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The Teichmüller space of a closed oriented (real) surface of genus at least 2 is a moduli space of complex structures on the surface, but can also be defined as a space of certain representations of the fundamental group of the surface in the group of orientation-preserving isometries of the hyperbolic plane. As a consequence the tangent spaces of Teichmüller space admit two rather different descriptions. We use harmonic vector fields (defined as infinitesimal analogs of harmonic maps) on the hyperbolic plane to make a bridge between these descriptions.

1. Introduction

1.1. ***Teichmüller space.*** Let Σ be an oriented, connected, closed and smooth surface of genus ≥ 2. Write $\mathrm{diff}_0(\Sigma)$ for the group of diffeomorphisms $\Sigma \to \Sigma$ which are isotopic to the identity. By the Korn–Lichtenstein theorem, the choice of a complex structure on Σ is equivalent to the choice of a complex structure $J : T\Sigma \to T\Sigma$ on the tangent bundle. (J is a smooth vector bundle automorphism covering $\mathrm{id} : \Sigma \to \Sigma$, and it satisfies $J^2 = -\mathrm{id}$. Such a J can be called an *almost complex structure on* Σ.)

Definition 1.1.1. Teichmüller space $\mathscr{T}(\Sigma)$ is the space of complex structures J on $T\Sigma$, modulo the right action of $\mathrm{diff}_0(\Sigma)$ given by $(J \cdot f)_x := (df_x)^{-1} \circ J_{f(x)} \circ df_x$ for $f \in \mathrm{diff}_0(\Sigma)$ and $x \in \Sigma$.

We will rely more on the "metric" definition of $\mathscr{T}(\Sigma)$. Let $\Sigma' \to \Sigma$ be a universal covering with deck transformation group Γ. We also refer to Γ as the *fundamental group* of Σ. A complex structure on Σ determines a complex structure on Σ', and an embedding of Γ into the group of complex automorphisms of Σ'. By uniformization

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theory, Σ' is isomorphic as a complex manifold to the open unit disk $\mathbb{D} \subset \mathbb{C}$. In this way, a complex structure on Σ gives us an embedding ρ of Γ into the group of complex automorphisms of $\mathbb{D}$, which we can also view as the group $\mathrm{isom}_+(\mathbb{D})$ of orientation-preserving isometries of $\mathbb{D}$ equipped with the Poincaré metric. The homomorphism $\rho : \Gamma \to \mathrm{isom}_+(\mathbb{D})$ is not well defined as such, but it is well defined up to conjugation by an element of $\mathrm{isom}_+(\mathbb{D})$. This leads us to the metric definition of $\mathscr{T}(\Sigma)$.

Definition 1.1.2. Teichmüller space $\mathscr{T}(\Sigma)$ is the space of injective homomorphisms $\rho : \Gamma \to \mathrm{isom}_+(\mathbb{D})$ with discrete and cocompact image $\rho(\Gamma)$, modulo the left action of $\mathrm{isom}_+(\mathbb{D})$ by conjugation.

The equivalence of the two definitions of $\mathscr{T}(\Sigma)$ is well known. The metric definition does not *use* uniformization theory, although it is explained by uniformization theory. One has to do some work to show that $\mathscr{T}(\Sigma)$, according to that definition, is a smooth manifold of real dimension $-3\chi(\Sigma)$.

1.2. ***The tangent spaces of Teichmüller space.*** We continue in the notation of the previous section. In particular, $\Sigma' \to \Sigma$ is a universal covering, J is a complex structure on $T\Sigma$ and $\rho : \Gamma \to \mathrm{isom}_+(\mathbb{D})$ is an injective homomorphism with discrete and cocompact image.

Definition 1.2.1. A quadratic differential on (Σ, J) is a (continuous) section of the complex line bundle $\hom_{\mathbb{C}}(T\Sigma \otimes_{\mathbb{C}} T\Sigma, \mathbb{C})$ on Σ.

In one description, the tangent space of $\mathscr{T}(\Sigma)$ at a point J (complex structure on $T\Sigma$) "is" the vector space of holomorphic quadratic differentials on Σ. This can be justified as follows. There is an $\mathbb{R}$-linear injective map $\phi \mapsto \mathrm{Re}(\phi)$ from the complex 1-dimensional vector bundle $\hom_{\mathbb{C}}(T\Sigma \otimes_{\mathbb{C}} T\Sigma, \mathbb{C})$ on Σ to the real 3-dimensional vector bundle of symmetric $\mathbb{R}$-bilinear forms on $T\Sigma$. We have already seen that J determines a complex structure on $\Sigma' \cong \mathbb{D}$ and so a hyperbolic metric on Σ' invariant under the action of Γ, and so a hyperbolic (Riemannian) metric g on Σ itself. A holomorphic section ϕ of $\hom_{\mathbb{C}}(T\Sigma \otimes_{\mathbb{C}} T\Sigma, \mathbb{C})$, indeed any smooth section ϕ of that vector bundle, determines a 1-parameter family of Riemannian metrics on Σ by $t \mapsto g + t\mathrm{Re}(\phi)$ for $t \in \mathbb{R}$ close enough to 0. Each of the Riemannian metrics $g + t\mathrm{Re}(\phi)$ determines a conformal structure on Σ, hence a complex structure J_t on $T\Sigma$. This gives a (germ of a) smooth curve $t \mapsto J_t$ in the Teichmüller space, with $J_0 = J$. The velocity vector of that curve at $t = 0$ is the tangent vector which we associate to ϕ. This procedure gives an isomorphism from the vector space of holomorphic quadratic differentials on Σ to that tangent space. It uses Definition 1.1.1. See [Imayoshi and Taniguchi 1992] for more details.

The other popular description of the tangent spaces of $\mathscr{T}(\Sigma)$ relies on the metric definition of $\mathscr{T}(\Sigma)$. Instead of selecting a datum J, we begin with $\rho : \Gamma \to \mathrm{isom}_+(\mathbb{D})$.

Let $\mathfrak{g}$ be the tangent space of $\mathrm{isom}_+(\mathbb{D})$ at the identity element, a.k.a. the Lie algebra of $\mathrm{isom}_+(\mathbb{D})$. The homomorphism ρ determines a left action of Γ on $\mathfrak{g}$ by conjugation, $\gamma \cdot v := \mathrm{ad}(\rho(\gamma))(v)$ for $\gamma \in \Gamma$ and $v \in \mathfrak{g}$. We may write $\mathfrak{g}_\rho$ to specify this action.

We select a tangent vector to $\mathscr{T}(\Sigma)$ at ρ by choosing a smooth 1-parameter family $(\rho_t)_{t\in[-\varepsilon,+\varepsilon]}$ of homomorphisms $\Gamma \to \mathrm{isom}_+(\mathbb{D})$ such that $\rho_0 = \rho$. (We could insist that the homomorphisms ρ_t are all injective with discrete and cocompact image, like ρ_0, but by [Weil 1960] this is automatically satisfied for t close enough to 0.) Then we can form

$$\frac{d(\rho_t \cdot \rho_0^{-1})}{dt}\bigg|_{t=0},$$

which gives us a map from Γ to $\mathfrak{g}_\rho$. It turns out to be a 1-cocycle. Its class in $H^1(\Gamma; \mathfrak{g}_\rho)$ is well defined. We arrive at the following description of the tangent space of $\mathscr{T}(\Sigma)$ at the point determined by the homomorphism ρ: it is $H^1(\Gamma; \mathfrak{g}_\rho)$. It is a small disadvantage of this description that $H^1(\Gamma; \mathfrak{g}_\rho)$ seems to depend on ρ itself, not just on the representation (conjugacy class of homomorphisms) determined by ρ. We leave it to the reader to come to terms with this.

1.3. ***Harmonic vector fields on the hyperbolic plane.*** Let $f : M \to N$ be a smooth map between Riemannian manifolds. The map f has a "Laplacian" $\tau(f)$ which is a section, defined on M, of the vector bundle $f^*(TN)$. It has a coordinate-free definition as the "trace" of the total second derivative of f. (The total second derivative is a fiberwise bilinear map over M from $TM \times_M TM$ to $f^*(TN)$.) This definition comes from [Eells and Sampson 1964]. The letter τ stands for *tension* more than for *trace*. Following [loc. cit.], the map f is considered *harmonic* if $\tau(f)$ is everywhere zero. There is also a characterization of harmonic maps as critical points of the energy functional

$$f \mapsto \int_M \tfrac{1}{2}\|df\|^2 \, d\mu, \tag{1.3.1}$$

where μ is the measure on M determined by the Riemannian metric. That characterization needs exegesis if M is not compact. In the case $N = \mathbb{R}$, the Eells–Sampson definition of a harmonic map agrees with the standard definition of a harmonic function on M, and in the case $M = \mathbb{R}$, the harmonic maps are the geodesics in N.

Let ξ be a smooth vector field on M, where M is still a Riemannian manifold. It is always possible to find a smooth flow $(\phi_t : M \to M)_{t\in\mathbb{R}}$ such that

$$\frac{d\phi_t}{dt}\bigg|_{t=0} = \xi.$$

The vector field

$$\tau(\xi) := \frac{d\tau(\phi_t)}{dt}\bigg|_{t=0}$$

depends only on ξ, and we may view it as an infinitesimal variant of the Eells–Sampson Laplacian for maps. Consequently we say that ξ is *harmonic* if $\tau(\xi)$ is everywhere zero. (Warning: *Harmonic vector field* can mean very different things to different people, but here we use it in the spirit of [Eells and Sampson 1964]. See [Dodson et al. 2002] for some foundational results on harmonic vector fields.)

For us the case where M is an oriented Riemannian 2-manifold with Riemannian metric g is important. In that case M is also a complex 1-manifold. The 3-dimensional real vector bundle E of symmetric $\mathbb{R}$-bilinear forms on TM has a canonical splitting

$$E = E_1 + E_2,$$

where E_1 is a real line bundle and E_2 has a preferred structure of complex (holomorphic) line bundle. Namely, E_1 is the real line subbundle spanned by the everywhere nonzero section g of E, and E_2 is the image of the vector bundle monomorphism which was mentioned before: $\hom_{\mathbb{C}}(TM \otimes_{\mathbb{C}} TM, \mathbb{C}) \to E$ given by $\phi \mapsto \mathrm{Re}(\phi)$. We like to call E_1 the *scalar* summand of E, and E_2 the *trace-free* summand. The following recognition principle for harmonic vector fields appears to be well known, and so we state it without proof. It has an analog for smooth maps between 2-dimensional oriented Riemannian manifolds [Jost 1984, Lemma 1.1; Gerstenhaber and Rauch 1954a; 1954b].

Proposition 1.3.1. *A smooth vector field ξ on the 2-dimensional oriented Riemannian manifold (M, g) is harmonic if and only if the trace-free component of the Lie derivative $\mathscr{L}_\xi(g)$ is holomorphic*; *i.e.*, *if it is* $\mathrm{Re}(\phi)$ *for a* ***holomorphic*** *section ϕ of* $\hom_{\mathbb{C}}(TM \otimes_{\mathbb{C}} TM, \mathbb{C})$.

Remark 1.3.2. The complex line bundle $\hom_{\mathbb{C}}(TM \otimes_{\mathbb{C}} TM, \mathbb{C})$ in Proposition 1.3.1 comes with a preferred hermitian metric. We use this to equip each fiber with a norm. (It is well known that a hermitian inner product on a finite-dimensional complex vector space is determined by its real part, and so by the associated norm. For $z \in M$, the preferred norm on $\hom_{\mathbb{C}}(T_zM \otimes_{\mathbb{C}} T_zM, \mathbb{C})$ is given by $\|f\| := |f(v \otimes v)|$ for $f \in \hom_{\mathbb{C}}(T_zM \otimes_{\mathbb{C}} T_zM, \mathbb{C})$ and $v \in T_zM$ such that $\|v\| = 1$. This does not depend on the choice of v.)

Now we specialize by letting $M = \mathbb{D}$ (with the Poincaré metric, which will still be called g). In order to state our first main result, Theorem I, we introduce some more vocabulary. In the definition that follows, λ is the ordinary Lebesgue measure on $\mathbb{R}^2$ and λ_0 is the unnormalized Haar measure on S^1, so that $\lambda_0(S^1) = 2\pi$.

Definition 1.3.3. Let ξ be a continuous vector field on $\mathbb{D}$ and let ζ be an L^2-vector field (with values in $\mathbb{R}^2$) along S^1. (For the purposes of this definition, ξ and ζ could be regarded as functions from $\mathbb{D}$ and S^1, respectively, to $\mathbb{R}^2$.) We say that ζ is a *distributional boundary* for ξ if for every continuous vector field α on $\mathbb{D} \cup S^1$

the function

$$s \mapsto \int_{z\in\mathbb{D},\ |z|\le s} \xi(z)\cdot\alpha(z)\,d\lambda \tag{1.3.2}$$

defined on [0, 1) has an extension to all of [0, 1] which is differentiable at $s = 1$, with derivative there equal to

$$\int_{S^1} \zeta(z)\cdot\alpha(z)\,d\lambda_0.$$

In this situation, ζ is determined by ξ. If in addition ζ is tangential, which means that $\zeta(z)\cdot z = 0$ for (almost) all $z \in S^1$, then we say that ξ is *boundary controlled*.

The matching condition relating ξ and ζ in Definition 1.3.3 is invariant under the preferred right action(s) of $\mathrm{diff}(\mathbb{D}\cup S^1)$. See Proposition A.2.2. The preferred right action of $\mathrm{diff}(\mathbb{D}\cup S^1)$ on the space of continuous vector fields on $\mathbb{D}$ is given by $(\xi\cdot h)(x) := (dh(x))^{-1}\big(\xi(h(x))\big)$ for $x \in \mathbb{D}$ and $h \in \mathrm{diff}(\mathbb{D}\cup S^1)$. The preferred right action of $\mathrm{diff}(\mathbb{D}\cup S^1)$ on the space of L^2-vector fields on S^1 is similar. It can be enlightening to write $h^*\xi$ and $h^*\zeta$ instead of $\xi\cdot h$ and $\zeta\cdot h$.

Definition 1.3.4. A smooth vector field ξ on $\mathbb{D}$ is *conformal*, respectively *quasiconformal*, if the trace-free component of $\mathscr{L}_\xi(g)$ is zero everywhere, respectively uniformly bounded in the norm of Remark 1.3.2.

Examples. The conformal vector fields on $\mathbb{D}$ are the holomorphic vector fields. See Lemma A.2.1. Every conformal vector field is harmonic. Every Killing vector field ξ on $\mathbb{D}$ (element of $\mathfrak{g}$) is conformal and boundary controlled. Indeed, ξ has a smooth extension to a vector field on $\mathbb{D}\cup S^1$ whose restriction to S^1 is tangential to S^1.

Theorem I. *For every holomorphic quadratic differential ϕ on $\mathbb{D}$ which is uniformly bounded in the norm of Remark 1.3.2, there exists a smooth and boundary controlled vector field ξ on $\mathbb{D}$ such that the trace-free component of $\mathscr{L}_\xi(g)$ is* $\mathrm{Re}(\phi)$. *In the case where $\phi \equiv 0$, the vector field ξ must be a Killing vector field.*

The theorem can be reformulated as follows. There is a short exact sequence of real vector spaces and $\mathbb{R}$-linear maps

$$0 \to \mathfrak{g} \xrightarrow{\text{(a)}} U_\infty \xrightarrow{\text{(b)}} V_\infty \to 0, \tag{1.3.3}$$

where V_∞ is the space of all holomorphic quadratic differentials on $\mathbb{D}$ which are bounded in the norm of Remark 1.3.2, and U_∞ is the space of all harmonic, boundary controlled and quasiconformal smooth vector fields on $\mathbb{D}$. The map (b) takes $\xi \in U_\infty$ to ϕ, where $\mathrm{Re}(\phi)$ is the trace-free component of $\mathscr{L}_\xi(g)$. The map (a) is an inclusion.

The proof of Theorem I takes up most of Section 2. Using the theorem, we can explain (*without* relying on uniformization theory) how the two descriptions of the tangent space to $\mathscr{T}(\Sigma)$ at the point determined by some $\rho : \Gamma \to \mathrm{isom}_+(\mathbb{D})$ are related. We identify Σ with the orbit space $\mathbb{D}/\rho = \mathbb{D}/\rho(\Gamma)$, so that a holomorphic quadratic differential on Σ is tantamount to a holomorphic quadratic differential ϕ on $\mathbb{D}$ which is invariant under the subgroup $\rho(\Gamma)$ of $\mathrm{isom}_+(\mathbb{D})$. Such a ϕ is automatically bounded! Choose ξ as in Theorem I such that the trace-free component of $\mathscr{L}_\xi(g)$ is $\mathrm{Re}(\phi)$. Then the formula $h \mapsto h^*\xi - \xi$ (for $h \in \rho(\Gamma) \cong \Gamma$) defines a 1-cocycle on Γ with values in $\mathfrak{g}_\rho$. Its class in $H^1(\Gamma; \mathfrak{g}_\rho)$ does not depend on the choice of ξ. In the case where that class is zero, it is easy to see that we can reconsider the choice of ξ so as to make it invariant under $\rho(\Gamma)$. Then ξ descends to a harmonic vector field on Σ. By [Dodson et al. 2002, Theorem 3.1] this implies $\xi \equiv 0$ and so $\phi \equiv 0$. In other words, we have an *injective* linear map from the vector space of holomorphic quadratic differentials on $\Sigma = \mathbb{D}/\rho$ to $H^1(\Gamma; \mathfrak{g}_\rho)$. By dimension counting, it must be a linear isomorphism.

The interesting aspect of Theorem I is that it makes only a boundedness assumption on ϕ, not an assumption of invariance under a discrete subgroup of $\mathrm{isom}_+(\mathbb{D})$. It is reminiscent of universal Teichmüller theory. See, for example, [Markovic and Šarić 2009; Markovic 2017].

1.4. ***Vector fields from boundary data.*** We have a partial converse to Theorem I which is inspired by the Poisson formula, as in [Ransford 1995, Theorem 1.2.4]. It is our second main result.

Theorem II. *There is a unique continuous linear map F from the vector space of continuous tangential vector fields on S^1 to the vector space of continuous vector fields on $\mathbb{D}$ satisfying the following conditions.*

(i) *$F(\xi)$ is harmonic, for every continuous tangential vector field ξ on S^1.*

(ii) *ξ and $F(\xi)$ together make up a continuous vector field on $\mathbb{D} \cup S^1$.*

(iii) *F is equivariant for the actions of* $\mathrm{isom}(\mathbb{D})$ *on domain and codomain.*

Moreover F extends uniquely to a continuous linear map from the vector space of tangential L^2-vector fields on S^1 to the vector space of continuous vector fields on $\mathbb{D}$. In this setting property (ii) *turns into the following*: *$F(\xi)$ is boundary controlled with distributional boundary ξ. Properties* (i) *and* (iii) *remain intact.*

Remark. In the first part of the statement (before "Moreover…"), the vector space W of continuous tangential vector fields on S^1 is equipped with the compact-open C^0 topology. The vector space Y of continuous vector fields on $\mathbb{D}$ is also equipped with the compact-open C^0 topology (throughout). In the second part of the statement, the vector space $\mathcal{X}$ of tangential L^2-vector fields on S^1 is viewed as a Hilbert space. The statement "Moreover F extends…" is slightly imprecise because W is not a subspace of $\mathcal{X}$. There is an inclusion $W \hookrightarrow \mathcal{X}$ of sets which is linear and continuous as a map of topological vector spaces. The image of W in $\mathcal{X}$ is dense; therefore the "extension" of F from W to $\mathcal{X}$ is certainly unique, if it exists. Properties (i) and (iii) are enough to characterize $F : W \to Y$ up to multiplication by a real scalar. We do not know whether properties (i) and (ii) are enough to characterize $F : W \to Y$.

Lemma 1.4.1. *Let $\xi \in \mathfrak{g}$ be a Killing vector field on $\mathbb{D}$. Let ζ be the matching tangential vector field on S^1, so that ξ and ζ together define a smooth vector field on $\mathbb{D} \cup S^1$. Then $F(\zeta) = \xi$.*

See the Appendix for the proof.

As before, let W be the vector space of continuous vector fields on $\mathbb{D}$. If we have $\rho : \Gamma \to \mathrm{isom}_+(\mathbb{D})$, then we have a preferred right action of Γ on W determined by ρ, and we may write W_ρ to specify the action. The following is again well known, and we omit the proof.

Lemma 1.4.2. *The map $H^1(\Gamma; \mathfrak{g}_\rho) \to H^1(\Gamma; W_\rho)$ induced by the inclusion of the Killing vector fields, $\mathfrak{g} \hookrightarrow W$, is zero.*

Proposition 1.4.3. *Let ψ be a continuous vector field on $\mathbb{D}$ such that $h^*\psi - \psi$ is in $\mathfrak{g}$, for all $h \in \rho(\Gamma)$. Then ψ is boundary controlled. The distributional boundary depends only on the 1-cocycle $h \mapsto h^*\psi - \psi$.*

Proof. (This uses Theorem I.) Let C be the cochain complex normally used to define the cohomology groups $H^j(\Gamma; W_\rho)$ for $j \geq 0$, and write $\delta : C^j \to C^{j+1}$ for the differential in C. Let $D \subset C$ be the cochain subcomplex corresponding to $\mathfrak{g} \subset W$. We have $\psi \in C^0$ and we are assuming $\delta\psi \in D^1$. Theorem I and the dimension-counting argument at the end of Section 1.3 imply that for the 1-cocycle $\delta\psi$ in D there exists $\xi \in C^0$, harmonic and boundary controlled, such that $\delta\xi = \delta\psi$. Therefore $\xi - \psi \in C^0$ is a 0-cocycle. This means that it is invariant under Γ. It follows by inspection that $\xi - \psi$ is boundary controlled with distributional boundary zero. Therefore ψ is boundary controlled and has the same distributional boundary as ξ. □

Now we explain very briefly how Theorem II can help us to make the passage from $H^1(\Gamma; \mathfrak{g}_\rho)$ to the vector space of holomorphic quadratic differentials on $\Sigma = \mathbb{D}/\rho$. We begin with some $v \in H^1(\Gamma; \mathfrak{g}_\rho)$. By Lemma 1.4.2, and in the notation used in the proof of Proposition 1.4.3, the class v can be represented by a cocycle (with values in $\mathfrak{g}_\rho$) of the form $\delta\psi$, where ψ is a continuous vector field on $\mathbb{D}$. By Proposition 1.4.3, the vector field ψ is boundary controlled. Let ζ be its distributional boundary. Then $\xi := F(\zeta)$ is a *harmonic* vector field on $\mathbb{D}$. On the basis of Lemma 1.4.1 and Theorem II it is easy to verify that $\delta\xi$ is a 1-cocycle with values in $\mathfrak{g}$. Moreover it agrees with $\delta\psi$ since both $\delta\xi$ and $\delta\psi$ are "matches" for $\delta\zeta$. Therefore ξ can be viewed as an improvement on ψ. The trace-free component of $\mathscr{L}_\xi(g)$ is $\mathrm{Re}(\phi)$ for a quadratic differential ϕ on $\mathbb{D}$ which is holomorphic by Proposition 1.3.1. The quadratic differential ϕ is also invariant under the group $\rho(\Gamma) \subset \mathrm{isom}_+(\mathbb{D})$ because of Lemma 1.4.1 and condition (iii) in Theorem II.

(Remember that $\delta\xi = \delta\psi$.) Therefore we have a holomorphic quadratic differential on $\Sigma = \mathbb{D}/\rho$.

The above procedure based on Theorem II which takes us from $H^1(\Gamma;\mathfrak{g}_\rho)$ to the vector space of holomorphic quadratic differentials on $\mathbb{D}/\rho$ is the inverse of the other one, based on Theorem I. The verification should be mechanical.

2. Constructing harmonic vector fields from quadratic differentials

2.1. ***Harmonic vector fields in isothermal coordinates.*** Suppose that ξ is defined on $U \subset \mathbb{R}^2$ and U is equipped with a Riemannian metric of the form $ds^2 = \lambda(x, y)(dx^2 + dy^2)$.

If the flow $(\phi_t)_{t\in[0,\varepsilon)}$ and the vector field ξ are related as above, then we can describe ϕ_t to first order in terms of ξ:

$$\phi_t(z) \approx z + t\xi(z) \quad \text{for } z \in U.$$

We define a family of Riemannian metrics on U as follows:

$$t \mapsto \rho_t = \phi_t^* g. \tag{2.1.1}$$

More precisely the map in (2.1.1) has the form

$$t \mapsto (D\phi_t : T_pU \to T_{\phi_t(p)}\mathbb{H})^* g. \tag{2.1.2}$$

To the first order, (2.1.2) can be expressed as follows:

$$t \mapsto \big(\mathrm{id} + t\cdot D\xi : T_pU \to T_{\phi_t(p)}\mathbb{H}\big)^* g,$$

where $D\xi$ is the total derivative of ξ (the latter being viewed as a smooth map from an open set in $\mathbb{R}$ or $\mathbb{C}$ to $\mathbb{R}^2$ or $\mathbb{C}$). Continuing in this manner, we get

$$\begin{aligned}\rho_t &\approx (\mathrm{id} + t\cdot D\xi)^T (g + t\cdot Dg(\xi))\,(\mathrm{id} + t\cdot D\xi)\\ &\approx g + t\cdot D\xi^T g + t\,Dg(\xi) + t\cdot D\xi\cdot g\\ &= g + (t\cdot D\xi^T + t\cdot D\xi)\cdot g + t\,Dg(\xi).\end{aligned}$$

Calculating

$$\frac{d\rho_t}{dt}\Big|_{t=0}$$

gives us a section of the vector bundle of (real) symmetric bilinear forms on TU and this is denoted by $\mathscr{L}_\xi(g)$, the Lie derivative of g along ξ. Therefore,

$$\mathscr{L}_\xi g = (D\xi^T + D\xi)g + Dg(\xi) \tag{2.1.3}$$

in our preferred coordinates. We can write $D\xi$ in matrix form,

$$D\xi = \begin{bmatrix} \xi_x^1 & \xi_y^1 \\ \xi_x^2 & \xi_y^2 \end{bmatrix}.$$

Then (2.1.3) turns into

$$\mathscr{L}_\xi(g) = \lambda \begin{bmatrix} 2\xi_x^1 & \xi_y^1 + \xi_x^2 \\ \xi_x^2 + \xi_y^1 & 2\xi_y^2 \end{bmatrix} + \begin{bmatrix} \langle D\lambda, \xi\rangle & 0 \\ 0 & \langle D\lambda, \xi\rangle \end{bmatrix}$$

$$= \underbrace{\lambda \begin{bmatrix} \xi_x^1 - \xi_y^2 & \xi_y^1 + \xi_x^2 \\ \xi_y^1 + \xi_x^2 & \xi_y^2 - \xi_x^1 \end{bmatrix}}_{\text{TF}} + \lambda \begin{bmatrix} \xi_x^1 + \xi_y^2 & 0 \\ 0 & \xi_x^1 + \xi_y^2 \end{bmatrix} + \begin{bmatrix} \langle D\lambda, \xi\rangle & 0 \\ 0 & \langle D\lambda, \xi\rangle \end{bmatrix}.$$

Recall from Section 1.3 that the vector bundle E of (real) symmetric $\mathbb{R}$-bilinear forms on TU splits into a 1-dimensional real vector subbundle (the scalar summand) spanned by the everywhere nonzero section g and a 1-dimensional complex line bundle (the trace-free summand) which is the image of the embedding

$$\hom_{\mathbb{C}}(TU \otimes_{\mathbb{C}} TU, \mathbb{C}) \to E$$

taking ϕ to $\operatorname{Re}(\phi)$. Since the local coordinates that we are using here are in agreement with the conformal structure determined by the Riemannian metric g, our calculation implies that the trace-free component of $\mathscr{L}_\xi(g)$ is the summand with the label TF,

$$\lambda \begin{bmatrix} \xi_x^1 - \xi_y^2 & \xi_y^1 + \xi_x^2 \\ \xi_y^1 + \xi_x^2 & \xi_y^2 - \xi_x^1 \end{bmatrix}. \tag{2.1.4}$$

Indeed, TF is $\operatorname{Re}(f \cdot (dz \otimes dz))$, for $f := \text{TF}_{11} - \iota\text{TF}_{12}$, where $\iota = \sqrt{-1}$.

Proposition 2.1.1. *Let ξ be a smooth vector field on $U \subset \mathbb{R}^2$, and let g be a Riemannian metric on U of the form $ds^2 = \lambda(x, y)(dx^2 + dy^2)$. Then the trace-free component of $\mathscr{L}_\xi(g)$ is* $\operatorname{Re}(f \cdot (dz \otimes_{\mathbb{C}} dz))$, *where $f : U \to \mathbb{C}$ is determined by*

$$\overline{f(z)} = 2\lambda \frac{\partial \xi}{\partial \bar{z}}(z). \tag{2.1.5}$$

Proof. Most of this has already been established. The formula for $\overline{f(z)}$ needs to be unraveled. The Wirtinger derivative (for differentiable functions from U to $\mathbb{C}$) is

$$\partial/\partial\bar{z} = \tfrac{1}{2}(\partial/\partial x + \iota\, \partial/\partial y), \tag{2.1.6}$$

where we allow ourselves to write $z = x + \iota y$. In particular

$$2\lambda \frac{\partial \xi}{\partial \bar{z}} = \lambda \cdot (\xi_x + \iota \xi_y).$$

It is understood that $\xi_x = \xi_x^1 + \iota\xi_x^2$ and $\xi_y = \xi_y^1 + \iota\xi_y^2$, so that we obtain

$$2\lambda \frac{\partial \xi}{\partial \bar{z}} = \lambda(\xi_x^1 + \iota\xi_x^2 - \xi_y^2 + \iota\xi_1^y) = \lambda(\xi_x^1 - \xi_y^2 + \iota(\xi_x^2 + \xi_y^1)) = \text{TF}_{11} + \iota\text{TF}_{12},$$

which is conjugate to $\text{TF}_{11} - \iota\text{TF}_{12}$, our definition of f, given just after (2.1.4). $\square$

Remark 2.1.2. Think of Proposition 2.1.1 as a complement to Proposition 1.3.1.

2.2. ***Solving the potential equation.*** Taking $U = \mathbb{H}$ and $\lambda(x, y) = y^{-2}$ in (2.1.5), we obtain

$$\overline{f(z)} = \frac{-8}{(z-\bar{z})^2} \frac{\partial \xi}{\partial \bar{z}}(z). \tag{2.2.1}$$

This is well known in Teichmüller theory. Scott Wolpert [1987, §2.1] calls it *the potential equation*. He has it in the form

$$(z-\bar{z})^2 \overline{f(z)} = -8 \frac{\partial \xi}{\partial \bar{z}}(z) \tag{2.2.2}$$

except for the factor -8 in the right-hand side, which he does not have. (One may ask why he does not have it. We believe this is explained by different conventions, e.g., regarding the preferred metric on $\mathbb{H}$.) He calls the left-hand side a *harmonic Beltrami differential*, and probably he views both sides as Beltrami differentials, consciously or unconsciously multiplying both sides with a standard Beltrami differential denoted $d\bar{z}/dz$ for better or worse.

Wolpert [1987, §2.3] has a very elegant solution for the potential equation. (Here we assume that f is "known", defined on all of $\mathbb{H}$ and holomorphic. The vector field ξ on $\mathbb{H}$ is the unknown. By Proposition 2.1.1 it must be a harmonic vector field.) His solution, in our notation, is

$$\xi(z) = -\frac{1}{8} \overline{\int_{z_0}^{z} (\bar{z}-s)^2 f(s)\, ds}, \tag{2.2.3}$$

where $z_0 \in \mathbb{H}$ is fixed. The integral is a complex path integral, along some path in $\mathbb{H}$ from z_0 to z. Since the integrand is holomorphic, the value of the integral is independent of the path selected. He writes: *the potential equation* (2.2.2) *is an immediate consequence of differentiation under the integral*. He is right.

For us, (2.2.3) is not the perfect solution. We need a solution which extends to the ideal boundary of $\mathbb{H}$ under conditions on f which we find reasonable. Therefore we will adopt a more laborious method to solve (2.2.2).

Let η be the constant vector field on $\mathbb{H}$ defined by $\eta(z) = 1 \in \mathbb{C}$ for all z.

Lemma 2.2.1. *Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function and let c be a positive real number. The vector field ξ_c defined by*

$$\xi_c(z) = \left(\int_{\operatorname{Im}(z)}^{c} \imath t^2\, \overline{f(\bar{z}+2\imath t)}\, dt \right) \eta(z)$$

for z subject to $\operatorname{Im}(z) < 2c$ *solves* (2.2.2).

We have to assume $\operatorname{Im}(z) < 2c$ in the definition of $\xi_c(z)$ to ensure that $f(\bar{z}+2\imath t)$ is defined for all $t \in [c, \operatorname{Im}(z)]$.

Proof. Write ξ instead of ξ_c in this proof (and drop the constant factor η, so that ξ becomes a function with values in $\mathbb{C}$). Write D for the total derivative acting on such functions, so that the values of D are real 2×2-matrices. Write $\Phi(z, t)$ for the integrand in the definition of ξ.

The function ξ is a composition $u \circ v$, where $v(z) = (\mathrm{Im}(z), z)$ for $z \in \mathbb{H}$ and u is a function of a real and a complex variable:

$$u(r, z) = \int_r^c \Phi(z, t)\, dt.$$

Applying the chain rule in this situation gives

$$D\xi(z) = -\Phi(z, \mathrm{Im}(z)) \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} + \int_{\mathrm{Im}(z)}^c D(\Phi(z, t))\, dt \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.2.4}$$

where $\Phi(z, \mathrm{Im}(z))$ must be viewed as a real $2{\times}1$-matrix. The Wirtinger derivative $\partial/\partial\bar{z}$ is $Q(D)$ for a certain linear map W_2 acting on real $2{\times}2$-matrices,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \frac{1}{2} \begin{bmatrix} a - d \\ b + c \end{bmatrix}.$$

Therefore

$$\frac{\partial\xi}{\partial\bar{z}}(z) = W_2\big(-\Phi(z, \mathrm{Im}(z)) \cdot \begin{bmatrix} 0 & 1 \end{bmatrix}\big) + \int_{\mathrm{Im}(z)}^c W_2(D(\Phi(z, t))\, dt.$$

But $\Phi(z, t)$ is a holomorphic function of z, so that $W_2(D(\Phi(z, t))$ is everywhere zero. So we obtain

$$W_2(D\xi(z)) = W_2\big(-\Phi(z, \mathrm{Im}(z)) \cdot \begin{bmatrix} 0 & 1 \end{bmatrix}\big).$$

Now $\Phi(z, \mathrm{Im}(z)) = \iota y^2 \overline{f(z)}$, where we have written y for $\mathrm{Im}(z)$. Therefore

$$-\Phi(z, \mathrm{Im}(z)) \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} = y^2 \begin{bmatrix} 0 & -\mathrm{Im}(f(z)) \\ 0 & -\mathrm{Re}(f(z)) \end{bmatrix},$$

and so

$$\frac{\partial\xi}{\partial\bar{z}}(z) = W_2(D\xi(z)) = \frac{1}{2} y^2 \begin{bmatrix} \mathrm{Re}(f(z)) \\ -\mathrm{Im}(f(z)) \end{bmatrix} = \frac{1}{2} y^2\, \overline{f(z)} = -\frac{1}{8}(z - \bar{z})^2\, \overline{f(z)}. \quad \square$$

Theorem 2.2.2. *Let f be a holomorphic function on $\mathbb{H}$ which satisfies the condition $|f(z)| \cdot \mathrm{Im}(z)^2 \le b_0$, where b_0 is a positive constant, $z \in \mathbb{H}$ arbitrary. Then the formula*

$$\xi^{\mathrm{reg}}(z) := \lim_{c \to \infty} \left(\xi_c(z) - \left(\xi_c(\iota) + \frac{\partial \xi_c}{\partial z}(\iota) \cdot (z - \iota) \right) \right), \tag{2.2.5}$$

where ξ_c is as in Lemma 2.2.1, defines a smooth vector field ξ^{reg} on $\mathbb{H}$ which solves (2.2.2).

Remarks. The superscript reg is for *regularization*, the art of making divergent things convergent. The formula for ξ^{reg} uses the Wirtinger derivative $\partial/\partial z = \frac{1}{2}(\partial/\partial x - \iota\, \partial/\partial y)$. The "regularizing" term $\xi_c(\iota) + (\partial \xi_c/\partial z)(\iota) \cdot (z - \iota)$ which we subtract from $\xi_c(z)$ can be regarded as the holomorphic part of the first Taylor approximation to ξ_c at $z = \iota$.

The function $z \mapsto |f(z)| \cdot \mathrm{Im}(z)^2$ on $\mathbb{H}$ is bounded if and only if the quadratic differential $f \cdot (dz \otimes_{\mathbb{C}} dz)$ on $\mathbb{H}$ is bounded in the metric of $\mathbb{H}$ and Remark 1.3.2. Indeed the pointwise norm of the 1-form dz at $z_0 \in \mathbb{H}$ is $\mathrm{Im}(z_0)$, and the pointwise norm of $dz \otimes_{\mathbb{C}} dz$ at z_0 is therefore $\mathrm{Im}(z_0)^2$.

The vector field ξ^{reg} is automatically harmonic if it solves (2.2.2), since f is holomorphic by assumption.

The proof of Theorem 2.2.2 is quite long. We begin by isolating some technicalities and generalities.

Lemma 2.2.3. *If $f : \mathbb{H} \to \mathbb{C}$ is holomorphic and satisfies $|f(z)| \cdot \mathrm{Im}(z)^2 \le b_0$ for all $z \in \mathbb{H}$, where b_0 is a positive constant, then there exist positive constants $b_1, b_2, b_3, \dots$ such that $|f^{(n)}(z)| \cdot \mathrm{Im}(z)^{n+2} \le b_n$ for all $z \in \mathbb{H}$ and $n = 1, 2, 3, \dots$.*

Proof. Fix $z_0 \in \mathbb{H}$ and let γ be a smooth curve describing (counterclockwise) a circle of radius $r = \mathrm{Im}(z_0)/2$ about z_0. Then by the Cauchy integral formulas,

$$f^{(n)}(z_0) = \frac{n!}{2\pi\iota} \oint_\gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

The values of $|f|$ on the circle are bounded by $b_0 \cdot (\mathrm{Im}(z_0)/2)^{-2}$. The circumference of the circle is $\pi\,\mathrm{Im}(z_0)$. Therefore

$$\begin{aligned} |f^{(n)}(z_0)| &\le \frac{n!}{2\pi}(\pi\,\mathrm{Im}(z_0)) \cdot b_0(\mathrm{Im}(z_0)/2)^{-2} \cdot (\mathrm{Im}(z_0)/2)^{-(n+1)} \\ &= 2^{n+2} n!\, \pi b_0 (\mathrm{Im}(z_0))^{-(n+2)}. \end{aligned}$$

We can take $b_n = 2^{n+2} n!\, \pi b_0$. □

Lemma 2.2.4 (notation of Lemma 2.2.1). *The Wirtinger derivative $\partial/\partial\bar{z}$ of ξ_c is independent of c, where defined.*

Proof. Make two choices for c, say c_1 and c_2, where $c_2 > c_1 > \frac{1}{2}$. Then

$$\xi_{c_2}(z) - \xi_{c_1}(z) = \left(\int_{c_1}^{c_2} \iota t^2\, \overline{f(\bar{z} + 2\iota t)}\, dt \right) \eta(z)$$

is a holomorphic vector field. It is defined for z with $\mathrm{Im}(z) < 2c_1$. □

Remark 2.2.5. Let $\lambda : U \to \mathbb{C}$ be a differentiable function, where U is open in $\mathbb{C}$. The Wirtinger derivatives $\partial/\partial z$ and $\partial/\partial\bar{z}$ of λ together determine the total derivative $D\lambda$. The relationship is

$$D\lambda = \frac{\partial\lambda}{\partial z} + \frac{\partial\lambda}{\partial\bar{z}} \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

This is meant as an equation between functions taking values in the algebra of real 2×2-matrices. (A complex number determines a real 2×2-matrix since multiplication by that number is an $\mathbb{R}$-linear operator on $\mathbb{R}^2$.)

Proposition 2.2.6. *Let $(\zeta_n)_{n\geq 0}$ be a sequence of harmonic vector fields defined on an open set $U \subset \mathbb{H}$, and converging in the compact-open C^0 topology to a vector field ζ_∞. Suppose that the trace-free component of $\mathscr{L}_{\zeta_n}(g)$ is the same for all n. Then the sequence (ζ_n) converges to ζ_∞ in the compact-open C^∞-topology. Hence ζ_∞ is harmonic, and the trace-free component of $\mathscr{L}_{\zeta_\infty}(g)$ agrees with the trace-free component of $\mathscr{L}_{\zeta_n}(g)$ for all n.*

Proof. The sequence $(\zeta_n - \zeta_0)_{n\geq 0}$ is a sequence of holomorphic vector fields which can also be viewed as a sequence of holomorphic functions. Therefore the Weierstrass convergence theorem applies. □

Proof of Theorem 2.2.2. Write D for total derivatives (of $\mathbb{C}$-valued functions, with respect to a variable $z \in \mathbb{H}$ or $z \in \mathbb{C}$). Write $\kappa(z) = \max\{1, \mathrm{Im}(z)\}$. As in the proof of Lemma 2.2.1 write

$$\Phi(z,t) := \iota t^2\, \overline{f(\bar{z} + 2\iota t)}.$$

We do not insist on the distinction between vector fields (on open subsets of $\mathbb{H}$) and $\mathbb{C}$-valued functions, because it is irrelevant here. In particular ξ_c will be viewed as a function. The first step is to show

$$\text{(2.2.6)}\quad \xi_c(z) - \left(\xi_c(\iota) + \frac{\partial\xi_c}{\partial z}(\iota)\cdot(z-\iota)\right) = \int_{\kappa(z)}^{c} \Phi(z,t) - \Phi(\iota,t) - D\Phi(\iota,t)\cdot(z-\iota)\,dt + F(z),$$

where F is a continuous function of the variable z, independent of c. As a matter of language, we might say (in this proof) that two continuous functions of z and c are *equivalent* if their difference is a function of z only. Then $(z,c) \mapsto \xi_c(z)$ is equivalent to

$$(z,c) \mapsto \int_{\kappa(z)}^{c} \Phi(z,t)\,dt$$

and $(z,c) \mapsto \xi_c(\iota)$ is equivalent to

$$(z,c) \mapsto \int_{\kappa(z)}^{c} \Phi(\iota,t)\,dt.$$

By Lemma 2.2.4 and Remark 2.2.5, the function $(z, c) \mapsto (\partial\xi_c/\partial z)(\iota) \cdot (z - \iota)$ is equivalent to $(z, c) \mapsto D\xi_c(\iota) \cdot (z - \iota)$. As in the proof of Lemma 2.2.1 we have

$$D\xi_c(\iota) = -\Phi(\iota, 1) \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} + \int_1^c D(\Phi(\iota, t))\, dt,$$

so that $(z, c) \mapsto D\xi_c(\iota) \cdot (z - \iota)$ is equivalent to

$$(z, c) \mapsto \int_{\kappa(z)}^{c} D(\Phi(\iota, t)) \cdot (z - \iota)\, dt.$$

Using all these equivalences, we obtain (2.2.6). And now the second step is clearly to show that the improper integral

$$\int_{\kappa(z)}^{\infty} \Phi(z, t) - \big(\Phi(\iota, t) + D\Phi(\iota, t) \cdot (z - \iota)\big)\, dt \tag{2.2.7}$$

exists. It is a good idea to think of the expression $\Phi(\iota, t) + D\Phi(\iota, t) \cdot (z - \iota)$ as the first Taylor polynomial at $z = \iota$ of the function $z \mapsto \Phi(z, t)$, for fixed t. The "integral form of the remainder" in Taylor's formula gives us

$$\Phi(z, t) - \big(\Phi(\iota, t) + D\Phi(\iota, t) \cdot (z - \iota)\big) = \frac{1}{2} \int_0^1 u''(s) \cdot (1 - s)\, ds,$$

where $u(s) := \Phi(\iota + s(z - \iota), t) = \iota t^2\, \overline{f(-\iota + s(\bar{z} + \iota) + 2\iota t)}$ for fixed t and z. Now Lemma 2.2.3 gives the estimate

$$|u''(s)| \le t^2 \frac{b_2 |z - \iota|^2}{|\iota + s(z - \iota) - 2\iota t|^4}.$$

Here $s \in [0, 1]$, and z is fixed, and b_2 is a positive constant. If $t \ge 1 + |z|$, then we may conclude

$$|u''(s)| \le \frac{b_2 |z - \iota|^2}{t^2}.$$

Therefore if $1 + |z| \le A_1 \le A_2$, then

$$\begin{aligned} &\left| \int_{A_1}^{A_2} \Phi(z, t) - \big(\Phi(\iota, t) + D\Phi(\iota, t) \cdot (z - \iota)\big)\, dt \right| \\ &\qquad \le \int_{A_1}^{A_2} \frac{1}{2} \int_0^1 \frac{b_2 |z - \iota|^2}{t^2} \cdot (1 - s)\, ds\, dt \\ &\qquad = \frac{1}{4} b_2 |z - \iota|^2 \int_{A_1}^{A_2} t^{-2}\, dt = \frac{1}{4} b_2 |z - \iota|^2 (A_1^{-1} - A_2^{-1}) \le \frac{b_2 |z - \iota|^2}{4A_1}. \end{aligned}$$

This means that the improper integral (2.2.7) converges. In other words, $\xi^{\mathrm{reg}}(z)$ is well defined by means of formula (2.2.5) for every $z \in \mathbb{H}$. But we have shown more: the convergence is uniform on compact sets. More precisely, writing

$$V(c, z) := \xi_c(z) - \left(\xi_c(\iota) + \frac{\partial \xi_c}{\partial z}(\iota) \cdot (z - \iota) \right),$$

we have shown that

$$|V(c_1, z) - V(c, z)| \leq \frac{b_2|z - \iota|^2}{4c} \tag{2.2.8}$$

under the condition $c_1 \geq c > 1 + |z|$. Now Proposition 2.2.6 can be applied. It follows that ξ^{reg} is a harmonic vector field on $\mathbb{H}$ and a solution to (2.2.2). ☐

2.3. ***Boundary values.*** The standard isometry from the upper half-plane model $\mathbb{H}$ of the hyperbolic plane to the disk model $\mathbb{D}$ is the Cayley transform C, defined by $C(z) := (z - \iota)/(z + \iota)$. The (complex) derivative of C is $z \mapsto (z + \iota)^{-2}$.

Theorem 2.3.1. *The vector field $C_*(\xi^{\text{reg}})$ on $\mathbb{D}$ obtained by applying the Cayley transform to ξ^{reg} of Theorem 2.2.2 admits a continuous extension to $\mathbb{D} \cup S^1$. The value of the extended vector field at $1 \in S^1$ is 0.*

Remark 2.3.2. The restriction to S^1 of the extended vector field on $\mathbb{D} \cup S^1$ is not claimed to be tangential to S^1. From our point of view this is an issue. It will be addressed in Section 2.4, which has no other purpose.

As a preparation for the proof, we translate the statement so that we can continue to work in the upper half-plane setting. It is convenient to show first that ξ^{reg} extends to a continuous vector field defined on all of $\mathbb{H} \cup \mathbb{R}$. Under the Cayley transform this corresponds to a continuous vector field on $\mathbb{D} \cup S^1 \smallsetminus \{1\}$. We extend this to all of $\mathbb{D} \cup S^1$ by *defining* the missing value (at $1 \in S^1$) to be 0. Then we still have to establish continuity at $1 \in \mathbb{D} \cup S^1$. By the formula for the derivative of C, this is equivalent to the following statement:

$$\lim_{z \in \mathbb{H},\, |z| \to \infty} \frac{\xi^{\text{reg}}(z)}{|z|^2} = 0. \tag{2.3.1}$$

Proof of Theorem 2.3.1. First we show that ξ^{reg} extends to a continuous vector field on $\mathbb{H} \cup \mathbb{R}$. Select some $z \in \mathbb{R}$. The integral

$$\int_{\text{Im}(z)}^{c} \iota t^2\, \overline{f(\bar{z} + 2\iota t)}\, dt \tag{2.3.2}$$

is an improper integral because the integrand is not defined for $t = 0 = \text{Im}(z)$. But for $t > 0$ the integrand is defined and moreover

$$|\iota t^2 \overline{f(\bar{z} + 2\iota t)}| \leq |\bar{z} + 2\iota t|^2 |f(\bar{z} + 2\iota t)| \leq b_0$$

by our condition on f. It follows that, for $z \in \mathbb{R}$, the improper integral (2.3.2) converges. So $\xi_c(z)$ is defined or definable for *all* z such that $0 \leq \text{Im}(z) < 2c$, and is continuous as a function of z. Inequality (2.2.8) remains meaningful and valid if we allow $z \in \mathbb{H} \cup \mathbb{R}$. Therefore $\xi^{\mathrm{reg}}(z)$ has a continuous extension to all of $\mathbb{H} \cup \mathbb{R}$.

It remains to prove the claim (2.3.1). Fix some $z \in \mathbb{H}$. We will use (2.2.8), but we have some freedom in choosing c, and we decide $c = 2 + |z|$. This gives

$$\left|\xi^{\mathrm{reg}}(z) - \xi_c(z) - \left(\xi_c(\iota) + \frac{\partial \xi_c}{\partial z}(\iota) \cdot (z - \iota)\right)\right| \leq \frac{b_2 |z - \iota|^2}{4c} = \frac{b_2 |z - \iota|^2}{8 + 4|z|}.$$

Since the right-hand side of this, divided by $|z|^2$, tends to zero for $|z| \to \infty$, the same is true for the left-hand side. Therefore it is enough to show that the fraction

$$\frac{\xi_c(z) - \big(\xi_c(\iota) + (\partial \xi_c / \partial z)(\iota) \cdot (z - \iota)\big)}{|z^2|}, \tag{2.3.3}$$

evaluated *only* on pairs (c, z) where $c = 2 + |z|$, tends to zero for $|z| \to \infty$. By Lemma 2.2.3 and elementary integration (see Remark 2.3.3 for more details), the following estimates are available:

$$|\xi_c(z)| < A \cdot |z|, \quad |\xi_c(\iota)| < A \cdot |z|, \quad \left|\frac{\partial \xi_c}{\partial z}(\iota)\right| < A \ln |z| + B$$

for some positive constants A and B. Therefore

$$\left|\frac{\partial \xi_c}{\partial z}(\iota) \cdot (z - \iota)\right| < (A \ln |z| + B) \cdot (|z| + 1).$$

Using these estimates in (2.3.3), we see that it does tend to zero for $|z| \to \infty$. □

Remark 2.3.3. Estimate for $\xi_c(z)$: We have

$$\xi_c(z) = \int_{\mathrm{Im}(z)}^{c} \iota t^2 \, \overline{f(\bar{z} + 2\iota t)} \, dt,$$

where $|f(\bar{z} + 2\iota t)| \leq b_0 |\bar{z} + 2\iota t|^{-2}$. Since $t \in [\mathrm{Im}(z), c]$, where $c = 2 + |z|$, we have $t \geq \mathrm{Im}(z)$ and so $|\bar{z} + 2\iota t| \geq \mathrm{Im}(\bar{z} + 2\iota t) = 2t - \mathrm{Im}(z) \geq t$. Therefore

$$|\xi_c(z)| \leq \int_{\mathrm{Im}(z)}^{c} t^2 b_0 |\bar{z} + 2\iota t|^{-2} \, dt \leq b_0 \int_{\mathrm{Im}(z)}^{c} 1 \, dt = b_0 (c - \mathrm{Im}(z)) \leq b_0 (2 + |z|).$$

Estimate for $\xi_c(\iota)$: It must be remembered that $c = 2 + |z|$, where z has little to do with ι. We have

$$\xi_c(\iota) = \int_{1}^{c} \iota t^2 \, \overline{f(\bar{\iota} + 2\iota t)} \, dt,$$

where $|f(\bar{\iota} + 2\iota t)| \leq b_0 |2t - 1|^{-2}$. Since $t \geq 1$ we have $|2t - 1| \geq t$ and so

$$|\xi_c(\iota)| \leq \int_{1}^{c} t^2 b_0 |\bar{\iota} + 2\iota t|^{-2} \, dt \leq b_0 \int_{1}^{c} 1 \, dt = b_0 (c - 1) = b_0 (1 + |z|).$$

Estimate for $(\partial\xi_c/\partial z)(\iota)$: Again it must be remembered that $c = 2 + |z|$, where z has little to do with ι. We try the total derivative $D\xi_c$ first. In the notation and conventions of (2.2.4) we have

$$D\xi_c(\iota) = -\Phi(\iota, 1) \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} + \int_1^c D(\Phi(\iota, t))\, dt,$$

which means

$$D\xi_c(\iota) = -\iota\overline{f(\iota)} \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} + \int_1^c \iota t^2 (Df_t)(\iota)\, dt, \tag{2.3.4}$$

where f_t is the (holomorphic) function $z \mapsto \overline{f(\bar{z} + 2\iota t)}$. We note therefore that the second summand in the right-hand side of (2.3.4) will contribute in full to $(\partial\xi_c/\partial z)(\iota)$; the other one may not contribute in full but it will make a constant contribution (independent of c). Therefore we have to estimate the second summand only. By the chain rule, $(Df_t)(\iota) = (Df_0)(\iota - 2\iota t)$. This is a complex number (which can also be viewed as a real 2×2-matrix). Now $|(Df_0)(\iota - 2\iota t)| = |Df(2\iota t - \iota)|$ and by Lemma 2.2.3 we have $|Df(2\iota t - \iota)| =\leq b_1|2t - 1|^{-3} \leq b_1 t^{-3}$. Therefore

$$\left| \int_1^c \iota t^2 (Df_t)(\iota)\, dt \right| \leq \int_1^c b_1 t^{-1}\, dt = b_1 \ln c = b_1 \ln(2 + |z|).$$

Remark 2.3.4. It seems to us that Wolpert's solution (2.2.3) of (2.2.2) admits a continuous extension to $\mathbb{H} \cup \mathbb{R}$ if f satisfies the conditions of Theorem 2.2.2. But we were unable to show that it has good growth behavior for $z \to \infty$, e.g., similar to (2.3.1), especially where z approaches ∞ on a horizontal through z_0.

2.4. ***Hardy space to the rescue.*** Let $\mathcal{H} = L^2(S^1, \mathbb{C})$ be the complex Hilbert space of square integrable complex valued functions on S^1. The hermitian inner product is

$$\langle v, w \rangle = \frac{1}{2\pi} \int_0^{2\pi} \langle v(e^{\iota t}), w(e^{\iota t}) \rangle\, dt$$

for $v, w \in \mathcal{H}$. Consequently $\mathcal{H}$ has a standard Hilbert basis consisting of the vectors $v_k := (z \mapsto z^k)$ for $k \in \mathbb{Z}$. It has an orthogonal splitting $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1$ is the closure of the span of the v_k for $k \geq 0$, and $\mathcal{H}_2$ is the closure of the span of the v_k for $k < 0$. The closed subspace $\mathcal{H}_1$ is known as the *Hardy* space. Each $w \in \mathcal{H}_1$ has a canonical "extension" to a function $w^e : \mathbb{D} \cup S^1 \to \mathbb{C}$ which is holomorphic in $\mathbb{D}$. Namely, if $w = \sum_{k=0}^{\infty} a_k v_k$ in $\mathcal{H}_1$, then the power series $\sum_{k=0}^{\infty} a_k z_k$ converges locally uniformly in $\mathbb{D}$ and represents a holomorphic function there. Note that w can be recovered from the restriction of w^e to $\mathbb{D}$. It is the limit (in the metric of $\mathcal{H}_1$) of the functions $z \mapsto w^e(sz)$, for $z \in S^1$, as $s \in [0, 1)$ tends to 1.

Let $\mathcal{X} \subset \mathcal{H}$ be the closed $\mathbb{R}$-linear subspace consisting of the functions on S^1 which have the form $z \mapsto \iota z \cdot u(z)$, where u is a *real*-valued square-integrable function on S^1. It is an exercise in linear algebra to show that $\mathcal{X}+\mathcal{H}_1=\mathcal{H}$ and $\dim_{\mathbb{R}}(\mathcal{X}\cap\mathcal{H}_1)=3$. In fact $\mathcal{X}\cap\mathcal{H}_1$ has an orthonormal (real) basis consisting of the functions

$$z\mapsto \iota z, \quad z\mapsto \tfrac{1}{2}\iota z\cdot\iota(z^{-1}-z), \quad z\mapsto \tfrac{1}{2}\iota z\cdot(z^{-1}+z).$$

This fact has a more illuminating formulation if we think of $\mathcal{H}$ as the vector space of L^2-vector fields on S^1 (where the vectors in the vector fields are elements of $\mathbb{C}$). Then $\mathcal{X}$ is precisely the space of *tangential* L^2-vector fields on S^1. The more illuminating formulation is as follows.

Lemma 2.4.1. $\mathcal{X}+\mathcal{H}_1=\mathcal{H}$ *and* $\mathcal{X}\cap\mathcal{H}_1=\mathfrak{g}$, *where* $\mathfrak{g}$ *is the space of Killing vector fields on* S^1 (see explanation below).

It makes sense to speak of Killing vector fields on $\mathbb{D}$; these are the "infinitesimal isometries" of $\mathbb{D}$ with the Poincaré metric. They form a 3-dimensional Lie algebra $\mathfrak{g}$ under the Lie product, which we identify with the Lie algebra of $\mathrm{isom}_+(\mathbb{D})$. It is well known that the Killing vector fields extend to smooth vector fields on $\mathbb{D}\cup S^1$ which are tangential to S^1. Restricting these to S^1 is a faithful operation, and so we may allow ourselves occasionally to think of $\mathfrak{g}$ as a 3-dimensional Lie algebra of tangential smooth vector fields on S^1. (The Lie product will not be of any importance here.)

Proof of Theorem I in Section 1. Let h be the Poincaré metric on $\mathbb{D}$. Let ϕ be a quadratic differential on $\mathbb{D}$ which is uniformly bounded in the norm of Remark 1.3.2. By Theorems 2.2.2 and 2.3.1, there exists a continuous vector field ζ on $\mathbb{D}\cup S^1$ which is smooth on $\mathbb{D}$, and such that the trace-free component of $\mathscr{L}_\zeta(h)$ is ϕ. (In more detail: Let $C^*(\phi)$ be the pullback of ϕ under the Cayley transform C, a holomorphic quadratic differential on $\mathbb{H}$. Write $C^*(\phi)=f\cdot(dz\otimes_{\mathbb{C}}dz)$, so that f is holomorphic and satisfies the conditions of Theorem 2.2.2. Find ξ^{reg} as in the said theorem. Let $C_*(\xi^{\mathrm{reg}})$ be the corresponding vector field on $\mathbb{D}$. Let ζ be the extension of $C_*(\xi^{\mathrm{reg}})$ to $\mathbb{D}\cup S^1$ which exists by Theorem 2.3.1.)

The vector field ζ need not be tangential along S^1. But by Lemma 2.4.1 there exists $\psi\in\mathcal{H}_1$ such that

$$\zeta|_{S^1}-\psi\in\mathcal{X}.$$

Because ψ belongs to the Hardy space $\mathcal{H}_1$, it has a canonical extension ψ^e to $\mathbb{D}\cup S^1$ which is holomorphic on $\mathbb{D}$. It turns out that $\xi:=\zeta-\psi^e$, restricted to $\mathbb{D}$, is a smooth and boundary-controlled vector field on $\mathbb{D}$ such that the trace-free component of $\mathcal{L}_\xi(g)$ is $\mathrm{Re}(\phi)$. We will verify the conditions one by one.

(i) The trace-free component of the Lie derivative of the Riemannian metric h along $\zeta-\psi^e$ is ϕ because the trace-free component of $\mathscr{L}_\zeta(h)$ is ϕ and ψ^e is holomorphic on $\mathbb{D}$.

(ii) The vector field $\zeta - \psi^e$ (restricted to $\mathbb{D}$) is boundary controlled because we have the extension to $\mathbb{D} \cup S^1$ by construction, and the extension is tangential along S^1 by construction. The matching condition which must be satisfied by the restrictions of $\zeta - \psi$ to $\mathbb{D}$ and S^1, respectively, is indeed satisfied because it is satisfied separately for ζ and ψ^e.

It remains to understand what happens if $\phi \equiv 0$. In this case, ξ must be a holomorphic vector field on $\mathbb{D}$. It can be written uniquely as a power series $\sum_{k=0}^{\infty} a_k z^k$ which converges locally uniformly in $\mathbb{D}$. Since ξ is boundary controlled, it has a distributional boundary $\beta \in \mathcal{X}$. The matching condition relating ξ and β means that the Fourier coefficients of β are precisely the numbers $a_0, a_1, a_2, \ldots$, the Taylor coefficients of ξ. It follows that $\beta \in \mathcal{H}_1$, and so $\beta \in \mathcal{H}_1 \cap \mathcal{X}$. Therefore β is a Killing vector field on S^1 by Lemma 2.4.1. This has consequences for the Fourier coefficients $a_0, a_1, \ldots$ of β (for example, only a_0, a_1, a_2 can be nonzero). Then we can conclude $\xi \in \mathfrak{g}$. □

3. Constructing harmonic vector fields from boundary data

3.1. ***Symmetry properties of the classical Poisson formula.*** The symmetry group that we have in mind is $\mathrm{isom}(\mathbb{D})$, where $\mathbb{D}$ has the Poincaré metric as usual. Of course this acts on $\mathbb{D}$, but it also acts on S^1 and (continuously) on the compact manifold with boundary $\mathbb{D} \cup S^1$. It acts (on the right, by precomposition) on the vector spaces $C^0(S^1; \mathbb{R})$ and $C^0(\mathbb{D}; \mathbb{R})$. (Both $C^0(S^1; \mathbb{R})$ and $C^0(\mathbb{D}; \mathbb{R})$ are to be equipped with the compact-open C^0-topology. One of them is a Banach space, the other is just a topological vector space.)

Let $\mathscr{F}$ be the real vector space of continuous, $\mathbb{R}$-linear and $\mathrm{isom}(\mathbb{D})$-equivariant maps from $C^0(S^1; \mathbb{R})$ to $C^0(\mathbb{D}; \mathbb{R})$

Proposition 3.1.1. $\dim_{\mathbb{R}}(\mathscr{F}) = 1.$

Proof. Let F be a nonzero element of $\mathscr{F}$. For each $z \in \mathbb{D}$, let ev_z from $C^0(\mathbb{D}; \mathbb{R})$ to $\mathbb{R}$ be the map *evaluation at* z, a linear functional. Because $\mathrm{isom}(\mathbb{D})$ acts transitively on $\mathbb{D}$, the $\mathrm{isom}(\mathbb{D})$-equivariant map F is determined by the composition $\mathrm{ev}_0 \circ F$. The map $\mathrm{ev}_0 \circ F$ can no longer be claimed to be equivariant for the action(s) of $\mathrm{isom}(\mathbb{D})$, but it is equivariant w.r.t. the subgroup $\mathrm{O}(2) \subset \mathrm{isom}(\mathbb{D})$ consisting of the elements which fix $0 \in \mathbb{D}$. Therefore $\mathrm{ev}_0 \circ F$ is an $\mathrm{O}(2)$-invariant linear functional on $C^0(S^1; \mathbb{R})$. It is well known that the real vector space of these is 1-dimensional, generated by the Haar integral. Therefore $\dim_{\mathbb{R}}(\mathscr{F}) \leq 1$.

On the other hand, we can use the equivariance condition to construct $P \in \mathscr{F}$ such that $\mathrm{ev}_0 \circ P$ is the nonzero linear functional taking $v \in C^0(S^1; \mathbb{R})$ to

$$\frac{1}{2\pi} \int_0^{2\pi} v(e^{it})\, dt.$$

Choose $h \in \mathrm{isom}(\mathbb{D})$. By equivariance we must have $(P(v))(h(0)) = (P(v \circ h))(0)$, which comes down to

$$P(v)(h(0)) = \frac{1}{2\pi} \int_0^{2\pi} v(h(e^{\iota t}))\, dt. \tag{3.1.1}$$

The value of the integral depends only on $h(0)$, since h is determined by $h(0)$ up to precomposition with an element of $\mathrm{O}(2) \subset \mathrm{isom}(\mathbb{D})$. Therefore we have a definition of P in (3.1.1). It follows that $\dim(\mathscr{F}) \geq 1$. □

Remark 3.1.2. It is not obvious from (3.1.1) that $P(v)$ is harmonic, and that v and $P(v)$ together define a *continuous* function on $\mathbb{D} \cup S^1$. We are not going to justify these statements fully here (because they are well known). But in Section 3.2 we will encounter similar statements in a slightly different setup, and we will have to justify those. Therefore we sketch an argument. Let us try to make sense of $K := P(\delta_1)$, where δ_1 is a Dirac distribution at $1 \in S^1$. Think of δ_1 as the limit of a sequence of step functions w_n for $n \geq 2$, where w_n is zero outside the short arc I_n in S^1 with endpoints $e^{-\iota/2n}$ and $e^{\iota/2n}$, and has the constant value n for points in the arc. Then $P(\delta_1)$ should be the limit of the $P(w_n)$, and for $P(w_n)$ we expect

$$P(w_n)(h(0)) = \frac{1}{2\pi} \int_0^{2\pi} w_n(h(e^{\iota t}))\, dt$$

although w_n is not continuous. Here we may assume $h \in \mathrm{isom}_+(\mathbb{D})$, which implies that h is holomorphic so that we can use complex calculus notation. Since $w_n \circ h$ is constant on $h^{-1}(I_n)$ with constant value n, and zero elsewhere, the integral is n times the length of $h^{-1}(I_n)$. The length of $h^{-1}(I_n)$ is well approximated by the modulus of the complex number $(h^{-1})'(1)$, times the length of I_n, which is $2\pi/n$. We arrive at

$$K(h(0)) = P(\delta_1)(h(0)) = |(h^{-1})'(1)| = \frac{1}{|h'(h^{-1}(1))|}.$$

This is well defined, which means: dependent only on $h(0)$. Next, it is an interesting exercise to show that the map taking $v \in C^0(S^1; \mathbb{R})$ to the convolution of v and K is an element of $\mathscr{F}$. Therefore we have found another definition of P. The two claims about P can now be reformulated as claims about K. In other words it remains to show that K is harmonic, and that for each $v \in C^0(S^1; \mathbb{R})$, the Hilbert space inner product of v and the function $z \mapsto K(sz)$ (for $z \in S^1$ variable and $s \in [0, 1)$ fixed) tends to $v(1)$ as s tends to 1.

3.2. ***Vector fields on the circle as boundary data.*** Let $C^0_v(S^1; TS^1)$ be the topological real vector space (with the compact-open topology) of continuous tangential vector fields on S^1, a.k.a. the space of continuous sections of $TS^1 \to S^1$. An element ψ of $C^0_v(S^1; TS^1)$ can also be viewed as a continuous map from S^1 to $\mathbb{C}$, subject to some conditions, because S^1 is a smooth submanifold of $\mathbb{C}$. This point of view is used in the next lemma. Note that U(1) acts on the right of $C^0_v(S^1; TS^1)$ by $(\psi, A) \mapsto A^*\psi = A^{-1}\cdot(\psi \circ A)$ for $A \in \mathrm{U}(1)$. (Of course A is nothing but a complex number of modulus 1, but we are tend to think of it as a $\mathbb{C}$-linear endomorphism of $\mathbb{C}$. And although it is sometimes convenient to view ψ as a function with values in $\mathbb{C}$, the formula for $A^*\psi$ is what it is because ψ is a vector field after all.)

Lemma 3.2.1. *Every continuous and $\mathbb{R}$-linear map $\Lambda : C^0_v(S^1; TS^1) \to \mathbb{C}$ which has $\Lambda(A^*\psi) = A^{-1}\Lambda(\psi)$ for all $\psi \in C^0_v(S^1; TS^1)$ and $A \in \mathrm{U}(1)$ is of the form*

$$\Lambda(\psi) = a \cdot \int_0^{2\pi} \psi(e^{\iota t})\, dt \tag{3.2.1}$$

for some $a \in \mathbb{C}$.

Proof. Given such Λ, we extend it to a map $\Lambda^{\mathbb{C}} : C^0_v(S^1; TS^1 \otimes_{\mathbb{R}} \mathbb{C}) \to \mathbb{C}$ as follows. Elements of $C^0_v(S^1; TS^1 \otimes_{\mathbb{R}} \mathbb{C})$ can be written uniquely in the form $\psi = \psi_1 + \iota\psi_2$, where both ψ_1 and ψ_2 are tangential to S^1. Then we let

$$\Lambda^{\mathbb{C}}(\psi_1 + \iota\psi_2) := \Lambda(\psi_1) + \iota\Lambda(\psi_2).$$

There are now two commuting actions of U(1) on $C^0_v(S^1; TS^1 \otimes_{\mathbb{R}} \mathbb{C})$. One is given by $(\psi, A) \mapsto A^*\psi$. The other is pointwise, $(\psi, A) \mapsto A \cdot \psi$. By our assumption on Λ, the map $\Lambda^{\mathbb{C}}$ intertwines the first action with the conjugate of the standard action of U(1) on $\mathbb{C}$. By construction, it intertwines the second action with the standard action of U(1) on $\mathbb{C}$. It follows that $\Lambda^{\mathbb{C}}$ is *invariant* under the operation

$$\psi \mapsto A \cdot A^*\psi,$$

where $\psi \in C^0_v(S^1; TS^1 \otimes_{\mathbb{R}} \mathbb{C})$ and $A \in \mathrm{U}(1)$. Here we can also write $A \cdot A^*\psi = \psi \circ A$ if we think of ψ as a function with values in $\mathbb{C}$. (The inclusion of the tangent space T_zS^1 in $\mathbb{C}$ extends uniquely to a $\mathbb{C}$-linear isomorphism $T_zS^1 \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$.) Briefly, $\Lambda^{\mathbb{C}}$ satisfies $\Lambda^{\mathbb{C}}(\psi \circ A) = \Lambda^{\mathbb{C}}(\psi)$ for all ψ and all $A \in \mathrm{U}(1)$. Because it is also $\mathbb{C}$-linear by construction, it must have the form

$$\psi \mapsto a \cdot \int_0^{2\pi} \psi(e^{\iota t})\, dt$$

for some $a \in \mathbb{C}$. In particular this is valid for $\psi \in C^0_v(S^1; TS^1)$. □

Remark 3.2.2. Clearly $\Lambda : C^0_v(S^1; TS^1) \to \mathbb{C}$ satisfies the condition $\Lambda(A^*\psi) = A^{-1}\Lambda(\psi)$ for all $\psi \in C^0_v(S^1; TS^1)$ and $A \in \mathrm{U}(1)$ if it is of the form (3.2.1). But if the complex number a in (3.2.1) is a real number, and only then, it will satisfy the same condition for all $A \in \mathrm{O}(2)$. The proof is by direct verification.

Let $C^0_v(\mathbb{D}; T\mathbb{D})$ be the real vector space of continuous vector fields on $\mathbb{D}$, equipped with the compact-open C^0 topology. The group $\mathrm{isom}(\mathbb{D})$ acts on the right of $C^0_v(\mathbb{D}; T\mathbb{D})$ by $(\xi, h) \mapsto h^*\xi$, which means $(h_*\xi)(z) = Dh(z)^{-1}\big(\xi(h(z))\big)$ for $z \in \mathbb{D}$. The action is $\mathbb{R}$-linear.

Let $\mathscr{F}_{vf}$ be the real vector space of continuous, $\mathbb{R}$-linear and $\mathrm{isom}(\mathbb{D})$-equivariant maps from $C^0_v(S^1; TS^1)$ to $C^0_v(\mathbb{D}; T\mathbb{D})$.

Corollary 3.2.3. $\dim_{\mathbb{R}}(\mathscr{F}_{vf}) = 1.$

Proof. Let F be a nonzero element of $\mathscr{F}_{vf}$. For each $z \in \mathbb{D}$, let ev_z from $C^0_v(\mathbb{D}; T\mathbb{D})$ to $\mathbb{C} = T_0\mathbb{D}$ be the map *evaluation at z*, an $\mathbb{R}$-linear map. Because $\mathrm{isom}(\mathbb{D})$ acts transitively on $\mathbb{D}$, the $\mathrm{isom}(\mathbb{D})$-equivariant map F is determined by the composition $\mathrm{ev}_0 \circ F$. The map $\Lambda := \mathrm{ev}_0 \circ F$ can no longer claim to be equivariant for the action(s) of $\mathrm{isom}(\mathbb{D})$, but it satisfies the condition of Lemma 3.2.1 for $A \in \mathrm{U}(1)$, and the stronger condition of Remark 3.2.2 which allows $A \in \mathrm{O}(2)$. Therefore $\Lambda = \mathrm{ev}_0 \circ F$ has the form (3.2.1) for some *real* number a. It follows that $\dim_{\mathbb{R}}(\mathscr{F}_{vf}) \le 1$.

We can use the equivariance condition to construct an $F \in \mathscr{F}_{vf}$ such that $\mathrm{ev}_0 \circ F$ is the nonzero linear functional taking $\psi \in C^0_v(S^1; TS^1)$ to

$$\frac{1}{2\pi}\int_0^{2\pi} \psi(e^{\iota t})\,dt.$$

Choose $h \in \mathrm{isom}(\mathbb{D})$. By equivariance we must have $(Dh(0))^{-1}\big((F(\psi))(h(0))\big) = (F(h^*\psi))(0)$, which comes down to

$$F(\psi)(h(0)) = \frac{1}{2\pi}(Dh(0))\bigg(\int_0^{2\pi} (h^*\psi)(e^{\iota t})\,dt\bigg). \tag{3.2.2}$$

It is easy to see that the right-hand side depends only on $h(0)$, because h is determined by $h(0)$ up to precomposition with an element of $\mathrm{O}(2) \subset \mathrm{isom}(\mathbb{D})$). Therefore F in (3.2.2) is well defined. It is easily seen to be nonzero. It follows that $\dim(\mathscr{F}_{vf}) \ge 1$. □

We want to show that $F(\psi)$, with the definition of F in (3.2.2), is always a harmonic vector field and that ψ and $c \cdot F(\psi)$ together define a continuous vector field on $\mathbb{D} \cup S^1$, where c is a positive real constant factor independent of ψ. Imitating the strategy outlined in Remark 3.1.2, we begin by making sense of

$$K_{vf} := F(\delta_1 \cdot \omega),$$

where δ_1 is the Dirac distribution and $\omega \in C^0_v(S^1, TS^1)$ is the tangential vector field defined by $\omega(z) = \iota z$ for $z \in S^1$. To simplify this, we remark that for every $z \in \mathbb{D}$ there exists a *unique* $h \in \mathrm{isom}_+(\mathbb{D})$ such that $h(0) = z$ and $h(1) = 1$. For such an $h \in \mathrm{isom}_+(\mathbb{D})$ we first need to make sense of $h^*(\delta_1 \cdot \omega) = h^*\delta_1 \cdot h^*\omega$. It is not hard to see (and the calculation of $h^*\delta_1$ can be seen in Remark 3.1.2) that this is $(h'(1))^{-2}\delta_1 \cdot \omega$. Therefore we make the *definition*

$$K_{vf}(h(0)) = \iota h'(0) \cdot (h'(1))^{-2} \tag{3.2.3}$$

on the understanding that $h \in \mathrm{isom}_+(\mathbb{D})$ and $h(1) = 1$. If we specify $z = h(0)$, then we can write

$$h(w) = \frac{yw + z}{yw\bar{z} + 1},$$

where $y = (1 - z)/(1 - \bar{z}) = (1 - z^2)/|1 - z|^2$. Then $h'(w) = y(1 - |z|^2)/(yw\bar{z} + 1)^2$. Therefore $h'(0) = y(1 - |z|^2)$ and

$$h'(1) = \frac{y(1 - |z|^2)}{(y\bar{z} + 1)^2} = \frac{y(1 - |z|^2)}{(z + y)^2}$$

(where we have used $h(w) = 1$, so that $y\bar{z} + 1 = z + y$) and

$$h'(0)/(h'(1))^2 = y(1 - |z|^2) \cdot \frac{(z + y)^4}{y^2(1 - |z|^2)^2} = \frac{(z + y)^4}{y(1 - |z|^2)} = \frac{|1 - z|^2(z + y)^4}{(1 - z)^2(1 - |z|^2)}.$$

Now we use $z + y = (z - |z|^2 + 1 - z)/(1 - \bar{z}) = (1 - |z|^2)/(1 - \bar{z})$ and obtain

$$h'(0)/(h'(1))^2 = \frac{(1 - |z|^2)^3}{(1 - \bar{z})^2|1 - z|^2},$$

so that

$$K_{vf}(z) = \iota \cdot \frac{(1 - |z|^2)^3}{(1 - \bar{z})^2|1 - z|^2}. \tag{3.2.4}$$

Lemma 3.2.4. *Suppose that* $h \in \mathrm{isom}_+(\mathbb{D})$ *satisfies* $h(1) = 1$. *Then*

$$h^* K_{vf} = c \cdot K_{vf},$$

where $c = (h'(1))^{-2}$.

Proof. Loosely and provisionally we defined ξ as $F(\delta_1 \cdot \omega)$. Therefore $h^*\xi = h^* F(\delta_1 \cdot \omega) = F(h^*(\delta_1 \cdot \omega))$ by the equivariance property of F. Here we can see that $h^*(\delta_1 \cdot \omega) = c \cdot \delta_1 \cdot \omega$ for some constant c, which turns out to be $(h'(1))^{-2}$. A more orderly proof can be given using the definition (3.2.3) and the chain rule. Then we have to allow two $h_1, h_2 \in \mathrm{isom}_+(\mathbb{D})$ such that $h_1(1) = h_2(1) = 1$. We obtain

$$\begin{aligned}
(h_2^*\xi)(h_1(0)) &= \big(Dh_2(h_1(0))\big)^{-1} F\big(h_2(h_1(0))\big) \\
&= \big(Dh_2(h_1(0))\big)^{-1} \iota(h_2h_1)'(0)/((h_2h_1)'(1))^2 \\
&= \iota h_1'(0)/((h_2h_1)'(1))^2 \\
&= c \cdot \iota h_1'(0)/(h_1'(1))^2 \\
&= c \cdot \xi(h_1(0)),
\end{aligned}$$

where $c = (h_2'(1))^{-2}$. □

Theorem 3.2.5. *The vector field* K_{vf} *is harmonic on* $\mathbb{D}$.

Proof. The first step will be to show that $\xi := K_{vf}$ is harmonic at the origin in $\mathbb{D}$. For that we can use the second order Taylor approximation of ξ at the origin:

$$\begin{aligned}
\xi(z) &= \frac{\iota(1-|z|^2)^3}{|1-\bar{z}|^2\cdot(1-\bar{z})^2} \\
&= \iota(1-|z|^2)^3(1-\bar{z})^{-3}(1-z)^{-1} \\
&\approx \iota(1-3|z|^2)(1+\bar{z}+\bar{z}^2)^3(1+z+z^2) \\
&\approx \iota(1-3|z|^2)(1+3\bar{z}+3\bar{z}^2+3\bar{z}^2)(1+z+z^2) \\
&\approx \iota(1+3\bar{z}+6\bar{z}^2-3|z|^2)(1+z+z^2) \\
&\approx \iota(1+3\bar{z}+6\bar{z}^2-3|z|^2+z+3|z|^2+z^2) \\
&= \iota(1+z+3\bar{z}+z^2+6\bar{z}^2) \\
&= \iota\big(1+(x+\iota y)+3(x-\iota y)+x^2-y^2+2\iota xy+6(x^2-y^2)-12\iota xy\big) \\
&= \iota(1+4x-2\iota y+7x^2-7y^2-10\iota xy) \\
&= (2y+10xy,\, 1+4x+7x^2-7y^2).
\end{aligned}$$

Let g be the Poincaré metric on $\mathbb{D}$, which we regard as a function from $\mathbb{D}$ to the vector space of symmetric 2×2-matrices. It has the form $g(w)=u(w)\cdot g^E$, where $u:\mathbb{D}\to\mathbb{R}$ is a smooth function which has $u(0)=2$ and g^E is the Euclidean (Riemannian) metric, i.e., a constant, the constant value being the identity 2×2-matrix. The product rule gives us

$$\mathscr{L}_\xi(g)=\mathscr{L}_\xi(u)\cdot g^E+u\cdot\mathscr{L}_\xi(g^E).$$

The first summand is a contribution to the scalar summand. We may neglect it. As to the second summand, we are only interested in the first Taylor approximation at 0, and since the first Taylor approximation of u at 0 is a constant 2, we can replace the second summand by $2\mathscr{L}_\xi(g^E)$. Now we can use (2.1.3):

$$\mathscr{L}_\xi(g^E)=(D\xi^T+D\xi)g^E+Dg(\xi)=D\xi^T+D\xi,$$

which in terms of the above Taylor approximation turns into

$$\begin{bmatrix} 20y & 6+24x \\ 6+24x & -28y \end{bmatrix}.$$

The trace-free part is

$$\begin{bmatrix} 24y & 6+24x \\ 6+24x & -24y \end{bmatrix},$$

which, as a symmetric bilinear form, agrees with $\mathrm{Re}((24y-\iota(6+24)x)\,dz\otimes_{\mathbb{C}}dz)$. This completes the verification that $\xi=K_{vf}$ is harmonic at the origin, because the first order polynomial map $z\mapsto 24y-\iota(6+24)x$ (for $z=x+\iota y$) is holomorphic.

To finish the proof, we want to argue that "harmonic at the origin" is enough. For other $z \in \mathbb{D}$ we can find $h \in \mathrm{isom}_+(\mathbb{D})$ such that $h(0) = z$ and $h(1) = 1$. Showing that ξ is harmonic at z is equivalent to showing that $h^*\xi$ is harmonic at 0. By Lemma 3.2.4, we can write $h^*\xi = c \cdot \xi$. □

Our next goal is to show that the linear map F in (3.2.2) has another description as something very close to *convolution with* K_{vf}. Let $M_t : \mathbb{C} \to \mathbb{C}$ be the multiplication by $e^{\iota t} \in S^1$; we will also view this as an element of $\mathrm{isom}_+(\mathbb{D})$. Let ψ be a tangential vector field on S^1. Write $\psi = u \cdot \omega$, where ω is the standard tangential vector field $z \mapsto \iota z$ and $u : S^1 \to \mathbb{R}$ is a continuous function. (It is allowed to write $u := \psi/\omega$.) Write $\eta(t) := e^{\iota t}$ for $t \in \mathbb{R}$. The new formula for F that we have in mind is

$$(F(\psi))(z) \overset{?}{=} \frac{1}{2\pi} \int_0^{2\pi} (M_{-t}^* K_{vf})(z) \cdot u(e^{\iota t})\, dt. \tag{3.2.5}$$

Unraveling the right-hand side, we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} e^{\iota t} \cdot K_{vf}(e^{-\iota t} z) \cdot u(e^{\iota t})\, dt = \frac{1}{2\pi} \int_0^{2\pi} K_{vf}(e^{-\iota t} z) \cdot e^{\iota t} u(e^{\iota t})\, dt.$$

But $e^{\iota t} u(e^{\iota t})$ is the same as $-\iota \cdot \psi(e^{\iota t})$. So we may write

$$-\iota \frac{1}{2\pi} \int_0^{2\pi} K_{vf}(e^{-\iota t} z) \cdot \psi(e^{\iota t})\, dt$$

and then also $-\iota \cdot (K_{vf} * \psi)(z)$, where the star is for *convolution*.

Proposition 3.2.6. *The linear map F in* (3.2.2) *has the alternative description*

$$\psi \mapsto -\iota \cdot (K_{vf} * \psi).$$

Proof. Take $z \in \mathbb{D}$. Then

$$-\iota \cdot (K_{vf} * \psi)(z) = -\iota \frac{1}{2\pi} \int_0^{2\pi} K_{vf}(e^{-\iota t} z) \cdot \psi(e^{\iota t} z)\, dt.$$

Choose h such that $h(0) = z$ and $h(1) = 1$. If we choose $s \in \mathbb{R}$ appropriately, depending on t, then $M_{-t} h M_s(0) = M_{-t}(z) = e^{-\iota t} z$ and $M_{-t} h M_s(1) = 1$. (We can determine s later.) Therefore by (3.2.3),

$$\begin{aligned} K_{vf}(e^{-\iota t} z) &= \iota (M_{-t} h M_s)'(0) \cdot ((M_{-t} h M_s)'(1))^{-2} \\ &= \iota e^{\iota(s-t)} h'(0) \cdot (h'(e^{\iota s}))^{-2} \cdot e^{-2\iota(s-t)} \\ &= \iota e^{\iota(t-s)} h'(0) \cdot (h'(e^{\iota s}))^{-2}, \end{aligned}$$

so that the formula for $-\iota \cdot (K_{vf} * \psi)(z)$ simplifies to

$$\frac{1}{2\pi} \int_0^{2\pi} e^{\iota(t-s)} h'(0) \cdot (h'(e^{\iota s}))^{-2} \cdot \psi(e^{\iota t})\, dt.$$

Now is the moment to observe that s and t are related by $h(e^{\iota s}) = e^{\iota t}$. Therefore it is "locally" justified to write $t = -\iota \cdot \ln(h(e^{\iota s}))$ and in any case

$$\frac{dt}{ds} = -\iota \frac{h'(e^{\iota s}) \cdot e^{\iota s} \cdot \iota}{h(e^{\iota s})} = \frac{h'(e^{\iota s}) \cdot e^{\iota s}}{h(e^{\iota s})} = h'(e^{\iota s}) \cdot e^{\iota(s-t)}.$$

With the substitution of $h(e^{\iota s})$ for $e^{\iota t}$ and $h'(e^{\iota s}) \cdot e^{\iota(s-t)}\, ds$ for dt the above integral turns into

$$\frac{1}{2\pi} \int_0^{2\pi} h'(0) \cdot (h'(e^{\iota s}))^{-1} \cdot \psi(h(e^{\iota s}))\, ds.$$

This agrees with $F(\psi)(z) = (F(\psi))(h(0))$ according to (3.2.2). □

Corollary 3.2.7. *$F(\psi)$ is harmonic, for every $\psi \in C^0_v(S^1; TS^1)$.*

Proof. By Proposition 3.2.6 and the discussion preceding it, (3.2.5) is correct. By Theorem 3.2.5, the "kernel" K_{vf} is harmonic and so $M^*_{-t}K_{vf}$ is also harmonic, for arbitrary $t \in \mathbb{R}$. Hence the right-hand side of (3.2.5) is harmonic. □

Lemma 3.2.8. *Let ψ be a continuous tangential vector field on S^1. Then $F(\psi)$ and ψ together make up a continuous vector field on the closed unit disk $\mathbb{D} \cup S^1$.*

Proof. For $s \in [0, 1)$, we define a continuous (but not tangential) vector field κ_s on S^1 by $\kappa_s(z) := K_{vf}(sz)$. Then for $\varepsilon \in (0, 1]$ and $z \in S^1$ we have

$$\begin{aligned}\kappa_{1-\varepsilon}(z) &= \frac{\iota(1-(1-\varepsilon)^2)^3}{(1-(1-\varepsilon)\bar{z})^3 \cdot (1-(1-\varepsilon)z)} \\ &= \frac{\iota z^3(1-(1-\varepsilon)^2)^3}{(z-(1-\varepsilon))^3 \cdot (1-(1-\varepsilon)z)} \\ &= \frac{8\iota z^3 \varepsilon^3 V(\varepsilon)}{(1-(1-\varepsilon)z) \cdot (z-(1-\varepsilon))^3},\end{aligned}$$

where V is a real polynomial of degree 3 with constant coefficient 1.

Let $\lambda_z = |z-(1-\varepsilon)|$. Then $\lambda_z \geq \varepsilon$ and

$$|\kappa_{1-\varepsilon}(z)| = \frac{8\varepsilon^3 V(\varepsilon)}{\lambda_z^4} \leq \frac{8V(\varepsilon)}{\varepsilon}. \tag{3.2.6}$$

This gives us an upper bound for $|\kappa_{1-\varepsilon}(z)|$ which is independent of z, but more importantly it tells us that $\kappa_{1-\varepsilon}$ is very small outside the arc of length $(2\varepsilon)^{1/2}$ centered at 1. Therefore it is enough to show that

$$\lim_{\varepsilon \to 0} \left(\frac{1}{2\pi} \int_0^{2\pi} \kappa_{1-\varepsilon}(e^{\iota t})\, dt \right) = \iota.$$

Since $\kappa_{1-\varepsilon}$ is so very small outside the arc of length $\sqrt{2\varepsilon}$ centered at 1, we may replace the ordinary integral by the complex path integral

$$\oint_\gamma \kappa_{1-\varepsilon}(z)\,dz$$

(where γ is a smooth curve describing the unit circle) at the price of dividing by ι. We may also write s for $1-\varepsilon$. Then

$$\begin{aligned}\lim_{\varepsilon\to 0}\left(\frac{1}{2\pi}\int_0^{2\pi}\kappa_{1-\varepsilon}(e^{\iota t})\,dt\right) &= \lim_{\varepsilon\to 0}\left(\frac{1}{2\pi\iota}\oint_\gamma \kappa_{1-\varepsilon}(z)\,dz\right)\\ &= \lim_{\varepsilon\to 0}\left(\frac{1}{2\pi\iota}\oint_\gamma \frac{8\iota z^3\varepsilon^3 V(\varepsilon)}{(1-sz)\cdot(z-s)^3}\,dz\right)\\ &= \lim_{\varepsilon\to 0}\left(\frac{8\iota\varepsilon^3 V(\varepsilon)}{2\pi\iota}\big(2\pi\iota\cdot \operatorname{Res}(f,s)\big)\right)\\ &= \lim_{\varepsilon\to 0}\big(8\iota\varepsilon^3(\operatorname{Res}(f,s))\big),\end{aligned}$$

where $f(z)=(z^3)(1-sz)^{-1}(z-s)^{-3}$. Now

$$\operatorname{Res}(f,s)=\frac{6s-12s^3+8s^5-2s^7}{2(1-s^2)^4}=\frac{4\varepsilon+2\varepsilon^2+2\varepsilon^3-30\varepsilon^4+34\varepsilon^5-14\varepsilon^6+2\varepsilon^7}{2(16\varepsilon^4-32\varepsilon^5+20\varepsilon^6-8\varepsilon^7+\varepsilon^8)}.$$

Therefore

$$\begin{aligned}&\lim_{\varepsilon\to 0}\big(\iota 8\varepsilon^3(\operatorname{Res}(f,s))\big)\\ &\quad=\lim_{\varepsilon\to 0}\left(8\iota\cdot\frac{4\varepsilon+2\varepsilon^2+2\varepsilon^3-30\varepsilon^4+34\varepsilon^5-14\varepsilon^6+2\varepsilon^7}{2(16\varepsilon-32\varepsilon^2+20\varepsilon^3-8\varepsilon^4+\varepsilon^5)}\right)=\iota. \qquad\square\end{aligned}$$

Lemma 3.2.9. *The map F in* (3.2.2) *has a Lipschitz property. More precisely, there is $c>0$ such that the following holds: if $\psi\in C^0(S^1;TS^1)$ satisfies $\|\psi(z)\|\le 1$ for all $z\in S^1$, in the euclidean norm, then $\|F(\psi)(z)\|\le c$ for all $z\in\mathbb{D}$, again in the euclidean norm.*

Proof. In the proof of Lemma 3.2.8 we learned

$$\lim_{s\to 1}\frac{1}{2\pi}\int_0^{2\pi}|\kappa_s(e^{\iota t})|\,dt=1.$$

This implies that the function

$$s\mapsto \frac{1}{2\pi}\int_0^{2\pi}|\kappa_s(e^{\iota t})|\,dt$$

is defined and continuous for $s\in[0,1]$. Then it has a maximum $c>0$ on the interval. Now let ψ be a continuous tangential vector field on S^1 and suppose that

$\|\psi(z)\| \le 1$ for all $z \in S^1$. By Proposition 3.2.6 we have, for $z \in S^1$ and $s \in [0,1)$:

$$(F(\psi))(sz) = -\iota \frac{1}{2\pi} \int_0^{2\pi} \kappa_s(e^{-\iota t} z) \cdot \psi(e^{\iota t})\, dt.$$

Therefore

$$\|(F(\psi))(sz)\| \le \frac{1}{2\pi} \int_0^{2\pi} |\kappa_s(e^{-\iota t} z)|\, dt \le c. \qquad \square$$

We come to the last part of Theorem II (in the introduction). This is about extending F in (3.2.2) and Proposition 3.2.6 to a continuous map defined on the Hilbert space $\mathcal{X}$ of tangential L^2-vector fields on S^1. The extension formula as such is obvious: both (3.2.2) and the formula in Proposition 3.2.6 tolerate a tangential L^2-vector field ψ instead of a continuous one. Both of these extensions are clearly continuous, and because they agree on a dense subspace of $\mathcal{X}$ they agree on all of $\mathcal{X}$.

Lemma 3.2.10. *For $\zeta \in \mathcal{X}$, let $\xi := F(\zeta) \in C^0_v(\mathbb{D}; T\mathbb{D})$. Then ξ is boundary controlled and ζ is the distributional boundary of ξ; see Definition 1.3.3.*

Proof. We use a method from [Shubin 2020, Section 5.3]. Fix α, continuous vector field on $\mathbb{D} \cup S^1$. Write

$$w(s) := \frac{1}{2\pi} \int_0^{2\pi} s \cdot \xi(se^{\iota t}) \cdot \alpha(se^{\iota t})\, dt$$

for $s \in [0,1)$. It is more than enough to show that

$$\lim_{s \to 1^-} w(s) = \frac{1}{2\pi} \int_0^{2\pi} \zeta(e^{\iota t}) \cdot \alpha(e^{\iota t})\, dt,$$

because w is, except for the constant factor $(2\pi)^{-1}$, the derivative of the function in (1.3.2). Write $\alpha_s : S^1 \to \mathbb{R}^2$ for the map $z \mapsto \alpha(sz)$, where $s \in [0,1]$, and write $\kappa_s : S^1 \to \mathbb{R}^2$ for the map $z \mapsto K_{vf}(sz)$ as in the proof of Lemma 3.2.8, assuming $s \in [0,1)$. By (3.2.2) in the general form which allows a tangential L^2-vector field as input for F, we can write

$$w(s) = \langle -\iota s(\kappa_s * \zeta), \alpha_s \rangle.$$

This notation uses the standard real L^2 inner product (based on the standard inner product in $\mathbb{R}^2$, a.k.a. real part of the hermitian inner product in $\mathbb{C}$). Therefore

$$w(s) = \langle \zeta, -\iota s(\kappa_s * \alpha_s) \rangle.$$

Here we can say that

$$\lim_{s \to 1^-} -\iota s(\kappa_s * \alpha_s) = \lim_{s \to 1^-} -\iota(\kappa_s * \alpha_1) = \alpha_1$$

by Lemmas 3.2.9 and 3.2.8. (These limits take place in $C^0_v(\mathbb{D}, T\mathbb{D})$ or in $C^0(\mathbb{D}, \mathbb{R}^2)$ depending on point of view.) It follows that

$$\lim_{s\to 1^-} w(s) = \langle \zeta, \alpha_1 \rangle. \qquad \square$$

Appendix: Odds and ends

A.1. ***Connections on the Teichmüller bundle.*** A *connection* on a smooth fiber bundle $f : E \to M$ with vertical tangent bundle T^vE is a smooth vector subbundle T^hE (the *horizontal* tangent bundle) of the tangent bundle TE such that $T^hE \oplus T^vE = TE$. Equivalently, a connection on $f : E \to M$ is a smooth vector bundle homomorphism $f^*TM \to TE$ such that the composition

$$f^*TM \to TE \to TE/T^vE$$

is the identity. We are interested in connections on the Teichmüller surface bundle, a.k.a. *universal Teichmüller curve* (the fibers can either be viewed as real surfaces or as complex curves). To describe the bundle we fix $\Gamma = \Gamma_g$ (fundamental group of a surface Σ of genus $g \geq 2$) and $G = \mathrm{isom}_+(\mathbb{H})$ as usual. Let $\hom_0(\Gamma, G)$ be the space of injective homomorphisms with discrete image and compact quotient space $G/\rho(\Gamma)$. Let $\mathrm{rep}_0(\Gamma, G)$ be the quotient of $\hom_0(\Gamma, G)$ obtained by passing to orbits for the conjugation action of G. (This was called $\mathscr{T}(\Sigma)$ in Definition 1.1.2.) There are two commuting left actions of Γ and G, respectively, on the product $\hom_0(\Gamma, G) \times \mathbb{H}$.

The action of Γ is given by $\gamma \cdot (\rho, z)) := (\rho, \rho(\gamma)(z))$ for $\gamma \in \Gamma$. The action of G is given by $A \cdot (\rho, z) := (A\rho A^{-1}, A(z))$ for $A \in G$. Therefore we obtain a commutative diagram

$$\text{(A.1.1)} \qquad \begin{array}{ccc} E := \dfrac{\hom_0(\Gamma, G) \times \mathbb{H}}{\Gamma} & \xrightarrow{\text{proj.}} & \hom_0(\Gamma, G) \\ \downarrow & & \downarrow \\ E_G := \dfrac{\hom_0(\Gamma, G) \times \mathbb{H}}{G \times \Gamma} & \xrightarrow{\text{proj.}} & \mathrm{rep}_0(\Gamma, G) \end{array}$$

where the vertical arrows are quotient maps and the horizontal ones are appropriate projections. The two horizontal arrows are surface bundle projections (or complex curve bundle projections). The vertical arrows are principal G-bundle projections. The diagram is a pullback diagram. In the top row, the fiber over $\rho \in \hom_0(\Gamma, G)$ is the surface $\mathbb{H}/\rho(\Gamma)$.

The lower horizontal arrow $E_G \to \mathrm{rep}_0(\Gamma, G)$ in (A.1.1) is the Teichmüller bundle. We take the view that we can construct connections on it by constructing connections on the bundle defined by the upper horizontal arrow $E \to \hom_0(\Gamma, G)$

and imposing conditions which ensure that these connections descend to connections on the bundle defined by the lower horizontal arrow.

By the general remarks on connections in smooth fiber bundles, choosing a connection on the trivial bundle

$$\mathrm{hom}_0(\Gamma, G) \times \mathbb{H} \to \mathrm{hom}_0(\Gamma, G) \tag{A.1.2}$$

amounts to choosing a smooth vector field $\xi(\rho, c, -)$ on $\mathbb{H}$ for every $\rho \in \mathrm{hom}_0(\Gamma, G)$ and 1-cocycle $c \in T_\rho(\mathrm{hom}_0(\Gamma, G)) \cong Z^1(\Gamma; \mathfrak{g}_\rho)$. (This should ideally depend smoothly on ρ and c.) But we want to choose a connection that respects the Γ-action on each fiber of the bundle in (A.1.2), since we want a connection for $E \to \mathrm{hom}_0(\Gamma, G)$ in diagram (A.1.1). This translates into the following condition on $\xi(\rho, c, -)$:

(i) $\delta\xi(\rho, c, -) = c$, where δ is the coboundary operator associated with the $\mathbb{R}\Gamma$-module of smooth vector fields on $\mathbb{H}$. (The module structure depends on ρ.)

Proposition A.1.1. *A connection* $\xi = (\xi(\rho, c, -))$ *for* $E \to \mathrm{hom}_0(\Gamma, G)$ *in* (A.1.1) *descends to a connection for* $E_G \to \mathrm{rep}_0(\Gamma, G)$ *if and only if it satisfies the following additional conditions*:

(ii) *it is invariant under the left action of* G;

(iii) $\xi(\rho, \delta\kappa, -) = \kappa$ *for every* $\kappa \in \mathfrak{g}_\rho$.

Here $\kappa \in \mathfrak{g}_\rho$ should be viewed as a Killing vector field on $\mathbb{H}$. Condition (iii) does not follow from (ii) and (i). It is easy to produce counterexamples.

Proof. Suppose the connection ξ for $E \to \mathrm{hom}_0(\Gamma, G)$ descends. Then condition (ii) is satisfied. Let K be any G-orbit in $\mathrm{hom}_0(\Gamma, G)$. Then the connection ξ restricted to $E|_K \to K$ is the connection determined by the trivialization of $E|_K \to K$ produced by the action of G on $E|_K$ (which is free). This translates into condition (iii). Conversely, suppose that (ii) and (iii) are satisfied by a connection $\xi = (\xi(\rho, c, -))$ on $E \to \mathrm{hom}_0(\Gamma, G)$. Then, by (iii), the restricted connection on $E|_K \to K$ reflects the trivialization of $E|_K \to K$ produced by the action of G on E_K, or equivalently, by the composition $E|_K \hookrightarrow E \to E_G$. Choose a smooth section

$$s : \mathrm{rep}_0(\Gamma, G) \to \mathrm{hom}_0(\Gamma, G)$$

of the projection $\mathrm{hom}_0(\Gamma, G) \to \mathrm{rep}_0(\Gamma, G)$. The section s is covered uniquely by a smooth map $\bar{s} : E_G \to E$ which is a section of the projection $E \to E_G$. The pullback along s and $\bar{s}$ of the connection ξ on $E \to \mathrm{hom}_0(\Gamma, G)$ is a connection θ on $E_G \to \mathrm{rep}_0(\Gamma, G)$. Conditions (ii) and (iii) ensure that ξ is also the pullback of θ along the projections $E \to E_G$ and $\mathrm{hom}_0(\Gamma, G) \to \mathrm{rep}_0(\Gamma, G)$. □

We can meet all of these conditions as follows. Given ρ and c, find a smooth vector field ψ as promised in Lemma 1.4.2 such that $\delta\psi = c$. Let ζ^c be the distributional boundary of ψ as promised in Proposition 1.4.3 and let $\xi(\rho, c, -) := F(\zeta^c)$, with F as in Theorem II. This is well defined and the conditions are easily verified. We authors believe that $\xi(\rho, c, -)$ depends continuously, indeed smoothly, on ρ and c, but the proof could be laborious and perhaps it deserves a separate treatment. (We also believe that this candidate for a connection ξ is identical with the connection which is standard in Teichmüller theory. This can be seen, for example, in [Wolpert 1986, §5]. Evidence for the suspected agreement was given in Section 2.2.)

A.2. *Some postponed proofs.*

Lemma A.2.1. *The conformal vector fields on $\mathbb{D}$ or open subsets of $\mathbb{D}$ are precisely the holomorphic vector fields.*

Proof. By definition, a vector field ξ on $\mathbb{D}$ (or on an open subset of $\mathbb{D}$) is conformal if and only if the trace-free component of $\mathscr{L}_\xi(g)$ is zero. By (2.1.3) and the calculations immediately following it, this happens if and only if $\xi^1_x = \xi^2_y$ and $\xi^1_y = -\xi^2_x$ (where the subscripts indicate partial derivatives). These are exactly the Cauchy–Riemann equations for ξ. □

Proof of Lemma 1.4.1. Let F_1 be the restriction of F in (3.2.2) to the 3-dimensional real vector space of Killing vector fields on S^1. Let F_2 be the linear map which is defined on the same vector space and which takes a Killing vector field on S^1 to the unique matching Killing vector field on $\mathbb{D}$. We need to show $F_1 = F_2$. Since both F_1 and F_2 are equivariant for the right actions of $\mathrm{isom}_+(\mathbb{D})$, it is enough to show that $(F_1(\psi))(0) = (F_2(\psi))(0)$ for all Killing vector fields on S^1. (Follow the reasoning in the proof of Corollary 3.2.3.) By definition, $(F_1(\psi))(0)$ is

$$\frac{1}{2\pi}\int_0^{2\pi} \psi(e^{it})\,dt.$$

In other words it is the mean value of ψ on the circle. But $(F_2(\psi))(0)$ is also the mean value of ψ on the circle. Indeed, ψ and $F_2(\psi)$ are the restrictions to S^1 and $\mathbb{D}$, respectively, of one and the same holomorphic function (alias holomorphic vector field) on $\mathbb{C}$, and this must satisfy the Cauchy integral formula. □

Proposition A.2.2. *The matching condition relating the vector fields ξ and ζ in Definition 1.3.3 is invariant under the canonical right action(s) of the diffeomorphism group of $\mathbb{D} \cup S^1$.*

Proof. We will write $S(\xi, \zeta)$ for the statement *the matching condition holds for ξ and ζ*. Therefore we assume $S(\xi, \zeta)$, and we have to show $S(h^*\xi, h^*\zeta)$, where h is an arbitrary diffeomorphism from $\mathbb{D} \cup S^1$ to itself.

The diffeomorphism h can be written as a composition $h_a \circ h_n$, where h_n is norm-preserving in a neighborhood of the boundary S^1, and h_a is argument-preserving. (By *argument-preserving*, we mean that there is a smooth function

$N : S^1 \times [0, 1] \to [0, 1]$ such that $h_a(sz) = N(z, s) \cdot z$ for all $(z, s) \in S^1 \times [0, 1]$. For convenience we also require $N(z, s) = s$ for s close to 0.) Using integration by substitution, it is easy to show that $S(h^*\xi, h^*\zeta)$ is equivalent to $S(h_a^*\xi, h_a^*\zeta)$. It is also obvious that $h_a^*\zeta = \zeta$. Therefore we may assume from now on that h is argument-preserving, and we have to show $S(h^*\xi, \zeta)$, knowing that $S(\xi, \zeta)$ holds.

Let $G_\xi : S^1 \times [0, 1) \to \mathbb{R}^2$ be defined by $(z, s) \mapsto \int_0^s t \cdot \xi(tz)\, dt$. Then we have

$$\int_{|z|=s} G_\xi \, d\lambda_0 = \int_{z \in \mathbb{D}_s} \xi(z)\, d\lambda$$

for $s < 1$. We use this to reformulate $S(\xi, \zeta)$. Namely, it is equivalent to the following, which we denote by $T(\xi, \zeta)$.

> *The map G_ξ has an extension to $S^1 \times [0, 1]$ which admits a distributional partial derivative $\partial/\partial s$ along $\{(z, s) \mid s = 1\}$, and the latter is equal to ζ.*

This is rather concise. To clarify, the map $z \mapsto G_\xi(z, 1)$ from S^1 to $\mathbb{R}^2$ has two coordinate functions. They are meant to be Lebesgue measurable and integrable functions, and a such defined for almost all $z \in S^1$.

To show that $S(\xi, \zeta)$ implies $T(\xi, \zeta)$, we may use the coordinate functions ξ^1 and ξ^2. We can write $\xi^1 = \xi^1_+ - \xi^1_-$, where ξ^1_+ and ξ^1_- are nonnegative everywhere, and similarly $\xi^2 = \xi^2_+ - \xi^2_-$. For fixed $s \in [0, 1)$, define a function $g^1_{s,+}$ on S^1 by

$$z \mapsto \int_0^s t \cdot \xi^1_+(tz)\, dt.$$

Its integral over S^1 is equal to $\int_{z \in \mathbb{D}_s} \xi^1_+(z)\, d\lambda$. The limit of this for $s \to 1$ exists and is finite. In fact by $S(\xi, \zeta)$ it is equal to

$$\int_{z \in S^1} \zeta^1_+(z)\, d\lambda_0.$$

Therefore we can apply the monotone convergence theorem (Beppo Levi) and conclude that $g^1_{s,+}$ is also defined for $s = 1$, as a measurable and Lebesgue integrable nonnegative function on S^1. We can proceed similarly for ξ^1_-, ξ^2_+ and ξ^2_-. Then we define

$$G_\xi(z, 1) := \begin{bmatrix} g^1_{s,+}(z, 1) - g^1_{s,-}(z, 1) \\ g^2_{s,+}(z, 1) - g^2_{s,-}(z, 1) \end{bmatrix}$$

for $z \in S^1$. The statement concerning the distributional partial derivative $\partial/\partial s$ along $\{(z, s) \mid s = 1\}$ is then clear. The implication $T(\xi, \zeta) \Rightarrow S(\xi, \zeta)$ is also clear.

If h is an argument-preserving diffeomorphism $\mathbb{D} \cup S^1 \to \mathbb{D} \cup S^1$, we can write $h(sz) = N(z, s) \cdot z$ as in the definition of *argument-preserving*. Let H be the diffeomorphism from $S^1 \times [0, 1]$ to $S^1 \times [0, 1]$ defined by $H(z, s) = (z, N(z, s))$. (This satisfies $q \circ H = h \circ q$, where $q : S^1 \times [0, 1] \to \mathbb{D} \cup S^1$ is defined by $(z, s) \mapsto sz$.)

Integration by substitution implies that

$$G_{h^*\xi} = G_{f\cdot\xi} \circ H$$

for a "suitable" smooth map $f : \mathbb{D} \cup S^1 \to \mathrm{End}_{\mathbb{R}}(\mathbb{R}^2)$. The formula for f is complicated. It is easier to describe $f \circ h \circ q$: namely,

$$f\big(h(q(z,s))\big) = \frac{s}{N(z,s)} \cdot \left(\frac{dN}{ds}(z,s)\right)^{-1} \cdot (dh(sz))^{-1}$$

for $z \in S^1$ and $s \in [0,1]$. (The first two factors in this product of three are real numbers, but the third one is a linear map.) The important features are: $f(z)$ is the unit of $\mathrm{End}_{\mathbb{R}}(\mathbb{R}^2)$ for z close to the origin, $f(z)$ always respects the linear subspace of $\mathbb{R}^2$ spanned by z, and if $|z| = 1$ it also respects the linear subspace perpendicular to z and restricts to the identity there.

This allows us to argue as follows: $T(\xi, \zeta)$ implies $T(f\cdot\xi, f\cdot\zeta)$ easily, and from there we can deduce $T(h^*\xi, \zeta)$ by an application of the chain rule. □

A.3. *Questions and suggestions.*

A.3.1. Let ξ be a harmonic vector field on $\mathbb{D}$ (with the Poincaré metric) which is boundary controlled (notation as in Definition 1.3.3). If the distributional boundary is identically zero, does it follow that ξ is identically zero?

A.3.2. Find a more direct proof of Proposition 1.4.3, i.e., one which does not rely on Theorem I. (Do use Lemma 1.4.2 and look up a proof of this.)

A.3.3. Find a practical characterization of the tangential L^2-vector fields ζ on S^1 such that the vector field $F(\zeta)$ on $\mathbb{D}$ (as in Theorem II) is quasiconformal (Definition 1.3.4).

Authorship

This work is in all essentials the PhD thesis of Divya Sharma (PhD 2021, University of Münster), lightly revised by Weiss, who was the research supervisor at the time. Since Sharma is no longer engaged in mathematics research, it fell to Weiss to make arrangements for getting the work published.

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