

*Pacific  
Journal of  
Mathematics*

**GROMOV–WITTEN THEORY OF HILBERT SCHEMES OF  
POINTS ON ELLIPTIC SURFACES WITH MULTIPLE FIBERS**

MAZEN M. ALHWAIMEL AND ZHENBO QIN

# GROMOV–WITTEN THEORY OF HILBERT SCHEMES OF POINTS ON ELLIPTIC SURFACES WITH MULTIPLE FIBERS

MAZEN M. ALHWAIMEL AND ZHENBO QIN

**We study the Gromov–Witten theory of Hilbert schemes of points on elliptic surfaces with multiple fibers. We prove a vanishing theorem for the Gromov–Witten invariants of these Hilbert schemes, and compute the exceptional genus-0 case for the Hilbert schemes of two points on elliptic surfaces with exactly one multiple fiber. The strategy is to use the theory of cosection localization and compute a certain obstruction sheaf.**

## 1. Introduction

Hilbert schemes are classical objects in algebraic geometry [11]. It is well-known [7; 14] that the Hilbert schemes of points on a smooth projective surface are smooth and irreducible. The investigation of the Gromov–Witten theory of these Hilbert schemes is important and has been extremely active. It began with the computation of 1-point genus-0 extremal Gromov–Witten invariants in [19]. Motivated by the Gromov–Witten and Donaldson–Thomas correspondence [24; 25], Okounkov and Pandharipande [29] studied the equivariant Gromov–Witten theory of the Hilbert schemes of points in the affine plane. More generally, Maulik and Oblomkov [23] determined the equivariant quantum cohomology of the Hilbert scheme of points on surface resolutions associated to type  $A_n$  singularities. For the Hilbert schemes of points on K3 surfaces, Oberdieck [27] considered the reduced Gromov–Witten theory. Via cosection localization [15; 16; 17], the quantum boundary operator and the 2-point genus-0 extremal Gromov–Witten invariants of the Hilbert schemes of points on an arbitrary smooth projective surface are obtained in [18], and the structure of the 3-point genus-0 extremal Gromov–Witten invariants are analyzed in [20; 12]. Moreover, when the surface admits a nontrivial holomorphic differential two-form, a vanishing theory for the Gromov–Witten invariants is proved in [13]. For the Hilbert schemes of points on elliptic surfaces without multiple fibers, the Gromov–Witten invariants are calculated in [1; 28].

---

*MSC2020:* 14C05, 14N35.

*Keywords:* Gromov–Witten invariant, Hilbert scheme, cosection localization, elliptic surface, stable map.

In this paper, we continue our previous work [1], and study the Gromov–Witten theory of the Hilbert schemes of points on an elliptic surface  $X$  with multiple fibers. Let  $X^{[n]}$  denote the Hilbert schemes of  $n$  points on  $X$ . For  $\beta \in H_2(X^{[n]}, \mathbb{Z})$ , let  $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$  be the moduli space parametrizing the degree- $\beta$   $r$ -pointed genus- $g$  stable maps to  $X^{[n]}$ . For a smooth curve  $C$  in  $X$  and for fixed distinct points  $x_1, \dots, x_{n-1} \in X$ , define the following two curves in  $X^{[n]}$ :

$$\begin{aligned}\beta_n &= \{\xi + x_2 + \dots + x_{n-1} \in X^{[n]} \mid \text{Supp}(\xi) = \{x_1\}\}, \\ \beta_C &= \{x + x_1 + \dots + x_{n-1} \in X^{[n]} \mid x \in C\}\end{aligned}$$

which, by abuse of notation, also denote their corresponding homology classes.

Our first result generalizes [13, Corollary 3.5] from the case of elliptic surfaces without multiple fibers to the case of elliptic surfaces with multiple fibers.

**Theorem 3.3.** *Let  $X$  be a regular elliptic surface with  $p_g \geq 1$  such that every singular fiber is either irreducible reduced or a multiple fiber with smooth reduction. Let  $f$  be a smooth fiber in  $X$  and  $\beta \in H_2(X^{[n]}, \mathbb{Z})$ . Then all the Gromov–Witten invariants of  $X^{[n]}$  defined via the moduli space  $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$  vanish except possibly when  $g \leq 1$  and  $\beta = d_0\beta_f + d\beta_n$  for some integer  $d$  and rational number  $d_0 \geq 0$ .*

When  $n = 2$  and the elliptic surface  $X$  contains exactly one multiple fiber with smooth reduction, the theorem can be strengthened.

**Theorem 4.5.** *Let  $X$  be a regular elliptic surface with  $p_g \geq 1$  and exactly one multiple fiber such that every singular fiber is either irreducible reduced or the unique multiple fiber with smooth reduction  $F$ . Let  $\beta \in H_2(X^{[2]}, \mathbb{Z})$ . Then all the Gromov–Witten invariants of  $X^{[2]}$  defined via the moduli space  $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, \beta)$  vanish except possibly when  $g \leq 1$  and  $\beta = d_0\beta_F + d\beta_2$  for some integers  $d_0$  and  $d$  satisfying  $d_0 \geq 0$  and  $d \geq -2d_0$ .*

Next, we compute the 1-point genus-0 Gromov–Witten invariants  $\langle \alpha \rangle_{0,d(\beta_F - 2\beta_2)}^{X^{[2]}}$  for  $d \geq 1$  and  $\alpha \in H^4(X^{[2]}, \mathbb{C})$ , which are among the exceptional cases in Theorem 4.5. Let

$$[\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\text{vir}}$$

be the virtual fundamental class of the moduli space  $\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))$ . The computation of the invariants  $\langle \alpha \rangle_{0,d(\beta_F - 2\beta_2)}^{X^{[2]}}$  is equivalent to determining the cycle

$$\text{ev}_{1*}([\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\text{vir}})$$

where  $\text{ev}_1$  is the evaluation map from  $\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))$  to  $X^{[2]}$ .

**Theorem 4.8.** *Let  $X$  be a regular elliptic surface with  $p_g \geq 1$  and exactly one multiple fiber such that every singular fiber is either irreducible reduced or the*

unique multiple fiber with smooth reduction  $F$ . Let  $m$  be the multiplicity of the unique multiple fiber, and  $1 \leq d < m$ . Then,

$$(1-1) \quad \text{ev}_{1*} \left( \left[ \overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2)) \right]^{\text{vir}} \right) = \frac{1}{d^2} \cdot [F^{(2)}] \in A_2(X^{[2]}).$$

The proof of [Theorem 3.3](#) involves cosection localization and a modification of the proof of [\[13, Corollary 3.5\]](#). The modification takes care of the presence of the multiple fibers in the elliptic surface  $X$ . To prove [Theorem 4.5](#), we analyze the homology classes of curves contained in

$$M_2(F) = \{\xi \in X^{[2]} \mid \text{Supp}(\xi) \text{ is a point in } F\}.$$

When  $1 \leq d < m$ , we show that the images of the stable maps parametrized by  $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$  must be contained in the symmetric product  $F^{(2)} \subset X^{[2]}$ . After expressing  $\left[ \overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2)) \right]^{\text{vir}}$  in terms of the Chern class of certain tautological bundle over  $\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))$ , we verify [Theorem 4.8](#).

To put [Theorems 4.5](#) and [4.8](#) into perspective, we propose the following regarding the 1-point genus-0 Gromov–Witten invariants and the 0-point genus-1 Gromov–Witten invariants respectively.

**Conjecture 1.1.** *Keep the notation from [Theorem 4.8](#). Then (1-1) holds for every integer  $d \geq 1$ .*

**Problem 1.2.** *Keep the notation from [Theorem 4.8](#). For every integer  $d \geq 1$ , compute the genus-1 Gromov–Witten invariant  $\langle \rangle_{1,d(\beta_F - 2\beta_2)}^{X^{[2]}}$ .*

In order to confirm [Conjecture 1.1](#), one has to understand the homology classes of curves in  $(f_s)^{(2)}$  and  $M_2(f_s)$  for a singular non-multiple fiber  $f_s$  in the elliptic surface  $X$ . As for [Problem 1.2](#), the genus-1 invariants  $\langle \rangle_{1,d(\beta_F - 2\beta_2)}^{X^{[2]}}$  are among the exceptional cases in [Theorem 4.5](#). Partial progress has been made in [Remark 4.9](#). We will leave [Conjecture 1.1](#) and [Problem 1.2](#) to interested readers.

Finally, the paper is organized as follows. In [Section 2](#), we collect basic facts about stable maps, Gromov–Witten invariants, the Hilbert schemes of points on surfaces, and the Heisenberg operators of Grojnowski and Nakajima. We prove [Theorem 3.3](#) in [Section 3](#), and [Theorems 4.5](#) and [4.8](#) in [Section 4](#).

**Conventions.** In this paper, an elliptic surface means a smooth projective complex surface which is minimal and admits an elliptic fibration over a smooth curve. For a smooth projective surface  $X$ , let  $K_X$  be the canonical divisor of  $X$  and

$$q = h^1(X, \mathcal{O}_X), \quad p_g = h^2(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X(K_X)).$$

## 2. Preliminaries

In this section, we will recall the standard concepts and notions regarding stable maps, Gromov–Witten invariants, the Hilbert schemes of points on surfaces, and the Heisenberg operators of Grojnowski [10] and Nakajima [26]. We will use these operators to describe the (co)homology groups of the Hilbert schemes and the homology classes of certain special curves in the Hilbert schemes.

**2.1. Stable maps and Gromov–Witten invariants.** Let  $Y$  be a smooth projective variety. An  $r$ -pointed stable map to  $Y$  consists of a complete nodal curve  $D$  with  $r$  distinct ordered smooth points  $p_1, \dots, p_r$  and a morphism  $\mu : D \rightarrow Y$  such that the data  $(\mu, D, p_1, \dots, p_r)$  has only finitely many automorphisms. In this case, the stable map is denoted by

$$[\mu : (D; p_1, \dots, p_r) \rightarrow Y],$$

or simply by  $[\mu : D \rightarrow Y]$ . For  $\beta \in H_2(Y, \mathbb{Z})$ , let  $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$  be the coarse moduli space parametrizing the stable maps  $[\mu : (D; p_1, \dots, p_r) \rightarrow Y]$  such that  $\mu_*[D] = \beta$  and the arithmetic genus of  $D$  is  $g$ . Then, we have the  $i$ -th evaluation map:

$$(2-1) \quad \text{ev}_i : \overline{\mathfrak{M}}_{g,r}(Y, \beta) \rightarrow Y$$

defined by  $\text{ev}_i([\mu : (D; p_1, \dots, p_r) \rightarrow Y]) = \mu(p_i)$ . It is known [21; 22; 5] that the coarse moduli space  $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$  is projective and has a virtual fundamental class  $[\overline{\mathfrak{M}}_{g,r}(Y, \beta)]^{\text{vir}} \in A_{\mathfrak{d}}(\overline{\mathfrak{M}}_{g,r}(Y, \beta))$ , where

$$(2-2) \quad \mathfrak{d} = -(K_Y \cdot \beta) + (\dim(Y) - 3)(1 - g) + r$$

is the expected complex dimension of  $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$ , and  $A_{\mathfrak{d}}(\overline{\mathfrak{M}}_{g,r}(Y, \beta))$  is the Chow group of  $\mathfrak{d}$ -dimensional cycles in the moduli space  $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$ .

The Gromov–Witten invariants are defined by using the virtual fundamental class  $[\overline{\mathfrak{M}}_{g,r}(Y, \beta)]^{\text{vir}}$ . Recall that an element

$$\alpha \in H^*(Y, \mathbb{C}) \stackrel{\text{def}}{=} \bigoplus_{j=0}^{2 \dim_{\mathbb{C}}(Y)} H^j(Y, \mathbb{C})$$

is *homogeneous* if  $\alpha \in H^j(Y, \mathbb{C})$  for some  $j$ ; in this case, we take  $|\alpha| = j$ . Let  $\alpha_1, \dots, \alpha_r \in H^*(Y, \mathbb{C})$  such that every  $\alpha_i$  is homogeneous and

$$(2-3) \quad \sum_{i=1}^r |\alpha_i| = 2\mathfrak{d}.$$

Then, we have the  $r$ -point genus- $g$  Gromov–Witten invariant defined by

$$(2-4) \quad \langle \alpha_1, \dots, \alpha_r \rangle_{g,\beta}^Y = \int_{[\overline{\mathfrak{M}}_{g,r}(Y,\beta)]^{\text{vir}}} \text{ev}_1^*(\alpha_1) \otimes \cdots \otimes \text{ev}_r^*(\alpha_r).$$

In particular, when  $r = 1$ , we see from the projection formula that

$$(2-5) \quad \langle \alpha \rangle_{g,\beta}^Y = \int_{\text{ev}_{1*}([\overline{\mathfrak{M}}_{g,1}(Y,\beta)]^{\text{vir}})} \alpha.$$

Next, we recall that *the excess dimension* is the difference between the dimension of  $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$  and the expected dimension  $\mathfrak{d}$  in (2-2). Let  $T_Y$  stand for the tangent sheaf of  $Y$ . For  $0 \leq i < r$ , we shall use

$$(2-6) \quad f_{r,i} : \overline{\mathfrak{M}}_{g,r}(Y, \beta) \rightarrow \overline{\mathfrak{M}}_{g,i}(Y, \beta)$$

to stand for the forgetful map obtained by forgetting the last  $(r - i)$  marked points and contracting all the unstable components. It is known that  $f_{r,i}$  is flat when  $\beta \neq 0$  and  $0 \leq i < r$ . The following can be found in [9, Proposition 2.5].

**Proposition 2.1.** *Let  $\beta \in H_2(Y, \mathbb{Z})$  and  $\beta \neq 0$ . Let  $e$  be the excess dimension of  $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$ . If  $R^1(f_{r+1,r})_*(\text{ev}_{r+1})^*T_Y$  is a rank- $e$  locally free sheaf, then*

$$[\overline{\mathfrak{M}}_{g,r}(Y, \beta)]^{\text{vir}} = c_e(R^1(f_{r+1,r})_*(\text{ev}_{r+1})^*T_Y).$$

Finally, the fundamental class axiom of Gromov–Witten theory states that

$$(2-7) \quad [\overline{\mathfrak{M}}_{g,r}(Y, \beta)]^{\text{vir}} = (f_{r,r-1})_*[\overline{\mathfrak{M}}_{g,r-1}(Y, \beta)]^{\text{vir}}$$

if either  $r + 2g \geq 4$  or  $\beta \neq 0$  and  $r \geq 1$ . The *Divisor Axiom* states that

$$(2-8) \quad \langle \alpha_1, \dots, \alpha_{r-1}, \alpha_r \rangle_{g,\beta}^Y = \int_{\beta} \alpha_r \cdot \langle \alpha_1, \dots, \alpha_{r-1} \rangle_{g,\beta}^Y$$

if  $\alpha_r \in H^2(Y, \mathbb{C})$ , and if either  $r + 2g \geq 4$  or  $\beta \neq 0$  and  $r \geq 1$ .

**2.2. Hilbert schemes of points on surfaces.** Let  $X$  be a smooth projective complex surface, and  $X^{[n]}$  be the Hilbert scheme of  $n$ -points in  $X$ . An element in  $X^{[n]}$  is represented by a length- $n$  0-dimensional closed subscheme  $\xi$  of  $X$ . For  $\xi \in X^{[n]}$ , let  $I_\xi$  and  $\mathcal{O}_\xi$  be the corresponding sheaf of ideals and structure sheaf respectively. It is known from [7; 14] that  $X^{[n]}$  is a smooth irreducible variety of dimension  $2n$ . In fact, the Hilbert–Chow morphism  $X^{[n]} \rightarrow X^{(n)}$ , mapping an element in  $X^{[n]}$  to its support in the  $n$ -th symmetric product  $X^{(n)}$ , is a crepant resolution. The universal codimension-2 subscheme is

$$(2-9) \quad \mathcal{Z}_n = \{(\xi, x) \in X^{[n]} \times X \mid x \in \text{Supp}(\xi)\} \subset X^{[n]} \times X.$$

The boundary of  $X^{[n]}$  is defined to be the subset

$$B_n = \{ \xi \in X^{[n]} \mid |\text{Supp}(\xi)| < n \}.$$

Let  $C$  be a real-surface in  $X$ , and fix distinct points  $x_1, \dots, x_{n-1} \in X$  which are not contained in  $C$ . Define the subsets

$$(2-10) \quad \beta_n = \{ \xi + x_2 + \dots + x_{n-1} \in X^{[n]} \mid \text{Supp}(\xi) = \{x_1\} \},$$

$$(2-11) \quad \beta_C = \{ x + x_1 + \dots + x_{n-1} \in X^{[n]} \mid x \in C \},$$

$$(2-12) \quad D_C = \{ \xi \in X^{[n]} \mid \text{Supp}(\xi) \cap C \neq \emptyset \}.$$

Note that  $\beta_C$  (respectively,  $D_C$ ) is a curve (respectively, a divisor) in  $X^{[n]}$  when  $C$  is a smooth algebraic curve in  $X$ . We extend the notions  $\beta_C$  and  $D_C$  to all the divisors  $C$  in  $X$  by linearity. For a subset  $Y \subset X$ , define

$$(2-13) \quad M_n(Y) = \{ \xi \in X^{[n]} \mid \text{Supp}(\xi) \text{ is a point in } Y \}.$$

Grojnowski [10] and Nakajima [26] geometrically constructed a Heisenberg algebra action on the cohomology of the Hilbert schemes  $X^{[n]}$ . Denote the Heisenberg operators by  $\mathfrak{a}_m(\alpha)$  where  $m \in \mathbb{Z}$  and  $\alpha \in H^*(X, \mathbb{C})$ . Put

$$\mathbb{H}_X = \bigoplus_{n=0}^{+\infty} H^*(X^{[n]}, \mathbb{C}).$$

The operators  $\mathfrak{a}_m(\alpha) \in \text{End}(\mathbb{H}_X)$  satisfy the commutation relation

$$(2-14) \quad [\mathfrak{a}_m(\alpha), \mathfrak{a}_n(\beta)] = -m \cdot \delta_{m, -n} \cdot \langle \alpha, \beta \rangle \cdot \text{Id}_{\mathbb{H}_X}$$

where we have used  $\delta_{m, -n}$  to denote 1 if  $m = -n$  and 0 otherwise. The space  $\mathbb{H}_X$  is an irreducible representation of the Heisenberg algebra generated by the operators  $\mathfrak{a}_m(\alpha)$  with the highest weight vector being

$$|0\rangle = 1 \in H^*(X^{[0]}, \mathbb{C}) = \mathbb{C}.$$

Let  $\{\alpha_1, \dots, \alpha_s\}$  be a basis of

$$H^{\text{even}}(X, \mathbb{C}) = H^0(X, \mathbb{C}) \oplus H^2(X, \mathbb{C}) \oplus H^4(X, \mathbb{C})$$

and  $\{\alpha_{s+1}, \dots, \alpha_{s+t}\}$  be a basis of

$$H^{\text{odd}}(X, \mathbb{C}) = H^1(X, \mathbb{C}) \oplus H^3(X, \mathbb{C})$$

where every  $\alpha_i$  is homogeneous. Then, a basis of the  $n$ -th component  $H^*(X^{[n]}, \mathbb{C})$  in  $\mathbb{H}_X$  consists of the *Heisenberg monomial classes*

$$(2-15) \quad \left( \prod_{i=1}^{s+t} \mathfrak{a}_{-n_{i,1}}(\alpha_i) \cdots \mathfrak{a}_{-n_{i,k_i}}(\alpha_i) \right) |0\rangle$$

where  $k_i \geq 0$  for each  $i$ , every  $n_{i,j}$  is a positive integer,  $\sum_{i,j} n_{i,j} = n$ , and the integers  $n_{i,1}, \dots, n_{i,k_i}$  are mutually distinct for every  $i \in \{s+1, \dots, s+t\}$ . In particular, a basis of the  $H^{4n-2}(X^{[n]}, \mathbb{C})$  in  $\mathbb{H}_X$  consists of the classes

$$(2-16) \quad \mathfrak{a}_{-2}(x)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle, \quad \mathfrak{a}_{-1}(\alpha_i)\mathfrak{a}_{-1}(x)^{n-1}|0\rangle, \quad \mathfrak{a}_{-1}(\alpha_j)\mathfrak{a}_{-1}(\alpha_k)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle,$$

where  $|\alpha_i| = 2$ ,  $|\alpha_j| = |\alpha_k| = 3$  with  $j < k$ , and by abusing notation, we have used  $x$  to denote the cohomology class Poincaré dual to a point  $x \in X$ . Also,

$$(2-17) \quad \beta_n = \mathfrak{a}_{-2}(x)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle,$$

$$(2-18) \quad \beta_C = \mathfrak{a}_{-1}(C)\mathfrak{a}_{-1}(x)^{n-1}|0\rangle,$$

$$(2-19) \quad B_n = \frac{1}{(n-2)!}\mathfrak{a}_{-1}(1_X)^{n-2}\mathfrak{a}_{-2}(1_X)|0\rangle,$$

$$(2-20) \quad D_C = \frac{1}{(n-1)!}\mathfrak{a}_{-1}(1_X)^{n-1}\mathfrak{a}_{-1}(C)|0\rangle$$

where  $1_X$  denotes the fundamental cohomology class of  $X$ , and for simplicity, we do not distinguish a homology class and its Poincaré dual.

**Definition 2.2.** Let  $n \geq 2$ . We define  $\widetilde{H}_2(X^{[n]}, \mathbb{C})$  to be the linear subspace of  $H_2(X^{[n]}, \mathbb{C})$  spanned by the Poincaré duals of the basis classes

$$\mathfrak{a}_{-1}(\alpha_j)\mathfrak{a}_{-1}(\alpha_k)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle$$

in (2-16), where  $|\alpha_j| = |\alpha_k| = 3$  with  $j < k$ .

Next, we recall the homology class of an irreducible curve in the Hilbert scheme  $X^{[n]}$ . Let  $\Gamma$  be an irreducible curve in  $X^{[n]}$ . Define

$$(2-21) \quad \mathcal{Z}_\Gamma = \Gamma \times_{X^{[n]}} \mathcal{Z}_n.$$

Then,  $\mathcal{Z}_\Gamma \subset \mathcal{Z}_n \subset X^{[n]} \times X$ . By [13, Lemma 5.1] and its proof there, we have the following (see also [27, Lemma 1] and [30, Lemma 3.19]).

**Lemma 2.3.** *Let  $n \geq 2$ . Let  $\Gamma$  be an irreducible curve in  $X^{[n]}$ . Then,*

$$(2-22) \quad \Gamma \equiv \beta_{\pi_{2*}[\mathcal{Z}_\Gamma]} + d\beta_n \pmod{\widetilde{H}_2(X^{[n]}, \mathbb{C})}$$

for some integer  $d$ , where  $\pi_2$  is the second projection of  $X^{[n]} \times X$ .

Finally, let  $C$  be a smooth irreducible curve in  $X$  with genus  $g_C$ . Let  $C^{(n)}$  denote the  $n$ -th symmetric product of  $C$ . We regard  $C^{(n)} \subset X^{[n]}$  whenever necessary. For a fixed point  $p \in C$ , let  $\mathfrak{E}$  denote the divisor  $p + C^{(n-1)} \subset C^{(n)}$ . Let

$$\text{AJ} : C^{(n)} \rightarrow \text{Jac}_n(C)$$

be the Abel–Jacobi map sending  $\xi \in C^{(n)}$  to the corresponding degree- $n$  divisor class in  $\text{Jac}_n(C)$ . For an element  $\delta \in \text{Jac}_n(C)$ , the fiber  $\text{AJ}^{-1}(\delta)$  is the complete

line system  $|\delta|$ . Let  $\Theta$  be the pullback via AJ of a theta divisor on  $\text{Jac}_n(C)$ . It is well-known that theta divisors on  $\text{Jac}_n(C)$  are ample.

**Lemma 2.4** [30, Lemma 3.20]. *Let  $n \geq 2$ , and  $\tilde{H}_2(X^{[n]}, \mathbb{C})$  be from Definition 2.2. Let  $C$  be a smooth curve in  $X$ , and  $\Gamma \subset C^{(n)}$  be a curve. Then,*

$$(2-23) \quad \Gamma \equiv (\Xi \cdot \Gamma)\beta_C + (-(n + g_C - 1)(\Xi \cdot \Gamma) + (\Theta \cdot \Gamma))\beta_n \pmod{\tilde{H}_2(X^{[n]}, \mathbb{C})}.$$

*In addition, for every line  $\Gamma_0$  in a positive-dimensional fiber  $\text{AJ}^{-1}(\delta)$ , we have*

$$(2-24) \quad \Gamma_0 \equiv \beta_C - (n + g_C - 1)\beta_n \pmod{\tilde{H}_2(X^{[n]}, \mathbb{C})}.$$

### 3. The vanishing of certain Gromov–Witten invariants of the Hilbert schemes of points on elliptic surfaces with multiple fibers

The goal of this section is to show that certain Gromov–Witten invariants of the Hilbert schemes of points on elliptic surfaces with multiple fibers are equal to 0. Theorem 3.3 below generalizes [13, Corollary 3.5] to the case when the surface is an elliptic surface with multiple fibers. We prove our Theorem 3.3 by modifying the cosection localization method in the proof of [13, Theorem 3.3].

Let  $X$  be a (minimal) elliptic surface. By [8, Corollary 7.5 on p. 113], up to deformation, we may assume that every singular fiber of  $X$  is either an irreducible reduced rational curve with one node or a multiple fiber with smooth reduction. If  $p_g = h^0(X, \mathcal{O}_X(K_X)) \geq 1$ , then by [17, Proposition 6.1 and Remark 6.2], up to deformation, we may further assume that  $|K_X|$  contains a member of the form

$$(3-1) \quad \sum_{i=1}^s f_i + \sum_{j=1}^t (m_j - 1)F_j$$

where  $f_1, \dots, f_s$  are distinct smooth fibers, and  $F_1, \dots, F_t$  are distinct smooth multiple fibers with multiplicities  $m_1, \dots, m_t$  respectively. Therefore, we fix the following assumption throughout this section unless otherwise specified.

**Assumption 3.1.**  $X$  is an elliptic surface with  $p_g \geq 1$  and with the elliptic fibration  $\pi : X \rightarrow C$  over a smooth curve  $C$  such that

- (i) every singular fiber of  $\pi$  is either irreducible reduced or a multiple fiber with smooth reduction;
- (ii)  $H^0(X, \Omega_X^2) = H^0(X, \mathcal{O}_X(K_X))$  contains a holomorphic differential two-form  $\theta$  whose zero-set is of the form (3-1).

By the results of Beauville [3; 4], the holomorphic differential two-form  $\theta$  induces a holomorphic two-form  $\theta^{[n]}$  of the Hilbert scheme  $X^{[n]}$  which can also be regarded as a map  $\theta^{[n]} : T_{X^{[n]}} \rightarrow \Omega_{X^{[n]}}$ . For simplicity, put

$$\overline{\mathfrak{M}} = \overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta).$$

Define the degeneracy locus  $\overline{\mathfrak{M}}(\theta)$  to be the subset of  $\overline{\mathfrak{M}}$  consisting of all the stable maps  $u : \Gamma \rightarrow X^{[n]}$  such that the composite

$$(3-2) \quad u^*(\theta^{[n]}) \circ du : T_{\Gamma_{\text{reg}}} \rightarrow u^*T_{X^{[n]}}|_{\Gamma_{\text{reg}}} \rightarrow u^*\Omega_{X^{[n]}}|_{\Gamma_{\text{reg}}}$$

is trivial over the regular locus  $\Gamma_{\text{reg}}$  of  $\Gamma$ . By the results of Kiem–Li [15; 16],  $\theta^{[n]}$  defines a regular cosection of the obstruction sheaf of  $\overline{\mathfrak{M}}$ :

$$(3-3) \quad \eta : \mathcal{O}b_{\overline{\mathfrak{M}}} \longrightarrow \mathcal{O}_{\overline{\mathfrak{M}}}$$

where  $\mathcal{O}b_{\overline{\mathfrak{M}}}$  is the obstruction sheaf and  $\mathcal{O}_{\overline{\mathfrak{M}}}$  is the structure sheaf of  $\overline{\mathfrak{M}}$ . Moreover, the cosection  $\eta$  is surjective away from the degeneracy locus  $\overline{\mathfrak{M}}(\theta)$ , and there exists a localized virtual cycle  $[\overline{\mathfrak{M}}]_{\text{loc}}^{\text{vir}} \in A_*(\overline{\mathfrak{M}}(\theta))$  such that

$$(3-4) \quad [\overline{\mathfrak{M}}]^{\text{vir}} = \iota_*[\overline{\mathfrak{M}}]_{\text{loc}}^{\text{vir}} \in A_*(\overline{\mathfrak{M}})$$

where  $\iota : \overline{\mathfrak{M}}(\theta) \rightarrow \overline{\mathfrak{M}}$  stands for the inclusion map.

**Lemma 3.2.** *Let  $X$  be an elliptic surface satisfying [Assumption 3.1](#). Let  $f$  be a smooth fiber in  $X$ , and let  $\tilde{H}_2(X^{[n]}, \mathbb{C}) \subset H_2(X^{[n]}, \mathbb{C})$  be from [Definition 2.2](#). If the subset  $\overline{\mathfrak{M}}(\theta)$  of  $\overline{\mathfrak{M}} = \overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$  is nonempty, then*

$$(3-5) \quad \beta \equiv d_0\beta_f + d\beta_n \pmod{\tilde{H}_2(X^{[n]}, \mathbb{C})}$$

for some integer  $d$  and some rational number  $d_0 \geq 0$ .

*Proof.* Let  $u : \Gamma \rightarrow X^{[n]}$  be a stable map in  $\overline{\mathfrak{M}}(\theta)$ , and let  $\Gamma_0$  be any irreducible component of  $\Gamma$ . By [Assumption 3.1\(ii\)](#), the zero-set of  $\theta$  is supported on

$$\left(\bigcup_{i=1}^s f_i\right) \cup \left(\bigcup_{j=1}^t F_j\right).$$

By [13, Lemma 3.1], there exists  $\xi_1 \in X^{[n_0]}$  for some  $n_0$  such that

$$\text{Supp}(\xi_1) \cap \left(\left(\bigcup_{i=1}^s f_i\right) \cup \left(\bigcup_{j=1}^t F_j\right)\right) = \emptyset,$$

$$(3-6) \quad u(\Gamma_0) \subset \xi_1 + \left\{ \xi_2 \mid \text{Supp}(\xi_2) \subset \left(\bigcup_{i=1}^s f_i\right) \cup \left(\bigcup_{j=1}^t F_j\right) \right\}.$$

By [Lemma 2.3](#), there exist integers  $a_{f_i} \geq 0$ ,  $a_{F_j} \geq 0$  and  $d'$  such that

$$(3-7) \quad u(\Gamma_0) \equiv \sum_{i=1}^s a_{f_i}\beta_{f_i} + \sum_{j=1}^t a_{F_j}\beta_{F_j} + d'\beta_n \pmod{\tilde{H}_2(X^{[n]}, \mathbb{C})}.$$

Since  $f = f_i = m_j F_j$  as divisors, we conclude that

$$(3-8) \quad u(\Gamma_0) \equiv d'_0\beta_f + d'\beta_n \pmod{\tilde{H}_2(X^{[n]}, \mathbb{C})}$$

for some rational number  $d'_0 \geq 0$ . Since  $\beta = u_*[\Gamma] = \sum_{\Gamma_0 \subset \Gamma} u_*[\Gamma_0]$ , our lemma follows from (3-8).  $\square$

**Theorem 3.3.** *Let  $X$  be a regular elliptic surface with  $p_g \geq 1$  such that every singular fiber is either irreducible reduced or a multiple fiber with smooth reduction. Let  $f$  be a smooth fiber in  $X$  and  $\beta \in H_2(X^{[n]}, \mathbb{Z})$ . Then all the Gromov–Witten invariants of  $X^{[n]}$  defined via the moduli space  $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$  vanish except possibly when  $g \leq 1$  and  $\beta = d_0\beta_f + d\beta_n$  for some integer  $d$  and rational number  $d_0 \geq 0$ .*

*Proof.* Since  $X$  is regular, the elliptic fibration is of the form  $\pi : X \rightarrow \mathbb{P}^1$ . So Assumption 3.1(ii) holds. Again, since  $X$  is regular,  $\tilde{H}_2(X^{[n]}, \mathbb{C}) = 0$  by Definition 2.2. Moreover, by (2-15), all odd cohomology groups of  $X^{[n]}$  vanish.

If  $\beta \neq d_0\beta_f + d\beta_n$  for some integer  $d$  and rational number  $d_0 \geq 0$ , then we see from Lemma 3.2 that  $\overline{\mathfrak{M}}(\theta) = \emptyset$  and  $[\overline{\mathfrak{M}}]_{\text{loc}}^{\text{vir}} = 0$ . By (3-4),  $[\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)]^{\text{vir}} = 0$  and all the Gromov–Witten invariants defined via  $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$  vanish.

Next, assume that  $g > 1$  and  $\beta = d_0\beta_f + d\beta_n$  for some integer  $d$  and rational number  $d_0 \geq 0$ . Since  $K_{X^{[n]}} = D_{K_X}$ , we see from (2-2) that the expected dimension of  $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$  is equal to  $(2n-3)(1-g) + r < r$ . By (2-3) and the fundamental class axiom (2-7), we conclude that all the Gromov–Witten invariants of  $X^{[n]}$  defined via the moduli space  $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$  vanish.  $\square$

**Remark 3.4.** For an arbitrary smooth projective surface  $X$ , the case  $g = 0$  and  $\beta = d\beta_n$  is studied in [20], and the case  $g = 1$  and  $\beta = d\beta_n$  is discussed in [12].

If the regular elliptic surface  $X$  in Theorem 3.3 has exactly one multiple fiber, then we can prove that the rational number  $d_0$  in Theorem 3.3 is an integer.

**Corollary 3.5.** *Let  $X$  be a regular elliptic surface with  $p_g \geq 1$  and exactly one multiple fiber such that every singular fiber is either irreducible reduced or the unique multiple fiber with smooth reduction  $F$ . Let  $\beta \in H_2(X^{[n]}, \mathbb{Z})$ . Then all the Gromov–Witten invariants of  $X^{[n]}$  defined via the moduli space  $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$  vanish except possibly when  $g \leq 1$  and  $\beta = d_0\beta_F + d\beta_n$  for some integers  $d$  and  $d_0 \geq 0$ .*

*Proof.* By [8, Corollary 2.3 on p.158],  $X$  must be simply connected. Let  $m$  denote the multiplicity of the unique multiple fiber  $F$ . Then  $f_i = mF$  in (3-7), and (3-8) becomes  $u(\Gamma_0) = d'_0\beta_F + d'\beta_n$  for some integers  $d'_0 \geq 0$  and  $d'$ . Accordingly, (3-5) becomes  $\beta = d_0\beta_F + d\beta_n$  for some integers  $d$  and  $d_0 \geq 0$ . Now the same proof of Theorem 3.3 yields our corollary.  $\square$

#### 4. The exceptional cases for $X^{[2]}$

In this section, we will investigate the exceptional cases in Corollary 3.5 when  $n = 2$ . First of all, we strengthen Corollary 3.5 by proving that the integers  $d$  and

$d_0 \geq 0$  in [Corollary 3.5](#) must satisfy  $d \geq -2d_0$ . Then, we compute the exceptional 1-point genus-0 Gromov–Witten invariants

$$(4-1) \quad \langle \alpha \rangle_{0,d(\beta_F - 2\beta_2)}^{X^{[2]}}$$

where  $\alpha \in H^4(X^{[2]}, \mathbb{C})$ ,  $F$  is the reduction of the unique multiple fiber in  $X$ , and  $1 \leq d < m$  with  $m$  being the multiplicity of the unique multiple fiber. By [\(2-5\)](#), the computation of [\(4-1\)](#) is equivalent to determining the cycle

$$\text{ev}_{1*}([\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\text{vir}}) \in A_2(X^{[2]}).$$

**4.1. The homology classes of certain curves in  $X^{[2]}$ .** In this subsection,  $X$  stands for an arbitrary elliptic surface. We will investigate the homology classes of certain curves in  $X^{[2]}$  related to a smooth multiple fiber  $f$  in the elliptic surface  $X$ . These include curves in  $M_2(f)$  (see [\(2-13\)](#) for the notation) and the fibers of the Abel–Jacobi map  $\text{AJ} : f^{(2)} \rightarrow \text{Jac}_2(f) \cong f$ . The results here generalize [\[1, Lemma 4.1 and Lemma 4.3\]](#).

**Lemma 4.1.** *Let  $X$  be an elliptic surface with  $f$  being a smooth fiber or the reduction of a smooth multiple fiber, and  $\tilde{H}_2(X^{[2]}, \mathbb{C}) \subset H_2(X^{[2]}, \mathbb{C})$  be from [Definition 2.2](#). Let  $\Gamma \subset M_2(f)$  be an irreducible curve in  $X^{[2]}$ . Then, there exist nonnegative integers  $d$  and  $d_0$  not both zero such that*

$$(4-2) \quad \Gamma \equiv 2d\beta_f + d_0\beta_2 \pmod{\tilde{H}_2(X^{[2]}, \mathbb{C})}.$$

*Proof.* When  $f$  is a smooth fiber of  $X$ , this is [\[1, Lemma 4.1\]](#). In the following, assume that  $f$  is the reduction of a smooth multiple fiber with multiplicity  $m$ . We will modify the proof of [\[1, Lemma 4.1\]](#). Note that

$$B_2 = M_2(X) \cong \mathbb{P}(T_X^\vee).$$

For convenience, we simply write  $B_2 = M_2(X) = \mathbb{P}(T_X^\vee)$ . Then, we have

$$M_2(f) = \mathbb{P}((T_X|_f)^\vee) \cong \mathbb{P}((T_X|_f)^\vee \otimes \mathcal{O}_f(f)).$$

From  $0 \rightarrow \mathcal{O}_f \rightarrow T_X|_f \rightarrow \mathcal{O}_f(f) \rightarrow 0$ , we obtain the exact sequence

$$0 \rightarrow \mathcal{O}_f \rightarrow (T_X|_f)^\vee \otimes \mathcal{O}_f(f) \rightarrow \mathcal{O}_f(f) \rightarrow 0,$$

where  $\mathcal{O}_f(f)$  is a torsion of order  $m$ . Therefore, we conclude that

$$(4-3) \quad \Gamma = d'\tilde{\sigma} + d'_0f' \in H_2(M_2(f), \mathbb{C}),$$

where  $f'$  is a fiber of the ruling  $M_2(f) \rightarrow f$ ,  $\tilde{\sigma}$  is a section to the ruling with

$$(4-4) \quad \tilde{\sigma}^2 = \deg((T_X|_f)^\vee \otimes \mathcal{O}_f(f)) = \deg \mathcal{O}_f(f) = 0,$$

and  $d'$  and  $d'_0$  are nonnegative integers. In addition, we have

$$(4-5) \quad \mathcal{O}_{M_2(f)}(\tilde{\sigma}) = \mathcal{O}_{M_2(f)}(1) \otimes \phi^* \mathcal{O}_f(f),$$

where  $\phi : M_2(f) \rightarrow f$  denotes the ruling on  $M_2(f)$ .

Since  $f' = \beta_2 \in H_2(X^{[2]}, \mathbb{C})$ , we see from (4-3) that it remains to show

$$(4-6) \quad \tilde{\sigma} \equiv 2\beta_f \pmod{\tilde{H}_2(X^{[2]}, \mathbb{C})}.$$

Indeed, by Lemma 2.3, we have

$$(4-7) \quad \tilde{\sigma} \equiv 2\beta_f + d_2\beta_2 \pmod{\tilde{H}_2(X^{[2]}, \mathbb{C})}$$

for some integer  $d_2$ . Since  $\mathcal{O}_{B_2}(B_2) = \mathcal{O}_{B_2}(-2)$  and  $\mathcal{O}_f(f)$  is a torsion,

$$B_2 \cdot \tilde{\sigma} = B_2|_{M_2(f)} \cdot \tilde{\sigma} = \mathcal{O}_{M_2(f)}(-2) \cdot \tilde{\sigma} = -2\tilde{\sigma} \cdot \tilde{\sigma} = 0$$

where we have used (4-5) and (4-4) in the last two steps. On the other hand, since

$$B_2 \cdot \beta_f = B_2 \cdot w = 0$$

for every class  $w \in \tilde{H}_2(X^{[2]}, \mathbb{C})$  and  $B_2 \cdot \beta_2 = -2$ , we obtain  $B_2 \cdot \tilde{\sigma} = -2d_2$  from (4-7). Therefore, we have  $d_2 = 0$  and (4-6) follows from (4-7).  $\square$

Let  $f$  be a smooth fiber in the elliptic surface  $X$ . Since  $f$  is an elliptic curve, we see that the Abel–Jacobi map

$$\text{AJ} : f^{(2)} \rightarrow \text{Jac}_2(f) \cong f$$

exhibits  $f^{(2)}$  as a ruled surface over  $f$ . The fiber  $\text{AJ}^{-1}(\delta)$  over an element  $\delta \in \text{Jac}_2(f)$  is the complete linear system  $|\delta| \cong \mathbb{P}^1$ . The following is [1, Lemma 4.3].

**Lemma 4.2.** *Let  $X$  be an elliptic surface with  $f$  being a smooth fiber. Let  $\Gamma = \text{AJ}^{-1}(\delta)$  be a fiber of the ruling  $\text{AJ} : f^{(2)} \rightarrow \text{Jac}_2(f)$ . Regard  $\Gamma$  as a curve in  $X^{[2]}$  via  $\Gamma \subset f^{(2)} \subset X^{[2]}$ . Let  $N_{\Gamma \subset X^{[2]}}$  be the normal bundle of  $\Gamma$  in  $X^{[2]}$ . Then,*

- (i)  $N_{\Gamma \subset X^{[2]}} = \mathcal{O}_{\Gamma}(-2) \oplus \mathcal{O}_{\Gamma} \oplus \mathcal{O}_{\Gamma}$ ;
- (ii)  $T_{X^{[2]}}|_{\Gamma} = \mathcal{O}_{\Gamma}(2) \oplus \mathcal{O}_{\Gamma}(-2) \oplus \mathcal{O}_{\Gamma} \oplus \mathcal{O}_{\Gamma}$ ;
- (iii)  $\Gamma = \beta_f - 2\beta_2 \in H_2(X^{[2]}, \mathbb{C})$ .

Our next goal is to prove an analogue of Lemma 4.2 when the smooth fiber  $f$  in Lemma 4.2 is replaced by the reduction of a smooth multiple fiber of the elliptic surface  $X$ . We begin with an elementary lemma.

**Lemma 4.3.** *Let  $f$  be a smooth elliptic curve and  $\phi : f \rightarrow \mathbb{P}^1$  be a double cover. If  $L$  is a non-trivial degree-0 invertible sheaf over  $f$ , then*

$$\phi_* L = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

*Proof.* Note that both  $\phi_*\mathcal{O}_f$  and  $\phi_*L$  are rank-2 locally free sheaf over  $\mathbb{P}^1$ . By the Grothendieck–Riemann–Roch Theorem,  $\deg \phi_*L = \deg \phi_*\mathcal{O}_f$ . Since  $\phi$  is branched over exactly 4 points in  $\mathbb{P}^1$ , we obtain

$$(4-8) \quad \phi_*\mathcal{O}_f = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}.$$

It follows that  $\deg \phi_*L = \deg \phi_*\mathcal{O}_f = -2$ . The rank-2 locally free sheaf  $\phi_*L$  is the direct sum of two invertible sheaves on  $\mathbb{P}^1$ . Since  $\deg \phi_*L = -2$  and  $H^0(\mathbb{P}^1, \phi_*L) \cong H^0(f, L) = 0$ , we must have  $\phi_*L = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ .  $\square$

**Lemma 4.4.** *Let  $X$  be an elliptic surface and  $f$  be the reduction of a smooth multiple fiber in  $X$ . Let  $\Gamma = \text{AJ}^{-1}(\delta)$  be a fiber of the ruling  $\text{AJ} : f^{(2)} \rightarrow \text{Jac}_2(f)$ . Regard  $\Gamma$  as a curve in  $X^{[2]}$  via  $\Gamma \subset f^{(2)} \subset X^{[2]}$ . Then,*

- (i)  $N_{\Gamma \subset X^{[2]}} = \mathcal{O}_{\Gamma}(-1) \oplus \mathcal{O}_{\Gamma}(-1) \oplus \mathcal{O}_{\Gamma}$ ;
- (ii)  $T_{X^{[2]}}|_{\Gamma} = \mathcal{O}_{\Gamma}(2) \oplus \mathcal{O}_{\Gamma}(-1) \oplus \mathcal{O}_{\Gamma}(-1) \oplus \mathcal{O}_{\Gamma}$ ;
- (iii)  $\Gamma = \beta_f - 2\beta_2 \in H_2(X^{[2]}, \mathbb{C})$ .

*Proof.* (i) We prove first that  $N_{f^{(2)} \subset X^{[2]}}|_{\Gamma} = \mathcal{O}_{\Gamma}(-1) \oplus \mathcal{O}_{\Gamma}(-1)$ . Recall the universal subscheme  $\mathcal{Z}_2 \subset X^{[2]} \times X$  from (2-9). Let  $\pi_1 : \mathcal{Z}_2 \rightarrow X^{[2]}$  and  $\pi_2 : \mathcal{Z}_2 \rightarrow X$  be the natural projections. It is known from [2] that

$$(4-9) \quad N_{f^{(2)} \subset X^{[2]}} = \pi_{1*}\pi_2^*\mathcal{O}_X(f)|_{f^{(2)}}.$$

Let  $\mathcal{Z}_{\Gamma} = \pi_1^{-1}(\Gamma) \subset \mathcal{Z}_2$ . Note that  $\pi_2(\mathcal{Z}_{\Gamma}) = f$ . Put  $\tilde{\pi}_1 = \pi_1|_{\mathcal{Z}_{\Gamma}} : \mathcal{Z}_{\Gamma} \rightarrow \Gamma$  and  $\tilde{\pi}_2 = \pi_2|_{\mathcal{Z}_{\Gamma}} : \mathcal{Z}_{\Gamma} \rightarrow f$ . Then,  $\tilde{\pi}_2$  is an isomorphism. Up to an isomorphism,  $\tilde{\pi}_1$  is the double cover  $f \rightarrow \mathbb{P}^1$  corresponding to the complete linear system  $|\delta|$ . Since  $\mathcal{O}_X(f)|_f$  is a non-trivial torsion, we see from Lemma 4.3 that

$$(4-10) \quad \begin{aligned} N_{f^{(2)} \subset X^{[2]}}|_{\Gamma} &= \pi_{1*}\pi_2^*\mathcal{O}_X(f)|_{\Gamma} \\ &= \tilde{\pi}_{1*}\tilde{\pi}_2^*(\mathcal{O}_X(f)|_f) = \mathcal{O}_{\Gamma}(-1) \oplus \mathcal{O}_{\Gamma}(-1). \end{aligned}$$

Next, the normal bundle of  $\Gamma$  in  $f^{(2)}$  is  $N_{\Gamma \subset f^{(2)}} = \mathcal{O}_{\Gamma}$ . So the exact sequence

$$0 \rightarrow T_{\Gamma} \rightarrow T_{f^{(2)}}|_{\Gamma} \rightarrow N_{\Gamma \subset f^{(2)}} \rightarrow 0$$

becomes  $0 \rightarrow \mathcal{O}_{\Gamma}(2) \rightarrow T_{f^{(2)}}|_{\Gamma} \rightarrow \mathcal{O}_{\Gamma} \rightarrow 0$  which splits. Thus,

$$(4-11) \quad T_{f^{(2)}}|_{\Gamma} = \mathcal{O}_{\Gamma}(2) \oplus \mathcal{O}_{\Gamma}.$$

Similarly, the exact sequence

$$0 \rightarrow N_{\Gamma \subset f^{(2)}} \rightarrow N_{\Gamma \subset X^{[2]}} \rightarrow N_{f^{(2)} \subset X^{[2]}}|_{\Gamma} \rightarrow 0$$

becomes  $0 \rightarrow \mathcal{O}_{\Gamma} \rightarrow N_{\Gamma \subset X^{[2]}} \rightarrow \mathcal{O}_{\Gamma}(-1) \oplus \mathcal{O}_{\Gamma}(-1) \rightarrow 0$ , which splits. It follows that  $N_{\Gamma \subset X^{[2]}} = \mathcal{O}_{\Gamma}(-1) \oplus \mathcal{O}_{\Gamma}(-1) \oplus \mathcal{O}_{\Gamma}$ .

(ii) From  $T_\Gamma = \mathcal{O}_\Gamma(2)$ , the exact sequence

$$0 \rightarrow T_\Gamma \rightarrow T_{X^{[2]}|_\Gamma} \rightarrow N_{\Gamma \subset X^{[2]}} \rightarrow 0$$

and  $N_{\Gamma \subset X^{[2]}} = \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma$ , we obtain

$$T_{X^{[2]}|_\Gamma} = \mathcal{O}_\Gamma(2) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma.$$

(iii) This follows from the same proof (given in [1]) of Lemma 4.2(iii).  $\square$

**4.2. Calculation of the 1-point Gromov–Witten invariants  $\langle \alpha \rangle_{0,d(\beta_F - 2\beta_2)}^{X^{[2]}}$ .** In this subsection, our surface  $X$  is from Corollary 3.5, i.e.,  $X$  is a regular elliptic surface with  $p_g \geq 1$  and exactly one multiple fiber such that every singular fiber is either irreducible reduced or the unique multiple fiber with smooth reduction  $F$ . We will compute the Gromov–Witten invariants  $\langle \alpha \rangle_{0,d(\beta_F - 2\beta_2)}^{X^{[2]}}$  when  $1 \leq d < m$  where  $m$  is the multiplicity of the unique multiple fiber in the elliptic surface  $X$ . These invariants belong to the exceptional cases in Corollary 3.5.

We begin with a theorem which strengthens Corollary 3.5 in the case  $n = 2$ .

**Theorem 4.5.** *Let  $X$  be a regular elliptic surface with  $p_g \geq 1$  and exactly one multiple fiber such that every singular fiber is either irreducible reduced or the unique multiple fiber with smooth reduction  $F$ . Let  $\beta \in H_2(X^{[2]}, \mathbb{Z})$ . Then all the Gromov–Witten invariants of  $X^{[2]}$  defined via the moduli space  $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, \beta)$  vanish except possibly when  $g \leq 1$  and  $\beta = d_0\beta_F + d\beta_2$  for some integers  $d_0$  and  $d$  satisfying  $d_0 \geq 0$  and  $d \geq -2d_0$ .*

*Proof.* Recall from the proofs of Theorem 3.3 and Corollary 3.5 that the elliptic fibration is of the form  $\pi : X \rightarrow \mathbb{P}^1$  and  $X$  is simply connected. Let  $m$  be the multiplicity of the unique multiple fiber in  $X$ . Fix a holomorphic differential two-form  $\theta$  in  $H^0(X, \Omega_X^2) = H^0(X, \mathcal{O}_X(K_X))$  such that the zero-set of  $\theta$  is equal to

$$(4-12) \quad \sum_{i=1}^{p_g-1} f_i + (m-1)F,$$

where  $f_1, \dots, f_{p_g-1}$  are distinct smooth fibers of  $\pi$ . Then Assumption 3.1 is satisfied. In view of the proof of Corollary 3.5, it remains to prove that if the subset  $\overline{\mathfrak{M}}(\theta)$  of  $\overline{\mathfrak{M}} = \overline{\mathfrak{M}}_{g,r}(X^{[2]}, \beta)$  is nonempty, then

$$(4-13) \quad \beta = d_0\beta_F + d\beta_2$$

for some integers  $d_0$  and  $d$  satisfying  $d_0 \geq 0$  and  $d \geq -2d_0$ .

Note that (4-13) is an improvement of Lemma 3.2. We will modify the proof of Lemma 3.2 by adopting the notation from there. Let  $u : \Gamma \rightarrow X^{[2]}$  be a stable map in  $\overline{\mathfrak{M}}(\theta) \subset \overline{\mathfrak{M}} = \overline{\mathfrak{M}}_{g,r}(X^{[2]}, \beta)$ , and let  $\Gamma_0$  denote any irreducible component of  $\Gamma$ . By (3-6), we have the following six cases.

Case 1:  $u(\Gamma_0)$  is a point in  $X^{[2]}$ . In this case, we get

$$(4-14) \quad [u(\Gamma_0)] = 0.$$

Case 2:  $u(\Gamma_0) = x + f_i$  for some  $1 \leq i \leq p_g - 1$  and some fixed point  $x \in X$  not lying in any  $f_i$  ( $1 \leq i \leq p_g - 1$ ) or in  $F$ . In this case, we obtain

$$(4-15) \quad [u(\Gamma_0)] = \beta_{f_i} = m\beta_F.$$

Case 3:  $u(\Gamma_0) = x + F$  for some fixed point  $x \in X - \left(\bigcup_{i=1}^{p_g-1} f_i\right) \cup F$ . In this case,

$$(4-16) \quad [u(\Gamma_0)] = \beta_F.$$

Case 4:  $u(\Gamma_0) \subset M_2(f_i) \cup (f_i)^{(2)}$  for some  $1 \leq i \leq p_g - 1$ . If  $u(\Gamma_0) \subset M_2(f_i)$ , then

$$(4-17) \quad [u(\Gamma_0)] = d_i \beta_{f_i} + \tilde{d}_i \beta_2 = d_i m \beta_F + \tilde{d}_i \beta_2$$

by [Lemma 4.1](#), where  $d_i$  and  $\tilde{d}_i$  are nonnegative integers not both zero. If  $u(\Gamma_0) \subset (f_i)^{(2)}$ , then we see from (2-23) that

$$(4-18) \quad [u(\Gamma_0)] = d'_i \beta_{f_i} + (-2d'_i + \Theta \cdot u(\Gamma_0)) \beta_2 = d'_i m \beta_F + (-2d'_i + \Theta \cdot u(\Gamma_0)) \beta_2,$$

where  $\Theta$  is the pullback of a theta divisor via  $AJ : (f_i)^{(2)} \rightarrow \text{Jac}_2(f_i)$  and  $d'_i$  is a nonnegative integer.

Case 5:  $u(\Gamma_0) \subset M_2(F) \cup F^{(2)}$ . If  $u(\Gamma_0) \subset M_2(F)$ , then by [Lemma 4.1](#) again,

$$(4-19) \quad [u(\Gamma_0)] = d_F \beta_F + \tilde{d}_F \beta_2$$

where  $d_F$  and  $\tilde{d}_F$  are nonnegative integers not both zero. If  $u(\Gamma_0) \subset F^{(2)}$ , then

$$(4-20) \quad [u(\Gamma_0)] = d'_F \beta_F + (-2d'_F + \Theta \cdot u(\Gamma_0)) \beta_2$$

where  $\Theta$  is the pullback of a theta divisor via  $AJ : F^{(2)} \rightarrow \text{Jac}_2(F)$  and  $d'_F$  is a nonnegative integer.

Case 6:  $u(\Gamma_0) \subset f_i + f_j$  or  $u(\Gamma_0) \subset f_i + F$  for some  $1 \leq i \neq j \leq p_g - 1$ . In this case,  $u(\Gamma_0) \cdot B_2 = 0$  since  $u(\Gamma_0) \cap B_2 = \emptyset$ . By (2-22),

$$(4-21) \quad [u(\Gamma_0)] = \beta_{\pi_{2*}[Z_\Gamma]} = \tilde{d}'_i \beta_{f_i} + \tilde{d}'_F \beta_F = (\tilde{d}'_i m + \tilde{d}'_F) \beta_F$$

for some nonnegative integers  $\tilde{d}'_i$  and  $\tilde{d}'_F$  not both zero.

Finally, since  $\beta = u_*[\Gamma] = \sum_{\Gamma_0 \subset \Gamma} u_*[\Gamma_0]$ , (4-13) follows from (4-14)–(4-21).  $\square$

For  $g \leq 1$  and  $\beta = d_0 \beta_F + d \beta_2$  where the integers  $d_0$  and  $d$  satisfy  $d_0 \geq 0$  and  $d \geq -2d_0$ , the expected dimension of  $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, \beta)$  is equal to  $1 - g + r$ . By the Divisor Axiom (2-8), the exceptional cases in [Theorem 4.5](#) for  $X^{[2]}$  can be reduced to the computation of the following two types of invariants: (i)  $\langle \alpha \rangle_{0,\beta}^{X^{[2]}}$ , with  $\alpha \in H^4(X^{[2]}, \mathbb{C})$ , and (ii)  $\langle \rangle_{1,\beta}^{X^{[2]}}$ .

Our next goal is to calculate the 1-point genus-0 Gromov–Witten invariants

$$(4-22) \quad \langle \alpha \rangle_{0,d(\beta_F - 2\beta_2)}^{X^{[2]}}$$

with  $\alpha \in H^4(X^{[2]}, \mathbb{C})$  and  $1 \leq d < m$  where  $m$  is the multiplicity of the unique multiple fiber in  $X$ . By (2-5), this is equivalent to determining

$$\text{ev}_{1*}([\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\text{vir}}).$$

The lemma below deals with the stable maps in  $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$ .

**Lemma 4.6.** *Let  $m$  be the multiplicity of the unique multiple fiber in  $X$ . Let*

$$[\mu : D \rightarrow X^{[2]}] \in \overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$$

with  $1 \leq d < m$ . Then,  $\mu(D)$  is a fiber of the ruling  $\text{AJ} : F^{(2)} \rightarrow \text{Jac}_2(F)$ , and the degree of the morphism  $\mu : D \rightarrow \mu(D)$  is equal to  $d$ .

*Proof.* Let  $\Gamma_1, \dots, \Gamma_t$  be the irreducible components of  $\mu(D)$ . Recall that  $X$  is simply connected. By Lemma 2.3, re-ordering  $\Gamma_1, \dots, \Gamma_t$  if necessary, there exists some  $t_0$  with  $0 \leq t_0 \leq t$  such that for  $1 \leq i \leq t_0$ ,  $\Gamma_i = \beta_{\pi_{2*}[\mathcal{Z}_{\Gamma_i}]} + \tilde{d}_i \beta_2$  with  $\pi_{2*}[\mathcal{Z}_{\Gamma_i}] \neq 0$ , and that  $\Gamma_j = \beta_2$  for  $t_0 + 1 \leq j \leq t$ . For  $1 \leq i \leq t$ , let  $m_i$  be the degree of the restriction  $\mu|_{\mu^{-1}(\Gamma_i)} : \mu^{-1}(\Gamma_i) \rightarrow \Gamma_i$ . Then,

$$(4-23) \quad \begin{aligned} d(\beta_F - 2\beta_2) &= \mu_*[D] = \sum_{i=1}^t m_i [\Gamma_i] \\ &= \sum_{i=1}^{t_0} m_i (\beta_{\pi_{2*}[\mathcal{Z}_{\Gamma_i}]} + \tilde{d}_i \beta_2) + \sum_{i=t_0+1}^t m_i \beta_2. \end{aligned}$$

So  $dF = \sum_{i=1}^{t_0} m_i \cdot \pi_{2*}[\mathcal{Z}_{\Gamma_i}]$ ,  $t_0 \geq 1$ , and  $f \cdot \pi_{2*}[\mathcal{Z}_{\Gamma_i}] = 0$  for  $1 \leq i \leq t_0$ . Since  $1 \leq d < m$  and every fiber in  $X$  is either irreducible reduced or the unique multiple fiber, we conclude that for  $1 \leq i \leq t_0$ , the only 1-dimensional irreducible component in  $\pi_2(\mathcal{Z}_{\Gamma_i})$  is  $F$ . If  $\pi_2(\mathcal{Z}_{\Gamma_i})$  contains an isolated point  $x \in X - F$ , then  $\pi_2(\mathcal{Z}_{\Gamma_i}) = F \amalg \{x\}$ ,  $\Gamma_i = F + x$  and

$$(4-24) \quad [\Gamma_i] = \beta_F.$$

Assume that  $\pi_2(\mathcal{Z}_{\Gamma_i})$  does not contain any isolated point in  $X - F$ . Then,  $\pi_2(\mathcal{Z}_{\Gamma_i}) = F$  and  $\text{Supp}(\xi) \subset F$  for every  $\xi \in \Gamma_i$ . So

$$\Gamma_i \subset M_2(F) \cup F^{(2)}.$$

If  $\Gamma_i \subset M_2(F)$ , then by Lemma 4.1, we obtain

$$(4-25) \quad [\Gamma_i] = 2d_i \beta_F + d_{i,0} \beta_2$$

for some integers  $d_i \geq 1$  and  $d_{i,0} \geq 0$ . If  $\Gamma_i \subset F^{(2)}$ , then we see from (2-23) that

$$(4-26) \quad [\Gamma_i] = d_i(\beta_F - 2\beta_2) + (\Theta \cdot \Gamma_i)\beta_2$$

where  $\Theta$  is the pullback of a theta divisor via  $\text{AJ} : F^{(2)} \rightarrow \text{Jac}_2(F)$ . Combining (4-24), (4-25), (4-26) and

$$d(\beta_F - 2\beta_2) = \sum_{i=1}^t m_i [\Gamma_i]$$

from (4-23), we deduce that  $t_0 = t$  and that for every  $1 \leq i \leq t$ ,  $\Gamma_i \subset F^{(2)}$  and  $\Theta \cdot \Gamma_i = 0$ . Thus, every  $\Gamma_i$  is a fiber of the ruling  $\text{AJ} : F^{(2)} \rightarrow \text{Jac}_2(F)$ . Since  $\Gamma_1, \dots, \Gamma_t$  are the irreducible components of  $\mu(D)$  and  $\mu(D)$  is connected, we conclude that  $t = 1$ ,  $\mu(D) = \Gamma_1$  is a fiber of the ruling  $\text{AJ} : F^{(2)} \rightarrow \text{Jac}_2(F)$ , and

$$d(\beta_F - 2\beta_2) = m_1 [\Gamma_1].$$

By Lemma 4.4(iii),  $[\Gamma_1] = \beta_F - 2\beta_2$ . It follows that  $m_1 = d$ , i.e., the degree of the morphism  $\mu : D \rightarrow \mu(D) = \Gamma_1$  is equal to  $d$ .  $\square$

Let  $1 \leq d < m$ . By Lemma 4.6, we obtain the commutative diagram

$$(4-27) \quad \begin{array}{ccc} \overline{\mathfrak{M}}_{g,r+1}(X^{[2]}, d(\beta_F - 2\beta_2)) & \xrightarrow{\text{ev}_{r+1}} & F^{(2)} \subset X^{[2]} \\ \downarrow f_{r+1,r} & & \downarrow \text{AJ} \\ \overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2)) & \xrightarrow{\Phi_r} & \text{Jac}_2(F) \end{array}$$

where  $f_{r+1,r}$  is the forgetful map forgetting the last marked point, and  $\Phi_r$  maps  $[\mu : D \rightarrow X^{[2]}] \in \overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$  to  $\text{AJ}(\mu(D))$ . We have

$$(4-28) \quad \Phi_r^{-1}(\delta) \cong \overline{\mathfrak{M}}_{g,r}(\text{AJ}^{-1}(\delta), d) \cong \overline{\mathfrak{M}}_{g,r}(\mathbb{P}^1, d)$$

for  $\delta \in \text{Jac}_2(F)$ . So the dimension of  $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$  is equal to

$$(4-29) \quad \dim \overline{\mathfrak{M}}_{g,r}(\mathbb{P}^1, d) + 1 = 2d + 2g + r - 1.$$

Next, when  $1 \leq d < m$ , we determine the virtual fundamental class

$$[\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\text{vir}}.$$

**Lemma 4.7.** *Let  $m$  be the multiplicity of the unique multiple fiber in the elliptic surface  $X$  from Theorem 4.5. Let  $1 \leq d < m$  and  $0 \leq g \leq 1$ . Then,*

- (i)  $R^1(f_{r+1,r})_*(\text{ev}_{r+1})^*T_{X^{[2]}}$  is a rank- $(2d + 3g - 2)$  locally free sheaf.
- (ii)  $[\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\text{vir}} = c_{2d+3g-2}(R^1(f_{r+1,r})_*(\text{ev}_{r+1})^*T_{X^{[2]}})$ .

*Proof.* (i) Let  $[\mu : D \rightarrow X^{[2]}] \in \overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$ . By Lemma 4.6,  $\mu(D)$  is a fiber of the ruling  $\text{AJ} : F^{(2)} \rightarrow \text{Jac}_2(F)$ . By Lemma 4.4(ii),

$$T_{X^{[2]}|\mu(D)} = \mathcal{O}_{\mu(D)}(2) \oplus \mathcal{O}_{\mu(D)}(-1) \oplus \mathcal{O}_{\mu(D)}(-1) \oplus \mathcal{O}_{\mu(D)}.$$

Since  $d \geq 1$  and  $0 \leq g \leq 1$ , we conclude that

$$h^1(D, \mu^* T_{X^{[2]}}) = 2h^1(D, \mu^* \mathcal{O}_{\mu(D)}(-1)) + h^1(D, \mathcal{O}_D) = 2d + 3g - 2.$$

Hence  $R^1(f_{r+1,r})_*(\text{ev}_{r+1})^* T_{X^{[2]}}$  is a rank- $(2d + 3g - 2)$  locally free sheaf.

(ii) By (2-2), the expected dimension of  $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$  is equal to  $1 - g + r$ . By (4-29), the excess dimension of  $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$  is  $2d + 3g - 2$ . So our result follows immediately from (i) and Proposition 2.1.  $\square$

Finally, we determine the cycle  $\text{ev}_{1*}([\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\text{vir}}$ .

**Theorem 4.8.** *Let  $X$  be a regular elliptic surface with  $p_g \geq 1$  and exactly one multiple fiber such that every singular fiber is either irreducible reduced or the unique multiple fiber with smooth reduction  $F$ . Let  $m$  be the multiplicity of the unique multiple fiber, and  $1 \leq d < m$ . Then,*

$$\text{ev}_{1*}([\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\text{vir}}) = \frac{1}{d^2} \cdot [F^{(2)}] \in A_2(X^{[2]}).$$

*Proof.* By (2-2), the expected dimension of  $\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))$  is equal to 2. By Lemma 4.6,  $\text{ev}_1(\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))) \subset F^{(2)}$ . So

$$(4-30) \quad \text{ev}_{1*}([\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\text{vir}}) = \kappa \cdot [F^{(2)}]$$

for some number  $\kappa$ . To determine  $\kappa$ , we take a complex 2-dimensional cycle  $W \subset X^{[2]}$  and pretend that  $W$  and  $F^{(2)}$  intersect transversally at a unique point  $\xi \in F^{(2)}$ . Let  $\delta = \text{AJ}(\xi)$  and  $\Gamma = \text{AJ}^{-1}(\delta)$ . Then  $\xi \in \Gamma \cong \mathbb{P}^1$ . Intersecting both sides of (4-30) with  $[W]$ , we get

$$(4-31) \quad \begin{aligned} \kappa &= \text{ev}_{1*}([\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\text{vir}}) \cdot [W] \\ &= [\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\text{vir}} \cdot \text{ev}_1^*[W] \\ &= [\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\text{vir}} \cdot \text{ev}_1^*[\xi] \\ &= [\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\text{vir}}|_{\overline{\mathfrak{M}}_{0,1}(\Gamma, d)} \cdot (\tilde{\text{ev}}_1)^*[\xi] \end{aligned}$$

where in the last step, we have used

$$(\text{ev}_1)^{-1}(\xi) \subset \overline{\mathfrak{M}}_{0,1}(\Gamma, d) \subset \overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2)),$$

and  $\tilde{\text{ev}}_1 : \overline{\mathfrak{M}}_{0,1}(\Gamma, d) \rightarrow \Gamma$  is the evaluation map.

By Lemma 4.7(i),  $R^1(f_{1,0})_*(\text{ev}_1)^*T_{X^{[2]}}$  is a rank- $(2d-2)$  locally free sheaf on  $\overline{\mathfrak{M}}_{0,0}(X^{[2]}, d(\beta_F - 2\beta_2))$  where  $f_{1,0}$  and  $\text{ev}_1$  are the forgetful map and the evaluation map on  $\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))$  respectively. For simplicity, put

$$\Omega = R^1(f_{1,0})_*(\text{ev}_1)^*T_{X^{[2]}}.$$

By (2-7) and Lemma 4.7(ii), we conclude that

$$\begin{aligned} [\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\text{vir}} &= (f_{1,0})^*([\overline{\mathfrak{M}}_{0,0}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\text{vir}}) \\ &= (f_{1,0})^*(c_{2d-2}(\Omega)). \end{aligned}$$

Combining with (4-31), we see that

$$\begin{aligned} (4-32) \quad \kappa &= (f_{1,0})^*(c_{2d-2}(\Omega))|_{\overline{\mathfrak{M}}_{0,1}(\Gamma, d)} \cdot (\tilde{\text{ev}}_1)^*[\xi] \\ &= (\tilde{f}_{1,0})^*(c_{2d-2}(\Omega)|_{\overline{\mathfrak{M}}_{0,0}(\Gamma, d)}) \cdot (\tilde{\text{ev}}_1)^*[\xi], \end{aligned}$$

where  $\tilde{f}_{1,0} : \overline{\mathfrak{M}}_{0,1}(\Gamma, d) \rightarrow \overline{\mathfrak{M}}_{0,0}(\Gamma, d)$  is the forgetful map. Note that

$$\Omega|_{\overline{\mathfrak{M}}_{0,0}(\Gamma, d)} \cong R^1(\tilde{f}_{1,0})_*(\tilde{\text{ev}}_1)^*(T_{X^{[2]}|_{\Gamma}}).$$

By Lemma 4.4(ii),  $T_{X^{[2]}|_{\Gamma}} = \mathcal{O}_{\Gamma}(2) \oplus \mathcal{O}_{\Gamma}(-1) \oplus \mathcal{O}_{\Gamma}(-1) \oplus \mathcal{O}_{\Gamma}$ . Thus,

$$\Omega|_{\overline{\mathfrak{M}}_{0,0}(\Gamma, d)} \cong R^1(\tilde{f}_{1,0})_*(\tilde{\text{ev}}_1)^*(\mathcal{O}_{\Gamma}(-1) \oplus \mathcal{O}_{\Gamma}(-1)).$$

By [6, Theorem 9.2.3],

$$c_{2d-2}(\Omega|_{\overline{\mathfrak{M}}_{0,0}(\Gamma, d)}) = \frac{1}{d^3} \cdot [\eta]$$

where  $[\eta]$  denotes the class of a generic stable map  $\eta \in \overline{\mathfrak{M}}_{0,0}(\Gamma, d)$ . By (4-32),

$$\kappa = \frac{1}{d^3} \cdot (\tilde{f}_{1,0})^*[\eta] \cdot (\tilde{\text{ev}}_1)^*[\xi] = \frac{1}{d^2}$$

since for a generic  $\eta = [\mu : D \rightarrow \Gamma] \in \overline{\mathfrak{M}}_{0,0}(\Gamma, d)$ , the map  $\mu$  is  $d$  to 1.  $\square$

**Remark 4.9.** As for the exceptional genus-1 invariant  $\langle \rangle_{1,d(\beta_F - 2\beta_2)}^{X^{[2]}}$ , we see from Lemma 4.7(ii) that if  $1 \leq d < m$  where  $m$  is the multiplicity of the unique multiple fiber in the elliptic surface  $X$  from Theorem 4.5, then

$$\langle \rangle_{1,d(\beta_F - 2\beta_2)}^{X^{[2]}} = c_{2d+1}(R^1(f_{1,0})_*(\text{ev}_1)^*T_{X^{[2]}}).$$

where  $f_{1,0}$  and  $\text{ev}_1$  are the forgetful map and the evaluation map on the moduli space  $\overline{\mathfrak{M}}_{1,1}(X^{[2]}, d(\beta_F - 2\beta_2))$  respectively. However, it is unclear how to compute the right-hand side explicitly.

## Acknowledgment

The authors would like to thank the two referees for carefully reading the paper and providing valuable comments. The first author would like to thank the Mathematics Department of the University of Missouri for its hospitality during his research leave from August 2024 to April 2025.

## References

- [1] M. M. Alhwaimel and Z. Qin, “Gromov–Witten invariants of Hilbert schemes of two points on elliptic surfaces”, *Internat. J. Math.* **33**:10-11 (2022), art. id. 2250079, 27 pp. [MR](#)
- [2] A. B. Altman, A. Iarrobino, and S. L. Kleiman, “Irreducibility of the compactified Jacobian”, pp. 1–12 in *Real and complex singularities* (Math., Oslo, 1976), Sijthoff & Noordhoff, Alphen aan den Rijn, Netherlands, 1977. [MR](#)
- [3] A. Beauville, “Variétés Kähleriennes dont la première classe de Chern est nulle”, *J. Differential Geom.* **18**:4 (1983), 755–782. [MR](#)
- [4] A. Beauville, “Variétés kähleriennes compactes avec  $c_1 = 0$ ”, pp. 181–192 in *Geometry of K3 surfaces: moduli and periods* (Palaiseau, 1981/1982), Astérisque **126**, 1985. [MR](#)
- [5] K. Behrend and B. Fantechi, “The intrinsic normal cone”, *Invent. Math.* **128**:1 (1997), 45–88. [MR](#)
- [6] D. A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, Mathematical Surveys and Monographs **68**, American Mathematical Society, Providence, RI, 1999. [MR](#)
- [7] J. Fogarty, “Algebraic families on an algebraic surface”, *Amer. J. Math.* **90** (1968), 511–521. [MR](#)
- [8] R. Friedman and J. W. Morgan, *Smooth four-manifolds and complex surfaces*, Ergebnisse der Mathematik (3) **27**, Springer, 1994. [MR](#)
- [9] E. Getzler, “Intersection theory on  $\overline{M}_{1,4}$  and elliptic Gromov–Witten invariants”, *J. Amer. Math. Soc.* **10**:4 (1997), 973–998. [MR](#)
- [10] I. Grojnowski, “Instantons and affine algebras, I: The Hilbert scheme and vertex operators”, *Math. Res. Lett.* **3**:2 (1996), 275–291. [MR](#)
- [11] A. Grothendieck, “Techniques de construction et théorèmes d’existence en géométrie algébrique, IV: Les schémas de Hilbert”, in *Séminaire Bourbaki 1960/1961* (exposé 221), W. A. Benjamin, New York, 1966. Reprinted as pp. 249–276 in *Séminaire Bourbaki 6*, Soc. Math. France, Paris, 1995. [MR](#) [Zbl](#)
- [12] J. Hu and Z. Qin, “Extremal Gromov–Witten invariants of the Hilbert scheme of 3 points”, *Forum Math. Sigma* **11** (2023), art. id. e21, 40 pp. [MR](#)
- [13] J. Hu, W.-P. Li, and Z. Qin, “The Gromov–Witten invariants of the Hilbert schemes of points on surfaces with  $p_g > 0$ ”, *Internat. J. Math.* **26**:1 (2015), art. id. 1550009, 26 pp. [MR](#)
- [14] A. A. Iarrobino, *Punctual Hilbert schemes*, Mem. Amer. Math. Soc. **188**, 1977. [MR](#)
- [15] Y.-H. Kiem and J. Li, “Gromov–Witten invariants of varieties with holomorphic 2-forms”, preprint, 2007. [arXiv 0707.2986](#)
- [16] Y.-H. Kiem and J. Li, “Localizing virtual cycles by cosections”, *J. Amer. Math. Soc.* **26**:4 (2013), 1025–1050. [MR](#)
- [17] J. Lee and T. H. Parker, “A structure theorem for the Gromov–Witten invariants of Kähler surfaces”, *J. Differential Geom.* **77**:3 (2007), 483–513. [MR](#)

- [18] J. Li and W.-P. Li, “Two point extremal Gromov–Witten invariants of Hilbert schemes of points on surfaces”, *Math. Ann.* **349**:4 (2011), 839–869. [MR](#)
- [19] W.-P. Li and Z. Qin, “On 1-point Gromov–Witten invariants of the Hilbert schemes of points on surfaces”, *Turkish J. Math.* **26**:1 (2002), 53–68. [MR](#)
- [20] W.-P. Li and Z. Qin, “The cohomological crepant resolution conjecture for the Hilbert–Chow morphisms”, *J. Differential Geom.* **104**:3 (2016), 499–557. [MR](#)
- [21] J. Li and G. Tian, “Virtual moduli cycles and Gromov–Witten invariants of algebraic varieties”, *J. Amer. Math. Soc.* **11**:1 (1998), 119–174. [MR](#)
- [22] J. Li and G. Tian, “Virtual moduli cycles and Gromov–Witten invariants of general symplectic manifolds”, pp. 47–83 in *Topics in symplectic 4-manifolds* (Irvine, CA, 1996), First Int. Press Lect. Ser. **1**, International Press, Cambridge, MA, 1998. [MR](#)
- [23] D. Maulik and A. Oblomkov, “Quantum cohomology of the Hilbert scheme of points on  $\mathcal{A}_n$ -resolutions”, *J. Amer. Math. Soc.* **22**:4 (2009), 1055–1091. [MR](#)
- [24] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, “Gromov–Witten theory and Donaldson–Thomas theory, I”, *Compos. Math.* **142**:5 (2006), 1263–1285. [MR](#)
- [25] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, “Gromov–Witten theory and Donaldson–Thomas theory, II”, *Compos. Math.* **142**:5 (2006), 1286–1304. [MR](#)
- [26] H. Nakajima, “Heisenberg algebra and Hilbert schemes of points on projective surfaces”, *Ann. of Math. (2)* **145**:2 (1997), 379–388. [MR](#)
- [27] G. Oberdieck, “Gromov–Witten invariants of the Hilbert schemes of points of a K3 surface”, *Geom. Topol.* **22**:1 (2018), 323–437. [MR](#)
- [28] G. Oberdieck and A. Pixton, “Quantum cohomology of the Hilbert scheme of points on an elliptic surface”, 2023. [arXiv 2312.13188](#)
- [29] A. Okounkov and R. Pandharipande, “Quantum cohomology of the Hilbert scheme of points in the plane”, *Invent. Math.* **179**:3 (2010), 523–557. [MR](#)
- [30] Z. Qin, *Hilbert schemes of points and infinite dimensional Lie algebras*, Mathematical Surveys and Monographs **228**, American Mathematical Society, Providence, RI, 2018. [MR](#)

Received September 20, 2024. Revised July 4, 2025.

MAZEN M. ALHWAIMEL  
DEPARTMENT OF MATHEMATICS  
COLLEGE OF SCIENCE, QASSIM UNIVERSITY  
BURAYDAH 51452  
SAUDI ARABIA  
[4200@qu.edu.sa](mailto:4200@qu.edu.sa)

ZHENBO QIN  
DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF MISSOURI  
COLUMBIA, MO 65211  
UNITED STATES  
[qinz@missouri.edu](mailto:qinz@missouri.edu)

# PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

[msp.org/pjm](http://msp.org/pjm)

## EDITORS

Don Blasius (Managing Editor)  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[blasius@math.ucla.edu](mailto:blasius@math.ucla.edu)

Matthias Aschenbrenner  
Fakultät für Mathematik  
Universität Wien  
Vienna, Austria  
[matthias.aschenbrenner@univie.ac.at](mailto:matthias.aschenbrenner@univie.ac.at)

Vyjayanthi Chari  
Department of Mathematics  
University of California  
Riverside, CA 92521-0135  
[chari@math.ucr.edu](mailto:chari@math.ucr.edu)

Atsushi Ichino  
Department of Mathematics  
Kyoto University  
Riverside, CA 92521-0135  
[atsushi.ichino@gmail.com](mailto:atsushi.ichino@gmail.com)

Robert Lipshitz  
Department of Mathematics  
University of Oregon  
Eugene, OR 97403  
[lipshitz@uoregon.edu](mailto:lipshitz@uoregon.edu)

Kefeng Liu  
School of Sciences  
Chongqing University of Technology  
Chongqing 400054, China  
[liu@math.ucla.edu](mailto:liu@math.ucla.edu)

Sucharit Sarkar  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[sucharit@math.ucla.edu](mailto:sucharit@math.ucla.edu)

Dimitri Shlyakhtenko  
Department of Mathematics  
University of California  
Los Angeles, CA 90095-1555  
[shlyakht@ipam.ucla.edu](mailto:shlyakht@ipam.ucla.edu)

Ruixiang Zhang  
Department of Mathematics  
University of California  
Berkeley, CA 94720-3840  
[ruixiang@berkeley.edu](mailto:ruixiang@berkeley.edu)

## PRODUCTION

Silvio Levy, Scientific Editor, [production@msp.org](mailto:production@msp.org)

---

See inside back cover or [msp.org/pjm](http://msp.org/pjm) for submission instructions.

---

The subscription price for 2025 is US \$677/year for the electronic version, and \$917/year for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [PASCAL CNRS Index](#), [Referativnyi Zhurnal](#), [Current Mathematical Publications](#) and [Web of Knowledge \(Science Citation Index\)](#).


---

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

---

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**  
nonprofit scientific publishing

<http://msp.org/>

© 2025 Mathematical Sciences Publishers

# PACIFIC JOURNAL OF MATHEMATICS

Volume 339    No. 1    November 2025

---

|   |     |
|---|-----|
| Gromov–Witten theory of Hilbert schemes of points on elliptic surfaces with multiple fibers                 | 1   |
| MAZEN M. ALHWAIMEL and ZHENBO QIN   |     |
| On Kazhdan–Yom Din asymptotic orthogonality for $K$ -finite matrix coefficients of tempered representations | 23  |
| ANNE-MARIE AUBERT and ALFIO FABIO LA ROSA   |     |
| A functoriality property for supercuspidal $L$ -packets   | 73  |
| ADÈLE BOURGEOIS and PAUL MEZO   |     |
| The local Gross–Prasad conjecture over Archimedean local fields   | 133 |
| CHENG CHEN  |     |
| Volume bounds for hyperbolic rod complements in the 3-torus   | 167 |
| NORMAN DO, CONNIE ON YU HUI and JESSICA S. PURCELL  |     |
| A remark on the Lewark–Zibrowius invariant  | 191 |
| MIHAI MARIAN  |     |