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DEFECT RELATION OF $n+1$ COMPONENTS THROUGH THE GCD METHOD

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We study the defect relation through the GCD method. In particular, among other results, we extend the defect relation result of Chen, Huynh, Sun and Xie (2025) to moving targets. The truncated defect relation is also studied. Furthermore, we obtain the degeneracy locus, which can be determined effectively and is independent of the maps under the consideration.

1. Motivation

Let $f:\mathbb{C}\to\mathbb{P}^n(\mathbb{C})$ be a holomorphic map. It is known that (see [10; 14; 15]) if $f(\mathbb{C})$ omits $n+2$ smooth hypersurfaces D_j, $1\le j\le n+2$, of $\mathbb{P}^n(\mathbb{C})$, where $D:=D_1+\cdots+D_{n+2}$ is located in general position, then f must be algebraically degenerate (that is, $f(\mathbb{C})$ is contained in a proper subvariety of $\mathbb{P}^n(\mathbb{C})$). In [11], J. Noguchi, J. Winkelmann, and K. Yamanoi showed that the number $n+2$ of the omitting hypersurfaces could be reduced to $n+1$ when $\deg D\ge n+2$. Their proof relies on their earlier result for holomorphic maps from $\mathbb{C}$ in the semiabelian variety $A:=(\mathbb{C}^*)^n$, which is stated as follows.

Theorem A [12]. *Let D be an effective divisor on $A:=(\mathbb{C}^*)^n$. Let $f:\mathbb{C}\to A$ be an algebraically nondegenerate holomorphic map. Then there exists a smooth compactification of A independent of f, such that, for any $\epsilon>0$,*

$$N_f(D,r)-N_f^{(1)}(D,r)\le_{\text{exc}}\epsilon T_{f,\overline{D}}(r). \tag{1-1}$$

Using the above theorem, Noguchi, Winkelmann, and Yamanoi showed that one can reduce the number of omitting divisors by one (i.e., from $n+2$ to $n+1$). Their argument is similar to ours described in Section 5. We briefly outline the argument here: Assume that $D_j=\{Q_j=0\}$ and $f(\mathbb{C})$ omits D_j for $1\le j\le n+1$. Consider a morphism $\pi:\mathbb{P}^n(\mathbb{C})\to\mathbb{P}^n(\mathbb{C})$ given by $\boldsymbol{x}\mapsto[Q_1^{a_1}(\boldsymbol{x}):\cdots:Q_{n+1}^{a_{n+1}}(\boldsymbol{x})]$, where

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$a_i := \mathrm{lcm}(\deg Q_1, \dots, \deg Q_{n+1})/\deg Q_i$. Let

$$G := \det\left(\frac{\partial Q_i}{\partial x_j}\right)_{\substack{1\le i\le n+1\\0\le j\le n}} \in \mathbb{C}[x_0, \dots, x_n].$$

By taking out a nonconstant irreducible factor $\tilde{G}$ of G in $\mathbb{C}[x_0, \dots, x_n]$, one produces an additional hypersurface $\tilde{D}_{n+2} = \{\tilde{G} = 0\}$ in $\mathbb{P}^n(\mathbb{C})$. Furthermore, one can show that $D_1, \dots, D_{n+1}, \tilde{D}_{n+2}$ are located in general position, and, by using (1-1), one can show that $N_f(\tilde{D}_{n+2}, r) \le_{\mathrm{exc}} \epsilon T_f(r)$. Thus we can apply the second main theorem obtained by the first author [15] to get the conclusion.

In a recent manuscript by Z. Chen, D. T. Huynh, R. Sun and S. Y. Xie (see [1]), the result of Noguchi, Winkelmann, and Yamanoi mentioned above was further extended to the following defect relation.

Theorem B (Chen, Huynh, Sun and Xie [1]). *Let $\{D_i\}_{i=1}^{n+1}$ be $n+1$ hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ with total degree $\sum_{i=1}^{n+1} \deg D_i \ge n+2$ satisfying one precise generic condition* (*see* (4.3) *in* [1]). *Then, for every algebraically nondegenerate entire holomorphic curve $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$, the following defect relation holds*:

$$\sum_{i=1}^{n+1} \delta_f(D_j) < n+1.$$

In the omitting case, we have that $Q_j(f)$ is nowhere zero for all $1 \le j \le n+1$ where $D_j = \{Q_j = 0\}$, so that one can reduce it to the semiabelian variety case $(\mathbb{C}^*)^n$ by considering

$$F := \left(\frac{Q_1(f)}{Q_{n+1}(f)}, \dots, \frac{Q_n(f)}{Q_{n+1}(f)}\right) \in (\mathbb{C}^*)^n,$$

assuming that $\deg Q_1 = \cdots = \deg Q_{n+1}$. Hence Theorem A could be applied directly. However, in their "proof-by-contradiction" argument of the proof of Theorem B, the condition that $\sum_{i=1}^{n+1} \delta_f(D_j) = n+1$ only implies that $N_f(r, D_i) = o(T_f(r))$ (rather than $f(\mathbb{C})$ omitting D_j). To overcome this difficulty, they used the "parabolic Nevanlinna theory" developed by M. Păun and N. Sibony (see [13]), by considering the holomorphic mapping $f : Y \to \mathbb{P}^n(\mathbb{C})$ with $Y := \mathbb{C} \setminus f^{-1}(D)$, which leads to the omitting case after restricting f to Y. The key ingredient in their paper is to show that Y is an open parabolic Riemann surface with exhaustion function σ satisfying

$$\limsup_{r\to\infty} \frac{\mathfrak{X}_\sigma(r)}{T_f(r)} = 0.$$

While the method of Chen, Huynh, Sun, and Xie is very interesting and creative, it still relies on the result of Noguchi, Winkelmann, and Yamanoi (Theorem A), which greatly depends on the geometry of semiabelian varieties. For example, it is very hard to generalize the result to the moving target case.

This paper studies the defect relation through the GCD method. We don't use Theorem A. Indeed, we give and prove a variant and more general version of Theorem A by using the GCD theorem established by Aaron Levin and the second author [9]. This allows us to get a much more general defect relation (for example, the moving target case). The method was initiated by P. Corvaja and U. Zannier (see [2]), where they studied the $n = 2$ case. After Aaron Levin and the second author [9] established the general GCD theorem, it has been successfully used in a series papers by the second author and her coauthors; see [4; 5; 6; 7]. The purpose of this paper is to further use the ideas developed in [4; 5; 6; 7] to extend the defect relation, for example, to the moving target case, by using the GCD method. Furthermore, the truncated defect relation is also studied. We also pay attention on the degenerate locus. In particular, we can relax the condition that f is algebraically nondegenerate to the condition that the image of f is not contained in a subvariety Z which can be effectively predetermined and is independent of f, in the spirit of the strong Green–Griffiths–Lang conjecture.

2. Statement of the results

We use the standard notation in Nevanlinna theory (see [16] or [4; 5; 6; 7]). Let $\boldsymbol{g} = (g_0, \dots, g_n) : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve, where $g_0, \dots, g_n$ are entire functions without common zero. We recall that the *small function field with respect to* $\boldsymbol{g}$ is given by

$$K_{\boldsymbol{g}} := \{a : a \text{ is a meromorphic function on } \mathbb{C} \text{ with } T_a(r) = o(T_{\boldsymbol{g}}(r))\}. \tag{2-1}$$

Let K be a subfield of the field $\mathcal{M}$ of meromorphic functions. We say that *Z is a Zariski closed subset in $\mathbb{P}^n$ defined over K* if there exists a nonconstant homogeneous polynomial $F \in K[x_0, \dots, x_n]$ such that

$$Z = \{[f_0 : \cdots : f_n] \in \mathbb{P}^n(\mathcal{M}) : F(f_0, \dots, f_n) \equiv 0\}.$$

We say a holomorphic map $\boldsymbol{g} : \mathbb{C} \to \mathbb{P}^n$ is not contained in Z if $F(\boldsymbol{g})$ is not identically zero. In particular, when $K = \mathbb{C}$, the Zariski closed set is defined over $\mathbb{C}$, that is, $F \in \mathbb{C}[x_0, \dots, x_n]$, and $\boldsymbol{g}$ is not contained in Z is equivalent to $F(\boldsymbol{g}) \not\equiv 0$. For each homogeneous polynomial $G = \sum_I a_I \boldsymbol{x}^I \in K[x_0, \dots, x_n]$, where $I = (i_0, \dots, i_n) \in \mathbb{Z}_{\geq 0}^{n+1}$ and $\boldsymbol{x}^I = x_0^{i_0} \cdots x_n^{i_n}$, we define $G(z_0) := \sum_I a_I(z_0)\boldsymbol{x}^I$ if all the coefficients of G are holomorphic at z_0 and do not vanish simultaneously at z_0. Let $G_1, \dots, G_q$ be nonconstant homogeneous polynomials in $K[x_0, \dots, x_n]$. We say that they are *in weakly general position* if there exists a point $z_0 \in \mathbb{C}$ such that each $G_i(z_0)$, $1 \leq i \leq q$, can be defined as above and the union of the zero loci of $G_i(z_0)$ (as a divisor in $\mathbb{P}^n(\mathbb{C})$), $1 \leq i \leq q$, is in general position.

Theorem 1. *Let K be a subfield of the field $\mathcal{M}$ of meromorphic functions. Let F be a nonconstant homogeneous polynomial in $K[x_0, \dots, x_n]$ with no monomial factors and no repeated factors. Denote by H_i, $0 \le i \le n$, the coordinate hyperplanes of $\mathbb{P}^n(\mathbb{C})$. Then, for any $\epsilon > 0$, there exists a proper Zariski closed subset Z of $\mathbb{P}^n$ defined over K such that for any nonconstant holomorphic curve $\boldsymbol{g} : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ with $K \subset K_{\boldsymbol{g}}$, $N_{\boldsymbol{g}}(H_i, r) = o(T_{\boldsymbol{g}}(r))$ for $0 \le i \le n$ and $\boldsymbol{g}$ not contained in Z, we have*

$$N_{\boldsymbol{g}}([F=0], r) - N_{\boldsymbol{g}}^{(1)}([F=0], r) \le_{\text{exc}} \epsilon T_{\boldsymbol{g}}(r). \tag{2-2}$$

If we assume furthermore that the hypersurface defined by F in $\mathbb{P}^n$ and the coordinate hyperplanes are in weakly general position, then

$$N_{\boldsymbol{g}}^{(1)}([F=0], r) \ge_{\text{exc}} (\deg F - \epsilon) \cdot T_{\boldsymbol{g}}(r). \tag{2-3}$$

Moreover, the exceptional set Z can be expressed as the zero locus of a finite set $\Sigma \subset K[x_0, \dots, x_n]$ with the following properties:

(Z1) *Σ depends on ϵ and F only and can be determined explicitly*;

(Z2) *the degree of each polynomial in Σ can be effectively bounded from above in terms of ϵ, n, and the degree of F.*

We apply Theorem 1 to derive the following version of the strong Green–Griffiths–Lang conjecture for moving targets.

Theorem 2. *Let K be a subfield of the field of meromorphic functions. Let F_i, $1 \le i \le n+1$, be homogeneous irreducible polynomials of positive degree in $K[x_0, \dots, x_n]$ such that $\sum_{i=1}^{n+1} \deg F_i \ge n+2$. Assume that there exists $z_0 \in \mathbb{C}$ such that all the coefficients of all F_i, $1 \le i \le n+1$, are holomorphic at z_0 and the zero locus of F_i evaluated at z_0, $1 \le i \le n+1$, intersect transversally. Then there exists a nontrivial homogeneous polynomial $B \in K[x_0, \dots, x_n]$ such that for any nonconstant holomorphic map $\boldsymbol{f} : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ with $K \subset K_{\boldsymbol{f}}$ and $N_{F_i(\boldsymbol{f})}(0, r) = o(T_{\boldsymbol{f}}(r))$ for $1 \le i \le n+1$, we have $B(\boldsymbol{f}) \equiv 0$. Furthermore, B can be determined effectively and its degree can be effectively bounded from above in terms of n, and the degrees of F_i, $1 \le i \le n+1$.*

As a consequence, we obtain the following defect relation for moving targets.

Corollary 3 (defect relation for moving targets). *With the same notation and assumptions as in Theorem 2, let $D_i = [F_i = 0]$ for $1 \le i \le n+1$. Then for any nonconstant holomorphic map $\boldsymbol{f} : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ with $K \subset K_{\boldsymbol{f}}$ and $B(\boldsymbol{f}) \not\equiv 0$, the following defect inequality holds*:

$$\sum_{i=1}^{n+1} \delta_{\boldsymbol{f}}(D_i) < n+1,$$

where $D_i = [F_i = 0]$. *Additionally, if* $n = 2$, *then*

$$\sum_{i=1}^{3} \delta_f^{(1)}(D_i) < 3,$$

where, for a divisor D *with* $d = \deg D$,

$$\delta_f^{(1)}(D) = 1 - \limsup_{r\to\infty} \frac{N_f^{(1)}(D, r)}{dT_f(r)}.$$

When $K = \mathbb{C}$, the following strong defect relation improves Theorem B by giving an explicit exceptional set and a truncated defect bound when $n = 2$.

Corollary 4. *Let* D_i, $1 \le i \le n+1$, *be* $n+1$ *hypersurfaces in* $\mathbb{P}^n(\mathbb{C})$, *not all being hyperplanes. Assume* D_i, $1 \le i \le n+1$, *intersect transversally. Then there exists a Zariski closed subset* Z *in* $\mathbb{P}^n(\mathbb{C})$, *which can be determined effectively and its degree can be effectively bounded from above in terms of* n, *and the degree of* D_i, *such that for any nonconstant holomorphic map* $\boldsymbol{f} : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ *whose image is not contained in* Z, *the following defect inequality holds*:

$$\sum_{i=1}^{n+1} \delta_f(D_i) < n+1.$$

Additionally, if $n = 2$, *then*

$$\sum_{i=1}^{3} \delta_f^{(1)}(D_i) < 3.$$

3. Some preliminaries and the GCD theorem

3.1. ***Preliminaries.*** We now introduce some basic notation and definitions from Nevanlinna theory, and recall fundamental results. For further details, we refer the reader to [16]. Let f be a meromorphic function, $z \in \mathbb{C}$ be a complex number, and m be a positive integer. Define the valuation functions $v_z(f) := \mathrm{ord}_z(f)$,

$$v_z^+(f) := \max\{0, v_z(f)\}, \quad \text{and} \quad v_z^-(f) := -\min\{0, v_z(f)\}.$$

Let $n_f(\infty, r)$ (respectively, $n_f^{(m)}(\infty, r)$) denote the number of poles of f in the set $\{z : |z| \le r\}$, counting multiplicity (respectively, ignoring multiplicity larger than $m \in \mathbb{N}$). The associated *counting function* and *truncated counting function* of f of order m at ∞ are

$$N_f(\infty, r) := \int_0^r \frac{n_f(\infty, t) - n_f(\infty, 0)}{t}\, dt + n_f(\infty, 0)\log r,$$

$$N_f^{(m)}(\infty, r) := \int_0^r \frac{n_f^{(m)}(\infty, t) - n_f^{(m)}(\infty, 0)}{t}\, dt + n_f^{(m)}(\infty, 0)\log r.$$

For $a \in \mathbb{C}$, the *counting function* and *truncated counting function* of f with respect to a are defined as

$$N_f(a, r) := N_{1/(f-a)}(r, \infty) \quad \text{and} \quad N_f^{(m)}(a, r) := N_{1/(f-a)}^{(m)}(\infty, r).$$

The *proximity function* $m_f(\infty, r)$ is given by

$$m_f(\infty, r) := \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi},$$

where $\log^+ x = \max\{0, \log x\}$ for $x \geq 0$. The *characteristic function* is defined by

$$T_f(r) := m_f(\infty, r) + N_f(\infty, r).$$

Let $f_1, \ldots, f_n$ be meromorphic functions with $n \geq 2$. Define the local gcd multiplicity function by

$$n(f_1, \ldots, f_n, r) := \sum_{|z| \leq r} \min_{1 \leq i \leq n} \{v_z^+(f_i)\}$$

and the associated gcd counting function by

$$N_{\text{gcd}}(f_1, \ldots, f_n, r) := \int_0^r \frac{n(f_1, \ldots, f_n, t) - n(f_1, \ldots, f_n, 0)}{t} \, dt + n(f_1, \ldots, f_n, 0) \log r.$$

Let $\boldsymbol{f} : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map and $(f_0, \ldots, f_n)$ be a reduced representation of $\boldsymbol{f}$, i.e., $f_0, \ldots, f_n$ are entire functions on $\mathbb{C}$ without common zeros. The *Nevanlinna–Cartan characteristic function* $T_{\boldsymbol{f}}(r)$ is defined by

$$T_{\boldsymbol{f}}(r) = \int_0^{2\pi} \log \max\{|f_0(re^{i\theta})|, \ldots, |f_n(re^{i\theta})|\} \frac{d\theta}{2\pi}.$$

Let $D = [F = 0]$ be a divisor in $\mathbb{P}^n(\mathbb{C})$ defined by a homogeneous polynomial $F \in \mathbb{C}[x_0, \ldots, x_n]$. The counting function with respect to D is defined by $N_{\boldsymbol{f}}(D, r) = N_{F(\boldsymbol{f})}(0, r)$.

We will make use of the following elementary inequality (see [16]).

Proposition 5. *Let $\boldsymbol{f} = (f_0, \ldots, f_n) : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be holomorphic curve, where $f_0, \ldots, f_n$ are entire functions without common zeros. Assume that f_0 is not identically zero. Then*

$$T_{f_j/f_0}(r) + O(1) \leq T_{\boldsymbol{f}}(r) \leq \sum_{j=1}^n T_{f_j/f_0}(r) + O(1).$$

Combining Proposition 5 with [17, Theorem 2.1], we obtain the following result.

Theorem 6 [17, Theorem 2.1]. *Let $f_0, \dots, f_n$ be entire functions with no common zeros. Assume that f_{n+1} is the holomorphic function such that*

$$f_0 + \cdots + f_n + f_{n+1} = 0.$$

If $\sum_{i \in I} f_i \neq 0$ for any proper subset $I \subset \{0, \dots, n+1\}$, then

$$T_{f_j/f_i}(r) \leq T_{\boldsymbol{f}}(r) + O(1) \leq_{\mathrm{exc}} \sum_{i=0}^{n+1} N_{f_i}^{(n)}(0, r) + O(\log T_{\boldsymbol{f}}(r))$$

for any pair $0 \leq i, j \leq n$, where $\boldsymbol{f} := (f_0, \dots, f_n)$.

We will use the following version of the Hilbert Nullstellensatz, reformulated from [8, Chapter IX, Theorem 3.4]. See also [3, Proposition 2.1; 18, Chapter XI].

Proposition 7. *Let K be a subfield of the field of meromorphic functions. Let $\{Q_i\}_{i=1}^{n+1}$ be a set of homogeneous polynomials in $K[x_0, \dots, x_n]$ in weakly general position and with* $\deg Q_j = d_j \geq 1$. *Then there exist a positive integer s, an element $R \in K$ which is not identically zero and $P_{ji} \in K[x_0, \dots, x_n]$, $1 \leq i, j \leq n+1$, such that, for each $0 \leq j \leq n$,*

$$x_j^s \cdot R = \sum_{i=1}^{n+1} P_{ji} Q_i.$$

The following is a version of the Borel lemma for small functions. The proof can easily be obtained with some slightly modifications from [4, Lemma 3.3].

Lemma 8. *Let $f_0, \dots, f_n$ be nontrivial entire functions with no common zero and let $\boldsymbol{f} := (f_0, \dots, f_n)$. Assume that*

$$N_{f_i}^{(1)}(0, r) = o(T_{\boldsymbol{f}}(r)) \quad \text{for } 0 \leq i \leq n.$$

If $f_0, \dots, f_n$ are linearly dependent over $K_{\boldsymbol{f}}$, then for each $i \in \{0, \dots, n\}$ there exists $j \in \{0, \dots, n\}$ with $j \neq i$ such that $f_i/f_j \in K_{\boldsymbol{f}}$.

3.2. *The GCD theorem.*

Theorem 9 (the GCD theorem). *Let $g_0, g_1, \dots, g_n$ be entire functions without common zeros and let $\boldsymbol{g} = [g_0 : g_1 : \cdots : g_n]$. Let $F, G \in K_{\boldsymbol{g}}[x_0, \dots, x_n]$ be nonconstant coprime homogeneous polynomials. Assume that one of the following holds*:

(a) $N_{g_i}(0, r) = o(T_{\boldsymbol{g}}(r))$ *for* $0 \leq i \leq n$;

(b) $N_{g_i}^{(1)}(0, r) = o(T_{\boldsymbol{g}}(r))$ *for $0 \leq i \leq n$ and one of the hypersurfaces defined by $G = 0$ or $F = 0$ in $\mathbb{P}^n(K)$ is in weakly general position with the $n+1$ coordinate hyperplanes.*

Then, for any $\epsilon > 0$, there exists a positive integer m independent of $\mathbf{g}$ such that we have either

$$N_{\mathrm{gcd}}(F(g_0,\ldots,g_n), G(g_0,\ldots,g_n), r) \le_{\mathrm{exc}} \epsilon T_{\mathbf{g}}(r), \tag{3-1}$$

or

$$T_{(g_1/g_0)^{m_1}\cdots(g_n/g_0)^{m_n}}(r) = o(T_{\mathbf{g}}(r)) \tag{3-2}$$

for some nontrivial tuple of integers $(m_1,\ldots,m_n)$ with $|m_1|+\cdots+|m_n| \le 2m$.

For the convenience of later application, we state the following result for $n = 1$.

Proposition 10. *Let g_0, g_1 be entire functions without common zeros and let $\mathbf{g} = (g_0, g_1)$. Assume that $\mathbf{g}$ is not constant. Let $F, G \in K_{\mathbf{g}}[x_0, x_1]$ be nonconstant coprime homogeneous polynomials. Then*

$$N_{\mathrm{gcd}}(F(g_0,g_1), G(g_0,g_1), r) \le o(T_{\mathbf{g}}(r)). \tag{3-3}$$

Proof. Since F and G are coprime homogeneous polynomials in $K_{\mathbf{g}}[x_0, x_1]$, we may apply Proposition 7 to find an integer s, $R \in K_{\mathbf{g}} \setminus \{0\}$ and $H_i \in K_{\mathbf{g}}[x_0, x_1]$, $1 \le i \le 4$, such that

$$x_0^s \cdot R = H_1F + H_2G \quad \text{and} \quad x_1^s \cdot R = H_3F + H_4G. \tag{3-4}$$

Here, we may assume that H_i, $1 \le i \le 4$, are homogeneous polynomials with degree equal to $s - \deg F$. By evaluating (3-4) at (g_0, g_1), we have

$$\begin{aligned} g_0^s \cdot R &= H_1(g_0,g_1)F(g_0,g_1) + H_2(g_0,g_1)G(g_0,g_1),\\ g_1^s \cdot R &= H_3(g_0,g_1)F(g_0,g_1) + H_4(g_0,g_1)G(g_0,g_1). \end{aligned} \tag{3-5}$$

Since g_0 and g_1 have no common zeros, we observe that

$$\min\{v_z^+(F(g_0,g_1)), v_z^+(G(g_0,g_1))\} \le v_z^+(R) + \sum_{\alpha\in I} v_z^-(\alpha) \tag{3-6}$$

for each $z \in \mathbb{C}$. Here I is the set of nontrivial coefficients of H_i, $1 \le i \le 4$. Hence,

$$N_{\mathrm{gcd}}(F(g_0,g_1), G(g_0,g_1), r) \le N_R(0,r) + \sum_{\alpha\in I} N_\alpha(\infty, r) \le o(T_{\mathbf{g}}(r)), \tag{3-7}$$

as R and the coefficients of F_i are in $K_{\mathbf{g}}$. ☐

To prove Theorem 9, we use the following fundamental result by Levin and the second author for $n \ge 2$.

Theorem 11 [9, Theorem 5.7]. *Let $g_0, g_1, \ldots, g_n$ be entire functions without common zeros with $n \ge 2$ and let $\mathbf{g} = [g_0 : g_1 : \cdots : g_n]$. Let $F, G \in K_{\mathbf{g}}[x_0, x_1, \ldots, x_n]$ be coprime homogeneous polynomials of the same degree $d > 0$. Let I be the set of exponents $\boldsymbol{i}$ such that $\mathbf{x}^{\boldsymbol{i}}$ appears with a nonzero coefficient in either F or G. Let*

$m \geq d$ be a positive integer. Suppose that $\{g_0^{i_0}\cdots g_n^{i_n} : i_0+\cdots+i_n = m\}$ is linearly independent over $K_{\boldsymbol{g}}$. Then, for any $\epsilon > 0$, there is a positive integer L such that

$$\begin{aligned}(3\text{-}8)\quad & MN_{\mathrm{gcd}}(F(\boldsymbol{g}), G(\boldsymbol{g}), r) \\ &\leq_{\mathrm{exc}} c_{m,n,d}\sum_{i=1}^{n} N_{g_i}^{(L)}(0,r) + \left(\frac{m}{n+1}\binom{m+n}{n} - c_{m,n,d} - M'm\right)\sum_{i=1}^{n} N_{g_i}(0,r) \\ &\quad + \binom{m+n-2d}{n} N_{\mathrm{gcd}}(\{\boldsymbol{g}^{\boldsymbol{i}}\}_{i\in I}, r) + \left(M'mn + \epsilon m + \frac{M\epsilon}{2}\right) T_{\boldsymbol{g}}(r) + o(T_{\boldsymbol{g}}(r)),\end{aligned}$$

where $c_{m,n,d} = 2\binom{m+n-d}{n+1} - \binom{m+n-2d}{n+1}$, $M = 2\binom{m+n-d}{n} - \binom{m+n-2d}{n}$, and M' is an integer of order $O(m^{n-2})$, where $\leq_{\mathrm{exc}}$ means the inequality holds for all $r \in (0,\infty)$ except for a set E of finite measure.

We note that $M' := \dim K_{\boldsymbol{g}}[x_0,\ldots,x_n]_m/(F,G)_m \leq d^2\binom{m+n-2}{n-2}$.

Proof of Theorem 9. Without loss of generality, we assume that $\deg F = \deg G$. We first prove when $n \geq 2$. Let $\epsilon > 0$. To establish (3-1) or (3-2), we can assume that ϵ is sufficiently small. We can choose a real $C_1 \geq 1$ independent of ϵ and $\boldsymbol{g}$ such that $m = C_1\epsilon^{-1} \geq 2d$,

$$(3\text{-}9)\quad \frac{M'mn}{M} \leq \frac{\epsilon}{4}, \quad \text{and} \quad \frac{1}{M}\left(\frac{m}{n+1}\binom{m+n}{n} - c_{m,n,d} - M'm\right) \leq \frac{\epsilon}{4(n+1)}.$$

We may assume that each g_i is not identically zero; otherwise, (3-2) holds trivially. Suppose that the set $\{g_0^{i_0}\cdots g_n^{i_n} : i_0+\cdots+i_n = m\}$ is linearly independent over $K_{\boldsymbol{g}}$. We aim at concluding (3-1) under assumption (a) or (b). Suppose (a) holds, i.e., $N_{g_i}(0,r) = o(T_{\boldsymbol{g}}(r))$ for $0 \leq i \leq n$. Then (3-8) implies that

$$(3\text{-}10)\quad N_{\mathrm{gcd}}(F(\boldsymbol{g}), G(\boldsymbol{g}), r) \leq_{\mathrm{exc}} \left(\frac{M'mn}{M} + \epsilon\frac{m}{M} + \frac{\epsilon}{2}\right) T_{\boldsymbol{g}}(r) + o(T_{\boldsymbol{g}}(r)) < \epsilon T_{\boldsymbol{g}}(r).$$

If (b) holds, then

$$N_{g_i}^{(L)}(0,r) \leq L N_{g_i}^{(1)}(0,r) = o(T_{\boldsymbol{g}}(r))$$

for $0 \leq i \leq n$. The assumption that one of $[G = 0]$ or $[F = 0]$ is in weakly general position with the $n+1$ coordinate hyperplanes in $\mathbb{P}^n$ implies that the set $\{(d,0,\ldots,0),\ldots,(0,\ldots,0,d)\}$ is a subset of I. Since $g_0,\ldots,g_n$ are entire function with no common zero, we have

$$N_{\mathrm{gcd}}(\{\boldsymbol{g}^{\boldsymbol{i}}\}_{i\in I}, r) = 0$$

when (b) holds. Then by (3-8), (3-9) and that $N_{g_i}(0,r) \leq T_{\boldsymbol{g}}(r)$, we obtain (3-1).

Finally, if the set $\{g_0^{i_0}\cdots g_n^{i_n} : i_0+\cdots+i_n = m\}$ is dependent over $K_{\boldsymbol{g}}$, then we may apply Lemma 8 to derive that there exists a nontrivial n-tuple of integers $(j_1,\ldots,j_n)$ with $|j_1|+\cdots+|j_n| \leq 2m$ such that

$$T_{(g_1/g_0)^{j_1}\cdots(g_n/g_0)^{j_n}}(r) = o(T_{\boldsymbol{g}}(r)). \qquad \square$$

4. Proof of Theorem 1

4.1. ***Some lemmas.*** We recall some lemmas from [7].

Lemma 12. *Let $n \geq 2$ and let $(m_1, \dots, m_n)$ be a nonzero vector in $\mathbb{Z}^n$ with* $\gcd(m_1, \dots, m_n) = 1$. *Then there exist $\mathbf{v}_i = (v_{i,1}, \dots, v_{i,n}) \in \mathbb{Z}^n$ for $1 \leq i \leq n-1$ such that*

$$|v_{i,j}| \leq \max\{|m_j|, 1\} \quad \text{for } 1 \leq j \leq n$$

and $(m_1, \dots, m_n)$ together with the $\mathbf{v}_i$'s form a basis of $\mathbb{Z}^n$.

Let k be a field and let q and r be positive integers. We write $\boldsymbol{t} := (t_1, \dots, t_q)$ and $\boldsymbol{x} := (x_1, \dots, x_r)$. For $\boldsymbol{i} = (i_1, \dots, i_r) \in \mathbb{Z}^r$, define $\boldsymbol{x}^{\boldsymbol{i}} = x_1^{i_1} \cdots x_r^{i_r}$ and $\boldsymbol{t}^{\boldsymbol{i}} = t_1^{i_1} \cdots t_r^{i_r}$. For $\Sigma \subseteq k[\boldsymbol{t}]$, let $\mathcal{Z}(\Sigma) = \{\lambda \in k^q : f(\lambda) = 0 \text{ for every } f \in \Sigma.\}$.

Lemma 13. *Assume that k is infinite. Let $f(\boldsymbol{t}, \boldsymbol{x}) \in k[\boldsymbol{t}, \boldsymbol{x}]$ be a polynomial with no monomial factor and no repeated irreducible factor in $k[\boldsymbol{t}, \boldsymbol{x}]$. Then there exists an effectively computable nonempty finite set $\Sigma \subset k[\boldsymbol{t}] \setminus \{0\}$ such that for every $\lambda \in k^q \setminus \mathcal{Z}(\Sigma)$, the polynomial $f(\lambda, \boldsymbol{x})$ has no monomial or repeated irreducible factor. Moreover, the cardinality of Σ and the degree of each polynomial in Σ can be bounded effectively in terms of q, r, and the degree of f. Furthermore, if $f(\boldsymbol{t}, \boldsymbol{x}) \in k_0[\boldsymbol{t}, \boldsymbol{x}]$ for k_0 being a subfield of k, then Σ is defined over k_0.*

4.2. ***Preliminary theorem.*** Let $\boldsymbol{g} = (g_0, \dots, g_n)$, where $g_i \not\equiv 0$, $0 \leq i \leq n$, are entire functions without common zeros. Let $u_i = g_i/g_0$, for $1 \leq i \leq n$. We observe that

$$\max_{1 \leq j \leq n} \{T_{u_j}(r)\} \leq T_{\boldsymbol{g}}(r) \leq n \max_{1 \leq j \leq n} \{T_{u_j}(r)\}, \tag{4-1}$$

and

$$N_{u_i}(0, r) + N_{u_i}(\infty, r) \leq N_{g_i}(0, r) + N_{g_0}(0, r) \tag{4-2}$$

for each $1 \leq i \leq n$.

Recall that

$$K_{\boldsymbol{g}} := \{a : a \text{ is a meromorphic function on } \mathbb{C} \text{ with } T_a(r) \leq o(T_{\boldsymbol{g}}(r))\},$$

which is the field of meromorphic functions of slow growth with respect to $\boldsymbol{g}$. We note that $a' \in K_{\boldsymbol{g}}$ if $a \in K_{\boldsymbol{g}}$. Furthermore, $u_i'/u_i \in K_{\boldsymbol{g}}$ if

$$N_{u_i}^{(1)}(0, r) + N_{u_i}^{(1)}(\infty, r) \leq o\Big(\max_{1 \leq j \leq n} \{T_{u_j}(r)\}\Big).$$

Let $\boldsymbol{x} := (x_1, \dots, x_n)$ and $\boldsymbol{u} = (u_1, \dots, u_n)$. For $\boldsymbol{i} = (i_1, \dots, i_n) \in \mathbb{Z}^n$, we let $\boldsymbol{x}^{\boldsymbol{i}} := x_1^{i_1} \cdots x_n^{i_n}$ and $\boldsymbol{u}^{\boldsymbol{i}} := u_1^{i_1} \cdots u_n^{i_n}$. For a nonconstant polynomial

$$F(\boldsymbol{x}) = \sum_{\boldsymbol{i}} a_{\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}} \in K_{\boldsymbol{g}}[\boldsymbol{x}] := K_{\boldsymbol{g}}[x_1, \dots, x_n],$$

we define

$$D_{\boldsymbol{u}}(F)(\boldsymbol{x}) := \sum_{\boldsymbol{i}} \frac{(a_{\boldsymbol{i}}\boldsymbol{u}^{\boldsymbol{i}})'}{\boldsymbol{u}^{\boldsymbol{i}}}\boldsymbol{x}^{\boldsymbol{i}} = \sum_{\boldsymbol{i}} \Big(a_{\boldsymbol{i}}' + a_{\boldsymbol{i}} \cdot \sum_{j=1}^{n} i_j \frac{u_j'}{u_j}\Big)\boldsymbol{x}^{\boldsymbol{i}} \in K_{\boldsymbol{g}}[\boldsymbol{x}]. \tag{4-3}$$

A direct computation shows that

$$F(\boldsymbol{u})' = D_{\boldsymbol{u}}(F)(\boldsymbol{u}), \tag{4-4}$$

and that the product rule

$$D_{\boldsymbol{u}}(FG) = D_{\boldsymbol{u}}(F)G + FD_{\boldsymbol{u}}(G) \tag{4-5}$$

holds for $F, G \in K_{\boldsymbol{g}}[\boldsymbol{x}]$.

Lemma 14 [5, Lemma 3.1]. *Let F be a nonconstant polynomial in $K_{\boldsymbol{g}}[\boldsymbol{x}]$ with no monomial factors and no repeated factors. Assume that*

$$N_{u_i}^{(1)}(0, r) + N_{u_i}^{(1)}(\infty, r) = o\big(\max_{1\le j\le n}\{T_{u_j}(r)\}\big)$$

for each $1 \le i \le n$. Then F and $D_{\boldsymbol{u}}(F)$ are coprime in $K_{\boldsymbol{g}}[\boldsymbol{x}]$ unless there exists a nontrivial tuple of integers $(m_1, \dots, m_n)$ with $\sum_{i=1}^{n} |m_i| \le 2 \deg F$ such that $T_{u_1^{m_1}\cdots u_n^{m_n}}(r) = o\big(\max_{1\le j\le n}\{T_{u_j}(r)\}\big)$.

We now state a preliminary theorem in affine form.

Theorem 15. *Let K be a subfield of the field of meromorphic functions. Let G be a nonconstant polynomial in $K[x_1, \dots, x_n]$ with no monomial factors and no repeated factors. Assume one of the following holds*:

(a) *$N_{u_i}(0, r) + N_{u_i}(\infty, r) = o\big(\max_{1\le j\le n}\{T_{u_j}(r)\}\big)$ for each $1 \le i \le n$, or*

(b) *$N_{u_i}^{(1)}(0, r) + N_{u_i}^{(1)}(\infty, r) = o\big(\max_{1\le j\le n}\{T_{u_j}(r)\}\big)$ for each $1 \le i \le n$, and that $[G = 0]$ and the $n+1$ coordinate hyperplanes are in weakly general position in $\mathbb{P}^n$.*

For any $\epsilon > 0$, there exists a positive integer m such that for any n-tuple of meromorphic functions $\boldsymbol{u} = (u_1, \dots, u_n)$ satisfying $K \subset K_{\boldsymbol{g}}$, where $\boldsymbol{g} = [1 : u_1 : \cdots : u_n]$, we have either

$$T_{{u_1}^{m_1}\cdots{u_n}^{m_n}}(r) = o\big(\max_{1\le j\le n}\{T_{u_j}(r)\}\big) \tag{4-6}$$

for a nontrivial n-tuple $(m_1, \dots, m_n)$ of integers with $\sum_{i=0}^{n} |m_i| \le m$, or

$$N_{G(\boldsymbol{u})}(0, r) - N_{G(\boldsymbol{u})}^{(1)}(0, r) \le_{\mathrm{exc}} \epsilon \max_{1\le j\le n}\{T_{u_j}(r)\}. \tag{4-7}$$

Proof. Let $z_0 \in \mathbb{C}$. If $v_{z_0}(G(\boldsymbol{u})) \ge 2$, then it follows from (4-4) that $v_{z_0}(D_{\boldsymbol{u}}(G)(\boldsymbol{u})) = v_{z_0}(G(\boldsymbol{u})) - 1$. Hence,

$$\min\{v_{z_0}^+(G(\boldsymbol{u})), v_{z_0}^+(D_{\boldsymbol{u}}(G)(\boldsymbol{u}))\} \ge v_{z_0}^+(G(\boldsymbol{u})) - \min\{1, v_{z_0}^+(G(\boldsymbol{u}))\}.$$

Consequently,

$$N_{\mathrm{gcd}}(G(\boldsymbol{u}), D_{\boldsymbol{u}}(G)(\boldsymbol{u}), r) \geq N_{G(\boldsymbol{u})}(0, r) - N^{(1)}_{G(\boldsymbol{u})}(0, r). \tag{4-8}$$

By Lemma 14, G and $D_{\boldsymbol{u}}(G)$ are either coprime or (4-6) holds for $m = 2 \deg G$. Therefore, we assume that G and $D_{\boldsymbol{u}}(G)$ are coprime. By Theorem 9, we find a positive integer m depending only on ϵ, n and $\deg G$ such that either (4-6) holds or

$$N_{\mathrm{gcd}}(G(\boldsymbol{u}), D_{\boldsymbol{u}}(G)(\boldsymbol{u}), r) \leq_{\mathrm{exc}} \epsilon \max_{1 \leq j \leq n} \{T_{u_j}(r)\}. \tag{4-9}$$

Together with (4-8), we obtain (4-7). □

4.3. ***Further refinement.*** We will prove the following theorem by finding an exceptional set in Theorem 15.

Theorem 16. *Let K be a subfield of the field of meromorphic functions. Let G be a nonconstant polynomial in $K[x_1, \ldots, x_n]$ with no monomial factors and no repeated factors. For any $\epsilon > 0$, there exists a nonconstant polynomial H in $K[x_1, \ldots, x_n]$ such that for any n-tuple of meromorphic functions $\boldsymbol{u} = (u_1, \ldots, u_n)$ satisfying*

$$N_{u_i}(0, r) + N_{u_i}(\infty, r) = o\Big(\max_{1 \leq j \leq n} \{T_{u_j}(r)\}\Big) \quad \text{for each } 1 \leq i \leq n, \tag{4-10}$$

and $K \subset K_{\boldsymbol{g}}$, where $\boldsymbol{g} = [1 : u_1 : \cdots : u_n]$, we have either $H(\boldsymbol{u}) \equiv 0$ or

$$N_{G(\boldsymbol{u})}(0, r) - N^{(1)}_{G(\boldsymbol{u})}(0, r) \leq_{\mathrm{exc}} \epsilon \max_{1 \leq j \leq n} \{T_{u_j}(r)\}. \tag{4-11}$$

Moreover, H can be determined effectively and the degree of H can be bounded effectively in terms of ϵ, n and the degree of G.

Remark. The effectiveness of determining H follows from the application of Lemma 13 in the induction process. Moreover, the estimate for the degree of H depends on the fact that the constant m in Theorem 9 can be determined effectively, as well as on the proof of Theorem 9 and the content of Lemma 12. While a rough bound for the degree of H can, in principle, be obtained by tracing these dependencies, carrying this out would involve substantial additional technical detail beyond the scope of the present work.

Proof. The proof of [7, Theorem 4] can be adapted to suit the current situation. We will closely adhere to their arguments and notation. We first fix some notation:

(i) For a matrix $A = (a_{ij})$ with complex-valued entries, let

$$\|A\|_\infty = \max_i \sum_j |a_{ij}|$$

be the maximum of the absolute row sums.

(ii) We say that a nontrivial meromorphic function β *has small zeros and poles with respect to* $\boldsymbol{g}$ if $N_\beta(0, r) + N_\beta(\infty, r) = o(T_{\boldsymbol{g}}(r))$.

Let $G \in K[x_1, \ldots, x_n] \setminus K$ with no monomial factors and no repeated factors. Let $\epsilon > 0$. In the following we consider a n-tuple of meromorphic functions $\boldsymbol{u} = (u_1, \ldots, u_n)$ satisfying

$$N_{u_i}(0, r) + N_{u_i}(\infty, r) = o\big(\max_{1 \le j \le n} \{T_{u_j}(r)\}\big) = o(T_{\boldsymbol{g}}(r))$$

for each $1 \le i \le n$, and $K \subset K_{\boldsymbol{g}}$, where $\boldsymbol{g} = [1 : u_1 : \cdots : u_n]$. We note that $\lambda \in K_{\boldsymbol{g}}$ if and only if $T_\lambda(r) = o\big(\max_{1 \le j \le n}\{T_{u_j}(r)\}\big)$ by (4-1).

When $n = 1$, the theorem is a direct consequence of Theorem 15 since u_1 is constant if (4-6) holds.

From this point, we let $n \ge 2$. We will effectively construct a nonconstant polynomial H in $K[x_1, \ldots, x_n]$ such that (4-11) holds if $H(u_1, \ldots, u_n) \not\equiv 0$.

The arguments are carried out inductively in several steps. In the following, the $c_{i,j}$'s and M_i's denote positive real numbers depending only on ϵ, n, $\deg G$, and the previously defined $c_{i',j'}$ and $M_{i'}$.

Step 1: We apply Theorem 15. The condition (a) in Theorem 15 holds under our assumption, so if (4-7) holds then we are done. Otherwise, there exists an n-tuple of integers $(m_1, \ldots, m_n) \ne (0, \ldots, 0)$ with $\sum |m_i| \le M_1$ such that

$$\lambda_1 := u_1^{m_1} \cdots u_n^{m_n} \in K_{\boldsymbol{g}}. \tag{4-12}$$

We may assume $\gcd(m_1, \ldots, m_n) = 1$. By Lemma 12, $(m_1, \ldots, m_n)$ extends to a basis $(m_1, \ldots, m_n)$, $(a_{21}, \ldots, a_{2n})$, $\ldots$, $(a_{n1}, \ldots, a_{nn})$ of $\mathbb{Z}^n$ such that

$$|a_{i1}| + \cdots + |a_{in}| \le M_1 + n \quad \text{for } 2 \le i \le n. \tag{4-13}$$

Consider the change of variables

$$\Lambda_1 := x_1^{m_1} \cdots x_n^{m_n} \qquad \text{and} \qquad X_{1,i} := x_1^{a_{i1}} \cdots x_n^{a_{in}} \quad \text{for } 2 \le i \le n \tag{4-14}$$

and put

$$\beta_{1,i} = u_1^{a_{i1}} \cdots u_n^{a_{in}} \quad \text{for } 2 \le i \le n. \tag{4-15}$$

Let A_1 denote the $n \times n$ matrix whose rows are the above basis of $\mathbb{Z}^n$. Then we formally express the above identities as

$$(\Lambda_1, X_{1,2}, \ldots, X_{1,n}) = (x_1, \ldots, x_n)^{A_1}, \quad (\lambda_1, \beta_{1,2}, \ldots, \beta_{1,n}) = (u_1, \ldots, u_n)^{A_1}. \tag{4-16}$$

Let $B_1 = A_1^{-1}$. The entries of B_1 can be bounded from above in terms of M_1 and n. We have

$$(x_1, \ldots, x_n) = (\Lambda_1, X_{1,2}, \ldots, X_{1,n})^{B_1}, \quad (u_1, \ldots, u_n) = (\lambda_1, \beta_{1,2}, \ldots, \beta_{1,n})^{B_1}. \tag{4-17}$$

Let $G_1(\Lambda_1, X_{1,2}, \dots, X_{1,n}) \in K[\Lambda_1, X_{1,2}, \dots, X_{1,n}]$ with no monomial factors and

$$G((\Lambda_1, X_{1,2}, \dots, X_{1,n})^{B_1}) = \Lambda_1^{d_1} X_{1,2}^{d_2} \cdots X_{1,n}^{d_n} G_1(\Lambda_1, X_{1,2}, \dots, X_{1,n}) \tag{4-18}$$

for some integers d_i, $1 \le i \le n$. Since the transformations in (4-16) and (4-17) are invertible of each other and G has no repeated irreducible factors, we have that G_1 has no repeated irreducible factors either. The coefficients of G_1 are the same as the coefficients of G and $\deg G_1$ can be bounded from above explicitly in terms of M_1, n, and $\deg G$. Consider $G_1(\lambda_1, X_{1,2}, \dots, X_{1,n}) \in K(\lambda_1)[X_{1,2}, \dots, X_{1,n}]$; by using (4-12) we have

$$K(\lambda_1) \subset K_{\boldsymbol{g}}. \tag{4-19}$$

For the particular change of variables in (4-16), (4-17), and (4-18) (that depends on the matrix A_1), we apply the Lemma 13 with k being the field of meromorphic functions $\mathcal{M}$ and $k_0 = K$ and (4-14) to find a nonconstant polynomial $H_1' \in K[x_1, \dots, x_n]$ such that $G_1(\lambda_1, X_{1,2}, \dots, X_{1,n})$ has neither monomial nor repeated irreducible factors if $H_1'(u_1, \dots, u_n) \not\equiv 0$. We now take H_1 to be the product of all such H_1' where A_1 ranges over the finitely many elements of $\mathrm{GL}_n(\mathbb{Z})$ with $\|A_1\|_\infty \le M_1 + n$. From Lemma 13, $\deg H_1$ depends only on ϵ, n and $\deg G$.

Since the u_i's, λ_1, and $\beta_{1,j}$'s have small zero and pole with respect to $\boldsymbol{g}$, we have

$$\begin{aligned} N_{G(\boldsymbol{u})}(0, r) - N^{(1)}_{G(\boldsymbol{u})}(0, r) \\ = N_{G_1(\lambda_1, \beta_{1,2}, \dots, \beta_{1,n})}(0, r) - N^{(1)}_{G_1(\lambda_1, \beta_{1,2}, \dots, \beta_{1,n})}(0, r) + o(T_{\boldsymbol{g}}(r)) \end{aligned} \tag{4-20}$$

by (4-16) and (4-18). From (4-16), (4-17) and (4-12), we have

$$\max_{1 \le i \le n} \{T_{u_i}(r)\} = O\big(\max\{T_{\lambda_1}(r), T_{\beta_{1,2}}(r), \dots, T_{\beta_{1,n}}(r)\}\big) = O\big(\max_{2 \le i \le n} \{T_{\beta_{1,i}}(r)\}\big). \tag{4-21}$$

In conclusion, at the end of this step we have

$$\max_{2 \le i \le n} \{T_{\beta_{1,i}}(r)\} = O\big(\max_{1 \le i \le n} \{T_{u_i}(r)\}\big). \tag{4-22}$$

Furthermore, it remains to consider the case when

$$N_{G_1(\lambda_1, \beta_{1,2}, \dots, \beta_{1,n})}(0, r) - N^{(1)}_{G_1(\lambda_1, \beta_{1,2}, \dots, \beta_{1,n})}(0, r) <_{\mathrm{exc}} \epsilon \max_{1 \le i \le n} \{T_{u_i}(r)\} \tag{4-23}$$

fails to hold under the assumption that $H_1(u_1, \dots, u_n) \not\equiv 0$.

There are $n-1$ many steps in total. Hence if $n \ge 3$, we proceed with the following $n-2$ many more steps.

Step 2: We include this step in order to illustrate the transition from Step $s-1$ to Step s below. Since the various estimates and constructions are similar to those in Step 1, we skip some of the details. Suppose $H_1(u_1, \dots, u_n) \not\equiv 0$ so that $G_1(\lambda_1, X_{1,2}, \dots, X_{1,n})$ has neither monomial nor repeated factors.

We apply Theorem 15, assuming (4-23) fails to hold for $G_1(\lambda_1, X_{1,2}, \dots, X_{1,n})$ and $(\beta_{1,2}, \dots, \beta_{1,n})$, and use (4-19), (4-22), to get an $(n-1)$-tuple $(m'_2, \dots, m'_n) \neq (0, \dots, 0)$ with $\sum |m'_i| \leq M_2$ such that

$$\lambda_2 := \beta_{1,2}^{m'_2} \cdots \beta_{1,n}^{m'_n} \in K_{\boldsymbol{g}}. \tag{4-24}$$

We may assume $\gcd(m'_2, \dots, m'_n) = 1$. By Lemma 12, $(m'_2, \dots, m'_n)$ extends to a basis of $\mathbb{Z}^{n-1}$ in which each vector has ℓ_1-norm at most $M_2 + n$.

Let A'_2 be the $(n-1)\times(n-1)$ matrix whose rows are the above basis of $\mathbb{Z}^{n-1}$. We make the transformation

$$(\Lambda_2, X_{2,3}, \dots, X_{2,n}) = (X_{1,2}, \dots, X_{1,n})^{A'_2}, \quad (\lambda_2, \beta_{2,3}, \dots, \beta_{2,n}) = (\beta_{1,2}, \dots, \beta_{1,n})^{A'_2}.$$

Let $A_2 = (1) \oplus A'_2$ be the $n\times n$ block diagonal matrix with the (1, 1)-entry 1 and the matrix A'_2 in the remaining $(n-1)\times(n-1)$ block. We have

$$(\Lambda_1, \Lambda_2, \dots, X_{2,n}) = (\Lambda_1, X_{1,2}, \dots, X_{1,n})^{A_2},$$
$$(\lambda_1, \lambda_2, \dots, \beta_{2,n}) = (\lambda_1, \beta_{1,2}, \dots, \beta_{1,n})^{A_2}.$$

Combining this with (4-16), we have

$$\begin{aligned}(\Lambda_1, \Lambda_2, X_{2,3}, \dots, X_{2,n}) &= (x_1, \dots, x_n)^{A_2 A_1},\\ (\lambda_1, \lambda_2, \beta_{2,3}, \dots, \beta_{2,n}) &= (u_1, \dots, u_n)^{A_2 A_1}.\end{aligned} \tag{4-25}$$

Let $B_2 = (A_2 A_1)^{-1}$. Let $G_2(\Lambda_1, \Lambda_2, X_{2,3}, \dots, X_{2,n})$ be the polynomial with no monomial factors such that

$$G_0((\Lambda_1, \Lambda_2, X_{2,3}, \dots, X_{2,n})^{B_2}) = \Lambda_1^{d'_1} \Lambda_2^{d'_2} X_{2,3}^{d'_3} \cdots X_{2,n}^{d'_n} G_2(\Lambda_1, \Lambda_2, X_{2,3}, \dots, X_{2,n})$$

for some $d'_1, \dots, d'_n \in \mathbb{Z}$. We have that $\deg G_2$ can be bounded from above explicitly in terms of M_2, M_1, n, and $\deg G$. As before, we regard $G_2(\lambda_1, \lambda_2, X_{2,3}, \dots, X_{2,n})$ as a polynomial in $X_{2,3}, \dots, X_{2,n}$ with coefficients in $K_{\boldsymbol{g}}$ using (4-12) and (4-24).

For a particular A_1 and A_2, we apply Lemma 13 with $k = \mathcal{M}$ and $k_0 = K$ and use (4-12) and (4-24) to get a nonconstant polynomial H'_2 in $K[x_1, \dots, x_n]$ such that $G_2(\lambda_1, \lambda_2, X_{2,3}, \dots, X_{2,n})$ has neither monomial nor repeated factors. We now take H_2 to be the product of all such H'_2 where A_1 and A_2 range over the finitely many unimodular matrices with $\|A_1\|_\infty \leq M_1 + n$ and $\|A_2\|_\infty \leq M_2 + n$. By using similar estimates, at the end of this step, we have

$$\max_{3\leq i\leq n} \{T_{\beta_{1,i}}(r)\} = O\big(\max_{1\leq i\leq n} \{T_{u_i}(r)\}\big) \tag{4-26}$$

and

$$N_{G_2(\lambda_1,\lambda_2,\beta_{2,3},\dots,\beta_{2,n})}(0, r) - N^{(1)}_{G_2(\lambda_1,\lambda_2,\beta_{2,3},\dots,\beta_{2,n})}(0, r) <_{\text{exc}} \epsilon \max_{1\leq i\leq n} \{T_{u_i}(r)\} \tag{4-27}$$

fails to hold.

Let $2 \le s \le n-1$ and suppose that we have completed Step $s-1$. This includes the construction of $H_{s-1} \in K[x_1, \dots, x_n]$ with degree depends on ϵ, n and $\deg G$ only. We then complete Step s in the same manner Step 2 is carried out after Step 1. The last one is Step $n-1$ resulting in $H_{n-1} \in K[x_1, \dots, x_n]$. We now define $H = H_1 \cdots H_{n-1}$. Then $\deg H$ depends only on ϵ, n and $\deg G$ since each H_i does so. Suppose $H(u_1, \dots, u_n) \not\equiv 0$. Assume we go through all the above $n-1$ steps to get the polynomial

$$P(X_{n-1,n}) := G_{n-1}(\lambda_1, \dots, \lambda_{n-1}, X_{n-1,n}) \in K_{\boldsymbol{g}}[X_{n-1,n}]$$

such that its degree can be bounded explicitly in terms of $M_{n-1}, \dots, M_1$, n, and $\deg G$. At the end of Step $n-1$, we have that $\beta_{n-1,n}$ has small zero and pole with respect to $\boldsymbol{g}$, so it satisfies

$$T_{\beta_{n-1,n}}(r) = O\Big(\max_{1\le i\le n}\{T_{u_i}(r)\}\Big). \tag{4-28}$$

If

$$N_{P(\beta_{n-1,n})}(0,r) - N^{(1)}_{P(\beta_{n-1,n})}(0,r) <_{\text{exc}} \epsilon \max_{1\le i\le n}\{T_{u_i}(r)\}, \tag{4-29}$$

then we are done. Otherwise, since $H_{n-1}(u_1, \dots, u_n) \not\equiv 0$, the polynomial $P(X_{n-1,n})$ has neither monomial nor repeated irreducible factors, according to Theorem 15, there exists a nonzero integer m such that, by using (4-28),

$$T_{\beta^m_{n-1,n}}(r) = o(T_{\beta_{n-1,n}}(r)), \tag{4-30}$$

which is not possible since $\beta_{n-1,n}$ is not constant. □

4.4. *Proof of Theorem 1.*

Proof of Theorem 1. Let $F \in K[x_0, \dots, x_n]$. Consider a holomorphic curve $\boldsymbol{g} = (g_0, \dots, g_n)$, where $g_0, \dots, g_n$ are entire functions with no common zeros, such that $K \subset K_{\boldsymbol{g}}$ and $N_{\boldsymbol{g}}(H_i, r) = o(T_{\boldsymbol{g}}(r))$ for $0 \le i \le n$. Let $u_i = g_i/g_0$ for $0 \le i \le n$, $\boldsymbol{u} = (u_1, \dots, u_n)$, and $G := F(1, x_1, \dots, x_n) \in K[x_1, \dots, x_n]$. Then

$$\begin{aligned} N_{u_i}(0,r) + N_{u_i}(\infty,r) &\le N_{g_i}(0,r) + N_{g_0}(0,r) \\ &= N_{\boldsymbol{g}}(H_i,r) + N_{\boldsymbol{g}}(H_0,r) = o(T_{\boldsymbol{g}}(r)) \end{aligned} \tag{4-31}$$

for each $1 \le i \le n$, and, by (4-1),

$$\max_{1\le i\le n}\{T_{u_i}(r)\} = O(T_{\boldsymbol{g}}(r)). \tag{4-32}$$

Since $F(\boldsymbol{g}) = F(g_0, \dots, g_n) = g_0^d G(\boldsymbol{u})$, we have

$$N_{F(\boldsymbol{g})}(0,r) = N_{G(\boldsymbol{u})}(0,r) + o(T_{\boldsymbol{g}}(r)), \quad N^{(1)}_{F(\boldsymbol{g})}(0,r) = N^{(1)}_{G(\boldsymbol{u})}(0,r) + o(T_{\boldsymbol{g}}(r)). \tag{4-33}$$

Consequently, we may apply Theorem 16 for any given positive real ϵ to find a nontrivial polynomial $Q \in K[x_1, \dots, x_n]$ such that (2-2) holds, that is,

$$N_{F(\boldsymbol{g})}(0,r) - N^{(1)}_{F(\boldsymbol{g})}(0,r) \le \epsilon T_{\boldsymbol{g}}(r), \tag{4-34}$$

when $Q(\boldsymbol{u}) \not\equiv 0$. In addition, the polynomial Q can be determined effectively and the degree of Q can be bounded effectively in terms of ϵ, n and the degree of F. At this step, we take Z to be the zero locus of the homogeneous polynomial

$$x_0^{\deg Q} \cdot Q\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in K[x_0, \dots, x_n].$$

Let $F = \sum_{\boldsymbol{i} \in I_F} \alpha_{\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}} \in K[x_0, \dots, x_n]$, and let W be the Zariski closed subset that is the union of hypersurfaces of $\mathbb{P}^n$ of the form $\sum_{\boldsymbol{i} \in J} \alpha_{\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}} = 0$, where J is a nonempty subset of I_F. The Zariski closed set $Z \cup W$ satisfies (Z1) and (Z2) since both Z and W do so. We now prove (2-3) holds (after possibly enlarging Z) by further assuming that the hypersurface $[F = 0]$ and the coordinate hyperplanes in $\mathbb{P}^n$ are in weakly general position. Therefore, we may write

$$F(\boldsymbol{g}) = \sum_{0 \le i \le n} \alpha_{\boldsymbol{i}_i} g_i^d + \sum_{\boldsymbol{i} \in I_G \setminus I} \alpha_{\boldsymbol{i}} \boldsymbol{g}^{\boldsymbol{i}}, \tag{4-35}$$

where $\alpha_{\boldsymbol{i}_i} \ne 0$ for $0 \le i \le n$ and $I = \{\boldsymbol{i}_0 := (d, 0, \dots, 0), \dots, \boldsymbol{i}_n := (0, \dots, 0, d)\}$.

For $\boldsymbol{g}$ with $\boldsymbol{g}(\mathbb{C})$ not contained in $Z \cup W$, we may use Theorem 6 to show that

$$dT_{\boldsymbol{g}}(r) \le N_{F(\boldsymbol{g})}(0,r) + o(T_{\boldsymbol{g}}(r))$$

since $\alpha_{\boldsymbol{i}_i} \in K_{\boldsymbol{g}}$ and $N_{g_i}(0,r) = o(T_{\boldsymbol{g}}(r))$ for $0 \le i \le n$. Together with (4-34), we arrive at $N^{(1)}_{F(\boldsymbol{g})}(0,r) \ge (d-\epsilon)T_{\boldsymbol{g}}(r)$. By letting $Z \cup W$ be the desired exceptional set Z, we finish the proof. □

5. Proof of Theorem 2

We will adapt the proof strategy employed in [6, Theorem 1.2] to suit the current situation and subsequently apply Theorem 1.

Proof. Let $z_0 \in \mathbb{C}$ such that all the coefficients of all F_i, $1 \le i \le n+1$, are holomorphic at z_0 and the zero locus of F_i, $1 \le i \le n+1$, evaluating at z_0, denoted by $D_i(z_0)$, intersect transversally. These conditions imply that z_0 is not a common zero of the coefficients of F_i, for each $1 \le i \le n+1$.

Since the zero locus of $F_i(z_0)$, $1 \le i \le n+1$, intersect transversally, they are in general position; thus the set of polynomials F_i, $1 \le i \le n+1$, is in weakly general position. Then Proposition 7 implies that the only $(x_0, \dots, x_n) \in \mathcal{M}^{n+1}$ with $F_i(x_0, \dots, x_n) \equiv 0$ for each $1 \le i \le n+1$ is $(0, \dots, 0)$. Thus the association

$\boldsymbol{x} \mapsto [F_1^{a_1}(\boldsymbol{x}) : \cdots : F_{n+1}^{a_{n+1}}(\boldsymbol{x})]$, where $a_i := \operatorname{lcm}(\deg F_1, \ldots, \deg F_{n+1})/\deg F_i$, defines a morphism $\pi : \mathbb{P}^n(\mathcal{M}) \to \mathbb{P}^n(\mathcal{M})$ over K. Let

$$G := \det\left(\frac{\partial F_i}{\partial x_j}\right)_{\substack{1\le i\le n+1\\ 0\le j\le n}} \in K[x_0, \ldots, x_n].$$

Define $\pi|_{z_0} = [F_1^{a_1}(z_0) : \cdots : F_{n+1}^{a_{n+1}}(z_0)] : \mathbb{P}^n(\mathbb{C}) \to \mathbb{P}^n(\mathbb{C})$, which is a morphism since $F_1(z_0), \ldots, F_{n+1}(z_0)$ are in general position. As proved in [6, Theorem 1.2], we have that $[G(z_0) = 0]$ (the zero locus of $G(z_0)$), $D_1(z_0), \ldots, D_{n+1}(z_0)$ are in general position (in $\mathbb{P}^n(\mathbb{C})$). Hence, there is a nonconstant irreducible factor $\tilde{G}$ of G in $K[x_0, \ldots, x_n]$ such that $\tilde{G}$, $F_1, \ldots, F_{n+1}$ is in weakly general position. Denote by Y the zero locus of $\tilde{G}$ in $\mathbb{P}^n(\overline{K})$. We note that Y is contained in the ramification divisor of π since $\tilde{G}$ is a factor of the determinant of the Jacobian matrix associated with the map π. Then there exists an irreducible homogeneous polynomial $A \in K[y_0, \ldots, y_n]$ such that the vanishing order of π^*A along Y is at least 2. Then this construction gives $\pi^* \circ A = \tilde{G}^2 H$ for some $H \in K[x_0, \ldots, x_n]$. Since the divisors defined by $\tilde{G}(z_0)$, $F_1(z_0), \ldots, F_{n+1}(z_0)$ are in general position, their images are also in general position. Therefore, A and y_i, $0 \le i \le n$, are in weakly general position.

Now let $\boldsymbol{f} = (f_0, \ldots, f_n) : \mathbb{C} \to \mathbb{P}^n$ be a holomorphic map, where $f_0, \ldots, f_n$ are entire functions without common zeros. Assume that $K \subset K_{\boldsymbol{f}}$. Let $\boldsymbol{g} := \pi(\boldsymbol{f}) = (F_1(\boldsymbol{f})^{a_1}, \ldots, F_{n+1}(\boldsymbol{f})^{a_{n+1}})$, where each $F_i(\boldsymbol{f})^{a_i}$, $1 \le i \le n+1$, is an entire function with no zeros. Then

$$T_{\boldsymbol{g}}(r) = d_1 T_{\boldsymbol{f}}(r) + o(T_{\boldsymbol{f}}(r)), \tag{5-1}$$

where $d_1 = \deg F_1 \cdot a_1$. From $A(\boldsymbol{g}) = (\pi^* \circ A)(\boldsymbol{f}) = \tilde{G}^2(\boldsymbol{f}) H(\boldsymbol{f})$, it follows that for each $z \in \mathbb{C}$ with $v_z(\tilde{G}(\boldsymbol{f})) > 0$, we have

$$\begin{aligned} \max\{0, v_z(A(\boldsymbol{g}))\} &\ge 2 v_z(\tilde{G}(\boldsymbol{f})) + \min\{0, v_z(H(\boldsymbol{f}))\} \\ &\ge v_z(\tilde{G}(\boldsymbol{f})) + 1 + \min\{0, v_z(H(\boldsymbol{f}))\}. \end{aligned} \tag{5-2}$$

Since $f_0, \ldots, f_n$ are entire functions, the nonnegative number $-\min\{0, v_z(H(\boldsymbol{f}))\}$ is bounded by the number of poles of the coefficients of H at z. Since the coefficients of H are in K and $N_\beta(\infty, r) \le T_\beta(r) + O(1) = o(T_{\boldsymbol{f}}(r))$ for any $\beta \in K$, it follows from (5-2) that

$$N_{\tilde{G}(\boldsymbol{f})}(0, r) \le N_{A(\boldsymbol{g})}(0, r) - N^{(1)}_{A(\boldsymbol{g})}(0, r) + o(T_{\boldsymbol{f}}(r)). \tag{5-3}$$

Assume furthermore that $N_{F_i(\boldsymbol{f})}(0, r) = o(T_{\boldsymbol{f}}(r))$ for $1 \le i \le n+1$. Then $N_{\boldsymbol{g}}(H_i, r) = o(T_{\boldsymbol{f}}(r))$ $(= o(T_{\boldsymbol{g}}(r))$ by (5-1)) for coordinate hyperplanes H_i, $0 \le i \le n$, of $\mathbb{P}^n$. We now apply Theorem 1 for $\epsilon = 1/(4d_1)$. Then we can find a homogeneous polynomial $B_0 \in K[y_0, \ldots, y_n]$ such that for any nonconstant holomorphic map

$\boldsymbol{f} = (f_0, \dots, f_n) : \mathbb{C} \to \mathbb{P}^n$ such that $K \subset K_{\boldsymbol{f}}$ and $N_{F_i(\boldsymbol{f})}(0, r) = o(T_{\boldsymbol{f}}(r))$ for $1 \le i \le n+1$, with $B_0(\boldsymbol{g}) = B_0(\pi(\boldsymbol{f}))$ not identically zero, we have

$$N_{A(\boldsymbol{g})}(0, r) - N^{(1)}_{A(\boldsymbol{g})}(0, r) \le_{\text{exc}} \epsilon T_{\boldsymbol{g}}(r) \tag{5-4}$$

and

$$N^{(1)}_{A(\boldsymbol{g})}(0, r) \ge_{\text{exc}} (\deg A - \epsilon) \cdot T_{\boldsymbol{g}}(r). \tag{5-5}$$

Combining (5-3) and (5-4), we have

$$N_{\tilde{G}(\boldsymbol{f})}(0, r) \le_{\text{exc}} \epsilon T_{\boldsymbol{g}}(r). \tag{5-6}$$

Since $[\tilde{G} = 0] \le \pi^*([A = 0])$ as divisors, we can derive, from the functorial property of Weil functions,

$$m_{\boldsymbol{f}}([\tilde{G} = 0], r) \le m_{\boldsymbol{g}}([A = 0], r) = \deg A \cdot T_{\boldsymbol{g}}(r) - N_{A(\boldsymbol{g})}(0, r) + o(T_{\boldsymbol{g}}(r)). \tag{5-7}$$

Then by (5-5),we have

$$m_{\boldsymbol{f}}([\tilde{G} = 0], r) \le_{\text{exc}} \epsilon T_{\boldsymbol{g}}(r). \tag{5-8}$$

Combining (5-6), (5-8) and (5-1), we have

$$T_{[\tilde{G}=0], \boldsymbol{f}}(r) \le_{\text{exc}} 2\epsilon T_{\boldsymbol{g}}(r) = 2\epsilon \cdot d_1 T_{\boldsymbol{f}}(r) + o(T_{\boldsymbol{f}}(r)), \tag{5-9}$$

On the other hand, the first main theorem implies that

$$\deg \tilde{G} \cdot T_{\boldsymbol{f}}(r) = T_{[\tilde{G}=0], \boldsymbol{f}}(r) + o(T_{\boldsymbol{f}}(r)). \tag{5-10}$$

Therefore, we have

$$T_{\boldsymbol{f}}(r) \le_{\text{exc}} 2\epsilon \cdot d_1 T_{\boldsymbol{f}}(r) + o(T_{\boldsymbol{f}}(r)), \tag{5-11}$$

which is not possible since $\epsilon = 1/(4d_1)$. This shows that $B_0(\boldsymbol{g})$ is identically zero. Let $B := \pi^*(B_0) = B_0(F_1^{a_1}, \dots, F_{n+1}^{a_{n+1}}) \in K[x_0, \dots, x_n]$, which is not identically zero since π is a finite morphism. Then $B(\boldsymbol{f})$ is identically zero as asserted. □

The defect relation stated in Corollary 3 directly follows from Theorem 2 by noticing that $\sum_{i=1}^{n+1} \delta_{\boldsymbol{f}}(D_i) = n+1$ if and only if $N_{\boldsymbol{f}}(D_i, r) = o(T_{\boldsymbol{f}}(r))$ for each i. To establish the truncated defect relation for $n = 2$, we relax the assumption to $N^{(1)}_{\boldsymbol{g}}(H_i, r) = o(T_{\boldsymbol{g}}(r))$ for $0 \le i \le 2$. In order to apply this relaxed condition (b) in Theorem 15, one must assume that the hypersurface $[G = 0]$ and the $n+1$ coordinate hyperplanes are in weakly general position in $\mathbb{P}^n$. Unfortunately, this geometric condition does not persist under the induction process. We state a modified version of Theorem 1 below to demonstrate that Theorem 2 remains valid under these relaxed assumptions.

Theorem 17. *Let K be a subfield of the field $\mathcal{M}$ of meromorphic functions. Let G be a nonconstant homogeneous polynomial in $K[x_0, x_1, x_2]$ with no monomial factors and no repeated factors. Let $H_i = [x_{i-1} = 0]$, $1 \le i \le 3$, be the coordinate hyperplane divisors of $\mathbb{P}^2$. Assume that the plane curve $[G = 0]$ and H_i, $1 \le i \le 3$, are in weakly general position. Then for any $\epsilon > 0$, there exists a proper Zariski closed subset Z of $\mathbb{P}^2$ defined over K such that for any nonconstant holomorphic curve $\boldsymbol{g} = (g_0, g_1, g_2) : \mathbb{C} \to \mathbb{P}^2(\mathbb{C})$ such that $N^{(1)}_{\boldsymbol{g}}(H_i, r) = o(T_{\boldsymbol{g}}(r))$ for $0 \le i \le 2$ with $\boldsymbol{g}$ not contained in Z, we have*

$$N_{G(\boldsymbol{g})}(0, r) - N^{(1)}_{G(\boldsymbol{g})}(0, r) \le_{\text{exc}} \epsilon T_{\boldsymbol{g}}(r) \tag{5-12}$$

and

$$N^{(1)}_{G(\boldsymbol{g})}(0, r) \ge_{\text{exc}} (\deg G - \epsilon) \cdot T_{\boldsymbol{g}}(r). \tag{5-13}$$

Furthermore, the exceptional set Z is a finite union of closed subsets given by homogenization equations of the form $x_1^{n_1} x_2^{n_2} = \lambda$, where $\lambda \in K^*$ and (n_1, n_2) is a pair of integers with $\max\{|n_1|, |n_2|\}$ bounded from above by an effectively computable integer m.

Proof of Corollary 3. Since $0 \le \delta_{\boldsymbol{f}}(D_i) \le 1$ for $1 \le i \le n+1$, it is clear that $\sum_{i=1}^{n+1} \delta_{\boldsymbol{f}}(D_i) = n+1$ if and only if $\delta_{\boldsymbol{f}}(D_i) = 1$ for each i. On the other hand, $\delta_{\boldsymbol{f}}(D_i) = 1$ if and only if $N_{\boldsymbol{f}}(D_i, r) = o(T_{\boldsymbol{f}}(r))$. Therefore, we have either $\sum_{i=1}^{n+1} \delta_{\boldsymbol{f}}(D_i) < n+1$ or there exists a homogeneous polynomial $B \in K[x_0, \dots, x_n]$ as described Theorem 2 such that $B(\boldsymbol{f})$ is identically zero.

When $n = 2$, the conclusion of Theorem 2 holds under a weaker assumption that $N^{(1)}_{\boldsymbol{f}}(D_i, r) = o(T_{\boldsymbol{f}}(r))$ for $i = 0, 1, 2$ by replacing the use of Theorem 1 with Theorem 17. Therefore, the above arguments show that $\sum_{i=1}^{3} \delta^{(1)}_{\boldsymbol{f}}(D_i) < 3$ or there exists a homogeneous polynomial $B \in K[x_0, x_1, x_2]$ as described Theorem 2 such that $B(\boldsymbol{f})$ is identically zero. □

Proof of Theorem 17. Let $\boldsymbol{g} = (g_0, g_1, g_2)$ with $N^{(1)}_{g_i}(0, r) = o(T_{\boldsymbol{g}}(r))$, $0 \le i \le 2$, where g_0, g_1, g_2 have no common zeros. We prove (5-12) first. Under our assumption, the condition (b) in Theorem 15 holds. Hence, by Theorem 15, we only need to consider the case that

$$T_{(g_1/g_0)^{n_1}(g_2/g_0)^{n_2}}(r) = o(T_{\boldsymbol{g}}(r)). \tag{5-14}$$

We may assume that n_1 and n_2 are coprime. Consequently, there exist integers a and b such that $n_1 a + n_2 b = 1$. Consider the variables

$$\Lambda = X^{n_1} Y^{n_2} \quad \text{and} \quad T = X^b Y^{-a}. \tag{5-15}$$

Then, we may express

$$X = \Lambda^a T^{n_2} \quad \text{and} \quad Y = \Lambda^b T^{-n_1}. \tag{5-16}$$

Let $G_1(X, Y) = G(1, X, Y)$. Define $B(\Lambda, T) \in K[\Lambda, T]$ as the polynomial with no monomial factors and such that

$$G_1(X, Y) = G_1(\Lambda^a T^{n_2}, \Lambda^b T^{-n_1}) = T^{M_1}\Lambda^{M_2} B(\Lambda, T) \tag{5-17}$$

for some integers M_1 and M_2.

Let $u_1 = g_1/g_0$, $u_2 = g_2/g_0$, and $\lambda := u_1^{n_1} u_2^{n_2}$. Then we have

$$T_\lambda(r) = o(T_{\boldsymbol{g}}(r)). \tag{5-18}$$

To prove (5-12), we will reduce the problem to one-variable polynomials $B(\lambda, T)$ for all possible $\lambda \in K$ that satisfy (5-18) but not (5-12). Our objective is to eliminate those λ values with $B(\lambda, T)$ containing a factor of T or having repeated factors, so that we can apply the GCD theorem after eliminating those λ. Since T is not a factor of $B(\Lambda, T)$, it follows that $B(\Lambda, 0) \in K[\Lambda]$ is not identically zero. Consequently, there exist at most finite $\gamma_1, \dots, \gamma_s \in K$ such that $B(\gamma_i, 0) = 0$ for $1 \le i \le s$. Therefore, T is not a factor of $B(\lambda, T)$ if $\lambda \ne \gamma_i$, $1 \le i \le s$. Regarding repeated factors, let's express $B(\Lambda, T) = B_\Lambda(T) \in K[\Lambda][T]$. Since the transformation in (5-15) establishes to a bijection between the sets $\{X^{t_1}Y^{t_2} : t_1, t_2 \in \mathbb{Z}\}$ and $\{\Lambda^{a_1}T^{a_2} : a_1, a_2 \in \mathbb{Z}\}$, it is evident that $B(\Lambda, T) \in K[\Lambda, T]$ is square free, given that G is square free. Consequently, the resultant $R(B_\Lambda, B'_\Lambda)$ of B_Λ and $B'_\Lambda(T)$ is a polynomial in $K[\Lambda]$, which is not identically zero. Let

(5-19) $\alpha_i \in K$, $1 \le i \le t$, be the zeros of the resultant $R(B_\Lambda, B'_\Lambda)$.

It is clear that $B(\lambda, T)$ has no multiple factors in $K[\lambda][T]$ if $\lambda \ne \alpha_i$ for any $1 \le i \le t$. Therefore, it is clear that we need to consider those λ with $\lambda \ne \alpha_i$ for any $1 \le i \le t$ and $\lambda \ne \gamma_j$ for any $1 \le j \le s$. Assuming such, let $B(T) := \lambda^{M_2} B(\lambda, T)$ as in (5-17). Let $\beta := u_1^b u_2^{-a}$ and define $D_\beta(B) \in K_{\boldsymbol{g}}[T]$ as in (4-3). By Lemma 14, the polynomials B and $D_\beta(B)$ are coprime in $K_{\boldsymbol{g}}[T]$. Let $\tilde{B} \in K(\lambda)[Z, U]$ and $\tilde{D}_\beta(B)$ be the homogenization of B and $D_\beta(B)$, respectively. Write $\beta = \beta_1/\beta_0$, where β_0 and β_1 are entire functions without common zeros. Then by Proposition 10

$$N_{\mathrm{gcd}}(\tilde{B}(\beta_0, \beta_1), \tilde{D}_\beta(B)(\beta_0, \beta_1), r) \le o(T_{\boldsymbol{g}}(r)) \tag{5-20}$$

since β is not constant. On the other hand, from the proof of [4, Proposition 5.3], there exists a proper Zariski closed set W of $\mathbb{P}^2(\mathbb{C})$, independent of $\boldsymbol{g}$, such that, if image of $\boldsymbol{g}$ is contained in W,

$$N_{G(\boldsymbol{g})}(0, r) - N^{(1)}_{G(\boldsymbol{g})}(0, r) \le_{\mathrm{exc}} N_{\mathrm{gcd}}(\tilde{B}(\beta_0, \beta_1), \tilde{D}_\beta(B)(\beta_0, \beta_1), r). \tag{5-21}$$

Furthermore, W can be described in Theorem 17. We conclude the proof of (5-12) by combining (5-20) and (5-21).

We now proceed to prove (5-13). Let $G = \sum_{\boldsymbol{i} \in I_G} \alpha_{\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}} \in K[x_0, x_1, x_2]$. Since the hypersurface $[G = 0]$ and the coordinate hyperplanes in $\mathbb{P}^2$ are in weakly general position, may write

$$G(\boldsymbol{g}) = \sum_{0\le i\le 2} \alpha_{\boldsymbol{i}_i} g_i^d + \sum_{\boldsymbol{i}\in I_G\setminus I} \alpha_{\boldsymbol{i}} \boldsymbol{g}^{\boldsymbol{i}}, \tag{5-22}$$

where $\alpha_{\boldsymbol{i}_i} \neq 0$ for $0 \le i \le 2$ and $I = \{(d, 0, 0), (0, d, 0), (0, 0, d)\}$.

Let's express $B(\Lambda, T)$ in the form

$$B(\Lambda, T) = \sum_{\boldsymbol{i}\in I_B} b_{\boldsymbol{i}}(\Lambda) T^{\boldsymbol{i}} \in K[\Lambda][T], \tag{5-23}$$

where $b_{\boldsymbol{i}} \neq 0$ if $\boldsymbol{i} \in I_B$. We define $J \subset K[\Lambda]$ as the finite set containing all $b_{\boldsymbol{i}}(\Lambda)$ for $\boldsymbol{i} \in I_B$ and all of their proper subsums. Set $\mathcal{R} := \{r \in K \mid h(r) = 0 \text{ for some } h \in J\}$. It is crucial that the proof of Theorem 1 has already demonstrated that (5-13) holds if neither $G(\boldsymbol{g})$ nor any proper subsum of (5-22) is zero. Therefore, when evaluating $B(\Lambda, T)$ at $\Lambda = \lambda \notin \mathcal{R}$ and $T = \beta$, we need to consider equations of the type

$$\sum_{\boldsymbol{i}\in I_B} a_{\boldsymbol{i}}(\lambda)\beta^{\boldsymbol{i}} = 0, \tag{5-24}$$

where $a_{\boldsymbol{i}}(\Lambda)$ is a subsum of $b_{\boldsymbol{i}}(\Lambda)$, and there are at least two nontrivial $a_{\boldsymbol{i}}$ in the left-hand side of (5-24) since $\lambda \notin \mathcal{R}$. Hence,

$$T_\beta(r) \le c_3 T_\lambda(r) = o(T_{\boldsymbol{g}}(r)).$$

This, however, leads to a contradiction. □

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