu_j \cap \nu_{>j})$ if $\pi(\psi) \cong \pi_\lambda$ by Lemma 3.2 and Corollaries 3.6–3.7. Suppose first that $\beta < \lambda_{p+1} + \frac{1}{2}(N-1)$. Then, there exists $x \in \nu$ such that $x > \beta$. In this case, the $(1,2)$-th entry in $\pi(\psi)$ is $\beta$ by $q_j > \#(\nu_j \cap \nu_{<j})$. If $\pi(\psi) \cong \pi_\lambda$, there exists no $x \in \nu$ with $x > \lambda_{p+1} + \frac{1}{2}(N-1)$, i.e., $f = 0$, and $\beta$ is the maximal member in $\nu$ with multiplicity two. Consider the number of boxes from the top to the box filled with $\lambda_{p+1} + \frac{1}{2}(N-1) - q'$ in the first column. For $\pi_\lambda$, by $f = 0$, this number is $q'$, but for $\pi(\psi)$, it is strictly less than $q'$ by $\beta < \lambda_{p+1} + \frac{1}{2}(N-1)$ unless $\beta = \lambda_p - \frac{1}{2}(N+1) + p'$. If $\beta = \lambda_p - \frac{1}{2}(N+1) + p'$, consider the location of the box filled with the unique entry $\lambda_{p+1} + \frac{1}{2}(N-1) - q'$. This is in the first column for $\mathrm{Ann}(\pi_\lambda)$ and in the second column for $\mathrm{Ann}(\pi(\psi))$. Hence, the representations $\pi_\lambda$ and $\mathscr{A}_{\underline{d}_0}(\psi)$ are different. Suppose next that $\beta > \lambda_{p+1} + \frac{1}{2}(N-1)$. Then, $f > 0$. If there exists $x \in \nu$ with $x > \beta$, the $(1,2)$-th entry in $\pi(\psi)$ is $\beta$. Then, one has $\beta = \lambda_{p+1} + \frac{1}{2}(N-1)$. This is a contradiction. When there exists no $x \in \nu$ with $x > \beta$, we compare the $\nu$-antitableaux of $\pi_\lambda$ and $\pi(\psi)$. In this case, for $\pi_\lambda$, the number of boxes from the top to the box filled with $\lambda_{p+1} + \frac{1}{2}(N+1) - q'$ in the first column is at most $\max\{-q + q' + (\lambda_p - \lambda_{p+1}), q'\}$, that is strictly less than $f+q'$ by $f \neq 0$. Note that $-q+q'+(\lambda_p-\lambda_{p+1})$ is equal to the integer $a$ such that $\lambda_a+\frac{1}{2}(p-q+1)-a=\lambda_{p+1}+\frac{1}{2}(N+1)-q'$. For $\pi(\psi)$, the number of boxes from the top to the box filled with $\lambda_{p+1}+\frac{1}{2}(N+1)-q'$ in the first column is $f+q'$, since $\nu_j$ contains $I$. Hence, the tableaux are different. We conclude that $\beta$ is equal to $\lambda_{p+1}+\frac{1}{2}(N-1)$ if $\pi_\lambda \cong \mathscr{A}_{\underline{d}_0}(\psi)$. In the following, we may assume $\beta=\lambda_{p+1}+\frac{1}{2}(N-1)$.

It remains to show $\alpha=\lambda_p-\frac{1}{2}(N-1)$. If there exists no $x \in \nu$ with $x > \lambda_{p+1}+\frac{1}{2}(N-1)$, the first column of $\mathrm{Ann}(\pi(\psi))$ consists of

$$\lambda_{p+1}+\tfrac{1}{2}(N-1)=\beta, \dots, \alpha, \dots$$

from top to bottom. Then, the entry next to $\alpha$ is strictly less than $\alpha-1$ since $q_j > \#(\nu_j \cap \nu_{<j})$. For $\pi_\lambda$, in the first column, the entry next to $\lambda_p-\frac{1}{2}(N-1)$ is strictly less than $\lambda_p-\frac{1}{2}(N-1)-1$ and the entry next to $x$ with $\lambda_p-\frac{1}{2}(N-1) < x \leq \lambda_{p+1}+\frac{1}{2}(N-1)$ is $x-1$. Hence, if $\pi_\lambda \cong \mathscr{A}_{\underline{d}_0}(\psi)$, we have $\alpha=\lambda_p-\frac{1}{2}(N-1)$. We may assume that there exists $x \in \nu$ with $x > \lambda_{p+1}+\frac{1}{2}(N-1)$, i.e., $f \neq 0$. Recall that by $\pi_\lambda \cong \mathscr{A}_{\underline{d}_0}(\psi)$, we have $p_j > \#(\nu_4')$. This shows that the entry next to $\alpha$ in the first column is strictly less than $\alpha-1$ if the box exists. By the description of $\mathrm{Ann}(\pi_\lambda)$ and the same discussion above, we have $\alpha=\lambda_p-\frac{1}{2}(N-1)$. □

For the converse, apply Lemma 3.2(1) and Corollaries 3.6–3.7. We then have $\pi_\lambda \cong \mathscr{A}_{\underline{d}_0}(\psi)$. This completes the proof.

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