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GROMOV–WITTEN THEORY OF HILBERT SCHEMES OF POINTS ON ELLIPTIC SURFACES WITH MULTIPLE FIBERS

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We study the Gromov–Witten theory of Hilbert schemes of points on elliptic surfaces with multiple fibers. We prove a vanishing theorem for the Gromov–Witten invariants of these Hilbert schemes, and compute the exceptional genus-0 case for the Hilbert schemes of two points on elliptic surfaces with exactly one multiple fiber. The strategy is to use the theory of cosection localization and compute a certain obstruction sheaf.

1. Introduction

Hilbert schemes are classical objects in algebraic geometry [11]. It is well-known [7; 14] that the Hilbert schemes of points on a smooth projective surface are smooth and irreducible. The investigation of the Gromov–Witten theory of these Hilbert schemes is important and has been extremely active. It began with the computation of 1-point genus-0 extremal Gromov–Witten invariants in [19]. Motivated by the Gromov–Witten and Donaldson–Thomas correspondence [24; 25], Okounkov and Pandharipande [29] studied the equivariant Gromov–Witten theory of the Hilbert schemes of points in the affine plane. More generally, Maulik and Oblomkov [23] determined the equivariant quantum cohomology of the Hilbert scheme of points on surface resolutions associated to type A_n singularities. For the Hilbert schemes of points on K3 surfaces, Oberdieck [27] considered the reduced Gromov–Witten theory. Via cosection localization [15; 16; 17], the quantum boundary operator and the 2-point genus-0 extremal Gromov–Witten invariants of the Hilbert schemes of points on an arbitrary smooth projective surface are obtained in [18], and the structure of the 3-point genus-0 extremal Gromov–Witten invariants are analyzed in [20; 12]. Moreover, when the surface admits a nontrivial holomorphic differential two-form, a vanishing theory for the Gromov–Witten invariants is proved in [13]. For the Hilbert schemes of points on elliptic surfaces without multiple fibers, the Gromov–Witten invariants are calculated in [1; 28].

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In this paper, we continue our previous work [1], and study the Gromov–Witten theory of the Hilbert schemes of points on an elliptic surface X with multiple fibers. Let $X^{[n]}$ denote the Hilbert schemes of n points on X. For $\beta \in H_2(X^{[n]}, \mathbb{Z})$, let $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ be the moduli space parametrizing the degree-β r-pointed genus-g stable maps to $X^{[n]}$. For a smooth curve C in X and for fixed distinct points $x_1, \ldots, x_{n-1} \in X$, define the following two curves in $X^{[n]}$:

$$\beta_n = \left\{\xi + x_2 + \cdots + x_{n-1} \in X^{[n]} \mid \mathrm{Supp}(\xi) = \{x_1\}\right\},$$
$$\beta_C = \left\{x + x_1 + \cdots + x_{n-1} \in X^{[n]} \mid x \in C\right\}$$

which, by abuse of notation, also denote their corresponding homology classes.

Our first result generalizes [13, Corollary 3.5] from the case of elliptic surfaces without multiple fibers to the case of elliptic surfaces with multiple fibers.

Theorem 3.3. *Let X be a regular elliptic surface with $p_g \geq 1$ such that every singular fiber is either irreducible reduced or a multiple fiber with smooth reduction. Let f be a smooth fiber in X and $\beta \in H_2(X^{[n]}, \mathbb{Z})$. Then all the Gromov–Witten invariants of $X^{[n]}$ defined via the moduli space $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ vanish except possibly when $g \leq 1$ and $\beta = d_0\beta_f + d\beta_n$ for some integer d and rational number $d_0 \geq 0$.*

When $n = 2$ and the elliptic surface X contains exactly one multiple fiber with smooth reduction, the theorem can be strengthened.

Theorem 4.5. *Let X be a regular elliptic surface with $p_g \geq 1$ and exactly one multiple fiber such that every singular fiber is either irreducible reduced or the unique multiple fiber with smooth reduction F. Let $\beta \in H_2(X^{[2]}, \mathbb{Z})$. Then all the Gromov–Witten invariants of $X^{[2]}$ defined via the moduli space $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, \beta)$ vanish except possibly when $g \leq 1$ and $\beta = d_0\beta_F + d\beta_2$ for some integers d_0 and d satisfying $d_0 \geq 0$ and $d \geq -2d_0$.*

Next, we compute the 1-point genus-0 Gromov–Witten invariants $\langle\alpha\rangle^{X^{[2]}}_{0,d(\beta_F - 2\beta_2)}$ for $d \geq 1$ and $\alpha \in H^4(X^{[2]}, \mathbb{C})$, which are among the exceptional cases in Theorem 4.5. Let

$$\left[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\right]^{\mathrm{vir}}$$

be the virtual fundamental class of the moduli space $\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)$. The computation of the invariants $\langle\alpha\rangle^{X^{[2]}}_{0,d(\beta_F - 2\beta_2)}$ is equivalent to determining the cycle

$$\mathrm{ev}_{1*}\Big(\left[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\right]^{\mathrm{vir}}\Big)$$

where ev_1 is the evaluation map from $\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)$ to $X^{[2]}$.

Theorem 4.8. *Let X be a regular elliptic surface with $p_g \geq 1$ and exactly one multiple fiber such that every singular fiber is either irreducible reduced or the* *unique multiple fiber with smooth reduction F. Let m be the multiplicity of the unique multiple fiber*, *and* $1 \le d < m$. *Then*,

$$\mathrm{ev}_{1*}\Big(\big[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\big]^{\mathrm{vir}}\Big) = \frac{1}{d^2} \cdot \big[F^{(2)}\big] \quad \in A_2(X^{[2]}). \tag{1-1}$$

The proof of Theorem 3.3 involves cosection localization and a modification of the proof of [13, Corollary 3.5]. The modification takes care of the presence of the multiple fibers in the elliptic surface X. To prove Theorem 4.5, we analyze the homology classes of curves contained in

$$M_2(F) = \{\xi \in X^{[2]} \mid \mathrm{Supp}(\xi) \text{ is a point in } F\}.$$

When $1 \le d < m$, we show that the images of the stable maps parametrized by $\overline{\mathfrak{M}}_{g,r}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)$ must be contained in the symmetric product $F^{(2)} \subset X^{[2]}$. After expressing $\big[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\big]^{\mathrm{vir}}$ in terms of the Chern class of certain tautological bundle over $\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)$, we verify Theorem 4.8.

To put Theorems 4.5 and 4.8 into perspective, we propose the following regarding the 1-point genus-0 Gromov–Witten invariants and the 0-point genus-1 Gromov–Witten invariants respectively.

Conjecture 1.1. *Keep the notation from Theorem 4.8. Then* (1-1) *holds for every integer* $d \ge 1$.

Problem 1.2. *Keep the notation from Theorem 4.8. For every integer* $d \ge 1$, *compute the genus*-1 *Gromov–Witten invariant* $\langle\ \rangle^{X^{[2]}}_{1, d(\beta_F - 2\beta_2)}$.

In order to confirm Conjecture 1.1, one has to understand the homology classes of curves in $(f_s)^{(2)}$ and $M_2(f_s)$ for a singular non-multiple fiber f_s in the elliptic surface X. As for Problem 1.2, the genus-1 invariants $\langle\rangle^{X^{[2]}}_{1, d(\beta_F - 2\beta_2)}$ are among the exceptional cases in Theorem 4.5. Partial progress has been made in Remark 4.9. We will leave Conjecture 1.1 and Problem 1.2 to interested readers.

Finally, the paper is organized as follows. In Section 2, we collect basic facts about stable maps, Gromov–Witten invariants, the Hilbert schemes of points on surfaces, and the Heisenberg operators of Grojnowski and Nakajima. We prove Theorem 3.3 in Section 3, and Theorems 4.5 and 4.8 in Section 4.

Conventions. In this paper, an elliptic surface means a smooth projective complex surface which is minimal and admits an elliptic fibration over a smooth curve. For a smooth projective surface X, let K_X be the canonical divisor of X and

$$q = h^1(X, \mathcal{O}_X), \quad p_g = h^2(X, \mathcal{O}_X) = h^0\big(X, \mathcal{O}_X(K_X)\big).$$

2. Preliminaries

In this section, we will recall the standard concepts and notions regarding stable maps, Gromov–Witten invariants, the Hilbert schemes of points on surfaces, and the Heisenberg operators of Grojnowski [10] and Nakajima [26]. We will use these operators to describe the (co)homology groups of the Hilbert schemes and the homology classes of certain special curves in the Hilbert schemes.

2.1. ***Stable maps and Gromov–Witten invariants.*** Let Y be a smooth projective variety. An r-pointed stable map to Y consists of a complete nodal curve D with r distinct ordered smooth points $p_1, \dots, p_r$ and a morphism $\mu : D \to Y$ such that the data $(\mu, D, p_1, \dots, p_r)$ has only finitely many automorphisms. In this case, the stable map is denoted by

$$[\mu : (D; p_1, \dots, p_r) \to Y],$$

or simply by $[\mu : D \to Y]$. For $\beta \in H_2(Y, \mathbb{Z})$, let $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$ be the coarse moduli space parametrizing the stable maps $[\mu : (D; p_1, \dots, p_r) \to Y]$ such that $\mu_*[D] = \beta$ and the arithmetic genus of D is g. Then, we have the i-th evaluation map:

$$\mathrm{ev}_i : \overline{\mathfrak{M}}_{g,r}(Y, \beta) \to Y \tag{2-1}$$

defined by $\mathrm{ev}_i([\mu : (D; p_1, \dots, p_r) \to Y]) = \mu(p_i)$. It is known [21; 22; 5] that the coarse moduli space $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$ is projective and has a virtual fundamental class $[\overline{\mathfrak{M}}_{g,r}(Y, \beta)]^{\mathrm{vir}} \in A_{\mathfrak{d}}(\overline{\mathfrak{M}}_{g,r}(Y, \beta))$, where

$$\mathfrak{d} = -(K_Y \cdot \beta) + (\dim(Y) - 3)(1 - g) + r \tag{2-2}$$

is the expected complex dimension of $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$, and $A_{\mathfrak{d}}(\overline{\mathfrak{M}}_{g,r}(Y, \beta))$ is the Chow group of $\mathfrak{d}$-dimensional cycles in the moduli space $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$.

The Gromov–Witten invariants are defined by using the virtual fundamental class $[\overline{\mathfrak{M}}_{g,r}(Y, \beta)]^{\mathrm{vir}}$. Recall that an element

$$\alpha \in H^*(Y, \mathbb{C}) \stackrel{\text{def}}{=} \bigoplus_{j=0}^{2\dim_{\mathbb{C}}(Y)} H^j(Y, \mathbb{C})$$

is *homogeneous* if $\alpha \in H^j(Y, \mathbb{C})$ for some j; in this case, we take $|\alpha| = j$. Let $\alpha_1, \dots, \alpha_r \in H^*(Y, \mathbb{C})$ such that every α_i is homogeneous and

$$\sum_{i=1}^{r} |\alpha_i| = 2\mathfrak{d}. \tag{2-3}$$

Then, we have the r-point genus-g Gromov–Witten invariant defined by

$$\langle \alpha_1, \dots, \alpha_r \rangle_{g,\beta}^Y = \int_{[\overline{\mathfrak{M}}_{g,r}(Y,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^*(\alpha_1) \otimes \cdots \otimes \mathrm{ev}_r^*(\alpha_r). \tag{2-4}$$

In particular, when $r = 1$, we see from the projection formula that

$$\langle \alpha \rangle_{g,\beta}^Y = \int_{\mathrm{ev}_{1*}([\overline{\mathfrak{M}}_{g,1}(Y,\beta)]^{\mathrm{vir}})} \alpha. \tag{2-5}$$

Next, we recall that *the excess dimension* is the difference between the dimension of $\overline{\mathfrak{M}}_{g,r}(Y,\beta)$ and the expected dimension $\mathfrak{d}$ in (2-2). Let T_Y stand for the tangent sheaf of Y. For $0 \le i < r$, we shall use

$$f_{r,i} : \overline{\mathfrak{M}}_{g,r}(Y,\beta) \to \overline{\mathfrak{M}}_{g,i}(Y,\beta) \tag{2-6}$$

to stand for the forgetful map obtained by forgetting the last $(r - i)$ marked points and contracting all the unstable components. It is known that $f_{r,i}$ is flat when $\beta \neq 0$ and $0 \le i < r$. The following can be found in [9, Proposition 2.5].

Proposition 2.1. *Let $\beta \in H_2(Y, \mathbb{Z})$ and $\beta \neq 0$. Let e be the excess dimension of $\overline{\mathfrak{M}}_{g,r}(Y,\beta)$. If $R^1(f_{r+1,r})_*(\mathrm{ev}_{r+1})^*T_Y$ is a rank-e locally free sheaf, then*

$$[\overline{\mathfrak{M}}_{g,r}(Y,\beta)]^{\mathrm{vir}} = c_e\big(R^1(f_{r+1,r})_*(\mathrm{ev}_{r+1})^*T_Y\big).$$

Finally, the fundamental class axiom of Gromov–Witten theory states that

$$[\overline{\mathfrak{M}}_{g,r}(Y,\beta)]^{\mathrm{vir}} = (f_{r,r-1})^*[\overline{\mathfrak{M}}_{g,r-1}(Y,\beta)]^{\mathrm{vir}} \tag{2-7}$$

if either $r + 2g \ge 4$ or $\beta \neq 0$ and $r \ge 1$. The *Divisor Axiom* states that

$$\langle \alpha_1, \dots, \alpha_{r-1}, \alpha_r \rangle_{g,\beta}^Y = \int_\beta \alpha_r \cdot \langle \alpha_1, \dots, \alpha_{r-1} \rangle_{g,\beta}^Y \tag{2-8}$$

if $\alpha_r \in H^2(Y, \mathbb{C})$, and if either $r + 2g \ge 4$ or $\beta \neq 0$ and $r \ge 1$.

2.2. ***Hilbert schemes of points on surfaces.*** Let X be a smooth projective complex surface, and $X^{[n]}$ be the Hilbert scheme of n-points in X. An element in $X^{[n]}$ is represented by a length-n 0-dimensional closed subscheme ξ of X. For $\xi \in X^{[n]}$, let I_ξ and $\mathcal{O}_\xi$ be the corresponding sheaf of ideals and structure sheaf respectively. It is known from [7; 14] that $X^{[n]}$ is a smooth irreducible variety of dimension $2n$. In fact, the Hilbert-Chow morphism $X^{[n]} \to X^{(n)}$, mapping an element in $X^{[n]}$ to its support in the n-th symmetric product $X^{(n)}$, is a crepant resolution. The universal codimension-2 subscheme is

$$\mathcal{Z}_n = \{(\xi, x) \in X^{[n]} \times X \mid x \in \mathrm{Supp}(\xi)\} \quad \subset X^{[n]} \times X. \tag{2-9}$$

The boundary of $X^{[n]}$ is defined to be the subset

$$B_n = \left\{ \xi \in X^{[n]} \mid |\mathrm{Supp}(\xi)| < n \right\}.$$

Let C be a real-surface in X, and fix distinct points $x_1, \ldots, x_{n-1} \in X$ which are not contained in C. Define the subsets

$$\beta_n = \left\{ \xi + x_2 + \cdots + x_{n-1} \in X^{[n]} \;\middle|\; \mathrm{Supp}(\xi) = \{x_1\} \right\}, \tag{2-10}$$

$$\beta_C = \left\{ x + x_1 + \cdots + x_{n-1} \in X^{[n]} \;\middle|\; x \in C \right\}, \tag{2-11}$$

$$D_C = \left\{ \xi \in X^{[n]} \;\middle|\; \mathrm{Supp}(\xi) \cap C \neq \varnothing \right\}. \tag{2-12}$$

Note that β_C (respectively, D_C) is a curve (respectively, a divisor) in $X^{[n]}$ when C is a smooth algebraic curve in X. We extend the notions β_C and D_C to all the divisors C in X by linearity. For a subset $Y \subset X$, define

$$M_n(Y) = \{\xi \in X^{[n]} \mid \mathrm{Supp}(\xi) \text{ is a point in } Y\}. \tag{2-13}$$

Grojnowski [10] and Nakajima [26] geometrically constructed a Heisenberg algebra action on the cohomology of the Hilbert schemes $X^{[n]}$. Denote the Heisenberg operators by $\mathfrak{a}_m(\alpha)$ where $m \in \mathbb{Z}$ and $\alpha \in H^*(X, \mathbb{C})$. Put

$$\mathbb{H}_X = \bigoplus_{n=0}^{+\infty} H^*(X^{[n]}, \mathbb{C}).$$

The operators $\mathfrak{a}_m(\alpha) \in \mathrm{End}(\mathbb{H}_X)$ satisfy the commutation relation

$$[\mathfrak{a}_m(\alpha), \mathfrak{a}_n(\beta)] = -m \cdot \delta_{m,-n} \cdot \langle \alpha, \beta \rangle \cdot \mathrm{Id}_{\mathbb{H}_X} \tag{2-14}$$

where we have used $\delta_{m,-n}$ to denote 1 if $m = -n$ and 0 otherwise. The space $\mathbb{H}_X$ is an irreducible representation of the Heisenberg algebra generated by the operators $\mathfrak{a}_m(\alpha)$ with the highest weight vector being

$$|0\rangle = 1 \in H^*(X^{[0]}, \mathbb{C}) = \mathbb{C}.$$

Let $\{\alpha_1, \ldots, \alpha_s\}$ be a basis of

$$H^{\mathrm{even}}(X, \mathbb{C}) = H^0(X, \mathbb{C}) \oplus H^2(X, \mathbb{C}) \oplus H^4(X, \mathbb{C})$$

and $\{\alpha_{s+1}, \ldots, \alpha_{s+t}\}$ be a basis of

$$H^{\mathrm{odd}}(X, \mathbb{C}) = H^1(X, \mathbb{C}) \oplus H^3(X, \mathbb{C})$$

where every α_i is homogeneous. Then, a basis of the n-th component $H^*(X^{[n]}, \mathbb{C})$ in $\mathbb{H}_X$ consists of the *Heisenberg monomial classes*

$$\left(\prod_{i=1}^{s+t} \mathfrak{a}_{-n_{i,1}}(\alpha_i) \cdots \mathfrak{a}_{-n_{i,k_i}}(\alpha_i) \right) |0\rangle \tag{2-15}$$

where $k_i \geq 0$ for each i, every $n_{i,j}$ is a positive integer, $\sum_{i,j} n_{i,j} = n$, and the integers $n_{i,1}, \dots, n_{i,k_i}$ are mutually distinct for every $i \in \{s+1, \dots, s+t\}$. In particular, a basis of the $H^{4n-2}(X^{[n]}, \mathbb{C})$ in $\mathbb{H}_X$ consists of the classes

$$\mathfrak{a}_{-2}(x)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle,\ \mathfrak{a}_{-1}(\alpha_i)\mathfrak{a}_{-1}(x)^{n-1}|0\rangle,\ \mathfrak{a}_{-1}(\alpha_j)\mathfrak{a}_{-1}(\alpha_k)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle, \tag{2-16}$$

where $|\alpha_i| = 2$, $|\alpha_j| = |\alpha_k| = 3$ with $j < k$, and by abusing notation, we have used x to denote the cohomology class Poincaré dual to a point $x \in X$. Also,

$$\beta_n = \mathfrak{a}_{-2}(x)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle, \tag{2-17}$$

$$\beta_C = \mathfrak{a}_{-1}(C)\mathfrak{a}_{-1}(x)^{n-1}|0\rangle, \tag{2-18}$$

$$B_n = \frac{1}{(n-2)!}\mathfrak{a}_{-1}(1_X)^{n-2}\mathfrak{a}_{-2}(1_X)|0\rangle, \tag{2-19}$$

$$D_C = \frac{1}{(n-1)!}\mathfrak{a}_{-1}(1_X)^{n-1}\mathfrak{a}_{-1}(C)|0\rangle \tag{2-20}$$

where 1_X denotes the fundamental cohomology class of X, and for simplicity, we do not distinguish a homology class and its Poincaré dual.

Definition 2.2. Let $n \geq 2$. We define $\widetilde{H}_2(X^{[n]}, \mathbb{C})$ to be the linear subspace of $H_2(X^{[n]}, \mathbb{C})$ spanned by the Poincaré duals of the basis classes

$$\mathfrak{a}_{-1}(\alpha_j)\mathfrak{a}_{-1}(\alpha_k)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle$$

in (2-16), where $|\alpha_j| = |\alpha_k| = 3$ with $j < k$.

Next, we recall the homology class of an irreducible curve in the Hilbert scheme $X^{[n]}$. Let Γ be an irreducible curve in $X^{[n]}$. Define

$$\mathcal{Z}_\Gamma = \Gamma \times_{X^{[n]}} \mathcal{Z}_n. \tag{2-21}$$

Then, $\mathcal{Z}_\Gamma \subset \mathcal{Z}_n \subset X^{[n]} \times X$. By [13, Lemma 5.1] and its proof there, we have the following (see also [27, Lemma 1] and [30, Lemma 3.19]).

Lemma 2.3. *Let $n \geq 2$. Let Γ be an irreducible curve in $X^{[n]}$. Then,*

$$\Gamma \equiv \beta_{\pi_{2*}[\mathcal{Z}_\Gamma]} + d\beta_n \pmod{\widetilde{H}_2(X^{[n]}, \mathbb{C})} \tag{2-22}$$

for some integer d, where π_2 is the second projection of $X^{[n]} \times X$.

Finally, let C be a smooth irreducible curve in X with genus g_C. Let $C^{(n)}$ denote the n-th symmetric product of C. We regard $C^{(n)} \subset X^{[n]}$ whenever necessary. For a fixed point $p \in C$, let Ξ denote the divisor $p + C^{(n-1)} \subset C^{(n)}$. Let

$$\mathrm{AJ} : C^{(n)} \to \mathrm{Jac}_n(C)$$

be the Abel–Jacobi map sending $\xi \in C^{(n)}$ to the corresponding degree-n divisor class in $\mathrm{Jac}_n(C)$. For an element $\delta \in \mathrm{Jac}_n(C)$, the fiber $\mathrm{AJ}^{-1}(\delta)$ is the complete line system $|\delta|$. Let Θ be the pullback via AJ of a theta divisor on $\mathrm{Jac}_n(C)$. It is well-known that theta divisors on $\mathrm{Jac}_n(C)$ are ample.

Lemma 2.4 [30, Lemma 3.20]. *Let $n \geq 2$, and $\widetilde{H}_2(X^{[n]}, \mathbb{C})$ be from Definition 2.2. Let C be a smooth curve in X, and $\Gamma \subset C^{(n)}$ be a curve. Then,*

$$\Gamma \equiv (\Xi \cdot \Gamma)\beta_C + \big(-(n+g_C-1)(\Xi \cdot \Gamma) + (\Theta \cdot \Gamma)\big)\beta_n \pmod{\widetilde{H}_2(X^{[n]}, \mathbb{C})}. \tag{2-23}$$

In addition, for every line Γ_0 in a positive-dimensional fiber $\mathrm{AJ}^{-1}(\delta)$, *we have*

$$\Gamma_0 \equiv \beta_C - (n+g_C-1)\beta_n \pmod{\widetilde{H}_2(X^{[n]}, \mathbb{C})}. \tag{2-24}$$

3. The vanishing of certain Gromov–Witten invariants of the Hilbert schemes of points on elliptic surfaces with multiple fibers

The goal of this section is to show that certain Gromov–Witten invariants of the Hilbert schemes of points on elliptic surfaces with multiple fibers are equal to 0. Theorem 3.3 below generalizes [13, Corollary 3.5] to the case when the surface is an elliptic surface with multiple fibers. We prove our Theorem 3.3 by modifying the cosection localization method in the proof of [13, Theorem 3.3].

Let X be a (minimal) elliptic surface. By [8, Corollary 7.5 on p. 113], up to deformation, we may assume that every singular fiber of X is either an irreducible reduced rational curve with one node or a multiple fiber with smooth reduction. If $p_g = h^0\big(X, \mathcal{O}_X(K_X)\big) \geq 1$, then by [17, Proposition 6.1 and Remark 6.2], up to deformation, we may further assume that $|K_X|$ contains a member of the form

$$\sum_{i=1}^{s} f_i + \sum_{j=1}^{t} (m_j - 1) F_j \tag{3-1}$$

where $f_1, \dots, f_s$ are distinct smooth fibers, and $F_1, \dots, F_t$ are distinct smooth multiple fibers with multiplicities $m_1, \dots, m_t$ respectively. Therefore, we fix the following assumption throughout this section unless otherwise specified.

Assumption 3.1. X is an elliptic surface with $p_g \geq 1$ and with the elliptic fibration $\pi : X \to C$ over a smooth curve C such that

(i) every singular fiber of π is either irreducible reduced or a multiple fiber with smooth reduction;

(ii) $H^0(X, \Omega_X^2) = H^0(X, \mathcal{O}_X(K_X))$ contains a holomorphic differential two-form θ whose zero-set is of the form (3-1).

By the results of Beauville [3; 4], the holomorphic differential two-form θ induces a holomorphic two-form $\theta^{[n]}$ of the Hilbert scheme $X^{[n]}$ which can also be regarded as a map $\theta^{[n]} : T_{X^{[n]}} \to \Omega_{X^{[n]}}$. For simplicity, put

$$\overline{\mathfrak{M}} = \overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta).$$

Define the degeneracy locus $\overline{\mathfrak{M}}(\theta)$ to be the subset of $\overline{\mathfrak{M}}$ consisting of all the stable maps $u : \Gamma \to X^{[n]}$ such that the composite

$$u^*(\theta^{[n]}) \circ du : \quad T_{\Gamma_{\text{reg}}} \to u^* T_{X^{[n]}}|_{\Gamma_{\text{reg}}} \to u^* \Omega_{X^{[n]}}|_{\Gamma_{\text{reg}}} \tag{3-2}$$

is trivial over the regular locus Γ_{reg} of Γ. By the results of Kiem–Li [15; 16], $\theta^{[n]}$ defines a regular cosection of the obstruction sheaf of $\overline{\mathfrak{M}}$:

$$\eta : \mathcal{O}b_{\overline{\mathfrak{M}}} \longrightarrow \mathcal{O}_{\overline{\mathfrak{M}}} \tag{3-3}$$

where $\mathcal{O}b_{\overline{\mathfrak{M}}}$ is the obstruction sheaf and $\mathcal{O}_{\overline{\mathfrak{M}}}$ is the structure sheaf of $\overline{\mathfrak{M}}$. Moreover, the cosection η is surjective away from the degeneracy locus $\overline{\mathfrak{M}}(\theta)$, and there exists a localized virtual cycle $[\overline{\mathfrak{M}}]^{\text{vir}}_{\text{loc}} \in A_*(\overline{\mathfrak{M}}(\theta))$ such that

$$[\overline{\mathfrak{M}}]^{\text{vir}} = \iota_*[\overline{\mathfrak{M}}]^{\text{vir}}_{\text{loc}} \in A_*(\overline{\mathfrak{M}}) \tag{3-4}$$

where $\iota : \overline{\mathfrak{M}}(\theta) \to \overline{\mathfrak{M}}$ stands for the inclusion map.

Lemma 3.2. *Let X be an elliptic surface satisfying Assumption 3.1. Let f be a smooth fiber in X, and let $\widetilde{H}_2(X^{[n]}, \mathbb{C}) \subset H_2(X^{[n]}, \mathbb{C})$ be from Definition 2.2. If the subset $\overline{\mathfrak{M}}(\theta)$ of $\overline{\mathfrak{M}} = \overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ is nonempty, then*

$$\beta \equiv d_0\beta_f + d\beta_n \pmod{\widetilde{H}_2(X^{[n]}, \mathbb{C})} \tag{3-5}$$

for some integer d and some rational number $d_0 \geq 0$.

Proof. Let $u : \Gamma \to X^{[n]}$ be a stable map in $\overline{\mathfrak{M}}(\theta)$, and let Γ_0 be any irreducible component of Γ. By Assumption 3.1(ii), the zero-set of θ is supported on

$$\Big(\bigcup_{i=1}^{s} f_i\Big) \cup \Big(\bigcup_{j=1}^{t} F_j\Big).$$

By [13, Lemma 3.1], there exists $\xi_1 \in X^{[n_0]}$ for some n_0 such that

$$\text{Supp}(\xi_1) \cap \Big(\Big(\bigcup_{i=1}^{s} f_i\Big) \cup \Big(\bigcup_{j=1}^{t} F_j\Big)\Big) = \varnothing,$$

$$u(\Gamma_0) \subset \xi_1 + \Big\{\xi_2 \mid \text{Supp}(\xi_2) \subset \Big(\bigcup_{i=1}^{s} f_i\Big) \cup \Big(\bigcup_{j=1}^{t} F_j\Big)\Big\}. \tag{3-6}$$

By Lemma 2.3, there exist integers $a_{f_i} \geq 0$, $a_{F_j} \geq 0$ and d' such that

$$u(\Gamma_0) \equiv \sum_{i=1}^{s} a_{f_i}\beta_{f_i} + \sum_{j=1}^{t} a_{F_j}\beta_{F_j} + d'\beta_n \pmod{\widetilde{H}_2(X^{[n]}, \mathbb{C})}. \tag{3-7}$$

Since $f = f_i = m_j F_j$ as divisors, we conclude that

$$u(\Gamma_0) \equiv d_0'\beta_f + d'\beta_n \pmod{\widetilde{H}_2(X^{[n]}, \mathbb{C})} \tag{3-8}$$

for some rational number $d_0' \geq 0$. Since $\beta = u_*[\Gamma] = \sum_{\Gamma_0 \subset \Gamma} u_*[\Gamma_0]$, our lemma follows from (3-8). □

Theorem 3.3. *Let X be a regular elliptic surface with $p_g \geq 1$ such that every singular fiber is either irreducible reduced or a multiple fiber with smooth reduction. Let f be a smooth fiber in X and $\beta \in H_2(X^{[n]}, \mathbb{Z})$. Then all the Gromov–Witten invariants of $X^{[n]}$ defined via the moduli space $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ vanish except possibly when $g \leq 1$ and $\beta = d_0\beta_f + d\beta_n$ for some integer d and rational number $d_0 \geq 0$.*

Proof. Since X is regular, the elliptic fiberation is of the form $\pi : X \to \mathbb{P}^1$. So Assumption 3.1(ii) holds. Again, since X is regular, $\widetilde{H}_2(X^{[n]}, \mathbb{C}) = 0$ by Definition 2.2. Moreover, by (2-15), all odd cohomology groups of $X^{[n]}$ vanish.

If $\beta \neq d_0\beta_f + d\beta_n$ for some integer d and rational number $d_0 \geq 0$, then we see from Lemma 3.2 that $\overline{\mathfrak{M}}(\theta) = \varnothing$ and $[\overline{\mathfrak{M}}]^{\text{vir}}_{\text{loc}} = 0$. By (3-4), $[\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)]^{\text{vir}} = 0$ and all the Gromov–Witten invariants defined via $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ vanish.

Next, assume that $g > 1$ and $\beta = d_0\beta_f + d\beta_n$ for some integer d and rational number $d_0 \geq 0$. Since $K_{X^{[n]}} = D_{K_X}$, we see from (2-2) that the expected dimension of $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ is equal to $(2n-3)(1-g)+r < r$. By (2-3) and the fundamental class axiom (2-7), we conclude that all the Gromov–Witten invariants of $X^{[n]}$ defined via the moduli space $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ vanish. □

Remark 3.4. For an arbitrary smooth projective surface X, the case $g = 0$ and $\beta = d\beta_n$ is studied in [20], and the case $g = 1$ and $\beta = d\beta_n$ is discussed in [12].

If the regular elliptic surface X in Theorem 3.3 has exactly one multiple fiber, then we can prove that the rational number d_0 in Theorem 3.3 is an integer.

Corollary 3.5. *Let X be a regular elliptic surface with $p_g \geq 1$ and exactly one multiple fiber such that every singular fiber is either irreducible reduced or the unique multiple fiber with smooth reduction F. Let $\beta \in H_2(X^{[n]}, \mathbb{Z})$. Then all the Gromov–Witten invariants of $X^{[n]}$ defined via the moduli space $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ vanish except possibly when $g \leq 1$ and $\beta = d_0\beta_F + d\beta_n$ for some integers d and $d_0 \geq 0$.*

Proof. By [8, Corollary 2.3 on p.158], X must be simply connected. Let m denote the multiplicity of the unique multiple fiber F. Then $f_i = mF$ in (3-7), and (3-8) becomes $u(\Gamma_0) = d_0'\beta_F + d'\beta_n$ for some integers $d_0' \geq 0$ and d'. Accordingly, (3-5) becomes $\beta = d_0\beta_F + d\beta_n$ for some integers d and $d_0 \geq 0$. Now the same proof of Theorem 3.3 yields our corollary. □

4. The exceptional cases for $X^{[2]}$

In this section, we will investigate the exceptional cases in Corollary 3.5 when $n = 2$. First of all, we strengthen Corollary 3.5 by proving that the integers d and

$d_0 \geq 0$ in Corollary 3.5 must satisfy $d \geq -2d_0$. Then, we compute the exceptional 1-point genus-0 Gromov–Witten invariants

$$\langle \alpha \rangle^{X^{[2]}}_{0,d(\beta_F - 2\beta_2)} \tag{4-1}$$

where $\alpha \in H^4(X^{[2]}, \mathbb{C})$, F is the reduction of the unique multiple fiber in X, and $1 \leq d < m$ with m being the multiplicity of the unique multiple fiber. By (2-5), the computation of (4-1) is equivalent to determining the cycle

$$\mathrm{ev}_{1*}\big(\big[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\big]^{\mathrm{vir}}\big) \in A_2(X^{[2]}).$$

4.1. ***The homology classes of certain curves in $X^{[2]}$.*** In this subsection, X stands for an arbitrary elliptic surface. We will investigate the homology classes of certain curves in $X^{[2]}$ related to a smooth multiple fiber f in the elliptic surface X. These include curves in $M_2(f)$ (see (2-13) for the notation) and the fibers of the Abel–Jacobi map $\mathrm{AJ} : f^{(2)} \to \mathrm{Jac}_2(f) \cong f$. The results here generalize [1, Lemma 4.1 and Lemma 4.3].

Lemma 4.1. *Let X be an elliptic surface with f being a smooth fiber or the reduction of a smooth multiple fiber, and $\widetilde{H}_2(X^{[2]}, \mathbb{C}) \subset H_2(X^{[2]}, \mathbb{C})$ be from Definition 2.2. Let $\Gamma \subset M_2(f)$ be an irreducible curve in $X^{[2]}$. Then, there exist nonnegative integers d and d_0 not both zero such that*

$$\Gamma \equiv 2d\beta_f + d_0\beta_2 \ (\mathrm{mod}\ \widetilde{H}_2(X^{[2]}, \mathbb{C})). \tag{4-2}$$

Proof. When f is a smooth fiber of X, this is [1, Lemma 4.1]. In the following, assume that f is the reduction of a smooth multiple fiber with multiplicity m. We will modify the proof of [1, Lemma 4.1]. Note that

$$B_2 = M_2(X) \cong \mathbb{P}(T_X^\vee).$$

For convenience, we simply write $B_2 = M_2(X) = \mathbb{P}(T_X^\vee)$. Then, we have

$$M_2(f) = \mathbb{P}\big((T_X|_f)^\vee\big) \cong \mathbb{P}\big((T_X|_f)^\vee \otimes \mathcal{O}_f(f)\big).$$

From $0 \to \mathcal{O}_f \to T_X|_f \to \mathcal{O}_f(f) \to 0$, we obtain the exact sequence

$$0 \to \mathcal{O}_f \to (T_X|_f)^\vee \otimes \mathcal{O}_f(f) \to \mathcal{O}_f(f) \to 0,$$

where $\mathcal{O}_f(f)$ is a torsion of order m. Therefore, we conclude that

$$\Gamma = d'\tilde{\sigma} + d_0' f' \quad \in H_2(M_2(f), \mathbb{C}), \tag{4-3}$$

where f' is a fiber of the ruling $M_2(f) \to f$, $\tilde{\sigma}$ is a section to the ruling with

$$\tilde{\sigma}^2 = \deg\big((T_X|_f)^\vee \otimes \mathcal{O}_f(f)\big) = \deg \mathcal{O}_f(f) = 0, \tag{4-4}$$

and d' and d'_0 are nonnegative integers. In addition, we have

$$\mathcal{O}_{M_2(f)}(\tilde{\sigma}) = \mathcal{O}_{M_2(f)}(1) \otimes \phi^* \mathcal{O}_f(f), \tag{4-5}$$

where $\phi : M_2(f) \to f$ denotes the ruling on $M_2(f)$.

Since $f' = \beta_2 \in H_2(X^{[2]}, \mathbb{C})$, we see from (4-3) that it remains to show

$$\tilde{\sigma} \equiv 2\beta_f \pmod{\widetilde{H}_2(X^{[2]}, \mathbb{C})}. \tag{4-6}$$

Indeed, by Lemma 2.3, we have

$$\tilde{\sigma} \equiv 2\beta_f + d_2\beta_2 \pmod{\widetilde{H}_2(X^{[2]}, \mathbb{C})} \tag{4-7}$$

for some integer d_2. Since $\mathcal{O}_{B_2}(B_2) = \mathcal{O}_{B_2}(-2)$ and $\mathcal{O}_f(f)$ is a torsion,

$$B_2 \cdot \tilde{\sigma} = B_2|_{M_2(f)} \cdot \tilde{\sigma} = \mathcal{O}_{M_2(f)}(-2) \cdot \tilde{\sigma} = -2\tilde{\sigma} \cdot \tilde{\sigma} = 0$$

where we have used (4-5) and (4-4) in the last two steps. On the other hand, since

$$B_2 \cdot \beta_f = B_2 \cdot w = 0$$

for every class $w \in \widetilde{H}_2(X^{[2]}, \mathbb{C})$ and $B_2 \cdot \beta_2 = -2$, we obtain $B_2 \cdot \tilde{\sigma} = -2d_2$ from (4-7). Therefore, we have $d_2 = 0$ and (4-6) follows from (4-7). $\square$

Let f be a smooth fiber in the elliptic surface X. Since f is an elliptic curve, we see that the Abel–Jacobi map

$$\mathrm{AJ} : f^{(2)} \to \mathrm{Jac}_2(f) \cong f$$

exhibits $f^{(2)}$ as a ruled surface over f. The fiber $\mathrm{AJ}^{-1}(\delta)$ over an element $\delta \in \mathrm{Jac}_2(f)$ is the complete linear system $|\delta| \cong \mathbb{P}^1$. The following is [1, Lemma 4.3].

Lemma 4.2. *Let X be an elliptic surface with f being a smooth fiber. Let $\Gamma = \mathrm{AJ}^{-1}(\delta)$ be a fiber of the ruling $\mathrm{AJ} : f^{(2)} \to \mathrm{Jac}_2(f)$. Regard Γ as a curve in $X^{[2]}$ via $\Gamma \subset f^{(2)} \subset X^{[2]}$. Let $N_{\Gamma \subset X^{[2]}}$ be the normal bundle of Γ in $X^{[2]}$. Then,*

(i) $N_{\Gamma \subset X^{[2]}} = \mathcal{O}_\Gamma(-2) \oplus \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma$;

(ii) $T_{X^{[2]}}|_\Gamma = \mathcal{O}_\Gamma(2) \oplus \mathcal{O}_\Gamma(-2) \oplus \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma$;

(iii) $\Gamma = \beta_f - 2\beta_2 \in H_2(X^{[2]}, \mathbb{C})$.

Our next goal is to prove an analogue of Lemma 4.2 when the smooth fiber f in Lemma 4.2 is replaced by the reduction of a smooth multiple fiber of the elliptic surface X. We begin with an elementary lemma.

Lemma 4.3. *Let f be a smooth elliptic curve and $\phi : f \to \mathbb{P}^1$ be a double cover. If L is a non-trivial degree-0 invertible sheaf over f, then*

$$\phi_* L = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Proof. Note that both $\phi_*\mathcal{O}_f$ and ϕ_*L are rank-2 locally free sheaf over $\mathbb{P}^1$. By the Grothendieck–Riemann–Roch Theorem, $\deg\phi_*L = \deg\phi_*\mathcal{O}_f$. Since ϕ is branched over exactly 4 points in $\mathbb{P}^1$, we obtain

$$\phi_*\mathcal{O}_f = \mathcal{O}_{\mathbb{P}^1}(-2)\oplus\mathcal{O}_{\mathbb{P}^1}. \tag{4-8}$$

It follows that $\deg\phi_*L = \deg\phi_*\mathcal{O}_f = -2$. The rank-2 locally free sheaf ϕ_*L is the direct sum of two invertible sheaves on $\mathbb{P}^1$. Since $\deg\phi_*L = -2$ and $H^0(\mathbb{P}^1, \phi_*L)\cong H^0(f, L) = 0$, we must have $\phi_*L = \mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$. □

Lemma 4.4. *Let X be an elliptic surface and f be the reduction of a smooth multiple fiber in X. Let $\Gamma = \mathrm{AJ}^{-1}(\delta)$ be a fiber of the ruling $\mathrm{AJ} : f^{(2)} \to \mathrm{Jac}_2(f)$. Regard Γ as a curve in $X^{[2]}$ via $\Gamma\subset f^{(2)}\subset X^{[2]}$. Then,*

(i) $N_{\Gamma\subset X^{[2]}} = \mathcal{O}_\Gamma(-1)\oplus\mathcal{O}_\Gamma(-1)\oplus\mathcal{O}_\Gamma$;

(ii) $T_{X^{[2]}}|_\Gamma = \mathcal{O}_\Gamma(2)\oplus\mathcal{O}_\Gamma(-1)\oplus\mathcal{O}_\Gamma(-1)\oplus\mathcal{O}_\Gamma$;

(iii) $\Gamma = \beta_f - 2\beta_2 \in H_2(X^{[2]}, \mathbb{C})$.

Proof. (i) We prove first that $N_{f^{(2)}\subset X^{[2]}}|_\Gamma = \mathcal{O}_\Gamma(-1)\oplus\mathcal{O}_\Gamma(-1)$. Recall the universal subscheme $\mathcal{Z}_2\subset X^{[2]}\times X$ from (2-9). Let $\pi_1 : \mathcal{Z}_2\to X^{[2]}$ and $\pi_2 : \mathcal{Z}_2\to X$ be the natural projections. It is known from [2] that

$$N_{f^{(2)}\subset X^{[2]}} = \pi_{1*}\pi_2^*\mathcal{O}_X(f)|_{f^{(2)}}. \tag{4-9}$$

Let $\mathcal{Z}_\Gamma = \pi_1^{-1}(\Gamma)\subset\mathcal{Z}_2$. Note that $\pi_2\big(\mathcal{Z}_\Gamma\big) = f$. Put $\tilde{\pi}_1 = \pi_1|_{\mathcal{Z}_\Gamma} : \mathcal{Z}_\Gamma\to\Gamma$ and $\tilde{\pi}_2 = \pi_2|_{\mathcal{Z}_\Gamma} : \mathcal{Z}_\Gamma\to f$. Then, $\tilde{\pi}_2$ is an isomorphism. Up to an isomorphism, $\tilde{\pi}_1$ is the double cover $f\to\mathbb{P}^1$ corresponding to the complete linear system $|\delta|$. Since $\mathcal{O}_X(f)|_f$ is a non-trivial torsion, we see from Lemma 4.3 that

$$\begin{aligned} N_{f^{(2)}\subset X^{[2]}}|_\Gamma &= \pi_{1*}\pi_2^*\mathcal{O}_X(f)|_\Gamma \\ &= \tilde{\pi}_{1*}\tilde{\pi}_2^*\big(\mathcal{O}_X(f)|_f\big) = \mathcal{O}_\Gamma(-1)\oplus\mathcal{O}_\Gamma(-1). \end{aligned} \tag{4-10}$$

Next, the normal bundle of Γ in $f^{(2)}$ is $N_{\Gamma\subset f^{(2)}} = \mathcal{O}_\Gamma$. So the exact sequence

$$0\to T_\Gamma\to T_{f^{(2)}}|_\Gamma\to N_{\Gamma\subset f^{(2)}}\to 0$$

becomes $0\to\mathcal{O}_\Gamma(2)\to T_{f^{(2)}}|_\Gamma\to\mathcal{O}_\Gamma\to 0$ which splits. Thus,

$$T_{f^{(2)}}|_\Gamma = \mathcal{O}_\Gamma(2)\oplus\mathcal{O}_\Gamma. \tag{4-11}$$

Similarly, the exact sequence

$$0\to N_{\Gamma\subset f^{(2)}}\to N_{\Gamma\subset X^{[2]}}\to N_{f^{(2)}\subset X^{[2]}}|_\Gamma\to 0$$

becomes $0\to\mathcal{O}_\Gamma\to N_{\Gamma\subset X^{[2]}}\to\mathcal{O}_\Gamma(-1)\oplus\mathcal{O}_\Gamma(-1)\to 0$, which splits. It follows that $N_{\Gamma\subset X^{[2]}} = \mathcal{O}_\Gamma(-1)\oplus\mathcal{O}_\Gamma(-1)\oplus\mathcal{O}_\Gamma$.

(ii) From $T_\Gamma = \mathcal{O}_\Gamma(2)$, the exact sequence

$$0 \to T_\Gamma \to T_{X^{[2]}}|_\Gamma \to N_{\Gamma\subset X^{[2]}} \to 0$$

and $N_{\Gamma\subset X^{[2]}} = \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma$, we obtain

$$T_{X^{[2]}}|_\Gamma = \mathcal{O}_\Gamma(2) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma.$$

(iii) This follows from the same proof (given in [1]) of Lemma 4.2(iii). □

4.2. *Calculation of the* 1*-point Gromov–Witten invariants* $\langle\alpha\rangle_{0,d(\beta_F-2\beta_2)}^{X^{[2]}}$. In this subsection, our surface X is from Corollary 3.5, i.e., X is a regular elliptic surface with $p_g \ge 1$ and exactly one multiple fiber such that every singular fiber is either irreducible reduced or the unique multiple fiber with smooth reduction F. We will compute the Gromov–Witten invariants $\langle\alpha\rangle_{0,d(\beta_F-2\beta_2)}^{X^{[2]}}$ when $1 \le d < m$ where m is the multiplicity of the unique multiple fiber in the elliptic surface X. These invariants belong to the exceptional cases in Corollary 3.5.

We begin with a theorem which strengthens Corollary 3.5 in the case $n = 2$.

Theorem 4.5. *Let X be a regular elliptic surface with $p_g \ge 1$ and exactly one multiple fiber such that every singular fiber is either irreducible reduced or the unique multiple fiber with smooth reduction F. Let $\beta \in H_2(X^{[2]}, \mathbb{Z})$. Then all the Gromov–Witten invariants of $X^{[2]}$ defined via the moduli space $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, \beta)$ vanish except possibly when $g \le 1$ and $\beta = d_0\beta_F + d\beta_2$ for some integers d_0 and d satisfying $d_0 \ge 0$ and $d \ge -2d_0$.*

Proof. Recall from the proofs of Theorem 3.3 and Corollary 3.5 that the elliptic fibration is of the form $\pi : X \to \mathbb{P}^1$ and X is simply connected. Let m be the multiplicity of the unique multiple fiber in X. Fix a holomorphic differential two-form θ in $H^0(X, \Omega_X^2) = H^0(X, \mathcal{O}_X(K_X))$ such that the zero-set of θ is equal to

$$\sum_{i=1}^{p_g-1} f_i + (m-1)F, \tag{4-12}$$

where $f_1, \dots, f_{p_g-1}$ are distinct smooth fibers of π. Then Assumption 3.1 is satisfied. In view of the proof of Corollary 3.5, it remains to prove that if the subset $\overline{\mathfrak{M}}(\theta)$ of $\overline{\mathfrak{M}} = \overline{\mathfrak{M}}_{g,r}(X^{[2]}, \beta)$ is nonempty, then

$$\beta = d_0\beta_F + d\beta_2 \tag{4-13}$$

for some integers d_0 and d satisfying $d_0 \ge 0$ and $d \ge -2d_0$.

Note that (4-13) is an improvement of Lemma 3.2. We will modify the proof of Lemma 3.2 by adopting the notation from there. Let $u : \Gamma \to X^{[2]}$ be a stable map in $\overline{\mathfrak{M}}(\theta) \subset \overline{\mathfrak{M}} = \overline{\mathfrak{M}}_{g,r}(X^{[2]}, \beta)$, and let Γ_0 denote any irreducible component of Γ. By (3-6), we have the following six cases.

Case 1: $u(\Gamma_0)$ is a point in $X^{[2]}$. In this case, we get

$$[u(\Gamma_0)] = 0. \tag{4-14}$$

Case 2: $u(\Gamma_0) = x + f_i$ for some $1 \le i \le p_g - 1$ and some fixed point $x \in X$ not lying in any f_i $(1 \le i \le p_g - 1)$ or in F. In this case, we obtain

$$[u(\Gamma_0)] = \beta_{f_i} = m\beta_F. \tag{4-15}$$

Case 3: $u(\Gamma_0) = x + F$ for some fixed point $x \in X - \big(\big(\bigcup_{i=1}^{p_g-1} f_i\big) \cup F\big)$. In this case,

$$[u(\Gamma_0)] = \beta_F. \tag{4-16}$$

Case 4: $u(\Gamma_0) \subset M_2(f_i) \cup (f_i)^{(2)}$ for some $1 \le i \le p_g - 1$. If $u(\Gamma_0) \subset M_2(f_i)$, then

$$[u(\Gamma_0)] = d_i\beta_{f_i} + \tilde{d}_i\beta_2 = d_i m\beta_F + \tilde{d}_i\beta_2 \tag{4-17}$$

by Lemma 4.1, where d_i and $\tilde{d}_i$ are nonnegative integers not both zero. If $u(\Gamma_0) \subset (f_i)^{(2)}$, then we see from (2-23) that

$$[u(\Gamma_0)] = d_i'\beta_{f_i} + (-2d_i' + \Theta \cdot u(\Gamma_0))\beta_2 = d_i' m\beta_F + (-2d_i' + \Theta \cdot u(\Gamma_0))\beta_2, \tag{4-18}$$

where Θ is the pullback of a theta divisor via $\mathrm{AJ} : (f_i)^{(2)} \to \mathrm{Jac}_2(f_i)$ and d_i' is a nonnegative integer.

Case 5: $u(\Gamma_0) \subset M_2(F) \cup F^{(2)}$. If $u(\Gamma_0) \subset M_2(F)$, then by Lemma 4.1 again,

$$[u(\Gamma_0)] = d_F\beta_F + \tilde{d}_F\beta_2 \tag{4-19}$$

where d_F and $\tilde{d}_F$ are nonnegative integers not both zero. If $u(\Gamma_0) \subset F^{(2)}$, then

$$[u(\Gamma_0)] = d_F'\beta_F + (-2d_F' + \Theta \cdot u(\Gamma_0))\beta_2 \tag{4-20}$$

where Θ is the pullback of a theta divisor via $\mathrm{AJ} : F^{(2)} \to \mathrm{Jac}_2(F)$ and d_F' is a nonnegative integer.

Case 6: $u(\Gamma_0) \subset f_i + f_j$ or $u(\Gamma_0) \subset f_i + F$ for some $1 \le i \ne j \le p_g - 1$. In this case, $u(\Gamma_0) \cdot B_2 = 0$ since $u(\Gamma_0) \cap B_2 = \varnothing$. By (2-22),

$$[u(\Gamma_0)] = \beta_{\pi_{2*}[\mathcal{Z}_\Gamma]} = \tilde{d}_i'\beta_{f_i} + \tilde{d}_F'\beta_F = (\tilde{d}_i' m + \tilde{d}_F')\beta_F \tag{4-21}$$

for some nonnegative integers $\tilde{d}_i'$ and $\tilde{d}_F'$ not both zero.

Finally, since $\beta = u_*[\Gamma] = \sum_{\Gamma_0 \subset \Gamma} u_*[\Gamma_0]$, (4-13) follows from (4-14)–(4-21). □

For $g \le 1$ and $\beta = d_0\beta_F + d\beta_2$ where the integers d_0 and d satisfy $d_0 \ge 0$ and $d \ge -2d_0$, the expected dimension of $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, \beta)$ is equal to $1 - g + r$. By the Divisor Axiom (2-8), the exceptional cases in Theorem 4.5 for $X^{[2]}$ can be reduced to the computation of the following two types of invariants: (i) $\langle \alpha \rangle_{0,\beta}^{X^{[2]}}$, with $\alpha \in H^4(X^{[2]}, \mathbb{C})$, and (ii) $\langle\ \rangle_{1,\beta}^{X^{[2]}}$.

Our next goal is to calculate the 1-point genus-0 Gromov–Witten invariants

$$\langle \alpha \rangle_{0, d(\beta_F - 2\beta_2)}^{X^{[2]}} \tag{4-22}$$

with $\alpha \in H^4(X^{[2]}, \mathbb{C})$ and $1 \le d < m$ where m is the multiplicity of the unique multiple fiber in X. By (2-5), this is equivalent to determining

$$\mathrm{ev}_{1*}\big(\big[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\big]^{\mathrm{vir}}\big).$$

The lemma below deals with the stable maps in $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$.

Lemma 4.6. *Let m be the multiplicity of the unique multiple fiber in X. Let*

$$[\mu : D \to X^{[2]}] \in \overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$$

with $1 \le d < m$. Then, $\mu(D)$ is a fiber of the ruling $\mathrm{AJ} : F^{(2)} \to \mathrm{Jac}_2(F)$, *and the degree of the morphism $\mu : D \to \mu(D)$ is equal to d.*

Proof. Let $\Gamma_1, \ldots, \Gamma_t$ be the irreducible components of $\mu(D)$. Recall that X is simply connected. By Lemma 2.3, re-ordering $\Gamma_1, \ldots, \Gamma_t$ if necessary, there exists some t_0 with $0 \le t_0 \le t$ such that for $1 \le i \le t_0$, $\Gamma_i = \beta_{\pi_{2*}[\mathcal{Z}_{\Gamma_i}]} + \tilde{d}_i \beta_2$ with $\pi_{2*}[\mathcal{Z}_{\Gamma_i}] \ne 0$, and that $\Gamma_j = \beta_2$ for $t_0 + 1 \le j \le t$. For $1 \le i \le t$, let m_i be the degree of the restriction $\mu|_{\mu^{-1}(\Gamma_i)} : \mu^{-1}(\Gamma_i) \to \Gamma_i$. Then,

$$\begin{aligned} d(\beta_F - 2\beta_2) = \mu_*[D] &= \sum_{i=1}^{t} m_i [\Gamma_i] \\ &= \sum_{i=1}^{t_0} m_i (\beta_{\pi_{2*}[\mathcal{Z}_{\Gamma_i}]} + \tilde{d}_i \beta_2) + \sum_{i=t_0+1}^{t} m_i \beta_2. \end{aligned} \tag{4-23}$$

So $dF = \sum_{i=1}^{t_0} m_i \cdot \pi_{2*}[\mathcal{Z}_{\Gamma_i}]$, $t_0 \ge 1$, and $f \cdot \pi_{2*}[\mathcal{Z}_{\Gamma_i}] = 0$ for $1 \le i \le t_0$. Since $1 \le d < m$ and every fiber in X is either irreducible reduced or the unique multiple fiber, we conclude that for $1 \le i \le t_0$, the only 1-dimensional irreducible component in $\pi_2(\mathcal{Z}_{\Gamma_i})$ is F. If $\pi_2(\mathcal{Z}_{\Gamma_i})$ contains an isolated point $x \in X - F$, then $\pi_2(\mathcal{Z}_{\Gamma_i}) = F \amalg \{x\}$, $\Gamma_i = F + x$ and

$$[\Gamma_i] = \beta_F. \tag{4-24}$$

Assume that $\pi_2(\mathcal{Z}_{\Gamma_i})$ does not contain any isolated point in $X - F$. Then, $\pi_2(\mathcal{Z}_{\Gamma_i}) = F$ and $\mathrm{Supp}(\xi) \subset F$ for every $\xi \in \Gamma_i$. So

$$\Gamma_i \subset M_2(F) \cup F^{(2)}.$$

If $\Gamma_i \subset M_2(F)$, then by Lemma 4.1, we obtain

$$[\Gamma_i] = 2d_i \beta_F + d_{i,0} \beta_2 \tag{4-25}$$

for some integers $d_i \geq 1$ and $d_{i,0} \geq 0$. If $\Gamma_i \subset F^{(2)}$, then we see from (2-23) that

$$[\Gamma_i] = d_i(\beta_F - 2\beta_2) + (\Theta \cdot \Gamma_i)\beta_2 \tag{4-26}$$

where Θ is the pullback of a theta divisor via $\mathrm{AJ} : F^{(2)} \to \mathrm{Jac}_2(F)$. Combining (4-24), (4-25), (4-26) and

$$d(\beta_F - 2\beta_2) = \sum_{i=1}^{t} m_i[\Gamma_i]$$

from (4-23), we deduce that $t_0 = t$ and that for every $1 \leq i \leq t$, $\Gamma_i \subset F^{(2)}$ and $\Theta \cdot \Gamma_i = 0$. Thus, every Γ_i is a fiber of the ruling $\mathrm{AJ} : F^{(2)} \to \mathrm{Jac}_2(F)$. Since $\Gamma_1, \ldots, \Gamma_t$ are the irreducible components of $\mu(D)$ and $\mu(D)$ is connected, we conclude that $t = 1$, $\mu(D) = \Gamma_1$ is a fiber of the ruling $\mathrm{AJ} : F^{(2)} \to \mathrm{Jac}_2(F)$, and

$$d(\beta_F - 2\beta_2) = m_1[\Gamma_1].$$

By Lemma 4.4(iii), $[\Gamma_1] = \beta_F - 2\beta_2$. It follows that $m_1 = d$, i.e., the degree of the morphism $\mu : D \to \mu(D) = \Gamma_1$ is equal to d. □

Let $1 \leq d < m$. By Lemma 4.6, we obtain the commutative diagram

$$\begin{array}{ccc} \overline{\mathfrak{M}}_{g,r+1}(X^{[2]}, d(\beta_F - 2\beta_2)) & \xrightarrow{\mathrm{ev}_{r+1}} & F^{(2)} \subset X^{[2]} \\ \downarrow f_{r+1,r} & & \downarrow \mathrm{AJ} \\ \overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2)) & \xrightarrow{\Phi_r} & \mathrm{Jac}_2(F) \end{array} \tag{4-27}$$

where $f_{r+1,r}$ is the forgetful map forgetting the last marked point, and Φ_r maps $[\mu : D \to X^{[2]}] \in \overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$ to $\mathrm{AJ}(\mu(D))$. We have

$$\Phi_r^{-1}(\delta) \cong \overline{\mathfrak{M}}_{g,r}(\mathrm{AJ}^{-1}(\delta), d) \cong \overline{\mathfrak{M}}_{g,r}(\mathbb{P}^1, d) \tag{4-28}$$

for $\delta \in \mathrm{Jac}_2(F)$. So the dimension of $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$ is equal to

$$\dim \overline{\mathfrak{M}}_{g,r}(\mathbb{P}^1, d) + 1 = 2d + 2g + r - 1. \tag{4-29}$$

Next, when $1 \leq d < m$, we determine the virtual fundamental class

$$[\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\mathrm{vir}}.$$

Lemma 4.7. *Let m be the multiplicity of the unique multiple fiber in the elliptic surface X from Theorem 4.5. Let $1 \leq d < m$ and $0 \leq g \leq 1$. Then,*

(i) $R^1(f_{r+1,r})_*(\mathrm{ev}_{r+1})^* T_{X^{[2]}}$ *is a rank-$(2d + 3g - 2)$ locally free sheaf.*

(ii) $[\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\mathrm{vir}} = c_{2d+3g-2}\big(R^1(f_{r+1,r})_*(\mathrm{ev}_{r+1})^* T_{X^{[2]}}\big).$

Proof. (i) Let $[\mu : D \to X^{[2]}] \in \overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$. By Lemma 4.6, $\mu(D)$ is a fiber of the ruling $\mathrm{AJ} : F^{(2)} \to \mathrm{Jac}_2(F)$. By Lemma 4.4(ii),

$$T_{X^{[2]}}|_{\mu(D)} = \mathcal{O}_{\mu(D)}(2) \oplus \mathcal{O}_{\mu(D)}(-1) \oplus \mathcal{O}_{\mu(D)}(-1) \oplus \mathcal{O}_{\mu(D)}.$$

Since $d \ge 1$ and $0 \le g \le 1$, we conclude that

$$h^1(D, \mu^* T_{X^{[2]}}) = 2h^1(D, \mu^* \mathcal{O}_{\mu(D)}(-1)) + h^1(D, \mathcal{O}_D) = 2d + 3g - 2.$$

Hence $R^1(f_{r+1,r})_*(\mathrm{ev}_{r+1})^* T_{X^{[2]}}$ is a rank-$(2d + 3g - 2)$ locally free sheaf.

(ii) By (2-2), the expected dimension of $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$ is equal to $1-g+r$. By (4-29), the excess dimension of $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$ is $2d+3g-2$. So our result follows immediately from (i) and Proposition 2.1. □

Finally, we determine the cycle $\mathrm{ev}_{1*}\big([\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\mathrm{vir}}\big)$.

Theorem 4.8. *Let X be a regular elliptic surface with $p_g \ge 1$ and exactly one multiple fiber such that every singular fiber is either irreducible reduced or the unique multiple fiber with smooth reduction F. Let m be the multiplicity of the unique multiple fiber, and $1 \le d < m$. Then,*

$$\mathrm{ev}_{1*}\big([\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\mathrm{vir}}\big) = \frac{1}{d^2} \cdot [F^{(2)}] \quad \in A_2(X^{[2]}).$$

Proof. By (2-2), the expected dimension of $\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))$ is equal to 2. By Lemma 4.6, $\mathrm{ev}_1\big(\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))\big) \subset F^{(2)}$. So

$$\mathrm{ev}_{1*}\big([\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\mathrm{vir}}\big) = \kappa \cdot [F^{(2)}] \tag{4-30}$$

for some number κ. To determine κ, we take a complex 2-dimensional cycle $W \subset X^{[2]}$ and pretend that W and $F^{(2)}$ intersect transversally at a unique point $\xi \in F^{(2)}$. Let $\delta = \mathrm{AJ}(\xi)$ and $\Gamma = \mathrm{AJ}^{-1}(\delta)$. Then $\xi \in \Gamma \cong \mathbb{P}^1$. Intersecting both sides of (4-30) with $[W]$, we get

$$\begin{aligned} \kappa &= \mathrm{ev}_{1*}\big([\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\mathrm{vir}}\big) \cdot [W] \\ &= [\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\mathrm{vir}} \cdot \mathrm{ev}_1^*[W] \\ &= [\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\mathrm{vir}} \cdot \mathrm{ev}_1^*[\xi] \\ &= [\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\mathrm{vir}}|_{\overline{\mathfrak{M}}_{0,1}(\Gamma, d)} \cdot (\widetilde{\mathrm{ev}}_1)^*[\xi] \end{aligned} \tag{4-31}$$

where in the last step, we have used

$$(\mathrm{ev}_1)^{-1}(\xi) \subset \overline{\mathfrak{M}}_{0,1}(\Gamma, d) \subset \overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big),$$

and $\widetilde{\mathrm{ev}}_1 : \overline{\mathfrak{M}}_{0,1}(\Gamma, d) \to \Gamma$ is the evaluation map.

By Lemma 4.7(i), $R^1(f_{1,0})_*(\mathrm{ev}_1)^*T_{X^{[2]}}$ is a rank-$(2d-2)$ locally free sheaf on $\overline{\mathfrak{M}}_{0,0}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)$ where $f_{1,0}$ and ev_1 are the forgetful map and the evaluation map on $\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)$ respectively. For simplicity, put

$$\Omega = R^1(f_{1,0})_*(\mathrm{ev}_1)^*T_{X^{[2]}}.$$

By (2-7) and Lemma 4.7(ii), we conclude that

$$\begin{aligned}\big[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\big]^{\mathrm{vir}} &= (f_{1,0})^*\big(\big[\overline{\mathfrak{M}}_{0,0}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\big]^{\mathrm{vir}}\big) \\ &= (f_{1,0})^*\big(c_{2d-2}(\Omega)\big).\end{aligned}$$

Combining with (4-31), we see that

$$\begin{aligned}\kappa &= (f_{1,0})^*\big(c_{2d-2}(\Omega)\big)|_{\overline{\mathfrak{M}}_{0,1}(\Gamma,d)} \cdot (\widetilde{\mathrm{ev}}_1)^*[\xi] \\ &= (\tilde{f}_{1,0})^*\big(c_{2d-2}(\Omega|_{\overline{\mathfrak{M}}_{0,0}(\Gamma,d)})\big) \cdot (\widetilde{\mathrm{ev}}_1)^*[\xi],\end{aligned} \tag{4-32}$$

where $\tilde{f}_{1,0} : \overline{\mathfrak{M}}_{0,1}(\Gamma, d) \to \overline{\mathfrak{M}}_{0,0}(\Gamma, d)$ is the forgetful map. Note that

$$\Omega|_{\overline{\mathfrak{M}}_{0,0}(\Gamma,d)} \cong R^1(\tilde{f}_{1,0})_*(\widetilde{\mathrm{ev}}_1)^*(T_{X^{[2]}}|_\Gamma).$$

By Lemma 4.4(ii), $T_{X^{[2]}}|_\Gamma = \mathcal{O}_\Gamma(2) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma$. Thus,

$$\Omega|_{\overline{\mathfrak{M}}_{0,0}(\Gamma,d)} \cong R^1(\tilde{f}_{1,0})_*(\widetilde{\mathrm{ev}}_1)^*(\mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1)).$$

By [6, Theorem 9.2.3],

$$c_{2d-2}(\Omega|_{\overline{\mathfrak{M}}_{0,0}(\Gamma,d)}) = \frac{1}{d^3} \cdot [\eta]$$

where $[\eta]$ denotes the class of a generic stable map $\eta \in \overline{\mathfrak{M}}_{0,0}(\Gamma, d)$. By (4-32),

$$\kappa = \frac{1}{d^3} \cdot (\tilde{f}_{1,0})^*[\eta] \cdot (\widetilde{\mathrm{ev}}_1)^*[\xi] = \frac{1}{d^2}$$

since for a generic $\eta = [\mu : D \to \Gamma] \in \overline{\mathfrak{M}}_{0,0}(\Gamma, d)$, the map μ is d to 1. □

Remark 4.9. As for the exceptional genus-1 invariant $\langle\rangle^{X^{[2]}}_{1,d(\beta_F-2\beta_2)}$, we see from Lemma 4.7(ii) that if $1 \le d < m$ where m is the multiplicity of the unique multiple fiber in the elliptic surface X from Theorem 4.5, then

$$\langle\rangle^{X^{[2]}}_{1,d(\beta_F-2\beta_2)} = c_{2d+1}\big(R^1(f_{1,0})_*(\mathrm{ev}_1)^*T_{X^{[2]}}\big).$$

where $f_{1,0}$ and ev_1 are the forgetful map and the evaluation map on the moduli space $\overline{\mathfrak{M}}_{1,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)$ respectively. However, it is unclear how to compute the right-hand side explicitly.

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