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Kaletha constructed L-packets for supercuspidal L-parameters of tame p-adic groups. These L-packets consist entirely of supercuspidal representations, which are explicitly described. In the setting of quasisplit reductive groups, we show that Kaletha's L-packets satisfy a functoriality property for homomorphisms with central kernel and abelian cokernel.

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1. Introduction

Let G be a connected reductive algebraic group defined over a nonarchimedean local field F. The local Langlands correspondence (LLC) is a conjectural map

$$\varphi \mapsto \Pi_\varphi$$

from *L-parameters* to *L-packets* [Borel 1979, Chapter III]. The latter are finite sets of (equivalence classes of) irreducible representations of the group of F-points, $G(F)$. The LLC is expected to satisfy numerous additional properties which give it content. We focus on only two properties. The first property concerns *supercuspidal* representations. A large class of supercuspidal representations have been grouped into L-packets by Kaletha [2019; 2021]. We shall *exclusively* be dealing with these supercuspidal representations. The second property concerns *functoriality* for homomorphisms with central kernel and abelian cokernel. This form of functoriality was introduced by Borel [1979, Desideratum 10.3(5)] and was later refined by Solleveld [2020, Corollary 2].

To describe the expected properties for supercuspidal representations, we recall that an L-parameter is an L-homomorphism

$$\varphi : W_F \times \mathrm{SL}_2 \to {}^L G$$

from the Weil–Deligne group into the L-group of G [Borel 1979, Section 8.2]. Following [Kaletha 2021, Section 4.1], the L-parameter φ is defined to be *supercuspidal* if it is trivial on SL_2, i.e.,

$$\varphi : W_F \to {}^L G,$$

and its image is not contained in a proper parabolic subgroup of ${}^L G$ [Borel 1979, Section 3.3]. As observed in [Kaletha 2021, Section 4.1], "compound" L-packets (or L-packets when G is quasisplit) consisting entirely of supercuspidal representations are conjectured to correspond precisely to supercuspidal L-parameters [DeBacker and Reeder 2009, Section 3.5; Aubert et al. 2018]. Kaletha [2019; 2021] provided an explicit construction for these conjectured L-packets, under the additional assumptions that G splits over a tamely ramified extension, and that the residual characteristic p of F does not divide the order of the Weyl group of G. He further proved that the L-packets satisfy some important properties (e.g., stability).

The first goal of this paper is to show that these L-packets satisfy the desired functorial property [Borel 1979, Desideratum 10.3(5)]. For this reason, and from now on, we work under the assumptions on G and the residual characteristic of F given in the previous paragraph. For the sake of simplicity, we also assume that G is quasisplit over F (see the discussion surrounding (1)). Let $\Phi_{\mathrm{sc}}(G)$ denote the set (of conjugacy classes) of supercuspidal L-parameters of G. Given $\varphi \in \Phi_{\mathrm{sc}}(G)$, we let Π_φ denote the associated supercuspidal L-packet obtained via Kaletha's construction.

Theorem (Theorem 4.1). *Suppose G is quasisplit and splits over a tamely ramified extension. Suppose further that the residual characteristic p of F does not divide the order of the Weyl group of G. Let $\eta : G \to \underline{G}$ be an F-morphism of connected reductive F-groups such that*

(i) *the kernel of $d\eta : \mathrm{Lie}(G) \to \mathrm{Lie}(\underline{G})$ is central,*

(ii) *the cokernel of η is an abelian F-group.*

Let $\underline{\varphi} \in \Phi_{\mathrm{sc}}(\underline{G})$ and set $\varphi = {}^L\eta \circ \underline{\varphi}$. Then for all $\underline{\pi} \in \Pi_{\underline{\varphi}}$, $\underline{\pi} \circ \eta$ is the direct sum of finitely many irreducible supercuspidal representations belonging to Π_{φ}.

Here, the map ${}^L\eta : {}^L\underline{G} \to {}^LG$ takes the form ${}^L\eta(g, w) = (\hat{\eta}(g), w)$ for all $g \in \widehat{\underline{G}}$, $w \in W_F$. The map $\hat{\eta} : \widehat{\underline{G}} \to \widehat{G}$ on the Langlands dual groups is recalled in Section 4.1.

The above theorem is a modified version of [Borel 1979, Desideratum 10.3(5)], in which η is required to have abelian kernel and cokernel. The hypothesis on η is precisely [Solleveld 2020, Condition 1], and is stronger [Solleveld 2020, Lemma 5.1] than that of [Borel 1979, Desideratum 10.3(5)]. It ensures that the root systems of G and $\underline{G}$ are identified through η in arbitrary characteristic (see [SGA 3_{III} 1970, Sections 6.8, 7.5]).

In addition to proving Theorem 4.1, we provide a description of the components of $\underline{\pi} \circ \eta$. The supercuspidal representations that make up the L-packets of Theorem 4.1 are constructed from *tame F-nonsingular elliptic pairs*, which consist of a particular kind of torus and a character thereof [Kaletha 2021, Definition 3.4.1]. Given such a pair $(\underline{S}, \underline{\theta})$ of $\underline{G}$, we let $\pi_{(\underline{S},\underline{\theta})}$ denote the attached supercuspidal representation of $\underline{G}(F)$, which is obtained from the *Kaletha–Yu construction*. This construction consists of applying the *J.-K. Yu construction* [2001] after unfolding $(\underline{S}, \underline{\theta})$ into an appropriate *$\underline{G}$-datum* [Kaletha 2019; 2021]. The representation $\pi_{(\underline{S},\underline{\theta})}$ may be reducible, and its irreducible components form part of an L-packet. The first big result of this paper is writing a decomposition formula for $\pi_{(\underline{S},\underline{\theta})} \circ \eta$.

Theorem (Theorem 3.1). *Let $(\underline{S}, \underline{\theta})$ and (S, θ) be tame F-nonsingular elliptic pairs for $\underline{G}$ and G, respectively. Assume that $\eta(S) \subset \underline{S}$ and $\theta = \underline{\theta} \circ \eta$. Then*

$$\pi_{(\underline{S},\underline{\theta})} \circ \eta \simeq \bigoplus_{\underline{c} \in \underline{C}} \pi_{(S,\theta)} \circ \underline{\mathrm{Ad}}(\underline{c}^{-1}),$$

where $\underline{C}$ is a set of coset representatives of $\eta(G(F))\backslash \underline{G}(F)/\underline{S}(F)$ and $\underline{\mathrm{Ad}}$ is the $\underline{G}(F)$-action on $G(F)$ described in Section 2.3.

The following three paragraphs sketch the main ideas required to prove this theorem, and its complete proof is given in Section 4.2.

The composition $\pi_{(\underline{S},\underline{\theta})} \circ \eta$ can be viewed as the restriction of $\pi_{(\underline{S},\underline{\theta})}$ to $\eta(G(F))$. Having abelian cokernel implies that $\eta(G)$ is a subgroup of $\underline{G}$ which contains the derived subgroup $[\underline{G}, \underline{G}]$. The kernel of η, which we denote by Z, is a central

$$(\pi_{(\underline{S},\underline{\theta})}, \underline{G}(F)) \xrightarrow{\text{[Bourgeois 2021]}} (\pi_{(\underline{S},\underline{\theta})}|_{G_Z(F)}, G_Z(F)) \xrightarrow{\text{Theorem 3.17}} (\pi_{(\underline{S},\underline{\theta})} \circ \eta, G(F))$$

Figure 1. Illustration of the two-step process required to compute $\pi_{(\underline{S},\underline{\theta})} \circ \eta$.

subgroup by [Solleveld 2020, Lemma 5.1]. We will write G_Z for $\eta(G)$ and use the Z in subscript for all objects attached to G_Z. In this notation, $G_Z \simeq G/Z$ and $\underline{G} = G_Z Z(\underline{G})$, where $Z(\underline{G})$ denotes the center of $\underline{G}$. We compute the restriction of $\pi_{(\underline{S},\underline{\theta})}$ to $\eta(G(F)) \simeq G(F)/Z(F)$ in two steps (as illustrated in Figure 1). First, by restricting to $\eta(G)(F) = G_Z(F)$, and second, by further restricting to $G(F)/Z(F)$. The group $\eta(G(F))$ is a normal subgroup of $G_Z(F)$ (Corollary 2.7) and the quotient is parameterized by a subgroup of a Galois cohomology group $H^1(F, Z)$ [Springer 2009, Proposition 12.3.4], which may be nontrivial. The restriction of supercuspidal representations to algebraic subgroups that contain the derived subgroup was extensively studied in [Bourgeois 2021]. We can apply the results therein to obtain a description for $\pi_{(\underline{S},\underline{\theta})}|_{G_Z(F)}$ (Theorem 3.19). The second restriction (Theorem 3.17) can be computed via Mackey theory, as the quotient $G_Z(F)/(G(F)/Z(F))$ is compact and abelian [Silberger 1979].

In order to describe the supercuspidal representations in the L-packets Π_φ and $\Pi_{\underline{\varphi}}$, one must know which tame F-nonsingular elliptic pairs to use. These pairs are provided by *supercuspidal L-packet data* [Kaletha 2021, Definition 4.1.4]. The supercuspidal L-packet data for φ and $\underline{\varphi}$ consist of tuples $(S, \hat{j}, \chi_0, \theta)$ and $(\underline{S}, \hat{\underline{j}}, \underline{\chi}_0, \underline{\theta})$, respectively. Unlike the previous paragraph, S and $\underline{S}$ are not subtori of G and $\underline{G}$. Rather, they are embedded into subtori of the respective groups. The elements $\hat{j}$ and $\hat{\underline{j}}$ specify families of *admissible embeddings* $S(F) \to G(F)$ and $\underline{S}(F) \to \underline{G}(F)$, denoted by $\mathcal{J}_F$ and $\underline{\mathcal{J}}_F$, respectively. Each embedding $j \in \mathcal{J}_F$ ($\underline{j} \in \underline{\mathcal{J}}_F$) is used to generate a tame F-nonsingular elliptic pair $(jS, j\theta)$ $((\underline{j}\underline{S}, \underline{j}\underline{\theta}))$. We let the components of $\pi_{(jS,j\theta)}$ $(\pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})})$ be elements of Π_φ $(\Pi_{\underline{\varphi}})$.

In order to apply our decomposition formula for $\pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})} \circ \eta$ and relate it to representations in Π_φ, we must first establish an appropriate relationship between the supercuspidal L-packet data and the admissible embeddings. This is given to us by Theorem 4.2, another key result of this paper, in which we show that for all $\underline{j} \in \underline{\mathcal{J}}_F$, there exists $j \in \mathcal{J}_F$ such that $\eta(jS) \subset \underline{j}\underline{S}$ and $j\theta = \underline{j}\underline{\theta} \circ \eta$. As such, we obtain a decomposition formula for $\pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})} \circ \eta$ in terms of certain conjugates of $\pi_{(jS,j\theta)}$. This completes the proof of Theorem 4.1.

The second goal of this paper is to provide a more detailed description of the decomposition of $\underline{\pi} \circ \eta$ in Theorem 4.1 in the special case that both φ and $\underline{\varphi}$ are *regular* supercuspidal L-parameters [Kaletha 2019, Definition 5.2.3]. The regularity assumptions on the L-parameters have several pleasant consequences. We list them for φ, with the understanding that their analogs hold for $\underline{\varphi}$. First, the representations $\pi_{(jS,j\theta)}$ are irreducible for all $j \in \mathcal{J}_F$. From this it follows that the set $\mathcal{J}_F$ parameterizes the representations in Π_φ. Second, the set $\mathcal{J}_F$ is in bijection with characters of the usual component group that one is accustomed to seeing in Langlands correspondences. More precisely, the component group of the centralizer of the image of φ is in bijection with the Galois-fixed subgroup of some torus $\widehat{S}$ [Kaletha 2019, Lemma 5.3.4], and certain characters of this component group are in bijection with $\mathcal{J}_F$ (see (17)). Since $\widehat{S}$ is abelian, so too is the component group.

The regularity assumptions consequently allow us to write

$$\underline{\pi} = \pi_{(\underline{j}\underline{S}, \underline{j}\underline{\theta})} = \pi_{(\underline{\varphi}, \underline{\varrho})},$$

where $\underline{j} \in \underline{\mathcal{J}}_F$ corresponds to a character $\underline{\varrho}$ of the component group for $\underline{\varphi}$. The map $\hat{\eta}$ sends the component group for $\underline{\varphi}$ to the one for φ. The precise description of $\underline{\pi} \circ \eta$ is given in Proposition 5.12. We summarize it here as follows.

Theorem 1.1. *Let $\eta : G \to \underline{G}$, $\underline{\varphi}$ and $\varphi = {}^L\eta \circ \underline{\varphi}$ be as in Theorem 4.1. Assume that φ and $\underline{\varphi}$ are regular. Then*

$$\pi_{(\underline{\varphi}, \underline{\varrho})} \circ \eta \simeq \bigoplus_{\varrho} \mathrm{Hom}(\underline{\varrho}, (\varrho \circ \hat{\eta})) \otimes \pi_{(\varphi, \varrho)},$$

where $\underline{\varrho}$ and ϱ are characters of the Langlands component groups.

Theorem 1.1 is the proof of a conjecture of Solleveld for regular supercuspidal L-parameters [2020, Conjecture 2]. Solleveld proved his conjecture in a variety of cases [2020, Theorem 3]. The only overlap of Theorem 1.1 with these cases is when G and $\underline{G}$ are inner forms of GL_n, SL_n or PGL_n.

One might hope that the regularity of $\varphi = {}^L\eta \circ \underline{\varphi}$ in Theorem 1.1 would follow from the regularity of $\underline{\varphi}$. While this is not true in general, as illustrated with a counterexample at the end of Section 5.1, the converse implication holds (Corollary 5.3). Furthermore, as explained after [Kaletha 2019, Definition 3.7.3], regular L-parameters are typical among all supercuspidal L-parameters.

Let us discuss how one might extend Theorem 1.1 to nonregular supercuspidal L-parameters. In this case, $\underline{\mathcal{J}}_F$ is no longer a parameterizing set for $\Pi_{\underline{\varphi}}$ since the representations $\pi_{(\underline{j}\underline{S}, \underline{j}\underline{\theta})}$, $\underline{j} \in \underline{\mathcal{J}}_F$, may be reducible. For each $\underline{j} \in \underline{\mathcal{J}}_F$, the irreducible components of $\pi_{(\underline{j}\underline{S}, \underline{j}\underline{\theta})}$ are parameterized by certain representations of $N(\underline{j}\underline{S}, \underline{G})(F)_{\underline{j}\underline{\theta}}$, the stabilizer of the pair $(\underline{j}\underline{S}, \underline{j}\underline{\theta})$ in $N(\underline{j}\underline{S}, \underline{G})(F)$ [Kaletha 2021, Corollary 3.4.7]. It appears that [Kaletha 2021, Proposition 4.3.2] serves as a bridge between $\{N(\underline{j}\underline{S}, \underline{G})(F)_{\underline{j}\underline{\theta}} : \underline{j} \in \underline{\mathcal{J}}_F\}$ and the component group of the centralizer of the image of $\underline{\varphi}$. Another key step in the proof of Theorem 1.1 is the decomposition formula for $\pi_{(\underline{j}\underline{S}, \underline{j}\underline{\theta})} \circ \eta$. Removing the regularity hypothesis means one would need to derive the decomposition formula of $\underline{\pi} \circ \eta$, where $\underline{\pi}$ is an irreducible component of $\pi_{(\underline{j}\underline{S}, \underline{j}\underline{\theta})}$. This would require a deeper study of the results in [Kaletha 2021, Section 3]. Once one has such a decomposition formula, we believe that similar arguments as the ones in Section 5.2 could be applied.

Let us briefly indicate what is required to extend Theorems 4.1 and 1.1 to nonquasisplit groups. Every connected reductive algebraic F-group G' is an inner form of a quasisplit form G. When char $F = 0$, the group G' may be assigned to a class of *rigid inner twists* for G [Kaletha 2016, Corollary 3.8 and Section 5.1]. This class is an element in a set of the form $H^1(u \to W, Z' \to G)$ which we shall not describe. For any $j \in \mathcal{J}_F$, there is a natural surjection

$$H^1(u \to W, Z' \to jS) \to H^1(u \to W, Z' \to G), \tag{1}$$

where jS is a maximal torus of G. The elements in a supercuspidal L-packet of G' are indexed by the fiber in $H^1(u \to W, Z' \to jS)$ over the class in $H^1(u \to W, Z' \to G)$ corresponding to G' [Kaletha 2019, Section 5.3]. If one is only interested in the quasisplit form, that is, $G' = G$, the classes of rigid inner twists may be chosen to equal the usual Galois cohomology sets (which we recall more precisely below), and a supercuspidal L-packet is indexed by the fiber of the more familiar map

$$H^1(F, jS) \to H^1(F, G) \tag{2}$$

over the trivial class. This fiber is in bijection with the set of admissible embeddings $\mathcal{J}_F$ above. In general, the fiber of (1) corresponding to G' is in bijection with the set $\mathcal{J}'_F$ of admissible embeddings into (the rigid inner twist for) $G'(F)$. When char $F \neq 0$, a parallel picture is given in [Dillery 2023]. The constructions and results of Sections 3 and 4 apply to $G'(F)$ and $\mathcal{J}'_F$ in exactly the same manner as they do to $G(F)$ and $\mathcal{J}_F$. More work is required to accommodate $G'(F)$ and $\mathcal{J}'_F$ in the constructions of Section 5. Rather than working with characters of $\pi_0(\widehat{S}^\Gamma/Z(\widehat{G})^\Gamma)$, one works with the characters of the larger group $\pi_0([\widehat{\bar{S}}]^+)$ which appear in [Kaletha 2019, Lemma 5.3.4; 2016, Corollary 5.4]. The admissible embeddings $\mathcal{J}'_F$ correspond to certain characters of $\pi_0([\widehat{\bar{S}}]^+)$ and these characters correspond to the representations in the L-packet for G'. A discussion of such matters may be found in [Kaletha 2016, Section 5.4]. In view of the length of this paper, which deals only with quasisplit groups, it seems prudent to leave the treatment of nonquasisplit groups to some future work.

The paper is organized as follows. Section 2 contains preliminaries, beginning with an outline of the notation and conventions used throughout this paper (Section 2.1). We also present results concerning the structure theory of G, G_Z and $\underline{G}$ (Sections 2.2 and 2.3), and provide summaries for the Kaletha–Yu construction of supercuspidal representations (Section 2.4) as well as Kaletha's construction of supercuspidal L-packets (Section 2.5). In Section 3, we prove Theorem 3.1, which describes the decomposition of $\pi_{(\underline{S},\underline{\theta})} \circ \eta$. Most of the section (Sections 3.1 and 3.2) focuses on proving the second part of the two-step restriction illustrated in Figure 1, that is, describing restrictions from $G_Z(F)$ to $G(F)/Z(F)$. In particular, a deep analysis of the Kaletha–Yu construction is required, and we show that the J.-K. Yu construction essentially commutes with η. In Section 4, we prove the functoriality of Kaletha's supercuspidal L-packets (Theorem 4.1). We first establish the relationship between the supercuspidal L-packet data associated to φ and $\underline{\varphi}$ (Section 4.1) and end with the proof of Theorem 4.1 (Section 4.2). In Section 5, we start by describing the regular supercuspidal L-parameters and their corresponding L-packet structure, as well as a discussion on when one might expect both parameters φ and $\underline{\varphi}$ to be regular (Section 5.1). We then proceed to reparameterize the L-packets in terms of characters of their corresponding component groups and proving Theorem 1.1 (Section 5.2).

2. Preliminaries

We set up results concerning the structure theory of G, G_Z and $\underline{G}$, and summarize the constructions that will be needed in this paper. We begin with notation and conventions in Section 2.1, after which we discuss how the structure theory of G_Z and that of G relate in Section 2.2. In Section 2.3, we describe an action of $\underline{G}(F)$ on representations of $G(F)$. In Section 2.4, we summarize the Kaletha–Yu construction, which produces supercuspidal representations from F-nonsingular elliptic pairs. We summarize Kaletha's construction of supercuspidal L-packets in Section 2.5.

2.1. ***Notation and conventions.*** Given the nonarchimedean local field F, we denote by $\mathcal{O}_F$ its ring of integers, $\mathfrak{p}_F$ the unique maximal ideal of $\mathcal{O}_F$ and $\mathfrak{f}$ its residue field of prime characteristic p. Let F^{un} be a maximal unramified extension of F. The residue field of F^{un} is an algebraic closure of $\mathfrak{f}$, so we denote it by $\bar{\mathfrak{f}}$. The Galois group $\mathrm{Gal}(F^{\mathrm{un}}/F)$ is canonically isomorphic to $\mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$, and we denote the Frobenius automorphism by Fr. Let $\Gamma = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ denote the Galois groups of F, where F^{sep} is a separable closure of F. We use the notation I_F and P_F for the inertia subgroup and wild inertia subgroup of the Weil group W_F, respectively. We also let E denote the tamely ramified extension of F over which G splits.

In this paper, we will encounter different types of cohomology groups. Given an algebraic group G' that is defined over a field F', we take $G'(F')$ to be the set of F'-points in the sense of [Springer 2009, Section 2.1]. For G' defined over F we write $H^1(F, G')$ for $H^1(\Gamma, G'(F^{\mathrm{sep}}))$. Similarly, given an algebraic group $\mathcal{G}'$ that is defined over $\mathfrak{f}$, we write $H^1(\mathfrak{f}, \mathcal{G}')$ for $H^1(\mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f}), \mathcal{G}'(\bar{\mathfrak{f}}))$. Furthermore, given a group $\widetilde{G}$ with $\mathrm{Gal}(F^{\mathrm{un}}/F)$-action, we write $H^1(\mathrm{Fr}, \widetilde{G})$ for $H^1(\mathrm{Gal}(F^{\mathrm{un}}/F), \widetilde{G})$ and $\widetilde{G}^{\mathrm{Fr}}$ for $\widetilde{G}^{\mathrm{Gal}(F^{\mathrm{un}}/F)}$.

Given a maximal torus T of G, we let $R(G, T)$ denote the root system of G with respect to T. Given $\alpha \in R(G, T)$, we denote the associated root subgroup by U_α. Letting $\underline{T}$ denote the maximal torus of $\underline{G}$ such that $\eta(T) = \underline{T} \cap \eta(G)$ (given by [Bourgeois 2021, Theorem 2.2]), the root systems $R(G, T)$ and $R(\underline{G}, \underline{T})$ are identified by η, and the Weyl groups of G and $\underline{G}$ coincide. We use $\mathfrak{g}$ and $\underline{\mathfrak{g}}$ for the Lie algebras of G and $\underline{G}$, respectively.

Furthermore, given groups $H_1 \subset H_2$, $h \in H_2$ and a representation γ of H_1, we let ${}^hH_1 := \mathrm{Ad}(h)(H_1) = hH_1h^{-1}$ and ${}^h\gamma := \gamma \circ \mathrm{Ad}(h^{-1})$.

The reader is assumed to be familiar with the structure theory of p-adic groups. Following the notation from [Kaletha 2019], we write $\mathcal{B}(G, F)$ for the reduced building of G over F and $\mathcal{A}(G, T, F)$ for the apartment associated to any maximal torus T of G which is maximally split. For each $x \in \mathcal{B}(G, F)$, we set $G(F)_x$ to be the stabilizer of x in $G(F)$. Furthermore, for $r > 0$, $G(F)_{x,r}$ denotes the Moy–Prasad filtration subgroup of the parahoric subgroup $G(F)_{x,0}$. We will be using Kaletha and Prasad's definitions [2023, Definition 13.2.1], which coincide with the ones of Moy and Prasad given our tameness assumption [2023, p. XXV]. In particular, we have $E_r^\times = 1 + \mathfrak{p}_E^{\lceil er \rceil}$, where e denotes the ramification degree of E/F. We also set $G(F)_{x,r^+} = \bigcup_{t>r} G(F)_{x,t}$. We use colons to abbreviate quotients, that is $G(F)_{x,r:t} = G(F)_{x,r}/G(F)_{x,t}$ for $t > r$. We have analogous filtrations of $\mathcal{O}_F$-submodules at the level of the Lie algebra.

For all $r > 0$, the quotient $G(F)_{x,r:r^+}$ is an abelian group and is isomorphic to its Lie algebra analog $\mathfrak{g}(F)_{x,r:r^+}$ via Adler's mock exponential map [1998]. The quotient $G(F)_{x,0:0^+}$ is also very important, as it results in the $\mathfrak{f}$-points of a reductive group $\mathcal{G}$, which we refer to as the *reductive subquotient of G at x*.

The construction of $\mathcal{G}$ is summarized in [Kaletha and Prasad 2023, Section 8.4.2]. One starts with the relative identity component $\mathcal{G}_x$ of a $\mathcal{O}_F$-group scheme associated to x, whose existence is guaranteed by [Kaletha and Prasad 2023, Proposition 8.3.1 and Section 9.2.5]. One then takes the special fiber $\overline{\mathcal{G}}_x$ of $\mathcal{G}_x$, and defines $\mathcal{G}$ to be the quotient by its unipotent radical, $\mathcal{G} := \overline{\mathcal{G}}_x / R_u(\overline{\mathcal{G}}_x)$. By [Kaletha and Prasad 2023, Theorem 8.3.13], $G(F^{\mathrm{un}})_{x,0} = \mathcal{G}_x(\mathcal{O}_{F^{\mathrm{un}}})$. The projection map

$$G(F^{\mathrm{un}})_{x,0} = \mathcal{G}_x(\mathcal{O}_{F^{\mathrm{un}}}) \to \overline{\mathcal{G}}_x(\bar{\mathfrak{f}}) \tag{3}$$

is surjective, and the preimage of $R_u(\overline{\mathcal{G}}_x)(\bar{\mathfrak{f}})$ under this map is equal to $G(F^{\mathrm{un}})_{x,0^+}$ [Kaletha and Prasad 2023, Corollary 8.4.12], whence $\mathcal{G}(\bar{\mathfrak{f}}) \simeq G(F^{\mathrm{un}})_{x,0:0^+}$.

There is a natural action of $\mathrm{Gal}(F^{\mathrm{un}}/F)$ on $\mathcal{G}_x(\mathcal{O}_{F^{\mathrm{un}}})$ and a natural action of $\mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$ on $\overline{\mathcal{G}}_x(\bar{\mathfrak{f}})$ and the map (3) is Galois-equivariant with respect to these actions. The Galois-equivariance passes to the isomorphism $G(F^{\mathrm{un}})_{x,0:0^+} \simeq \mathcal{G}(\bar{\mathfrak{f}})$.

All of the objects described above have their analogs in G_Z and $\underline{G}$, which we will denote using the subscript Z and the underline, respectively.

2.2. ***Structure theory of G_Z in relation to G.*** In this section, η is the map on G, as stated in the introduction (Theorem 4.1) and $G_Z = \eta(G) \simeq G/Z$. We let (S, θ) and (S_Z, θ_Z) be tame F-nonsingular elliptic pairs of G and G_Z, respectively, which satisfy $\eta(S) = S_Z$ and $\theta = \theta_Z \circ \eta$. The goal of this section is to compare certain subgroups of $G(F)$ to their analogs in $G_Z(F)$. In particular, we focus on Moy–Prasad filtration subgroups and reductive subquotients of G and G_Z, respectively.

These results will be useful when computing a decomposition formula for $\pi_{(S,\theta)} \circ \eta$ in Section 3.

First, we note that $\eta : G \to G_Z$ induces an equivariant isomorphism $\eta_{\mathcal{B}} : \mathcal{B}(G, F) \to \mathcal{B}(G_Z, F)$ by [Kaletha and Prasad 2023, Axiom 4.1.1]. In particular, this map satisfies $\eta_{\mathcal{B}}(gx) = \eta(g) \cdot \eta_{\mathcal{B}}(x)$ for all $g \in G(F)$, $x \in \mathcal{B}(G, F)$, where $\cdot$ refers to the action of $G(F)$ on $\mathcal{B}(G, F)$. For the rest of this section, let us fix $y \in \mathcal{B}(G, F)$ and set $y_Z = \eta_{\mathcal{B}}(y)$. We begin with a result stating a relationship between $G_Z(F)_{y_z}$ and $G(F)_y$.

Lemma 2.1. *We have $\eta(G(F)_y) = G_Z(F)_{y_Z} \cap \eta(G(F))$. Furthermore, $\eta(G(F)_y)$ is normal in $G_Z(F)_{y_Z}$.*

Proof. We start by showing that $\eta(G(F)_y)$ is a normal subgroup of $G_Z(F)_{y_Z}$. Let $g \in G(F)_y$. Then $g \cdot y = y$ by definition. It follows that $\eta_{\mathcal{B}}(g \cdot y) = \eta_{\mathcal{B}}(y)$, or equivalently $\eta(g) \cdot y_Z = y_Z$. Thus $\eta(G(F)_y) \subset G_Z(F)_{y_Z}$. For normality, take $g \in G(F)_y$ and $g_Z \in G_Z(F)_{y_Z}$. Since $\eta(G(F))$ is a normal subgroup of $G_Z(F)$, it follows that $g_Z \eta(g) g_Z^{-1} = \eta(g')$ for some $g' \in G(F)$. By what precedes, we also have $\eta(g') \in G_Z(F)_{y_Z}$, or equivalently $\eta(g') \cdot y_Z = y_Z$. Since the map $\eta_{\mathcal{B}}$ is a bijection, it follows that $g' \cdot y = y$. Thus, $g' \in G(F)_y$ and $g_Z \eta(g) g_Z^{-1} \in \eta(G(F)_y)$. For the intersection, it is clear that $G_Z(F)_{y_Z} \cap \eta(G(F)) \supset \eta(G(F)_y)$. Conversely, take $g_Z \in G_Z(F)_{y_Z} \cap \eta(G(F))$. Then $g_Z = \eta(g')$ for some $g' \in G(F)$ and $\eta(g') \cdot y_Z = y_Z$. Using the bijectivity of $\eta_{\mathcal{B}}$, it follows that $g' \in G(F)_y$, and thus $g_Z \in \eta(G(F)_y)$. □

Next, we describe the relationship between the Moy–Prasad filtration subgroups.

Lemma 2.2. *For all $r > 0$ we have $\eta(G(F)_{y,r}) = G_Z(F)_{y_Z,r}$.*

Proof. Let $r > 0$. Following the proof of [Kaletha 2019, Lemma 3.3.2], use [Bruhat and Tits 1972, Lemma 6.4.48] to write $G(F)_{y,r}$ as the direct product of (topological spaces) $T(F)_r$ and the appropriate affine root subgroups. Here T is a maximally unramified maximally split maximal torus, whose existence is guaranteed by [Bruhat and Tits 1984, Corollary 5.1.12]. Since η induces an isomorphism on the affine root subgroups, it suffices to show that $\eta(T(F)_r) = T_Z(F)_r$, where $T_Z = \eta(T) \simeq T/Z$. To do so, let Z° denote the identity component of Z. The map η factors as follows:

$$\begin{array}{ccc} T & \xrightarrow{\quad\eta\quad} & T_Z \simeq (T/Z^\circ)/(Z/Z^\circ) \\ & \searrow^{\eta^\circ} \quad \nearrow_{\bar{\eta}} & \\ & T/Z^\circ & \end{array}$$

We have that Z° is a torus by [Humphreys 1975, Theorem 16.2] as it is a closed and connected subgroup of the torus $Z(G)^\circ$. By [Kaletha 2019, Lemma 3.1.3], we have an exact sequence

$$1 \to Z^\circ(F)_r \to T(F)_r \to (T/Z^\circ)(F)_r \to 1,$$

implying that $(T/Z^\circ)(F)_r \simeq T(F)_r/Z^\circ(F)_r \simeq \eta^\circ(T(F)_r)$. Furthermore, since $\bar{\eta}: T/Z^\circ \to T_Z$ is an isogeny, [Kaletha 2019, Lemma 3.1.3] tells us that

$$\bar{\eta}((T/Z^\circ)(F)_r) = T_Z(F)_r.$$

Combining these two equations allows us to conclude that $\eta(T(F)_r) = T_Z(F)_r$. □

Remark 2.3. For all $r>0$, the map η induces a surjection $G(F)_{y,r:r^+} \to G_Z(F)_{y_Z,r:r^+}$, $0 \le i \le d$. At the depth-zero level, we can only guarantee an inclusion. In other words, $\eta(G(F))_{y,0} \subset G_Z(F)_{y_Z,0}$. This induces a homomorphism $G(F)_{y,0:0^+} \to G_Z(F)_{y_Z,0:0^+}$.

Now, let $\mathcal{G}$ and $\mathcal{G}_Z$ denote the reductive subquotients of G and G_Z at y and y_Z, respectively (see discussion surrounding (3)). In light of Remark 2.3, the map η induces a map between $\mathcal{G}$ and $\mathcal{G}_Z$, which we describe as follows.

Since η induces a homomorphism $G(F^{\text{un}})_{y,0} \to G_Z(F^{\text{un}})_{y_Z,0}$, we can compose with the quotient map to obtain a homomorphism

$$\begin{aligned} G(F^{\text{un}})_{y,0} &\to G_Z(F^{\text{un}})_{y_Z,0:0^+}, \\ g &\mapsto \eta(g)G_Z(F^{\text{un}})_{y,0^+}. \end{aligned}$$

The kernel of this homomorphism is $(Z \cap G(F^{\text{un}})_{y,0})G(F^{\text{un}})_{y,0^+}$, resulting in an embedding

$$G(F^{\text{un}})_{y,0}/(Z \cap G(F^{\text{un}})_{y,0})G(F^{\text{un}})_{y,0^+} \hookrightarrow G_Z(F^{\text{un}})_{y_Z,0:0^+}.$$

By the third isomorphism theorem, the domain of the previous embedding is isomorphic to

$$G(F^{\text{un}})_{y,0:0^+}/\big((Z \cap G(F^{\text{un}})_{y,0})G(F^{\text{un}})_{y,0^+}/G(F^{\text{un}})_{y,0^+}\big).$$

Given that $Z \subset Z(G)$, it follows that $(Z \cap G(F^{\text{un}})_{y,0})G(F^{\text{un}})_{y,0^+}/G(F^{\text{un}})_{y,0^+}$ is a closed central subgroup of $G(F^{\text{un}})_{y,0:0^+}$. As such, it corresponds to a closed central subgroup of $\mathcal{G}(\bar{\mathfrak{f}})$, which we denote by $\mathcal{Z}(\bar{\mathfrak{f}})$. Given that we identify reductive groups with their $\bar{\mathfrak{f}}$-points, what we have just described is an embedding

$$\bar{\eta}: \mathcal{G}/\mathcal{Z} \hookrightarrow \mathcal{G}_Z.$$

We expect this embedding to be surjective when the groups are split. Even though we do not have a counterexample, it is not clear to us whether it is surjective in general. Note that the groups of the embedding are defined over $\mathfrak{f}$. The embedding is $\text{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$-equivariant, as η is $\text{Gal}(F^{\text{un}}/F)$-equivariant. Furthermore, $\mathcal{Z}$ is defined over $\mathfrak{f}$ by [Springer 2009, Corollary 12.1.3]. Thus, the map between $\mathcal{G}$ and $\mathcal{G}_Z$

(which is defined over $\mathfrak{f}$) is given by

$$\begin{array}{ccccc} \mathcal{G} & \xrightarrow{\quad q \quad} & \mathcal{G}/\mathcal{Z} & \xhookrightarrow{\quad \bar{\eta} \quad} & \mathcal{G}_Z \\ \updownarrow\simeq & & \updownarrow\simeq & & \updownarrow\simeq \\ G(F^{\mathrm{un}})_{y,0:0+} & \longrightarrow & G(F^{\mathrm{un}})_{y,0}/(Z\cap G(F^{\mathrm{un}})_{y,0})G(F^{\mathrm{un}})_{y,0+} & \hookrightarrow & G_Z(F^{\mathrm{un}})_{y_Z,0:0+} \\ gG(F^{\mathrm{un}})_{y,0+} & & \longmapsto & & \eta(g)G_Z(F^{\mathrm{un}})_{y_Z,0+} \end{array} \tag{4}$$

where q is the obvious quotient map. To alleviate notation, we will keep the isomorphisms implicit and say that an element of $\mathcal{G}(\bar{\mathfrak{f}})$ (or $\mathcal{G}(\mathfrak{f})$) is of the form $gG(F^{\mathrm{un}})_{y,0+}$ for some $g\in G(F^{\mathrm{un}})_{y,0}$ (or $gG(F)_{y,0+}$ for some $g\in G(F)_{y,0}$).

Remark 2.4. The map $\bar{\eta}\circ q$ as defined above corresponds to the restriction of the map

$$G(F^{\mathrm{un}})_y/G(F^{\mathrm{un}})_{y,0+}\to G_Z(F^{\mathrm{un}})_{y_Z}/G_Z(F^{\mathrm{un}})_{y_Z,0+},$$
$$gG(F^{\mathrm{un}})_{y,0+}\mapsto \eta(g)G_Z(F^{\mathrm{un}})_{y_Z,0+}.$$

We will abuse notation and also denote this map by $\bar{\eta}\circ q$ when called upon.

We now prove two elementary results involving the maps $\bar{\eta}$ and q.

Lemma 2.5. *Let $\bar{\eta}$ and $\mathcal{Z}$ be as above. Then $\bar{\eta}(\mathcal{G}/\mathcal{Z})\supseteq[\mathcal{G}_Z,\mathcal{G}_Z]$.*

Proof. We identify the reductive groups with their $\bar{\mathfrak{f}}$-points. Based on the definitions above, we have

$$\bar{\eta}(\mathcal{G}(\bar{\mathfrak{f}})/\mathcal{Z}(\bar{\mathfrak{f}}))=\eta(G(F^{\mathrm{un}})_{y,0})G_Z(F^{\mathrm{un}})_{y_Z,0+}/G_Z(F^{\mathrm{un}})_{y_Z,0+}.$$

Let S'_Z be a maximal F^{un}-split torus of G_Z and let T_Z be its centralizer in G_Z. By definition (see, for example, [Fintzen 2021, Section 2.4] or [Kaletha and Prasad 2023, Definition 13.2.1]),

$$G_Z(F^{\mathrm{un}})_{y,0}=\langle T_Z(F^{\mathrm{un}})_0,\,U_\alpha(F^{\mathrm{un}}):\alpha\in R^{\mathrm{aff}}_{F^{\mathrm{un}}},\,\langle\alpha,\ y_Z\rangle\geq 0\rangle,$$

where $R^{\mathrm{aff}}_{F^{\mathrm{un}}}=\{\lambda+k:\lambda\in R(G_Z,T_Z)\text{ such that }\lambda|_{S'_Z}\neq 1,\ k\in\mathbb{R}\}$, and $U_\alpha(F^{\mathrm{un}})$ is the affine root subgroup associated to the affine root α. Given that root subgroups are normalized by toral elements, it follows that $[G_Z(F^{\mathrm{un}})_{y_Z,0},G_Z(F^{\mathrm{un}})_{y_Z,0}]$ consists only of products of root subgroup elements. Since η induces an isomorphism on the affine root subgroups of G and G_Z, we conclude that $[G_Z(F^{\mathrm{un}})_{y_Z,0},G_Z(F^{\mathrm{un}})_{y_Z,0}]\subseteq\eta(G(F^{\mathrm{un}})_{y,0})$. It follows that

$$\begin{aligned}[\mathcal{G}_Z(\bar{\mathfrak{f}}),\mathcal{G}_Z(\bar{\mathfrak{f}})]&=[G_Z(F^{\mathrm{un}})_{y_Z,0},G_Z(F^{\mathrm{un}})_{y_Z,0}]G_Z(F^{\mathrm{un}})_{y_Z,0+}/G_Z(F^{\mathrm{un}})_{y_Z,0+}\\&\subseteq\bar{\eta}(\mathcal{G}_Z(\bar{\mathfrak{f}})/\mathcal{Z}(\bar{\mathfrak{f}})).\end{aligned}$$ ∎

2.3. ***The action of $\underline{G}(F)$ on $G(F)$.*** The decomposition formula for $\pi_{(\underline{S},\underline{\theta})} \circ \eta$ involves an action of $\underline{G}(F)$ on representations of $G(F)$. The purpose of this section is to describe this action.

Let $\underline{g} \in \underline{G}(F)$. Using $\underline{G} = G_Z Z(\underline{G})$, write $\underline{g} = g_Z z$ for some $g_Z \in G_Z$, $z \in Z(\underline{G})$. Since G_Z is the image of η, there exists $g \in G$ such that $g_Z = \eta(g)$. It follows that $\mathrm{Ad}(\underline{g}) = \mathrm{Ad}(\eta(g))$. We also set $\underline{\mathrm{Ad}}(\underline{g}) := \mathrm{Ad}(g)$, an automorphism of G.

Lemma 2.6. *For all $\underline{g} \in \underline{G}(F)$, $\underline{\mathrm{Ad}}(\underline{g}) \in \mathrm{Aut}(G(F))$ is defined over F. Furthermore,*

$$\underline{\mathrm{Ad}} : \underline{G}(F) \to \mathrm{Aut}(G(F)),$$
$$\underline{g} \mapsto \underline{\mathrm{Ad}}(\underline{g}),$$

is a well-defined homomorphism.

Proof. It is clear that $\underline{\mathrm{Ad}}(\underline{g})$ maps G to G. We first prove that $\underline{\mathrm{Ad}}(\underline{g})$ is defined over F^{sep}. According to [Springer 2009, 12.3.3], a quotient map carries the F^{sep}-points of its domain surjectively onto the F^{sep}-points of its image. Now, the group $\underline{G}$ is the quotient of $G_Z \times Z(\underline{G})$ by $G_Z \cap Z(\underline{G})$. Therefore $\underline{g} = g_Z z$ where $g_Z \in G_Z(F^{\mathrm{sep}})$ and $z \in Z(\underline{G})(F^{\mathrm{sep}})$. Our map η is also a quotient map so g_Z can be written as $\eta(g)$ where $g \in G(F^{\mathrm{sep}})$. Consequently $\mathrm{Ad}(g) = \underline{\mathrm{Ad}}(\underline{g})$ is defined over F^{sep}.

To conclude that $\underline{\mathrm{Ad}}(\underline{g})$ is defined over F (and therefore maps $G(F)$ to $G(F)$), we show that $\underline{\mathrm{Ad}}(\underline{g}) \circ \sigma = \sigma \circ \underline{\mathrm{Ad}}(\underline{g})$ for all $\sigma \in \Gamma$. Recall that $\underline{\mathrm{Ad}}(\underline{g}) = \mathrm{Ad}(g)$, where $g \in G$ is such that $\underline{g} = \eta(g)z$ for some $z \in Z(\underline{G})$. Since η is defined over F and $\underline{g} \in \underline{G}(F)$, we have

$$\begin{aligned} \eta \circ \sigma \circ \underline{\mathrm{Ad}}(\underline{g}) &= \sigma \circ \mathrm{Ad}(\eta(g)) \circ \eta \\ &= \sigma \circ \mathrm{Ad}(\underline{g}) \circ \eta \\ &= \mathrm{Ad}(\underline{g}) \circ \eta \circ \sigma \\ &= \mathrm{Ad}(\eta(g)) \circ \eta \circ \sigma \\ &= \eta \circ \underline{\mathrm{Ad}}(\underline{g}) \circ \sigma. \end{aligned}$$

Given $x \in G$, the previous equality implies $(\sigma \circ \underline{\mathrm{Ad}}(\underline{g}))(x) = (\underline{\mathrm{Ad}}(\underline{g}) \circ \sigma)(x) z_x$ for some $z_x \in Z$. Define the map

$$f : G \to Z,$$
$$x \mapsto z_x.$$

This is a homomorphism, and is trivial on $Z(G)$. Furthermore, because Z is abelian, f is also trivial on $[G, G]$. Thus, f is trivial on $G = [G, G]Z(G)$, and $z_x = 1$ for all $x \in G$. We conclude that $\underline{\mathrm{Ad}}(\underline{g}) \circ \sigma = \sigma \circ \underline{\mathrm{Ad}}(\underline{g})$, as desired. To show that the map $\underline{\mathrm{Ad}}$ is well defined, assume $\underline{g} = \eta(g_1)z_1 = \eta(g_2)z_2$, where $g_1, g_2 \in G$, $z_1, z_2 \in Z(\underline{G})$. It follows that $\eta(g_1 g_2^{-1}) = z_1^{-1} z_2 \in Z(\underline{G}) \cap G_Z \subset Z(G_Z)$, and therefore $g_1 g_2^{-1} \in Z(G)$. We conclude that $\mathrm{Ad}(g_1) = \mathrm{Ad}(g_2)$, and thus $\underline{\mathrm{Ad}}(\underline{g})$ is well defined. It is straightforward to show that $\underline{\mathrm{Ad}}$ is a homomorphism. □

Corollary 2.7. *The group $\eta(G(F))$ is normal in $\underline{G}(F)$.*

Proof. Let $h \in G(F)$ and $\underline{g} \in \underline{G}(F)$. We show that $\mathrm{Ad}(\underline{g})(\eta(h)) \in \eta(G(F))$. Following the notation above, we have that $\mathrm{Ad}(\underline{g}) = \mathrm{Ad}(\eta(g))$ for some $g \in G$. It follows that

$$\mathrm{Ad}(\underline{g})(\eta(h)) = (\eta \circ \mathrm{Ad}(g))(h) = (\eta \circ \underline{\mathrm{Ad}}(\underline{g}))(h).$$

By the previous lemma, $\underline{\mathrm{Ad}}(\underline{g})$ is defined over F, which implies $\underline{\mathrm{Ad}}(\underline{g})(h) \in G(F)$. Thus, we conclude that

$$\mathrm{Ad}(\underline{g})(\eta(h)) \in \eta(G(F)). \qquad \square$$

The following lemma will also be useful in proving the main statements of this section.

Lemma 2.8. *Let π_Z be a representation of $G_Z(F)$ and $\underline{g} \in \underline{G}(F)$. Then*

$$^{\underline{g}}\pi_Z \circ \eta = \pi_Z \circ \eta \circ \underline{\mathrm{Ad}}(\underline{g}^{-1}).$$

Proof. We have $\underline{\mathrm{Ad}}(\underline{g}) = \mathrm{Ad}(g)$, where $g \in G$ satisfies $\underline{g} = \eta(g)z$ for some $z \in Z(\underline{G})$. For all $h \in G(F)$,

$$\begin{aligned}(\pi_Z \circ \eta \circ \underline{\mathrm{Ad}}(\underline{g}^{-1}))(h) &= (\pi_Z \circ \eta)(g^{-1}hg) \\ &= \pi_Z(\eta(g)^{-1}\eta(h)\eta(g)) \\ &= (^{\eta(g)}\pi_Z \circ \eta)(h) \\ &= (^{\underline{g}}\pi_Z \circ \eta)(h). \qquad \square\end{aligned}$$

2.4. ***Summary of the Kaletha–Yu construction.*** Let us recall the construction of supercuspidal representations from tame F-nonsingular elliptic pairs as per [Kaletha 2019; 2021], which we refer to as the Kaletha–Yu construction. For simplicity of notation, we will describe the construction over G, though it is also applied to $G_Z \simeq G/Z$ and $\underline{G}$.

The construction of the supercuspidal representation $\pi_{(S,\theta)}$ of G starts from a tame F-nonsingular elliptic pair (S, θ) in the sense of [Kaletha 2021, Definition 3.4.1]. The representation $\pi_{(S,\theta)}$ is obtained in two steps. One starts by unfolding the pair (S, θ) into a G-datum $\Psi_{(S,\theta)} = (\vec{G}, y, \vec{r}, \rho, \vec{\phi})$ in the sense of [Yu 2001, Section 3]. We will refer to $\Psi_{(S,\theta)}$ as the *corresponding G-datum* of (S, θ). The properties of S and θ provided by [Kaletha 2021, Definition 3.4.1] allow us to go to the reductive subquotient and use the theory of Deligne–Lusztig cuspidal representations in order to construct ρ, the so-called *depth-zero* part of the datum $\Psi_{(S,\theta)}$. The second step consists of applying the J.-K. Yu construction [2001] on the obtained G-datum. The unfolding of the tame F-nonsingular elliptic pair into a G-datum is given as follows.

2.4.1. *The twisted Levi sequence $\vec{G}$ and the sequence $\vec{r}$.* We recall how to construct a Levi sequence from S as per [Kaletha 2019, Section 3.6]. We consider the set $R_r := \{\alpha \in R(G, S) : \theta(N_{E/F}(\check{\alpha}(E_r^\times))) = 1\}$,where $\check{\alpha}$ is the coroot associated to α and $N_{E/F}$ is the norm of E/F, and set $R_{r^+} = \bigcap_{s>r} R_s$. There will be breaks in this filtration, $r_{d-1} > r_{d-2} > \cdots > r_0 > 0$. We set $r_{-1} = 0$ and $r_d = \mathrm{depth}(\theta)$. For all $0 \le i \le d$, $G^i := \langle S, U_\alpha : \alpha \in R_{r_{i-1}^+} \rangle$ is a tamely ramified twisted Levi subgroup of G [Kaletha 2019, Lemma 3.6.1]. These twisted Levi subgroups are what we use to form the twisted Levi sequence $\vec{G} = (G^0, \dots, G^d)$. We also set $G^{-1} = S$ and $\vec{r} = (r_0, \dots, r_d)$.

2.4.2. *The character sequence $\vec{\phi}$.* By [Kaletha 2019, Proposition 3.6.7], given the character θ of $S(F)$, there exists a Howe factorization, that is, a sequence of characters $\phi^i : G^i(F) \to \mathbb{C}^\times$ for $i = -1, \dots, d$ such that:

(1) $\theta = \prod_{i=-1}^{d} \phi^i|_{S(F)}$.

(2) For all $0 \le i \le d$, ϕ^i is trivial on the simply connected cover of G^i.

(3) For all $0 \le i < d$, ϕ^i is a $G^{i+1}(F)$-generic character of depth r_i in the sense of [Hakim and Murnaghan 2008, Definition 3.9]. For $i = d$, ϕ^d is trivial if $r_d = r_{d-1}$ and has depth r_d otherwise. For $i = -1$, ϕ^{-1} is trivial if $G^0 = S$ and otherwise satisfies $\phi^{-1}|_{S(F)_{0+}} = 1$.

Given such a factorization, we set $\vec{\phi} = (\phi^0, \dots, \phi^d)$.

2.4.3. *The point y.* Since (S, θ) is a tame F-nonsingular elliptic pair, the torus S is a maximally unramified elliptic maximal F-torus of G^0 in the sense of [Kaletha 2019, Definition 3.4.2]. As such, we can associate to it a vertex y of $\mathcal{B}(G^0, F) \subset \mathcal{B}(G, F)$ [Kaletha 2019, Lemma 3.4.3], which is the unique $\mathrm{Gal}(F^{\mathrm{un}}/F)$-fixed point of $\mathcal{A}(G^0, S, F^{\mathrm{un}})$ [Kaletha and Prasad 2023, Section 17.8].

2.4.4. *The representation ρ.* Let $\mathcal{G}^0$ denote the reductive subquotient of G^0 at y, that is, the connected reductive $\mathfrak{f}$-group such that $\mathcal{G}^0(\bar{\mathfrak{f}}) \simeq G^0(F^{\mathrm{un}})_{y,0:0^+}$ and $\mathcal{G}^0(\mathfrak{f}) \simeq G^0(F)_{y,0:0^+}$, as recalled at the end of Section 2.1.

By [Kaletha 2019, Lemma 3.4.4], there exists an elliptic maximal $\mathfrak{f}$-torus $\mathcal{S}$ of $\mathcal{G}^0$ such that for every unramified extension F' of F, the image of $S(F')_0$ in $G(F')_{y,0:0^+}$ is isomorphic to $\mathcal{S}(\mathfrak{f}')$. For every character $\bar{\chi}$ of $\mathcal{S}(\mathfrak{f})$, one can construct a virtual character of $\mathcal{G}(\mathfrak{f})$ as per [Deligne and Lusztig 1976], which we denote by $R_{\mathcal{S},\bar{\chi}}$. When $\bar{\chi}$ is nonsingular in the sense of [Deligne and Lusztig 1976, Definition 5.15], $\pm R_{\mathcal{S},\bar{\chi}}$ is a Deligne–Lusztig cuspidal representation of $\mathcal{G}(\mathfrak{f})$ [1976, Proposition 7.4, Theorem 8.3]. The sign $\pm$ refers to $(-1)^{r_{\mathfrak{f}}(\mathcal{G}^0) - r_{\mathfrak{f}}(\mathcal{S})}$, where $r_{\mathfrak{f}}(\mathcal{G}^0)$ and $r_{\mathfrak{f}}(\mathcal{S})$ denote the $\mathfrak{f}$-split ranks of $\mathcal{G}^0$ and $\mathcal{S}$, respectively. The character ϕ^{-1} factors through to a character $\bar{\phi}^{-1}$ of $S(F)/S(F)_{0^+}$, which restricts to a character of $\mathcal{S}(\mathfrak{f})$. By [Kaletha 2019, Lemma 3.4.14], this character of $\mathcal{S}(\mathfrak{f})$ is nonsingular, meaning that the virtual character $\pm R_{\mathcal{S},\bar{\phi}^{-1}}$ is a (possibly reducible) Deligne–Lusztig cuspidal

$$(\pm R_{\mathcal{S},\bar{\phi}^{-1}}, \mathcal{G}^0(\mathfrak{f})) \xmapsto{\text{pullback, extend}} (\kappa_{(S,\phi^{-1})}, S(F)G^0(F)_{y,0}) \xmapsto{\text{induce}} (\rho, G^0(F)_y)$$

Figure 2. Summary of the construction of ρ.

representation of $\mathcal{G}^0(\mathfrak{f})$. The pullback of $\pm R_{\mathcal{S},\bar{\phi}^{-1}}$ to $G^0(F)_{y,0}$ then gets extended to a representation $\kappa_{(S,\phi^{-1})}$ of $S(F)G^0(F)_{y,0}$. This extension process is explained in [Kaletha 2019, Section 3.4.4] as well as [Kaletha 2021, Remark 2.6.5, p. 35], and will be recalled when needed in Corollary 3.12. We then define

$$\rho := \operatorname{Ind}_{S(F)G^0(F)_{y,0}}^{G^0(F)_y} \kappa_{(S,\phi^{-1})}.$$

The construction of ρ is summarized in Figure 2. Note that we are following the notation from [Kaletha 2019] in the paragraph above. What we have denoted by ρ is denoted by $\kappa_{(S,\phi^{-1})}$ in [Kaletha 2021, Section 3.3].

Once we have the G-datum $\Psi_{(S,\theta)} = (\vec{G}, y, \vec{r}, \rho, \vec{\phi})$, we apply the J.-K. Yu construction to obtain the supercuspidal representation $\pi_{(S,\theta)}$. We do not recall all the details of this construction, but provide a summary in the form of a diagram (Figure 3). We invite the reader to consult [Bourgeois 2021, Section 3] for a brief description of the steps involved. We point out that it is sometimes convenient to write $\kappa_G(\Psi_{(S,\theta)})$ for $\kappa_{(S,\theta)}$ and $\pi_G(\Psi_{(S,\theta)})$ for $\pi_{(S,\theta)}$ to indicate that we are applying the J.-K. Yu construction on the G-datum $\Psi_{(S,\theta)}$.

The representation ρ above may be reducible, and its irreducible components are given by [Kaletha 2021, Theorem 2.7.7]. While the definition of a G-datum in [Yu 2001] requires ρ to be irreducible, we may still apply the steps of the J.-K. Yu construction on $(\vec{G}, y, \vec{r}, \rho, \vec{\phi})$ to obtain $\pi_{(S,\theta)}$, which is a completely reducible supercuspidal representation independent of the chosen Howe factorization [Kaletha 2021, Corollary 3.4.7]. We write $[\pi_{(S,\theta)}]$ for the set of irreducible components of $\pi_{(S,\theta)}$.

$$\begin{array}{cccccc}
 & (\phi^0, K^0) & \cdots & (\phi^{d-2}, K^{d-2}) & (\phi^{d-1}, K^{d-1}) & (\phi^d, K^d) \\
 & \downarrow \text{enlarge} & & \downarrow \text{enlarge} & \downarrow \text{enlarge} & \downarrow = \\
(\rho, K^0) & (\phi^{0'}, K^1) & \cdots & (\phi^{d-2'}, K^{d-1}) & (\phi^{d-1'}, K^d) & (\phi^{d'}, K^d) \\
\downarrow \text{inflate} & \downarrow \text{inflate} & & \downarrow \text{inflate} & \downarrow = & \downarrow = \\
(\kappa^{-1}, K^d) & (\kappa^0, K^d) & \cdots & (\kappa^{d-2}, K^d) & (\kappa^{d-1}, K^d) & (\kappa^d, K^d)
\end{array}$$

$$\pi_{(S,\theta)} := \operatorname{Ind}_{K^d}^G \kappa_{(S,\theta)}, \text{ where}$$

$$\kappa_{(S,\theta)} = \kappa^{-1} \otimes \kappa^0 \otimes \kappa^1 \otimes \kappa^2 \cdots \otimes \kappa^{d-2} \otimes \kappa^{d-1} \otimes \kappa^d$$

Figure 3. Summary of the J.-K. Yu construction for $\pi_{(S,\theta)}$, where $K^0 = G^0(F)_y$ and $K^{i+1} = K^0 G^1(F)_{y,r_0/2} \cdots G^{i+1}(F)_{y,r_i/2}$, $0 \le i \le d-1$.

From the pair (S, θ), one may also perform what we call a *twisted* J.-K. Yu construction. Indeed, following [Fintzen et al. 2023, Section 4.1 page 2259], let $\epsilon = \prod_{i=1}^{d} \epsilon^{G^i/G^{i-1}}$, where $\epsilon^{G^i/G^{i-1}}$ is the quadratic character of K^d that is trivial on $G^1(F)_{y,r_0/2} \cdots G^d(F)_{y,r_{d-1}/2}$ and whose restriction to K^0 is given by $\epsilon_y^{G^i/G^{i-1}}$ defined in [Fintzen et al. 2023, Definition 4.1.10]. The so-called twisted representation then refers to $\mathrm{Ind}_{K^d}^{G}(\kappa_{(S,\theta)} \cdot \epsilon)$, which is equivalent to constructing $\pi_{(S,\theta\cdot\epsilon)}$ via the above steps. Since one obtains the Levi sequence $\vec{G}$ from S, we simply say that ϵ is constructed from S.

2.5. ***The construction of supercuspidal L-packets.*** Recall that $\Phi_{\mathrm{sc}}(G)$ denotes the set (of conjugacy classes) of supercuspidal L-parameters of G. Given our hypothesis on p, every $\varphi \in \Phi_{\mathrm{sc}}(G)$ has the property that $\varphi(P_F)$ is contained in a maximal torus of $\widehat{G}$ [Kaletha 2021, Lemma 4.1.3]. Such parameters are called *torally wild* in [Kaletha 2021]. Since all supercuspidal parameters we consider in this paper are torally wild, we will omit these adjectives.

Given $\varphi \in \Phi_{\mathrm{sc}}(G)$, we let Π_φ denote the associated L-packet of [Kaletha 2021]. Kaletha provides an explicit parameterization for Π_φ, and elements therein consist entirely of supercuspidal representations obtained from the construction outlined in Section 2.4. Thus, when $\varphi \in \Phi_{\mathrm{sc}}(G)$, we shall refer to Π_φ as a *supercuspidal* L-packet.

In order to describe Kaletha's construction of supercuspidal L-packets, we must first familiarize ourselves with his notion of a supercuspidal L-packet datum. We start this section by recalling the definition below.

Definition 2.9 [Kaletha 2021, Definition 4.1.4]. A supercuspidal L-packet datum of G is a tuple $(S, \hat{\jmath}, \chi_0, \theta)$, where

(1) S is a torus of dimension equal to the absolute rank of G, defined over F and split over a tame extension of F;

(2) $\hat{\jmath} : \widehat{S} \to \widehat{G}$ is an embedding of complex reductive groups whose $\widehat{G}$-conjugacy class is Γ-stable;

(3) $\chi_0 = (\chi_{\alpha_0})_{\alpha_0}$ is tamely ramified χ-data for $R(G, S^0)$, where S^0 is a particular subtorus of S defined from R_{0+} as explained in [Kaletha 2021, p. 41];

(4) and $\theta : S(F) \to \mathbb{C}^\times$ is a character;

subject to the condition that (S, θ) is a tame F-nonsingular elliptic pair in the sense of [Kaletha 2021, Definition 3.4.1].

Despite appearances, the torus S does not actually live inside G. It is an abstract torus that will be embedded into G below.

The notion of χ-data was introduced in [Langlands and Shelstad 1987] and is recalled in [Kaletha 2019, Section 4.6]. It is not necessary for the reader to be familiar with χ-data in what follows. For our purposes, one can think of χ-data for $R(G, S^0)$ simply as a set of characters of unit groups of finite extensions of F which are indexed by roots.

By [Kaletha 2021, Proposition 4.1.8], there is a one-to-one correspondence between the $\widehat{G}$-conjugacy classes of supercuspidal L-parameters for G and isomorphism classes of supercuspidal L-packet data. Following the proof of [Kaletha 2021, Proposition 4.1.8], given $\varphi \in \Phi_{\mathrm{sc}}(G)$, one constructs a representative $(S, \hat{j}, \chi_0, \theta)$ of the corresponding isomorphism class of supercuspidal L-packet data as follows:

- S: Let $\widehat{M} = \mathrm{Cent}(\varphi(P_F), \widehat{G})^\circ$, $\widehat{C} = \mathrm{Cent}(\varphi(I_F), \widehat{G})^\circ$ and $\widehat{S} = \mathrm{Cent}(\widehat{C}, \widehat{M})$. By [Kaletha 2019, Lemma 5.2.2; 2021, Lemma 4.1.3], $\widehat{M}$ is Levi subgroup of $\widehat{G}$, $\widehat{C}$ is a torus of $\widehat{G}$ and $\widehat{S}$ is a maximal torus of $\widehat{G}$. The action of W_F (which extends to Γ) on $\widehat{S}$ is defined as $\mathrm{Ad}(\varphi(-))$. The torus S is then the torus dual to $\widehat{S}$.
- $\hat{j}$: One simply takes $\hat{j}$ to be the set inclusion $\widehat{S} \hookrightarrow \widehat{G}$.
- χ_0: One chooses tame χ-data χ_0 for S^0, which extends to χ-data for S by [Kaletha 2021, Remark 4.1.5].
- θ: Following [Langlands and Shelstad 1987, Section 2.6], the χ-data allow one to extend $\hat{j}$ to an embedding ${}^L j : {}^L S \to {}^L G$. The image of ${}^L j$ contains the image of φ so that we may write $\varphi = {}^L j \circ \varphi_S$ for some L-parameter φ_S of S. We let θ be the corresponding character of $S(F)$ via the LLC for tori.

We will say that $(S, \hat{j}, \chi_0, \theta)$ is the *supercuspidal L-packet datum associated to $\varphi \in \Phi_{\mathrm{sc}}(G)$*. The embedding $\hat{j}$ belongs to a Γ-stable $\widehat{G}$-conjugacy class $\widehat{\mathcal{J}}$ of embeddings $\widehat{S} \to \widehat{G}$, and from $\widehat{\mathcal{J}}$ we obtain a Γ-stable $G(F^{\mathrm{sep}})$-conjugacy class $\mathcal{J}$ of embeddings $S \to G$ (called *admissible embeddings*) as per [Kaletha 2019, Section 5.1; Dillery 2023, Sections 6.1 and 7.1]. We denote by $\mathcal{J}_F$ the set of $G(F)$-conjugacy classes of elements of $\mathcal{J}$ which are defined over F. For each $j \in \mathcal{J}_F$, we consider the torus $jS = j(S)$ and let $j\theta = \theta \circ j^{-1} \cdot \epsilon_j$, where ϵ_j is the specific character from [Fintzen et al. 2023, Section 4.1] constructed from jS, described at the end of Section 2.4. Each pair $(jS, j\theta)$ is a tame F-nonsingular elliptic pair from which we can construct a supercuspidal representation $\pi_{(jS, j\theta)}$ as described in Section 2.4. The supercuspidal L-packet Π_φ is then defined as

$$\Pi_\varphi := \{[\pi_{(jS, j\theta)}] : j \in \mathcal{J}_F\},$$

where j is identified with its $G(F)$-conjugacy class and $\pi_{(jS, j\theta)}$ is identified with its equivalence class. Similarly, given $\underline{\varphi} \in \Phi_{\mathrm{sc}}(\underline{G})$, we denote the associated supercuspidal L-packet datum by $(\underline{S}, \underline{\hat{j}}, \underline{\chi}_0, \underline{\theta})$, and let $\underline{\mathcal{J}}_F$ be the set of $\underline{G}(F)$-conjugacy classes of admissible embeddings obtained from the Γ-stable $\widehat{\underline{G}}$-conjugacy class of $\underline{\hat{j}}$ which are defined over F, so that

$$\Pi_{\underline{\varphi}} = \{[\pi_{(\underline{j}\underline{S}, \underline{j}\underline{\theta})}] : \underline{j} \in \underline{\mathcal{J}}_F\}.$$

What we have denoted by Π_φ is what Kaletha [2021] denotes as $\Pi_\varphi(G)$. Kaletha assigns the notation Π_φ to his "compound L-packet" which encompasses rigid inner forms of G.

3. The decomposition of $\pi_{(\underline{S},\underline{\theta})} \circ \eta$

Kaletha [2019; 2021] described a way to construct a supercuspidal representation $\pi_{(\underline{S},\underline{\theta})}$ of $\underline{G}$ from a tame F-nonsingular elliptic pair $(\underline{S},\underline{\theta})$ [2021, Definition 3.4.1] (recalled above in Section 2.4). Here $\underline{S}$ is a maximally unramified elliptic maximal torus and $\underline{\theta}$ is a character of $\underline{S}(F)$ satisfying a certain *nonsingularity* condition. As seen in Section 2.5, the irreducible components of these representations are what make up the supercuspidal L-packets. As such, finding a decomposition formula for $\pi_{(\underline{S},\underline{\theta})} \circ \eta$ is crucial in proving Theorem 4.1.

The main result of this section is precisely this decomposition formula, and is given by the following theorem.

Theorem 3.1. *Let $(\underline{S},\underline{\theta})$ and (S,θ) be tame F-nonsingular elliptic pairs for $\underline{G}$ and G, respectively. Assume that $\eta(S) \subset \underline{S}$ and $\theta = \underline{\theta} \circ \eta$. Then*

$$\pi_{(\underline{S},\underline{\theta})} \circ \eta \simeq \bigoplus_{\underline{c}\in\underline{C}} \pi_{(S,\theta)} \circ \underline{\mathrm{Ad}}(\underline{c}^{-1}),$$

where $\underline{C}$ is a set of coset representatives of $\eta(G(F))\backslash\underline{G}(F)/\underline{S}(F)$ and $\underline{\mathrm{Ad}}$ is the $\underline{G}(F)$-action on $G(F)$ described in Section 2.3.

The proof of Theorem 3.1 is done in two steps (as illustrated in Figure 1). Indeed, by noting that

$$\pi_{(\underline{S},\underline{\theta})} \circ \eta = \left(\mathrm{Res}^{\underline{G}(F)}_{G_Z(F)} \pi_{(\underline{S},\underline{\theta})}\right) \circ \eta,$$

we first seek a decomposition formula for $\mathrm{Res}^{\underline{G}(F)}_{G_Z(F)} \pi_{(\underline{S},\underline{\theta})}$. The results from [Bourgeois 2021] grant us such a formula, as G_Z is a normal subgroup of $\underline{G}$ that contains $[\underline{G},\underline{G}]$. This is stated in Theorem 3.19. The second step is describing the composition of a supercuspidal representation of $G_Z(F)$ with η. More specifically, we prove the following theorem.

Theorem (Theorem 3.17). *Let (S,θ) and (S_Z,θ_Z) be tame F-nonsingular elliptic pairs of G and G_Z, respectively. Assume that $\eta(S) = S_Z$ and $\theta = \theta_Z \circ \eta$. Then*

$$\pi_{(S_Z,\theta_Z)} \circ \eta \simeq \bigoplus_{d_Z\in D_Z} \pi_{(S,\theta)} \circ \underline{\mathrm{Ad}}(d_Z^{-1}),$$

where D_Z is a set of coset representatives of $\eta(G(F))\backslash G_Z(F)/S_Z(F)$ and $\underline{\mathrm{Ad}}$ is the $\underline{G}(F)$-action on $G(F)$ described in Section 2.3.

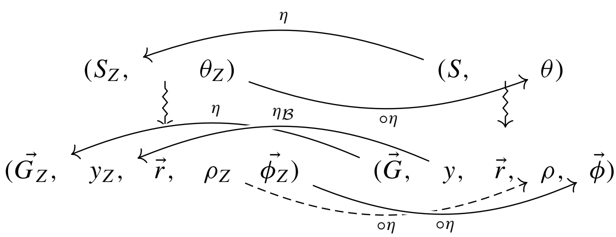

Figure 4. Relationship between the corresponding J.-K. Yu data given the relationship between the tame F-nonsingular elliptic pairs.

The strategy that we use to prove Theorem 3.17 is similar to the one from [Bourgeois 2021] in the sense that we go through the constructions of $\pi_{(S,\theta)}$ and $\pi_{(S_Z,\theta_Z)}$ step-by-step to make comparisons along the way.

Recall that the Kaletha–Yu construction involves two steps: producing a datum (in the sense of [Yu 2001, Section 3]) from the F-nonsingular elliptic pair, and then applying the J.-K. Yu construction to this datum. As such, the first thing we do is establish the relationship between the G-datum $\Psi_{(S,\theta)}$ and G_Z-datum $\Psi_{(S_Z,\theta_Z)}$ produced from (S,θ) and (S_Z,θ_Z), respectively. In particular, these data are related via the map η, a statement we will make precise with Theorem 3.2 and illustrate in Figure 4.

Once the relationship between $\Psi_{(S,\theta)}$ and $\Psi_{(S_Z,\theta_Z)}$ is established, we start comparing the representations at each step of the J.-K. Yu construction. What we show is that the steps of the construction commute with the composition with η (Propositions 3.14 and 3.15), as illustrated in Figure 5.

Establishing these relationships between the data and representations is a lengthy process. However, once complete, the proof of Theorem 3.17 reduces to an application of Mackey theory.

The rest of this section is organized as follows. Most of its contents are dedicated to the proof of Theorem 3.17. In Section 3.1, we establish the relationship between $\Psi_{(S,\theta)}$ and $\Psi_{(S_Z,\theta_Z)}$. In Section 3.2, we go through the steps of the J.-K. Yu

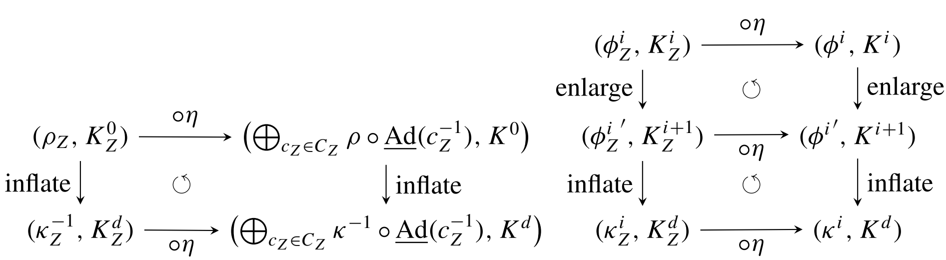

Figure 5. Commutativity of the composition with η with the steps of the J.-K. Yu construction. Here, C_Z denotes a set of coset representatives of $\eta(K^0)\backslash K_Z^0/S_Z(F)$.

construction to show that it commutes with the composition with η, and prove the statement of Theorem 3.17. Finally, in Section 3.3, we restate the results from [Bourgeois 2021] in the context of the Kaletha–Yu construction and provide the proof of Theorem 3.1.

3.1. ***Relationship between the G-datum and the G_Z-datum.*** In this section, η is the map on G, as stated in the introduction (Theorem 4.1) and $G_Z = \eta(G) \simeq G/Z$. We let (S, θ) and (S_Z, θ_Z) be tame F-nonsingular elliptic pairs of G and G_Z, respectively, which satisfy $\eta(S) = S_Z$ and $\theta = \theta_Z \circ \eta$. The goal of this section is to show that the corresponding G-datum and G_Z-datum are also related via the map η, a statement we illustrated in Figure 4 and will make precise with Theorem 3.2. First, we note that $\eta : G \to G_Z$ induces an equivariant isomorphism $\eta_{\mathcal{B}} : \mathcal{B}(G, F) \to \mathcal{B}(G_Z, F)$ by [Kaletha and Prasad 2023, Axiom 4.1.1]. In particular, this map satisfies $\eta_{\mathcal{B}}(gx) = \eta(g) \cdot \eta_{\mathcal{B}}(x)$ for all $g \in G(F)$, $x \in \mathcal{B}(G, F)$, where $\cdot$ refers to the action of $G(F)$ on $\mathcal{B}(G, F)$.

Theorem 3.2. *Let (S, θ) and (S_Z, θ_Z) be tame F-nonsingular elliptic pairs of G and G_Z, respectively, such that $\eta(S) = S_Z$ and $\theta = \theta_Z \circ \eta$. Let $\Psi_{(S,\theta)} = (\vec{G}, y, \vec{r}, \rho, \vec{\phi})$ and $\Psi_{(S_Z,\theta_Z)} = (\vec{G}_Z, y_Z, \vec{r}_Z, \rho_Z, \vec{\phi}_Z)$ be the corresponding J.-K. Yu data as described in Section 2.4. Then,*

(a) $\vec{r} = \vec{r}_Z$ *and* $\eta(\vec{G}) = \vec{G}_Z$,

(b) $y_Z = \eta_{\mathcal{B}}(y)$,

(c) $\vec{\phi} = \vec{\phi}_Z \circ \eta$, *and*

(d) $\rho_Z \circ \eta \simeq \bigoplus_{c_Z \in C_Z} \rho \circ \underline{\mathrm{Ad}}(c_Z^{-1})$, *where C_Z is a set of coset representatives of $\eta(K^0)\backslash K_Z^0 / S_Z(F)$.*

The proof of this theorem will be divided into four parts. Lemma 3.3 shows that $\eta(\vec{G}) = \vec{G}_Z$ and $\vec{r} = \vec{r}_Z$. Lemma 3.4 gives us $y_Z = \eta_{\mathcal{B}}(y)$. Proposition 3.8 allows us to set $\vec{\phi} = \vec{\phi}_Z \circ \eta$. Finally, we obtain the decomposition formula of $\rho_Z \circ \eta$ from Proposition 3.13. We note that proving part (d) is itself a multistep process, having to work over the reductive subquotients and call on Deligne–Lusztig theory.

Lemma 3.3 (Theorem 3.2(a)). *Let $\vec{G} = (G^0, \dots, G^d)$ and $\vec{G}_Z = (G_Z^0, \dots, G_Z^{d_Z})$ be the twisted Levi sequences obtained from S and S_Z, respectively, as per Section 2.4. Then $d = d_Z$ and $\eta(G^i) = G_Z^i$ for all $0 \le i \le d$.*

The previous lemma easily follows from the fact that the root systems $R(G, S)$ and $R(G_Z, S_Z)$ are identified by η. Furthermore, the induced sequence of numbers $\vec{r}$ is the same for both S and S_Z.

Lemma 3.4 (Theorem 3.2(b)). *Let y be the vertex of $\mathcal{B}(G, F)$ associated to S and y_Z be the vertex of $\mathcal{B}(G_Z, F)$ associated to S_Z as per* [Kaletha 2019, Lemma 3.4.3]. *Then $y_Z = \eta_{\mathcal{B}}(y)$.*

Proof. The bijection $\eta_{\mathcal{B}}$ maps $\mathcal{A}(G, S, F^{\mathrm{un}})$ to $\mathcal{A}(G_Z, S_Z, F^{\mathrm{un}})$. Recall that y is fixed by $\mathrm{Gal}(F^{\mathrm{un}}/F)$ by definition. Since $\eta_{\mathcal{B}}$ is also defined over F, $\eta_{\mathcal{B}}(y)$ is a $\mathrm{Gal}(F^{\mathrm{un}}/F)$-fixed point. Since S_Z is elliptic, this fixed point is unique [Kaletha and Prasad 2023, Section 17.8], and thus $y_Z = \eta_{\mathcal{B}}(y)$. □

Remark 3.5. Given that $\eta(G^i) = G^i_Z$ and that Z is a central subgroup of G^i for all $0 \le i \le d$, the results of Lemma 2.2 and Remark 2.3 apply to G and G_Z replaced by G^i and G^i_Z. In particular, $\eta(G^i(F)_{y,r_i}) = G^i_Z(F)_{y_Z,r_i}$, which induces a surjection $G^i(F)_{y,r_i:r_i^+} \to G^i_Z(F)_{y_Z,r_i:r_i^+}$, $0 \le i \le d$. Using a similar argument, we also have $\eta(J^{i+1}) = J^{i+1}_Z$ and $\eta(J^{i+1}_+) = J^{i+1}_{Z+}$ for all $0 \le i \le d-1$, where $J^{i+1} = (G^i, G^{i+1})(F)_{y,(r_i,r_i/2)}$ and $J^{i+1}_+ = (G^i, G^{i+1})(F)_{y,(r_i,r_i/2^+)}$ as per [Yu 2001, Section 1], and J^{i+1}_Z and J^{i+1}_{Z+} are defined analogously. Furthermore, one sees from [Yu 2001, Section 1] that $J^{i+1} \cap Z = J^{i+1}_+ \cap Z$ so that we have an isomorphism $J^{i+1}/J^{i+1}_+ \simeq J^{i+1}_Z/J^{i+1}_{Z+}$ for all $0 \le i \le d-1$.

Lemma 3.6. *Let $K^0_Z = G^0_Z(F)_{y_Z}$ and $K^i_Z = K^0_Z G^1_Z(F)_{y_Z,r_0/2} \cdots G^i_Z(F)_{y_Z,r_{i-1}/2}$ for all $0 \le i \le d-1$. Then $\eta(K^i) = K^i_Z \cap \eta(G(F))$ for all $0 \le i \le d$. Furthermore, $\eta(K^i)$ is normalized by K^0_Z for all $0 \le i \le d$.*

Proof. The case of $i = 0$ is given by Lemma 2.1, in which we replace G by G^0.

For $0 < i \le d$, we proceed as follows. Let $J = G^1(F)_{y,r_0/2} \cdots G^i(F)_{y,r_{i-1}/2}$ and $J_Z = G^1_Z(F)_{y_Z,r_0/2} \cdots G^i_Z(F)_{y_Z,r_{i-1}/2}$. By Remark 3.5, we have $\eta(J) = J_Z$. It follows that $K^i_Z = K^0_Z J_Z = K^0_Z \eta(J)$, and therefore

$$K^i_Z \cap \eta(G(F)) = \big(K^0_Z \cap \eta(G(F))\big)\eta(J) = \eta(K^0)\eta(J) = \eta(K^i).$$

Using the fact that ${}^g G^j_Z(F)_{y_Z,r} = G^j_Z(F)_{g\cdot y_Z,r}$ for all $g \in G^j_Z(F)$, one sees that $\eta(J) = J_Z$ is normalized by K^0_Z. We conclude, with what precedes for $i = 0$, that $\eta(K^i)$ is normalized by K^0_Z. □

Corollary 3.7. *Let $0 \le i \le d$. Then $\underline{\mathrm{Ad}}(k_Z)(K^i) = K^i$ for all $k_Z \in K^0_Z$.*

Proof. By the previous lemma, we have $k_Z \eta(K^i) k_Z^{-1} = \eta(K^i)$ for all $k_Z \in K^0_Z$. Given $k_Z \in K^0_Z \subset G^0_Z(F)$, $k_Z = \eta(g)$ for some $g \in G^0(F^{\mathrm{sep}})$ such that $\sigma(g)g^{-1} \in Z$ for all $\sigma \in \Gamma$. It follows that $\eta(gK^i g^{-1}) = \eta(K^i)$, with $gK^i g^{-1} \subset G(F)$ (or equivalently, $\sigma(gK^i g^{-1}) = gK^i g^{-1}$ for all $\sigma \in \Gamma$). As a map on $G(F)$, the kernel of η is $Z(F)$, which implies $gK^i Z(F) g^{-1} = K^i Z(F)$. By [Yu 2001, Lemma 3.3], one has $K^0 = N_{G^0(F)}(G^0(F)_{y,0})$, and hence $Z(F) \subset K^0 \subset K^i$. Thus, we conclude that $gK^i g^{-1} = K^i$, or equivalently, $\underline{\mathrm{Ad}}(k_Z)(K^i) = K^i$. □

Proposition 3.8 (Theorem 3.2(c)). *Let $(\phi^{-1}_Z, \phi^0_Z, \dots, \phi^d_Z)$ be a Howe factorization for θ_Z. For each $-1 \le i \le d$, set $\phi^i = \phi^i_Z \circ \eta$. Then $(\phi^{-1}, \phi^0, \dots, \phi^d)$ is a Howe factorization for θ.*

Proof. One sees that $(\phi^{-1}, \phi^0, \dots, \phi^d)$ satisfies the two first axioms to be a Howe factorization of θ, so it remains to verify the third axiom. Let $0 \leq i < d$. By [Kaletha 2019, Lemma 3.6.8], proving genericity is equivalent to showing that

$$\phi^i\big(N_{E/F}(\check{\alpha}(E_r^\times))\big) \neq 1$$

for all $\alpha \in R(G^{i+1}, S)\backslash R(G^i, S)$. Since the character ϕ_Z^i is generic, we have that

$$\phi_Z^i\big(N_{E/F}(\check{\alpha_Z}(E_r^\times))\big) \neq 1$$

for all $\alpha_Z \in R(G_Z^{i+1}, S)\backslash R(G_Z^i, S)$. Since η is defined over F, we have that

$$\phi^i\big(N_{E/F}(\check{\alpha}(E_r^\times))\big) = (\phi_Z^i \circ \eta)\big(N_{E/F}(\check{\alpha}(E_r^\times))\big) = \phi_Z^i\big(N_{E/F}((\eta \circ \check{\alpha})(E_r^\times))\big).$$

Since the root systems of G and G_Z are identified by η, we conclude from the genericity of ϕ_Z^i that ϕ^i is generic.

For $i = d$, we see that ϕ^d is trivial whenever ϕ_Z^d is. When $\phi_Z^d \neq 1$, ϕ^d must be of the same depth, as a consequence of the surjection $G(F)_{y,r_d:r_d^+} \to G_Z(F)_{y_Z,r_d:r_d^+}$.

Finally, for $i = -1$, it is clear that ϕ^{-1} is trivial in the case where ϕ_Z^{-1} is trivial, and that $\phi^{-1}|_{S(F)_{0+}} = 1$ whenever $\phi_Z^{-1}|_{S_Z(F)_{0+}} = 1$ as $\eta(S(F)_{0^+}) \subset S_Z(F)_{0^+}$. □

It now remains to prove part (d) of Theorem 3.2. Recall from Figure 2 that $\rho = \operatorname{Ind}_{S(F)G^0(F)_{y,0}}^{G^0(F)_y} \kappa_{(S,\phi^{-1})}$, where $\kappa_{(S,\phi^{-1})}$ is Kaletha's extension of the Deligne–Lusztig cuspidal representation $\pm R_{\mathcal{S},\bar{\phi}^{-1}}$ of $\mathcal{G}^0(\mathfrak{f})$ with $\mathcal{S}$ a maximal torus of $\mathcal{G}^0$ which satisfies $\mathcal{S}(\mathfrak{f}) \simeq S(F)_{0:0^+}$. Adopting similar notation for G_Z^0, we have that

$$\rho_Z = \operatorname{Ind}_{G_Z^0(F)_{y_Z,0}}^{G_Z^0(F)_{y_Z}} \kappa_{(S_Z,\phi_Z^{-1})},$$

where $\kappa_{(S_Z,\phi_Z^{-1})}$ is Kaletha's extension of the Deligne–Lusztig cuspidal representation $\pm R_{\mathcal{S}_Z,\bar{\phi}_Z^{-1}}$ of $\mathcal{G}_Z^0(\mathfrak{f})$, the reductive subquotient of G_Z^0 at y_Z, with $\mathcal{S}_Z$ a maximal torus of $\mathcal{G}_Z^0$ which satisfies $\mathcal{S}_Z(\mathfrak{f}) \simeq S_Z(F)_{0:0^+}$. To understand the relationship between ρ and ρ_Z, we start by studying the relationship between $\pm R_{\mathcal{S},\bar{\phi}^{-1}}$ and $\pm R_{\mathcal{S}_Z,\bar{\phi}_Z^{-1}}$.

Given that $\eta(G^0) = G_Z^0$ and that Z is a central subgroup of G^0 (Lemma 3.3), the map η induces a map

$$\mathcal{G}^0 \xrightarrow{q} \mathcal{G}^0/\mathcal{Z}^0 \xrightarrow{\bar{\eta}} \mathcal{G}_Z^0,$$

where $\mathcal{Z}^0$ is such that $\mathcal{Z}^0(\bar{\mathfrak{f}}) = (Z \cap G^0(F^{\mathrm{un}})_{y,0})G^0(F^{\mathrm{un}})_{y,0^+}$. This is precisely the map illustrated in (4), with G and G_Z replaced by G^0 and G_Z^0, respectively. The tori $\mathcal{S}$ and $\mathcal{S}_Z$ are related via this map, as per the following lemma.

Lemma 3.9. *One has* $(\bar{\eta} \circ q)(\mathcal{S}) = \mathcal{S}_Z \cap \bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)$.

Proof. We identify the reductive groups with their $\bar{\mathfrak{f}}$-points. Combining the lower map of (4) with the definitions above, we have

$$\begin{aligned}(\bar{\eta}\circ q)(\mathcal{S}(\bar{\mathfrak{f}})) &= (\bar{\eta}\circ q)\big(S(F^{\mathrm{un}})_0 G^0(F^{\mathrm{un}})_{y,0+}/G^0(F^{\mathrm{un}})_{y,0+}\big)\\ &= \eta(S(F^{\mathrm{un}})_0) G^0_Z(F^{\mathrm{un}})_{y_Z,0+}/G^0_Z(F^{\mathrm{un}})_{y_Z,0+}\\ &\subseteq \big(S_Z(F^{\mathrm{un}})_0 G^0_Z(F^{\mathrm{un}})_{y_Z,0+}/G^0_Z(F^{\mathrm{un}})_{y_Z,0+}\big)\\ &\qquad\qquad \cap \big(\eta(G^0(F^{\mathrm{un}})_{y,0}) G^0_Z(F^{\mathrm{un}})_{y_Z,0+}/G^0_Z(F^{\mathrm{un}})_{y_Z,0+}\big)\\ &= \mathcal{S}_Z(\bar{\mathfrak{f}})\cap\bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)(\bar{\mathfrak{f}}).\end{aligned}$$

Given that both tori are maximal, the equality follows. □

Lemma 3.10. *We have* $G^0_Z(F)_{y_Z,0} = S_Z(F)_0\,\eta(G^0(F)_{y,0})$.

Proof. Using Lemma 2.5, we have that $\mathcal{G}^0_Z = \mathcal{S}_Z\,\bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)$. Furthermore, the intersection of $\mathcal{S}_Z$ with $\bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)$ is a maximal torus of $\bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)$. As a consequence of Lang's theorem, $H^1(\mathfrak{f}, \mathcal{S}_Z\cap\bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)) = 1$. Combining this with the usual Galois cohomology sequence,

$$\mathcal{G}^0_Z(\mathfrak{f}) = \mathcal{S}_Z(\mathfrak{f})\bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)(\mathfrak{f}) = \mathcal{S}_Z(\mathfrak{f})\bar{\eta}((\mathcal{G}^0/\mathcal{Z}^0)(\mathfrak{f})). \tag{5}$$

We have that $(\mathcal{G}^0/\mathcal{Z}^0)(\mathfrak{f}) = C(\mathcal{G}^0(\mathfrak{f})/\mathcal{Z}^0(\mathfrak{f}))$, where C is a set of coset representatives of $(\mathcal{G}^0/\mathcal{Z}^0)(\mathfrak{f})/(\mathcal{G}^0(\mathfrak{f})/\mathcal{Z}^0(\mathfrak{f}))$.

Without loss of generality, we may assume that $C\subseteq q(\mathcal{S})(\mathfrak{f})$. To see this, consider the exact sequences

$$1\to\mathcal{Z}^0\to\mathcal{G}^0\to\mathcal{G}^0/\mathcal{Z}^0\to 1$$

and

$$1\to\mathcal{Z}^0\to\mathcal{S}\to q(\mathcal{S})\to 1.$$

Given that $\mathcal{G}^0$ and $\mathcal{S}$ are connected, Lang's theorem implies $H^1(\mathfrak{f},\mathcal{G}^0) = 1 = H^1(\mathfrak{f},\mathcal{S})$, giving us exact cohomology sequences [Springer 2009, Theorem 12.3.4]

$$1\to\mathcal{Z}^0(\mathfrak{f})\to\mathcal{G}^0(\mathfrak{f})\to(\mathcal{G}^0/\mathcal{Z}^0)(\mathfrak{f})\to H^1(\mathfrak{f},\mathcal{Z}^0)\to 1$$

and

$$1\to\mathcal{Z}^0(\mathfrak{f})\to\mathcal{S}(\mathfrak{f})\to q(\mathcal{S})(\mathfrak{f})\to H^1(\mathfrak{f},\mathcal{Z}^0)\to 1.$$

The exactness of the sequences implies

$$(\mathcal{G}^0/\mathcal{Z}^0)(\mathfrak{f})/(\mathcal{G}^0(\mathfrak{f})/\mathcal{Z}^0(\mathfrak{f}))\simeq H^1(\mathfrak{f},\mathcal{Z}^0)\simeq q(\mathcal{S})(\mathfrak{f})/(\mathcal{S}(\mathfrak{f})/\mathcal{Z}^0(\mathfrak{f})).$$

The definitions of the connecting homomorphisms $(\mathcal{G}^0/\mathcal{Z}^0)(\mathfrak{f})\to H^1(\mathfrak{f},\mathcal{Z}^0)$ and $q(\mathcal{S})(\mathfrak{f})\to H^1(\mathfrak{f},\mathcal{Z}^0)$ from [Springer 2009, Section 12.3.3] allow us to conclude that we may choose $C\subset q(\mathcal{S})(\mathfrak{f})$.

Having $C\subseteq q(\mathcal{S})(\mathfrak{f})$ allows us to rewrite (5) as

$$\mathcal{G}^0_Z(\mathfrak{f}) = \mathcal{S}_Z(\mathfrak{f})\bar{\eta}(C)\bar{\eta}(\mathcal{G}^0(\mathfrak{f})/\mathcal{Z}^0(\mathfrak{f})) = \mathcal{S}_Z(\mathfrak{f})\bar{\eta}(\mathcal{G}^0(\mathfrak{f})/\mathcal{Z}^0(\mathfrak{f})). \tag{6}$$

Given that $\mathcal{G}^0(\mathfrak{f}) = G^0(F)_{y,0:0^+}$, it follows that

$$\mathcal{Z}^0(\mathfrak{f}) = (Z \cap G^0(F)_{y,0})G^0(F)_{y,0^+}/G^0(F)_{y,0^+}.$$

Therefore,

$$\bar{\eta}(\mathcal{G}^0(\mathfrak{f})/\mathcal{Z}^0(\mathfrak{f})) = \eta(G^0(F)_{y,0})G^0_Z(F)_{y_Z,0^+}/G^0_Z(F)_{y_Z,0^+}.$$

Using $\mathcal{G}^0_Z(\mathfrak{f}) \simeq G^0_Z(F)_{y_Z,0:0^+}$ and $\mathcal{S}_Z(\mathfrak{f}) \simeq S_Z(F)_0 G^0_Z(F)_{y_Z,0^+}/G^0_Z(F)_{y_Z,0^+}$, we rewrite (6) as

$$G^0_Z(F)_{y_Z,0:0^+} = \big(S_Z(F)_0 G^0_Z(F)_{y_Z,0^+}/G^0_Z(F)_{y_Z,0^+}\big)\big(\eta(G^0(F)_{y,0})G^0_Z(F)_{y_Z,0^+}/G^0_Z(F)_{y_Z,0^+}\big),$$

from which we conclude that

$$G^0_Z(F)_{y_Z,0} = S_Z(F)_0\,\eta(G^0(F)_{y,0})G^0_Z(F)_{y_Z,0^+}.$$

Finally, we have that $G^0_Z(F)_{y_Z,0^+} = \eta(G^0(F)_{y,0^+})$. Indeed, using Lemma 2.2,

$$\begin{aligned} G^0_Z(F)_{y_Z,0^+} &= \bigcup_{r>0} G^0_Z(F)_{y_Z,r} = \bigcup_{r>0} \eta(G^0(F)_{y,0}) \\ &= \eta\Big(\bigcup_{r>0} G^0(F)_{y,0}\Big) = \eta(G^0(F)_{y,0^+}). \end{aligned}$$

Thus, we conclude that

$$G^0_Z(F)_{y_Z,0} = S_Z(F)_0\,\eta(G^0(F)_{y,0}). \qquad \square$$

Using the map $\bar{\eta} \circ q$, and Lemma 2.5, we can establish the following relationship between the virtual characters $\pm R_{\mathcal{S},\bar{\phi}^{-1}}$ and $\pm R_{\mathcal{S}_Z,\bar{\phi}_Z^{-1}}$.

Proposition 3.11. *Given the above notation, one has* $\pm R_{\mathcal{S},\bar{\phi}^{-1}} = \pm R_{\mathcal{S}_Z,\bar{\phi}_Z^{-1}} \circ (\bar{\eta} \circ q)$.

Proof. Let us recall the construction of $\pm R_{\mathcal{S},\bar{\phi}^{-1}}$. Following the notation of [Kaletha 2019, Section 3.4.4; 2021, Section 2.4], let $\mathcal{U}$ be the unipotent radical of a Borel subgroup $\mathcal{B}$ of $\mathcal{G}^0$ which contains $\mathcal{S}$ and define $Y^{\mathcal{G}^0}_{\mathcal{U}} = \{g \in \mathcal{G}^0/\mathcal{U} : g^{-1}\operatorname{Fr}(g) \in \mathcal{U}\operatorname{Fr}(\mathcal{U})\}$, where Fr is a generator of $\operatorname{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$. Let $d\mathcal{U}$ denote the number of hyperplanes separating the Weyl chambers of $\mathcal{U}$ and $\operatorname{Fr}(\mathcal{U})$, and consider the ℓ-adic cohomology group $H^{d\mathcal{U}}_c(Y^{\mathcal{G}^0}_{\mathcal{U}}, \overline{\mathbb{Q}}_\ell)$. The virtual character $\pm R_{\mathcal{S},\bar{\phi}^{-1}}$ is then defined to be the action of $\mathcal{G}^0(\mathfrak{f})$ on $H^{d\mathcal{U}}_c(Y^{\mathcal{G}^0}_{\mathcal{U}}, \overline{\mathbb{Q}}_\ell)_{\bar{\phi}^{-1}}$, the $\bar{\phi}^{-1}$-isotypic component of $H^{d\mathcal{U}}_c(Y^{\mathcal{G}^0}_{\mathcal{U}}, \overline{\mathbb{Q}}_\ell)$. Similarly, $\pm R_{\mathcal{S}_Z,\bar{\phi}_Z^{-1}}$ is the action of $\mathcal{G}^0_Z(\mathfrak{f})$ on $H^{d\mathcal{U}_Z}_c(Y^{\mathcal{G}^0_Z}_{\mathcal{U}_Z}, \overline{\mathbb{Q}}_\ell)_{\bar{\phi}_Z^{-1}}$, where $\mathcal{U}_Z$ is the unipotent radical of a Borel subgroup $\mathcal{B}_Z$ of $\mathcal{G}^0_Z$ containing $\mathcal{S}_Z$.

Using Lemma 3.9, we have that $\bar{\eta} \circ q(\mathcal{S}) = \mathcal{S}_Z \cap \bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)$. Since the virtual characters do not depend on the choice of Borel subgroup (see, e.g., [Deligne and Lusztig 1976, Corollary 4.3; Carter 1993, Proposition 7.3.6; Kaletha 2021, Section 2.5]), we may assume without loss of generality that $\bar{\eta} \circ q(\mathcal{B}) = \mathcal{B}_Z \cap \bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)$, and therefore $\bar{\eta} \circ q(\mathcal{U}) = \mathcal{U}_Z \cap \bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)$. Thus, $\bar{\eta} \circ q$ induces a map $Y^{\mathcal{G}^0}_{\mathcal{U}} \to Y^{\mathcal{G}^0_Z}_{\mathcal{U}_Z}$.

Furthermore, since we are setting $\phi_Z^{-1} \circ \eta = \phi^{-1}$ (Proposition 4.5), we have that $\overline{\phi}_Z^{-1} \circ (\bar{\eta} \circ q) = \overline{\phi}^{-1}$. Indeed, for all $g \in G^0(F)_{y,0}$, we obtain

$$\begin{aligned}\overline{\phi}_Z^{-1} \circ (\bar{\eta} \circ q)(gG^0(F)_{y,0^+}) = \overline{\phi}_Z^{-1}(\eta(g)G_Z^0(F)_{y,0^+}) &= \phi_Z^{-1}(\eta(g)) \\ &= \phi^{-1}(g) = \overline{\phi}^{-1}(gG^0(F)_{y,0^+}).\end{aligned}$$

By [Kaletha 2021, D.4], $\bar{\eta} \circ q$ induces an isomorphism

$$(\bar{\eta} \circ q)^* : H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}_Z^{-1}} \to H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}}.$$

Now, for all $g \in \mathcal{G}^0(\mathfrak{f})$, letting $L(g) : Y_{\mathcal{U}}^{\mathcal{G}^0} \to Y_{\mathcal{U}}^{\mathcal{G}^0}$ be the map corresponding to left multiplication by g, one sees that the following diagram commutes:

$$\begin{array}{ccc} Y_{\mathcal{U}}^{\mathcal{G}^0} & \xrightarrow{\bar{\eta}\circ q} & Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0} \\ {\scriptstyle L(g)}\downarrow & & \downarrow{\scriptstyle L_Z((\bar{\eta}\circ q)(g))} \\ Y_{\mathcal{U}}^{\mathcal{G}^0} & \xrightarrow{\bar{\eta}\circ q} & Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0} \end{array}$$

Given that ℓ-adic cohomology is functorial, we have that the following diagram also commutes for all $g \in \mathcal{G}^0(\mathfrak{f})$:

$$\begin{array}{ccc} H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}} & \xleftarrow{(\bar{\eta}\circ q)^*} & H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}_Z^{-1}} \\ {\scriptstyle \pm R_{\mathcal{S},\overline{\phi}^{-1}}(g)}\uparrow & & \uparrow{\scriptstyle \pm R_{\mathcal{S}_Z,\overline{\phi}_Z^{-1}}((\bar{\eta}\circ q)(g))} \\ H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}} & \xleftarrow[(\bar{\eta}\circ q)^*]{} & H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}_Z^{-1}} \end{array} \tag{7}$$

Thus, we conclude that $\pm R_{\mathcal{S},\overline{\phi}^{-1}} = \pm R_{\mathcal{S}_Z,\overline{\phi}_Z^{-1}} \circ (\bar{\eta} \circ q)$. □

Corollary 3.12. *Let $\kappa_{(S,\phi^{-1})}$ and $\kappa_{(S_Z,\phi_Z^{-1})}$ be the representations of $S(F)G^0(F)_{y,0}$ and $S_Z(F)G_Z^0(F)_{y_Z,0}$ (as in* [Kaletha 2019, Section 3.4.4]*) which extend the pullbacks of $\pm R_{\mathcal{S},\overline{\phi}^{-1}}$ and $\pm R_{\mathcal{S}_Z,\overline{\phi}_Z^{-1}}$, respectively. Then $\kappa_{(S,\phi^{-1})} \simeq \kappa_{(S_Z,\phi_Z^{-1})} \circ \eta$.*

Proof. Since we are building up from Proposition 3.11, let us follow the notation within its proof.

As in [Kaletha 2021, Section 3], we have an $\mathfrak{f}$-group scheme $\mathcal{S}'$, which satisfies $\mathcal{S}'(\mathfrak{f}) = S(F)/S(F)_{0^+}$. Every $s' \in \mathcal{S}'(\mathfrak{f})$ acts on $Y_{\mathcal{U}}^{\mathcal{G}^0}$ by conjugation, and induces an action on $H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}}$, denoted by $\mathrm{Ad}(s')$. As explained in [Kaletha 2019, Section 3.4.4; 2021, Remark 2.6.5], this allows us to define an action of $\mathcal{S}'(\mathfrak{f})\mathcal{G}^0(\mathfrak{f})$ on $H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}}$, denoted by κ, as

$$\kappa(s'g)(v) = \overline{\phi}^{-1}(s') \cdot (\pm R_{\mathcal{S},\overline{\phi}^{-1}}(g) \circ \mathrm{Ad}(s'))(v)$$

for all $s' \in \mathcal{S}'(\mathfrak{f})$, $g \in \mathcal{G}^0(\mathfrak{f})$, $v \in H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}}$. The action pulls back to an action of $S(F)G^0(F)_{y,0}$ on $H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}}$, which is the representation $\kappa_{(S,\phi^{-1})}$.

Similarly, $\kappa_{(S_Z,\phi_Z^{-1})}$ is the action of $S_Z(F)G_Z^0(F)_{y_Z,0}$ on $H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0},\overline{\mathbb{Q}}_\ell)_{\overline{\phi}_Z^{-1}}$, obtained by pulling back the action

$$\kappa_Z(s_Z'g_Z)(v_Z)=\overline{\phi}_Z^{-1}(s_Z')\cdot(\pm R_{\mathcal{S}_Z,\overline{\phi}_Z^{-1}}(g_Z)\circ\mathrm{Ad}(s_Z'))(v_Z)$$

for all $s_Z'\in\mathcal{S}_Z'(\mathfrak{f})=S_Z(F)/S_Z(F)_{0^+}$, $g_Z\in\mathcal{G}_Z^0(\mathfrak{f})$, $v_Z\in H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0},\overline{\mathbb{Q}}_\ell)_{\overline{\phi}_Z^{-1}}$.

Using Remark 2.4, one sees that the following diagram commutes for all $s'\in\mathcal{S}'(\mathfrak{f})$:

$$\begin{array}{ccc} Y_{\mathcal{U}}^{\mathcal{G}^0} & \xrightarrow{\bar{\eta}\circ q} & Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0} \\ {\scriptstyle \mathrm{Ad}(s')}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}((\bar{\eta}\circ q)(s'))} \\ Y_{\mathcal{U}}^{\mathcal{G}^0} & \xrightarrow{\bar{\eta}\circ q} & Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0} \end{array}$$

Given that ℓ-adic cohomology is functorial, we have that the following diagram also commutes for all $s'\in\mathcal{S}'(\mathfrak{f})$:

$$\begin{array}{ccc} H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0},\overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}} & \xleftarrow{(\bar{\eta}\circ q)^*} & H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0},\overline{\mathbb{Q}}_\ell)_{\overline{\phi}_Z^{-1}} \\ {\scriptstyle \mathrm{Ad}(s')}\uparrow & & \uparrow{\scriptstyle \mathrm{Ad}((\bar{\eta}\circ q)(s'))} \\ H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0},\overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}} & \xleftarrow[(\bar{\eta}\circ q)^*]{} & H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0},\overline{\mathbb{Q}}_\ell)_{\overline{\phi}_Z^{-1}} \end{array} \tag{8}$$

where $(\bar{\eta}\circ q)^*$ is an isomorphism.

Combining the diagrams (7) and (8), we obtain a final commutative diagram

$$\begin{array}{ccc} H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0},\overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}} & \xleftarrow{(\bar{\eta}\circ q)^*} & H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0},\overline{\mathbb{Q}}_\ell)_{\overline{\phi}_Z^{-1}} \\ {\scriptstyle \kappa(s'g)}\uparrow & & \uparrow{\scriptstyle (\kappa_Z\circ(\bar{\eta}\circ q))(s'g)} \\ H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0},\overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}} & \xleftarrow[(\bar{\eta}\circ q)^*]{} & H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0},\overline{\mathbb{Q}}_\ell)_{\overline{\phi}_Z^{-1}} \end{array}$$

for all $s'\in\mathcal{S}'(\mathfrak{f})$, $g\in\mathcal{G}^0(\mathfrak{f})$.

As such, we conclude that $\kappa\simeq\kappa_Z\circ(\bar{\eta}\circ q)$, and thus $\kappa_{(S,\phi^{-1})}\simeq\kappa_{(S_Z,\phi_Z^{-1})}\circ\eta$. □

Proposition 3.13 (Theorem 3.2(d)). *Let ρ and ρ_Z be the representations of $G^0(F)_y$ and $G_Z^0(F)_{y_Z}$ constructed from (S,θ) and (S_Z,θ_Z), respectively, as per Section 2.4. Then*

$$\rho_Z\circ\eta=\bigoplus_{c_Z\in C_Z}\rho\circ\underline{\mathrm{Ad}}(c_Z^{-1}),$$

where C_Z is a set of coset representatives of $\eta(K^0)\backslash K_Z^0/S_Z(F)$.

By Corollary 3.7, we have that $\underline{\mathrm{Ad}}(c_Z^{-1})(K^0)=K^0$ for all $c_Z\in K_Z^0$. Therefore, the direct sum decomposition above makes sense as a representation of K^0.

Proof of Proposition 3.13. From the Mackey decomposition, we have

$$\begin{aligned}\rho_Z\circ\eta &= \big(\mathrm{Ind}_{S_Z(F)G_Z^0(F)_{y_Z,0}}^{K_Z^0}\kappa_{(S_Z,\phi_Z^{-1})}\big)\circ\eta\\ &= \big(\mathrm{Res}_{\eta(K^0)}^{K_Z^0}\mathrm{Ind}_{S_Z(F)G_Z^0(F)_{y_Z,0}}^{K_Z^0}\kappa_{(S_Z,\phi_Z^{-1})}\big)\circ\eta\\ &= \bigoplus_{c_Z\in C_Z}\big[\big(\mathrm{Ind}_{\eta(K^0)\cap{}^{c_Z}(S_Z(F)G_Z^0(F)_{y_Z,0})}^{\eta(K^0)}\mathrm{Res}_{\eta(K^0)\cap{}^{c_Z}(S_Z(F)G_Z^0(F)_{y_Z,0})}^{{}^{c_Z}(S_Z(F)G_Z^0(F)_{y_Z,0})}{}^{c_Z}\kappa_{(S_Z,\phi_Z^{-1})}\big)\circ\eta\big],\end{aligned}$$

where C_Z is a set of coset representatives of $\eta(K^0)\backslash K_Z^0/S_Z(F)G_Z^0(F)_{y_Z,0}$, which is equal to $\eta(K^0)\backslash K_Z^0/S_Z(F)$ by Lemma 3.10. Given that $\eta(K^0)$ is a normal subgroup of K_Z^0 (Lemma 3.6) and that $\eta(K^0)\cap S_Z(F)G_Z^0(F)_{y_Z,0}=\eta(S(F)G^0(F)_{y,0})$ (as a consequence of Lemma 3.10), it follows that

$$\begin{aligned}\eta(K^0)\cap{}^{c_Z}(S_Z(F)G_Z^0(F)_{y_Z,0}) &= {}^{c_Z}(\eta(K^0)\cap S_Z(F)G_Z^0(F)_{y_Z,0})\\ &= {}^{c_Z}\eta(S(F)G^0(F)_{y,0})\\ &= \eta(\underline{\mathrm{Ad}}(c_Z)(S(F)G^0(F)_{y,0})).\end{aligned}$$

Thus, we may simplify the above expression and apply Proposition A.2 to obtain

$$\begin{aligned}\rho_Z\circ\eta &= \bigoplus_{c_Z\in C_Z}\big[\big(\mathrm{Ind}_{\eta(\underline{\mathrm{Ad}}(c_Z)(S(F)G^0(F)_{y,0}))}^{\eta(K^0)}\mathrm{Res}_{\eta(\underline{\mathrm{Ad}}(c_Z)(S(F)G^0(F)_{y,0}))}^{{}^{c_Z}(S_Z(F)G_Z^0(F)_{y_Z,0})}{}^{c_Z}\kappa_{(S_Z,\phi_Z^{-1})}\big)\circ\eta\big]\\ &\simeq \bigoplus_{c_Z\in C_Z}\mathrm{Ind}_{\underline{\mathrm{Ad}}(c_Z)(S(F)G^0(F)_{y,0})}^{\underline{\mathrm{Ad}}(c_Z)(K^0)}({}^{c_Z}\kappa_{(S_Z,\phi_Z^{-1})}\circ\eta).\end{aligned}$$

Applying Lemma 2.8, followed by Corollary 3.12 on the previous expression,

$$\begin{aligned}\rho_Z\circ\eta &\simeq \bigoplus_{c_Z\in C_Z}\mathrm{Ind}_{\underline{\mathrm{Ad}}(c_Z)(S(F)G^0(F)_{y,0})}^{\underline{\mathrm{Ad}}(c_Z)(K^0)}[(\kappa_{(S_Z,\phi_Z^{-1})}\circ\eta)\circ\underline{\mathrm{Ad}}(c_Z^{-1})]\\ &= \bigoplus_{c_Z\in C_Z}\mathrm{Ind}_{\underline{\mathrm{Ad}}(c_Z)(S(F)G^0(F)_{y,0})}^{\underline{\mathrm{Ad}}(c_Z)(K^0)}(\kappa_{(S,\phi^{-1})}\circ\underline{\mathrm{Ad}}(c_Z^{-1})).\end{aligned}$$

Finally, we apply Proposition A.1 to extract the $\underline{\mathrm{Ad}}$ map from the induction and get

$$\begin{aligned}\rho_Z\circ\eta &\simeq \bigoplus_{c_Z\in C_Z}\big(\mathrm{Ind}_{S(F)G^0(F)_{y,0}}^{K^0}\kappa_{(S,\phi^{-1})}\big)\circ\underline{\mathrm{Ad}}(c_Z^{-1})\\ &= \bigoplus_{c_Z\in C_Z}\rho\circ\underline{\mathrm{Ad}}(c_Z^{-1}).\end{aligned}$$ □

Proposition 3.13 completes the proof of Theorem 3.2.

3.2. ***Going through the steps of the J.-K. Yu construction.*** Let (S,θ) and (S_Z,θ_Z) be tame F-nonsingular elliptic pairs of G and G_Z, respectively, such that $\eta(S)=S_Z$ and $\theta=\theta_Z\circ\eta$. In the previous section, we have established the relationship between the corresponding J.-K. Yu data, $(\vec{G}, y, \vec{r}, \rho, \vec{\phi})$ and $(\vec{G}_Z, y_Z, \vec{r}, \rho_Z, \vec{\phi}_Z)$, respectively. It is from these data that we construct the representations $\pi_{(S,\theta)}$ and $\pi_{(S_Z,\theta_Z)}$ following the steps of the J.-K. Yu construction as outlined in Figure 3. To be consistent with notation, we will keep using subscript Z to differentiate between the construction over G_Z from that of G. Since we have that $\phi^i = \phi^i_Z \circ \eta$ for all $0 \le i \le d$ and $\rho_Z \circ \eta = \bigoplus_{c_Z \in C_Z} \rho \circ \underline{\mathrm{Ad}}(c_Z^{-1})$ with C_Z a set of coset representatives of $\eta(K^0)\backslash K^0_Z/S_Z(F)$, it is natural to expect that we also have $\phi^{i\,\prime} = \phi^{i\,\prime}_Z \circ \eta$, $\kappa_Z^{-1} \circ \eta = \bigoplus_{c_Z \in C_Z} \kappa^{-1} \circ \underline{\mathrm{Ad}}(c_Z^{-1})$ and $\kappa^i = \kappa^i_Z \circ \eta$ for all $0 \le i \le d$, as illustrated in Figure 5. Indeed, we prove these equalities and inclusion with Propositions 3.14 and 3.15. In particular, one can say that the J.-K. Yu construction commutes with the map η. The above results allow us to complete the proof of Theorem 3.17 at the end of this section.

In order to define the representation $\phi^{i\,\prime}$, we require the groups J^{i+1} and J^{i+1}_+, which were previously mentioned in Remark 3.5. The construction of $\phi^{i\,\prime}$ from ϕ^i is divided into two steps: the first step consists of extending ϕ^i to a character $\widehat{\phi^i}$ of $K^i G^{i+1}(F)_{y,s_i^+}$, where $s_i = r_i/2$. The character $\widehat{\phi^i}$ is the unique character of $K^i G^{i+1}(F)_{y,s_i^+}$ that agrees with ϕ^i on K^i and is trivial on $(G^i, G^{i+1})(F)_{y,(r_i^+,s_i^+)}$ [Hakim and Murnaghan 2008, Section 3.1]. When $J^{i+1} \neq J^{i+1}_+$, a second step is required to enlarge the character $\widehat{\phi^i}$ a little further to a representation of K^{i+1} by means of a Heisenberg–Weil lift. We adopt analogous notation to describe the construction of $\phi^{i\,\prime}_Z$ from ϕ^i_Z. We note that $J^{i+1} = J^{i+1}_+$ if and only if $J^{i+1}_Z = J^{i+1}_{Z+}$ (as a consequence of Remark 3.5), which ensures that the construction of $\phi^{i\,\prime}$ requires a Heisenberg–Weil lift if and only if that of $\phi^{i\,\prime}_Z$ does.

Proposition 3.14. *For all* $0 \le i \le d$ *we have* $\phi^{i\,\prime} = \phi^{i\,\prime}_Z \circ \eta$.

Proof. We have that $\widehat{\phi^i} = \widehat{\phi^i_Z} \circ \eta$. Indeed, given that

$$\eta(K^i) \subset K^i_Z \quad \text{and} \quad \eta((G^i, G^{i+1})(F)_{y,(r_i^+,s_i^+)}) = (G^i_Z, G^{i+1}_Z)(F)_{y_Z,(r_i^+,s_i^+)}$$

(Remark 3.5), one sees that $\widehat{\phi^i_Z} \circ \eta$ agrees with ϕ^i on K^i and that it is trivial on $(G^i, G^{i+1})(F)_{y,(r_i^+,s_i^+)}$.

If $J^{i+1} = J^{i+1}_+$, we have $\phi^{i\,\prime} = \widehat{\phi^i}$ and $\phi^{i\,\prime}_Z = \widehat{\phi^i_Z}$ and we are done. If $J^{i+1} \neq J^{i+1}_+$, we have that $\phi^{i\,\prime}$ is constructed using a Heisenberg–Weil lift ω^i, which is a representation of $K^i \ltimes \mathcal{H}^i$, where $\mathcal{H}^i = J^{i+1}/\ker(\xi^i)$ and $\xi^i = \widehat{\phi^i}|_{J^{i+1}_+}$. We then have $\phi^{i\,\prime}(kj) = \widehat{\phi^i}(k)\omega^i(k, j\ker(\xi^i))$ for all $k \in K^i$, $j \in J^{i+1}$. Since $J^{i+1} \neq J^{i+1}_+$ if and only if $J^{i+1}_Z \neq J^{i+1}_{Z+}$ (Remark 3.5), we also require a Heisenberg–Weil lift ω^i_Z, which is a representation of $K^i_Z \ltimes \mathcal{H}^i_Z$, where $\mathcal{H}^i_Z = J^{i+1}_Z/\ker(\xi^i_Z)$ and $\xi^i_Z = \widehat{\phi^i_Z}|_{J^{i+1}_{Z+}}$, and have that

$$\phi^{i\,\prime}_Z(k_Z j_Z) = \widehat{\phi^i_Z}(k)\omega^i_Z(k_Z, j_Z \ker(\xi^i_Z))$$

for all $k_Z \in K^i_Z$, $j_Z \in J^{i+1}_Z$.

Since we already know that $\widehat{\phi^i} = \widehat{\phi^i_Z} \circ \eta$, it then suffices to show that $\omega^i = \omega^i_Z \circ \eta$. The map η induces isomorphisms $\mathcal{H}^i \simeq \mathcal{H}^i_Z$ and $W^i \simeq W^i_Z$ (Remark 3.5), where $W^i = J^{i+1}/J^{i+1}_+$ and $W^i_Z = J^{i+1}_Z/J^{i+1}_{Z+}$. We then obtain that $\omega^i = \omega^i_Z \circ \eta$ as an application of [Nevins 2015, Proposition 3.2], in which we set $H_1 = \mathcal{H}^i$, $H_2 = \mathcal{H}^i_Z$, $W_1 = W^i$, $W_2 = W^i_Z$, $T_1 = K^i$, $T_2 = K^i_Z$, $\alpha = \delta = \eta$, ν_1 and ν_2 the corresponding special isomorphisms from [Hakim and Murnaghan 2008, Lemma 2.35], and f_1 and f_2 the homomorphisms coming from the actions by conjugation of K^i and K^i_Z on J^{i+1} and J^{i+1}_Z, respectively. □

Proposition 3.15. *For all* $0 \le i \le d$ *we have* $\kappa^i = \kappa^i_Z \circ \eta$. *Furthermore,*

$$\kappa_Z^{-1} \circ \eta = \bigoplus_{c_Z \in C_Z} \kappa^{-1} \circ \underline{\mathrm{Ad}}(c_Z^{-1}),$$

where C_Z *is a set of coset representatives of* $\eta(K^0)\backslash K^0_Z/S_Z(F)$.

Proof. Let $0 \le i \le d-1$. Let us briefly recall the process of inflation. We have that $K^d = K^{i+1}J$, where $J = J^{i+2} \cdots J^d$. Then, for all $k \in K^{i+1}$, $j \in J$, $\kappa^i(kj) = \phi^{i'}(k)$. Similarly, we have $K^d_Z = K^{i+1}_Z J_Z$, where $J_Z = J^{i+2}_Z \cdots J^d_Z$, and $\kappa^i_Z(k_Z j_Z) = \phi^{i}_Z{}'(k_Z)$ for all $k_Z \in K^{i+1}_Z$, $j_Z \in J_Z$.

Using these definitions, for all $k \in K^{i+1}$, $j \in J$, we have

$$\kappa^i(kj) = \phi^{i'}(k) = \phi^i_Z{}'(\eta(k)).$$

By Remark 3.5, we have that $\eta(k) \in K^{i+1}_Z$ and $\eta(j) \in J_Z$. Therefore,

$$\phi^i_Z{}'(\eta(k)) = \kappa^i_Z(\eta(k)\eta(j)) = \kappa^i_Z \circ \eta(kj).$$

Thus, we conclude that $\kappa^i = \kappa^i_Z \circ \eta$.

By a similar argument, we have that

$$\kappa_Z^{-1} \circ \eta = \bigoplus_{c_Z \in C_Z} \kappa^{-1} \circ \underline{\mathrm{Ad}}(c_Z^{-1})$$

as a consequence of having (Proposition 3.13)

$$\rho_Z \circ \eta = \bigoplus_{c_Z \in C_Z} \rho \circ \underline{\mathrm{Ad}}(c_Z^{-1}),$$

where C_Z is a set of coset representatives of $\eta(K^0)\backslash K^0_Z/S_Z(F)$. □

Combining the previous proposition with the Mackey decomposition formula, we obtain the following relationship between $\kappa_{(S,\theta)}$ and $\kappa_{(S_Z,\theta_Z)}$.

Proposition 3.16. *Let* (S, θ) *and* (S_Z, θ_Z) *be tame* F*-nonsingular elliptic pairs of* G *and* G_Z*, respectively. Assume that* $\eta(S) = S_Z$ *and* $\theta = \theta_Z \circ \eta$. *Then,* $\kappa_{(S_Z,\theta_Z)} \circ \eta = \bigoplus_{l_Z \in L_Z} \kappa_{(S,\theta)} \circ \underline{\mathrm{Ad}}(l_Z^{-1})$, *where* L_Z *is a set of coset representatives of* $\eta(K^0)\backslash K^0_Z/S_Z(F)$.

By Corollary 3.7, we have that $\underline{\mathrm{Ad}}(l_Z^{-1})(K^d) = K^d$ for all $l_Z \in L_Z \subseteq K_Z^0$. Therefore, the direct sum decomposition above makes sense as a representation of K^d.

Proof of Proposition 3.16. For all $l_Z \in L_Z$, we have that ${}^{l_Z}\phi_Z^i = \phi_Z^i$, as $K_Z^0 \subset G_Z^i(F)$ for all i. It follows that ${}^{l_Z}\kappa_Z^i \simeq \kappa_Z^i$, which implies ${}^{l_Z}\kappa_Z^i \circ \eta \simeq \kappa_Z^i \circ \eta$. Using Proposition 3.15, we conclude that $\kappa^i \circ \underline{\mathrm{Ad}}(l_Z^{-1}) \simeq \kappa_i$ for all $l_Z \in L_Z$. Furthermore, Proposition 3.15 also tells us that $\kappa_Z^{-1} \circ \eta = \bigoplus_{l_Z \in L_Z} \kappa^{-1} \circ \underline{\mathrm{Ad}}(l_Z^{-1})$. We thus obtain

$$\begin{aligned}
\kappa_{(S_Z,\theta_Z)} \circ \eta &= \Big[\kappa_Z^{-1} \otimes \Big(\bigotimes_{i=0}^{d} \kappa_Z^i\Big)\Big] \circ \eta \\
&= \kappa_Z^{-1} \circ \eta \otimes \Big(\bigotimes_{i=0}^{d} \kappa_Z^i \circ \eta\Big) \\
&= \Big(\bigoplus_{l_Z \in L_Z} \kappa^{-1} \circ \underline{\mathrm{Ad}}(l_Z^{-1})\Big) \otimes \bigotimes_{i=0}^{d} \kappa^i \\
&= \bigoplus_{l_Z \in L_Z} \Big(\kappa^{-1} \circ \underline{\mathrm{Ad}}(l_Z^{-1}) \otimes \bigotimes_{i=0}^{d} \kappa^i\Big) \\
&\simeq \bigoplus_{l_Z \in L_Z} \Big(\kappa^{-1} \otimes \bigotimes_{i=0}^{d} \kappa^i\Big) \circ \underline{\mathrm{Ad}}(l_Z^{-1}) \\
&= \bigoplus_{l_Z \in L_Z} \kappa_{(S,\theta)} \circ \underline{\mathrm{Ad}}(l_Z^{-1}). \qquad \square
\end{aligned}$$

We are now in a position to complete the proof of our main theorem.

Theorem 3.17. *Let (S, θ) and (S_Z, θ_Z) be tame F-nonsingular elliptic pairs of G and G_Z, respectively. Assume that $\eta(S) = S_Z$ and $\theta = \theta_Z \circ \eta$. Then*

$$\pi_{(S_Z,\theta_Z)} \circ \eta \simeq \bigoplus_{d_Z \in D_Z} \pi_{(S,\theta)} \circ \underline{\mathrm{Ad}}(d_Z^{-1}),$$

where D_Z is a set of coset representatives of $\eta(G(F))\backslash G_Z(F)/S_Z(F)$.

Proof. From the Mackey decomposition, we have

$$\begin{aligned}
\pi_{(S_Z,\theta_Z)} \circ \eta &= \big(\mathrm{Ind}_{K_Z^d}^{G_Z(F)} \kappa_{(S_Z,\theta_Z)}\big) \circ \eta \\
&= \big(\mathrm{Res}_{\eta(G(F))}^{G_Z(F)} \mathrm{Ind}_{K_Z^d}^{G_Z(F)} \kappa_{(S_Z,\theta_Z)}\big) \circ \eta \\
&= \bigoplus_{\ell_Z \in \mathcal{L}_Z} \big(\mathrm{Ind}_{\eta(G(F)) \cap {}^{\ell_Z}K_Z^d}^{\eta(G(F))} \mathrm{Res}_{\eta(G(F)) \cap {}^{\ell_Z}K_Z^d}^{{}^{\ell_Z}K_Z^d} {}^{\ell_Z}\kappa_{(S_Z,\theta_Z)}\big) \circ \eta,
\end{aligned}$$

where $\mathcal{L}_Z$ is a set of coset representatives of $\eta(G(F))\backslash G_Z(F)/K_Z^d$. Given that $\eta(G(F))$ is a normal subgroup of $G_Z(F)$ and $\eta(G(F)) \cap K_Z^d = \eta(K^d)$ (Lemma 3.6),

$$\eta(G(F)) \cap {}^{\ell_Z}K_Z^d = {}^{\ell_Z}(\eta(G(F)) \cap K_Z^d) = {}^{\ell_Z}\eta(K^d) = \eta(\underline{\mathrm{Ad}}(\ell_Z)(K^d)).$$

As a result, we may simplify the above expression and apply Proposition A.2 to obtain

$$\begin{aligned}\pi_{(S_Z,\theta_Z)}\circ\eta &= \bigoplus_{\ell_Z\in\mathcal{L}_Z}\big(\operatorname{Ind}_{\eta(\underline{\mathrm{Ad}}(\ell_Z)(K^d))}^{\ell_Z\eta(G(F))}\operatorname{Res}_{\eta(\underline{\mathrm{Ad}}(\ell_Z)(K^d))}^{\ell_Z K_Z^d}{}^{\ell_Z}\kappa_{(S_Z,\theta_Z)}\big)\circ\eta\\ &\simeq \bigoplus_{\ell_Z\in\mathcal{L}_Z}\operatorname{Ind}_{\underline{\mathrm{Ad}}(\ell_Z)(K^d)}^{\underline{\mathrm{Ad}}(\ell_Z)(G(F))}\big({}^{\ell_Z}\kappa_{(S_Z,\theta_Z)}\circ\eta\big).\end{aligned}$$

We then apply Lemma 2.8, followed by Proposition A.1 on the previous expression and get

$$\begin{aligned}\pi_{(S_Z,\theta_Z)}\circ\eta &\simeq \bigoplus_{\ell_Z\in\mathcal{L}_Z}\operatorname{Ind}_{\underline{\mathrm{Ad}}(\ell_Z)(K^d)}^{\underline{\mathrm{Ad}}(\ell_Z)(G(F))}[(\kappa_{(S_Z,\theta_Z)}\circ\eta)\circ\underline{\mathrm{Ad}}(\ell_Z^{-1})]\\ &\simeq \bigoplus_{\ell_Z\in\mathcal{L}_Z}\big(\operatorname{Ind}_{K^d}^{G(F)}\kappa_{(S_Z,\theta_Z)}\circ\eta\big)\circ\underline{\mathrm{Ad}}(\ell_Z^{-1}).\end{aligned}$$

Replacing $\kappa_{(S_Z,\theta_Z)}\circ\eta$ by its equivalent direct sum decomposition in Proposition 3.16, with L_Z a set of coset representatives of $\eta(K^0)\backslash K_Z^0/S_Z(F)$, it follows that

$$\begin{aligned}\pi_{(S_Z,\theta_Z)}\circ\eta &= \bigoplus_{\ell_Z\in\mathcal{L}_Z}\operatorname{Ind}_{K^d}^{G(F)}\Bigg(\bigoplus_{l_Z\in L_Z}\kappa_{(S,\theta)}\circ\underline{\mathrm{Ad}}(l_Z^{-1})\Bigg)\circ\underline{\mathrm{Ad}}(\ell_Z^{-1})\\ &= \bigoplus_{\ell_Z\in\mathcal{L}_Z}\bigoplus_{l_Z\in L_Z}\operatorname{Ind}_{\underline{\mathrm{Ad}}(l_Z)(K^d)}^{\underline{\mathrm{Ad}}(l_Z)(G(F))}(\kappa_{(S,\theta)}\circ\underline{\mathrm{Ad}}(l_Z^{-1}))\circ\underline{\mathrm{Ad}}(\ell_Z^{-1})\\ &\simeq \bigoplus_{\ell_Z\in\mathcal{L}_Z}\bigoplus_{l_Z\in L_Z}\big(\operatorname{Ind}_{K^d}^{G(F)}\kappa_{(S,\theta)}\big)\circ\underline{\mathrm{Ad}}(l_Z^{-1})\circ\underline{\mathrm{Ad}}(\ell_Z^{-1})\\ &= \bigoplus_{\ell_Z\in\mathcal{L}_Z}\bigoplus_{l_Z\in L_Z}\pi_{(S,\theta)}\circ\underline{\mathrm{Ad}}((\ell_Z l_Z)^{-1}).\end{aligned}$$

Finally, we claim that $\{\ell_Z l_Z : \ell_Z\in\mathcal{L}_Z,\ l_Z\in L_Z\}$ is a set of coset representatives of $\eta(G(F))\backslash G_Z(F)/S_Z(F)$, which we denote by D_Z, allowing us to write

$$\pi_{(S_Z,\theta_Z)}\circ\eta = \bigoplus_{d_Z\in D_Z}\pi_{(S,\theta)}\circ\underline{\mathrm{Ad}}(d_Z^{-1}).$$

To prove this last claim, we set $N=\eta(G(F))$, $A=G_Z(F)$, $B=K_Z^d$, $\overline{N}=\eta(K^0)$, $\overline{A}=K_Z^0$ and $\overline{B}=S_Z(F)$ and show that N, A, B, $\overline{N}$, $\overline{A}$, $\overline{B}$ satisfy the hypotheses of Lemma A.4(1). It is clear that N and $\overline{N}$ are normal subgroups of A and $\overline{A}$, respectively, and that $\overline{A}\subseteq B$ and $N\cap\overline{A}\subseteq\overline{N}$. It remains to show that $B/(N\cap B)\overline{B}\simeq\overline{A}/\overline{N}\overline{B}$.

Recall that $K_Z^d=K_Z^0 J_Z$ and $\eta(K^d)=\eta(K^0 J)=\eta(K^0)J_Z$, where J and J_Z are as in the proof of Lemma 3.6. Therefore,

$$B/(N\cap B)\overline{B}=K_Z^d/\eta(K^d)\overline{B}=K_Z^0 J_Z/\eta(K^0)J_Z\overline{B}.$$

Given that $\bar{B}$ is in the stabilizer of y_Z, it follows that $\bar{B}$ normalizes J_Z and thus $J_Z\bar{B} = \bar{B}J_Z$. This, in combination with Lemma A.3, allows us to obtain

$$B/(N \cap B)\bar{B} \simeq K_Z^0/\eta(K^0)\bar{B}(K_Z^0 \cap J_Z) = K_Z^0/\eta(K^0)\bar{B},$$

where the last equality follows from the fact that $K_Z^0 \cap J_Z = K_Z^0 \cap \eta(J) \subset \eta(K^0)$ (Lemma 3.6). Thus, we conclude from Lemma A.4(1) that $\{\ell_Z l_Z : \ell_Z \in \mathcal{L}_Z,\ l_Z \in L_Z\}$ is a set of coset representatives of $\eta(G(F))\backslash G_Z(F)/S_Z(F)$. □

3.3. ***The proof of Theorem 3.1.*** As mentioned earlier in this section, and illustrated in Figure 1, computing the decomposition of $\pi_{(\underline{S},\underline{\theta})} \circ \eta$ is done in two steps. So far, this section has focused on the second step. The first step consists of applying the results of [Bourgeois 2021] to compute the restriction of $\pi_{(\underline{S},\underline{\theta})}$ to $G_Z(F)$. We begin this section with a lemma on parahoric subgroups. We then summarize the results of [Bourgeois 2021] in the context of the Kaletha–Yu construction, and end with a proof of Theorem 3.1.

Lemma 3.18. *Let G be a reductive F-group, and H be an F-subgroup that contains $[G, G]$. Let S be a maximally unramified elliptic maximal torus of G, and let $S_H = S \cap H$, so that S_H is a maximally unramified elliptic maximal torus of H. Denote by y the point of the reduced building associated to S (and S_H) via* [Kaletha 2019, Lemma 3.4.3]*. Let G^0 and G_H^0 denote the smallest Levi subgroups of the Levi sequences obtained from S and S_H, respectively, and recall that $G_H^0 = G^0 \cap H$. Then*

$$G^0(F)_{y,0} = S(F)_0\, H^0(F)_{y,0}.$$

Proof. By [Kaletha 2019, Lemma 3.4.2], we have that S is the centralizer of a maximal F^{un}-split torus S' of G^0. It follows by definition (see, for example, [Fintzen 2021, Section 2.4] or [Kaletha and Prasad 2023, Definition 13.2.1]) that

$$G^0(F^{\mathrm{un}})_{y,0} = \langle S(F^{\mathrm{un}})_0,\, U_\alpha(F^{\mathrm{un}}) : \alpha \in R^{\mathrm{aff}}_{F^{\mathrm{un}}},\ \langle \alpha, y\rangle \geq 0\rangle,$$

where $R^{\mathrm{aff}}_{F^{\mathrm{un}}} = \{\lambda + k : \lambda \in R(G^0, S) \text{ such that } \lambda|_{S'} \neq 1,\ k \in \mathbb{R}\}$, and $U_\alpha(F^{\mathrm{un}})$ is the affine root subgroup associated to the affine root α. The affine root subgroups are normalized by $S(F^{\mathrm{un}})_0$, allowing us to write

$$\begin{aligned} G^0(F^{\mathrm{un}})_{y,0} &= S(F^{\mathrm{un}})_0\, \langle S_H(F^{\mathrm{un}})_0,\, U_\alpha(F^{\mathrm{un}}) : \alpha \in R^{\mathrm{aff}}_{F^{\mathrm{un}}},\ \langle \alpha, y\rangle \geq 0\rangle \\ &= S(F^{\mathrm{un}})_0\, H^0(F^{\mathrm{un}})_{y,0}, \end{aligned}$$

and

$$G^0(F)_{y,0} = (G^0(F^{\mathrm{un}})_{y,0})^{\mathrm{Fr}} = (S(F^{\mathrm{un}})_0\, H^0(F^{\mathrm{un}})_{y,0})^{\mathrm{Fr}}.$$

Using [Kaletha 2019, Lemma 3.4.6] and the definition of S_H, we have that

$$S(F^{\mathrm{un}})_0 \cap H^0(F^{\mathrm{un}})_{y,0} = S_H(F^{\mathrm{un}})_0.$$

Furthermore, $H^1(\mathrm{Fr}, S_H(F^{\mathrm{un}})_0)$ is trivial [Kaletha and Prasad 2023, Lemma 8.1.4]. It follows from the usual sequence of Galois cohomology [Springer 2009, Proposition 12.3.4] that

$$(S(F^{\mathrm{un}})_0 H^0(F^{\mathrm{un}})_{y,0})^{\mathrm{Fr}} = S(F^{\mathrm{un}})_0^{\mathrm{Fr}} (H^0(F^{\mathrm{un}})_{y,0})^{\mathrm{Fr}} = S(F)_0 H^0(F)_{y,0},$$

and therefore

$$G^0(F)_{y,0} = S(F)_0 H^0(F)_{y,0}. \qquad \square$$

Theorem 3.19. *Let (S,θ) be a tame F-nonsingular elliptic pair of G and let y be the vertex of $\mathcal{B}(G,F)$ associated to S. Let H be a closed connected F-subgroup of G that contains $[G,G]$. Set $S_H = S\cap H$ and $\theta_H = \theta|_{S_H}$. Then (S_H,θ_H) is a tame F-nonsingular elliptic pair of H and*

$$\pi_{(S,\theta)}|_{H(F)} = \bigoplus_{d\in D} {}^d\pi_{(S_H,\theta_H)},$$

where D is a set of coset representatives of $H(F)\backslash G(F)/S(F)$.

Proof. Let $\Psi_{(S,\theta)} = (\vec{G}, y, \vec{r}, \rho, \vec{\phi})$ be the G-datum obtained from the pair (S,θ) as in Section 2.4. Recall that we may write $\pi_G(\Psi_{(S,\theta)})$ for $\pi_{(S,\theta)}$ and $\kappa_G(\Psi_{(S,\theta)})$ for $\kappa_{(S,\theta)}$ to indicate that we are applying the J.-K. Yu construction to $\Psi_{(S,\theta)}$. Set $K_H^i = K^i \cap H$ for all $0\le i\le d$ and $\Psi^H_{(S,\theta)} = (\vec{H}, y, \vec{\tilde{r}}, \rho|_{K_H^0}, \vec{\phi}_H)$, where $\vec{H}$, $\vec{\tilde{r}}$ and $\vec{\phi}_H$ are as per [Bourgeois 2021, Theorem 4.1]. Then, it follows from [Bourgeois 2021, Theorems 5.7 and 5.8] that

$$\pi_{(S,\theta)}|_{H(F)} = \pi_G(\Psi_{(S,\theta)})|_{H(F)} \simeq \bigoplus_{l\in L} {}^l\pi_H(\Psi^H_{(S,\theta)}),$$

where L is a set of coset representatives of $H(F)\backslash G(F)/K^d$.

We have that the elements $\vec{H}$, y, $\vec{\tilde{r}}$ and $\vec{\phi}_H$ from the datum $\Psi^H_{(S,\theta)}$ also appear in the datum $\Psi_{(S_H,\theta_H)}$. Indeed, $\vec{H}$ is the twisted Levi sequence associated to S_H by [Bourgeois 2021, Theorem 2.3] and the discussion preceding it, and the point y is the vertex of $\mathcal{B}(H,F)$ associated to S_H by [Bourgeois 2021, Lemma 7.1]. The character sequence $\vec{\phi}_H$ clearly satisfies the first two axioms to be a Howe factorization of θ_H, and genericity is given by [Bourgeois 2021, Proposition 4.7]. Therefore, assembling these pieces along with the construction from Figure 2, we have

$$\Psi_{(S_H,\theta_H)} = \big(\vec{H}, y, \vec{\tilde{r}}, \mathrm{Ind}_{S_H(F)H^0(F)_{y,0}}^{K_H^0} \kappa_{(S_H,\theta_H)}, \vec{\phi}_H\big).$$

Applying the Mackey decomposition as in the proof of [Bourgeois 2021, Proposition 7.5], we have

$$\rho|_{K_H^0} \simeq \bigoplus_{\ell\in\mathcal{L}} {}^\ell\mathrm{Ind}_{S_H(F)H^0(F)_{y,0}}^{K_H^0} \kappa_{(S_H,\theta_H)},$$

where $\mathcal{L}$ is a set of coset representatives of $K_H^0\backslash K^0/S(F)G^0(F)_{y,0}$. Therefore,

$$\pi_H(\Psi^H_{(S,\theta)}) \simeq \bigoplus_{\ell\in\mathcal{L}} {}^{\ell}\pi_H(\Psi_{(S_H,\theta_H)}) = \bigoplus_{\ell\in\mathcal{L}} {}^{\ell}\pi_{(S_H,\theta_H)},$$

which implies

$$\pi_{(S,\theta)}|_{H(F)} \simeq \bigoplus_{l\in L}\bigoplus_{\ell\in\mathcal{L}} {}^{l\ell}\pi_{(S_H,\theta_H)}.$$

Using Lemma 3.18, one rewrites $\mathcal{L}$ as $K_H^0\backslash K^0/S(F)$. We claim that $L\mathcal{L} = \{l\ell : l\in L,\ \ell\in\mathcal{L}\}$ is a set of coset representatives of $H(F)\backslash G(F)/S(F)$, which we denote by D, allowing us to write

$$\pi_{(S,\theta)} \simeq \bigoplus_{d\in D} {}^{d}\pi_{(S_H,\theta_H)}.$$

To prove this last claim, we set $N = H(F)$, $A = G(F)$, $B = K^d$, $\bar{N} = K_H^0$, $\bar{A} = K^0$ and $\bar{B} = S(F)$, and show that N, A, B, $\bar{N}$, $\bar{A}$, $\bar{B}$ satisfy the hypotheses of Lemma A.4(1). It is clear that N and $\bar{N}$ are normal subgroups of A and $\bar{A}$, respectively, and that $\bar{A}\subseteq B$ and $N\cap\bar{A} = \bar{N}$. It remains to show that $B/(N\cap B)\bar{B} \simeq \bar{A}/\bar{N}\bar{B}$.

Setting $J_H = H^1(F)_{y,r_0/2}\cdots H^d(F)_{y,r_{d-1}/2}$, we have $K_H^d = K_H^0 J_H$ by definition, and $K^d = K^0 J_H$ as per [Bourgeois 2021, Proof of Proposition 5.1]. It follows that

$$B/(N\cap B)\bar{B} = K^d/K_H^d S(F) = K^0 J_H/K_H^0 J_H S(F).$$

Given that $S(F)$ is in the stabilizer of y, we have that ${}^{s}H^i(F)_{y,r} = H^i(F)_{s\cdot y,r} = H^i(F)_{y,r}$ for all $s\in S(F)$, $r\geq 0$, $0\leq i\leq d$. Therefore, $S(F)$ normalizes J_H and $J_H S(F) = S(F)J_H$, allowing us to write

$$B/(N\cap B)\bar{B} = K^0 J_H/(K_H^0 S(F))J_H.$$

Applying Lemma A.3, we obtain

$$B/(N\cap B)\bar{B} \simeq K^0/K_H^0 S(F)(K^0\cap J_H) = K^0/K_H^0 S(F),$$

where the last equality follows from the fact that $K^0\cap J_H \subset K_H^0$. Thus,

$$B/(N\cap B)\bar{B} \simeq \bar{A}/\bar{N}\bar{B}. \qquad \square$$

Proof of Theorem 3.1. Setting $G = \underline{G}$ and $H = G_Z$ in Theorem 3.19, we have

$$\pi_{(\underline{S},\underline{\theta})}\circ\eta = \left(\mathrm{Res}^{G(F)}_{G_Z(F)}\,\pi_{(\underline{S},\underline{\theta})}\right)\circ\eta \simeq \left(\bigoplus_{\underline{c}\in\underline{C}} {}^{\underline{c}}\pi_{(S_Z,\theta_Z)}\right)\circ\eta,$$

where $S_Z = \underline{S}\cap G_Z$, $\theta_Z = \theta|_{S_Z}$ and $\underline{C}$ is a set consisting of coset representatives of $G_Z(F)\backslash\underline{G}(F)/\underline{S}(F)$.

By Lemma 2.8, ${}^{c}\pi_{(S_Z,\theta_Z)} \circ \eta = \pi_{(S_Z,\theta_Z)} \circ \eta \circ \underline{\mathrm{Ad}}(\underline{c}^{-1})$. Using this last equality, and applying Theorem 3.17, it follows that

$$\begin{aligned}\pi_{(\underline{S},\underline{\theta})} \circ \eta &\simeq \bigoplus_{\underline{c}\in\underline{C}} (\pi_{(S_Z,\theta_Z)} \circ \eta) \circ \underline{\mathrm{Ad}}(\underline{c}^{-1}) \\ &\simeq \bigoplus_{\underline{c}\in\underline{C}} \Big(\bigoplus_{d_Z\in D_Z} \pi_{(S,\theta)} \circ \underline{\mathrm{Ad}}(d_Z^{-1}) \Big) \circ \underline{\mathrm{Ad}}(\underline{c}^{-1}) \\ &= \bigoplus_{\underline{c}\in\underline{C}} \bigoplus_{d_Z\in D_Z} \pi_{(S,\theta)} \circ \underline{\mathrm{Ad}}((\underline{c}d_Z)^{-1}),\end{aligned}$$

where D_Z is a set of coset representatives of $\eta(G(F))\backslash G_Z(F)/S_Z(F)$. Setting $N = G_Z(F)$, $A = \underline{G}(F)$, $B = \underline{S}(F)$, $\bar{N} = \eta(G(F))$, $\bar{A} = G_Z(F)$, $\bar{B} = S_Z(F)$, one sees from Lemma A.4(2) that $\{\underline{c}d_Z : \underline{c} \in \underline{C},\ d_Z \in D_Z\}$ is a set of coset representatives of $\eta(G(F))\backslash\underline{G}(F)/\underline{S}(F)$. The decomposition formula thus follows. □

4. Functoriality for supercuspidal L-packets

One of the main aims of this paper is to show that Kaletha's supercuspidal L-packets satisfy a certain functoriality property. More specifically, the goal of this section is to prove the following theorem.

Theorem 4.1. *Suppose G is quasisplit and splits over a tamely ramified extension. Suppose further that the residual characteristic p of F does not divide the order of the Weyl group of G. Let $\eta : G \to \underline{G}$ be an F-morphism of connected reductive F-groups such that*

(i) *the kernel of $d\eta : \mathrm{Lie}(G) \to \mathrm{Lie}(\underline{G})$ is central,*

(ii) *the cokernel of η is an abelian F-group.*

Let $\underline{\varphi} \in \Phi_{\mathrm{sc}}(\underline{G})$ and set $\varphi = {}^{L}\eta \circ \underline{\varphi}$. Then for all $\underline{\pi} \in \Pi_{\underline{\varphi}}$, $\underline{\pi} \circ \eta$ is the direct sum of finitely many irreducible supercuspidal representations belonging to Π_{φ}.

Recall from Section 2.5 that the packet $\Pi_{\underline{\varphi}}$ is constructed from $\underline{\varphi}$ by first taking its associated supercuspidal L-packet datum $(\underline{S}, \hat{\underline{j}}, \underline{\chi}_0, \underline{\theta})$, and then taking the irreducible components of $\pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})}$ as $\underline{j}$ varies over $\underline{\mathcal{J}}_F$. Similarly, we let (S, j, χ_0, θ) be the supercuspidal L-packet datum associated to φ so that Π_{φ} consists of the irreducible components of $\pi_{(jS,j\theta)}$ as j varies over $\mathcal{J}_F$

The strategy to prove Theorem 4.1 is to apply the decomposition formula from Section 3 on the representation $\pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})} \circ \eta$. However, to do so, we must first find an F-nonsingular elliptic pair of G that relates to $(\underline{j}\underline{S}, \underline{j}\underline{\theta})$ in the sense of Theorem 3.1. To achieve this, we establish the relationship between the supercuspidal L-packet data $(\underline{S}, \hat{\underline{j}}, \underline{\chi}_0, \underline{\theta})$ and (S, j, χ_0, θ). In particular, the relationship between these data is induced by the map ${}^{L}\eta$, a statement we will make precise with Theorem 4.2 and illustrate in Figure 6.

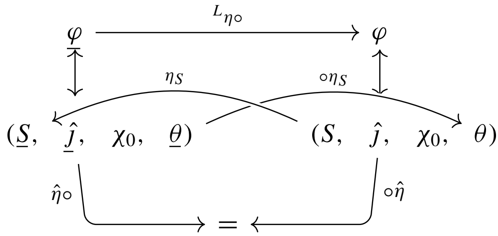

Figure 6. Summary of the relationship between the supercuspidal L-packet data associated to $\underline{\varphi}$ and φ. The map ${}^L\eta$ is constructed from a map $\hat{\eta}$, dual to η, which we recall in Section 4.1. The map η_S is the map dual to $\hat{\eta}|_{\underline{\hat{S}}}$.

In Section 4.1, we prove the relationship between the L-packet data, and then prove Theorem 3.1 in Section 4.2.

4.1. ***Matching the supercuspidal L-packet data.*** Let $\underline{\varphi} \in \Phi_{\mathrm{sc}}(\underline{G})$ and $\varphi = {}^L\eta \circ \underline{\varphi}$. The goal of this section is to match the corresponding parameterizing data for $\underline{\varphi}$ and φ through ${}^L\eta$. This matching is crucial for establishing the relationship between $\Pi_{\underline{\varphi}}$ and Π_{φ}.

The map ${}^L\eta$ is defined by ${}^L\eta(\underline{g}, w) = (\hat{\eta}(\underline{g}), w)$ for all $\underline{g} \in \widehat{\underline{G}}$ and $w \in W_F$, and so we begin with a review of the map $\hat{\eta} : \widehat{\underline{G}} \to \widehat{G}$ [Springer 1979, Sections 1 and 2]. Recall that we are assuming that G is quasisplit. We may therefore fix a Borel subgroup $B \subset G$ defined over F, and fix a maximally F-split maximal torus T of G with $T \subset B$. This choice of Borel pair (B, T) is equivalent to fixing a based root datum for G. The Γ-equivariant map η carries T to a maximally F-split torus $\eta(T)$ of $\eta(G) \supset [G, G]$. Set $\underline{T} = \eta(T)Z(\underline{G})$ and $\underline{B} = \eta(B)Z(\underline{G})$. Then $(\underline{B}, \underline{T})$ is a Borel pair for $\underline{G}$ which is defined over F [Springer 2009, Corollary 8.1.6]. Thus, we have fixed based root data for both G and $\underline{G}$, and η induces a homomorphism between them, in the sense of [Jantzen 2003, II.1.13]. This homomorphism of root data determines a homomorphism from the dual based root datum of $\underline{G}$ to the dual based root datum of G. These two dual based data may be identified with the based root data arising from Γ-invariant Borel pairs $(\widehat{\underline{B}}, \widehat{\underline{T}})$ and $(\widehat{B}, \widehat{T})$ which are implicit in the definition of ${}^L\underline{G}$ and ${}^LG$ [Borel 1979, 2.3]. Under this identification and the assumptions of Theorem 4.1, the homomorphism of dual based root data produces the homomorphism $\hat{\eta} : \widehat{\underline{G}} \to \widehat{G}$ of algebraic groups with abelian kernel and cokernel [Jantzen 2003, Proposition II.1.14]. It is only unique up to conjugation by $\widehat{T}$.

A change in the choice of based root data above has the effect of conjugating $\hat{\eta}$ by an element of $\widehat{G}$. This change has no effect on the equivalence classes of the objects under consideration. However, even for fixed based root data and a fixed choice of $\hat{\eta}$, there remains an ambiguity between η and $\hat{\eta}$. Indeed, conjugating η by an element $\underline{t} \in \underline{T}$ such that $\underline{t}Z(\underline{G}) \in (\underline{T}/Z(\underline{G}))(F)$ produces another F-homomorphism whose dual is identical to $\hat{\eta}$. As observed in [Solleveld 2020, Section 3], this ambiguity is consequential in matching the parameters of the L-packets under consideration.

In order to be more precise about these matters, we fix some pinnings. We extend the Γ-invariant Borel pair (B, T) to a Γ-invariant pinning $(T, B, \{X_\alpha\})$ of G. The application of η to this pinning fixes a Γ-invariant pinning $\big(\underline{T}, \underline{B}, \{\eta(X_\alpha)\}\big)$ of $\underline{G}$. It follows from [Solleveld 2020, Theorem 3.2] that $\eta : G \to \underline{G}$ is the unique F-homomorphism in its $(\underline{G}/Z(\underline{G}))(F)$-conjugacy class which carries the former pinning to the latter (see the discussion preceding [Solleveld 2020, Theorem 6.2]). Underlying the definition of ${}^L G$ and ${}^L \underline{G}$ are respective pinnings $(\widehat{T}, \widehat{B}, \{Y_{\hat{\alpha}}\})$ of $\widehat{G}$ and $(\widehat{\underline{T}}, \widehat{\underline{B}}, \{\underline{Y}_{\hat{\alpha}}\})$ of $\widehat{\underline{G}}$. The fixed pinnings determine $\hat{\eta}$ uniquely by the requirement that $\{\hat{\eta}(Y_{\hat{\alpha}})\} = \{\underline{Y}_{\hat{\alpha}}\}$. Furthermore, if $\eta' : G \to \underline{G}$ is another F-homomorphism satisfying the assumptions of Theorem 4.1 and $\widehat{\eta'} = \hat{\eta}$, then there is a unique element $\underline{t}' Z(\underline{G}) \in (\underline{T}/Z(\underline{G}))(F)$ such that $\eta' = \mathrm{Ad}(\underline{t}') \circ \eta$ [Solleveld 2020, Proposition 3.4].

Having set the foundation for the comparison between G and $\underline{G}$, let us return to the matching of the apposite parameters. They are to be matched as in Figure 6.

Theorem 4.2. *Let $\underline{\varphi} \in \Phi_{\mathrm{sc}}(\underline{G})$, $\varphi = {}^L\eta \circ \underline{\varphi}$ and $(\underline{S}, \underline{\hat{j}}, \underline{\chi}_0, \underline{\theta})$ and $(S, \hat{j}, \chi_0, \theta)$ be the associated supercuspidal L-packet data. Let $\mathcal{J}_F$ and $\underline{\mathcal{J}}_F$ be the sets of embeddings which parameterize Π_φ and $\Pi_{\underline{\varphi}}$, respectively. Then*

(a) $\hat{\eta}(\widehat{\underline{S}}) \subset \widehat{S}$, $\underline{\chi}_0 = \chi_0$ *and* $\hat{\eta} \circ \underline{\hat{j}} = \hat{j} \circ \hat{\eta}$,

(b) $\theta = \underline{\theta} \circ \eta_S$, *where η_S is the dual map of $\hat{\eta}|_{\widehat{\underline{S}}} : \widehat{\underline{S}} \to \widehat{S}$, and*

(c) *for all $\underline{j} \in \underline{\mathcal{J}}_F$, there exists $j \in \mathcal{J}_F$ such that $\eta(jS) \subset \underline{j}\underline{S}$ and $j\theta = \underline{j}\underline{\theta} \circ \eta$.*

The proof of this theorem will be divided into three parts. Proposition 4.4 will give the relationship between the tori, and consequently the embeddings and χ-data. Proposition 4.5 will give the relationship between the characters and Proposition 4.7 will provide the statement regarding the sets $\mathcal{J}_F$ and $\underline{\mathcal{J}}_F$.

Before proceeding to the statements of these propositions, we begin with a useful result, which can be shown using [Humphreys 1995, Theorem 2.2] and the presentation of reductive groups in terms of generators from [Humphreys 1975, Theorem 26.3].

Proposition 4.3. *Let T be a maximal torus of a connected reductive group G', and assume H is a subtorus of T. Then $T = \mathrm{Cent}(H, G')$ if and only if for every $\alpha \in R(G', T)$ there exists $h_\alpha \in H$ such that $\alpha(h_\alpha) \neq 1$.*

Proposition 4.4 (Theorem 4.2(a)). *Let $\underline{S}$ and S be as in Theorem 4.2. Then $\hat{\eta}(\widehat{\underline{S}}) \subset \widehat{S}$.*

Proof. Recall that $\widehat{S} = \mathrm{Cent}(\widehat{C}, \widehat{M})$, where $\widehat{M} = \mathrm{Cent}(\varphi(P_F), \widehat{G})^\circ$ and $\widehat{C} = \mathrm{Cent}(\varphi(I_F), \widehat{G})^\circ$. Similarly, $\widehat{\underline{S}} = \mathrm{Cent}(\widehat{\underline{C}}, \widehat{\underline{M}})$, where $\widehat{\underline{M}} = \mathrm{Cent}(\underline{\varphi}(P_F), \widehat{\underline{G}})^\circ$ and $\widehat{\underline{C}} = \mathrm{Cent}(\underline{\varphi}(I_F), \widehat{\underline{G}})^\circ$. We start by showing that

$$\hat{\eta}(\widehat{\underline{M}}) = \widehat{M} \cap \hat{\eta}(\widehat{\underline{G}}).$$

We have that $\underline{\varphi}(P_F)$ is contained in some maximal torus $\widehat{\underline{\mathcal{T}}}$ of $\widehat{\underline{G}}$ [Kaletha 2021, Lemma 4.1.3], and therefore $\varphi(P_F)$ is contained in the maximal torus $\hat{\eta}(\widehat{\underline{\mathcal{T}}})$ of $\hat{\eta}(\widehat{\underline{G}})$. Since $\hat{\eta}(\widehat{\underline{G}})$ contains $[\widehat{G}, \widehat{G}]$, we have that $\hat{\eta}(\widehat{\underline{\mathcal{T}}}) = \widehat{\mathcal{T}} \cap \hat{\eta}(\widehat{\underline{G}})$ for some maximal torus $\widehat{\mathcal{T}}$ of $\widehat{G}$ [Bourgeois 2021, Theorem 2.2]. By definition, we have

$$\widehat{\underline{M}} = \mathrm{Cent}(\underline{\varphi}(P_F), \widehat{\underline{G}})^\circ = \Bigg(\bigcap_{s \in \underline{\varphi}(P_F)} \mathrm{Cent}(s, \widehat{\underline{G}}) \Bigg)^\circ .$$

Using the description from [Humphreys 1995, Theorem 2.2] for each set $\mathrm{Cent}(s, \widehat{\underline{G}})$, $s \in \underline{\varphi}(P_F)$, it follows that

$$\widehat{\underline{M}} = \langle \widehat{\underline{\mathcal{T}}}, U_\beta : \beta(s) = 1 \text{ for all } s \in \underline{\varphi}(P_F) \rangle,$$

where U_α denotes the root subgroup of $\widehat{\underline{G}}$ associated to the root $\alpha \in R(\widehat{\underline{G}}, \widehat{\underline{\mathcal{T}}})$.

Using a similar argument, and given that the root systems of $\widehat{\underline{G}}$ and $\widehat{G}$ are identified, we have that

$$\widehat{M} = \langle \widehat{\mathcal{T}}, \hat{\eta}(U_\beta) : \beta(s) = 1 \text{ for all } s \in \varphi(P_F) \rangle.$$

We then deduce from [Bourgeois 2021, Section 2B] that $\hat{\eta}(\widehat{\underline{M}}) = \widehat{M} \cap \hat{\eta}(\widehat{\underline{G}})$. Analogously, one has $\hat{\eta}(\widehat{\underline{C}}) = \widehat{C} \cap \hat{\eta}(\widehat{\underline{G}})$.

Since $\widehat{\underline{S}} = \mathrm{Cent}(\widehat{\underline{C}}, \widehat{\underline{M}})$ it follows from Proposition 4.3 that for all $\alpha \in R(\widehat{\underline{M}}, \widehat{\underline{S}})$ there exists $c_\alpha \in \widehat{\underline{C}}$ such that $\alpha(c_\alpha) \neq 1$. Applying $\hat{\eta}$, we have that for all $\alpha \in R(\hat{\eta}(\widehat{\underline{M}}), \hat{\eta}(\widehat{\underline{S}}))$, there exists $\hat{\eta}(c_\alpha) \in \hat{\eta}(\widehat{\underline{C}})$ such that $\alpha(\hat{\eta}(c_\alpha)) \neq 1$. Reapplying Proposition 4.3, we obtain $\hat{\eta}(\widehat{\underline{S}}) = \mathrm{Cent}(\hat{\eta}(\widehat{\underline{C}}), \hat{\eta}(\widehat{\underline{M}}))$. It follows that

$$\widehat{S} \cap \hat{\eta}(\widehat{\underline{G}}) = \mathrm{Cent}(\widehat{C}, \widehat{M}) \cap \hat{\eta}(\widehat{\underline{G}}) = \mathrm{Cent}(\widehat{C}, \hat{\eta}(\widehat{\underline{M}})) \subset \mathrm{Cent}(\hat{\eta}(\widehat{\underline{C}}), \hat{\eta}(\widehat{\underline{M}})) = \hat{\eta}(\widehat{\underline{S}}).$$

Since both $\widehat{S} \cap \hat{\eta}(\widehat{\underline{G}})$ and $\hat{\eta}(\widehat{\underline{S}})$ are maximal tori of $\hat{\eta}(\widehat{\underline{G}})$, we conclude that they are equal, and therefore $\hat{\eta}(\widehat{\underline{S}}) \subset \widehat{S}$. □

Having $\hat{\eta}(\widehat{\underline{S}}) \subset \widehat{S}$ implies that the root systems $R(\widehat{\underline{G}}, \widehat{\underline{S}})$ and $R(\widehat{G}, \widehat{S})$, together with their Γ-actions, are identified, which allows us to choose $\chi_0 = \underline{\chi}_0$, as the χ-data are parameterized by roots. Also, $\underline{\hat{j}} : \widehat{\underline{S}} \to \widehat{\underline{G}}$ and $\hat{j} : \widehat{S} \to \widehat{G}$ are simply inclusions. This means we have the commutative diagram

$$\begin{array}{ccc} \widehat{\underline{S}} & \xrightarrow{\underline{\hat{j}}} & \widehat{\underline{G}} \\ \big\downarrow{\scriptstyle \hat{\eta}} & & \big\downarrow{\scriptstyle \hat{\eta}} \\ \widehat{S} & \xrightarrow{\hat{j}} & \widehat{G} \end{array}$$

Proposition 4.5 (Theorem 4.2(b)). *Let $\theta, \underline{\theta}$ and η_S be as in Theorem 4.2. Then, $\theta = \underline{\theta} \circ \eta_S$.*

Proof. Using the χ-data as in [Langlands and Shelstad 1987, Section 2.6], the above diagram extends into the commutative diagram

$$\begin{array}{ccc} \widehat{\underline{S}} \rtimes W_F & \xrightarrow{{}^L\underline{j}} & \widehat{\underline{G}} \rtimes W_F \\ \big\downarrow {}^L\eta & & \big\downarrow {}^L\eta \\ \widehat{S} \rtimes W_F & \xrightarrow{{}^Lj} & \widehat{G} \rtimes W_F \end{array}$$

where ${}^L\eta(g, w) = (\hat{\eta}(g), w)$ for all $g \in \widehat{\underline{G}}$, $w \in W_F$.

Following [Kaletha 2021, Proposition 4.1.8], $\mathrm{Im}(\underline{\varphi}) \subset \mathrm{Im}({}^L\underline{j})$ and $\mathrm{Im}(\phi) \subset \mathrm{Im}({}^Lj)$, meaning that $\underline{\varphi} = {}^L\underline{j} \circ \varphi_{\underline{S}}$ and $\varphi = {}^Lj \circ \varphi_S$ for some L-parameters $\varphi_{\underline{S}}$ and φ_S of $\underline{S}$ and S, respectively. We claim that $\varphi_S = {}^L\eta \circ \varphi_{\underline{S}}$. Indeed, by definition, $\varphi = {}^L\eta \circ \underline{\varphi}$, which implies ${}^Lj \circ \varphi_S = {}^L\eta \circ {}^L\underline{j} \circ \varphi_{\underline{S}}$. Using the commutative diagram above, it follows that ${}^Lj \circ \varphi_S = {}^Lj \circ {}^L\eta \circ \varphi_{\underline{S}}$. Given that ${}^Lj$ is an embedding, it is injective by definition, which implies that $\varphi_S = {}^L\eta \circ \varphi_{\underline{S}}$ as claimed.

By definition, $\underline{\theta}$ and θ are the characters which correspond to $\varphi_{\underline{S}}$ and φ_S, respectively, under the LLC for tori. Since L-packets of tori are singletons, we apply the functoriality property for the LLC of tori to conclude that $\theta = \underline{\theta} \circ \eta_S$. □

We now arrive to the final statement of Theorem 4.2 which matches the embeddings in $\mathcal{J}_F$ and $\underline{\mathcal{J}}_F$. The description of these embeddings depends on our fixed pinnings. We will require the following lemma for our proof.

Lemma 4.6. *Let (S, θ) and $(\underline{S}, \underline{\theta})$ be tame F-nonsingular elliptic pairs of G and $\underline{G}$, respectively, which satisfy $\eta(S) \subset \underline{S}$ and $\theta = \underline{\theta} \circ \eta$. Let ϵ and $\underline{\epsilon}$ be the characters from* [Fintzen et al. 2023, Section 4.1] *(recalled at the end of Section 2.4) constructed from S and $\underline{S}$, respectively. Then $\epsilon = \underline{\epsilon} \circ \eta$.*

Proof. Let $(G^0, \ldots, G^d)$ and $(\underline{G}^0, \ldots, \underline{G}^d)$ be the twisted Levi sequences obtained from S and $\underline{S}$, respectively. As defined in [Fintzen et al. 2023, p.2259], we have that $\epsilon = \prod_{i=1}^d \epsilon^{G^i/G^{i-1}}$, where $\epsilon^{G^i/G^{i-1}}$ is the quadratic character of K^d that is trivial on $G^1(F)_{y,r_0/2} \cdots G^d(F)_{y,r_d/2}$ and whose restriction to K^0 is given by $\epsilon_y^{G^i/G^{i-1}}$ defined in [Fintzen et al. 2023, Definition 4.1.10]. The character $\epsilon_y^{G^i/G^{i-1}}$ is essentially just a composition of a sign character constructed from the adjoint groups of G^i and G^{i-1}, and the adjoint map of G^i. The character $\underline{\epsilon}$ is defined similarly. Given that $\eta(G^i) = \underline{G}^i \cap \eta(G)$ (Lemma 3.3 and Theorem 3.19), it follows that G^i and $\underline{G}^i$ have the same adjoint group and that the adjoint map of G^i is the composition of the adjoint map of $\underline{G}^i$ with η. It follows that $\epsilon_y^{G^i/G^{i-1}} = \underline{\epsilon}_{\underline{y}}^{\underline{G}^i/\underline{G}^{i-1}} \circ \eta$ for all $1 \le i \le d$ and therefore $\epsilon = \underline{\epsilon} \circ \eta$. □

Proposition 4.7 (Theorem 4.2(c)). *For all $\underline{j} \in \underline{\mathcal{J}}_F$, there exists $j \in \mathcal{J}_F$ such that $\eta(jS) \subset \underline{j}\underline{S}$ and $j\theta = \underline{j}\underline{\theta} \circ \eta$.*

Proof. Our fixed Γ-invariant pinnings satisfy $\eta(T) = \underline{T} \cap \eta(G)$ and $\hat{\eta}(\widehat{\underline{T}}) = \widehat{T} \cap \hat{\eta}(\widehat{\underline{G}})$. Following [Kaletha 2019, Section 5.1], we may describe $\underline{\mathcal{J}}$ and $\mathcal{J}$ as follows. Choose $\hat{\underline{\imath}}$ in $\widehat{\underline{\mathcal{J}}}$ such that $\hat{\underline{\imath}}(\widehat{\underline{S}}) = \widehat{\underline{T}}$ and define $\underline{i}$ to be the inverse of the isomorphism $\underline{T} \to \underline{S}$ induced by $\hat{\underline{\imath}}$. We have that $\hat{\underline{\imath}} = \mathrm{Ad}(\hat{\underline{g}}) \circ \hat{\underline{\jmath}}$ for some $\hat{\underline{g}} \in \widehat{\underline{G}}$. Let $\hat{\imath} \in \widehat{\mathcal{J}}$ be defined by $\hat{\imath} = \mathrm{Ad}(\hat{\eta}(\hat{\underline{g}})) \circ \hat{\jmath}$. Since $\hat{\eta} \circ \hat{\underline{\jmath}} = \hat{\jmath} \circ \hat{\eta}$, we have the commutative diagram

$$
\begin{array}{ccc}
\widehat{\underline{S}} & \xrightarrow{\hat{\underline{\imath}}} & \widehat{\underline{T}} \\
\downarrow{\scriptstyle \hat{\eta}} & & \downarrow{\scriptstyle \hat{\eta}} \\
\widehat{S} & \xrightarrow{\hat{\imath}} & \widehat{T}
\end{array}
\tag{9}
$$

It follows that

$$\widehat{T} \cap \hat{\eta}(\widehat{\underline{G}}) = \hat{\eta}(\widehat{\underline{T}}) = \hat{\eta}(\hat{\underline{\imath}}(\widehat{\underline{S}})) = \hat{\imath}(\hat{\eta}(\widehat{\underline{S}})) \subset \hat{\imath}(\widehat{S}).$$

Since we know $\hat{\imath}(\widehat{S})$ has to be a maximal torus of $\widehat{G}$, we conclude from [Bourgeois 2021, Theorem 2.2] that $\hat{\imath}(\widehat{S}) = \widehat{T}$. Therefore, $\underline{\mathcal{J}}$ corresponds to the $\underline{G}(F^{\mathrm{sep}})$-conjugacy class of $\underline{i}$, and $\mathcal{J}$ corresponds to the $G(F^{\mathrm{sep}})$-conjugacy class of i.

Now, given $\underline{j} \in \underline{\mathcal{J}}_F$, we have that $\underline{j} = \mathrm{Ad}(\underline{g}) \circ \underline{i}$ for some $\underline{g} \in \underline{G}(F^{\mathrm{sep}})$. Using the fact that $\underline{G} = Z(\underline{G})G_Z$, we may assume without loss of generality that $\underline{g} \in G_Z(F^{\mathrm{sep}})$. Let g be any preimage of $\underline{g}$ in $G(F^{\mathrm{sep}})$ by η and set $j = \mathrm{Ad}(g) \circ i$. By taking the dual of diagram (9), we have

$$
\begin{array}{ccc}
\underline{S} & \xrightarrow{\underline{i}} & \underline{T} \\
{\scriptstyle \eta_S}\uparrow & & {\scriptstyle \eta}\uparrow \\
S & \xrightarrow{i} & T
\end{array}
\tag{10}
$$

Here, η_S is the dual map of $\hat{\eta}|_{\widehat{\underline{S}}} : \widehat{\underline{S}} \to \widehat{S}$. It follows that

$$\eta(jS) = \eta(giSg^{-1}) = \eta(g)\eta(iS)\eta(g)^{-1} = \underline{g}\underline{i}(\eta_S(S))\underline{g}^{-1} = \underline{j}(\eta_S(S)) \subset \underline{j}\underline{S}.$$

Since $\underline{j}$ and η are defined over F, we have that $j \in \mathcal{J}_F$. Indeed, by [Dillery 2023, Lemma 6.2], which generalizes [Kottwitz 1982, Corollary 2.2] to arbitrary local fields, there exists $h \in G(F^{\mathrm{sep}})$ such that $\mathrm{Ad}(h) \circ j$ is defined over F. Then $\mathrm{Ad}(\eta(h)) \circ \underline{j}$ is also defined over F, implying $\sigma\big(\mathrm{Ad}(\eta(h)) \circ \underline{j}\big)\sigma^{-1} = \mathrm{Ad}(\eta(h)) \circ \underline{j}$ for all $\sigma \in \Gamma$. Equivalently, $\eta(h)^{-1}\sigma(\eta(h)) = \eta(h^{-1}\sigma(h)) \in \underline{j}\underline{S}$ for all $\sigma \in \Gamma$. This implies $h^{-1}\sigma(h) \in jS$, and therefore $\mathrm{Ad}(h) \circ j = \mathrm{Ad}(\sigma(h)) \circ j$ for all $\sigma \in \Gamma$. Using the fact that $\mathrm{Ad}(h) \circ j$ is defined over F, we rewrite this last equality as $\sigma(\mathrm{Ad}(h) \circ j)\sigma^{-1} = \mathrm{Ad}(\sigma(h)) \circ j$ for all $\sigma \in \Gamma$. It follows that $\mathrm{Ad}(\sigma(h)) \circ \sigma j \sigma^{-1} = \mathrm{Ad}(\sigma(h)) \circ j$, and therefore $\sigma j \sigma^{-1} = j$ for all $\sigma \in \Gamma$.

It remains to show that $j\theta = \underline{j}\underline{\theta} \circ \eta$. We have that $j\theta = \theta \circ j^{-1} \cdot \epsilon_j$ and $\underline{j}\underline{\theta} = \underline{\theta} \circ \underline{j}^{-1} \cdot \epsilon_{\underline{j}}$, where ϵ_j and $\epsilon_{\underline{j}}$ are the characters from [Fintzen et al. 2023, Section 4.1] constructed from jS and $\underline{j}\underline{S}$, respectively, which we briefly recalled at the end of Section 2.4. By what precedes, we have that $\eta(jS) \subset \underline{j}\underline{S}$ and $\underline{\theta} \circ \underline{j}^{-1} \circ \eta = \underline{\theta} \circ \eta_S \circ j^{-1} = \theta \circ j^{-1}$. Using Lemma 4.6, we have $\epsilon_j = \epsilon_{\underline{j}} \circ \eta$, and thus $j\theta = \underline{j}\underline{\theta} \circ \eta$. □

4.2. *The proof of Theorem 4.1.* In this section, we begin with the statement of a lemma, after which we will combine the results of the previous section with the decomposition formula from Section 3 to prove Theorem 4.1.

Lemma 4.8. *Let (S, θ) be a tame F-nonsingular elliptic pair of G, and let $c \in G$ be such that* $\mathrm{Ad}(c)$ *is defined over F. Then, the following statements hold.*

(1) ${}^c\pi_{(S,\theta)} \simeq \pi_{({}^cS,{}^c\theta)}$.

(2) *If ϵ is the character from* [Fintzen et al. 2023, Section 4.1] *constructed from S, then ${}^c\epsilon$ is the character from* [Fintzen et al. 2023, Section 4.1] *constructed from ${}^cS$.*

Proof. We have that $\mathrm{Ad}(c) : G \to G$ is a map which satisfies hypotheses (i) and (ii) of Theorem 4.1, having trivial kernel and trivial cokernel.

For (1), we apply Theorem 3.17, in which we set $\eta = \mathrm{Ad}(c)$, $S_Z = \mathrm{Ad}(c)(S) = {}^cS$ and $\theta_Z = \theta \circ \mathrm{Ad}(c^{-1}) = {}^c\theta$. We obtain as a result $\pi_{({}^cS,{}^c\theta)} \circ \mathrm{Ad}(c) \simeq \pi_{(S,\theta)}$, or equivalently, $\pi_{({}^cS,{}^c\theta)} \simeq {}^c\pi_{(S,\theta)}$.

For (2), we apply Lemma 4.6, in which we set $\eta = \mathrm{Ad}(c)$, $\underline{S} = {}^cS$ and $\underline{\theta} = {}^c\theta$. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $(\underline{S}, \underline{\hat{j}}, \chi_0, \underline{\theta})$ and $(S, \hat{j}, \chi_0, \theta)$ be the supercuspidal L-packet data associated to $\underline{\varphi}$ and φ, respectively. By construction of $\Pi_{\underline{\varphi}}$, we have that $\underline{\pi} \subset \pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})}$ for some $\underline{j} \in \underline{\mathcal{J}}_F$. By Theorem 4.2, there exists $j \in \mathcal{J}_F$ such that $\eta(jS) \subset \underline{j}\underline{S}$ and $j\theta = \underline{j}\underline{\theta} \circ \eta$. By Theorem 3.1, it follows that

$$\underline{\pi} \circ \eta \subset \pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})} \circ \eta = \bigoplus_{\underline{c} \in \underline{C}} \pi_{(jS,j\theta)} \circ \underline{\mathrm{Ad}}(\underline{c}^{-1}), \tag{11}$$

where $\underline{C}$ is a set of coset representatives of $\eta(G(F))\backslash \underline{G}(F)/\underline{j}\underline{S}(F)$. Recall from Section 2.3 that $\underline{\mathrm{Ad}}(\underline{c}^{-1}) = \mathrm{Ad}(c^{-1})$, where $c \in G$ satisfies $\underline{c} = \eta(c)z$ for some $z \in Z(\underline{G})$. Using Lemma 4.8(1) one sees that

$$\pi_{(jS,j\theta)} \circ \underline{\mathrm{Ad}}(\underline{c}^{-1}) = {}^c\pi_{(jS,j\theta)} \simeq \pi_{({}^cjS,{}^cj\theta)},$$

where ${}^cjS = (\mathrm{Ad}(c) \circ j)S$ and

$${}^cj\theta = j\theta \circ \mathrm{Ad}(c^{-1}) = ((\theta \circ j^{-1}) \cdot \epsilon_j) \circ \mathrm{Ad}(c^{-1}) = (\theta \circ j^{-1} \circ \mathrm{Ad}(c^{-1})) \cdot (\epsilon_j \circ \mathrm{Ad}(c^{-1})).$$

By Lemma 4.8(2), $\epsilon_j \circ \mathrm{Ad}(c^{-1}) = \epsilon_{\mathrm{Ad}(c)\circ j}$ so that

$${}^cj\theta = \left(\theta \circ (\mathrm{Ad}(c) \circ j)^{-1}\right) \cdot \epsilon_{\mathrm{Ad}(c)\circ j} = (\mathrm{Ad}(c) \circ j)\theta.$$

Since $\mathrm{Ad}(c)$ is defined over F (Lemma 2.6), $\mathrm{Ad}(c) \circ j \in \mathcal{J}_F$, and therefore

$$[\pi_{(jS,j\theta)} \circ \underline{\mathrm{Ad}}(\underline{c}^{-1})] = [\pi_{((\mathrm{Ad}(c)\circ j)S,(\mathrm{Ad}(c)\circ j)\theta)}] \subset \Pi_\varphi$$

for all $\underline{c} \in \underline{C}$. Thus, all irreducible components of $\underline{\pi} \circ \eta$ belong to Π_φ. □

5. Specializing to regular supercuspidal parameters

Part of the local Langlands conjectures is a correspondence between the irreducible representations in an L-packet Π_φ and the irreducible representations of the component group of $\mathrm{Cent}(\varphi(W_F), \widehat{G})$ [Vogan 1993, Conjecture 1.15]. In this section, we review this correspondence for *regular* supercuspidal L-parameters.

The regular supercuspidal L-parameters [Kaletha 2019, Definition 5.2.3] are an important subclass of the supercuspidal L-parameters. They are also easier to study, as their corresponding L-packets are simpler to describe (see (12) below), and their corresponding component groups are abelian [Kaletha 2019, Lemma 5.3.4].

Under the assumption that φ and $\underline{\varphi}$ are both regular, we show how to reparameterize the elements of Π_φ and $\Pi_{\underline{\varphi}}$ in terms of characters of their respective component groups. From this reparameterization, we obtain an alternate formulation for the decomposition formula for $\underline{\pi} \circ \eta$, $\underline{\pi} \in \Pi_{\underline{\varphi}}$, obtained from Theorem 3.1. This reformulation amounts to a proof of [Solleveld 2020, Conjecture 2] for regular supercuspidal L-packets of quasisplit groups (Theorem 1.1, Proposition 5.12).

First we explicitly describe the regular supercuspidal L-parameters, their corresponding L-packet structure, and the relationship between the regularity of φ and $\underline{\varphi}$.

5.1. ***Regular L-Packets and conditions for regularity of φ and $\underline{\varphi}$.*** One way to describe the regular supercuspidal L-parameters is via the notion of regular supercuspidal L-packet data [Kaletha 2019, Definition 5.2.4]. A regular supercuspidal L-packet datum of G is a supercuspidal L-packet datum $(S, \hat{\jmath}, \chi_0, \theta)$ (see Definition 2.9), with the stronger condition that (S, θ) is an extra regular elliptic pair in the sense of [Kaletha 2019, Definition 3.7.5]. In particular, this means that the character $\theta|_{S(F)_0}$ has trivial stabilizer for the action of $\Omega(S, G)(F) := (N_G(S)/S)(F)$.

By [Kaletha 2019, Proposition 5.2.7], there is a one-to-one correspondence between the $\widehat{G}$-conjugacy classes of regular supercuspidal L-parameters for G and isomorphism classes of regular supercuspidal L-packet data. Given a regular supercuspidal L-parameter φ of G with associated regular supercuspidal L-packet datum $(S, \hat{\jmath}, \chi_0, \theta)$, the representations $\pi_{(jS, j\theta)}$ are irreducible for all $j \in \mathcal{J}_F$ [Kaletha 2019, Lemma 3.4.20]. Thus, the corresponding L-packet is

$$\Pi_\varphi = \{\pi_{(jS, j\theta)} : j \in \mathcal{J}_F\}, \tag{12}$$

where j is identified with its $G(F)$-conjugacy class and $\pi_{(jS, j\theta)}$ is identified with its equivalence class. Furthermore, as stated in [Kaletha 2019, Section 5.3; 2021, Section 4.2], the elements of Π_φ are in one-to-one correspondence with the elements of $\mathcal{J}_F$. The following lemma is a proof of this statement.

Lemma 5.1. *Let φ be a regular supercuspidal L-parameter of G with associated regular L-packet datum $(S, \hat{\jmath}, \chi_0, \theta)$. Then the map $j \mapsto \pi_{(jS, j\theta)}$ induces a bijection $\mathcal{J}_F \to \Pi_\varphi$.*

Proof. We prove an equivalent statement: $\pi_{(j_1 S, j_1\theta)} \simeq \pi_{(j_2 S, j_2\theta)}$ if and only if j_1 and j_2 are $G(F)$-conjugate.

Assume $\pi_{(j_1 S, j_1\theta)} \simeq \pi_{(j_2 S, j_2\theta)}$. Then, by [Kaletha 2019, Corollary 3.7.10], there exists $g \in G(F)$ such that $j_1 S = \mathrm{Ad}(g) j_2 S$ and $j_1\theta = {}^g j_2\theta$. Using [Kaletha 2019, Lemmas 3.4.10 and 3.4.12], there exists $j' \in \mathcal{J}_F$ such that $(j'S, j'\theta)$ is extra regular in the sense of [Kaletha 2019, Definition 3.7.5]. As in the proof of Proposition 4.7, $j_1 = \mathrm{Ad}(h_1) \circ j'$ and $j_2 = \mathrm{Ad}(h_2) \circ j'$ for some $h_1, h_2 \in G(F^{\mathrm{sep}})$ such that $\mathrm{Ad}(h_1)$ and $\mathrm{Ad}(h_2)$ (as maps of $j'S$) are defined over F. Thus, $j_1 S = \mathrm{Ad}(g) j_2 S$ and $j_1\theta = {}^g j_2\theta$ if and only if $h_1^{-1} g h_2 \in N_G(j'S)$ and $j'\theta = {}^{h_1^{-1} g h_2} j'\theta$. Because $\mathrm{Ad}(h_1^{-1} g h_2)$ is defined over F, it is an easy exercise to show that $\sigma(h_1^{-1} g h_2)^{-1}(h_1^{-1} g h_2) \in C_G(j'S) = j'S$ for all $\sigma \in \Gamma$. It follows that $(h_1^{-1} g h_2) j'S \in \Omega(j'S, G)(F) = (N_G(j'S)/j'S)(F)$. By the extra regularity of $j'\theta$, we conclude that $h_1^{-1} g h_2 \in j'S$, and therefore $\mathrm{Ad}(h_1^{-1} g h_2) \circ j' = j'$. Thus, $\mathrm{Ad}(g) \circ j_2 = j_1$.

The converse is a direct consequence of [Kaletha 2019, Corollary 3.7.10], and is built into the definition of Π_φ. □

Suppose as usual that $\underline{\varphi} \in \Phi_{\mathrm{sc}}(\underline{G})$ and $\varphi = {}^L\eta \circ \underline{\varphi} \in \Phi_{\mathrm{sc}}(G)$. It is natural to ask under what conditions φ and $\underline{\varphi}$ are both regular. The following lemma and corollary address this question from the perspective of regular supercuspidal L-packet data.

Lemma 5.2. *Let (S, θ) and $(\underline{S}, \underline{\theta})$ be F-nonsingular elliptic pairs of G and $\underline{G}$, respectively, satisfying $\eta(S) \subset \underline{S}$ and $\theta = \underline{\theta} \circ \eta$. If (S, θ) is extra regular, then $(\underline{S}, \underline{\theta})$ is also extra regular.*

Corollary 5.3. *Let $\underline{\varphi}$ be a supercuspidal L-parameter of $\underline{G}$ and let $\varphi = {}^L\eta \circ \underline{\varphi}$. If φ is regular, then $\underline{\varphi}$ is also regular.*

Before proving Lemma 5.2, recall from Section 2.5 that S is not a subtorus of G. Rather, as in [Kaletha 2019, p. 1145], we have a structure on S that is given to us by $\hat{j}$. This means that the action of $\Omega(S, G)$ on S corresponds to the action of $\Omega(T, G)$ twisted by $i : S \to T$. More precisely, given $w \in \Omega(S, G) = \Omega(T, G)$, ${}^w s = i^{-1}({}^w i(s))$ for all $s \in S$. Here, T is the maximal torus from our fixed Γ-invariant pinning, and i is as per the proof of Proposition 4.7. The same is also true of $\underline{S}$, for which we adopt analogous notation. Furthermore, since $\eta(T) = \eta(G) \cap \underline{T}$ and $\eta(G) \supset [\underline{G}, \underline{G}]$, one sees from [Bourgeois 2020, Proposition 2.1.24] that η induces a Γ-equivariant isomorphism

$$\eta_\Omega : \Omega(T, G) \xrightarrow{\eta} \Omega(\eta(T), \eta(G)) \to \Omega(\underline{T}, \underline{G}),$$

which sends gT to $\eta(g)\underline{T}$ for all $g \in N_G(T)$.

Proof of Lemma 5.2. It is clear that the first two conditions in the definition of extra regularity [Kaletha 2019, Definition 3.7.5] are satisfied for $(\underline{S}, \underline{\theta})$ if and only if they are satisfied for (S, θ). We focus our attention on the third and final condition.

That is, we assume that $\theta|_{S(F)_0}$ has trivial stabilizer for the action of $\Omega(S, G)(F)$ and show that $\underline{\theta}|_{\underline{S}(F)_0}$ has trivial stabilizer for the action of $\Omega(\underline{S}, \underline{G})(F)$.

Recall from Proposition 4.5 and (10) that $\theta = \tilde{\underline{\theta}} \circ \eta_S$ and $\underline{i} \circ \eta_S = \eta \circ i$. Using these equalities in combination with the definitions of the actions of $\Omega(S, G)$ and $\Omega(\underline{S}, \underline{G})$, we obtain

$$ {}^{w}\theta = {}^{\eta_\Omega(w)}\tilde{\underline{\theta}} \circ \eta_S \quad \text{for all } w \in \Omega(S, G)(F). \tag{13} $$

Assume that ${}^{\underline{w}}\underline{\theta}|_{\underline{S}(F)_0} = \underline{\theta}|_{\underline{S}(F)_0}$ for some $\underline{w} \in \Omega(\underline{S}, \underline{G})(F)$. By the above discussion, $\underline{w} = \eta_\Omega(w)$ for some $w \in \Omega(S, \mathrm{G})(F)$. Using (13), it follows that

$$ \theta|_{S(F)_0} = (\underline{\theta} \circ \eta_S)|_{S(F)_0} = ({}^{\underline{w}}\underline{\theta} \circ \eta_S)|_{S(F)_0} = {}^{w}\theta|_{S(F)_0}. $$

Given the assumption on θ, we conclude that $w = 1$. Thus $\underline{w} = 1$ and $\underline{\theta}|_{\underline{S}(F)_0}$ has trivial stabilizer for the action of $\Omega(\underline{S}, \underline{G})(F)$. □

The converse of Lemma 5.2 and Corollary 5.3 is not true in general. Consider the case of $G = \mathrm{SL}_2$ and $\underline{G} = \mathrm{GL}_2$, with η being the inclusion map. Then, all irreducible supercuspidal representations of $\underline{G}(F)$ are extra regular [Kaletha 2019, Lemma 3.7.7], whereas there exist irreducible supercuspidal representations of $G(F)$ which are not regular (e.g., the four *exceptional* supercuspidal representations from [Adler et al. 2011]). Given one such representation of $G(F)$, say π, the irreducible components of $\mathrm{Ind}_{G(F)}^{\underline{G}(F)}\pi$ are all extra regular, and thus correspond to extra regular elliptic pairs of $\underline{G}(F)$. We claim that the restrictions of these extra regular elliptic pairs to $G(F)$ can not be extra regular (or even regular). Indeed, given $\underline{\pi} = \pi_{(\underline{S},\underline{\theta})} \subset \mathrm{Ind}_{G(F)}^{\underline{G}(F)}\pi$, with $(\underline{S}, \underline{\theta})$ extra regular, Theorem 3.1 says that $\underline{\pi} \circ \eta$ is a sum of conjugates of $\pi_{(S,\theta)}$, where $S = \underline{S} \cap G$ and $\theta = \underline{\theta} \circ \eta$. Assuming (S, θ) is extra regular (or even regular) contradicts the nonregularity of $\pi \subset \underline{\pi} \circ \eta$, and thus (S, θ) cannot be extra regular.

It is worth pointing out that the instances for which the converse holds are not extremely rare. Indeed, assume that $\underline{\theta}$ is extra regular and that ${}^{w}\theta|_{S(F)_0} = \theta|_{S(F)_0}$ for some $w \in \Omega(S, G^0)(F)$. Then (13) tells us ${}^{\eta_\Omega(w)}\underline{\theta} \circ \eta_S|_{S(F)_0} = \underline{\theta} \circ \eta_S|_{S(F)_0}$, or equivalently, ${}^{\eta_\Omega(w)}\underline{\theta}|_{\eta_S(S(F)_0)} = \underline{\theta}|_{\eta_S(S(F)_0)}$. In order to conclude that $w = 1$ (i.e., θ is extra regular), we must have ${}^{\eta_\Omega(w)}\underline{\theta}|_{\underline{S}(F)_0} = \underline{\theta}|_{\underline{S}(F)_0}$. If the difference between $\underline{S}(F)_0$ and $\eta_S(S(F)_0)$ is centralized by $\Omega(\underline{S}, \underline{G})(F)$, then we can conclude that $w = 1$, and so θ is extra regular when $\underline{\theta}$ is extra regular. This happens, for example, if the equality $\underline{T} = Z(\underline{G})\eta(T)$ (or equivalently, $\underline{S} = \underline{i}^{-1}(Z(\underline{G}))\eta_S(S)$) remains true at the level of the F-points.

5.2. ***Functoriality and the characters of the dual component group.*** Let $\varphi \in \Phi_{\mathrm{sc}}(G)$ be a regular supercuspidal L-parameter. Up until now in this section, the elements of Π_φ have been parameterized by regular elliptic pairs. However, in Solleveld's conjecture [2020, Conjecture 2], the elements of the L-packets are parameterized by φ and characters of the associated component group. Thus, the first step in proving Theorem 1.1 is to reconcile the two parameterizations.

As discussed in Section 5.1, the L-parameter φ corresponds to a regular supercuspidal L-packet datum $(S, \hat{j}, \chi_0, \theta)$, and the (equivalence classes of) irreducible representations in Π_φ correspond bijectively to the ($G(F)$-conjugacy classes of) admissible embeddings in $\mathcal{J}_F$

$$\Pi_\varphi \longleftrightarrow \mathcal{J}_F. \tag{14}$$

Thus far, there has been no need to specify a particular irreducible representation or admissible embedding in (14) as being special. However, it shall be necessary to specify a particular representation and embedding to correspond to the identity character of the dual component group. Kaletha [2019, Sections 5.3, 6.2; 2021, Lemma 4.2.1] discusses what the particular choice of representation and embedding should be upon fixing a Whittaker datum, and when F is of characteristic zero the particular representation and embedding is fixed on [Fintzen et al. 2023, page 2273]. For F of characteristic zero we may fix a Whittaker datum for G and thereby an embedding $j \in \mathcal{J}_F$ as on [Fintzen et al. 2023, page 2273]. At the time of writing, a preferred choice of embedding does not appear to be available for regular supercuspidal L-packets in positive characteristic. In this case, we arbitrarily fix $j \in \mathcal{J}_F$ and thereby its corresponding irreducible representation $\pi_{(jS, j\theta)} \in \Pi_\varphi$. The bijection in (14) is now an isomorphism of pointed sets.

The dual group attached to $\varphi \in \Phi_{\mathrm{sc}}(G)$ is $\mathrm{Cent}(\varphi(W_F), \widehat{G})$, and according to [Kaletha 2019, Lemma 5.3.4], it is naturally isomorphic to the fixed-point subgroup $\widehat{S}^\Gamma$. We denote the finite abelian component group $\widehat{S}^\Gamma/(\widehat{S}^\Gamma)^\circ$ by $\pi_0(\widehat{S}^\Gamma)$, and denote its group of characters by $\pi_0(\widehat{S}^\Gamma)^D$. What we wish to do here is supplement (14) with an inclusion

$$\Pi_\varphi \longleftrightarrow \mathcal{J}_F \hookrightarrow \pi_0(\widehat{S}^\Gamma)^D \tag{15}$$

and to describe the image of this inclusion.

Recall from (2) that $\mathcal{J}_F$ is in bijection with $\ker(H^1(F, jS) \to H^1(F, G))$. An arbitrary element of $\ker(H^1(F, jS) \to H^1(F, G))$ is represented by a cocycle

$$z_g(\sigma) = g^{-1}\sigma(g), \quad \sigma \in \Gamma,$$

for some $g \in G(F^{\mathrm{sep}})$. By fixing $j \in \mathcal{J}_F$ as we have above, we fix a bijection from $\ker(H^1(F, jS) \to H^1(F, G))$ to $\mathcal{J}_F$ given by

$$z_g \mapsto \mathrm{Ad}(g) \circ j. \tag{16}$$

The desired inclusion of (15) is given through this fixed bijection and the commutative diagram

$$\begin{array}{ccc} H^1(F, jS) & \longrightarrow & \pi_0(\widehat{S}^\Gamma)^D \\ \downarrow & & \downarrow \\ H^1(F, G) & \longrightarrow & \pi_0(Z(\widehat{G})^\Gamma)^D \end{array} \tag{17}$$

of [Kottwitz 1986, Theorem 1.2; Thang 2011, Theorem 2.1]. In this diagram the upper and lower maps are bijections which arise from perfect pairings in Tate–Nakayama duality. The map on the left is given by the inclusion $jS \subset G$, and the map on the right is given by restriction to $Z(\widehat{G})^\Gamma \subset \widehat{S}^\Gamma$. Since (17) is commutative, $\ker(H^1(F, jS) \to H^1(F, G))$ is in bijection with the kernel of the restriction map on the right of the diagram. The kernel of the restriction map is isomorphic to $\pi_0(\widehat{S}^\Gamma/Z(\widehat{G})^\Gamma)^D$.

Combining these observations with (16), we obtain a bijection between $\mathcal{J}_F$ and $\pi_0(\widehat{S}^\Gamma/Z(\widehat{G})^\Gamma)^D$ given by the map

$$\mathrm{Ad}(g) \circ j \mapsto \tau_g \tag{18}$$

in which $\mathrm{Ad}(g) \circ j$ is a representative of a $G(F)$-conjugacy class in $\mathcal{J}_F$ and $\tau_g \in \pi_0(\widehat{S}^\Gamma/Z(\widehat{G})^\Gamma)^D$ is obtained through z_g and Tate–Nakayama duality.

In summary, the desired arrangement of (15) takes the shape of three bijections

$$\Pi_\varphi \longleftrightarrow \mathcal{J}_F \longleftrightarrow \big(\ker(H^1(F, jS) \to H^1(F, G))\big) \longleftrightarrow \pi_0(\widehat{S}^\Gamma/Z(\widehat{G})^\Gamma)^D. \tag{19}$$

On the level of elements, the bijections have the form

$$\pi_{({}^g jS, {}^g j\theta)} \longleftrightarrow \mathrm{Ad}(g) \circ j \longleftrightarrow z_g \longleftrightarrow \tau_g,$$

where $g \in G(F^{\mathrm{sep}})$ and $z_g \in Z^1(F, jS)$. We can go one step further and obtain an alternative description for $\ker(H^1(F, jS) \to H^1(F, G))$ as follows.

Given a maximal torus S' of G which is defined over F, we have in particular that S' is a closed subgroup of G. The quotient G/S' is therefore a variety defined over F. An element $gS' \in (G/S')(F^{\mathrm{sep}})$ belongs to $(G/S')(F)$ if and only if $gS' = \sigma(g)S'$ for all $\sigma \in \Gamma$. The group $G(F)$ acts on $(G/S')(F)$ by left multiplication. Let $G(F)\backslash(G/S')(F)$ denote the set of $G(F)$-orbits. The following lemma is a special case of [Serre 2002, I.5.4 Corollary 1].

Lemma 5.4. *Let $gS' \in (G/S')(F)$. Then the map $z_g : \Gamma \to S'$ defined by*

$$z_g(\sigma) = g^{-1}\sigma(g), \quad \sigma \in \Gamma,$$

is a cocycle in $Z^1(F, S')$. In addition, the map $g \mapsto z_g$ induces a bijection from $G(F)\backslash(G/S')(F)$ to $\ker(H^1(F, S') \to H^1(F, G))$.

We may now rewrite (19) using the bijection of Lemma 5.4 with $S' = jS$:

$$\begin{array}{ccccccc} \Pi_\varphi & \longleftrightarrow & \mathcal{J}_F & \longleftrightarrow & G(F)\backslash(G/jS)(F) & \longleftrightarrow & \pi_0(\widehat{S}^\Gamma/Z(\widehat{G})^\Gamma)^D, \\ \pi_{({}^{g^{-1}}jS, {}^{g^{-1}}j\theta)} & \longleftrightarrow & \mathrm{Ad}(g) \circ j & \longleftrightarrow & gjS & \longleftrightarrow & \tau_g. \end{array} \tag{20}$$

Assume now that we have two regular supercuspidal L-parameters $\underline{\varphi} \in \Phi_{\mathrm{sc}}(\underline{G})$ and $\varphi = {}^L\eta \circ \underline{\varphi}$. In light of the discussion after Corollary 5.3, it is not sufficient to assume that only $\underline{\varphi}$ is regular. Our next objective is to describe a commutative diagram

$$
\begin{array}{ccccccc}
\Pi_{\underline{\varphi}} & \longleftrightarrow & \underline{\mathcal{J}}_F & \overset{\mathrm{Ad}(\,\cdot\,)\circ \underline{j}}{\longleftrightarrow} & \underline{G}(F)\backslash(\underline{G}/\underline{j}\underline{S})(F) & \overset{\underline{\tau}}{\longleftrightarrow} & \pi_0(\widehat{\underline{S}}^{\Gamma}/Z(\widehat{\underline{G}})^{\Gamma})^D \\
\uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \bar{\eta} & & \uparrow\!\!\uparrow \bar{\eta} & & \uparrow\!\!\uparrow \circ\hat{\eta} \\
\Pi_{\varphi} & \longleftrightarrow & \mathcal{J}_F & \underset{\mathrm{Ad}(\,\cdot\,)\circ j}{\longleftrightarrow} & G(F)\backslash(G/jS)(F) & \underset{\tau}{\longleftrightarrow} & \pi_0(\widehat{S}^{\Gamma}/Z(\widehat{G})^{\Gamma})^D
\end{array}
\tag{21}
$$

Once this diagram is in place, the decomposition formula for $\underline{\pi}_{(\underline{j}\underline{S},\underline{j}\underline{\theta})} \circ \eta$ that one would obtain by applying Theorem 3.1 may be transferred to the right-hand square of the diagram, which is key for proving Theorem 1.1.

The starting point is the top row of (21). Replacing G with $\underline{G}$, we fix (the $\underline{G}(F)$-conjugacy class of) an admissible embedding $\underline{j} \in \underline{\mathcal{J}}_F$ relative to a fixed Whittaker datum for $\underline{G}$ when char $F = 0$. When char $F \neq 0$ we fix $\underline{j} \in \underline{\mathcal{J}}_F$ arbitrarily. The top row is then the sequence of bijections in (20), in which S and j are replaced by $\underline{S}$ and $\underline{j}$. By Proposition 4.7, there exists $j \in \mathcal{J}_F$ such that $\eta(jS) \subset \underline{j}\underline{S}$ and $j\theta = \underline{j}\underline{\theta} \circ \eta$. The embedding j in Proposition 4.7 is specified by the alignment of the Γ-invariant pinnings of $\underline{G}$ and G through η. This alignment preserves the simple root spaces and therefore transfers a fixed Whittaker datum of $\underline{G}$ to a Whittaker datum for G. Consequently, when char $F = 0$, the embedding j may be chosen as on [Fintzen et al. 2023, page 2273] relative to the latter Whittaker datum. Otherwise, we fix j as in Proposition 4.7 arbitrarily. In any case, the bottom row is now given by (20).

We continue by describing the middle two vertical maps of (21). Since $\eta(jS) \subset \underline{j}\underline{S}$ the map $\bar{\eta}$ sending $gjS \in G/jS$ to $\eta(g)\underline{j}\underline{S} \in \underline{G}/\underline{j}\underline{S}$ is well defined. Furthermore, η is defined over F so that, for $g \in G(F^{\mathrm{sep}})$,

$$
\sigma(\bar{\eta}(gjS)) = \eta(\sigma(g))\underline{j}\underline{S} = \bar{\eta}(\sigma(gjS)), \quad \sigma \in \Gamma.
$$

This means that the restriction of $\bar{\eta}$ to $G(F^{\mathrm{sep}})/jS(F^{\mathrm{sep}})$ is defined over F, and passes to a map $G(F)\backslash(G/jS)(F) \to \underline{G}(F)\backslash(\underline{G}/\underline{j}\underline{S})(F)$. This defines the second map from the right in (21). We define the map $\mathcal{J}_F \to \underline{\mathcal{J}}_F$ to its left as the map which takes $\mathrm{Ad}(g) \circ j$ to $\mathrm{Ad}(\eta(g)) \circ \underline{j}$. In this way the middle square in (21) commutes. The arguments in the proof of Proposition 4.7 imply the surjectivity of the two vertical maps in the middle square.

The vertical map $\circ\,\hat{\eta}$ on the right of (21) is defined by composition with $\hat{\eta}$ (see Proposition 4.4). The commutativity of the right-hand square of (21) may be explained as follows. The functoriality of [Kottwitz 1986, Theorem 1.2; Thang

2011, Theorem 2.1] imply that the following diagram commutes:

$$\begin{array}{ccc} H^1(F, \underline{j}\underline{S}) & \longrightarrow & \pi_0(\widehat{\underline{S}}^{\Gamma})^D \\ {\scriptstyle \eta\circ}\uparrow & & \uparrow{\scriptstyle \circ\hat{\eta}} \\ H^1(F, jS) & \longrightarrow & \pi_0(\widehat{S}^{\Gamma})^D \end{array}$$

Therefore, the restriction

$$\begin{array}{ccc} \ker(H^1(F, \underline{j}\underline{S}) \to H^1(F, \underline{G})) & \longrightarrow & \pi_0(\widehat{\underline{S}}^{\Gamma}/Z(\widehat{\underline{G}})^{\Gamma})^D \\ {\scriptstyle \eta\circ}\uparrow & & \uparrow{\scriptstyle \circ\hat{\eta}} \\ \ker(H^1(F, jS) \to H^1(F, G)) & \longrightarrow & \pi_0(\widehat{S}^{\Gamma}/Z(\widehat{G})^{\Gamma})^D \end{array}$$

also commutes. Applying Lemma 5.4 then gives us the right-hand square of (21) as a commutative diagram. As a consequence, the surjectivity of $\circ\,\hat{\eta}$ follows from the surjectivity of $\bar{\eta}$.

The leftmost vertical arrow of (21) is defined as the unique map which makes the leftmost square of (21) commute. It is defined by

$$\pi_{({}^{g}jS,{}^{g}j\theta)} \mapsto \pi_{({}^{\eta(g)}\underline{j}\underline{S},{}^{\eta(g)}\underline{j}\underline{\theta})}$$

for all $\mathrm{Ad}(g)\circ j \in \mathcal{J}_F$. This map is surjective, as $\bar{\eta}$ is surjective. Given $\mathrm{Ad}(\underline{g})\circ \underline{j} \in \underline{\mathcal{J}}_F$ the preimage of $\pi_{((\mathrm{Ad}(\underline{g})\circ\underline{j})\underline{S},(\mathrm{Ad}(\underline{g})\circ\underline{j})\underline{\theta})} \in \Pi_{\underline{\varphi}}$ under this map is

$$\{\pi_{({}^{g}jS,{}^{g}j\theta)} : \mathrm{Ad}(g)\circ j \in \bar{\eta}^{-1}(\mathrm{Ad}(\underline{g})\circ \underline{j})\}. \tag{22}$$

5.3. ***The proof of Theorem 1.1.*** The commutative diagram (21) from the previous section will allow us to rewrite the decomposition formula provided in the proof of Theorem 4.1 with respect to the parameterization in terms of characters of the component groups.

Let φ, $\underline{\varphi}$, $\underline{j}$, j be as in the previous section. In the proof of Theorem 4.1, equation (11) takes the form

$$\pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})}\circ\eta \simeq \bigoplus_{\underline{c}\in\underline{C}} \pi_{((\mathrm{Ad}(c^{-1})\circ j)S,(\mathrm{Ad}(c^{-1})\circ j)\theta)}, \tag{23}$$

where $\underline{C}$ is a set of coset representatives of $\eta(G(F))\backslash\underline{G}(F)/\underline{j}\underline{S}(F)$ and $c\in G(F^{\mathrm{sep}})$ is such that $\underline{c} = \eta(c)z$ for some $z \in Z(\underline{G})(F^{\mathrm{sep}})$. Note that, by construction, $\mathrm{Ad}(c^{-1})\circ j \in \bar{\eta}^{-1}(\mathrm{Ad}(\underline{c}^{-1})\circ \underline{j})$ and $\mathrm{Ad}(\underline{c}^{-1})\circ \underline{j}$ and $\underline{j}$ belong to the same $\underline{G}(F)$-equivalence class.

The following proposition tells us that the set of representations (22) coincides with the irreducible components of the decomposition formula (23).

Proposition 5.5. *Suppose* $\underline{g}\underline{j}\underline{S} \in (\underline{G}/\underline{j}\underline{S})(F)$. *Then*

$$\pi_{({}^{\underline{g}}\underline{j}\underline{S},{}^{\underline{g}}\underline{j}\underline{\theta})} \circ \eta \cong \bigoplus_{gjS \in \bar{\eta}^{-1}(\underline{g}\underline{j}\underline{S})} \pi_{({}^{g}jS,{}^{g}j\theta)}. \tag{24}$$

Equivalently, an irreducible representation π *of* $G(F)$ *is a subrepresentation of* $\pi_{({}^{\underline{g}}\underline{j}\underline{S},{}^{\underline{g}}\underline{j}\underline{\theta})} \circ \eta$ *if and only if* $\pi \cong \pi_{({}^{g}jS,{}^{g}j\theta)}$ *for* $gjS \in \bar{\eta}^{-1}(\underline{g}\underline{j}\underline{S})$ (*i.e.*, $\mathrm{Ad}(g) \circ j$ *in the fiber of* $\bar{\eta}$ *over* $\mathrm{Ad}(\underline{g}) \circ \underline{j}$).

The key to proving (24) is to identify the fiber over $\underline{j}$ with $\eta(G(F))\backslash\underline{G}(F)/\underline{j}\underline{S}(F)$ in (23), which is done via the following lemma.

Lemma 5.6. *Let* $\bar{\eta}^{-1}(\underline{j})$ *be the fiber of* $\mathcal{J}_F \xrightarrow{\bar{\eta}} \underline{\mathcal{J}}_F$ *over* $\underline{j}$. *The map of Lemma 5.4 and the map* η *induce horizontal maps in the commutative diagram*

$$\begin{array}{ccccc}
\mathcal{J}_F & \longrightarrow & G(F)\backslash(G/jS)(F) & \longrightarrow & \eta(G(F))\backslash(\underline{G}/\underline{j}\underline{S})(F) \\
\uparrow & & \uparrow & & \uparrow \\
\bar{\eta}^{-1}(\underline{j}) & \longrightarrow & G(F)\backslash\bar{\eta}^{-1}(\{\underline{g}\underline{j}\underline{S} : \underline{g} \in \underline{G}(F)\}) & \longrightarrow & \eta(G(F))\backslash\underline{G}(F)/\underline{j}\underline{S}
\end{array}$$

In addition, the horizontal maps are bijections.

Proof. We first prove the assertion for the square on the left. The upper map is bijective by Lemma 5.4 and the vertical maps are inclusions. For the lower horizontal map, suppose $\mathrm{Ad}(g) \circ j \in \mathcal{J}$ is a representative of some $G(F)$-orbit in $\mathcal{J}_F$ where $gjS \in (G/jS)(F)$, and $\bar{\eta}(\mathrm{Ad}(g) \circ j) = \mathrm{Ad}(\underline{g}) \circ \underline{j}$ for some $\underline{g} \in \underline{G}(F)$. The bijectivity of the lower horizontal map follows from the equivalences

$$\begin{aligned}
\mathrm{Ad}(\underline{g}^{-1}\eta(g)) \circ \underline{j} = \underline{j} \iff & \underline{g}^{-1}\eta(g) \in \underline{j}\underline{S} \\
\iff & \eta(g)\underline{j}\underline{S} = \underline{g}\underline{j}\underline{S} \iff gjS \in \bar{\eta}^{-1}(\mathrm{Ad}(\underline{G}(F))\underline{j}).
\end{aligned}$$

We continue by examining the upper horizontal map in the square on the right. This map may be described as

$$G(F)gjS \mapsto \eta(G(F))\tilde{\eta}(g\underline{j}\underline{S}), \quad \text{where } \tilde{\eta}(gjS) = \eta(g)\underline{j}\underline{S},\ gjS \in G/jS.$$

Clearly, the upper horizontal map is bijective if $\tilde{\eta} : G/jS \to \underline{G}/\underline{j}\underline{S}$ yields an isomorphism on F-points. Let us prove that $\tilde{\eta}$ is an isomorphism. The injectivity of $\tilde{\eta}$ follows from $jS = \eta^{-1}(\underline{j}\underline{S})$. Suppose $\underline{g}\underline{j}\underline{S} \in \underline{G}/\underline{j}\underline{S}$. We may assume that $\underline{g} \in [\underline{G}, \underline{G}]$ [Springer 2009, Corollary 8.1.6]. Since $\eta(G) \supset [\underline{G}, \underline{G}]$, there exists $g \in G$ such that $\eta(g) = \underline{g}$ and $\tilde{\eta}(gjS) = \underline{g}\underline{j}\underline{S}$. This proves the surjectivity of $\tilde{\eta}$. It also proves that η induces a transitive G-action on $\underline{G}/\underline{j}\underline{S}$. In other words, $\underline{G}/\underline{j}\underline{S}$ is a homogeneous space for G. Thus, $\tilde{\eta}$ is a bijective G-equivariant morphism of homogeneous spaces. According to [Springer 2009, Theorem 5.3.2 (ii)], the bijective morphism $\tilde{\eta}$ is an isomorphism if and only if its differential $d\tilde{\eta} : \mathfrak{g}/dj\mathfrak{s} \to \underline{\mathfrak{g}}/d\underline{j}\underline{\mathfrak{s}}$ is an isomorphism. We observe that

$$d\tilde{\eta}(X + dj\mathfrak{s}) = d\eta(X) + d\underline{j}\underline{\mathfrak{s}}, \quad X \in \mathfrak{g}.$$

By hypothesis (Theorem 4.1(i)), the kernel of $d\eta$ is central in $\mathfrak{g}$. The one-dimensional root space $\mathfrak{g}_\alpha$, $\alpha \in R(G, S)$, is not central in $\mathfrak{g}$. Consequently $d\eta$ carries $\mathfrak{g}_\alpha$ onto the root space $\underline{\mathfrak{g}}_\alpha$ of $\underline{\mathfrak{g}}$. As the diagram

$$\begin{array}{ccc} \mathfrak{g}/dj\mathfrak{s} & \xrightarrow{d\tilde{\eta}} & \underline{\mathfrak{g}}/d\underline{j}\underline{\mathfrak{s}} \\ \cong\downarrow & & \downarrow\cong \\ \bigoplus_\alpha \mathfrak{g}_\alpha & \underset{d\eta}{\xrightarrow{\cong}} & \bigoplus_\alpha \underline{\mathfrak{g}}_\alpha \end{array}$$

commutes, we see in turn that $d\tilde{\eta}$ is an isomorphism and that $\tilde{\eta}$ is an isomorphism. Since η is defined over F, so is $\tilde{\eta}$, and it follows that $\tilde{\eta}$ yields an isomorphism $(G/jS)(F) \cong (\underline{G}/\underline{j}\underline{S})(F)$. This proves the bijectivity of the upper horizontal map.

Finally, the lower horizontal map on the right of the main diagram is defined by restricting the upper right horizontal map. This yields a bijection

$$G(F)\backslash\bar{\eta}^{-1}(\{\underline{g}\underline{j}\underline{S} : \underline{g} \in \underline{G}(F)\}) \to \eta(G(F))\backslash\underline{G}(F)\underline{j}\underline{S}/\underline{j}\underline{S},$$

and the set on the right is $\eta(G(F))\backslash\underline{G}(F)/\underline{j}\underline{S}(F)$. □

The following lemma is an immediate consequence of Lemma 5.6 and decomposition formula (23).

Lemma 5.7. *An irreducible representation π of $G(F)$ is a subrepresentation of $\pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})} \circ \eta$ if and only if $\pi \cong \pi_{({}^g jS, {}^g j\theta)}$ for $gjS \in \bar{\eta}^{-1}(\underline{j}\underline{S})$ $(\mathrm{Ad}(g) \circ j$ is in the fiber of $\bar{\eta}$ over $\underline{j})$. In particular, (24) holds for $\underline{g} = 1$.*

We are now ready to prove Proposition 5.5

Proof of Proposition 5.5. Let $\underline{j}' = \mathrm{Ad}(\underline{g}) \circ \underline{j}$. Arguing as in the proof of Proposition 4.7 there exists $g'S \in (G/S)(F)$ such that $j' = \mathrm{Ad}(g') \circ j$ is sent to $\underline{j}'$ under $\bar{\eta}$. We may replace $\underline{j}$ with $\underline{j}'$, and j with j' in the earlier results. Lemma 5.7 then tells us that the irreducible subrepresentations of

$$\pi_{({}^{\underline{g}}\underline{j}\underline{S}, {}^{\underline{g}}\underline{j}\underline{\theta})} \circ \eta = \pi_{(\underline{j}'\underline{S}, \underline{j}'\underline{\theta})} \circ \eta$$

are $\pi_{({}^h j'S, {}^h j'\theta)} = \pi_{({}^{hg}jS, {}^{hg}j\theta)}$, where $hj'S \in \bar{\eta}^{-1}(\underline{j}'\underline{S})$. The corollary now follows from

$$\begin{aligned} hj'S \in \bar{\eta}^{-1}(\underline{j}'\underline{S}) &\iff \eta(hg'jS(g')^{-1}) \subset \underline{g}\underline{j}\underline{S}\underline{g}^{-1} \\ &\iff \eta(hg')\underline{j}\underline{S} = \underline{g}\underline{j}\underline{S} \iff (hg')jS \in \bar{\eta}^{-1}(\underline{g}\underline{j}\underline{S}) \end{aligned}$$

and setting $g = hg'$. □

A simple consequence of Proposition 5.5 and diagram (21) is the following corollary, analogous to [Solleveld 2020, Corollary 5.8].

Corollary 5.8. *The L-packet Π_φ consists of the irreducible representations appearing on the right of (24). More precisely,*

$$\Pi_\varphi = \coprod_{\mathrm{Ad}(\underline{g})\circ \underline{j}\in \underline{\mathcal{J}}_F} \{\pi_{({}^g jS,{}^g j\theta)} : \mathrm{Ad}(g)\circ j \in \bar{\eta}^{-1}(\mathrm{Ad}(\underline{g})\circ \underline{j})\},$$

or equivalently, $\Pi_\varphi = \{[\underline{\pi}\circ\eta] : \underline{\pi}\in \Pi_{\underline{\varphi}}\}$.

Proof. The corollary follows from the commutativity of (21) and the partition

$$\mathcal{J}_F = \coprod_{\mathrm{Ad}(\underline{g})\circ \underline{j}\in \underline{\mathcal{J}}_F} \bar{\eta}^{-1}(\mathrm{Ad}(\underline{g})\circ \underline{j}). \qquad \square$$

We continue by expressing Proposition 5.5, which concerns the left-hand side of diagram (21), in terms of the characters of the dual groups, which appear on the right of the diagram. Recall that the characters τ and $\underline{\tau}$ on the right of the diagram are defined in (18).

Corollary 5.9. *Suppose $\underline{g}\,\underline{j}\,\underline{S}\in(\underline{G}/\underline{j}\,\underline{S})(F)$ and $gjS\in(G/jS)(F)$. Then $\pi_{({}^g jS,{}^g j\theta)}$ is a subrepresentation of $\pi_{({}^{\underline{g}}\underline{j}\,\underline{S},{}^{\underline{g}}\underline{j}\,\underline{\theta})}\circ\eta$ if and only if $\tau_g\circ\hat{\eta} = \underline{\tau}_{\underline{g}}$. In addition,*

$$\pi_{({}^{\underline{g}}\underline{j}\,\underline{S},{}^{\underline{g}}\underline{j}\,\underline{\theta})}\circ\eta \cong \bigoplus_{g\in G(F)\backslash (G/jS)(F)} \mathrm{Hom}(\underline{\tau}_{\underline{g}},\tau_g\circ\hat{\eta})\otimes \pi_{({}^g jS,{}^g j\theta)}.$$

Remark 5.10. Since $\underline{\tau}_{\underline{g}}$ and τ_g are characters, $\dim\mathrm{Hom}(\underline{\tau}_{\underline{g}},\tau_g\circ\hat{\eta})$ is either equal to 1 or 0. As such, $\pi_{({}^{\underline{g}}\underline{j}\,\underline{S},{}^{\underline{g}}\underline{j}\,\underline{\theta})}\circ\eta$ is multiplicity free for all $\underline{g}\,\underline{j}\,\underline{S}\in(\underline{G}/\underline{j}\,\underline{S})(F)$. One can also prove that the decomposition is multiplicity free using tools directly from the classification theory of supercuspidal representations such as [Hakim and Murnaghan 2008; Murnaghan 2011] (as done, for instance, in [Bourgeois 2021, Section 6]).

Yet another manner of expressing the decompositions of the corollaries is to set $\underline{\varrho}=\underline{\tau}_{\underline{g}}$ and set $\pi(\underline{\varphi},\underline{\varrho})=\pi_{({}^{\underline{g}}\underline{j}\,\underline{S},{}^{\underline{g}}\underline{j}\,\underline{\theta})}$. Then the decomposition of Corollary 5.9 reads as

$$\pi(\underline{\varphi},\underline{\varrho})\circ\eta \cong \bigoplus_{\varrho\in\pi_0(\widehat{S}^\Gamma/Z(\widehat{G})^\Gamma)^D} \mathrm{Hom}(\underline{\varrho},\varrho\circ\hat{\eta})\otimes\pi(\varphi,\varrho), \tag{25}$$

thus completing the proof of Theorem 1.1.

This form of the decomposition is the one proposed by Solleveld [2020, Conjecture 2] when η is chosen to preserve fixed pinnings of G and $\underline{G}$ as done in Section 4.1.

In Section 4.1 we also remarked that dropping the requirement of preserving the pinnings, but keeping the dual homomorphism $\hat{\eta}$ fixed, allows one to replace η with

$\eta' = \mathrm{Ad}(\underline{t}') \circ \eta$ where $\underline{t}' Z(\underline{G}) \in (\underline{T}/Z(\underline{G}))(F)$. It is convenient to write $\underline{t} = (\underline{t}')^{-1}$, for in this arrangement

$$\pi_{({}^{\underline{g}}\underline{j}\underline{S}, {}^{\underline{g}}\underline{j}\underline{\theta})} \circ \eta' = \pi_{({}^{\underline{g}}\underline{j}\underline{S}, {}^{\underline{g}}\underline{j}\underline{\theta})} \circ \mathrm{Ad}(\underline{t}') \circ \eta = \pi_{({}^{\underline{tg}}\underline{j}\underline{S}, {}^{\underline{tg}}\underline{j}\underline{\theta})} \circ \eta$$

and Corollary 5.9 yields

$$\pi_{({}^{\underline{g}}\underline{j}\underline{S}, {}^{\underline{g}}\underline{j}\underline{\theta})} \circ \eta' \cong \bigoplus_{g \in G(F)\backslash(G/jS)(F)} \mathrm{Hom}(\underline{\tau}_{\underline{tg}}, \tau_g \circ \hat{\eta}) \otimes \pi_{({}^{g}jS, {}^{g}j\theta)}. \tag{26}$$

This decomposition can be rephrased in terms of characters on the dual group as in (25). The introduction of $\mathrm{Ad}(\underline{t}')$ on the left of (25) introduces the character $\underline{\tau}_{\underline{t}'}$ on the right, as one sees in the following corollary.

Corollary 5.11. *Suppose* $\underline{g} \in (\underline{G}/\underline{j}\underline{S})(F)$ *and* $\eta' = \mathrm{Ad}(\underline{t}') \circ \eta$, *where* $\underline{t}' Z(\underline{G}) \in (\underline{T}/Z(\underline{G}))(F)$. *Set* $\underline{\varrho} = \underline{\tau}_{\underline{g}}$, $\pi(\underline{\varphi}, \underline{\varrho}) = \pi_{({}^{\underline{g}}\underline{j}\underline{S}, {}^{\underline{g}}\underline{j}\underline{\theta})}$. *Then*

$$\pi(\underline{\varphi}, \underline{\varrho}) \circ \eta' \cong \bigoplus_{\varrho \in \pi_0(\widehat{S}^{\Gamma}/Z(\widehat{G})^{\Gamma})^D} \mathrm{Hom}(\underline{\varrho}, (\varrho \circ \hat{\eta}) \otimes \underline{\tau}_{\underline{t}'}) \otimes \pi(\varphi, \varrho).$$

Proof. The character $\underline{\tau}_{\underline{tg}}$ appearing in (26) corresponds to the element $\underline{tg} \in (\underline{G}/\underline{j}\underline{S})$. According to Lemma 5.4 (applied to $\underline{G}$ and $\underline{j}\underline{S}$), the element $\underline{tg}$ represents the cocycle $z_{\underline{tg}}$ defined by

$$z_{\underline{tg}}(\sigma) = (\underline{tg})^{-1}\sigma(\underline{tg}), \quad \sigma \in \Gamma.$$

Since $\underline{t} Z(\underline{G}) \in (\underline{T}/Z(\underline{G}))(F)$, we have in turn that $\underline{t}^{-1}\sigma(\underline{t}) \in Z(\underline{G})$ and

$$(\underline{tg})^{-1}\sigma(\underline{tg}) = \underline{g}^{-1}(\underline{t}^{-1}\sigma(\underline{t}))\sigma(\underline{g}) = \underline{t}^{-1}\sigma(\underline{t})\underline{g}^{-1}\sigma(\underline{g}).$$

Therefore $z_{\underline{tg}} = z_{\underline{t}} z_{\underline{g}}$ in the group $Z^1(F, jS)$. Applying $\underline{\tau}$, we obtain $\underline{\tau}_{\underline{tg}} = \underline{\tau}_{\underline{t}} \otimes \underline{\tau}_{\underline{g}}$. A similar argument leads to $1 = \underline{\tau}_{\underline{t}} \otimes \tau_{\underline{t}^{-1}}$, and so $\underline{\tau}_{\underline{t}}^{-1} = \underline{\tau}_{\underline{t}'}$. By setting $\varrho = \tau_g$ in (26) we see that

$$\mathrm{Hom}(\underline{\tau}_{\underline{tg}}, \tau_g \circ \hat{\eta}) = \mathrm{Hom}(\underline{\tau}_{\underline{t}}\underline{\tau}_{\underline{g}}, \varrho \circ \hat{\eta}) = \mathrm{Hom}(\underline{\varrho}, (\varrho \circ \hat{\eta}) \otimes \underline{\tau}_{\underline{t}'}).$$

The corollary now follows from (26) and the commutativity of (21). □

The decomposition of Corollary 5.11 resembles the one appearing in Solleveld's conjecture [2020, Conjecture 2]. The only difference is that in place of the term $(\varrho \circ \hat{\eta}) \otimes \underline{\tau}_{\underline{t}'}$ in Corollary 5.11, Solleveld [2020, (5.4)] has a term ${}^{S}\eta'^{*}(\varrho)$. Translated into our setting, Solleveld's term is expressed as

$${}^{S}\eta'^{*}(\varrho) = (\varrho \circ \hat{\eta}') \otimes \tau_{\underline{\varphi}}(t') = (\varrho \circ \hat{\eta}) \otimes \tau_{\underline{\varphi}}(t'),$$

where $\tau_{\underline{\varphi}} : \underline{G}(F)\backslash(\underline{G}/Z(\underline{G}))(F) \to \pi_0(\widehat{\underline{S}}^{\Gamma}/Z(\widehat{\underline{G}})^{\Gamma})^D$ is a homomorphism defined in [Solleveld 2020, (2.12), Lemma 2.1].

Proposition 5.12. *Under the regularity assumptions on φ and $\underline{\varphi}$, Corollary 5.11 coincides with* [Solleveld 2020, Conjecture 2]. *That is,* $(\varrho \circ \hat{\eta}) \otimes \underline{\tau}_{\underline{t}'} = {}^{S}\eta'^{*}(\varrho)$, *and therefore,*

$$\pi(\underline{\varphi}, \underline{\varrho}) \circ \eta' \cong \bigoplus_{\varrho \in \pi_0(\widehat{S}^{\Gamma}/Z(\widehat{G})^{\Gamma})^D} \operatorname{Hom}(\underline{\varrho}, (\varrho \circ \hat{\eta}) \otimes \underline{\tau}_{\underline{t}'}) \otimes \pi(\varphi, \varrho)$$
$$= \bigoplus_{\varrho \in \pi_0(\widehat{S}^{\Gamma}/Z(\widehat{G})^{\Gamma})^D} \operatorname{Hom}(\underline{\varrho}, {}^{S}\eta'^{*}(\varrho)) \otimes \pi(\varphi, \varrho).$$

Proof. All we need to show is

$$\tau_{\underline{\varphi}}(\underline{t}') = \underline{\tau}_{\underline{t}'}. \tag{27}$$

Under the assumption that the characteristic of F is zero, this identity is given in the proof of [2013, Lemma 4.2] in which Kaletha writes $\tau_{\underline{\varphi}}(\underline{t}')$ as $(\mathfrak{w}, \mathfrak{w}')$ for a pair of Whittaker data conjugate under $\underline{t}'$ [2013, pp. 2454–2455], and $\underline{\tau}_{\underline{t}'}$ is expressed in terms of the cocycle $z = z_{\underline{t}'}$ and the Tate–Nakayama pairing (Lemma 5.4). His argument references the work in [Kottwitz and Shelstad 1999, Appendix A] on the hypercohomology of complexes of tori of length two. The argument is the same in positive characteristic, and to convince the reader that nothing runs awry we offer a sketch.

We write a complex of F-tori of length two simply as $\mathcal{T} \to \mathcal{S}$, concentrated in degrees 0 and 1. Let $\mathcal{T}$ be a maximal torus in $\underline{G}$ which is defined over F. Let $\underline{Z} = Z(\underline{G})$ and $\mathcal{T}_{\text{ad}} = \mathcal{T}/\underline{Z}$. Then $\mathcal{T}_{\text{ad}}$ is a maximal torus in $\underline{G}_{\text{ad}} = \underline{G}/\underline{Z}$, and its Langlands dual is $(\widehat{\mathcal{T}})_{\text{sc}}$, a maximal torus in the simply connected dual group $(\widehat{\underline{G}})_{\text{sc}} = \widehat{\underline{G}_{\text{ad}}}$. The map $(\widehat{\underline{G}})_{\text{sc}} \to [\widehat{\underline{G}}, \widehat{\underline{G}}] \to \widehat{G}$ induces a map $(\widehat{\mathcal{T}})_{\text{sc}} \to \widehat{\mathcal{T}}$ with kernel denoted by $\widehat{\underline{Z}}$.

The sequence

$$0 \to (0 \to \mathcal{T}_{\text{ad}}) \xrightarrow{(0,id)} (\mathcal{T} \to \mathcal{T}_{\text{ad}}) \xrightarrow{(id,0)} (\mathcal{T} \to 0) \to 0$$

is a short exact sequence, and therefore gives rise to a long exact sequence of Galois hypercohomology. The first hypercohomology portion of this long exact sequence appears in the second row of the diagram

$$\begin{array}{ccccc}
\mathcal{T}_{\text{ad}}(F) & \longrightarrow & H^1(F, \underline{Z}) & \longrightarrow & H^1(F, \mathcal{T}) \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
H^1(F, 0 \to \mathcal{T}_{\text{ad}}) & \longrightarrow & H^1(F, \mathcal{T} \to \mathcal{T}_{\text{ad}}) & \longrightarrow & H^1(F, \mathcal{T} \to 0) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(W_F, (\widehat{\mathcal{T}})_{\text{sc}} \to 0)^D & \longrightarrow & H^1(W_F, (\widehat{\mathcal{T}})_{\text{sc}} \to \widehat{\mathcal{T}})^D & \longrightarrow & H^1(W_F, 0 \to \widehat{\mathcal{T}})^D \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
H^1(W_F, (\widehat{\mathcal{T}})_{\text{sc}})^D & \longrightarrow & H^1(W_F, \widehat{\underline{Z}})^D & \longrightarrow & (\widehat{\mathcal{T}}^{W_F})^D
\end{array} \tag{28}$$

The third row of the diagram is the Pontryagin dual of the analogous sequence of the dual tori together with the action of the Weil group on the dual tori. The vertical arrows between the second and third rows are given by the pairing [Kottwitz and Shelstad 1999, (A.3.15)] (see [Dillery 2022, Proposition A.4]).

The first row of the diagram is defined as follows. Any element in $\mathcal{T}_{\text{ad}}(F)$ may be written as $t'\underline{Z}$ where $t' \in \mathcal{T}$. The first horizontal map sends $t'\underline{Z}$ to the class of the 1-cocycle $z_{t'}$ defined by

$$z_{t'}(\sigma) = (t')^{-1}\sigma(t'), \quad \sigma \in \Gamma. \tag{29}$$

The second horizontal map in the top row carries $z_{t'}$ to itself. The vertical isomorphisms between the first and second rows are canonical and left as exercises (see [Kottwitz and Shelstad 1999, A.1]).

The maps of the fourth row and the isomorphisms with the third row follow just as the ones for the first and second rows. Starting with $h \in \widehat{\mathcal{T}}^{W_F}$, we choose $h' \in (\widehat{\mathcal{T}})_{\text{sc}}$ so that $h = h'\widehat{\underline{Z}}$. We then define an element $c_{h'}$ in $H^1(W_F, \widehat{\underline{Z}})$ or $H^1(W_F, (\widehat{\mathcal{T}})_{\text{sc}})$ by imitating (29) (see [Solleveld 2020, (2.8)]). The maps in the fourth row are the ones dual to those just defined.

Diagram (28) is commutative, due to the functoriality of the vertical morphisms (see [Kottwitz and Shelstad 1999, (A.3.5)]). By making a comparison with the cohomology crossed modules, one can also see that the map $H^1(F, \underline{Z}) \to H^1(W_F, \widehat{\underline{Z}})^D$ in the middle of (28) is independent of the choice of maximal torus $\mathcal{T}$ (see the proof of [Kaletha 2015, Proposition 5.19]). Combining these facts with $\mathcal{T} = \underline{T}$ and $\mathcal{T} = j\underline{S}$, we obtain the commutative diagram

$$\begin{array}{ccccc}
\underline{T}_{\text{ad}}(F) & \longrightarrow & H^1(F, \underline{Z}) & \longrightarrow & H^1(F, j\underline{S}) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(W_F, (\widehat{\underline{T}})_{\text{sc}})^D & \longrightarrow & H^1(W_F, \widehat{\underline{Z}})^D & \longrightarrow & (\widehat{\underline{S}}^{\varphi(W_F)})^D
\end{array}$$

Consider the element $t'\underline{Z} \in \underline{T}_{\text{ad}}(F)$ in the top left of this diagram. If we trace $t' \in \underline{T}$ through the vertical map on the left and the lower horizontal maps, we arrive at $\tau_{\underline{\varphi}}(t')$ [Kottwitz and Shelstad 1999, (A.3.13); Solleveld 2020, (2.11)]. Alternatively, tracing $t' \in \underline{T}$ through the upper horizontal maps followed by the vertical map on the right we arrive at $\tau_{t'}$ [Kottwitz and Shelstad 1999, (A.3.14)]. The commutativity of the diagram yields the desired identity (27). □

Appendix: Intertwining maps and coset identities

This appendix groups together some important results that were cited in the main text, but whose proofs did not necessarily fit with the flow of the main narrative.

Proposition A.1. *Let G and $\underline{G}$ be as per Theorem 4.1, and let $G_1 \subseteq G_2 \subseteq G(F)$. Let $\underline{g} \in \underline{G}(F)$ and let $\underline{\text{Ad}}(\underline{g})$ be the automorphism of $G(F)$ as defined in Section 2.3.*

Then, given a representation π of G_1,

$$\operatorname{Ind}_{\underline{\mathrm{Ad}}(\underline{g})(G_1)}^{\underline{\mathrm{Ad}}(\underline{g})(G_2)}(\pi \circ \underline{\mathrm{Ad}}(\underline{g}^{-1})) \simeq \operatorname{Ind}_{G_1}^{G_2} \pi \circ \underline{\mathrm{Ad}}(\underline{g}^{-1}).$$

Proof. Let V denote the vector space on which the representations π and $\pi \circ \underline{\mathrm{Ad}}(\underline{g}^{-1})$ act. Then, $\operatorname{Ind}_{\underline{\mathrm{Ad}}(\underline{g})(G_1)}^{\underline{\mathrm{Ad}}(\underline{g})(G_2)}(\pi \circ \underline{\mathrm{Ad}}(\underline{g}^{-1}))$ and $\operatorname{Ind}_{G_1}^{G_2} \pi \circ \underline{\mathrm{Ad}}(\underline{g}^{-1})$ act on the vector spaces

$$\underline{W} = \{\underline{f} : \underline{\mathrm{Ad}}(\underline{g})(G_2) \to V \text{ locally constant} \mid \underline{f}(gh) = (\pi \circ \underline{\mathrm{Ad}}(\underline{g}))(h)^{-1} \underline{f}(g) \text{ for all } h \in \underline{\mathrm{Ad}}(\underline{g})(G_1)\}$$

and

$$W = \{f : G_2 \to V \text{ locally constant} \mid f(gh) = \pi(h)^{-1} f(g) \text{ for all } h \in G_1\},$$

respectively. Define a linear map $\mathcal{F}$ as

$$\mathcal{F} : W \to \underline{W}, \quad f \mapsto f \circ \underline{\mathrm{Ad}}(\underline{g}^{-1}).$$

The map $\mathcal{F}$ is bijective as a consequence of $\underline{\mathrm{Ad}}(\underline{g}^{-1}) : \underline{\mathrm{Ad}}(\underline{g})(G_2) \to G_2$ being bijective. It is then an easy computation to verify that $\mathcal{F}$ intertwines the representations $\operatorname{Ind}_{\underline{\mathrm{Ad}}(\underline{g})(G_1)}^{\underline{\mathrm{Ad}}(\underline{g})(G_2)}(\pi \circ \underline{\mathrm{Ad}}(\underline{g}^{-1}))$ and $\operatorname{Ind}_{G_1}^{G_2} \pi \circ \underline{\mathrm{Ad}}(\underline{g}^{-1})$. ☐

Proposition A.2. *Let $\mu : H_2' \to H_2$ be a morphism of locally profinite groups, $H_1 \subset H_2$ and $H_1' \subset H_2'$ subgroups such that $\mu(H_2') \cap H_1 = \mu(H_1')$ and $\ker(\mu) \subset H_1'$. Let π be a representation of H_1. Then,*

$$\operatorname{Ind}_{H_1'}^{H_2'}(\pi \circ \mu) \simeq \left(\operatorname{Ind}_{\mu(H_1')}^{\mu(H_2')} \operatorname{Res}_{\mu(H_1')}^{H_1} \pi\right) \circ \mu.$$

Proof. Let V denote the vector space on which the representations π and $\pi \circ \mu$ act. Then, the representations $\left(\operatorname{Ind}_{\mu(H_1')}^{\mu(H_2')} \operatorname{Res}_{\mu(H_1')}^{H_1} \pi\right) \circ \mu$ and $\operatorname{Ind}_{H_1'}^{H_2'}(\pi \circ \mu)$ act on the vector spaces

$$W_\mu = \{f_\mu : \mu(H_2') \to V \text{ locally constant} \mid f_\mu(gh) = \pi(h)^{-1} f_\mu(g) \text{ for all } h \in \mu(H_1')\}$$

and

$$W = \{f : H_2' \to V \text{ locally constant} \mid f(gh) = (\pi \circ \mu)(h)^{-1} f(g) \text{ for all } h \in H_1'\},$$

respectively. Define a linear map $\mathcal{F}$ as

$$\mathcal{F} : W_\mu \to W, \quad f_\mu \mapsto f_\mu \circ \mu.$$

One sees that the map $\mathcal{F}$ is injective, as $\mu : H_2' \to \mu(H_2')$ is surjective.

Next, we show that $\mathcal{F}$ is surjective. Given $f \in W$, we have that f is constant on the coset $h \ker \mu$ for all $h \in H_2'$. Indeed, for all $z \in \ker \mu \subset H_1'$, we have $f(hz) = (\pi \circ \mu)(z)^{-1} f(h) = f(h)$. This allows us to define a map

$$f_\mu : H_2'/\ker \mu \to V, \quad h \ker \mu \mapsto f(h),$$

which we view as an element of W_μ under the isomorphism $\mu(H_2') \simeq H_2'/\ker\mu$. By construction we have $\mathcal{F}(f_\mu) = f$, which proves surjectivity.

It is then an easy computation to verify that $\mathcal{F}$ intertwines the representations $\big(\mathrm{Ind}_{\mu(H_1')}^{\mu(H_2')} \mathrm{Res}_{\mu(H_1')}^{H_1} \pi\big) \circ \mu$ and $\mathrm{Ind}_{H_1'}^{H_2'}(\pi \circ \mu)$. □

Lemma A.3. *Let A, B, C be subgroups of a fixed group such that $B \subseteq A$, and both A and B normalize C. Then there is a natural bijection*

$$AC/BC \simeq A/B(A \cap C).$$

Proof. Note that $B \subseteq A$ normalizes both A and C, so that B normalizes $A \cap C$. Consequently, $B(A \cap C)$ is a group, and $A/B(A \cap C)$ is a set of cosets. Consider the map $A/B(A \cap C) \to AC/BC$ of cosets defined by

$$aB(A \cap C) \mapsto aBC, \quad a \in A.$$

This map is clearly well defined and surjective. Furthermore, if $a_1BC = a_2BC$ for $a_1, a_2 \in A$, then $a_1 = a_2bc$, where $b \in B$ and $c = b^{-1}a_2^{-1}a_1 \in A \cap C$. This proves that the map is injective. □

Lemma A.4. *Let N, A, B, $\bar{N}$, $\bar{A}$, $\bar{B}$ be groups that satisfy the conditions*

$$\begin{array}{ccccccc} N & \lhd & A & & \bar{A} & \rhd & \bar{N} \\ & & \text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} & & \\ & & B & \supseteq & \bar{B} & & \end{array}$$

Let L be a set of coset representatives of $N\backslash A/B$, $\mathcal{L}$ be a set of coset representatives of $\bar{N}\backslash\bar{A}/\bar{B}$, and suppose that both L and $\mathcal{L}$ are finite. Write $L\mathcal{L} = \{l\ell : l \in L,\ \ell \in \mathcal{L}\}$.

(1) *Suppose $\bar{A} \subseteq B$, $N \cap \bar{A} \subseteq \bar{N}$, and $B/(N \cap B)\bar{B} \simeq \bar{A}/\bar{N}\bar{B}$. Then $L\mathcal{L}$ is a set of coset representatives of $N\backslash A/\bar{B}$.*

(2) *If $\bar{N} \subseteq N$, $N \cap B = \bar{B}$, $N = \bar{A}$ and $\bar{N}$ is a normal subgroup of A, then $L\mathcal{L}$ is a set of coset representatives of $\bar{N}\backslash A/B$.*

Proof. Since N is normal in A, and $\bar{N}$ is normal in $\bar{A}$, we note that $N\backslash A/B = A/NB$, $N\backslash A/\bar{B} = A/N\bar{B}$ and $\bar{N}\backslash\bar{A}/\bar{B} = \bar{A}/\bar{N}\bar{B}$.

To prove (1), we use the normality of N, the inclusion $\bar{B} \subseteq B$ and the isomorphism $B/(N \cap B)\bar{B} \simeq \bar{A}/\bar{N}\bar{B}$, along with the second and third isomorphism theorems to show that

$$A/NB \simeq A/N\bar{B}/\bar{A}/\bar{N}\bar{B}.$$

As both L and $\mathcal{L}$ are finite it follows that

$$|A/NB| \cdot |\bar{A}/\bar{N}\bar{B}| = |A/N\bar{B}|.$$

We then use the inclusions $\bar{A} \subseteq B$ and $N \cap \bar{A} \subseteq \bar{N}$ to show that the map

$$\mu : L \times \mathcal{L} \to A/N\bar{B}, \quad (l, \ell) \mapsto l\ell N\bar{B},$$

is injective.

The proof of (2) follows the exact same strategy as the proof of (1). □

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References

[Adler 1998] J. D. Adler, "Refined anisotropic K-types and supercuspidal representations", *Pacific J. Math.* **185**:1 (1998), 1–32. MR

[Adler et al. 2011] J. D. Adler, S. DeBacker, P. J. Sally, Jr., and L. Spice, "Supercuspidal characters of SL_2 over a p-adic field", pp. 19–69 in *Harmonic analysis on reductive*, *p-adic groups*, edited by R. S. Doran et al., Contemp. Math. **543**, Amer. Math. Soc., Providence, RI, 2011. MR

[Aubert et al. 2018] A.-M. Aubert, A. Moussaoui, and M. Solleveld, "Generalizations of the Springer correspondence and cuspidal Langlands parameters", *Manuscripta Math.* **157**:1-2 (2018), 121–192. MR

[Borel 1979] A. Borel, "Automorphic L-functions", pp. 27–61 in *Automorphic forms*, *representations and L-functions*, *part* 2 (Corvallis, OR, 1977), edited by A. Borel and W. Casselman, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, RI, 1979. MR

[Bourgeois 2020] A. Bourgeois, *On the restriction of supercuspidal representations: an in-depth exploration of the data*, Ph.D. thesis, University of Ottawa, 2020, available at http://hdl.handle.net/10393/40901.

[Bourgeois 2021] A. Bourgeois, "Restricting supercuspidal representations via a restriction of data", *Pacific J. Math.* **312**:1 (2021), 1–39. MR

[Bruhat and Tits 1972] F. Bruhat and J. Tits, "Groupes réductifs sur un corps local", *Inst. Hautes Études Sci. Publ. Math.* 41 (1972), 5–251. MR

[Bruhat and Tits 1984] F. Bruhat and J. Tits, "Groupes réductifs sur un corps local, II: Schémas en groupes; existence d'une donnée radicielle valuée", *Inst. Hautes Études Sci. Publ. Math.* 60 (1984), 5–184. MR

[Carter 1993] R. W. Carter, *Finite groups of Lie type*: *conjugacy classes and complex characters*, Wiley, Chichester, 1993. MR

[DeBacker and Reeder 2009] S. DeBacker and M. Reeder, "Depth-zero supercuspidal L-packets and their stability", *Ann. of Math.* (2) **169**:3 (2009), 795–901. MR

[Deligne and Lusztig 1976] P. Deligne and G. Lusztig, "Representations of reductive groups over finite fields", *Ann. of Math.* (2) **103**:1 (1976), 103–161. MR

[Dillery 2022] P. E. Dillery, *Rigid inner forms over function fields*, Ph.D. thesis, University of Michigan, Ann Arbor, MI, 2022, available at https://www.proquest.com/docview/2723382422. MR

[Dillery 2023] P. Dillery, "Rigid inner forms over local function fields", *Adv. Math.* **430** (2023), art. id. 109204. MR

[Fintzen 2021] J. Fintzen, "On the Moy–Prasad filtration", *J. Eur. Math. Soc.* (*JEMS*) **23**:12 (2021), 4009–4063. MR

[Fintzen et al. 2023] J. Fintzen, T. Kaletha, and L. Spice, "A twisted Yu construction, Harish-Chandra characters, and endoscopy", *Duke Math. J.* **172**:12 (2023), 2241–2301. MR

[Hakim and Murnaghan 2008] J. Hakim and F. Murnaghan, "Distinguished tame supercuspidal representations", *Int. Math. Res. Pap. IMRP* 2 (2008), art. id. rpn005. MR

[Humphreys 1975] J. E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics **21**, Springer, 1975. MR

[Humphreys 1995] J. E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Mathematical Surveys and Monographs **43**, Amer. Math. Soc., Providence, RI, 1995. MR

[Jantzen 2003] J. C. Jantzen, *Representations of algebraic groups*, 2nd ed., Mathematical Surveys and Monographs **107**, Amer. Math. Soc., Providence, RI, 2003. MR

[Kaletha 2013] T. Kaletha, "Genericity and contragredience in the local Langlands correspondence", *Algebra Number Theory* **7**:10 (2013), 2447–2474. MR

[Kaletha 2015] T. Kaletha, "Epipelagic L-packets and rectifying characters", *Invent. Math.* **202**:1 (2015), 1–89. MR

[Kaletha 2016] T. Kaletha, "Rigid inner forms of real and p-adic groups", *Ann. of Math.* (2) **184**:2 (2016), 559–632. MR

[Kaletha 2019] T. Kaletha, "Regular supercuspidal representations", *J. Amer. Math. Soc.* **32**:4 (2019), 1071–1170. MR

[Kaletha 2021] T. Kaletha, "Supercuspidal L-packets", preprint, 2021. arXiv 1912.03274

[Kaletha and Prasad 2023] T. Kaletha and G. Prasad, *Bruhat–Tits theory — a new approach*, New Mathematical Monographs **44**, Cambridge Univ. Press, 2023. MR

[Kottwitz 1982] R. E. Kottwitz, "Rational conjugacy classes in reductive groups", *Duke Math. J.* **49**:4 (1982), 785–806. MR

[Kottwitz 1986] R. E. Kottwitz, "Stable trace formula: elliptic singular terms", *Math. Ann.* **275**:3 (1986), 365–399. MR

[Kottwitz and Shelstad 1999] R. E. Kottwitz and D. Shelstad, *Foundations of twisted endoscopy*, Astérisque **255**, Soc. Math. de France, 1999. MR

[Langlands and Shelstad 1987] R. P. Langlands and D. Shelstad, "On the definition of transfer factors", *Math. Ann.* **278**:1-4 (1987), 219–271. MR

[Murnaghan 2011] F. Murnaghan, "Parametrization of tame supercuspidal representations", pp. 439–469 in *On certain L-functions*, edited by J. Arthur et al., Clay Math. Proc. **13**, Amer. Math. Soc., Providence, RI, 2011. MR

[Nevins 2015] M. Nevins, "Restricting toral supercuspidal representations to the derived group, and applications", *J. Pure Appl. Algebra* **219**:8 (2015), 3337–3354. MR

[Serre 2002] J.-P. Serre, *Galois cohomology*, Springer, 2002. MR

[SGA $3_{\rm III}$ 1970] M. Demazure and A. Grothendieck, *Schémas en groupes, Tome III: Structure des schémas en groupes réductifs, Exposés XIX–XXVI* (Séminaire de Géométrie Algébrique du Bois Marie 1962–1964), Lecture Notes in Math. **153**, Springer, 1970. MR

[Silberger 1979] A. J. Silberger, "Isogeny restrictions of irreducible admissible representations are finite direct sums of irreducible admissible representations", *Proc. Amer. Math. Soc.* **73**:2 (1979), 263–264. MR

[Solleveld 2020] M. Solleveld, "Langlands parameters, functoriality and Hecke algebras", *Pacific J. Math.* **304**:1 (2020), 209–302. MR

[Springer 1979] T. A. Springer, "Reductive groups", pp. 3–27 in *Automorphic forms*, *representations and L-functions*, *part* 1 (Corvallis, OR, 1977)), edited by A. Borel and W. Casselman, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, RI, 1979. MR

[Springer 2009] T. A. Springer, *Linear algebraic groups*, 2nd ed., Birkhäuser, Boston, MA, 2009. MR

[Thang 2011] N. Q. Thang, "On Galois cohomology and weak approximation of connected reductive groups over fields of positive characteristic", *Proc. Japan Acad. Ser. A Math. Sci.* **87**:10 (2011), 203–208. MR

[Vogan 1993] D. A. Vogan, Jr., "The local Langlands conjecture", pp. 305–379 in *Representation theory of groups and algebras*, edited by J. Adams et al., Contemp. Math. **145**, Amer. Math. Soc., Providence, RI, 1993. MR

[Yu 2001] J.-K. Yu, "Construction of tame supercuspidal representations", *J. Amer. Math. Soc.* **14**:3 (2001), 579–622. MR

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