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We prove a conjecture about the concordance invariant ϑ, defined in a recent paper by Lewark and Zibrowius. This result simplifies the relation between ϑ and Rasmussen's s-invariant. The proof relies on Bar-Natan's tangle version of Khovanov homology or, more precisely, on its distillation in the case of 4-ended tangles into the immersed curve theory of Kotelskiy, Watson and Zibrowius.

1. Introduction

Lewark and Zibrowius [2024] defined two new families of smooth concordance invariants,

$$\{\vartheta_c : \mathcal{C}_{\mathrm{sm}} \to \mathbb{Z}\} \quad \text{and} \quad \{\vartheta'_c : \mathcal{C}_{\mathrm{sm}} \to \mathbb{Z} \cup \{\infty\}\},$$

parametrized by a prime c. These invariants exploit the following linearity property of Rasmussen's invariant in characteristic c. Given a knot $K \subset S^3$ and a pattern $P \subset D^2 \times S^1$ of wrapping number 2, the function

$$t \mapsto s_c(P_t(K))$$

is the restriction to $\mathbb{Z}$ of a piecewise affine function $\mathbb{R} \to \mathbb{R}$ of slope 1 or 0 that has at most one jump discontinuity. If the winding number of P is ± 2 then the function has slope 1, otherwise the winding number and slope are 0 and, in this latter case, the function does have a jump discontinuity. In the case of winding number ± 2, the invariant $\vartheta'_c(K)$ is defined to be the value of t for which

$$s_c(P_{\vartheta'_c(K)}(K)) = s_c(P_{\vartheta'_c(K)-1}(K)),$$

if it exists. If no such value exists because the piecewise affine function is affine, then $\vartheta'_c(K) := \infty$. Not only do Lewark and Zibrowius prove that ϑ_c and ϑ'_c are concordance invariants and that ϑ_c is a homomorphism $\mathcal{C}_{\mathrm{sm}} \to \mathbb{Z}$, but they also show that ϑ_c is a genuinely new invariant, in that it is not simply a multiple of s_c, in contrast to the τ-invariant [2024, §2.2].

The knots K with $\vartheta'_c(K) \neq \infty$ are of particular interest, and they are called ϑ_c-rational. We establish here a conjecture on the expected simplicity of ϑ'_c:

MSC2020: 57K18.
Keywords: Bar-Natan homology, s-invariant, tangles.

Theorem 1.1 [Lewark and Zibrowius 2024, Conjecture 2.24]. *If K is a ϑ_c-rational knot, then $\vartheta_c(K)=0$.*

Since ϑ_c agrees with ϑ_c' on the class of ϑ_c-rational knots [Lewark and Zibrowius 2024, Theorem 2.23], it follows that the second family of invariants $\{\vartheta_c'\}$ contains no more information than a single $\mathbb{Z}/2\mathbb{Z}$-valued invariant. A consequence noted by Lewark and Zibrowius [2024, p. 250] is the following simplification of their Theorem 2.23:

Corollary 1.2. *Let $K \subset S^3$ be a ϑ_c-rational knot and let P be a pattern with wrapping number 2 and winding number ± 2. Then*

$$s_c(P(K)) = s_c(P_{-\vartheta_c(K)}(U)) = s_c(P_0(U)). \qquad \square$$

Our argument uses the immersed curve theory of 4-ended tangles, constructed in [Kotelskiy et al. 2019] as a specialization of the theory developed in [Bar-Natan 2005], and a property of Lee's homology [2005].

2. Background

Tangles are considered modulo isotopy fixing the endpoints. Let $(K,*)$ be a pointed oriented knot and let T_K be the 4-ended tangle obtained by taking a copy of the long knot $K \setminus \{*\}$ together with its Seifert push-off, as in Figure 1. We generally also orient our tangles and mark an endpoint, as required for the theory in [Kotelskiy et al. 2019].

To specify notation for the cut-and-paste procedures used, let $n \in \mathbb{Z} \cup \{\infty\}$. First, the rational n-tangle Q_n is the one in Figure 2 for $n>0$. If $n<0$, then $Q_n = mQ_{-n}$, where m denotes the mirror. And if $n=0,\infty$, we set $Q_0 =$  and $Q_\infty =$ . Second, given two 4-ended tangles T_1 and T_2, the link $\mathcal{L}(T_1,T_2)$ is obtained by identifying endpoints as in Figure 2 below. Finally, let the n-closure $T(n)$ of a 4-ended tangle T be $\mathcal{L}(T, Q_{-n})$. By convention, diagrams for the tangle T_K are chosen so that their ∞-closure is the unknot, and the tangle is oriented compatibly with the 0-closure, as in Figure 1.

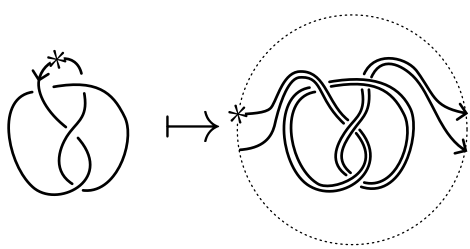


Figure 1. A pointed oriented knot $(K,*)$ and its associated double T_K.

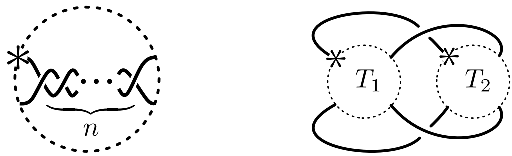


Figure 2. Left: The tangle Q_n. Right: The link $\mathcal{L}(T_1, T_2)$.

2.1. ***Bar-Natan homology.*** The Bar-Natan homology of a link is a version of Khovanov homology [2000] defined in [Bar-Natan 2005] with coefficients in the field with two elements $\mathbb{F}_2$, and later extended as a theory with coefficients in any prime field in [Mackaay et al. 2007]. It has been observed that varying the field characteristic results in interesting differences [Lewark and Zibrowius 2021], so let $\mathbb{F}_c$ be the prime field of characteristic c (in particular, $\mathbb{F}_0 = \mathbb{Q}$). We use the setup in [Kotelskiy et al. 2019, §3].

Given a link L, its Bar-Natan homology is a bigraded $\mathbb{F}_c[H]$-module $\mathrm{BN}(L; \mathbb{F}_c)$, where H is a formal variable that lowers the secondary (quantum) grading by 2. The shift operators for the homological and quantum gradings are denoted using square and curly brackets, respectively. For example,

$$\mathrm{BN}(L; \mathbb{F}_c)\{-1\}$$

is the Bar-Natan homology of L with coefficients in $\mathbb{F}_c$, but with quantum gradings formally reduced by 1.

If the link L is pointed, then there is a reduced theory $\widetilde{\mathrm{BN}}(L; \mathbb{F}_c)$, which is related to unreduced Bar-Natan homology by a short exact sequence of bigraded $\mathbb{F}_c[H]$-complexes:

$$0 \to \widetilde{\mathrm{CBN}}(D; \mathbb{F}_c)\{-1\} \to \mathrm{CBN}(D; \mathbb{F}_c) \to \widetilde{\mathrm{CBN}}(D; \mathbb{F}_c)\{1\} \to 0, \tag{1}$$

where D is a choice of diagram for L.

Notation. Free summands of the bigraded $\mathbb{F}_c[H]$-module $\widetilde{\mathrm{BN}}(L; \mathbb{F}_c)$ are called towers. The grading of a tower refers to the grading of a corresponding free generator.

2.2. ***Lee's deformation.*** Rasmussen [2010] used the work in [Lee 2005] to define the s-invariant of a knot. While the s-invariant can also be defined for links, as in [Beliakova and Wehrli 2008; Pardon 2012], this construction is not used as much, and Lewark and Zibrowius arranged so that their work only dealt with s-invariants of knots. This subsection recalls an aspect of the definition of the s-invariant for links in Lemma 2.1 below. This result is known to the experts and is the main observation needed to prove Theorem 1.1. See also [Lee 2005, Proposition 4.3].

Lemma 2.1. *Let L be an oriented 2-component pointed link. If* $\mathrm{lk}(L) \neq 0$, *then there is a unique tower* $\mathbb{F}_c[H] \hookrightarrow \widetilde{\mathrm{BN}}(L; \mathbb{F}_c)$ *in homological grading* 0. *Otherwise, if* $\mathrm{lk}(L) = 0$, *then both towers have homological grading* 0.

Proof. The idea is that, by setting $H = 1$ in the chain complex $\mathrm{CBN}(L; \mathbb{F}_c)$, we obtain a chain complex $\mathrm{fCBN}(L; \mathbb{F}_c)$ that is no longer bigraded, but rather homologically graded and quantum filtered. Courtesy of the filtration, there is an induced spectral sequence

$$\mathrm{fCBN}(L; \mathbb{F}_c) \rightrightarrows H_*(\mathrm{fCBN}(L; \mathbb{F}_c)).$$

Theorem 2.2 of [Lipshitz and Sarkar 2014] establishes that the vector space $H_*(\mathrm{fCBN}(L; \mathbb{F}_c))$ is 4-dimensional, and there is a canonical identification between the set of orientations on L and a set of generators of $H_*(\mathrm{fCBN}(L; \mathbb{F}_c))$. To understand this identification, note that each orientation on L determines an oriented resolution of a diagram for L. Lee's argument applies in this context to show that each generator of $H_*(\mathrm{fCBN}(L; \mathbb{F}_c))$ is the homology class of an algebra element assigned to an oriented resolution of L by the TQFT defining fCBN; see [Lee 2005, Theorem 4.2] or [Rasmussen 2010, §2.4] for the construction and [Lipshitz and Sarkar 2014, Theorem 2.2] for the applicability of Lee's work in this slightly different context.

Now, as explained in [Kotelskiy et al. 2019, Proposition 3.8], the components of the differential $\partial_{\mathrm{CBN}(L)}$ that are given by $1 \mapsto H^l$ induce differentials on the l-th page of the spectral sequence above, and this implies that

$$\mathrm{BN}(L; \mathbb{F}_c) \cong (\mathbb{F}_c[H])^{\oplus 4} \oplus \mathrm{Tors},$$

where the towers in $\mathrm{BN}(L; \mathbb{F}_c)$ correspond to the generators of $H_*(\mathrm{fCBN}(L; \mathbb{F}_c)$. Moreover it follows from the short exact sequence (1) that there is a 2-to-1 correspondence that preserves homological grading between the towers of $\mathrm{BN}(L)$ and the towers of $\widetilde{\mathrm{BN}}(L)$.

Finally, fix an oriented diagram $(D, \mathfrak{o}_0)$ for L, where $\mathfrak{o}_0$ is the orientation on D induced from L. Let $n_+(\mathfrak{o}_0)$ and $n_-(\mathfrak{o}_0)$ be the number of positive and negative crossings in $(D, \mathfrak{o}_0)$. Pick a component K of L and let $\mathfrak{o}_1$ be the orientation on D which is obtained by reversing the orientation on K. Then the number of negative crossings in $(D, \mathfrak{o}_1)$ is

$$n_-(\mathfrak{o}_1) = n_-(\mathfrak{o}_0) + 2\,\mathrm{lk}(L).$$

It follows that, while the oriented resolution of $(D, \mathfrak{o}_0)$ lies in homological grading 0, the $\mathfrak{o}_1$-oriented resolution $D^{\mathfrak{o}_1}$ lies in homological grading $2\,\mathrm{lk}(L)$. □

2.3. ***The immersed curve theory.*** In [Kotelskiy et al. 2019], two equivalent invariants of pointed 4-ended oriented tangles are defined:

$$T \mapsto [\![T; \mathbb{F}_c]\!] \in \mathbf{Mod}^{\mathcal{B}} \quad \text{and} \quad T \mapsto \widetilde{\mathrm{BN}}(T; \mathbb{F}_c) \in \mathbf{Fuk}(S^2_{4,*}).$$

The first produces type-D structures over the Bar-Natan algebra $\mathcal{B}$, which we will describe in Section 4. The second lands in the (partially wrapped) Fukaya category of S^2, punctured at four points, one of which is marked $*$. In other words, $\widetilde{\mathrm{BN}}(T;\mathbb{F}_c)$ is an immersed curve in $S^2_{4,*}$, possibly carrying a nontrivial local system. This possibility does not occur for noncompact curves, which are the only curves of interest in what follows. Moreover, the invariants are bigraded in an appropriate sense. Our main tool is the following pairing theorem.

Theorem 2.2 [Kotelskiy et al. 2019, Theorem 7.2]. *Let T_1 and T_2 be two pointed 4-ended tangles, and let $L=\mathcal{L}(T_1,T_2)$. Then the Bar-Natan homology is isomorphic to the wrapped Lagrangian intersection Floer homology of the tangle invariants, as bigraded $\mathbb{F}_c[H]$-modules*:

$$\widetilde{\mathrm{BN}}(L;\mathbb{F}_c)\{-1\}\cong \mathrm{HF}(\widetilde{\mathrm{BN}}(mT_1;\mathbb{F}_c),\widetilde{\mathrm{BN}}(T_2;\mathbb{F}_c)).$$

3. The proof of Theorem 1.1

Suppose now that K is a ϑ_c-rational knot. Lewark and Zibrowius identified $\vartheta_c(K)$ with a certain slope of $\widetilde{\mathrm{BN}}(T_K;\mathbb{F}_c)$, and this allows us to reduce the proof to a simple statement that can be checked using Lemma 2.1. Let $\widetilde{\mathrm{BN}}_a(T;\mathbb{F}_c)$ consist of the noncompact component(s) of $\widetilde{\mathrm{BN}}(T;\mathbb{F}_c)$.

Proposition 3.1 [Lewark and Zibrowius 2024, Proposition 6.18]. *If K is ϑ_c-rational, then the immersed curve $\widetilde{\mathrm{BN}}_a(T_K;\mathbb{F}_c)$ is equal to the immersed curve of the rational tangle Q_n, for some choice of $n\in 2\mathbb{Z}$, up to some grading shift.*

We have then $\widetilde{\mathrm{BN}}_a(T_K;\mathbb{F}_c)=\widetilde{\mathrm{BN}}(Q_n;\mathbb{F}_c)$, for some $n\in 2\mathbb{Z}$, up to grading shift. The immersed curve invariants $\widetilde{\mathrm{BN}}(Q_n;\mathbb{F}_c)$ are calculated in [Kotelskiy et al. 2019]. It turns out that they are independent of the coefficient field, so we may drop it from the notation. These invariants are best described in the following covering space of the 4-punctured sphere:

$$\mathbb{R}^2\setminus\left(\tfrac{1}{2}\mathbb{Z}\right)^2\xrightarrow{\ \alpha\ }T^2_{4,*}\xrightarrow{\ \beta\ }S^2_{4,*},$$

where β is the double cover given by hyperelliptic involution and α is the universal Abelian cover of the punctured torus. The puncture $*$ lifts to the integer lattice $\mathbb{Z}^2\subset\frac{1}{2}\mathbb{Z}^2$. The lift of $\widetilde{\mathrm{BN}}(Q_n)$ is (isotopic to) a line of slope n, as depicted in Figure 3 in the cases $n=-2,0,2$.

Proposition 3.2 [Lewark and Zibrowius 2024, Corollary 6.14]. *Given a knot K in S^3, let σ_c be the slope of $\widetilde{\mathrm{BN}}_a(T_K;\mathbb{F}_c)$ near the bottom-right tangle end. Then $\vartheta_c(K)=\lceil\sigma_c\rceil$.*

Since the curve $\widetilde{\mathrm{BN}}(Q_n)$ lifts to a curve that is isotopic to a line of slope n, the above two propositions reduce the proof of Theorem 1.1 to proving that $\widetilde{\mathrm{BN}}_a(T_K;\mathbb{F}_c)=\widetilde{\mathrm{BN}}(Q_0)$, up to grading shift. Consider the Bar-Natan homology of the 0-closure $T_K(0)$. Since T_K is obtained by taking the union of a long knot with its

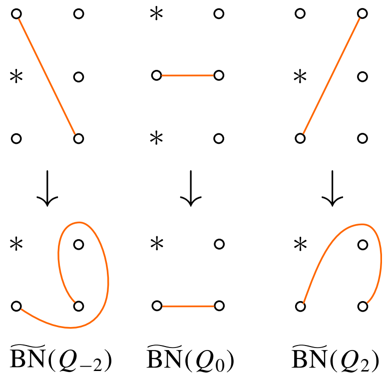


Figure 3. Some immersed curve invariants of Q_n and their lifts to the covering space $\mathbb{R}^2 \setminus \mathbb{Z}^2$.

Seifert push-off, the closure $T_K(0)$ has linking number 0. Thus, by Lemma 2.1, the Bar-Natan homology $\widetilde{\mathrm{BN}}(T_K(0); \mathbb{F}_c)$ has both $\mathbb{F}_c[H]$ towers in grading 0. We may compute this homology using Theorem 2.2:

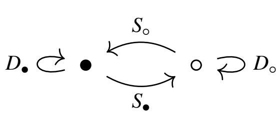
$$\begin{aligned}\widetilde{\mathrm{BN}}(T_K(0))\{-1\} &\cong \mathrm{HF}\Big(\widetilde{\mathrm{BN}}\Big(m\genfrac{}{}{0pt}{}{\smile}{\frown}\Big), \widetilde{\mathrm{BN}}(T_K)\Big)\\ &\cong \mathrm{HF}\Big(\widetilde{\mathrm{BN}}\Big(m\genfrac{}{}{0pt}{}{\smile}{\frown}\Big), \widetilde{\mathrm{BN}}_a(T_K)[h]\{q\}\Big) \oplus \mathrm{Tors}\\ &\cong \widetilde{\mathrm{BN}}(T(2,2n); \mathbb{F}_c)[h]\{q\} \oplus \mathrm{Tors},\end{aligned}$$

where Tors is a torsion $\mathbb{F}_c[H]$-module, $T(2, 2n)$ is the $(2, 2n)$-torus link and $[h]\{q\}$ is a possible bigrading shift. Clearly both towers of $\widetilde{\mathrm{BN}}(T_K(0))$ sit in a summand of the homology that is isomorphic to $\widetilde{\mathrm{BN}}(T(2, 2n))$, up to a grading shift. But the homology of 2-strand torus links is well understood — indeed, we will indicate how to compute it in the next section. In particular, the only way for both towers of $\widetilde{\mathrm{BN}}(T(2, 2n))$ to be in the same homological grading is if $n = 0$. □

4. Epilogue

Let us now indicate how to compute $\widetilde{\mathrm{BN}}(T(2, n); \mathbb{F}_c)$, using a technique that applies more generally and that is the honest source of the proof above. To that end, we will need to look under the hood of Theorem 2.2 and use the bigraded type-D structures $\mathcal{D}(Q_n; \mathbb{F}_c) \in \mathbf{Mod}^{\mathcal{B}}$. First, we will write $\Bbbk$ instead of $\mathbb{F}_c$ in what follows, since the characteristic does not matter and clutters the notation.

Definition 4.1. The Bar-Natan algebra $\mathcal{B}$ is the bigraded path algebra over $\Bbbk$ of the quiver



subject to the relations

$$D_\circ S_\bullet = S_\bullet D_\bullet = 0 \quad \text{and} \quad D_\bullet S_\circ = S_\circ D_\circ = 0,$$

and with bigrading given by

$$q(1_{\bullet}) = 0, \quad q(S_{\bullet}) = -1, \quad q(D_{\bullet}) = -2, \quad h(1_{\bullet}) = h(S_{\bullet}) = h(D_{\bullet}) = 0,$$

where $\bullet \in \{\circ, \bullet\}$.

Remark. Alternatively, consider the quiver above as describing an additive category with two objects and with four nonidentity morphisms indicated, and suppose that the composites DS and SD vanish. Then the algebra $\mathcal{B}$ is the collection of all morphisms of this category, where the algebra operation corresponds to composition of morphisms, and we formally set the composite of noncomposable morphisms to 0. This is a bigraded category in the sense of [Bar-Natan 2005].

Remark. By definition, path algebras have idempotent elements $1_{\bullet}$: the constant paths at each vertex. These correspond to identity morphisms in the categorical perspective. The idempotents generate a subring $\mathcal{I} := \Bbbk\langle 1_\circ, 1_\bullet\rangle \cong \Bbbk^2$, giving $\mathcal{B}$ the additional structure of an $\mathcal{I}$-algebra.

Now a type-D structure over $\mathcal{B}$ is, by definition, an $\mathcal{I}$-module M together with a map $\delta : M \to M \otimes_{\mathcal{I}} \mathcal{B}$ subject to an appropriate "$d^2 = 0$" condition:

$$(\mathrm{Id}_M \otimes m) \circ (\delta \otimes \mathrm{Id}_{\mathcal{B}}) \circ \delta = 0.$$

Notation. Type-D structures are described as labeled directed graphs, with vertices labeled by $\bullet$ or $\circ$, and edges labeled with elements of $\mathcal{B}$. The vertices correspond to homogeneous generators (with respect to the action of $\mathcal{I}$) and the edges are the homogeneous components of the differential δ. To avoid heavy use of brackets, we denote homological and quantum shifts by subscripts and left-superscripts, respectively. For example, ${}^{q}\bullet_{h}$ is a type-D structure generator fixed by $1_\bullet$ and in (homological, quantum)-bigrading (h, q).

The Π-invariants of Q_n are explicitly computed as Example 4.27 of [Kotelskiy et al. 2019] (where Q_n is oriented compatibly with the 0-closure): $\Pi(Q_0) = {}^{0}\bullet_{0}$ and, more generally,

$$\Pi(Q_n; k) = \begin{cases} \underbrace{{}^{3n-1}\circ_{n} \xrightarrow{X} \cdots \xrightarrow{D} \circ \xrightarrow{SS} \circ \xrightarrow{D} \circ \xrightarrow{S} {}^{n}\bullet_{0}}_{-n+1} & \text{if } n < 0, \\ \underbrace{{}^{n}\bullet_{0} \xrightarrow{S} \circ \xrightarrow{D} \circ \xrightarrow{SS} \circ \xrightarrow{D} \cdots \xrightarrow{X} {}^{3n-1}\circ_{n}}_{n+1} & \text{if } n > 0, \end{cases}$$

where the algebra element X is D if n is even and SS if n is odd.

Finally, the following element is defined in $\mathcal{B}$:

$$H := SS_\bullet - D_\bullet + SS_\circ - D_\circ.$$

This gives the Bar-Natan algebra the structure of a $\Bbbk[H]$-algebra, and, by design, this structure is compatible with the $\Bbbk[H]$-module structure of Bar-Natan homology:

Theorem 4.2 [Kotelskiy et al. 2019, Proposition 4.31]. *Let T_1 and T_2 be two pointed oriented 4-ended tangles. Then there is a homotopy*

$$\widetilde{\mathrm{CBN}}(\mathcal{L};\Bbbk)\{-1\} \simeq \mathrm{Mor}(\text{Д}(mT_1;\Bbbk), \text{Д}(T_2;\Bbbk)) \tag{2}$$

of bigraded chain complexes of $\Bbbk[H]$-modules, where m denotes the mirror, and the bifunctor $\mathrm{Mor}(-,-)$ *above is the internal Hom in the category of bigraded type-D structures.*

The type-D structure of $\mathrm{Mor}(\text{Д}_1, \text{Д}_2)$ is defined in [Kotelskiy et al. 2019, §2]. Briefly, $\mathrm{Mor}(\text{Д}_1, \text{Д}_2)$ consists of all morphisms $\text{Д}_1 \to \text{Д}_2$, not just the grading-preserving ones. Given generators $x_i \in \text{Д}_i$ the quantum and homological grading of a morphism is given by

$$\mathrm{gr}(x_1 \xrightarrow{f} x_2) = \mathrm{gr}(x_2) - \mathrm{gr}(x_1) + \mathrm{gr}(f).$$

Finally, a differential D on $\mathrm{Mor}(\text{Д}_1, \text{Д}_2)$ is given on morphisms between generators by pre- and post-composing with the δ_i differentials on Д_i:

$$D(x_1 \xrightarrow{f} x_2) = f \circ \delta_1 - \delta_2 \circ f.$$

For our purposes, note the computations

$$\mathrm{Mor}({}^i\bullet_j, {}^k\circ_l) = \Bbbk[H]\langle {}^i\bullet_j \xrightarrow{S_\bullet} {}^k\circ_l\rangle \cong {}^{k-i-1}(\Bbbk[H])_{l-j},$$
$$\mathrm{Mor}({}^i\bullet_j, {}^k\bullet_l) = \Bbbk[H]\langle {}^i\bullet_j \xrightarrow{1_\bullet} {}^k\bullet_l, {}^i\bullet_j \xrightarrow{D_\bullet} {}^k\bullet_l\rangle \cong {}^{k-i}(\Bbbk[H])_{l-j} \oplus {}^{k-i-2}(\Bbbk[H])_{l-j}.$$

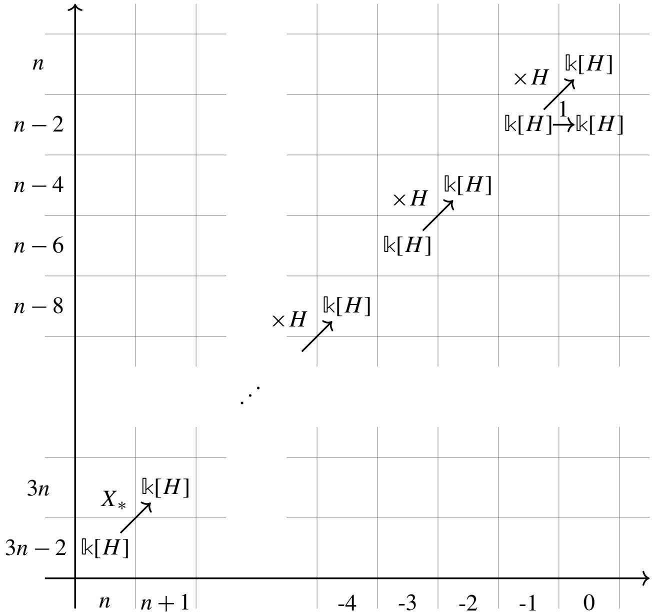
To give the simplest application of Theorem 4.2, the unknot U is $\mathcal{L}(\asymp,)()$. Thus

$$\widetilde{\mathrm{BN}}(U)\{-1\} \cong H_*[\mathrm{Mor}({}^0\bullet_0, {}^0\circ_0)] \cong {}^{-1}(\Bbbk[H])_0.$$

Now we can give a rapid computation of $\widetilde{\mathrm{BN}}(T(2,n)) = \widetilde{\mathrm{BN}}(\mathcal{L}(\asymp, Q_n))$. If $n < 0$, then

$$\begin{aligned}\widetilde{\mathrm{BN}}(T(2,n))\{-1\} &\cong H_*[\mathrm{Mor}({}^0\bullet_0, {}^{3n-1}\circ_n \xrightarrow{X} \circ \xrightarrow{D} \circ \xrightarrow{SS} \cdots \to {}^n\bullet_0)] \\ &\cong H_*\big[\mathrm{Mor}({}^0\bullet_0, {}^{3n-1}\circ_n) \xrightarrow{X_*} \mathrm{Mor}({}^0\bullet_0, \circ) \xrightarrow{D_*} \mathrm{Mor}({}^0\bullet_0, \circ) \\ &\qquad \xrightarrow{SS_*} \cdots \to \mathrm{Mor}({}^0\bullet_0, {}^n\bullet_0)\big],\end{aligned}$$

where the maps above are the ones induced by postcomposing with the components of the differential on $\text{Д}(Q_n)$. It is convenient to organize the above complex in a grid as follows:



Here, the horizontal and vertical axes measure the homological and quantum grading, respectively, and only the nonzero components of the differential are indicated. These components are easy to compute: every morphism group, except for the last one, is generated over $\Bbbk[H]$ by an $S_\bullet$, which D_* takes to 0 and SS_* takes to $SSS = HS$. The last morphism group is generated by $1_\bullet$ and $D_\bullet$ and the incoming differential is $S_\bullet \mapsto S_\circ S_\bullet = H1_\bullet + D_\bullet$.

Taking homology of the above bigraded complex of free $\Bbbk[H]$-modules yields $\widetilde{\mathrm{BN}}(T(2,n);\Bbbk)$. In particular, when n is even, the two towers are in homological grading n and 0, in accordance with Lemma 2.1. The computation for $n \geq 0$ is analogous.

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