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Following the approach of C. Mœglin and J.-L. Waldspurger, this article proves the local Gross–Prasad conjecture over $\mathbb{R}$ and $\mathbb{C}$ based on the tempered cases of Luo and the author.

1. Introduction

In [19; 20], B. Gross and D. Prasad formulated a conjecture on the local multiplicity for Bessel models of special orthogonal groups over a local field of characteristic 0, known as the *local Gross–Prasad conjecture*. When the local field is non-Archimedean, the conjecture was proved in [29] based on the tempered cases proved in [36; 37; 38; 39; 40]. This paper proves the local Gross–Prasad conjecture over Archimedean local fields. The proof over the real field follows Mœglin and Waldspurger's approach and is based on the tempered cases proved in [28; 10].

There are some recent applications of the local Gross–Prasad conjecture. The paper [22] takes it as an input to prove one direction of the global Gross–Prasad

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conjecture, and the paper [24] uses the local Gross–Prasad conjecture to develop the theory of arithmetic wavefront sets for irreducible admissible representations of classical groups. We refer to the ICM report of R. Beuzart-Plessis [6] for a general discussion of the significance of the local Gross–Prasad conjecture in arithmetic.

The local Gross–Prasad conjecture is set up as follows: Let F be a local field of characteristic 0, and (W, V) be a pair of nondegenerate quadratic spaces over F such that the orthogonal complement $W^{\perp}$ of W in V is odd-dimensional and split over F. We let G be the algebraic group $\mathsf{SO}(W) \times \mathsf{SO}(V)$ over F and take its subgroup $\mathsf{H} = \Delta\mathsf{SO}(W) \ltimes \mathsf{N}$, where $\Delta\mathsf{SO}(W)$ is the image of the diagonal embedding $\mathsf{SO}(W) \hookrightarrow \mathsf{SO}(W) \times \mathsf{SO}(V)$ and N is the unipotent part of a parabolic subgroup stabilizing a full totally isotropic flag on $W^{\perp}$. We fix a generic character ξ_N of $N = \mathsf{N}(F)$ that uniquely extends to a character ξ of $H = \mathsf{H}(F)$. For every irreducible admissible representation π of $G = \mathsf{G}(F)$ (we require the representation to be Casselman–Wallach when F is Archimedean), we define the multiplicity

$$m(\pi) := \dim \operatorname{Hom}_H(\pi|_H, \xi).$$

It was proved in [1; 16; 40] over non-Archimedean fields and in [34; 23] over Archimedean fields that

$$m(\pi) \leqslant 1.$$

This result is known as the *multiplicity-one theorem*. The local Gross–Prasad conjecture is a refinement of the multiplicity-one theorem that takes representations of pure inner forms of G into consideration.

For every $\alpha \in H^1(F, \mathsf{H}) \hookrightarrow H^1(F, \mathsf{G})$, the inner twists of G, H by α give pure inner forms G_α, H_α, respectively. Then $\mathsf{G}_\alpha = \mathsf{SO}(W_\alpha) \times \mathsf{SO}(V_\alpha)$ and $\mathsf{H}_\alpha = \Delta\mathsf{SO}(W_\alpha) \ltimes \mathsf{N}$, where W_α is the inner twist of W by $\alpha \in H^1(F, \mathsf{H}) = H^1(F, \mathsf{SO}(W_\alpha))$ and $V_\alpha = W_\alpha \perp S$. Let ξ_α be the character of $H_\alpha = \mathsf{H}_\alpha(F)$ obtained by the extension of ξ_N. For every irreducible admissible representation π of $G_\alpha = \mathsf{G}_\alpha(F)$ (we require the representation to be Casselman–Wallach when F is Archimedean), we extend the definition of multiplicity by setting

$$m(\pi) := \dim \operatorname{Hom}_{H_\alpha}(\pi|_{H_\alpha}, \xi_\alpha).$$

For every local L-parameter $\phi : \mathcal{W}_F \to {}^L\mathsf{G}$, we denote by $\Pi_{F,\phi}(\mathsf{G})$ the corresponding L-packet, which consists of finitely many irreducible admissible representations of $\mathsf{G}(F)$, which are Casselman–Wallach when F is Archimedean. For every $\alpha \in H^1(F, \mathsf{G})$. the Langlands dual group ${}^L\mathsf{G}_\alpha$ of G_α is isomorphic to that of G, so ϕ also represents a local L-parameter of G_α. Following D. Vogan [35], we can define the Vogan L-packet associated to ϕ as

$$\Pi_{F,\phi}^{\text{Vogan}} := \bigsqcup_{\alpha \in H^1(F,\mathsf{G})} \Pi_\phi(\mathsf{G}_\alpha).$$

The L-parameter ϕ is called *tempered* if $\mathrm{Im}(\phi)$ is bounded. The L-parameter ϕ is called *generic* if there is a generic representation in $\Pi^{\mathrm{Vogan}}_{F,\phi}$. In particular, tempered parameters are generic.

When ϕ is generic, it was conjectured by Vogan and known over Archimedean local fields [35, Theorem 6.3], that, fixing a Whittaker datum of $\{\mathsf{G}_\alpha\}_{\alpha\in H^1(F,\mathsf{G})}$, there is a bijection

$$\pi \in \Pi^{\mathrm{Vogan}}_{F,\phi} \longleftrightarrow \eta_\pi \in \widehat{\mathcal{S}_\phi}.$$

Here $\widehat{\mathcal{S}_\phi}$ is the set of (complex) characters of component group

$$\mathcal{S}_\phi := \pi_0(\mathrm{Cent}_{\widehat{G}}(\mathrm{Im}(\phi))),$$

where $\mathrm{Cent}_{\widehat{G}}(\mathrm{Im}(\phi))$ is the centralizer of the image $\mathrm{Im}(\phi)$ in the dual group $\widehat{\mathsf{G}}$. Gross and Prasad suggested that one may consider the *relevant Vogan packet*, defined as

$$\Pi^{\mathrm{Vogan}}_{F,\phi,\mathrm{rel}} := \bigsqcup_{\alpha\in H^1(F,\mathsf{H})} \Pi_{F,\phi}(\mathsf{G}_\alpha) \subset \Pi^{\mathrm{Vogan}}_{F,\phi}.$$

In particular, the multiplicity $m(\pi)$ is well-defined for representations in $\Pi^{\mathrm{Vogan}}_{F,\phi,\mathrm{rel}}$.

Conjecture 1 [19; 20]. With the notions above, the following two statements hold.

(1) (multiplicity one) For every generic parameter ϕ of G, we have

$$\sum_{\pi\in\Pi^{\mathrm{Vogan}}_{F,\phi,\mathrm{rel}}} m(\pi) = 1.$$

This implies that there is an unique representation $\pi \in \Pi^{\mathrm{Vogan}}_{F,\phi,\mathrm{rel}}$ such that $m(\pi) = 1$.

(2) (epsilon dichotomy) Fix the Whittaker datum of $\{\mathsf{G}_\alpha\}_{\alpha\in H^1(F,\mathsf{G})}$ as [20, (6.3)]. The unique representation $\pi \in \Pi^{\mathrm{Vogan}}_{F,\phi,\mathrm{rel}}$ such that $m(\pi) = 1$ can be characterized as

$$\eta_\pi = \eta_\phi,$$

where η_ϕ is defined in (2.3.2).

When F is non-Archimedean and ϕ is tempered, Waldspurger proved the conjecture in [36; 37; 38; 39; 40]. Mœglin and Waldspurger completed the proof of Conjecture 1 for generic parameters based on the results in the tempered cases.

When $F = \mathbb{R}$ and the parameter ϕ is tempered, Z. Luo proved the multiplicity-one part of Conjecture 1 in [28] following the work of R. Beuzart-Plessis in [5]. The author and Luo proved the epsilon-dichotomy part of Conjecture 1 in [10] by a simplification of Waldspurger's approach.

The main result of the paper is the following.

Theorem 1.0.1. *When $F = \mathbb{R}$ or $\mathbb{C}$, Conjecture 1 holds for generic parameters.*

The proof over $\mathbb{C}$ is done by construction based on results in [18] and the proof over $\mathbb{R}$ follows the strategy in [29]. The proof consists of a structure theorem (Proposition 4.0.5) for representations in generic packets and a multiplicity formula (Theorem 5.0.1). With these results, we can reduce all situations of the conjecture into the tempered cases.

In Section 4, we prove the structure theorem using the standard module conjecture. The proof of the multiplicity formula, however, is more intricate. Following [29], this requires a formula for reduction to basic cases and two multiplicity formulas that establish inequalities needed to prove the basic cases.

In the basic case, one inequality of the multiplicity formula is proved using orbit analysis (Section 5.3). The proof of the other inequality is expected to be completed using harmonic analysis in Section 5.4. The formula for reduction to the basic cases, which is an equality, can be established by proving two inequalities in a manner similar to the inequalities in the basic case. The non-Archimedean counterpart is discussed in [29, Section 2], [29, Sections 1.4 1.6], and [29, Sections 1.7 1.8].

There is a parallel conjecture for unitary groups, formulated by W. Gan, Gross, and Prasad. Over non-Archimedean local fields, the conjecture for tempered parameters was treated by Beuzart-Plessis in [3; 4]; Based on the tempered cases, Gan and A. Ichino proved the conjecture for generic parameters in [15]. Over Archimedean local fields, Beuzart-Plessis proved the multiplicity-one part of the conjecture in [5] for tempered parameters using local trace formula and endoscopy. Xue completed the proof for tempered cases in [43] using theta correspondence and proved the generic cases in [42].

Although it is not necessary for the proof for the local Gan–Gross–Prasad conjecture, the multiplicity formula (Theorem 5.0.1) also works for reducible representations obtained from parabolic induction. This result can be applied to the study of local descents in my joint work with D. Jiang, D. Liu, L. Zhang [12].

Organization. In Section 2, we recall the statement of the local Gross–Prasad conjecture following [19; 20]. In Section 3, we work over the complex field $\mathbb{C}$. We follow the observation in [19, §11] and prove the conjecture by constructing an explicit functional of the representation $\pi_V \boxtimes \pi_W$ using the results in [18].

In Sections 4 5, we work over the real field $\mathbb{R}$. Section 4 provides a structure theorem for representations in generic packets, using a sufficient condition for irreducibility. In Section 5, we reduce the conjecture to the tempered cases by employing a multiplicity formula, following the approach in [29].

For the basic case of the multiplicity formula, we prove one inequality using representation theory and orbit analysis (Section 5.3) and the other using harmonic analysis (Section 5.4). Additionally, in Sections 5.3 5.4, we establish a formula that reduces the multiplicity to the basic cases.

2. Local Gross–Prasad Conjecture

In this section, we review the local Gross–Prasad conjecture over Archimedean local fields following [19] and [20].

2.1. ***Gross–Prasad triples.*** Let $F=\mathbb{R}$ or $\mathbb{C}$ and (W,V) be a pair of nondegenerate quadratic spaces over F. The pair (W,V) is called *relevant* if and only if there exists an anisotropic line D and a nondegenerate even-dimensional split quadratic space Z over F such that

$$V=W\perp D\perp Z.$$

We set $r=\frac{\dim Z}{2}$. There exists a basis $\{z_i\}_{i=\pm1}^{\pm r}$ of Z such that

$$q(z_i,z_j)=\delta_{i,-j},\quad \forall i,j\in\{\pm1,\dots,\pm r\},$$

where q is the quadratic form on V. We denote by P_V the parabolic subgroup of the special orthogonal group $\mathsf{SO}(V)$ stabilizing the totally isotropic flag

$$\langle z_r\rangle\subset\langle z_r,z_{r-1}\rangle\subset\cdots\subset\langle z_r,\dots,z_1\rangle. \tag{2.1.1}$$

We take $\mathsf{P}_V=\mathsf{M}_V\cdot\mathsf{N}$ to be its Levi decomposition. In particular, the Levi subgroup $\mathsf{M}_V\simeq\mathsf{SO}(W\oplus D)\times\mathrm{GL}_1^r$.

Let $\mathsf{G}=\mathsf{SO}(W)\times\mathsf{SO}(V)$. We identify N as a subgroup of G via the embedding $\mathsf{SO}(V)\hookrightarrow 1\times\mathsf{SO}(V)$. We set $\Delta\mathsf{SO}(W)$ as the image of the diagonal embedding $\mathsf{SO}(W)\hookrightarrow\mathsf{G}$. Then $\Delta\mathsf{SO}(W)$ acts on N by adjoint action of $\mathsf{SO}(W)\subset\mathsf{M}_V$. We set

$$\mathsf{H}=\Delta\mathsf{SO}(W)\ltimes\mathsf{N}.$$

We define a morphism $\lambda:\mathsf{N}\to\mathbb{G}_a$ by

$$\lambda(n)=\sum_{i=0}^{r-1}q(z_{-i-1},nz_i),\qquad n\in\mathsf{N}.$$

Then λ is $\Delta\mathsf{SO}(W)$-conjugation invariant and hence λ admits an unique extension to H that is trivial on $\Delta\mathsf{SO}(W)$. We still denote this character by λ. Let $\lambda_F:\mathsf{H}(F)\to F$ be the induced morphism on F-rational points. We define an unitary character of $H=\mathsf{H}(F)$ by

$$\xi(h)=\lambda_F(h),\qquad h\in H,$$

where ψ is a fixed additive (unitary) character ψ of F. The triple $(\mathsf{G},\mathsf{H},\xi)$ is called the *Gross–Prasad triple* associated with the relevant pair (W,V).

2.2. ***Vogan L-packets.*** We now recall the notion of Vogan L-packets for special orthogonal groups over Archimedean local fields following [35] and review the definition of the relevant Vogan L-packet following [19; 16].

For any reductive algebraic group G over a local field F, we denote by $\widehat{\mathsf{G}}$ the dual group of G and by ${}^L\mathsf{G}$ the Langlands dual group of G. It was established by Langlands in [27] that every local L-parameter $\phi : \mathcal{L}_F \to {}^L\mathsf{G}$ gives a local L-packet $\Pi_{F,\phi}(\mathsf{G})$, which consists of a finite set of irreducible admissible representations of $G = \mathsf{G}(F)$. In particular, when F is Archimedean, the representations in the packet are Casselman–Wallach [7; 41], which means that they are smooth Fréchet representations of moderate growth and the associated Harish-Chandra modules are admissible.

A *pure inner form* G_α is an inner twist of G by $\alpha \in H^1(F, \mathsf{G})$. Since pure inner forms of G share the same dual group, every local L-parameter $\phi : \mathcal{L}_F \to {}^L\mathsf{G}$ of G can be viewed as an L-parameter for any pure inner form G_α. Hence, one can define the *Vogan L-packet* as

$$\Pi^{\mathrm{Vogan}}_{F,\phi} := \bigsqcup_{\alpha \in H^1(F,\mathsf{G})} \Pi_{F,\phi}(\mathsf{G}').$$

Now we consider reductive group G with a quasisplit pure inner form. A *Whittaker datum* $\mathfrak{w}$ for G is a triple $(\mathsf{G}', \mathsf{B}', \psi')$ where G' is a quasisplit pure inner form of G, B' is a Borel subgroup of G', and ψ' is a generic character of the unipotent radical $N' = \mathsf{N}'(F)$ of $\mathsf{B}'(F)$. A representation π' of $\mathsf{G}'(F)$ is called $\mathfrak{w}$-generic if $\mathrm{Hom}_{N'}(\pi'|_{N'}, \xi') \neq 0$. An L-parameter ϕ is called ($\mathfrak{w}$-)*generic* if the Vogan L-packet contains a generic representation. As argued in [16, §18], the genericity of an L-parameter is independent of the choice of the Whittaker datum.

From [35], when F is Archimedean, fixing a generic L-parameter ϕ and a Whittaker datum $\mathfrak{w}$ of G, there is a bijection

$$\pi \in \Pi^{\mathrm{Vogan}}_{F,\phi} \mapsto \eta_\pi \in \Pi(\mathcal{S}_\phi), \tag{2.2.1}$$

where $\Pi(\mathcal{S}_\phi)$ is the set of characters of the *component group*

$$\mathcal{S}_\phi := \pi_0(\mathrm{Cent}_{\widehat{\mathsf{G}}}(\mathrm{Im}(\phi))).$$

Therefore, we can parametrize representations in Vogan packets with characters $\eta : \mathcal{S}_\phi \to \{\pm 1\}$.

Now we return to the setting in Section 2.1. For $\alpha \in H^1(F, \mathsf{H}) = H^1(F, \mathrm{SO}(W))$, we denote by W_α the inner twist of W by α and set $V_\alpha = W_\alpha \perp D \perp Z$. Then the inner twists of G and H by $\alpha \in H^1(F, \mathsf{H}) \subset H^1(F, \mathsf{G})$ are

$$\mathsf{G}_\alpha = \mathrm{SO}(V_\alpha) \times \mathrm{SO}(W_\alpha) \text{ and } \mathsf{H}_\alpha = \Delta\mathrm{SO}(W_\alpha) \ltimes \mathsf{N}.$$

Together with the character $\xi_\alpha : \mathsf{N}(F) \to \mathbb{C}$ obtained by the extension of ξ_N, we obtain the Gross–Prasad triple associated to the relevant pair (W_α, V_α). The *relevant*

Vogan packet is defined by

$$\Pi^{\mathrm{Vogan}}_{F,\phi,\mathrm{rel}} := \bigsqcup_{\alpha\in H^1(F,\mathsf{H})} \Pi_\phi(\mathsf{G}_\alpha). \tag{2.2.2}$$

It is a subset of $\Pi^{\mathrm{Vogan}}_{F,\phi}$ and thus can be parametrized with a subset of $\Pi(\mathcal{S}_\phi)$ via (2.2.1).

2.3. ***The conjecture.*** In this subsection, we review the statement of the local Gross–Prasad conjecture formulated in [19; 20].

Let (W, V) be a relevant pair over an Archimedean local field F and $(\mathsf{G}, \mathsf{H}, \xi)$ be the Gross–Prasad triple associate to it. For an irreducible Casselman–Wallach representation π of $G = \mathsf{G}(F)$, we set $H = \mathsf{H}(F)$ and define the multiplicity

$$m(\pi) := \dim \mathrm{Hom}_H(\pi, \xi). \tag{2.3.1}$$

From the multiplicity-one theorem established in [34; 23], we have

$$m(\pi) \leqslant 1.$$

The local Gross–Prasad conjecture (Conjecture 1) studies the refinement behavior of the multiplicity $m(\pi)$ in a relevant Vogan L-packet, which shows that there is exactly one representation π_ϕ in $\Pi^{\mathrm{Vogan}}_{F,\mathrm{rel},\phi}$ with multiplicity equal to 1 and the character $\eta_{\pi_\phi} : \mathcal{S}_\phi \to \{\pm 1\}$ attached to π_ϕ is equal to an explicit character η_ϕ.

For a generic character $\phi = \phi_V \times \phi_W$ of G, the character

$$\eta_\phi = \eta^W_{\phi_V} \times \eta^V_{\phi_W} : \mathcal{S}_{\phi_V} \times \mathcal{S}_{\phi_W} \to \{\pm 1\}$$

was constructed explicitly in [19, §10]. For every element $s \in \mathcal{S}_{\phi_W} \times \mathcal{S}_{\phi_V}$, set

$$\begin{aligned} \eta^W_{\phi_V}(s_V) &= \det(\mathrm{M}_V^{s_V=-1})(-1)^{\frac{\dim \mathrm{M}_W}{2}} \det(\mathrm{M}_W)(-1)^{\frac{\dim \mathrm{M}_V^{s_V=-1}}{2}} \varepsilon\big(\tfrac{1}{2}, \mathrm{M}_V^{s_V=-1} \otimes \mathrm{M}_W, \psi\big), \\ \eta^V_{\phi_W}(s_W) &= \det(\mathrm{M}_W^{s_W=-1})(-1)^{\frac{\dim \mathrm{M}_V}{2}} \det(\mathrm{M}_V)(-1)^{\frac{\dim \mathrm{M}_W^{s_W=-1}}{2}} \varepsilon\big(\tfrac{1}{2}, \mathrm{M}_W^{s_W=-1} \otimes \mathrm{M}_V, \psi\big). \end{aligned} \tag{2.3.2}$$

Here M_V and M_W are the spaces of the standard representation of ${}^L\mathrm{SO}(V)$ and ${}^L\mathsf{SO}(W)$, respectively. The notion $\det(\,\cdot\,)$ makes a finite-dimensional representation into a character and the $\det(\,\cdot\,)(-1)$ means its value at $-1 \in \mathcal{W}_{\mathbb{R}}^{ab} \cong \mathbb{R}^\times$, equivalently, $\det(\,\cdot\,)(j)$ for $j \in \mathcal{W}_{\mathbb{R}}$. The space $\mathrm{M}_V^{s_V=-1}$ denotes the $s_V = (-1)$-eigenspace of M_V and $\varepsilon(\dots)$ is the local root number defined by the Rankin–Selberg integral [21].

When $F = \mathbb{C}$, the relevant Vogan L-packet $\Pi^{\mathrm{Vogan}}_{F,\phi,\mathrm{rel}}$ contains only one element. Hence, part (1) of the conjecture implies part (2) of the conjecture. We will prove the following theorem by constructing a nonzero element in $\mathrm{Hom}_H(\pi, \xi)$ in Section 3.

Theorem 2.3.1. *When $F = \mathbb{C}$, Conjecture 1 holds.*

When $F = \mathbb{R}$, in [28], following the work of Waldspurger [36; 38] and Beuzart-Plessis [5], Luo proved part (1) of Conjecture 1 when the parameter ϕ is tempered. In [10], by simplifying Waldspurger's approach [36; 37; 38; 39; 40], the author and Luo proved part (2) of Conjecture 1 when the parameter ϕ is tempered. The main result in Section 5 is to prove Theorem 5.0.1 that implies the following theorem based on the Conjecture 1 for tempered parameters.

Theorem 2.3.2. *When $F = \mathbb{R}$, Conjecture 1 holds.*

3. Integral method and the proof for the complex case

One of the main tools for proving Conjecture 1 is the integral method. In particular, this is the only tool we would apply to prove Conjecture 1 when $F = \mathbb{C}$. When $F = \mathbb{C}$ and $\dim V = \dim W + 1$, Conjecture 1 was proved by J. Möllers in [14] using an equivalent method. In Section 3, we use some computation in [14] and present the proof using the integral method following [18].

Let $F = \mathbb{R}, \mathbb{C}$. Let G be a quasisplit group over F and H be a closed subgroup of G such that G/H is absolutely spherical. Suppose there is a Borel subgroup B of G such that

$$\mathsf{B} \cap \mathsf{H} = 1.$$

Let T be the Levi component of B. We set

$$G = \mathsf{G}(F), \quad H = \mathsf{H}(F), \quad B = \mathsf{B}(F), \quad T = \mathsf{T}(F).$$

Fix a unitary character ψ of F. For an algebraic character $\lambda : \mathsf{H} \to \mathbb{G}_a$, we set $\xi = \psi \circ \lambda_F$, which is a unitary character of H.

As a consequence of the integral method in [18], we have the following theorem.

Theorem 3.0.1. *Let G, H, B, T as above. For every character σ of T, we have*

$$\dim \mathrm{Hom}_H(\mathrm{Ind}_B^G(\sigma), \xi) \geqslant 1.$$

First, we construct a measure μ on $B \cdot H \subset G$ by setting $\mu = f(bh)dbdh$ where

$$f(bh) := \delta_B^{-1/2}(b)\sigma^{-1}(b)\xi(h), \quad b \in B,\ h \in H.$$

We can express the function f in the form of

$$f(bh) = t^{\mu_1}\bar{t}^{\mu_2}e^{is_1\mathrm{Re}(\lambda(h)) + s_2\mathrm{Im}(\lambda(h))} \quad \forall b = t \cdot n \in B = T \cdot N,\ h \in H$$

for certain $s_1, s_2 \in \mathbb{R}$ and $\mu_1, \mu_2 \in \mathrm{Hom}(\mathsf{T}, \mathbb{G}_m)$. Hence, for every differential operator D on $B \times H$, the growth of $|Df|$ can be controlled by a polynomial. Therefore, μ is a tempered measure on $B \cdot H$, which is left-$(B, \delta_B^{1/2}\sigma)$-equivariant and right-$(\mathsf{H}(F), \xi)$-equivariant. Because B is solvable, from [18, Theorem B], one can construct a left-$(B, \delta_{\mathsf{B}}^{1/2}\sigma)$-equivariant and right-$(H, \xi)$-equivariant distribution on G.

From [13] and the compactness of $B\backslash G$, there is a one-to-one correspondence between $\mathrm{Hom}(\mathrm{Ind}_B^G(\sigma), \xi)$ and the space of left-$(B, \delta_B^{1/2}\sigma)$-equivariant and right-$(H, \xi)$-equivariant distributions on G.

Now we return to the Gross–Prasad conjecture over $F = \mathbb{C}$. As argued in [19, §11], since there is exactly one representation in the relevant Vogan L-packet and this representation is a principal series, it suffices to verify that $m(\pi) \geqslant 1$ for every principal series representation $\pi = \mathrm{Ind}_B^G(\sigma)$. For this purpose, we verify $\mathsf{B} \cap \mathsf{H} = 1$ when $(\mathsf{G}, \mathsf{H}, \xi)$ is the Gross–Prasad triple associated to a relevant pair (W, V).

Set $\mathsf{P}_V = \mathsf{M}_V \cdot \mathsf{N}$ be the parabolic subgroup stabilizing the totally isotropic flag (2.1.1) and the Levi subgroup M_V can be decomposed as $\mathsf{M}_V = \prod_{i=1}^r \mathrm{GL}(\mathbb{C} \cdot z_i) \times \mathrm{SO}(V \oplus D)$. Let $\overline{\mathsf{P}}_V = \mathsf{M}_V \cdot \overline{\mathsf{N}}$ be the opposite parabolic subgroup of P_V.

Let $(\mathsf{G}', \mathsf{H}', \xi')$ be the Gross–Prasad triple associated to the relevant pair $(W, W \oplus D)$. From [14, §6.2.4], there exists a Borel subgroup B' of $\mathsf{G}' = \mathrm{SO}(W \oplus D) \times \mathrm{SO}(W)$ such that $\mathsf{B}' \cap \mathsf{H}' = 1$. We set $\mathsf{B} = \mathsf{B}' \cdot \prod_{i=1}^r \mathrm{GL}(\mathbb{C} \cdot z_i) \cdot \mathsf{B}' \cdot (\bar{\mathsf{N}} \times 1)$. Consider the parabolic subgroup $\mathsf{P} = \mathsf{P}_V \times \mathrm{SO}(W) = \mathsf{M} \cdot (\mathsf{N} \times 1)$ of G. Since $\prod_{i=1}^r \mathrm{GL}(\mathbb{C} \cdot z_i)\mathsf{B}'$ and H' are subgroups of $\mathsf{M} = \mathsf{M}_V \times \mathrm{SO}(W)$ such that

$$\prod_{i=1}^{r} \mathrm{GL}(\mathbb{C} \cdot z_i)\mathsf{B}' \cap \mathsf{H}' = 1,$$

we have

$$\mathsf{B} \cap \mathsf{H} = \overline{\mathsf{N}} \cdot \prod_{i=1}^{r} \mathrm{GL}(\mathbb{C} \cdot z_i)\mathsf{B}' \cap \mathsf{H}' \cdot \mathsf{N} = \mathrm{GL}(\mathbb{C} \cdot z_i)\mathsf{B}' \cap \mathsf{H}' = 1.$$

This completes the proof for Theorem 2.3.1.

4. Representations in generic packets

In this section, we prove that, for every parameter ϕ of a special orthogonal group over $\mathbb{R}$, there is a tempered L-parameter ϕ_0 of a smaller special orthogonal group with decomposition $\phi = \phi^{\mathrm{GL}} \oplus \phi_0 \oplus (\phi^{\mathrm{GL}})^\vee$, such that the parabolic induction

$$\pi_0 \mapsto \sigma \ltimes \pi_0$$

induces isomorphism before $\Pi_{\phi_0}^{\mathrm{Vogan}}$ and $\Pi_{\phi}^{\mathrm{Vogan}}$, where σ is the unique representation in the packet $\Pi_{\phi^{\mathrm{GL}}}$.

Let V be a nondegenerate quadratic space over $\mathbb{R}$. It is well-known that an L-parameter ϕ_V of $\mathrm{SO}(V)$ is generic if and only if the adjoint L-function $L(s, \phi_V, \mathrm{Ad})$ is holomorphic at $s = 1$ (see [19, Conjecture 2.6] and the remark after it). Based on this property, we first compute an equivalent condition for ϕ_V to be generic.

Definition 4.0.1. Given a generic L-parameter $\phi_V : \mathcal{W}_{\mathbb{R}} \to {}^L\mathrm{SO}(V)$, we denote by ϕ_V^{ss} the *semisimplification* of ϕ_V, that is, the semisimple representation on M_V

defined by the composition ϕ_V with the standard representation $\mathrm{std}_V : {}^L\mathrm{SO}(V) \to \mathrm{GL}(M_V)$.

Given an L-parameter ϕ_V, its semisimplification ϕ_V^{ss} can be decomposed as

$$\phi_V^{\mathrm{ss}} = \bigoplus |\cdot|^{s^1_{V,i}} \phi^1_{l_{V,i}} + \bigoplus |\cdot|^{s^2_{V,i}} \phi^2_{m_{V,i}}. \tag{4.0.1}$$

Here $\phi^1_{l_{V,i}}$ ($l_{V,i} \in \mathbb{Z}$) is a one-dimensional representation of $\mathcal{W}_\mathbb{R} = \mathbb{C} \cup \mathbb{C} \cdot j$ ($j^2 = -1$) defined by

$$\phi^1_{l_{V,i}}(z) = 1, \quad \phi^2_{l_{V,i}}(z \cdot j) = (-1)^{l_{V,i}}, \quad z \in \mathbb{C},$$

and $\phi^2_{m_{V,i}}$ ($m_{V,i} \in \mathbb{N}$) is the two-dimensional representation of $\mathcal{W}_\mathbb{R}$ with basis u, v satisfying

$$\begin{aligned} \phi^2_{m_{V,i}}(z)u &= u, \quad \phi^2_{m_{V,i}}(z \cdot j)u = (-1)^{m_{V,i}} v, \\ \phi^2_{m_{V,i}}(z)v &= v, \quad \phi^2_{m_{V,i}}(z \cdot j)v = u. \end{aligned}$$

The adjoint L-function $L(s, \phi_V, \mathrm{Ad}) = L(s, \phi_V^{\mathrm{ss}} \otimes \phi_V^{\mathrm{ss},\vee})$ is a product of factors

$$\begin{aligned} &L(s, \phi_V, |\cdot|^{s^1_{V,i}} \phi^1_{l_{V,i}} \otimes (|\cdot|^{s^1_{V,j}} \phi^1_{l_{V,j}})^\vee), \quad L(s, \phi_V, |\cdot|^{s^1_{V,i}} \phi^1_{l_{V,i}} \otimes (|\cdot|^{s^2_{V,j}} \phi^2_{m_{V,j}})^\vee), \\ &L(s, \phi_V, |\cdot|^{s^2_{V,i}} \phi^2_{m_{V,i}} \otimes (|\cdot|^{s^1_{V,j}} \phi^1_{l_{V,j}})^\vee), \quad L(s, \phi_V, |\cdot|^{s^2_{V,i}} \phi^2_{l_{V,i}} \otimes (|\cdot|^{s^2_{V,j}} \phi^2_{m_{V,j}})^\vee). \end{aligned}$$

From [25], we can compute the value of these L-functions and obtain that:

(1) $L(s, \phi_V, |\cdot|^{s^1_{V,i}} \phi^1_{l_{V,i}} \otimes (|\cdot|^{s^1_{V,j}} \phi^1_{l_{V,j}})^\vee)$ has a pole at $s = 1$ if and only if $\frac{1}{2}(1 + s^1_{V,i} - s^1_{V,j} + (1 - (-1)^{l_{V,i}+l_{V,j}})/2)$ is a nonpositive integer.

(2) $L(s, \phi_V, |\cdot|^{s^1_{V,i}} \phi^1_{m_{V,i}} \otimes (|\cdot|^{s^2_{V,j}} \phi^2_{m_{V,j}})^\vee)$ has a pole at $s = 1$ if and only if $1 + s^1_{V,i} - s^2_{V,j} + \frac{1}{2} m_{V,j}$ is a nonpositive integer.

(3) $L(s, \phi_V, |\cdot|^{s^2_{V,i}} \phi^2_{m_{V,i}} \otimes (|\cdot|^{s^1_{V,j}} \phi^1_{l_{V,j}})^\vee)$ has a pole at $s = 1$ if and only if $1 + s^2_{V,i} - s^1_{V,j} + \frac{1}{2} m_{V,i}$ is a nonpositive integer.

(4) $L(s, \phi_V, |\cdot|^{s^2_{V,i}} \phi^2_{m_{V,i}} \otimes (|\cdot|^{s^2_{V,j}} \phi^2_{m_{V,j}})^\vee)$ has a pole at $s = 1$ if and only if $1 + s^2_{V,i} - s^2_{V,j} + \frac{1}{2}(m^2_{V,i} + m_{V,j})$ or $1 + s^2_{V,i} - s^2_{V,j} + \frac{1}{2}(|m_{V,i} - m_{V,j}|)$ is a nonpositive integer.

Lemma 4.0.2. *A parameter ϕ_V with semisimplification ϕ_V^{ss} in* (4.0.1) *is generic if and only if none of*

$$\begin{aligned} &\tfrac{1}{2}(1 + s^1_{V,i} - s^1_{V,j} + (1 - (-1)^{l_{V,i}+l_{V,j}})/2),\ 1 + s^1_{V,i} - s^2_{V,j} + \tfrac{1}{2} m_{V,j}, \\ &1 + s^2_{V,i} - s^1_{V,j} + \tfrac{1}{2} l_{V,i},\ 1 + s^2_{V,i} - s^2_{V,j} + \tfrac{1}{2}(|m_{V,i} - m_{V,j}|) \end{aligned}$$

is a nonpositive integer.

Irreducibility criteria. B. Speh and D. Vogan gave a sufficient condition for the irreducibility of limits of generalized principal series representations in [32, Theorem 6.19]. We apply this result to prove the irreducibility of standard models for representations in generic packets.

Definition 4.0.3. Given $\sigma_1 \in \Pi(\mathrm{GL}_{n_1}), \dots, \sigma_r \in \Pi(\mathrm{GL}_{n_r})$ and $\pi_{V_0} \in \Pi(\mathrm{SO}(p,q))$. We denote by

$$\sigma_1 \times \cdots \times \sigma_r \ltimes \pi_{V_0}$$

the normalized parabolic induction

$$\mathrm{I}_{P_{n_1,\cdot,n_r,p+q}}^{\mathrm{SO}(p+n,q+n)}(\sigma_1 \otimes \cdots \otimes \sigma_r \otimes \pi_{V_0}) \in \Pi(\mathrm{SO}(p+n,q+n)), \quad n = n_1 + \cdots + n_r.$$

Lemma 4.0.4. *Fix a generic parameter* $\phi_V = \phi_V^{\mathrm{GL}} \oplus \phi_{V_0} \oplus (\phi_V^{\mathrm{GL}})^\vee$ *of* $\mathrm{SO}(p,q)$ $(p > q)$. *For* $\sigma \in \Pi_{\phi_V^{\mathrm{GL}}}$ *and* $\pi_{V_0} \in \Pi_{\phi_{V_0}}^{\mathrm{Vogan}}$, *the representation* $\sigma \ltimes \pi_{V_0}$ *is irreducible.*

Proof. From [25, Theorem 14.2], we may write the tempered representation π_{V_0} as a parabolic induction from a limit of discrete series representations. Then we can express $\sigma \ltimes \pi_{V_0}$ as

$$\sigma_1 \times \cdots \times \sigma_l \ltimes \pi_{V_0'} \quad \sigma_i \in \Pi(\mathrm{GL}_{n_{V,i}}) \tag{4.0.2}$$

where $\pi_{V_0}' \in \Pi(\mathrm{SO}(V_0'))$ is a limit of discrete series representation and

$$\sigma_i = |\cdot|^{s_{V,i}^1} \mathrm{sgn}^i \text{ or } \sigma_i = |\det|^{s_{V,i}^2} D_{m_{V,i}}.$$

Following [32, Theorem 6.19], it suffices to check the following conditions:

(4.0.3) For every root α such that

$$n_\alpha = \langle \alpha, \nu \rangle / \langle \alpha, \alpha \rangle \in \mathbb{Z},$$

(1) if α is a complex root ($\alpha \neq -\theta\alpha$), then $\langle \alpha, \nu \rangle \langle \theta\alpha, \nu \rangle \geqslant 0$;

(2) if α is a real root ($\alpha = -\theta\alpha$), then

$$(-1)^{n_\alpha + 1} = \epsilon_\alpha \cdot \lambda(m_\alpha)$$

Here λ is the central character of σ, m_α is the image of $\rho_\alpha(-I_2)$ in G for the embedding $\rho_\alpha : \mathrm{SL}_2(\mathbb{R}) \to G(\mathbb{R})$ determined by α and $\epsilon_\alpha = -1$.

Then we check them using Lemma 4.0.2.

(1) For every complex root α such that $n_\alpha \in \mathbb{Z}$,

(a) if α is a root of $\mathrm{SO}(p-q)$, then $\langle \alpha, \nu \rangle \langle \theta\alpha, \nu \rangle = 0$;

(b) otherwise, $\theta\alpha = \alpha$, and then $\langle \alpha, \nu \rangle \langle \theta\alpha, \nu \rangle = \langle \alpha, \nu \rangle^2 \geqslant 0$.

(2) For every real root $\beta_{ab} = e_a - e_b$ such that $n_{\beta_{ab}} \in \mathbb{Z}$.

(a) If E_{aa} is in the GL_1-block $\mathrm{GL}_{n_{V,i}}$ and E_{bb} is the in a GL_1-block $\mathrm{GL}_{n_{V,j}}$ (in the inducing datum in (4.0.2)), then $n_{\beta_{ab}} = \frac{1}{2}(s^1_{V,i} - s^1_{V,j})$ is an integer, and both

$$\tfrac{1}{2}(1+s^1_{V,i}-s^1_{V,j}+(1-(-1)^{l_{V,i}+l_{V,j}})/2) \text{ and } \tfrac{1}{2}(1+s^1_{V,j}-s^1_{V,i}+(1-(-1)^{l_{V,i}+l_{V,j}})/2)$$

are not nonpositive integers. If $l_{V,i} + l_{V,j}$ is odd, the sum is equal to 2, then $s^1_{V,i} = s^1_{V,j}$ or $s^1_{V,i} - s^1_{V,j}$ is odd. If $l_{V,i} + l_{V,j}$ is even, the sum is equal to $3/2$, then $s^1_{V,i} - s^1_{V,j}$ is even.

(b) If E_{aa} is in the GL_1-block $\mathrm{GL}_{n_{V,i}}$ and E_{bb} is the in a GL_2-block $\mathrm{GL}_{n_{V,j}}$, Lemma 4.0.2 implies

$$s^2_{V,j} - \tfrac{1}{2}l_{V,j} \leqslant s^1_{V,i} \leqslant s^2_{V,j} + \tfrac{1}{2}l_{V,j}.$$

(c) If E_{aa} is in the GL_2-block $\mathrm{GL}_{n_{V,i}}$ and E_{bb} is the in a GL_2-block $\mathrm{GL}_{n_{V,j}}$, we may assume $l_{V,j} \geqslant l_{V,i}$, Lemma 4.0.2 implies

$$s^2_{V,j} - \tfrac{1}{2}l_{V,j} \leqslant s^2_{V,i} - \tfrac{1}{2}l_{V,i} \leqslant s^2_{V,i} + \tfrac{1}{2}l_{V,i} \leqslant s^2_{V,j} + \tfrac{1}{2}l_{V,j}.$$

Therefore, we have checked cases (b) and (c) following an understanding of the parity condition in [30, Theorem 2]. For case (a), parity holds unless $l_{V,i} + l_{V,j}$ is odd and $s^1_{V,i} = s^1_{V,j}$. In this situation

$$|\cdot|^{s^1_{V,i}}\mathrm{sgn}^{l_{V,i}} \times |\cdot|^{s^1_{V,j}}\mathrm{sgn}^{l_{V,j}} = |\cdot|^{s^1_{V,i}}\mathrm{sgn}^{l_{V,i}}(1 \times \mathrm{sgn})$$

And $1 \times \mathrm{sgn}$ is the limit of a discrete series representation with parameter ϕ^2_0, which can be treated as in cases (b) and (c). ☐

Representations in generic packets. The classification of representations of $\mathcal{W}_{\mathbb{R}}$ [25] shows the following factorization into irreducible representations:

$$\phi^{\mathrm{ss}}_V = \phi^{\mathrm{GL}}_V \oplus \phi_{V_0} \oplus (\phi^{\mathrm{GL}}_V)^\vee, \tag{4.0.4}$$

where ϕ_{V_0} is tempered and

$$\phi^{\mathrm{GL}}_V = \bigoplus_{i=1}^{l_V} |\cdot|^{s_i}\phi^{\mathrm{GL}}_{V,i} \quad \text{where } \mathrm{Re}(s_i) > 0 \text{ for } 1 \leqslant i \leqslant l_V$$

for discrete series parameter $\phi_{V,i}$ (i.e., the image of $\phi_{V,i}$ is bounded and does not lie in any proper Levi).

It is straightforward that ϕ^{GL}_V is unpaired. Let $n_{V,i} = \dim \phi^{\mathrm{GL}}_{V,i}$, $n_V = \dim \phi^{\mathrm{GL}}_V$ and $\sigma_{V,i}$ be the unique representation of GL_n in the L-packet $\Pi_{\phi^{\mathrm{GL}}_{V,i}}(\mathrm{GL}_{n_{V,i}})$, then

$$\Pi_{\phi^{\mathrm{GL}}_V}(\mathrm{GL}_{n_V}) = \{\sigma_V\} \quad \text{where } \sigma_V = |\det|^{s_1}\sigma_{V,1} \times \cdots \times |\det|^{s_{l_V}}\sigma_{V,l_V} \tag{4.0.5}$$

By Lemma 4.0.4, there is an injective map

$$\Pi^{\mathrm{Vogan}}_{\phi_{V_0}} \to \Pi^{\mathrm{Vogan}}_{\phi_V}, \quad \pi_{V_0} \mapsto \sigma_V \ltimes \pi_{V_0}. \tag{4.0.6}$$

Since ϕ_V^{GL} is unpaired, $|\mathcal{S}_{\phi_{V_0}}| = |\mathcal{S}_{\phi_V}|$ and thus $|\Pi^{\mathrm{Vogan}}_{\phi_{V_0}}| = |\Pi^{\mathrm{Vogan}}_{\phi_V}|$. This implies that the above map is an isomorphism and we have the following result.

Proposition 4.0.5. *For a generic L-parameter $\phi_V = \phi_V^{\mathrm{GL}} \oplus \phi_{V_0} \oplus (\phi_V^{\mathrm{GL}})^\vee$, every representation π_V in $\Pi^{\mathrm{Vogan}}_{\phi_V}$ can be expressed as $\pi_V = \sigma_V \ltimes \pi_{V_0}$ where $\pi_{V_0} \in \Pi^{\mathrm{Vogan}}_{\phi_{V_0}}$ and σ_V given in* (4.0.5).

This result shows that representations in the generic packets are in the form

$$\pi_V = \sigma_V \ltimes \pi_{V_0}, \quad \sigma_V = |\det|^{s_{V,1}}\sigma_{V,1} \times \cdots \times |\det|^{s_{V,r}}\sigma_{V,r}, , \tag{4.0.7}$$

where $\mathrm{Re}(s_{V,1}) \geqslant \mathrm{Re}(s_{V,2}) \geqslant \cdots \geqslant \mathrm{Re}(s_{V,r}) > 0$, and tempered $\pi_0 \in \mathrm{Irr}(\mathrm{SO}(V_0))$. And $\sigma_{V,i} = \mathrm{sgn}^{l_{V,i}}$ for $l_{V,i} = 0, 1$ or $\sigma_{V_i} = D_{m_{V,i}}$ for $m_i \in \mathbb{N}_+$.

For π_V in the form of (4.0.7), we define the following notions.

Definition 4.0.6. We parametrize the infinitesimal character of π_V with the *Harish-Chandra parameter* for π_V in (4.0.7) is defined as

$$\nu = (\nu_1, \dots, \nu_r, \nu_{\pi_{V_0}})$$

where $\nu_{\pi_{V_0}}$ is the Harish-Chandra parameter of the tempered representation π_{V_0}, $\nu_i = s_i$ when $\rho_{V,i} = \mathrm{sgn}^{l_{V,i}}$, and $\nu_i = \big(s_{V,i} + \frac{1}{2}m_{V,i}, s_{V,i} - \frac{1}{2}m_{V,i}\big)$ when $\rho_{V,i} = D_{m_{V,i}}$.

Definition 4.0.7. We define the *leading index* of π_V as the largest number among $\mathrm{Re}(s_{V,i})$. We denote it by $\mathrm{LI}(\pi_V)$.

5. Proof for the real case

We now complete the proof of the local Gross–Prasad conjecture (Conjecture 1) over the real field based on the tempered cases. More specifically, following the approach in [29], we prove a multiplicity formula for the reduction to the tempered cases and conclude the conjecture with the tempered cases proved in [10].

The proof uses the idea of Mackey's theory. Let G be a reductive group over $\mathbb{R}$, H is a closed subgroup of G and P is a parabolic subgroup of G with Levi decomposition $\mathsf{P} = \mathsf{MN}$. We denote by $G = \mathsf{G}(\mathbb{R})$, $H = \mathsf{H}(\mathbb{R})$ and $P = \mathsf{P}(\mathbb{R})$. For a representation σ of $M = \mathsf{M}(\mathbb{R})$, we study the space $\mathrm{Hom}_H(\mathrm{Ind}_P^G(\sigma), 1_H)$ by analyzing the double coset $P\backslash G/H$. Since $P\backslash G$ is compact, the smooth induction $\mathrm{Ind}_P^G(\sigma)$ is equal to the Schwartz induction in the sense of [13]. In order to use the analytic tools established in [13] and [11], we work within the category of almost linear Nash groups [33, Definition 1.1] and consider the category of Nash manifolds [33, Definition 2.1], with the possible action of certain almost linear Nash groups.

In particular, for a linear algebraic group G over $\mathbb{R}$, $\mathsf{G}(\mathbb{R})$ can be treated as an almost linear Nash group.

Let G be an almost linear Nash group. We denote by $\mathcal{SF}(G)$ the categories of smooth Fréchet G-representations of moderate growth. We denote by $\mathcal{CW}(G)$ the subcategory of $\mathcal{SF}(G)$ consisting of representations with admissible $(\mathfrak{g}_{\mathbb{C}}, K)$-modules, that is, the category of Casselman–Wallach representations of G. We use $\mathrm{Irr}(G)$ to denote the set of irreducible Casselman–Wallach representations of G.

Our main result in this section is the following theorem.

Theorem 5.0.1 (multiplicity formula). *Let V, W be quadratic spaces with decompositions $V = V_0 \perp (X_V + X_V^\vee)$, $W = W_0 \perp (X_W + X_W^\vee)$. Let $\pi_{V_0} \in \mathrm{Irr}(\mathrm{SO}(V_0))$, $\pi_{W_0} \in \mathrm{Irr}(\mathrm{SO}(W_0))$ be tempered representations and $\sigma_V \in \mathcal{CW}(\mathrm{SO}(V))$, $\sigma_W \in \mathcal{CW}(\mathrm{SO}(W))$ such that*

$$\begin{aligned} \sigma_V &= |\det|^{s_{V,1}}\sigma_{V,1} \times \cdots \times |\det|^{s_{V,r_V}}\sigma_{V,r_V}, \\ \sigma_W &= |\det|^{s_{W,1}}\sigma_{W,1} \times \cdots \times |\det|^{s_{W,r_W}}\sigma_{W,r_W}, \end{aligned} \tag{5.0.1}$$

for $\mathrm{Re}(s_{V,i}), \mathrm{Re}(s_{W,i}) > 0$ and tempered representations $\sigma_{V,i} \in \mathrm{Irr}(\mathrm{GL}_{n_{V,i}}(F))$ $(i = 1, \ldots, r_V)$, $\sigma_{W,i} \in \mathrm{Irr}(\mathrm{GL}_{n_{W,i}}(F))$ $(j = 1, \ldots, r_W)$; here $n_{V,i}, n_{W,i}$ are integers such that $\sum_{i=1}^{r_V} n_{V,i} = \dim X_V$ and $\sum_{i=1}^{r_W} n_{W,i} = \dim X_W$. Then we have

$$m((\sigma_V \ltimes \pi_{V_0}) \boxtimes (\sigma_W \ltimes \pi_{W_0})) = m(\pi_{V_0} \boxtimes \pi_{W_0}).$$

Note that in the theorem, the representations $\sigma_V \ltimes \pi_{V_0}$ and $\sigma_W \ltimes \pi_{W_0}$ can be reducible. The reducible case of the multiplicity formula is actually necessary when it is applied in [12]. In this article, to complete the proof for the real case of the Gross–Prasad conjecture, we only use the formula when both $\sigma_V \ltimes \pi_{V_0}$ and $\sigma_W \ltimes \pi_{W_0}$ are irreducible.

Proof of Theorem 2.3.2 given Theorem 5.0.1. Given generic parameters ϕ_V, ϕ_W, from Proposition 4.0.5, we can express the parameters as

$$\phi_V = \phi_V^{\mathrm{GL}} + \phi_{V_0} + (\phi_V^{\mathrm{GL}})^\vee, \quad \phi_W = \phi_W^{\mathrm{GL}} + \phi_{W_0} + (\phi_W^{\mathrm{GL}})^\vee \tag{5.0.2}$$

such that ϕ_V^{GL} has no self-dual subrepresentation.

Let σ_V be the unique representation in $\Pi^{\mathrm{Vogan}}_{\phi_V^{\mathrm{GL}}}$ and σ_W be the unique representation in $\Pi^{\mathrm{Vogan}}_{\phi_W^{\mathrm{GL}}}$. For every $\pi_V \boxtimes \pi_W \in \Pi^{\mathrm{Vogan}}_{\phi_V \times \phi_W}$, there exists $\pi_{V_0} \boxtimes \pi_{W_0} \in \Pi^{\mathrm{Vogan}}_{\phi_V \times \phi_W}$ such that

$$\pi_V = \sigma_V \ltimes \pi_{V_0}, \quad \pi_W = \sigma_W \ltimes \pi_{W_0}.$$

Therefore, the maps

$$\begin{aligned} &\Pi^{\mathrm{Vogan}}_{\phi_{V_0}} \to \Pi^{\mathrm{Vogan}}_{\phi_V}, \quad \pi_{V_0} \mapsto \sigma_V \ltimes \pi_{V_0}, \quad \text{and} \\ &\Pi^{\mathrm{Vogan}}_{\phi_{W_0}} \to \Pi^{\mathrm{Vogan}}_{\phi_W}, \quad \pi_{W_0} \mapsto \sigma_W \ltimes \pi_{W_0} \end{aligned}$$

are isomorphisms. Hence, we can identify the component group $\mathcal{S}_{\phi_{V_0}\times\phi_{W_0}}$ with $\mathcal{S}_{\phi_V\times\phi_W}$. Under this identification, it can be easily verified that for ϕ_V, ϕ_W, ϕ_{V_0}, ϕ_{W_0}, we have

$$\eta_{\phi_{V_0}\times\phi_{W_0}} = \eta_{\phi_V\times\phi_W}.$$

Theorem 5.0.1 reduces Conjecture 1 for ϕ_V, ϕ_W to that for ϕ_{V_0}, ϕ_{W_0}, which is a tempered case proved in [28; 10]. □

Following [29], there are three steps in our proof for Theorem 5.0.1: reduction to basic cases, the first inequalities, and the second inequalities.

A relevant pair (W, V) is called *basic* if $\dim V = \dim W + 1$. For a general relevant pair (W, V) with decomposition $V = W \perp Z \perp D$, we let D^+ be the anisotropic line with the opposite signature to D. We set $Z^+ = Z \perp (D + D^+)$ and set $(V, W^+) = (V, Z^+ \oplus W)$ and we call (V, W^+) the *basic relevant pair associate to* (W, V).

Definition 5.0.2. Let $s_1, s_2, \dots, s_{r+1}$ be complex numbers. We say the $(r+1)$-tuple $\underline{s} = (s_1, \dots, s_{r+1})$ are *in general position*, if $\underline{s} \in \mathbb{C}^{t+1}$ does not lie in the set of zeros of countably many polynomial functions on $\mathbb{C}^{t+1}$.

For the $(r+1)$-tuple $\underline{s} = (s_1, \dots, s_{r+1})$, we denote by $\sigma_{\underline{s}}$ the spherical principal series representation $|\cdot|^{s_1} \times \cdots \times |\cdot|^{s_{r+1}}$.

Lemma 5.0.3 (reduction to basic cases). *For every* $\pi_V \in \mathrm{Irr}(\mathrm{SO}(V))$ *and* $\pi_W \in \mathrm{Irr}(\mathrm{SO}(W))$, *we have*

$$m(\pi_V \boxtimes \pi_W) = m((\sigma_{\underline{s}} \ltimes \pi_W) \boxtimes \pi_V)$$

for $\underline{s} = (s_1, \dots, s_{r+1}) \in \mathbb{C}^{r+1}$ *in general position.*

With this, we find such a spherical principal series $\sigma_{\underline{s}}$ and reduce Theorem 5.0.1 to the case for a relevant pair $(V, W \oplus Z^+)$ and representations $\sigma_{\underline{s}} \ltimes \pi_W$, π_V that can be expressed in the parabolic induction form as in (4.0.7), which is a basic case.

Proposition 5.0.4 (basic case of the multiplicity formula). *Given a basic relevant pair* (W, V), *let* $\pi_V \in \mathcal{CW}(\mathrm{SO}(V))$ *and* $\pi_W \in \mathcal{CW}(\mathrm{SO}(W))$ *as in Theorem 5.0.1, we have*

$$m(\pi_V \boxtimes \pi_W) = m(\pi_{V_0} \boxtimes \pi_{W_0}).$$

The inequalities $m(\pi_V \boxtimes \pi_W) \geqslant m(\pi_{V_0} \boxtimes \pi_{W_0})$ and $m(\pi_V \boxtimes \pi_W) \leqslant m(\pi_{V_0} \boxtimes \pi_{W_0})$ are called “the first inequality” and “the second inequality” in [29]. Using a similar approach as [29], we prove the first inequality using mathematical induction with the following lemma as the building block (Section 5.3).

Lemma 5.0.5. *Let* π_V *be a representation in a generic packet and* $\pi_W \in \mathrm{Irr}(\mathrm{SO}(W))$.

(1) *When* $\dim V = \dim W + 1$ *and* $\mathrm{Re}(s) \geqslant \mathrm{LI}(\pi_V)$, *we have*

$$m(\pi_V \boxtimes \pi_W) \geqslant m((|\cdot|^s \mathrm{sgn}^m \ltimes \pi_W) \boxtimes \pi_V).$$

(2) *When* $\dim V = \dim W + 3$ *and* $\mathrm{Re}(s) \geqslant \mathrm{LI}(\pi_V)$, *we have*

$$m(\pi_V \boxtimes (|\cdot|^{s+\frac{m}{2}} \mathrm{sgn}^{m+1} \ltimes \pi_W)) \geqslant m((|\det|^s D_m \ltimes \pi_W) \boxtimes \pi_V),$$

where D_m *is the Langlands quotient of the induction* $|\cdot|^{-\frac{m}{2}} \times |\cdot|^{\frac{m}{2}} \mathrm{sgn}^{m+1}$.

The second inequality holds in a more general setup.

Lemma 5.0.6. *For* $\pi_V \in \mathcal{CW}(\mathrm{SO}(V))$, $\pi_W \in \mathcal{CW}(\mathrm{SO}(W))$ *and* σ_{X^+} *is a generic representation in* $\mathcal{CW}(\mathrm{GL}(X^+))$, *we have*

$$m(\pi_V \boxtimes \pi_W) \leqslant m((\sigma_{X^+} \ltimes \pi_W) \boxtimes \pi_V)$$

We prove one inequality of Lemma 5.0.3 and Lemma 5.0.5 in Section 5.3 and prove the other inequality of Lemma 5.0.3 and Lemma 5.0.6 in Section 5.4. It is worth mentioning that Lemma 5.0.3 can also be proved with Schwartz homology as in [43].

5.1. ***Some functors and vanishing theorems.*** In this section, we review some analytic tools established in [13] and [11] to study certain Fréchet spaces of moderate growth.

Schwartz induction. Let G be an almost linear Nash group.

Proposition 5.1.1. *For* $\pi \in \mathcal{CW}(G)$, *the projective tensor product* $\cdot \widehat{\otimes} \pi$ *is an exact functor in* $\mathcal{SF}(G)$.

Proof. From [2], the underlying Fréchet space of π is nuclear and the proposition follows from [8, Lemma A.3]. □

Let H be a Nash subgroup of G and $\pi_H \in \mathcal{SF}(H)$. We denote by $H\backslash(G \times \pi_H)$ the vector bundle over $H\backslash G$ obtained by $G \times \pi_H$ quotient by left H-action

$$h.(g, v) = (h \cdot g, \pi_H(h).v) \quad \text{for } h \in H, \quad g \in G \text{ and } v \in \pi_H. \tag{5.1.1}$$

This vector bundle is tempered. We define the *Schwartz induction* as the functor

$$\mathrm{Ind}_P^{\mathcal{S},G} : \mathcal{SF}(H) \to \mathcal{SF}(G), \quad \pi_H \mapsto \Gamma^{\mathcal{S}}(H\backslash G, \pi_H),$$

where $\Gamma^{\mathcal{S}}(H\backslash G, \pi_H)$ stands for the space of Schwartz sections over the tempered vector bundle $H\backslash(G \times \pi_H)$. In particular, when G is reductive and $P \subset G$ is a parabolic subgroup of it, $P\backslash G$ is compact, so the Schwartz induction $\mathrm{Ind}_P^{\mathcal{S},G}$ coincides with the smooth induction, and we denote by I_P^G the normalized induction $\mathrm{Ind}_P^{\mathcal{S},G}(\delta_P^{1/2} \cdot)$, where δ_P is the modular characters of P. We will use the following properties of Schwartz inductions.

Proposition 5.1.2. (1) [11, Propositon 7.1] $\mathrm{Ind}_H^{\mathcal{S},G}$ *is an exact functor* $\mathcal{SF}(H) \to \mathcal{SF}(G)$.

(2) [11, Proposition 7.2] *For a closed subgroup* H' *of* H, *we have*

$$\mathrm{Ind}_H^{\mathcal{S},G} \circ \mathrm{Ind}_{H'}^{\mathcal{S},H} = \mathrm{Ind}_{H'}^{\mathcal{S},G}.$$

(3) [11; 2, Proposition 7.4] *For* $\pi_G \in \mathcal{CW}(G)$ *and* $\pi_H \in \mathcal{SF}(H)$, *then*

$$\mathrm{Ind}_H^{\mathcal{S},G}(\pi_H \widehat{\otimes} \pi_G|_H) = \mathrm{Ind}_H^{\mathcal{S},G}(\pi_H) \widehat{\otimes} \pi_G.$$

The* Hom*-functor. For any category $\mathcal{C}$ and object M, it is well-known that the functor $\mathrm{Hom}(-, M)$ is left exact and invariant under projective limit. We first apply this result to the category $\mathcal{SF}(G)$ and obtain the following result.

Lemma 5.1.3. (1) *For an exact sequence* $0 \to \pi_1 \to \pi_2 \to \pi_3 \to 0$ *in* $\mathcal{SF}(G)$, *suppose* $\mathrm{Hom}_G(\pi_1, 1_G) = \mathrm{Hom}_G(\pi_3, 1_G) = 0$. *Then*

$$\mathrm{Hom}_G(\pi_2, 1_G) = 0.$$

(2) *For a directed set* I *and projective system* $(\pi_\alpha, f_{\alpha\beta})_{\alpha,\beta\in I}$ *in* $\mathcal{SF}(G)$, *and for* $I' \subset I$, *suppose* $\mathrm{Hom}_G(\pi_\alpha, 1_G) = 0$ *for all* $\alpha \in I'$. *Then*

$$\mathrm{Hom}(\varprojlim_{i\in I} \pi_\alpha, 1_G) = 0.$$

Definition 5.1.4. (1) For a countable directed set I and a Fréchet space V, a set $\{V_k\}_{k\in I}$ of subspaces of V is called a *complete decreasing filtration* of π if

(a) $V_j \subset V_i$ for $i < j$, and, denoting by f_{ji} the injection maps,

(b) $\{V_i, f_{ji}\}_{i<j\in I}$ is a complete projective system, that is,

$$\varprojlim_{i\in I} V/V_i = V.$$

(2) The *composition factors* of a complete decreasing filtration are

$$V_\alpha / V_{\alpha+}, \quad \alpha \in I,$$

where $\alpha+$ is the successor of α in I.

Corollary 5.1.5. *For an almost linear Nash group* G, $\pi \in \mathcal{SF}(G)$ *and a complete decreasing filtration* $\{\pi_k\}_{k\in I}$ *of* π, *suppose* $\mathrm{Hom}_G(V_\alpha / V_{\alpha+}, 1_G) = 0$ *for all* $\alpha \in I$. *Then we have*

$$\mathrm{Hom}_G(\pi, 1_G) = 0.$$

Proof. This can be obtained from Lemma 5.1.3 with the arguments in [42]. □

Propositions 8.2 and 8.3 of [11] provide a complete decreasing filtration that is helpful for distributional analysis.

Theorem 5.1.6. *Let $\mathcal{X}$ be a Nash manifold, $\mathcal{Z}$ be a closed Nash manifold of $\mathcal{X}$ and $\mathcal{U} = \mathcal{X} - \mathcal{Z}$. There is a decreasing complete filtration on $\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}, \mathcal{E})$, denoted by $\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}, \mathcal{E})_k$, whose composition factors are isomorphic to*

$$\Gamma^{\mathcal{S}}(\mathcal{Z}, \mathrm{Sym}^k \mathcal{N}^{\vee}_{\mathcal{Z}/\mathcal{X}} \otimes \mathcal{E}|_{\mathcal{Z}}), \quad k = 0, 1, \dots, \tag{5.1.2}$$

where $\mathcal{N}^{\vee}_{\mathcal{Z}/\mathcal{X}}$ is the conormal bundle over $\mathcal{Z}$ (see [11, Section 6.1].)

Vanishing by infinitesimal characters.

Definition 5.1.7. For an infinitesimal character $\chi : \mathcal{Z}(\mathcal{U}(\mathfrak{g}_{\mathbb{C}})) \to \mathbb{C}$, we denote by $\chi^{\vee}$ the infinitesimal character generated by the relation

$$\chi^{\vee}(X) = \chi(-X), \quad X \in \mathfrak{g}_{\mathbb{C}}.$$

Theorem 5.1.8. *For representations π_1, π_2 of G with infinitesimal characters $\chi_{\pi_1}, \chi_{\pi_2}$, satisfying $\chi_{\pi_1} \neq \chi^{\vee}_{\pi_2}$, we have*

$$\mathrm{Hom}_G(\pi_1 \widehat{\otimes} \pi_2, 1_G) = 0.$$

Proof. The existence of elements in $\mathrm{Hom}_G(\pi_1 \widehat{\otimes} \pi_2, 1_G)$ implies the existence of a homomorphism on $(\mathfrak{g}_{\mathbb{C}}, K)$-modules. This contradicts the relation of infinitesimal characters. ☐

We apply the above theorem in the following setup:

Corollary 5.1.9. *Suppose $\pi_{V_0} \in \mathcal{SF}(\mathrm{SO}(V_0))$ and $\pi_V \in \mathrm{Irr}(\mathrm{SO}(V))$.*

$$(\sigma_{\underline{s}} \ltimes \pi_{V_0}) \widehat{\otimes} \pi_V$$

for $\sigma_{\underline{s}} = |\cdot|^{s_1} \times \cdots \times |\cdot|^{s_r}$ and $\underline{s} = (s_1, \dots, s_r)$ in general positions.

Vanishing by leading index.

Definition 5.1.10. By the Langlands classification, for every $\pi_V \in \mathrm{Irr}(\mathrm{SO}(V))$, we can express π_V as the Langlands quotient of a certain induction

$$|\det|^{s_1} \rho_1 \times \cdots \times |\det|^{s_r} \rho_r \ltimes \pi_{V_0} \tag{5.1.3}$$

for $\mathrm{Re}(s_1) \geqslant \cdots \geqslant \mathrm{Re}(a_r) > 0$ and tempered representations $\rho_1, \dots, \rho_r, \pi_{V_0}$. We define the *leading index for Langlands quotient* as $\mathrm{LI}(\pi_V) = \mathrm{Re}(s_1)$. This definition is compatible with Definition 4.0.7 when the standard module (5.1.3) is irreducible. In particular, the definitions are compatible when π_V is in a generic packet.

Theorem 5.1.11 [9, Theorem A.1.1]. *If* $\mathrm{Re}(s) > \mathrm{LI}(\pi_V)$, *then*

$$\mathrm{Hom}_{\Delta \mathrm{SO}(V)}((|\det|^s \rho \ltimes \pi_{V_0}) \boxtimes \pi_V, 1_{\Delta \mathrm{SO}(V)}) = 0$$

for $\pi_{V_0} \in \mathcal{SF}(\mathrm{SO}(V_0))$ and $\pi_V \in \mathrm{Irr}(\mathrm{SO}(V))$.

5.2. ***The restriction of principal series to mirabolic subgroups.*** We now turn to the graded structure of the restriction of certain principal series of GL_n to the mirabolic subgroup $R_{n-1,1}$ as in [42, §5], that is, the subgroup of GL_n leaving V_n/V_{n-1} invariant, where V_n is the space of the standard representation of GL_n and V_{n-1} is an $(n-1)$-dimensional subspace of V_n. These results will be used in the distributional analysis of the open orbit in Section 5.3.

Graded structure of $|\cdot|^{-\frac{m}{2}} \times |\cdot|^{\frac{m}{2}}\mathrm{sgn}^{m+1}$**.** By definition, the discrete series D_m of $\mathrm{GL}_2(\mathbb{R})$ is the unique quotient of the induction $\pi_I = |\cdot|^{-\frac{m}{2}} \times |\cdot|^{\frac{m}{2}}\mathrm{sgn}^{m+1}$. We denote π_F the unique subrepresentation of this induction π_I, then π_m is an m-dimensional irreducible representation of $\mathrm{GL}_2(\mathbb{R})$.

- Let B_2 be the (upper-triangular) Borel subgroup of GL_2 with Levi decomposition $\mathsf{B}_2 = \mathsf{T}_2\mathsf{N}_2$. Let $K = \mathrm{SO}_2(\mathbb{R})$, $B_2 = \mathsf{B}_2(\mathbb{R})$, $T_2 = \mathsf{T}_2(\mathbb{R})$, $N_2 = \mathsf{N}_2(\mathbb{R})$ and $R_{1,1} = \mathsf{R}_{1,1}(\mathbb{R})$.
- We write
$$n_x = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \quad k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad w_2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix},$$
then $N_2 = \{n_x : x \in \mathbb{R}\}$ and $K = \{k_\theta : \theta \in [0, 2\pi)\}$.
- We write $\mathcal{X}_2 = B_2\backslash\mathrm{GL}_2(\mathbb{R})$, $\mathcal{U}_2 = B_2\backslash B_2 w_2 B_2 \subset \mathcal{X}_2$ and $\mathcal{Z}_2 = B_2\backslash B_2$.
- By definition,
$$\pi_I = \mathrm{Ind}_{B_2}^{\mathcal{S},\mathrm{GL}_2(\mathbb{R})}(|\cdot|^{\frac{m+1}{2}} \otimes |\cdot|^{\frac{m-1}{2}}\mathrm{sgn}^{m+1}).$$
We write $\chi_1 = |\cdot|^{-m+1}\mathrm{sgn}^{m+1}$ and $\chi_2 = |\cdot|^{\frac{m-1}{2}}\mathrm{sgn}^{m+1}$. Then
$$\pi_I = \mathrm{Ind}_{B_2}^{\mathcal{S},\mathrm{GL}_2(\mathbb{R})}(\chi_1\chi_2 \otimes \chi_2).$$

Lemma 5.2.1. (1) *The representation π_F is isomorphic to the n-dimensional* $\mathrm{GL}_2(\mathbb{R})$*-representation*
$$\chi_1\chi_2(\det(\,\cdot\,))\mathrm{Sym}^{n-1}(\mathbb{C}^2),$$
where $\mathbb{C}^2$ is the standard representation of $\mathrm{GL}_2(\mathbb{R})$.

(2) *The restriction $\pi_F|_{R_{1,1}}$ has irreducible components*
$$|\det(\,\cdot\,)|^k\mathrm{sgn}^k(\det(\,\cdot\,)), \textit{ for } k = 0, 1, \ldots, m-1.$$

Proof. Part (1) follows directly from [17, §2.3]. Part (2) follows from direct computation based on (1). □

Using the left quotient in the sense of (5.1.1), we define
$$\mathcal{E}_2 := B_2\backslash(\mathrm{GL}_2(\mathbb{R}) \times \chi_1\chi_2 \otimes \chi_2).$$

Extension by zero gives a natural embedding of $R_{1,1}$-representations

$$i_{UX} : \Gamma^{\mathcal{S}}(\mathcal{U}_2, \mathcal{E}_2) \to \Gamma^{\mathcal{S}}(\mathcal{X}_2, \mathcal{E}_2). \tag{5.2.1}$$

Lemma 5.2.2. *There is a complete decreasing filtration* $\{\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}_2, \mathcal{E}_2)_i\}_{i\in\mathbb{N}}$ *of submodules of* $\Gamma^{\mathcal{S}}(\mathcal{X}_2, \mathcal{E}_2)/\,\Gamma^{\mathcal{S}}(\mathcal{U}_2, \mathcal{E}_2)$ *such that the composition factors are* $R_{1,1}$-*isomorphic to*

$$\chi_1\chi_2(\det(\,\cdot\,))\mathrm{sgn}^k(\det(\,\cdot\,))|\det(\,\cdot\,)|^k\big|_{R_{1,1}}, \quad k \in \mathbb{N}.$$

Proof. This lemma follows from [11, Propositions 8.2, 8.3]. ☐

We identify $\Gamma^{\mathcal{S}}(\mathcal{U}_2, \mathcal{E}_2)$ as $\mathrm{Ind}^{\mathcal{S},R_{1,1}}_{\mathbb{R}^\times\times 1}(\chi_2)$ using the equation

$$\begin{aligned}\Gamma^{\mathcal{S}}(\mathcal{U}_2, \mathcal{E}_2) &= \Gamma^{\mathcal{S}}(B_2\backslash B_2 w_2 B_2, \mathcal{E}_2)\\ &= \Gamma^{\mathcal{S}}(T_2\backslash B_2, \chi_2\otimes\chi_1\chi_2)\\ &= \Gamma^{\mathcal{S}}(\mathbb{R}^\times\times 1\backslash R_{1,1}, \chi_2) = \mathrm{Ind}^{\mathcal{S},R_{1,1}}_{\mathbb{R}^\times\times 1}(\chi_2),\end{aligned}$$

and then define an $R_{1,1}$-homomorphism

$$T_d : \mathrm{Ind}^{\mathcal{S},R_{1,1}}_{\mathbb{R}^\times\times 1}(\chi_2) \to \pi_D$$

by composing the embedding (5.2.1) and the quotient map π_I to π_F:

$$T_d : \mathrm{Ind}^{\mathcal{S},R_{1,1}}_{\mathbb{R}^\times\times 1}(\chi_2) = \Gamma^{\mathcal{S}}(\mathcal{U}_2, \mathcal{E}_2) \hookrightarrow \Gamma^{\mathcal{S}}(\mathcal{X}_2, \mathcal{E}_2) = \pi_I \to \pi_I/\pi_F = \pi_D.$$

Lemma 5.2.3. *The homomorphism* T_d *is injective.*

Proof. Suppose T_d is not injective. Then there exist $\tilde{f} \in \Gamma^{\mathcal{S}}(\mathcal{U}, \chi_1\chi_2\otimes\chi_2)$ whose extension by zero $\tilde{f}_G$ in π_I is contained in π_F.

On the one hand, $f(x) = \tilde{f}(w_2 n_x)$ is a Schwartz function. For $\theta \in (0, \pi)$, we can compute $\tilde{f}$ with the decomposition

$$k_\theta = \begin{pmatrix} 1/\sin\theta & \cos\theta \\ & \sin\theta \end{pmatrix} w_2 \begin{pmatrix} 1 & -\cot\theta \\ & 1 \end{pmatrix}.$$

Then we have

$$\tilde{f}_G(k_\theta) = \tilde{f}(k_\theta) = \chi_1\chi_2(1/\sin\theta)\chi_2(\sin\theta)f(-\cot\theta) = o(\theta^l), \qquad \text{for every } l > 0.$$

Then $\left(\frac{d}{d\theta}\right)^l \tilde{f}_G(k_\theta)|_{\theta=0} = 0$ for every positive integer l.

On the other hand, from [17, Section 2.3], π_F is generated by the functions

$$\phi_{-m+1}, \phi_{-m+3}, \ldots, \phi_{m-1},$$

where $\phi_l\,(n_x \cdot t(a, b) \cdot k_\theta) = \chi_1\chi_2(a)\chi_2(b)e^{il\theta}$.

Then $\tilde{f}_G \in \pi_F$ is a linear combination of ϕ_k, that is, there is a nonzero n-tuple $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that $\tilde{f}_G = \sum_{k=1}^{n} \lambda_k \phi_{2k-n-1}$. Then we have

$$\left(\frac{d}{d\theta}\right)^l \tilde{f}_G(k_\theta)\Big|_{\theta=0} = \sum_{k=0}^{n-1} \lambda_k((2k-n-1)i)^l,$$

Hence, there exists l such that $\left(\frac{d}{d\theta}\right)^l (\tilde{f}_G(k_\theta))|_{\theta=0} \neq 0$, which leads to a contradiction. Therefore, the $R_{1,1}$-homomorphism T_d is injective. □

Proposition 5.2.4. Coker(T_d) *has a decreasing complete filtration* $\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}_2, \mathcal{E}_2)_k$ *with composition factors isomorphic to*

$$|\det(\cdot)|^{k+\frac{m-1}{2}} \operatorname{sgn}(\cdot)^k\big|_{R_{1,1}}, \textit{ for } k = 1, 2, \dots. \tag{5.2.2}$$

Proof. From Lemma 5.2.2, $\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}_2, \mathcal{E}_2) = \pi_I / \Gamma^{\mathcal{S}}(\mathcal{U}_2, \mathcal{E}_2)$ has a decreasing complete filtration $\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}_2, \mathcal{E}_2)_k$ with composition factors isomorphic to

$$|\det(\cdot)|^k \operatorname{sgn}(\cdot)^k \chi_1\chi_2(\det(\cdot))\big|_{R_{1,1}}, \text{ for } k = 0, 1, \dots. \tag{5.2.3}$$

From Lemma 5.2.1, the finite-dimensional representation π_F in π_I has $R_{1,1}$-composition factors with irreducible pieces

$$|\det(\cdot)|^k \operatorname{sgn}^k(\det(\cdot))\chi_1\chi_2(\det(\cdot))\big|_{R_{1,1}}, \text{ for } k = 0, 1, \dots, m-1.$$

Then the projection $\pi_I \to \pi_I / i_{UX}(\Gamma^{\mathcal{S}}(\mathcal{U}_2, \mathcal{E}_2))$ gives an isomorphism between π_F and $\overline{\pi}_F = \Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}_2, \mathcal{E}_2) / \Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}_2, \mathcal{E}_2)_n$, implying that

$$\Gamma^{\mathcal{S}}_{\mathcal{Z}_2}(\mathcal{X}_2, \mathcal{E}_2) = \pi_F \oplus \Gamma^{\mathcal{S}}_{\mathcal{Z}_2}(\mathcal{X}_2, \mathcal{E}_2)_m.$$

Therefore,

$$\operatorname{Coker}(T_d) = \pi_D / i_{UX}(\Gamma^{\mathcal{S}}(\mathcal{U}_2, \mathcal{E}_2)) = (\pi_I / \Gamma^{\mathcal{S}}(\mathcal{U}_2, \mathcal{E}_2))/\pi_F = \Gamma^{\mathcal{S}}_{\mathcal{Z}_2}(\mathcal{X}_2, \mathcal{E}_2)_m,$$

and thus Coker(T_d) has a decreasing complete filtration with composition factors isomorphic to

$$\sigma_k = |\det(\cdot)|^k \operatorname{sgn}^k(\det(\cdot))\chi_2(\det(\cdot))\big|_{R_{1,1}} = |\det(\cdot)|^{k+\frac{m-1}{2}} \operatorname{sgn}(\cdot)^k\big|_{R_{1,1}},$$

for $k = 1, 2, \dots$. □

Graded structure of spherical principal series. Let $(s_1, \dots, s_{r+1}) \in \mathbb{C}^{r+1}$, and set $\sigma_{X^+} = |\cdot|^{s_1} \times \cdots \times |\cdot|^{s_{r+1}}$, which is a spherical principal series. The computation in [42, Section 5.1] for the restriction of spherical principal series representations to the mirabolic subgroup $R_{r,1}$ can be generalized over the real field verbatim and we can obtain a proposition parallel to [42, Proposition 5.1].

Following [42, §5], we denote by $Q_{a,b,c}$ the intersection of the parabolic subgroup $P_{a,b,c}$ associated to the partition (a, b, c) in GL_{a+b+c} and the mirabolic subgroup

$R_{a+b+c-1}$. We let the "Levi part" $L_{a,b,c}$ of $Q_{a,b,c}$ to be the image of $\mathrm{GL}_a \times \mathrm{GL}_b \times R_{c-1,1}$ diagonally embedded into $\mathrm{GL}_{a,b,c}$. Then $Q_{a,b,c} = L_{a,b,c}U_{a,b,c}$ for the unipotent group associated to the partition (a, b, c).

Proposition 5.2.5. *When restricted to* $R_{r,1}$, *the representation* σ_{X^+} *has a subrepresentation* $\mathrm{Ind}_{N_{r+1}}^{\mathcal{S},R_{r,1}}(\psi_{r+1}^{-1})$. *Moreover, the quotient* $\sigma_{X^+}/\mathrm{Ind}_{N_{r+1}}^{\mathcal{S},\mathrm{GL}_{r+1}}(\psi_{r+1}^{-1})$ *admits an* $R_{r,1}$*-stable complete filtration whose composition factors have the shape*

$$\mathrm{Ind}_{Q_{a,b,c}}^{\mathcal{S},R_{r,1}}(\tau_a \boxtimes \tau_b \boxtimes \tau_c)$$

where $a + b + c = t + 1$, $a + b \neq 0$ *and the tensor* $\tau_a \boxtimes \tau_b \boxtimes \tau_c$ *is regarded as a* $Q_{a,b,c}$ *representation by trivial extension on* $N_{a,b,c}$.

(1) $\tau_a = \mathrm{Ind}_{B_a}^{\mathcal{S},\mathrm{GL}_a(\mathbb{R})}(\mathrm{sgn}^{m_1}|\cdot|^{s_{i_1}+k_1} \boxtimes \cdots \boxtimes \mathrm{sgn}^{m_a}|\cdot|^{s_{i_a}+k_a})$ *where* $1 \leqslant i_1, \ldots, i_a \leqslant t + 1$ *are integers,* $l_1, \ldots, l_a \in \mathbb{Z}$ *and* $k_1, \ldots, k_a \in \frac{1}{2}\mathbb{Z}$;

(2) $\tau_b = \tau_b' \otimes \rho$ *where* τ_b' *is a representation of the same form as* τ_a *and* ρ *is a finite-dimensional representation of* $\mathrm{GL}_b(\mathbb{R})$;

(3) $\tau_c = \mathrm{Ind}_{N_c}^{\mathcal{S},R_{c-1,1}}(\psi_c^{-1})$.

5.3. ***Multiplicity formula: first inequality.*** In this section, we prove Lemma 5.0.5 and one inequality of Lemma 5.0.3. More precisely, in the setting of Theorem 5.0.1, we prove the inequality

$$m(\pi_V \boxtimes \pi_W) \geqslant m((|\det|^s \sigma_{X^+} \ltimes \pi_W) \boxtimes \pi_V)$$

for a basic relevant pair (W^+, V) when

(1) $\sigma_{X^+} = \mathrm{sgn}^l$ and $s \geqslant \mathrm{LI}(\pi_V)$, or

(2) $\sigma_{X^+} = \sigma_{\underline{s}}$ for $\underline{s}$ in general positions.

With a similar approach, we show that

$$m(\pi_V \boxtimes (|\cdot|^{s+\frac{m}{2}} \mathrm{sgn}^{m+1} \ltimes \pi_W)) \geqslant m((|\det|^s \sigma_{X^+} \ltimes \pi_W) \boxtimes \pi_V)$$

when $\sigma_{X^+} = D_m$ and $s \geqslant \mathrm{LI}(\pi_V)$.

For a relevant pair (W, V) and we let (V, W^+) be the associated basic relevant pair with the decomposition $W^+ = W \perp (X^+ \oplus Y^+)$. We denote by $(\mathsf{G}^+, \mathsf{H}^+, \xi^+)$ the Gross–Prasad triple associated to (V, W^+).

Let P_{X^+} be the parabolic subgroup of $\mathrm{SO}(W^+)$ stabilizing X^+. For $\sigma_{X^+} \in \mathcal{SF}(\mathrm{GL}(X^+))$ and $\pi_W \in \mathcal{SF}(\mathrm{SO}(W))$, from Definition 4.0.3,

$$\sigma_{X^+} \ltimes \pi_W = \mathrm{Ind}_{P_{X^+}}^{\mathcal{S},G}(|\det|^s \sigma_{X^+} \ltimes \pi_W) = \Gamma^S(P_{X^+} \backslash \mathrm{SO}(W^+), \mathcal{E})$$

where

$$\mathcal{E} = \mathcal{E}_{\sigma_{X^+},\pi_W} = P_{X^+} \backslash \big(\mathrm{SO}(W^+) \times (\delta_{P_{X^+}}^{1/2} |\det|^s \sigma_{X^+} \boxtimes \pi_W)\big). \tag{5.3.1}$$

We first study the structure of the right-$\mathrm{SO}(V)$-orbits of $\mathcal{X} = P_{W^+}\backslash\mathrm{SO}(W^+)$.

(1) When $\dim W^+ > 2(r+1)$, $\mathcal{X}$ consists of all k-dimensional totally isotropic subspaces of V. When $\dim W^+ = 2(r+1)$, there are exactly two maximal totally isotropic spaces and $\mathcal{X}$ is exactly one of them.

(2) When $\dim W^+ > 2(r+1)$, there is an open $\mathrm{SO}(V)$-orbit $\mathcal{U}$ consisting of $(r+1)$-dimensional totally isotropic spaces that are not contained in V. Its complement $\mathcal{Z}$ is the space of $(r+1)$-dimensional totally isotropic spaces contained in V. When $\dim V = 2(r+1)$ and $X^+.g_0 \subset V$ for some $g_0 \in \mathrm{SO}(W^+)$, $\mathcal{Z}$ has two orbits and both of them are singletons, more precisely, $[X^+.g_0]$ and $[X^+.g_0 g]$ for any $g \in \mathrm{O}(V)\backslash\mathrm{SO}(V)$; when $\dim V = 2(r+1)$ and if $X^+.g_0 \not\subseteq V$ for all $g_0 \in \mathrm{SO}(W^+)$, $\mathcal{Z}$ is empty; otherwise, $\mathcal{Z}$ has just one orbit.

We can draw the following conclusion:

Lemma 5.3.1. (1) *$\mathcal{Z}$ is empty when* $\dim W^+ = 2(r+1)$ *or* $\dim V = 2(r+1)$ *and* $X^+.g_0 \not\subseteq V$ *for all* $g_0 \in \mathrm{SO}(W^+)$.

(2) *$\mathcal{Z}$ has two* $\mathrm{SO}(V)$*-orbits, when* $\dim V \neq 2(r+1)$.

(3) *$\mathcal{Z}$ has a single* $\mathrm{SO}(V)$*-orbit, when* $\dim V = 2(r+1)$ *and* $X^+.g_0 \subseteq V$ *for some* $g_0 \in \mathrm{SO}(W^+)$.

Let $\Gamma^S_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}) = \Gamma^S(\mathcal{X}, \mathcal{E}) / \Gamma^S(\mathcal{U}, \mathcal{E})$. From Proposition 5.1.1, there is a short exact sequence

$$0 \to \Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V \to \Gamma^S(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V \to \Gamma^S_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V \to 0. \tag{5.3.2}$$

Hence, we have the short exact sequence

$$\begin{aligned}0 \to \mathrm{Hom}_{H^+}(\Gamma^S_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}) &\to \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}) \\ &\to \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}).\end{aligned} \tag{5.3.3}$$

When $\mathrm{Hom}_{H^+}(\Gamma^S_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}) = 0$, we have

$$m((\sigma_{X^+} \ltimes \pi_W) \boxtimes \pi_V) \leqslant \dim \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}).$$

We first analyze the closed orbits on $\mathcal{Z}$ to prove

$$\mathrm{Hom}_{H^+}(\Gamma^S_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}) = 0$$

and then analyze the open orbit $\mathcal{U}$ to prove

$$\dim \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}) \leqslant m(\pi_V \boxtimes \pi_W),$$

under the given conditions.

Closed orbits. Suppose $\mathcal{Z}$ is nonempty. Let $\gamma \in \mathrm{SO}(W^+)$ be a representative of an orbit of $\mathcal{Z}$ such that $X^+.\gamma = X'$ where X' is a totally isotropic subspace of V satisfying $\dim X^+ = \dim X'$. Then the stabilizer group S_γ at $[X]$ is equal to $\gamma^{-1} P_{W^+}\gamma \cap \mathrm{SO}(V)$, which a parabolic subgroup of $\mathrm{SO}(V)$ with Levi decomposition $S_\gamma = M_\gamma N_\gamma$ and the Levi subgroup $M_\gamma = \mathrm{GL}(X') \times \mathrm{SO}(V_0)$. The cotangent bundles and their fibers at $[X']$ are

$$\begin{aligned} T^*_{\mathcal{Z}} &= \mathrm{SO}(V) \times_{S_\gamma} S_\gamma^\perp, & \mathrm{Fib}_{[X']}(T^*_{\mathcal{Z}}) &= S_\gamma^\perp \\ T^*_{\mathcal{X}} &= \mathrm{SO}(W^+) \times_{P_{W^+}} P_{W^+}^\perp, & \mathrm{Fib}_{[X']}(T^*_{\mathcal{X}}) &= P_{W^+}^\perp \end{aligned}$$

and S_γ acts by adjoint action. Then the fiber of the conormal bundle at $[X']$

$$\mathrm{Fib}_{[X']}(\mathcal{N}^\vee_{\mathcal{Z}/\mathcal{X}}) = \mathrm{Fib}_{[X']}(T^*_{\mathcal{X}})/\mathrm{Fib}_{[X']}(T^*_{\mathcal{Z}}) = P_{W^+}^\perp / S_\gamma^\perp,$$

which is $\dim(X')$-dimensional. The $\mathrm{SO}(V_0)$ and N_γ act trivially and $\mathrm{GL}(X')$ acts as the standard representations. Then

$$\begin{aligned} &\Gamma^{\mathcal{S}}(\mathrm{SO}(V).[X], \mathrm{Sym}^k \mathcal{N}^\vee_{\mathcal{Z}/\mathcal{X}} \otimes \mathcal{E}|_{\mathcal{Z}}) \\ &\qquad = \mathrm{Ind}_{S_\gamma}^{\mathcal{S},\mathrm{SO}(V)}(\mathrm{Fib}_{[X]}(\mathrm{Sym}^k \mathcal{N}_{\mathcal{Z}/\mathcal{X}} \otimes \mathcal{E}|_{\mathcal{Z}})) \\ &\qquad = \mathrm{I}_{S_\gamma}^{\mathrm{SO}(V)}((|\det(\cdot)|^{s+\frac{1}{2}}\sigma_{X^+} \otimes \mathrm{Sym}^k \rho_{X'}^{\mathrm{std}}) \boxtimes ({}^\gamma\pi_W|_{\mathrm{SO}(V_0)})) \end{aligned}$$

Therefore,

$$\begin{aligned} (5.3.4)\quad &\Gamma^{\mathcal{S}}(\mathcal{Z}, \mathrm{Sym}^k \mathcal{N}^\vee_{\mathcal{Z}/\mathcal{X}} \otimes \mathcal{E}|_{\mathcal{Z}}) \\ &\qquad = (\mathrm{I}_{S_\gamma}^{\mathrm{SO}(V)}((|\det(\cdot)|^{s+\frac{1}{2}}\sigma_{X^+} \otimes \mathrm{Sym}^k \rho_{X'}^{\mathrm{std}}) \boxtimes ({}^\gamma\pi_W|_{\mathrm{SO}(V_0)})))^{\oplus c} \end{aligned}$$

where $\rho_{X'}^{\mathrm{std}}$ is the standard representation of $\mathrm{GL}(X')$ and c is the number of $\mathrm{SO}(V)$-orbits in $\mathcal{Z}$.

Proposition 5.3.2. *We have*

$$\mathrm{Hom}_{H^+}(\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}) = 0$$

under any of the following conditions:

(1) $\sigma_{X^+} = \mathrm{sgn}^l$ $(l = 0, 1)$ *or* $\sigma_{X^+} = D_m$ $(m \in \mathbb{N}_+)$, *and* $s \geqslant \mathrm{LI}(\pi_V)$, *or*

(2) $\sigma_{X^+} = \sigma_{\underline{s}} \in \mathbb{C}^r$ *and* $\underline{s}$ *is in general position.*

Proof. By (5.3.4), we have

$$\begin{aligned} &\Gamma^{\mathcal{S}}(\mathcal{Z}, \mathrm{Sym}^k \mathcal{N}^\vee_{\mathcal{Z}/\mathcal{X}} \otimes \mathcal{E}|_{\mathcal{Z}}) \boxtimes \pi_V \\ &\qquad = (\mathrm{I}_{S_\gamma}^{\mathrm{SO}(V)}(|\det(\cdot)|^{s+\frac{1}{2}}(\sigma_{X^+} \otimes \mathrm{Sym}^k \rho_{X'}^{\mathrm{std}}) \boxtimes ({}^\gamma\pi_W|_{\mathrm{SO}(V_0)})))^{\oplus c} \boxtimes \pi_V. \end{aligned}$$

- When $\sigma_{X^+} = \mathrm{sgn}^m$, we have $\sigma_{X^+} \otimes \mathrm{Sym}^k \rho = |\det|^k \mathrm{sgn}^m$. When $\mathrm{Re}(s) \geqslant \mathrm{LI}(\pi_V)$, we have $s + \frac{1}{2} + k > \mathrm{LI}(\pi_V)$, from Theorem 5.1.11, we have

$$\mathrm{Hom}_{H^+}(\Gamma^{\mathcal{S}}(\mathcal{Z}, \mathrm{Sym}^k \mathcal{N}^{\vee}_{\mathcal{Z}/\mathcal{X}} \otimes \mathcal{E}|_{\mathcal{Z}}) \boxtimes \pi_V, 1_{H^+}) = 0.$$

- When $\sigma_{X^+} = D_m$, by computation with the base of D_{m+2a} in [17, §2.3], we have

$$\sigma_{X^+} \otimes \mathrm{Sym}^k \rho = \bigoplus_{a=0}^{k} D_{m+2a}.$$

When $\mathrm{Re}(s) \geqslant \mathrm{LI}(\pi_V)$, we have $s + \frac{1}{2} > \mathrm{LI}(\pi_V)$, from Theorem 5.1.11, we have

$$\mathrm{Hom}_{H^+}(\Gamma^{\mathcal{S}}(\mathcal{Z}, \mathrm{Sym}^k \mathcal{N}^{\vee}_{\mathcal{Z}/\mathcal{X}} \otimes \mathcal{E}|_{\mathcal{Z}}) \boxtimes \pi_V, 1_{H^+}) = 0.$$

- When $\sigma_{X^+} = |\cdot|^{s_1} \times \cdots \times |\cdot|^{s_r}$, from [26, Corollary 5.6], the Harish-Chandra parameter of the infinitesimal character of $\sigma_{X^+} \otimes \mathrm{Sym}^k \rho$ is

$$[(s_1 + a_1, \ldots, s_{r+1} + a_{r+1})],$$

where the a_i are nonnegative integers. From Corollary 5.1.9, we have

$$\mathrm{Hom}_{H^+}(\Gamma^{\mathcal{S}}(\mathcal{Z}, \mathrm{Sym}^k \mathcal{N}^{\vee}_{\mathcal{Z}/\mathcal{X}} \otimes \mathcal{E}|_{\mathcal{Z}}) \boxtimes \pi_V, 1_{H^+}) = 0$$

for $\underline{s} \in \mathbb{C}^{r+1}$ in general positions.

From Corollary 5.1.5, we can conclude that, under the conditions given in the proposition, we have

$$\mathrm{Hom}_{H^+}(\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}) = 0.$$

Hence, from (5.3.3), we have

$$\dim \mathrm{Hom}_{H^+}(\Gamma^{\mathcal{S}}(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}) \leqslant \dim \mathrm{Hom}_{H^+}(\Gamma^{\mathcal{S}}(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}). \quad \square$$

The open orbit. We study $\Gamma^{\mathcal{S}}(\mathcal{U}, \mathcal{E})$ and show that $\dim \mathrm{Hom}_{H^+}(\Gamma^{\mathcal{S}}(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+})$ is less than or equal to $m(\pi_V \boxtimes \pi_W)$ under the given conditions.

We introduce the following notations just for this section:

- Let $d = \dim V$, $r = \frac{1}{2}(\dim V - \dim W - 1)$. We can compute the modular character

$$\delta_{P_{X^+}}((m \times g_W) \ltimes n) = |\det(m)|^{d-1-r}, \quad m \in \mathrm{GL}(X^+),\ g_W \in \mathrm{SO}(W),\ n \in N.$$

- Let N_{r+1} be the unipotent subgroup of $\mathrm{GL}_{r+1}(\mathbb{R})$ consisting of upper-triangular unipotent matrices, and let $R_{r,1}$ be the mirabolic subgroup of GL_{r+1}. We denote by $N_{r,1}$ the unipotent radical of $R_{r,1}$.
- We define a generic character π_{r+1} of N_{r+1} by letting

$$\psi_{r+1}(n) = \psi\Big(\sum_{i=1}^{r+1} n_{i,i+1}\Big),$$

where $n_{i,j}$ is the entry of matrix n at i-th row and j-th column.

Recall the decomposition $V = W \perp D \perp Z$ in Section 2.1. Let $X = X^+ \cap Z$ and we have X is totally isotropic and $\dim X = \dim X^+ - 1$. Let N be the unipotent radical of the parabolic subgroup P_X of $\mathrm{SO}(V)$ stabilizing X. We define N'_V the subgroup of N stabilizing D, then $H = (N_{r+1} \times \Delta\mathrm{SO}(W)) \ltimes N'_V$.

From Frobenius reciprocity, we have

$$\mathrm{Hom}_H(\xi^{-1} \otimes (\pi_V \boxtimes \pi_W), 1_H) = \mathrm{Hom}_{H^+}\big(\mathrm{Ind}_H^{\mathcal{S},H^+}(\xi^{-1} \otimes (\pi_V \boxtimes \pi_W)), 1_{H^+}\big). \tag{5.3.5}$$

By definition, the dimension of the left-hand side of (5.3.5) is equal to $m(\pi_V \boxtimes \pi_W)$. The right-hand side of (5.3.5) can be expressed as

$$\begin{aligned}\mathrm{Ind}_H^{\mathcal{S},H^+}&(\xi^{-1} \otimes (\pi_V \boxtimes \pi_W)) \\ &= \mathrm{Ind}^{\mathcal{S},H^+}_{(N_{r+1} \times \Delta\mathrm{SO}(W)) \ltimes N'_V}(\xi^{-1} \otimes (\pi_V \boxtimes \pi_W)) \\ &= \mathrm{Ind}^{\mathcal{S},H^+}_{(R_{r,1} \times \Delta\mathrm{SO}(W)) \ltimes N'_V}(\mathrm{Ind}^{\mathcal{S},R_{r,1}}_{N_{r+1}}(\psi^{-1}_{r+1})|_{R_{r,1}} \boxtimes \pi_W \boxtimes \pi_V).\end{aligned} \tag{5.3.6}$$

Recall that the open orbit $\mathcal{U} = P_{W^+} \backslash P_{W^+}\mathrm{SO}(V)$ equals $(P_{W^+} \cap \mathrm{SO}(V)) \backslash \mathrm{SO}(V)$ and the stabilizer group can be decomposed as

$$P_{W^+} \cap \mathrm{SO}(V) = (\mathrm{GL}(X) \times 1 \times \mathrm{SO}(W)) \ltimes N = \mathrm{SO}(W) \ltimes (R_{r,1} \ltimes N'_V). \tag{5.3.7}$$

By definition, we have

$$\begin{aligned}\Gamma^S_{\mathcal{Z}}(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V &= \mathrm{Ind}^{\mathcal{S},\mathrm{SO}(V)}_{P_{W^+} \cap \mathrm{SO}(V)}(|\det|^{\frac{d-1-r}{2}} \sigma_{X^+} \otimes \pi_W|_{P_{W^+} \cap \mathrm{SO}(V)}) \boxtimes \pi_V \\ &= \mathrm{Ind}^{\mathcal{S},\mathrm{SO}(V)}_{P_{W^+} \cap \mathrm{SO}(V)}(|\det|^{\frac{d-1-r}{2}} \sigma_{X^+}|_{R_{r,1}} \boxtimes \pi_W) \boxtimes \pi_V \\ &= \mathrm{Ind}^{\mathcal{S},H^+}_{(R_{r,1} \times \Delta\mathrm{SO}(W)) \ltimes N'_V}(|\det|^{\frac{d-1-r}{2}} \sigma_{X^+}|_{R_{r,1}} \boxtimes \pi_W \boxtimes \pi_V)\end{aligned} \tag{5.3.8}$$

- When $r = 0$ and $\sigma_{X^+} = \mathrm{sgn}^l$, we have

$$\mathrm{Ind}^{\mathcal{S},R_{r,1}}_{N_{r+1}}(\psi^{-1}_{r+1})|_{R_{r,1}} = |\det|^{\frac{d-1-r}{2}} \sigma_{X^+}|_{R_{r,1}},$$

so the right sides of (5.3.8) and (5.3.6) are the same. Hence, we have

$$m(\pi_V \boxtimes \pi_W) = \dim \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}).$$

- When $\sigma_{X^+} = |\cdot|^{s_1} \times \cdots \times |\cdot|^{s_{r+1}}$ for $(s_1, \ldots, s_{r+1}) \in \mathbb{C}^n$, from Proposition 5.2.5, there is an $R_{r,1}$-equivariant embedding

$$\mathrm{Ind}^{\mathcal{S},R_{r,1}}_{N_{r+1}}(\psi^{-1}_{r+1}) \hookrightarrow |\det|^{\frac{d-1-r}{2}} \sigma_{X^+}. \tag{5.3.9}$$

Applying the quotient of (5.3.8) and (5.3.6), we obtain

$$\Gamma^S_{\mathcal{Z}}(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V / \mathrm{Ind}^{\mathcal{S},H^+}_H(\xi^{-1} \otimes (\pi_V \boxtimes \pi_W)) = \mathfrak{Q}, \tag{5.3.10}$$

where

$$\mathfrak{Q} = \mathrm{Ind}^{\mathcal{S},H^+}_{(R_{r,1}\times \Delta\mathrm{SO}(W))\ltimes N'_V}\big((|\det|^{\frac{d-1-r}{2}}\sigma_{X^+}|_{R_{r,1}}/\mathrm{Ind}^{\mathcal{S},R_{r,1}}_{N_{r+1}}\psi^{-1}_{r+1})\boxtimes\pi_W\boxtimes\pi_V\big).$$

Therefore, to conclude that $\dim \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U},\mathcal{E})\boxtimes\pi_V, 1_{H^+}) \leqslant m(\pi_V\boxtimes\pi_W)$, it suffices to prove that

$$\mathrm{Hom}_{H^+}(\mathfrak{Q}, 1_{H^+}) = 0. \tag{5.3.11}$$

Using Proposition 5.2.5, from the exactness of Schwartz induction (Proposition 5.1.2) and projective tensor product (Proposition 5.1.1), we obtain that the quotient $\mathfrak{Q}$ has composition factors

$$\mathrm{Ind}^{\mathcal{S},H^+}_{(R_{r,1}\times \Delta\mathrm{SO}(W))\ltimes N'_V}\big(\mathrm{Ind}^{\mathcal{S},R_{r,1}}_{Q_{a,b,c}}(\tau_a\boxtimes\tau_b\boxtimes\tau_c)\boxtimes\pi_W\big), \tag{5.3.12}$$

where $Q_{a,b,c} = P_{a,b,c}\cap R_{r,1}$ and τ_a, τ_b, τ_c are defined in Proposition 5.2.5. Since (5.3.12) can be expressed as the parabolic induction

$$\big(|\det|^{-\frac{d-1-r+c}{2}}(\tau_a\boxtimes\tau_b)\big)\ltimes\mathrm{Ind}^{\mathcal{S},\mathrm{SO}(W\oplus D\oplus X_c)}_{(R_{c-1,1}\times\mathrm{SO}(W))\ltimes N_{W^+,c}}(\xi_c^{-1}\otimes\pi_W),$$

based on Corollary 5.1.5 and the fact that $a+b\geqslant 1$, the Hom-space in (5.3.12) vanishes for $(s_1,\dots,s_{r+1})\in\mathbb{C}^n$ in general position.

- When $r=1$ and $\sigma_{X^+} = D_l$, instead of (5.3.6), we use the equality

$$\begin{aligned}\mathrm{Ind}^{\mathcal{S},H^+}_{\Delta\mathrm{SO}(W\oplus\mathbb{R})}&((|\cdot|^{s+\frac{m}{2}}\mathrm{sgn}^{m+1}\ltimes\pi_W)\boxtimes\pi_V)\\ &= \mathrm{Ind}^{\mathcal{S},H^+}_{(R_{1,1}\times\Delta\mathrm{SO}(W))\ltimes N'_V}(\mathrm{Ind}^{\mathcal{S},R_{1,1}}_{\mathbb{R}^\times\times 1}(\chi_2)\boxtimes\pi_W\boxtimes\pi_V).\end{aligned} \tag{5.3.13}$$

From Section 5.2, there is an injection $T_d : \mathrm{Ind}^{\mathcal{S},R_{1,1}}_{\mathbb{R}^\times\times 1}(\chi_2)\hookrightarrow D_m$, and it induces an injection

$$\mathrm{Ind}^{\mathcal{S},R_{1,1}}_{\mathbb{R}^\times\times 1}(|\cdot|^s\chi_2)\hookrightarrow |\det|^s D_m.$$

Applying the quotient of (5.3.8) and (5.3.13), we obtain

$$\begin{aligned}\Gamma^S_{\mathcal{Z}}(\mathcal{U},\mathcal{E})\boxtimes\pi_V/&\mathrm{Ind}^{\mathcal{S},H^+}_{\Delta\mathrm{SO}(W\oplus\mathbb{R})}((|\cdot|^{s+\frac{m}{2}}\mathrm{sgn}^{m+1}\ltimes\pi_W)\boxtimes\pi_V)\\ &= \mathrm{Ind}^{\mathcal{S},H^+}_{(R_{r,1}\times\Delta\mathrm{SO}(W))\ltimes N'_V}\big((|\det|^{\frac{d-2}{2}}\sigma_{X^+}|_{R_{1,1}}/\mathrm{Ind}^{\mathcal{S},R_{1,1}}_{N_2}(\psi_2^{-1}))\boxtimes\pi_W\boxtimes\pi_V\big).\end{aligned} \tag{5.3.14}$$

From Proposition 5.2.4, the quotient $|\det|^s\sigma_{X^+}|_{R_{1,1}}/\mathrm{Ind}^{\mathcal{S},R_{1,1}}_{\mathbb{R}^\times\times 1}(|\cdot|^s\chi_2)|_{R_{1,1}}$ has composition factors

$$\sigma_k := |\det(\cdot)|^{s+k+\frac{m-1}{2}}\mathrm{sgn}(\cdot)^k|_{R_{1,1}},\quad k=1,2,\dots$$

From the exactness of Schwartz induction (Proposition 5.1.2) and projective tensor product (Proposition 5.1.1), there is a decreasing complete filtration of

$$\mathrm{Ind}^{\mathcal{S},H^+}_{(R_{r,1}\times\Delta\mathrm{SO}(W))\ltimes N'_V}\big((|\det|^{s+\frac{d-2}{2}}\sigma_{X^+}|_{R_{1,1}}/\mathrm{Ind}^{\mathcal{S},R_{1,1}}_{N_2}(\psi_2^{-1})|_{R_{1,1}})\boxtimes\pi_W\boxtimes\pi_V\big)$$

with composition factors

$$\mathrm{Ind}_{(R_{1,1}\times\Delta\mathrm{SO}(W))\ltimes N_V'}^{\mathcal{S},H^+}(\sigma_k\boxtimes\pi_W\boxtimes\pi_V).$$

Notice that

$$\mathrm{Ind}_{(R_{1,1}\times\Delta\mathrm{SO}(W))\ltimes N_V'}^{\mathcal{S},H^+}(\sigma_k\boxtimes\pi_W\boxtimes\pi_V)=(|\cdot|^{s+\frac{m}{2}+k}\mathrm{sgn}^m\ltimes\mathrm{Ind}_{\mathrm{SO}(W)}^{\mathcal{S},\mathrm{SO}(W\oplus\mathbb{R})}(\pi_W))\boxtimes\pi_V.$$

Since we have assumed that $\mathrm{Re}(s)\geqslant\mathrm{LI}(\pi_V)$ and k is a positive integer, we have

$$s+\frac{m}{2}+k>\mathrm{LI}(\pi_V).$$

Then, from Theorem 5.1.11, we have

$$\mathrm{Hom}_{H^+}\big((|\cdot|\mathrm{sgn}^m\ltimes\mathrm{Ind}_{\mathrm{SO}(W)}^{\mathcal{S},\mathrm{SO}(W\oplus\mathbb{R})}(\pi_W))\boxtimes\pi_V,1_{H^+}\big)=0,\quad k=1,2,\ldots$$

From Corollary 5.1.5, this implies

$$\mathrm{Hom}_{H^+}\big(\Gamma_{\mathcal{Z}}^{S}(\mathcal{U},\mathcal{E})\boxtimes\pi_V/\mathrm{Ind}_{\Delta\mathrm{SO}(W\oplus\mathbb{R})}^{\mathcal{S},H^+}((|\cdot|^{s+\frac{m}{2}}\mathrm{sgn}^{m+1}\ltimes\pi_W)\boxtimes\pi_V),1_{H^+}\big)=0.$$

Hence, Lemma 5.1.3, we have

$$m(\pi_V\boxtimes(|\cdot|^{s+\frac{m}{2}}\mathrm{sgn}^{m+1}\ltimes\pi_W))\geqslant\dim\mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U},\mathcal{E})\boxtimes\pi_V,1_{H^+}).$$

Proof of the "first inequality". We now make use of Lemma 5.0.5 to prove one side of the equality in Proposition 5.0.4, namely

$$m(\pi_V\boxtimes\pi_W)\leqslant m(\pi_{V_0}\boxtimes\pi_{W_0}).\tag{5.3.15}$$

We express $\pi_V=\sigma_V\ltimes\pi_{V_0}$, $\pi_W=\sigma_W\ltimes\pi_{W_0}$ in the form of (4.0.7) and prove the inequality by induction on

$$N(\sigma_V,\sigma_W)=\sum_{\mathrm{Re}(s_{V,i})\neq0}n_{V,i}+\sum_{\mathrm{Re}(s_{W,i})\neq0}n_{W,i},$$

where $s_{V,i},s_{W,i},n_{V,i},n_{W,i}$ are defined as in (4.0.7).

If $N(\sigma_V,\sigma_W)=0$, both π_V and π_W are tempered; then the inequality follows from Conjecture 1 for tempered parameters, which was proved in [28; 10].

In other cases, we may assume

$$\mathrm{Re}(s_{V,1})\geqslant\mathrm{Re}(s_{V,2})\geqslant\cdots\geqslant\mathrm{Re}(s_{V,l})>0,$$
$$\mathrm{Re}(s_{W,1})\geqslant\mathrm{Re}(s_{W,2})\geqslant\cdots\geqslant\mathrm{Re}(s_{W,l})>0.$$

Suppose the proposition holds when $N(\sigma_V,\sigma_W)\leqslant k$, then when $N(\sigma_V,\sigma_W)=k+1$, we consider the following cases.

Case 1: If $l_V\neq0$ and $\mathrm{Re}(s_{V,1})\geqslant\mathrm{Re}(s_{W,1})$, then let $\widetilde{\sigma}_V=|\det(\cdot)|^{s_{V,2}}\sigma_{V,2}\times\cdots\times|\det(\cdot)|^{s_{V,l}}\sigma_{V,l}$.

(1) If $n_{V,1} = 1$, from Lemma 5.0.5(1) we have

$$m((\sigma_V \ltimes \pi_{V_0}) \boxtimes (\sigma_W \ltimes \pi_{W_0})) \leqslant m((\sigma_W \ltimes \pi_{W_0}) \boxtimes (\widetilde{\sigma}_V \ltimes \pi_{V_0})).$$

(2) If $n_{V,1} = 2$, let $\widehat{\sigma}_V = |\cdot|^{s_{V,1}+\frac{m_{V,1}}{2}} \mathrm{sgn}^{m_{V,1}+1} \ltimes \widetilde{\sigma}_V$. From Lemma 5.0.5(2), we have

$$m((\sigma_V \ltimes \pi_{V_0}) \boxtimes (\sigma_W \ltimes \pi_{W_0})) \leqslant m((\sigma_W \ltimes \pi_{W_0}) \boxtimes (\widehat{\sigma}_V \ltimes \pi_{V_0})).$$

Since $N(\widetilde{\sigma}_V, \sigma_W), N(\widehat{\sigma}_V, \sigma_W) \leqslant N(\sigma_V, \sigma_W) - 1 = k$, we have

$$m((\sigma_W \ltimes \pi_{W_0}) \boxtimes (\widetilde{\sigma}_V \ltimes \pi_{V_0})) \leqslant m(\pi_{V_0} \boxtimes \pi_{W_0}),$$
$$m((\sigma_W \ltimes \pi_{W_0}) \boxtimes (\widehat{\sigma}_V \ltimes \pi_{V_0})) \leqslant m(\pi_{V_0} \boxtimes \pi_{W_0}).$$

Therefore, we have

$$m((\sigma_V \ltimes \pi_{V_0}) \boxtimes (\sigma_W \ltimes \pi_{W_0})) \leqslant m(\pi_{V_0} \boxtimes \pi_{W_0}),$$

<u>Case 2:</u> If $l_V = 0$ or $\mathrm{Re}(s_{V,1}) < \mathrm{Re}(s_{W,1})$, we switch the order of V, W to reduce to Case 1. More explicitly, we take $\sigma_W^{(s')} = |\cdot|^{s'} \times \sigma_{W_0}$. There is an $s' \in i\mathbb{R}$ such that

$$m((\sigma_V \ltimes \pi_{V_0}) \boxtimes (\sigma_W \ltimes \pi_{W_0})) = m((\sigma_W^{(s')} \ltimes \pi_{W_0}) \boxtimes (\sigma_V \ltimes \pi_{V_0}))$$

From [32, Theorem 1.1] and Langlands classification, we may assume $\sigma_W^{(s')} \ltimes \pi_{W_0}$ is irreducible. Then the pair $(\sigma_W^{(s')}, \sigma_V)$ belongs to Case 1 and $N(\sigma_W^{(s')}, \sigma_V) = N(\sigma_V, \sigma_W) = k+1$, so

$$m((\sigma_W^{(s')} \ltimes \pi_{W_0}) \boxtimes (\sigma_V \ltimes \pi_{V_0})) \leqslant m(\pi_{V_0} \boxtimes \pi_{W_0}).$$

Therefore, we have

$$m((\sigma_V \ltimes \pi_{V_0}) \boxtimes (\sigma_W \ltimes \pi_{W_0})) \leqslant m(\pi_{V_0} \boxtimes \pi_{W_0}).$$

The proposition now follows by induction on $N(\sigma_V, \sigma_W)$. □

5.4. *Multiplicity formula: the second inequality.*

In this section, we complete the proof for the "second inequality" of Proposition 5.0.4.

A construction. We prove Lemma 5.0.6 by construction. Recall that, for a relevant pair (W, V), we can construct a basic relevant pair (V, W^+) by taking $W^+ = W \perp (X^+ \oplus Y^+)$ for certain totally isotropic spaces X^+ and Y^+. Let $\mathsf{G}^+ = \mathrm{SO}(W^+) \times \mathrm{SO}(V)$, $\mathsf{H}^+ = \Delta\mathrm{SO}(V)$, P^+ is the parabolic subgroup $\mathsf{P}_{X^+} \times \mathrm{SO}(V)$, where P_{X^+} is the parabolic subgroup of $\mathrm{SO}(W^+)$ stabilizing X^+. We note

$$G^+ = \mathsf{G}^+(\mathbb{R}), \quad H^+ = \mathsf{H}^+(\mathbb{R}), \quad P^+ = \mathsf{P}^+(\mathbb{R}).$$

From the multiplicity-one theorem [34], we have $m(\pi_V \boxtimes \pi_W) \leqslant 1$, so it suffices to prove the following proposition.

Proposition 5.4.1. *When $m(\pi_V \boxtimes \pi_W) \neq 0$ and σ_{X^+} is a generic representation of* $\mathrm{GL}(X^+)$, *then one can construct a nonzero element in*

$$\mathrm{Hom}_{H^+}((\sigma_{X^+} \ltimes \pi_W) \boxtimes \pi_V, 1_{H^+}).$$

The main idea for proving this proposition is from the following theorem.

Theorem 5.4.2 [18, Proposition 4.9]. *For a Casselman–Wallach representation σ^+ of P^+, suppose*:

1. *The complement $G^+ - P^+H^+$ is the zero set of a polynomial f^+ on G^+ that is left-H^+-invariant and right-(P^+, ψ_{P^+})-equivariant for an algebraic character ψ_{P^+} of P^+.*
2. *H^+ has finitely many orbits on the flag of a minimal parabolic subgroup of G^+*
3. *σ^+ admits a nonzero $(P^+ \cap H^+, \delta_{P^+\cap H^+}\delta_{H^+}^{-1})$-equivariant continuous linear functional, where $\delta_{P^+\cap H^+}$, δ_{H^+} are the modular characters of $P^+ \cap H^+$ and H^+ respectively.*

Then $\mathrm{Ind}_{P^+}^{\mathcal{S},G^+}(\sigma^+)$ *admits a nonzero H^+-invariant functional.*

We first verify (1) and (2) in the setup of Proposition 5.4.1.

1. Fix a basis $v_1, \dots, v_n$ of V and a basis $v_1^+, \dots, v_{r+1}^+$ of X^+. For every $(g_{W^+}, g_V) \in G^+$, $g \in G^+ - P^+H^+$ if and only if $Xg_{W^+} \subset V$, equivalently, the $(n+1) \times (n+1+r)$-matrix
$$A_g = \left[v_1 g_V, \dots, v_n g_V, v_1^X g_{W^+}^{-1}, \dots, v_{r+1}^X g_{W^+}^{-1}\right]$$
is of rank n. We let
$$f(g) = \det(A_g A_g^t); \tag{5.4.1}$$
then f is left-(P^+, ψ_{P^+})-equivariant and right-H^+-invariant, where
$$\psi_{P^+}(p_{X^+}, g_V) = \det(g_{X^+})^2 \quad \text{for } p_{X^+} = (g_{X^+}, g_W) \cdot n_{X^+} \in P_{W^+} \text{ and } g_V \in \mathrm{SO}(V).$$
2. Since G^+/H^+ is an absolutely spherical variety (Section 3), the Borel subgroup has finitely many orbits, so the complexification of the minimal parabolic also has finitely many orbits. Then condition (2) is a direct consequence of the finiteness of the first Galois cohomology for groups over local fields.

Therefore, to complete the proof for Proposition 5.4.1, it suffices to construct a nonzero $(P^+ \cap H^+, \delta_{P^+\cap H^+}\delta_{H^+}^{-1})$-equivariant continuous linear functional.

As computed in Section 5.3, we have

$$H \backslash P^+ \cap H^+ = N_{r+1} \backslash R_{r,1},$$

where N_{r+1} and $R_{r,1}$ are the unipotent group and mirabolic group defined in Section 5.3. Hence, from [31], the Rankin–Selberg integral

$$\begin{aligned} &F_s(v_{\pi_V}, v_{\pi_W}, v_{\sigma_{X^+}}) \\ &\quad := \int_{P^+ \cap H^+} \mu(\pi_V(p_{X^+})v_{\pi_V}, v_{\pi_W})\lambda(\sigma_{X^+}(p_{X^+})v_{\sigma_{X^+}})|\det(g_{X^+})|^s d(p_{X^+}, p_{X^+}) \end{aligned}$$

is absolutely convergent when $\mathrm{Re}(s)$ is large enough and extends to a meromorphic family in

$$F_s \in \mathrm{Hom}_{P^+ \cap H^+}(\pi_V \boxtimes \pi_W \boxtimes \sigma_{X^+}, |\det(g_X)|^{s-s_0}),$$

where $s_0 = \dim W - \dim X^+$, which is the real number satisfying $\delta_{P^+}(p_{X^+}) = |\det(g_{X^+})|^{s_0}$. From [18], we know

$$\frac{F_s}{(s-s_0)^{n_{s_0}}}\Big|_{s=s_0}$$

is a nonzero element

$$\mathrm{Hom}_{P^+ \cap H^+}(\pi_V \boxtimes \pi_W \boxtimes \sigma_{X^+}, 1_{P^+ \cap H^+}),$$

where n_{s_0} is the order of poles of F_s at $s = s_0$. This completes the proof for Proposition 5.4.1.

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