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GROMOV–WITTEN THEORY OF HILBERT SCHEMES OF POINTS ON ELLIPTIC SURFACES WITH MULTIPLE FIBERS

MAZEN M. ALHWAIMEL AND ZHENBO QIN

We study the Gromov–Witten theory of Hilbert schemes of points on elliptic surfaces with multiple fibers. We prove a vanishing theorem for the Gromov–Witten invariants of these Hilbert schemes, and compute the exceptional genus-0 case for the Hilbert schemes of two points on elliptic surfaces with exactly one multiple fiber. The strategy is to use the theory of cosection localization and compute a certain obstruction sheaf.

1. Introduction

Hilbert schemes are classical objects in algebraic geometry [11]. It is well-known [7; 14] that the Hilbert schemes of points on a smooth projective surface are smooth and irreducible. The investigation of the Gromov–Witten theory of these Hilbert schemes is important and has been extremely active. It began with the computation of 1-point genus-0 extremal Gromov–Witten invariants in [19]. Motivated by the Gromov–Witten and Donaldson–Thomas correspondence [24; 25], Okounkov and Pandharipande [29] studied the equivariant Gromov–Witten theory of the Hilbert schemes of points in the affine plane. More generally, Maulik and Oblomkov [23] determined the equivariant quantum cohomology of the Hilbert scheme of points on surface resolutions associated to type A_n singularities. For the Hilbert schemes of points on K3 surfaces, Oberdieck [27] considered the reduced Gromov–Witten theory. Via cosection localization [15; 16; 17], the quantum boundary operator and the 2-point genus-0 extremal Gromov–Witten invariants of the Hilbert schemes of points on an arbitrary smooth projective surface are obtained in [18], and the structure of the 3-point genus-0 extremal Gromov–Witten invariants are analyzed in [20; 12]. Moreover, when the surface admits a nontrivial holomorphic differential two-form, a vanishing theory for the Gromov–Witten invariants is proved in [13]. For the Hilbert schemes of points on elliptic surfaces without multiple fibers, the Gromov–Witten invariants are calculated in [1; 28].

MSC2020: 14C05, 14N35.

Keywords: Gromov–Witten invariant, Hilbert scheme, cosection localization, elliptic surface, stable map.

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In this paper, we continue our previous work [1], and study the Gromov–Witten theory of the Hilbert schemes of points on an elliptic surface X with multiple fibers. Let $X^{[n]}$ denote the Hilbert schemes of n points on X. For $\beta \in H_2(X^{[n]}, \mathbb{Z})$, let $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ be the moduli space parametrizing the degree-β r-pointed genus-g stable maps to $X^{[n]}$. For a smooth curve C in X and for fixed distinct points $x_1, \ldots, x_{n-1} \in X$, define the following two curves in $X^{[n]}$:

$$\beta_n = \left\{\xi + x_2 + \cdots + x_{n-1} \in X^{[n]} \mid \mathrm{Supp}(\xi) = \{x_1\}\right\},$$
$$\beta_C = \left\{x + x_1 + \cdots + x_{n-1} \in X^{[n]} \mid x \in C\right\}$$

which, by abuse of notation, also denote their corresponding homology classes.

Our first result generalizes [13, Corollary 3.5] from the case of elliptic surfaces without multiple fibers to the case of elliptic surfaces with multiple fibers.

Theorem 3.3. *Let X be a regular elliptic surface with $p_g \geq 1$ such that every singular fiber is either irreducible reduced or a multiple fiber with smooth reduction. Let f be a smooth fiber in X and $\beta \in H_2(X^{[n]}, \mathbb{Z})$. Then all the Gromov–Witten invariants of $X^{[n]}$ defined via the moduli space $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ vanish except possibly when $g \leq 1$ and $\beta = d_0\beta_f + d\beta_n$ for some integer d and rational number $d_0 \geq 0$.*

When $n = 2$ and the elliptic surface X contains exactly one multiple fiber with smooth reduction, the theorem can be strengthened.

Theorem 4.5. *Let X be a regular elliptic surface with $p_g \geq 1$ and exactly one multiple fiber such that every singular fiber is either irreducible reduced or the unique multiple fiber with smooth reduction F. Let $\beta \in H_2(X^{[2]}, \mathbb{Z})$. Then all the Gromov–Witten invariants of $X^{[2]}$ defined via the moduli space $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, \beta)$ vanish except possibly when $g \leq 1$ and $\beta = d_0\beta_F + d\beta_2$ for some integers d_0 and d satisfying $d_0 \geq 0$ and $d \geq -2d_0$.*

Next, we compute the 1-point genus-0 Gromov–Witten invariants $\langle \alpha \rangle^{X^{[2]}}_{0,d(\beta_F - 2\beta_2)}$ for $d \geq 1$ and $\alpha \in H^4(X^{[2]}, \mathbb{C})$, which are among the exceptional cases in Theorem 4.5. Let

$$\left[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\right]^{\mathrm{vir}}$$

be the virtual fundamental class of the moduli space $\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)$. The computation of the invariants $\langle \alpha \rangle^{X^{[2]}}_{0,d(\beta_F - 2\beta_2)}$ is equivalent to determining the cycle

$$\mathrm{ev}_{1*}\Big(\left[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\right]^{\mathrm{vir}}\Big)$$

where ev_1 is the evaluation map from $\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)$ to $X^{[2]}$.

Theorem 4.8. *Let X be a regular elliptic surface with $p_g \geq 1$ and exactly one multiple fiber such that every singular fiber is either irreducible reduced or the* *unique multiple fiber with smooth reduction F. Let m be the multiplicity of the unique multiple fiber, and $1 \le d < m$. Then,*

$$\mathrm{ev}_{1*}\Big(\big[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\big]^{\mathrm{vir}}\Big) = \frac{1}{d^2} \cdot \big[F^{(2)}\big] \quad \in A_2(X^{[2]}). \tag{1-1}$$

The proof of Theorem 3.3 involves cosection localization and a modification of the proof of [13, Corollary 3.5]. The modification takes care of the presence of the multiple fibers in the elliptic surface X. To prove Theorem 4.5, we analyze the homology classes of curves contained in

$$M_2(F) = \{\xi \in X^{[2]} \mid \mathrm{Supp}(\xi) \text{ is a point in } F\}.$$

When $1 \le d < m$, we show that the images of the stable maps parametrized by $\overline{\mathfrak{M}}_{g,r}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)$ must be contained in the symmetric product $F^{(2)} \subset X^{[2]}$. After expressing $\big[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\big]^{\mathrm{vir}}$ in terms of the Chern class of certain tautological bundle over $\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)$, we verify Theorem 4.8.

To put Theorems 4.5 and 4.8 into perspective, we propose the following regarding the 1-point genus-0 Gromov–Witten invariants and the 0-point genus-1 Gromov–Witten invariants respectively.

Conjecture 1.1. *Keep the notation from Theorem 4.8. Then (1-1) holds for every integer $d \ge 1$.*

Problem 1.2. *Keep the notation from Theorem 4.8. For every integer $d \ge 1$, compute the genus-1 Gromov–Witten invariant $\langle\ \rangle^{X^{[2]}}_{1,d(\beta_F-2\beta_2)}$.*

In order to confirm Conjecture 1.1, one has to understand the homology classes of curves in $(f_s)^{(2)}$ and $M_2(f_s)$ for a singular non-multiple fiber f_s in the elliptic surface X. As for Problem 1.2, the genus-1 invariants $\langle\rangle^{X^{[2]}}_{1,d(\beta_F-2\beta_2)}$ are among the exceptional cases in Theorem 4.5. Partial progress has been made in Remark 4.9. We will leave Conjecture 1.1 and Problem 1.2 to interested readers.

Finally, the paper is organized as follows. In Section 2, we collect basic facts about stable maps, Gromov–Witten invariants, the Hilbert schemes of points on surfaces, and the Heisenberg operators of Grojnowski and Nakajima. We prove Theorem 3.3 in Section 3, and Theorems 4.5 and 4.8 in Section 4.

Conventions. In this paper, an elliptic surface means a smooth projective complex surface which is minimal and admits an elliptic fibration over a smooth curve. For a smooth projective surface X, let K_X be the canonical divisor of X and

$$q = h^1(X, \mathcal{O}_X), \quad p_g = h^2(X, \mathcal{O}_X) = h^0\big(X, \mathcal{O}_X(K_X)\big).$$

2. Preliminaries

In this section, we will recall the standard concepts and notions regarding stable maps, Gromov–Witten invariants, the Hilbert schemes of points on surfaces, and the Heisenberg operators of Grojnowski [10] and Nakajima [26]. We will use these operators to describe the (co)homology groups of the Hilbert schemes and the homology classes of certain special curves in the Hilbert schemes.

2.1. ***Stable maps and Gromov–Witten invariants.*** Let Y be a smooth projective variety. An r-pointed stable map to Y consists of a complete nodal curve D with r distinct ordered smooth points $p_1, \dots, p_r$ and a morphism $\mu : D \to Y$ such that the data $(\mu, D, p_1, \dots, p_r)$ has only finitely many automorphisms. In this case, the stable map is denoted by

$$[\mu : (D; p_1, \dots, p_r) \to Y],$$

or simply by $[\mu : D \to Y]$. For $\beta \in H_2(Y, \mathbb{Z})$, let $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$ be the coarse moduli space parametrizing the stable maps $[\mu : (D; p_1, \dots, p_r) \to Y]$ such that $\mu_*[D] = \beta$ and the arithmetic genus of D is g. Then, we have the i-th evaluation map:

$$\mathrm{ev}_i : \overline{\mathfrak{M}}_{g,r}(Y, \beta) \to Y \tag{2-1}$$

defined by $\mathrm{ev}_i([\mu : (D; p_1, \dots, p_r) \to Y]) = \mu(p_i)$. It is known [21; 22; 5] that the coarse moduli space $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$ is projective and has a virtual fundamental class $[\overline{\mathfrak{M}}_{g,r}(Y, \beta)]^{\mathrm{vir}} \in A_{\mathfrak{d}}(\overline{\mathfrak{M}}_{g,r}(Y, \beta))$, where

$$\mathfrak{d} = -(K_Y \cdot \beta) + (\dim(Y) - 3)(1 - g) + r \tag{2-2}$$

is the expected complex dimension of $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$, and $A_{\mathfrak{d}}(\overline{\mathfrak{M}}_{g,r}(Y, \beta))$ is the Chow group of $\mathfrak{d}$-dimensional cycles in the moduli space $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$.

The Gromov–Witten invariants are defined by using the virtual fundamental class $[\overline{\mathfrak{M}}_{g,r}(Y, \beta)]^{\mathrm{vir}}$. Recall that an element

$$\alpha \in H^*(Y, \mathbb{C}) \stackrel{\mathrm{def}}{=} \bigoplus_{j=0}^{2\dim_{\mathbb{C}}(Y)} H^j(Y, \mathbb{C})$$

is *homogeneous* if $\alpha \in H^j(Y, \mathbb{C})$ for some j; in this case, we take $|\alpha| = j$. Let $\alpha_1, \dots, \alpha_r \in H^*(Y, \mathbb{C})$ such that every α_i is homogeneous and

$$\sum_{i=1}^{r} |\alpha_i| = 2\mathfrak{d}. \tag{2-3}$$

Then, we have the r-point genus-g Gromov–Witten invariant defined by

$$\langle \alpha_1, \ldots, \alpha_r \rangle_{g,\beta}^{Y} = \int_{[\overline{\mathfrak{M}}_{g,r}(Y,\beta)]^{\text{vir}}} \text{ev}_1^*(\alpha_1) \otimes \cdots \otimes \text{ev}_r^*(\alpha_r). \tag{2-4}$$

In particular, when $r = 1$, we see from the projection formula that

$$\langle \alpha \rangle_{g,\beta}^{Y} = \int_{\text{ev}_{1*}([\overline{\mathfrak{M}}_{g,1}(Y,\beta)]^{\text{vir}})} \alpha. \tag{2-5}$$

Next, we recall that *the excess dimension* is the difference between the dimension of $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$ and the expected dimension $\mathfrak{d}$ in (2-2). Let T_Y stand for the tangent sheaf of Y. For $0 \le i < r$, we shall use

$$f_{r,i} : \overline{\mathfrak{M}}_{g,r}(Y, \beta) \to \overline{\mathfrak{M}}_{g,i}(Y, \beta) \tag{2-6}$$

to stand for the forgetful map obtained by forgetting the last $(r - i)$ marked points and contracting all the unstable components. It is known that $f_{r,i}$ is flat when $\beta \ne 0$ and $0 \le i < r$. The following can be found in [9, Proposition 2.5].

Proposition 2.1. *Let $\beta \in H_2(Y, \mathbb{Z})$ and $\beta \ne 0$. Let e be the excess dimension of $\overline{\mathfrak{M}}_{g,r}(Y, \beta)$. If $R^1(f_{r+1,r})_*(\text{ev}_{r+1})^*T_Y$ is a rank-e locally free sheaf, then*

$$[\overline{\mathfrak{M}}_{g,r}(Y, \beta)]^{\text{vir}} = c_e\big(R^1(f_{r+1,r})_*(\text{ev}_{r+1})^*T_Y\big).$$

Finally, the fundamental class axiom of Gromov–Witten theory states that

$$[\overline{\mathfrak{M}}_{g,r}(Y, \beta)]^{\text{vir}} = (f_{r,r-1})^*[\overline{\mathfrak{M}}_{g,r-1}(Y, \beta)]^{\text{vir}} \tag{2-7}$$

if either $r + 2g \ge 4$ or $\beta \ne 0$ and $r \ge 1$. The *Divisor Axiom* states that

$$\langle \alpha_1, \ldots, \alpha_{r-1}, \alpha_r \rangle_{g,\beta}^{Y} = \int_\beta \alpha_r \cdot \langle \alpha_1, \ldots, \alpha_{r-1} \rangle_{g,\beta}^{Y} \tag{2-8}$$

if $\alpha_r \in H^2(Y, \mathbb{C})$, and if either $r + 2g \ge 4$ or $\beta \ne 0$ and $r \ge 1$.

2.2. ***Hilbert schemes of points on surfaces.*** Let X be a smooth projective complex surface, and $X^{[n]}$ be the Hilbert scheme of n-points in X. An element in $X^{[n]}$ is represented by a length-n 0-dimensional closed subscheme ξ of X. For $\xi \in X^{[n]}$, let I_ξ and $\mathcal{O}_\xi$ be the corresponding sheaf of ideals and structure sheaf respectively. It is known from [7; 14] that $X^{[n]}$ is a smooth irreducible variety of dimension $2n$. In fact, the Hilbert-Chow morphism $X^{[n]} \to X^{(n)}$, mapping an element in $X^{[n]}$ to its support in the n-th symmetric product $X^{(n)}$, is a crepant resolution. The universal codimension-2 subscheme is

$$\mathcal{Z}_n = \{(\xi, x) \in X^{[n]} \times X \mid x \in \text{Supp}(\xi)\} \quad \subset X^{[n]} \times X. \tag{2-9}$$

The boundary of $X^{[n]}$ is defined to be the subset

$$B_n = \left\{ \xi \in X^{[n]} \mid |\mathrm{Supp}(\xi)| < n \right\}.$$

Let C be a real-surface in X, and fix distinct points $x_1, \ldots, x_{n-1} \in X$ which are not contained in C. Define the subsets

$$\beta_n = \left\{ \xi + x_2 + \cdots + x_{n-1} \in X^{[n]} \,\middle|\, \mathrm{Supp}(\xi) = \{x_1\} \right\}, \tag{2-10}$$

$$\beta_C = \left\{ x + x_1 + \cdots + x_{n-1} \in X^{[n]} \,\middle|\, x \in C \right\}, \tag{2-11}$$

$$D_C = \left\{ \xi \in X^{[n]} \,\middle|\, \mathrm{Supp}(\xi) \cap C \neq \varnothing \right\}. \tag{2-12}$$

Note that β_C (respectively, D_C) is a curve (respectively, a divisor) in $X^{[n]}$ when C is a smooth algebraic curve in X. We extend the notions β_C and D_C to all the divisors C in X by linearity. For a subset $Y \subset X$, define

$$M_n(Y) = \{\xi \in X^{[n]} \mid \mathrm{Supp}(\xi) \text{ is a point in } Y\}. \tag{2-13}$$

Grojnowski [10] and Nakajima [26] geometrically constructed a Heisenberg algebra action on the cohomology of the Hilbert schemes $X^{[n]}$. Denote the Heisenberg operators by $\mathfrak{a}_m(\alpha)$ where $m \in \mathbb{Z}$ and $\alpha \in H^*(X, \mathbb{C})$. Put

$$\mathbb{H}_X = \bigoplus_{n=0}^{+\infty} H^*(X^{[n]}, \mathbb{C}).$$

The operators $\mathfrak{a}_m(\alpha) \in \mathrm{End}(\mathbb{H}_X)$ satisfy the commutation relation

$$[\mathfrak{a}_m(\alpha), \mathfrak{a}_n(\beta)] = -m \cdot \delta_{m,-n} \cdot \langle \alpha, \beta \rangle \cdot \mathrm{Id}_{\mathbb{H}_X} \tag{2-14}$$

where we have used $\delta_{m,-n}$ to denote 1 if $m = -n$ and 0 otherwise. The space $\mathbb{H}_X$ is an irreducible representation of the Heisenberg algebra generated by the operators $\mathfrak{a}_m(\alpha)$ with the highest weight vector being

$$|0\rangle = 1 \in H^*(X^{[0]}, \mathbb{C}) = \mathbb{C}.$$

Let $\{\alpha_1, \ldots, \alpha_s\}$ be a basis of

$$H^{\mathrm{even}}(X, \mathbb{C}) = H^0(X, \mathbb{C}) \oplus H^2(X, \mathbb{C}) \oplus H^4(X, \mathbb{C})$$

and $\{\alpha_{s+1}, \ldots, \alpha_{s+t}\}$ be a basis of

$$H^{\mathrm{odd}}(X, \mathbb{C}) = H^1(X, \mathbb{C}) \oplus H^3(X, \mathbb{C})$$

where every α_i is homogeneous. Then, a basis of the n-th component $H^*(X^{[n]}, \mathbb{C})$ in $\mathbb{H}_X$ consists of the *Heisenberg monomial classes*

$$\left(\prod_{i=1}^{s+t} \mathfrak{a}_{-n_{i,1}}(\alpha_i) \cdots \mathfrak{a}_{-n_{i,k_i}}(\alpha_i) \right) |0\rangle \tag{2-15}$$

where $k_i \geq 0$ for each i, every $n_{i,j}$ is a positive integer, $\sum_{i,j} n_{i,j} = n$, and the integers $n_{i,1}, \ldots, n_{i,k_i}$ are mutually distinct for every $i \in \{s+1, \ldots, s+t\}$. In particular, a basis of the $H^{4n-2}(X^{[n]}, \mathbb{C})$ in $\mathbb{H}_X$ consists of the classes

$$\mathfrak{a}_{-2}(x)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle,\ \mathfrak{a}_{-1}(\alpha_i)\mathfrak{a}_{-1}(x)^{n-1}|0\rangle,\ \mathfrak{a}_{-1}(\alpha_j)\mathfrak{a}_{-1}(\alpha_k)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle, \tag{2-16}$$

where $|\alpha_i| = 2$, $|\alpha_j| = |\alpha_k| = 3$ with $j < k$, and by abusing notation, we have used x to denote the cohomology class Poincaré dual to a point $x \in X$. Also,

$$\beta_n = \mathfrak{a}_{-2}(x)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle, \tag{2-17}$$

$$\beta_C = \mathfrak{a}_{-1}(C)\mathfrak{a}_{-1}(x)^{n-1}|0\rangle, \tag{2-18}$$

$$B_n = \frac{1}{(n-2)!}\mathfrak{a}_{-1}(1_X)^{n-2}\mathfrak{a}_{-2}(1_X)|0\rangle, \tag{2-19}$$

$$D_C = \frac{1}{(n-1)!}\mathfrak{a}_{-1}(1_X)^{n-1}\mathfrak{a}_{-1}(C)|0\rangle \tag{2-20}$$

where 1_X denotes the fundamental cohomology class of X, and for simplicity, we do not distinguish a homology class and its Poincaré dual.

Definition 2.2. Let $n \geq 2$. We define $\widetilde{H}_2(X^{[n]}, \mathbb{C})$ to be the linear subspace of $H_2(X^{[n]}, \mathbb{C})$ spanned by the Poincaré duals of the basis classes

$$\mathfrak{a}_{-1}(\alpha_j)\mathfrak{a}_{-1}(\alpha_k)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle$$

in (2-16), where $|\alpha_j| = |\alpha_k| = 3$ with $j < k$.

Next, we recall the homology class of an irreducible curve in the Hilbert scheme $X^{[n]}$. Let Γ be an irreducible curve in $X^{[n]}$. Define

$$\mathcal{Z}_\Gamma = \Gamma \times_{X^{[n]}} \mathcal{Z}_n. \tag{2-21}$$

Then, $\mathcal{Z}_\Gamma \subset \mathcal{Z}_n \subset X^{[n]} \times X$. By [13, Lemma 5.1] and its proof there, we have the following (see also [27, Lemma 1] and [30, Lemma 3.19]).

Lemma 2.3. *Let $n \geq 2$. Let Γ be an irreducible curve in $X^{[n]}$. Then,*

$$\Gamma \equiv \beta_{\pi_{2*}[\mathcal{Z}_\Gamma]} + d\beta_n \pmod{\widetilde{H}_2(X^{[n]}, \mathbb{C})} \tag{2-22}$$

for some integer d, where π_2 is the second projection of $X^{[n]} \times X$.

Finally, let C be a smooth irreducible curve in X with genus g_C. Let $C^{(n)}$ denote the n-th symmetric product of C. We regard $C^{(n)} \subset X^{[n]}$ whenever necessary. For a fixed point $p \in C$, let Ξ denote the divisor $p + C^{(n-1)} \subset C^{(n)}$. Let

$$\mathrm{AJ} : C^{(n)} \to \mathrm{Jac}_n(C)$$

be the Abel–Jacobi map sending $\xi \in C^{(n)}$ to the corresponding degree-n divisor class in $\mathrm{Jac}_n(C)$. For an element $\delta \in \mathrm{Jac}_n(C)$, the fiber $\mathrm{AJ}^{-1}(\delta)$ is the complete line system $|\delta|$. Let Θ be the pullback via AJ of a theta divisor on $\mathrm{Jac}_n(C)$. It is well-known that theta divisors on $\mathrm{Jac}_n(C)$ are ample.

Lemma 2.4 [30, Lemma 3.20]. *Let $n \geq 2$, and $\widetilde{H}_2(X^{[n]}, \mathbb{C})$ be from Definition 2.2. Let C be a smooth curve in X, and $\Gamma \subset C^{(n)}$ be a curve. Then,*

$$\Gamma \equiv (\Xi \cdot \Gamma)\beta_C + \big(-(n+g_C-1)(\Xi \cdot \Gamma) + (\Theta \cdot \Gamma)\big)\beta_n \pmod{\widetilde{H}_2(X^{[n]}, \mathbb{C})}. \tag{2-23}$$

In addition, for every line Γ_0 in a positive-dimensional fiber $\mathrm{AJ}^{-1}(\delta)$, we have

$$\Gamma_0 \equiv \beta_C - (n+g_C-1)\beta_n \pmod{\widetilde{H}_2(X^{[n]}, \mathbb{C})}. \tag{2-24}$$

3. The vanishing of certain Gromov–Witten invariants of the Hilbert schemes of points on elliptic surfaces with multiple fibers

The goal of this section is to show that certain Gromov–Witten invariants of the Hilbert schemes of points on elliptic surfaces with multiple fibers are equal to 0. Theorem 3.3 below generalizes [13, Corollary 3.5] to the case when the surface is an elliptic surface with multiple fibers. We prove our Theorem 3.3 by modifying the cosection localization method in the proof of [13, Theorem 3.3].

Let X be a (minimal) elliptic surface. By [8, Corollary 7.5 on p. 113], up to deformation, we may assume that every singular fiber of X is either an irreducible reduced rational curve with one node or a multiple fiber with smooth reduction. If $p_g = h^0\big(X, \mathcal{O}_X(K_X)\big) \geq 1$, then by [17, Proposition 6.1 and Remark 6.2], up to deformation, we may further assume that $|K_X|$ contains a member of the form

$$\sum_{i=1}^{s} f_i + \sum_{j=1}^{t} (m_j - 1)F_j \tag{3-1}$$

where $f_1, \dots, f_s$ are distinct smooth fibers, and $F_1, \dots, F_t$ are distinct smooth multiple fibers with multiplicities $m_1, \dots, m_t$ respectively. Therefore, we fix the following assumption throughout this section unless otherwise specified.

Assumption 3.1. X is an elliptic surface with $p_g \geq 1$ and with the elliptic fibration $\pi : X \to C$ over a smooth curve C such that

(i) every singular fiber of π is either irreducible reduced or a multiple fiber with smooth reduction;

(ii) $H^0(X, \Omega_X^2) = H^0(X, \mathcal{O}_X(K_X))$ contains a holomorphic differential two-form θ whose zero-set is of the form (3-1).

By the results of Beauville [3; 4], the holomorphic differential two-form θ induces a holomorphic two-form $\theta^{[n]}$ of the Hilbert scheme $X^{[n]}$ which can also be regarded as a map $\theta^{[n]} : T_{X^{[n]}} \to \Omega_{X^{[n]}}$. For simplicity, put

$$\overline{\mathfrak{M}} = \overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta).$$

Define the degeneracy locus $\overline{\mathfrak{M}}(\theta)$ to be the subset of $\overline{\mathfrak{M}}$ consisting of all the stable maps $u : \Gamma \to X^{[n]}$ such that the composite

$$u^*(\theta^{[n]}) \circ du : \quad T_{\Gamma_{\mathrm{reg}}} \to u^* T_{X^{[n]}}|_{\Gamma_{\mathrm{reg}}} \to u^* \Omega_{X^{[n]}}|_{\Gamma_{\mathrm{reg}}} \tag{3-2}$$

is trivial over the regular locus Γ_{reg} of Γ. By the results of Kiem–Li [15; 16], $\theta^{[n]}$ defines a regular cosection of the obstruction sheaf of $\overline{\mathfrak{M}}$:

$$\eta : \mathcal{O}b_{\overline{\mathfrak{M}}} \longrightarrow \mathcal{O}_{\overline{\mathfrak{M}}} \tag{3-3}$$

where $\mathcal{O}b_{\overline{\mathfrak{M}}}$ is the obstruction sheaf and $\mathcal{O}_{\overline{\mathfrak{M}}}$ is the structure sheaf of $\overline{\mathfrak{M}}$. Moreover, the cosection η is surjective away from the degeneracy locus $\overline{\mathfrak{M}}(\theta)$, and there exists a localized virtual cycle $[\overline{\mathfrak{M}}]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*(\overline{\mathfrak{M}}(\theta))$ such that

$$[\overline{\mathfrak{M}}]^{\mathrm{vir}} = \iota_* [\overline{\mathfrak{M}}]^{\mathrm{vir}}_{\mathrm{loc}} \in A_*(\overline{\mathfrak{M}}) \tag{3-4}$$

where $\iota : \overline{\mathfrak{M}}(\theta) \to \overline{\mathfrak{M}}$ stands for the inclusion map.

Lemma 3.2. *Let X be an elliptic surface satisfying Assumption 3.1. Let f be a smooth fiber in X, and let $\widetilde{H}_2(X^{[n]}, \mathbb{C}) \subset H_2(X^{[n]}, \mathbb{C})$ be from Definition 2.2. If the subset $\overline{\mathfrak{M}}(\theta)$ of $\overline{\mathfrak{M}} = \overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ is nonempty, then*

$$\beta \equiv d_0 \beta_f + d\beta_n \pmod{\widetilde{H}_2(X^{[n]}, \mathbb{C})} \tag{3-5}$$

for some integer d and some rational number $d_0 \geq 0$.

Proof. Let $u : \Gamma \to X^{[n]}$ be a stable map in $\overline{\mathfrak{M}}(\theta)$, and let Γ_0 be any irreducible component of Γ. By Assumption 3.1(ii), the zero-set of θ is supported on

$$\Big(\bigcup_{i=1}^{s} f_i\Big) \cup \Big(\bigcup_{j=1}^{t} F_j\Big).$$

By [13, Lemma 3.1], there exists $\xi_1 \in X^{[n_0]}$ for some n_0 such that

$$\mathrm{Supp}(\xi_1) \cap \bigg(\Big(\bigcup_{i=1}^{s} f_i\Big) \cup \Big(\bigcup_{j=1}^{t} F_j\Big)\bigg) = \varnothing,$$

$$u(\Gamma_0) \subset \xi_1 + \bigg\{\xi_2 \mid \mathrm{Supp}(\xi_2) \subset \Big(\bigcup_{i=1}^{s} f_i\Big) \cup \Big(\bigcup_{j=1}^{t} F_j\Big)\bigg\}. \tag{3-6}$$

By Lemma 2.3, there exist integers $a_{f_i} \geq 0$, $a_{F_j} \geq 0$ and d' such that

$$u(\Gamma_0) \equiv \sum_{i=1}^{s} a_{f_i} \beta_{f_i} + \sum_{j=1}^{t} a_{F_j} \beta_{F_j} + d'\beta_n \pmod{\widetilde{H}_2(X^{[n]}, \mathbb{C})}. \tag{3-7}$$

Since $f = f_i = m_j F_j$ as divisors, we conclude that

$$u(\Gamma_0) \equiv d_0' \beta_f + d'\beta_n \pmod{\widetilde{H}_2(X^{[n]}, \mathbb{C})} \tag{3-8}$$

for some rational number $d_0' \geq 0$. Since $\beta = u_*[\Gamma] = \sum_{\Gamma_0 \subset \Gamma} u_*[\Gamma_0]$, our lemma follows from (3-8). □

Theorem 3.3. *Let X be a regular elliptic surface with $p_g \geq 1$ such that every singular fiber is either irreducible reduced or a multiple fiber with smooth reduction. Let f be a smooth fiber in X and $\beta \in H_2(X^{[n]}, \mathbb{Z})$. Then all the Gromov–Witten invariants of $X^{[n]}$ defined via the moduli space $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ vanish except possibly when $g \leq 1$ and $\beta = d_0\beta_f + d\beta_n$ for some integer d and rational number $d_0 \geq 0$.*

Proof. Since X is regular, the elliptic fiberation is of the form $\pi : X \to \mathbb{P}^1$. So Assumption 3.1(ii) holds. Again, since X is regular, $\widetilde{H}_2(X^{[n]}, \mathbb{C}) = 0$ by Definition 2.2. Moreover, by (2-15), all odd cohomology groups of $X^{[n]}$ vanish.

If $\beta \neq d_0\beta_f + d\beta_n$ for some integer d and rational number $d_0 \geq 0$, then we see from Lemma 3.2 that $\overline{\mathfrak{M}}(\theta) = \varnothing$ and $[\overline{\mathfrak{M}}]^{\text{vir}}_{\text{loc}} = 0$. By (3-4), $[\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)]^{\text{vir}} = 0$ and all the Gromov–Witten invariants defined via $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ vanish.

Next, assume that $g > 1$ and $\beta = d_0\beta_f + d\beta_n$ for some integer d and rational number $d_0 \geq 0$. Since $K_{X^{[n]}} = D_{K_X}$, we see from (2-2) that the expected dimension of $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ is equal to $(2n-3)(1-g)+r < r$. By (2-3) and the fundamental class axiom (2-7), we conclude that all the Gromov–Witten invariants of $X^{[n]}$ defined via the moduli space $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ vanish. □

Remark 3.4. For an arbitrary smooth projective surface X, the case $g = 0$ and $\beta = d\beta_n$ is studied in [20], and the case $g = 1$ and $\beta = d\beta_n$ is discussed in [12].

If the regular elliptic surface X in Theorem 3.3 has exactly one multiple fiber, then we can prove that the rational number d_0 in Theorem 3.3 is an integer.

Corollary 3.5. *Let X be a regular elliptic surface with $p_g \geq 1$ and exactly one multiple fiber such that every singular fiber is either irreducible reduced or the unique multiple fiber with smooth reduction F. Let $\beta \in H_2(X^{[n]}, \mathbb{Z})$. Then all the Gromov–Witten invariants of $X^{[n]}$ defined via the moduli space $\overline{\mathfrak{M}}_{g,r}(X^{[n]}, \beta)$ vanish except possibly when $g \leq 1$ and $\beta = d_0\beta_F + d\beta_n$ for some integers d and $d_0 \geq 0$.*

Proof. By [8, Corollary 2.3 on p.158], X must be simply connected. Let m denote the multiplicity of the unique multiple fiber F. Then $f_i = mF$ in (3-7), and (3-8) becomes $u(\Gamma_0) = d_0'\beta_F + d'\beta_n$ for some integers $d_0' \geq 0$ and d'. Accordingly, (3-5) becomes $\beta = d_0\beta_F + d\beta_n$ for some integers d and $d_0 \geq 0$. Now the same proof of Theorem 3.3 yields our corollary. □

4. The exceptional cases for $X^{[2]}$

In this section, we will investigate the exceptional cases in Corollary 3.5 when $n = 2$. First of all, we strengthen Corollary 3.5 by proving that the integers d and

$d_0 \geq 0$ in Corollary 3.5 must satisfy $d \geq -2d_0$. Then, we compute the exceptional 1-point genus-0 Gromov–Witten invariants

$$\langle \alpha \rangle_{0,d(\beta_F - 2\beta_2)}^{X^{[2]}} \tag{4-1}$$

where $\alpha \in H^4(X^{[2]}, \mathbb{C})$, F is the reduction of the unique multiple fiber in X, and $1 \leq d < m$ with m being the multiplicity of the unique multiple fiber. By (2-5), the computation of (4-1) is equivalent to determining the cycle

$$\mathrm{ev}_{1*}\big(\big[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\big]^{\mathrm{vir}}\big) \in A_2(X^{[2]}).$$

4.1. ***The homology classes of certain curves in $X^{[2]}$.*** In this subsection, X stands for an arbitrary elliptic surface. We will investigate the homology classes of certain curves in $X^{[2]}$ related to a smooth multiple fiber f in the elliptic surface X. These include curves in $M_2(f)$ (see (2-13) for the notation) and the fibers of the Abel–Jacobi map $\mathrm{AJ} : f^{(2)} \to \mathrm{Jac}_2(f) \cong f$. The results here generalize [1, Lemma 4.1 and Lemma 4.3].

Lemma 4.1. *Let X be an elliptic surface with f being a smooth fiber or the reduction of a smooth multiple fiber, and $\widetilde{H}_2(X^{[2]}, \mathbb{C}) \subset H_2(X^{[2]}, \mathbb{C})$ be from Definition 2.2. Let $\Gamma \subset M_2(f)$ be an irreducible curve in $X^{[2]}$. Then, there exist nonnegative integers d and d_0 not both zero such that*

$$\Gamma \equiv 2d\beta_f + d_0\beta_2 \pmod{\widetilde{H}_2(X^{[2]}, \mathbb{C})}. \tag{4-2}$$

Proof. When f is a smooth fiber of X, this is [1, Lemma 4.1]. In the following, assume that f is the reduction of a smooth multiple fiber with multiplicity m. We will modify the proof of [1, Lemma 4.1]. Note that

$$B_2 = M_2(X) \cong \mathbb{P}(T_X^\vee).$$

For convenience, we simply write $B_2 = M_2(X) = \mathbb{P}(T_X^\vee)$. Then, we have

$$M_2(f) = \mathbb{P}\big((T_X|_f)^\vee\big) \cong \mathbb{P}\big((T_X|_f)^\vee \otimes \mathcal{O}_f(f)\big).$$

From $0 \to \mathcal{O}_f \to T_X|_f \to \mathcal{O}_f(f) \to 0$, we obtain the exact sequence

$$0 \to \mathcal{O}_f \to (T_X|_f)^\vee \otimes \mathcal{O}_f(f) \to \mathcal{O}_f(f) \to 0,$$

where $\mathcal{O}_f(f)$ is a torsion of order m. Therefore, we conclude that

$$\Gamma = d'\tilde{\sigma} + d_0' f' \quad \in H_2(M_2(f), \mathbb{C}), \tag{4-3}$$

where f' is a fiber of the ruling $M_2(f) \to f$, $\tilde{\sigma}$ is a section to the ruling with

$$\tilde{\sigma}^2 = \deg\big((T_X|_f)^\vee \otimes \mathcal{O}_f(f)\big) = \deg \mathcal{O}_f(f) = 0, \tag{4-4}$$

and d' and d_0' are nonnegative integers. In addition, we have

$$\mathcal{O}_{M_2(f)}(\tilde{\sigma}) = \mathcal{O}_{M_2(f)}(1) \otimes \phi^* \mathcal{O}_f(f), \tag{4-5}$$

where $\phi : M_2(f) \to f$ denotes the ruling on $M_2(f)$.

Since $f' = \beta_2 \in H_2(X^{[2]}, \mathbb{C})$, we see from (4-3) that it remains to show

$$\tilde{\sigma} \equiv 2\beta_f \pmod{\widetilde{H}_2(X^{[2]}, \mathbb{C})}. \tag{4-6}$$

Indeed, by Lemma 2.3, we have

$$\tilde{\sigma} \equiv 2\beta_f + d_2\beta_2 \pmod{\widetilde{H}_2(X^{[2]}, \mathbb{C})} \tag{4-7}$$

for some integer d_2. Since $\mathcal{O}_{B_2}(B_2) = \mathcal{O}_{B_2}(-2)$ and $\mathcal{O}_f(f)$ is a torsion,

$$B_2 \cdot \tilde{\sigma} = B_2|_{M_2(f)} \cdot \tilde{\sigma} = \mathcal{O}_{M_2(f)}(-2) \cdot \tilde{\sigma} = -2\tilde{\sigma} \cdot \tilde{\sigma} = 0$$

where we have used (4-5) and (4-4) in the last two steps. On the other hand, since

$$B_2 \cdot \beta_f = B_2 \cdot w = 0$$

for every class $w \in \widetilde{H}_2(X^{[2]}, \mathbb{C})$ and $B_2 \cdot \beta_2 = -2$, we obtain $B_2 \cdot \tilde{\sigma} = -2d_2$ from (4-7). Therefore, we have $d_2 = 0$ and (4-6) follows from (4-7). □

Let f be a smooth fiber in the elliptic surface X. Since f is an elliptic curve, we see that the Abel–Jacobi map

$$\mathrm{AJ} : f^{(2)} \to \mathrm{Jac}_2(f) \cong f$$

exhibits $f^{(2)}$ as a ruled surface over f. The fiber $\mathrm{AJ}^{-1}(\delta)$ over an element $\delta \in \mathrm{Jac}_2(f)$ is the complete linear system $|\delta| \cong \mathbb{P}^1$. The following is [1, Lemma 4.3].

Lemma 4.2. *Let X be an elliptic surface with f being a smooth fiber. Let $\Gamma = \mathrm{AJ}^{-1}(\delta)$ be a fiber of the ruling $\mathrm{AJ} : f^{(2)} \to \mathrm{Jac}_2(f)$. Regard Γ as a curve in $X^{[2]}$ via $\Gamma \subset f^{(2)} \subset X^{[2]}$. Let $N_{\Gamma \subset X^{[2]}}$ be the normal bundle of Γ in $X^{[2]}$. Then,*

(i) $N_{\Gamma \subset X^{[2]}} = \mathcal{O}_\Gamma(-2) \oplus \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma$;

(ii) $T_{X^{[2]}}|_\Gamma = \mathcal{O}_\Gamma(2) \oplus \mathcal{O}_\Gamma(-2) \oplus \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma$;

(iii) $\Gamma = \beta_f - 2\beta_2 \in H_2(X^{[2]}, \mathbb{C})$.

Our next goal is to prove an analogue of Lemma 4.2 when the smooth fiber f in Lemma 4.2 is replaced by the reduction of a smooth multiple fiber of the elliptic surface X. We begin with an elementary lemma.

Lemma 4.3. *Let f be a smooth elliptic curve and $\phi : f \to \mathbb{P}^1$ be a double cover. If L is a non-trivial degree-0 invertible sheaf over f, then*

$$\phi_* L = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1).$$

Proof. Note that both $\phi_*\mathcal{O}_f$ and ϕ_*L are rank-2 locally free sheaf over $\mathbb{P}^1$. By the Grothendieck–Riemann–Roch Theorem, $\deg\phi_*L = \deg\phi_*\mathcal{O}_f$. Since ϕ is branched over exactly 4 points in $\mathbb{P}^1$, we obtain

$$\phi_*\mathcal{O}_f = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}. \tag{4-8}$$

It follows that $\deg\phi_*L = \deg\phi_*\mathcal{O}_f = -2$. The rank-2 locally free sheaf ϕ_*L is the direct sum of two invertible sheaves on $\mathbb{P}^1$. Since $\deg\phi_*L = -2$ and $H^0(\mathbb{P}^1, \phi_*L) \cong H^0(f, L) = 0$, we must have $\phi_*L = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. ☐

Lemma 4.4. *Let X be an elliptic surface and f be the reduction of a smooth multiple fiber in X. Let $\Gamma = \mathrm{AJ}^{-1}(\delta)$ be a fiber of the ruling $\mathrm{AJ} : f^{(2)} \to \mathrm{Jac}_2(f)$. Regard Γ as a curve in $X^{[2]}$ via $\Gamma \subset f^{(2)} \subset X^{[2]}$. Then,*

(i) $N_{\Gamma\subset X^{[2]}} = \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma$;

(ii) $T_{X^{[2]}}|_\Gamma = \mathcal{O}_\Gamma(2) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma$;

(iii) $\Gamma = \beta_f - 2\beta_2 \in H_2(X^{[2]}, \mathbb{C})$.

Proof. (i) We prove first that $N_{f^{(2)}\subset X^{[2]}}|_\Gamma = \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1)$. Recall the universal subscheme $\mathcal{Z}_2 \subset X^{[2]} \times X$ from (2-9). Let $\pi_1 : \mathcal{Z}_2 \to X^{[2]}$ and $\pi_2 : \mathcal{Z}_2 \to X$ be the natural projections. It is known from [2] that

$$N_{f^{(2)}\subset X^{[2]}} = \pi_{1*}\pi_2^*\mathcal{O}_X(f)|_{f^{(2)}}. \tag{4-9}$$

Let $\mathcal{Z}_\Gamma = \pi_1^{-1}(\Gamma) \subset \mathcal{Z}_2$. Note that $\pi_2(\mathcal{Z}_\Gamma) = f$. Put $\tilde{\pi}_1 = \pi_1|_{\mathcal{Z}_\Gamma} : \mathcal{Z}_\Gamma \to \Gamma$ and $\tilde{\pi}_2 = \pi_2|_{\mathcal{Z}_\Gamma} : \mathcal{Z}_\Gamma \to f$. Then, $\tilde{\pi}_2$ is an isomorphism. Up to an isomorphism, $\tilde{\pi}_1$ is the double cover $f \to \mathbb{P}^1$ corresponding to the complete linear system $|\delta|$. Since $\mathcal{O}_X(f)|_f$ is a non-trivial torsion, we see from Lemma 4.3 that

$$\begin{aligned} N_{f^{(2)}\subset X^{[2]}}|_\Gamma &= \pi_{1*}\pi_2^*\mathcal{O}_X(f)|_\Gamma \\ &= \tilde{\pi}_{1*}\tilde{\pi}_2^*\big(\mathcal{O}_X(f)|_f\big) = \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1). \end{aligned} \tag{4-10}$$

Next, the normal bundle of Γ in $f^{(2)}$ is $N_{\Gamma\subset f^{(2)}} = \mathcal{O}_\Gamma$. So the exact sequence

$$0 \to T_\Gamma \to T_{f^{(2)}}|_\Gamma \to N_{\Gamma\subset f^{(2)}} \to 0$$

becomes $0 \to \mathcal{O}_\Gamma(2) \to T_{f^{(2)}}|_\Gamma \to \mathcal{O}_\Gamma \to 0$ which splits. Thus,

$$T_{f^{(2)}}|_\Gamma = \mathcal{O}_\Gamma(2) \oplus \mathcal{O}_\Gamma. \tag{4-11}$$

Similarly, the exact sequence

$$0 \to N_{\Gamma\subset f^{(2)}} \to N_{\Gamma\subset X^{[2]}} \to N_{f^{(2)}\subset X^{[2]}}|_\Gamma \to 0$$

becomes $0 \to \mathcal{O}_\Gamma \to N_{\Gamma\subset X^{[2]}} \to \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1) \to 0$, which splits. It follows that $N_{\Gamma\subset X^{[2]}} = \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma$.

(ii) From $T_\Gamma = \mathcal{O}_\Gamma(2)$, the exact sequence

$$0 \to T_\Gamma \to T_{X^{[2]}}|_\Gamma \to N_{\Gamma\subset X^{[2]}} \to 0$$

and $N_{\Gamma\subset X^{[2]}} = \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma$, we obtain

$$T_{X^{[2]}}|_\Gamma = \mathcal{O}_\Gamma(2) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma.$$

(iii) This follows from the same proof (given in [1]) of Lemma 4.2(iii). □

4.2. *Calculation of the* 1*-point Gromov–Witten invariants* $\langle\alpha\rangle^{X^{[2]}}_{0,d(\beta_F-2\beta_2)}$. In this subsection, our surface X is from Corollary 3.5, i.e., X is a regular elliptic surface with $p_g \ge 1$ and exactly one multiple fiber such that every singular fiber is either irreducible reduced or the unique multiple fiber with smooth reduction F. We will compute the Gromov–Witten invariants $\langle\alpha\rangle^{X^{[2]}}_{0,d(\beta_F-2\beta_2)}$ when $1 \le d < m$ where m is the multiplicity of the unique multiple fiber in the elliptic surface X. These invariants belong to the exceptional cases in Corollary 3.5.

We begin with a theorem which strengthens Corollary 3.5 in the case $n = 2$.

Theorem 4.5. *Let X be a regular elliptic surface with $p_g \ge 1$ and exactly one multiple fiber such that every singular fiber is either irreducible reduced or the unique multiple fiber with smooth reduction F. Let $\beta \in H_2(X^{[2]}, \mathbb{Z})$. Then all the Gromov–Witten invariants of $X^{[2]}$ defined via the moduli space $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, \beta)$ vanish except possibly when $g \le 1$ and $\beta = d_0\beta_F + d\beta_2$ for some integers d_0 and d satisfying $d_0 \ge 0$ and $d \ge -2d_0$.*

Proof. Recall from the proofs of Theorem 3.3 and Corollary 3.5 that the elliptic fibration is of the form $\pi : X \to \mathbb{P}^1$ and X is simply connected. Let m be the multiplicity of the unique multiple fiber in X. Fix a holomorphic differential two-form θ in $H^0(X, \Omega^2_X) = H^0(X, \mathcal{O}_X(K_X))$ such that the zero-set of θ is equal to

$$\sum_{i=1}^{p_g-1} f_i + (m-1)F, \tag{4-12}$$

where $f_1, \dots, f_{p_g-1}$ are distinct smooth fibers of π. Then Assumption 3.1 is satisfied. In view of the proof of Corollary 3.5, it remains to prove that if the subset $\overline{\mathfrak{M}}(\theta)$ of $\overline{\mathfrak{M}} = \overline{\mathfrak{M}}_{g,r}(X^{[2]}, \beta)$ is nonempty, then

$$\beta = d_0\beta_F + d\beta_2 \tag{4-13}$$

for some integers d_0 and d satisfying $d_0 \ge 0$ and $d \ge -2d_0$.

Note that (4-13) is an improvement of Lemma 3.2. We will modify the proof of Lemma 3.2 by adopting the notation from there. Let $u : \Gamma \to X^{[2]}$ be a stable map in $\overline{\mathfrak{M}}(\theta) \subset \overline{\mathfrak{M}} = \overline{\mathfrak{M}}_{g,r}(X^{[2]}, \beta)$, and let Γ_0 denote any irreducible component of Γ. By (3-6), we have the following six cases.

Case 1: $u(\Gamma_0)$ is a point in $X^{[2]}$. In this case, we get

$$[u(\Gamma_0)] = 0. \tag{4-14}$$

Case 2: $u(\Gamma_0) = x + f_i$ for some $1 \le i \le p_g - 1$ and some fixed point $x \in X$ not lying in any f_i $(1 \le i \le p_g - 1)$ or in F. In this case, we obtain

$$[u(\Gamma_0)] = \beta_{f_i} = m\beta_F. \tag{4-15}$$

Case 3: $u(\Gamma_0) = x + F$ for some fixed point $x \in X - \big((\bigcup_{i=1}^{p_g-1} f_i) \cup F\big)$. In this case,

$$[u(\Gamma_0)] = \beta_F. \tag{4-16}$$

Case 4: $u(\Gamma_0) \subset M_2(f_i) \cup (f_i)^{(2)}$ for some $1 \le i \le p_g - 1$. If $u(\Gamma_0) \subset M_2(f_i)$, then

$$[u(\Gamma_0)] = d_i\beta_{f_i} + \tilde{d}_i\beta_2 = d_i m\beta_F + \tilde{d}_i\beta_2 \tag{4-17}$$

by Lemma 4.1, where d_i and $\tilde{d}_i$ are nonnegative integers not both zero. If $u(\Gamma_0) \subset (f_i)^{(2)}$, then we see from (2-23) that

$$[u(\Gamma_0)] = d_i'\beta_{f_i} + (-2d_i' + \Theta \cdot u(\Gamma_0))\beta_2 = d_i' m\beta_F + (-2d_i' + \Theta \cdot u(\Gamma_0))\beta_2, \tag{4-18}$$

where Θ is the pullback of a theta divisor via $\mathrm{AJ} : (f_i)^{(2)} \to \mathrm{Jac}_2(f_i)$ and d_i' is a nonnegative integer.

Case 5: $u(\Gamma_0) \subset M_2(F) \cup F^{(2)}$. If $u(\Gamma_0) \subset M_2(F)$, then by Lemma 4.1 again,

$$[u(\Gamma_0)] = d_F\beta_F + \tilde{d}_F\beta_2 \tag{4-19}$$

where d_F and $\tilde{d}_F$ are nonnegative integers not both zero. If $u(\Gamma_0) \subset F^{(2)}$, then

$$[u(\Gamma_0)] = d_F'\beta_F + (-2d_F' + \Theta \cdot u(\Gamma_0))\beta_2 \tag{4-20}$$

where Θ is the pullback of a theta divisor via $\mathrm{AJ} : F^{(2)} \to \mathrm{Jac}_2(F)$ and d_F' is a nonnegative integer.

Case 6: $u(\Gamma_0) \subset f_i + f_j$ or $u(\Gamma_0) \subset f_i + F$ for some $1 \le i \ne j \le p_g - 1$. In this case, $u(\Gamma_0) \cdot B_2 = 0$ since $u(\Gamma_0) \cap B_2 = \varnothing$. By (2-22),

$$[u(\Gamma_0)] = \beta_{\pi_{2*}[\mathcal{Z}_\Gamma]} = \tilde{d}_i'\beta_{f_i} + \tilde{d}_F'\beta_F = (\tilde{d}_i' m + \tilde{d}_F')\beta_F \tag{4-21}$$

for some nonnegative integers $\tilde{d}_i'$ and $\tilde{d}_F'$ not both zero.

Finally, since $\beta = u_*[\Gamma] = \sum_{\Gamma_0 \subset \Gamma} u_*[\Gamma_0]$, (4-13) follows from (4-14)–(4-21). □

For $g \le 1$ and $\beta = d_0\beta_F + d\beta_2$ where the integers d_0 and d satisfy $d_0 \ge 0$ and $d \ge -2d_0$, the expected dimension of $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, \beta)$ is equal to $1 - g + r$. By the Divisor Axiom (2-8), the exceptional cases in Theorem 4.5 for $X^{[2]}$ can be reduced to the computation of the following two types of invariants: (i) $\langle \alpha \rangle_{0,\beta}^{X^{[2]}}$, with $\alpha \in H^4(X^{[2]}, \mathbb{C})$, and (ii) $\langle\ \rangle_{1,\beta}^{X^{[2]}}$.

Our next goal is to calculate the 1-point genus-0 Gromov–Witten invariants

$$\langle \alpha \rangle_{0,d(\beta_F - 2\beta_2)}^{X^{[2]}} \tag{4-22}$$

with $\alpha \in H^4(X^{[2]}, \mathbb{C})$ and $1 \le d < m$ where m is the multiplicity of the unique multiple fiber in X. By (2-5), this is equivalent to determining

$$\mathrm{ev}_{1*}\big([\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\mathrm{vir}}\big).$$

The lemma below deals with the stable maps in $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$.

Lemma 4.6. *Let m be the multiplicity of the unique multiple fiber in X. Let*

$$[\mu : D \to X^{[2]}] \in \overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$$

with $1 \le d < m$. Then, $\mu(D)$ is a fiber of the ruling $\mathrm{AJ} : F^{(2)} \to \mathrm{Jac}_2(F)$, and the degree of the morphism $\mu : D \to \mu(D)$ is equal to d.

Proof. Let $\Gamma_1, \dots, \Gamma_t$ be the irreducible components of $\mu(D)$. Recall that X is simply connected. By Lemma 2.3, re-ordering $\Gamma_1, \dots, \Gamma_t$ if necessary, there exists some t_0 with $0 \le t_0 \le t$ such that for $1 \le i \le t_0$, $\Gamma_i = \beta_{\pi_{2*}[\mathcal{Z}_{\Gamma_i}]} + \tilde{d}_i \beta_2$ with $\pi_{2*}[\mathcal{Z}_{\Gamma_i}] \ne 0$, and that $\Gamma_j = \beta_2$ for $t_0 + 1 \le j \le t$. For $1 \le i \le t$, let m_i be the degree of the restriction $\mu|_{\mu^{-1}(\Gamma_i)} : \mu^{-1}(\Gamma_i) \to \Gamma_i$. Then,

$$\begin{aligned} d(\beta_F - 2\beta_2) = \mu_*[D] &= \sum_{i=1}^{t} m_i [\Gamma_i] \\ &= \sum_{i=1}^{t_0} m_i (\beta_{\pi_{2*}[\mathcal{Z}_{\Gamma_i}]} + \tilde{d}_i \beta_2) + \sum_{i=t_0+1}^{t} m_i \beta_2. \end{aligned} \tag{4-23}$$

So $dF = \sum_{i=1}^{t_0} m_i \cdot \pi_{2*}[\mathcal{Z}_{\Gamma_i}]$, $t_0 \ge 1$, and $f \cdot \pi_{2*}[\mathcal{Z}_{\Gamma_i}] = 0$ for $1 \le i \le t_0$. Since $1 \le d < m$ and every fiber in X is either irreducible reduced or the unique multiple fiber, we conclude that for $1 \le i \le t_0$, the only 1-dimensional irreducible component in $\pi_2(\mathcal{Z}_{\Gamma_i})$ is F. If $\pi_2(\mathcal{Z}_{\Gamma_i})$ contains an isolated point $x \in X - F$, then $\pi_2(\mathcal{Z}_{\Gamma_i}) = F \amalg \{x\}$, $\Gamma_i = F + x$ and

$$[\Gamma_i] = \beta_F. \tag{4-24}$$

Assume that $\pi_2(\mathcal{Z}_{\Gamma_i})$ does not contain any isolated point in $X - F$. Then, $\pi_2(\mathcal{Z}_{\Gamma_i}) = F$ and $\mathrm{Supp}(\xi) \subset F$ for every $\xi \in \Gamma_i$. So

$$\Gamma_i \subset M_2(F) \cup F^{(2)}.$$

If $\Gamma_i \subset M_2(F)$, then by Lemma 4.1, we obtain

$$[\Gamma_i] = 2d_i \beta_F + d_{i,0} \beta_2 \tag{4-25}$$

for some integers $d_i \geq 1$ and $d_{i,0} \geq 0$. If $\Gamma_i \subset F^{(2)}$, then we see from (2-23) that

$$[\Gamma_i] = d_i(\beta_F - 2\beta_2) + (\Theta \cdot \Gamma_i)\beta_2 \tag{4-26}$$

where Θ is the pullback of a theta divisor via $\mathrm{AJ} : F^{(2)} \to \mathrm{Jac}_2(F)$. Combining (4-24), (4-25), (4-26) and

$$d(\beta_F - 2\beta_2) = \sum_{i=1}^{t} m_i[\Gamma_i]$$

from (4-23), we deduce that $t_0 = t$ and that for every $1 \leq i \leq t$, $\Gamma_i \subset F^{(2)}$ and $\Theta \cdot \Gamma_i = 0$. Thus, every Γ_i is a fiber of the ruling $\mathrm{AJ} : F^{(2)} \to \mathrm{Jac}_2(F)$. Since $\Gamma_1, \ldots, \Gamma_t$ are the irreducible components of $\mu(D)$ and $\mu(D)$ is connected, we conclude that $t = 1$, $\mu(D) = \Gamma_1$ is a fiber of the ruling $\mathrm{AJ} : F^{(2)} \to \mathrm{Jac}_2(F)$, and

$$d(\beta_F - 2\beta_2) = m_1[\Gamma_1].$$

By Lemma 4.4(iii), $[\Gamma_1] = \beta_F - 2\beta_2$. It follows that $m_1 = d$, i.e., the degree of the morphism $\mu : D \to \mu(D) = \Gamma_1$ is equal to d. □

Let $1 \leq d < m$. By Lemma 4.6, we obtain the commutative diagram

$$\begin{array}{ccc} \overline{\mathfrak{M}}_{g,r+1}(X^{[2]}, d(\beta_F - 2\beta_2)) & \xrightarrow{\mathrm{ev}_{r+1}} & F^{(2)} \subset X^{[2]} \\ \downarrow f_{r+1,r} & & \downarrow \mathrm{AJ} \\ \overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2)) & \xrightarrow{\Phi_r} & \mathrm{Jac}_2(F) \end{array} \tag{4-27}$$

where $f_{r+1,r}$ is the forgetful map forgetting the last marked point, and Φ_r maps $[\mu : D \to X^{[2]}] \in \overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$ to $\mathrm{AJ}(\mu(D))$. We have

$$\Phi_r^{-1}(\delta) \cong \overline{\mathfrak{M}}_{g,r}(\mathrm{AJ}^{-1}(\delta), d) \cong \overline{\mathfrak{M}}_{g,r}(\mathbb{P}^1, d) \tag{4-28}$$

for $\delta \in \mathrm{Jac}_2(F)$. So the dimension of $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$ is equal to

$$\dim \overline{\mathfrak{M}}_{g,r}(\mathbb{P}^1, d) + 1 = 2d + 2g + r - 1. \tag{4-29}$$

Next, when $1 \leq d < m$, we determine the virtual fundamental class

$$[\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\mathrm{vir}}.$$

Lemma 4.7. *Let m be the multiplicity of the unique multiple fiber in the elliptic surface X from Theorem 4.5. Let $1 \leq d < m$ and $0 \leq g \leq 1$. Then,*

(i) *$R^1(f_{r+1,r})_*(\mathrm{ev}_{r+1})^* T_{X^{[2]}}$ is a rank-$(2d + 3g - 2)$ locally free sheaf.*

(ii) *$[\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))]^{\mathrm{vir}} = c_{2d+3g-2}\big(R^1(f_{r+1,r})_*(\mathrm{ev}_{r+1})^* T_{X^{[2]}}\big)$.*

Proof. (i) Let $[\mu : D \to X^{[2]}] \in \overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$. By Lemma 4.6, $\mu(D)$ is a fiber of the ruling $\mathrm{AJ} : F^{(2)} \to \mathrm{Jac}_2(F)$. By Lemma 4.4(ii),

$$T_{X^{[2]}}|_{\mu(D)} = \mathcal{O}_{\mu(D)}(2) \oplus \mathcal{O}_{\mu(D)}(-1) \oplus \mathcal{O}_{\mu(D)}(-1) \oplus \mathcal{O}_{\mu(D)}.$$

Since $d \geq 1$ and $0 \leq g \leq 1$, we conclude that

$$h^1(D, \mu^* T_{X^{[2]}}) = 2h^1(D, \mu^* \mathcal{O}_{\mu(D)}(-1)) + h^1(D, \mathcal{O}_D) = 2d + 3g - 2.$$

Hence $R^1(f_{r+1,r})_*(\mathrm{ev}_{r+1})^* T_{X^{[2]}}$ is a rank-$(2d + 3g - 2)$ locally free sheaf.

(ii) By (2-2), the expected dimension of $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$ is equal to $1-g+r$. By (4-29), the excess dimension of $\overline{\mathfrak{M}}_{g,r}(X^{[2]}, d(\beta_F - 2\beta_2))$ is $2d+3g-2$. So our result follows immediately from (i) and Proposition 2.1. □

Finally, we determine the cycle $\mathrm{ev}_{1*}\big(\big[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\big]^{\mathrm{vir}}\big)$.

Theorem 4.8. *Let X be a regular elliptic surface with $p_g \geq 1$ and exactly one multiple fiber such that every singular fiber is either irreducible reduced or the unique multiple fiber with smooth reduction F. Let m be the multiplicity of the unique multiple fiber, and $1 \leq d < m$. Then,*

$$\mathrm{ev}_{1*}\big(\big[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\big]^{\mathrm{vir}}\big) = \frac{1}{d^2} \cdot \big[F^{(2)}\big] \quad \in A_2(X^{[2]}).$$

Proof. By (2-2), the expected dimension of $\overline{\mathfrak{M}}_{0,1}(X^{[2]}, d(\beta_F - 2\beta_2))$ is equal to 2. By Lemma 4.6, $\mathrm{ev}_1\big(\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\big) \subset F^{(2)}$. So

$$\mathrm{ev}_{1*}\big(\big[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\big]^{\mathrm{vir}}\big) = \kappa \cdot \big[F^{(2)}\big] \tag{4-30}$$

for some number κ. To determine κ, we take a complex 2-dimensional cycle $W \subset X^{[2]}$ and pretend that W and $F^{(2)}$ intersect transversally at a unique point $\xi \in F^{(2)}$. Let $\delta = \mathrm{AJ}(\xi)$ and $\Gamma = \mathrm{AJ}^{-1}(\delta)$. Then $\xi \in \Gamma \cong \mathbb{P}^1$. Intersecting both sides of (4-30) with $[W]$, we get

$$\begin{aligned} \kappa &= \mathrm{ev}_{1*}\big(\big[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\big]^{\mathrm{vir}}\big) \cdot [W] \\ &= \big[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\big]^{\mathrm{vir}} \cdot \mathrm{ev}_1^*[W] \\ &= \big[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\big]^{\mathrm{vir}} \cdot \mathrm{ev}_1^*[\xi] \\ &= \big[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big)\big]^{\mathrm{vir}}|_{\overline{\mathfrak{M}}_{0,1}(\Gamma, d)} \cdot (\widetilde{\mathrm{ev}}_1)^*[\xi] \end{aligned} \tag{4-31}$$

where in the last step, we have used

$$(\mathrm{ev}_1)^{-1}(\xi) \subset \overline{\mathfrak{M}}_{0,1}(\Gamma, d) \subset \overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F - 2\beta_2)\big),$$

and $\widetilde{\mathrm{ev}}_1 : \overline{\mathfrak{M}}_{0,1}(\Gamma, d) \to \Gamma$ is the evaluation map.

By Lemma 4.7(i), $R^1(f_{1,0})_*(\mathrm{ev}_1)^*T_{X^{[2]}}$ is a rank-$(2d-2)$ locally free sheaf on $\overline{\mathfrak{M}}_{0,0}\big(X^{[2]}, d(\beta_F-2\beta_2)\big)$ where $f_{1,0}$ and ev_1 are the forgetful map and the evaluation map on $\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F-2\beta_2)\big)$ respectively. For simplicity, put

$$\Omega = R^1(f_{1,0})_*(\mathrm{ev}_1)^*T_{X^{[2]}}.$$

By (2-7) and Lemma 4.7(ii), we conclude that

$$\begin{aligned}\big[\overline{\mathfrak{M}}_{0,1}\big(X^{[2]}, d(\beta_F-2\beta_2)\big)\big]^{\mathrm{vir}} &= (f_{1,0})^*\big(\big[\overline{\mathfrak{M}}_{0,0}\big(X^{[2]}, d(\beta_F-2\beta_2)\big)\big]^{\mathrm{vir}}\big)\\ &= (f_{1,0})^*\big(c_{2d-2}(\Omega)\big).\end{aligned}$$

Combining with (4-31), we see that

$$\begin{aligned}\kappa &= (f_{1,0})^*\big(c_{2d-2}(\Omega)\big)|_{\overline{\mathfrak{M}}_{0,1}(\Gamma,d)} \cdot (\widetilde{\mathrm{ev}}_1)^*[\xi]\\ &= (\tilde{f}_{1,0})^*\big(c_{2d-2}(\Omega|_{\overline{\mathfrak{M}}_{0,0}(\Gamma,d)})\big) \cdot (\widetilde{\mathrm{ev}}_1)^*[\xi],\end{aligned} \tag{4-32}$$

where $\tilde{f}_{1,0} : \overline{\mathfrak{M}}_{0,1}(\Gamma, d) \to \overline{\mathfrak{M}}_{0,0}(\Gamma, d)$ is the forgetful map. Note that

$$\Omega|_{\overline{\mathfrak{M}}_{0,0}(\Gamma,d)} \cong R^1(\tilde{f}_{1,0})_*(\widetilde{\mathrm{ev}}_1)^*(T_{X^{[2]}}|_\Gamma).$$

By Lemma 4.4(ii), $T_{X^{[2]}}|_\Gamma = \mathcal{O}_\Gamma(2) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma$. Thus,

$$\Omega|_{\overline{\mathfrak{M}}_{0,0}(\Gamma,d)} \cong R^1(\tilde{f}_{1,0})_*(\widetilde{\mathrm{ev}}_1)^*(\mathcal{O}_\Gamma(-1) \oplus \mathcal{O}_\Gamma(-1)).$$

By [6, Theorem 9.2.3],

$$c_{2d-2}(\Omega|_{\overline{\mathfrak{M}}_{0,0}(\Gamma,d)}) = \frac{1}{d^3} \cdot [\eta]$$

where $[\eta]$ denotes the class of a generic stable map $\eta \in \overline{\mathfrak{M}}_{0,0}(\Gamma, d)$. By (4-32),

$$\kappa = \frac{1}{d^3} \cdot (\tilde{f}_{1,0})^*[\eta] \cdot (\widetilde{\mathrm{ev}}_1)^*[\xi] = \frac{1}{d^2}$$

since for a generic $\eta = [\mu : D \to \Gamma] \in \overline{\mathfrak{M}}_{0,0}(\Gamma, d)$, the map μ is d to 1. □

Remark 4.9. As for the exceptional genus-1 invariant $\langle\rangle^{X^{[2]}}_{1,d(\beta_F-2\beta_2)}$, we see from Lemma 4.7(ii) that if $1 \le d < m$ where m is the multiplicity of the unique multiple fiber in the elliptic surface X from Theorem 4.5, then

$$\langle\rangle^{X^{[2]}}_{1,d(\beta_F-2\beta_2)} = c_{2d+1}\big(R^1(f_{1,0})_*(\mathrm{ev}_1)^*T_{X^{[2]}}\big).$$

where $f_{1,0}$ and ev_1 are the forgetful map and the evaluation map on the moduli space $\overline{\mathfrak{M}}_{1,1}\big(X^{[2]}, d(\beta_F-2\beta_2)\big)$ respectively. However, it is unclear how to compute the right-hand side explicitly.

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References

[1] M. M. Alhwaimel and Z. Qin, "Gromov–Witten invariants of Hilbert schemes of two points on elliptic surfaces", *Internat. J. Math.* **33**:10-11 (2022), art. id. 2250079, 27 pp. MR

[2] A. B. Altman, A. Iarrobino, and S. L. Kleiman, "Irreducibility of the compactified Jacobian", pp. 1–12 in *Real and complex singularities* (Math., Oslo, 1976), Sijthoff & Noordhoff, Alphen aan den Rijn, Netherlands, 1977. MR

[3] A. Beauville, "Variétés Kähleriennes dont la première classe de Chern est nulle", *J. Differential Geom.* **18**:4 (1983), 755–782. MR

[4] A. Beauville, "Variétés kählériennes compactes avec $c_1 = 0$", pp. 181–192 in *Geometry of K3 surfaces: moduli and periods* (Palaiseau, 1981/1982), Astérisque **126**, 1985. MR

[5] K. Behrend and B. Fantechi, "The intrinsic normal cone", *Invent. Math.* **128**:1 (1997), 45–88. MR

[6] D. A. Cox and S. Katz, *Mirror symmetry and algebraic geometry*, Mathematical Surveys and Monographs **68**, American Mathematical Society, Providence, RI, 1999. MR

[7] J. Fogarty, "Algebraic families on an algebraic surface", *Amer. J. Math.* **90** (1968), 511–521. MR

[8] R. Friedman and J. W. Morgan, *Smooth four-manifolds and complex surfaces*, Ergebnisse der Mathematik (3) **27**, Springer, 1994. MR

[9] E. Getzler, "Intersection theory on $\overline{\mathcal{M}}_{1,4}$ and elliptic Gromov–Witten invariants", *J. Amer. Math. Soc.* **10**:4 (1997), 973–998. MR

[10] I. Grojnowski, "Instantons and affine algebras, I: The Hilbert scheme and vertex operators", *Math. Res. Lett.* **3**:2 (1996), 275–291. MR

[11] A. Grothendieck, "Techniques de construction et théorèmes d'existence en géométrie algébrique, IV: Les schémas de Hilbert", in *Séminaire Bourbaki* 1960/1961 (exposé 221), W. A. Benjamin, New York, 1966. Reprinted as pp. 249–276 in *Séminaire Bourbaki* **6**, Soc. Math. France, Paris, 1995. MR Zbl

[12] J. Hu and Z. Qin, "Extremal Gromov–Witten invariants of the Hilbert scheme of 3 points", *Forum Math. Sigma* **11** (2023), art. id. e21, 40 pp. MR

[13] J. Hu, W.-P. Li, and Z. Qin, "The Gromov–Witten invariants of the Hilbert schemes of points on surfaces with $p_g > 0$", *Internat. J. Math.* **26**:1 (2015), art. id. 1550009, 26 pp. MR

[14] A. A. Iarrobino, *Punctual Hilbert schemes*, Mem. Amer. Math. Soc. **188**, 1977. MR

[15] Y.-H. Kiem and J. Li, "Gromov–Witten invariants of varieties with holomorphic 2-forms", preprint, 2007. arXiv 0707.2986

[16] Y.-H. Kiem and J. Li, "Localizing virtual cycles by cosections", *J. Amer. Math. Soc.* **26**:4 (2013), 1025–1050. MR

[17] J. Lee and T. H. Parker, "A structure theorem for the Gromov–Witten invariants of Kähler surfaces", *J. Differential Geom.* **77**:3 (2007), 483–513. MR

[18] J. Li and W.-P. Li, "Two point extremal Gromov–Witten invariants of Hilbert schemes of points on surfaces", *Math. Ann.* **349**:4 (2011), 839–869. MR

[19] W.-P. Li and Z. Qin, "On 1-point Gromov–Witten invariants of the Hilbert schemes of points on surfaces", *Turkish J. Math.* **26**:1 (2002), 53–68. MR

[20] W.-P. Li and Z. Qin, "The cohomological crepant resolution conjecture for the Hilbert–Chow morphisms", *J. Differential Geom.* **104**:3 (2016), 499–557. MR

[21] J. Li and G. Tian, "Virtual moduli cycles and Gromov–Witten invariants of algebraic varieties", *J. Amer. Math. Soc.* **11**:1 (1998), 119–174. MR

[22] J. Li and G. Tian, "Virtual moduli cycles and Gromov–Witten invariants of general symplectic manifolds", pp. 47–83 in *Topics in symplectic* 4*-manifolds* (Irvine, CA, 1996), First Int. Press Lect. Ser. **1**, International Press, Cambridge, MA, 1998. MR

[23] D. Maulik and A. Oblomkov, "Quantum cohomology of the Hilbert scheme of points on $\mathscr{A}_n$-resolutions", *J. Amer. Math. Soc.* **22**:4 (2009), 1055–1091. MR

[24] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, "Gromov–Witten theory and Donaldson–Thomas theory, I", *Compos. Math.* **142**:5 (2006), 1263–1285. MR

[25] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande, "Gromov–Witten theory and Donaldson–Thomas theory, II", *Compos. Math.* **142**:5 (2006), 1286–1304. MR

[26] H. Nakajima, "Heisenberg algebra and Hilbert schemes of points on projective surfaces", *Ann. of Math.* (2) **145**:2 (1997), 379–388. MR

[27] G. Oberdieck, "Gromov–Witten invariants of the Hilbert schemes of points of a K3 surface", *Geom. Topol.* **22**:1 (2018), 323–437. MR

[28] G. Oberdieck and A. Pixton, "Quantum cohomology of the Hilbert scheme of points on an elliptic surface", 2023. arXiv 2312.13188

[29] A. Okounkov and R. Pandharipande, "Quantum cohomology of the Hilbert scheme of points in the plane", *Invent. Math.* **179**:3 (2010), 523–557. MR

[30] Z. Qin, *Hilbert schemes of points and infinite dimensional Lie algebras*, Mathematical Surveys and Monographs **228**, American Mathematical Society, Providence, RI, 2018. MR

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ON KAZHDAN–YOM DIN ASYMPTOTIC ORTHOGONALITY FOR K-FINITE MATRIX COEFFICIENTS OF TEMPERED REPRESENTATIONS

Anne-Marie Aubert and Alfio Fabio La Rosa

Recently, D. Kazhdan and A. Yom Din conjectured the validity of an asymptotic form of Schur orthogonality for tempered, irreducible, unitary representations of semisimple groups defined over local fields. In the non-Archimedean case, they established it for K-finite matrix coefficients. In this article we prove the analogous result in the Archimedean case.

1. Introduction

Let G be a semisimple group over a local field and let K be a maximal compact subgroup of G. We fix a Haar measure on G, denoted by dg. If H is the Hilbert space underlying a unitary representation of G, let H_K denote the space of K-finite vectors and H^∞ the space of smooth vectors.

Recently, D. Kazhdan and A. Yom Din [10] conjectured the validity of an asymptotic version of Schur orthogonality relations. It should hold for matrix coefficients of tempered, irreducible, unitary representations of G, generalising well-known Schur orthogonality relations for discrete series.

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Following [10], we fix a norm on the Lie algebra $\mathfrak{g}$ of G. By [10, Claim 5.2], we can choose it so that Ad K acts unitarily on $\mathfrak{g}$. We define the function

$$\boldsymbol{r} : G \to \mathbb{R}_{\geq 0}, \quad \boldsymbol{r}(g) = \log \max\big\{\|\mathrm{Ad}(g)\|_{\mathrm{op}}, \|\mathrm{Ad}(g^{-1})\|_{\mathrm{op}}\big\}$$

so that, given $r \in \mathbb{R}_{>0}$, we can introduce the corresponding ball

$$G_{<r} := \{g \in G \mid \boldsymbol{r}(g) < r\}.$$

Given this setup, we can state their conjecture.

Conjecture 1.1 (Kazhdan–Yom Din, asymptotic Schur orthogonality relations). Let G be a semisimple group over a local field F and let (π, H) be a tempered, irreducible, unitary representation of G. Then there are $\boldsymbol{d}(\pi) \in \mathbb{Z}_{\geq 0}$ and $\boldsymbol{f}(\pi) \in \mathbb{R}_{>0}$ such that, for all $v_1, v_2, v_3, v_4 \in H$,

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \langle \pi(g)v_1, v_2\rangle \overline{\langle \pi(g)v_3, v_4\rangle}\, dg = \frac{1}{\boldsymbol{f}(\pi)} \langle v_1, v_3\rangle \overline{\langle v_2, v_4\rangle}.$$

Assuming that the matrix coefficients involved are K-finite, one has the following:

Theorem 1.2 [10, Theorem 1.7]. *Let G be a semisimple group defined over a local field F and let K be a maximal compact subgroup of G. Let (π, H) be a tempered, irreducible, unitary representation of G and let H_K denote the space of K-finite vectors in H. Then there exists $\boldsymbol{d}(\pi) \in \mathbb{Z}_{\geq 0}$ such that*:

(1) *If F is non-Archimedean, there is $\boldsymbol{f}(\pi) \in \mathbb{R}_{>0}$ such that, for all $v_1, v_2, v_3, v_4 \in H_K$,*

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \langle \pi(g)v_1, v_2\rangle \overline{\langle \pi(g)v_3, v_4\rangle}\, dg = \frac{1}{\boldsymbol{f}(\pi)} \langle v_1, v_3\rangle \overline{\langle v_2, v_4\rangle}.$$

(2) *If F is Archimedean, for any given nonzero $v_1, v_2 \in H_K$, there is $C(v_1, v_2) > 0$ such that*

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} |\langle \pi(g)v_1, v_2\rangle|^2\, dg = C(v_1, v_2).$$

In the non-Archimedean case, the proof of (1) is achieved by first establishing the validity of the analogous version of (2). The polarisation identity allows the authors of [10] to define a form

$$D(\cdot,\cdot,\cdot,\cdot) : H_K \times H_K \times H_K \times H_K \to \mathbb{C}$$

via the prescription

$$D(v_1, v_2, v_3, v_4) := \lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \langle \pi(g)v_1, v_2\rangle \overline{\langle \pi(g)v_3, v_4\rangle}\, dg.$$

In [10, Section 4.1], this form is shown to be G-invariant and one would like to invoke an appropriate form of the Schur lemma to argue as in the standard proof of

Schur orthogonality relations. That is, for fixed $v_2, v_4 \in H_K$, one defines the form

$$D(\cdot, v_2, \cdot, v_4) : H_K \times H_K \to \mathbb{C},$$

and, for fixed $v_1, v_3 \in H_K$, the form

$$D(v_1, \cdot, v_3, \cdot) : H_K \times H_K \to \mathbb{C}.$$

One applies the Schur lemma to these forms, which implies that each such form is a scalar multiple of the inner product on H. Upon comparing them, one obtains the desired orthogonality relations.

In the non-Archimedean case, it seems to us that the representations considered in [10] are implicitly assumed to be smooth [16, Définition III.1.1], otherwise it is not clear how the theory of asymptotic expansion can be applied.

The appropriate version of the Schur lemma in this case is a consequence of Dixmier's lemma [18, Lemma 0.5.2], which can be applied since in the non-Archimedean setting the subspace of K-finite vectors H_K and the subspace of smooth vectors H^∞ coincide: the latter is irreducible since H itself is irreducible. The required countability of the dimension of H_K follows from the admissibility [16, Théorème VI.2.2] of the irreducible smooth unitary representation (π, H) and by invoking [18, Lemma 0.5.2] in the proof of [16, III.1.9].

The purpose of this article is to prove that the analogue of (1) in Theorem 1.2 holds in the Archimedean case. As explained in [10, Section 4.2], it suffices to prove the result for real semisimple groups (Theorem 4.6).

Theorem 1.3. *Let (π, H) be a tempered, irreducible, unitary representation of a connected, semisimple Lie group G with finite centre. Let K be a maximal compact subgroup of G. Then there exists $\boldsymbol{f}(\pi) \in \mathbb{R}_{>0}$ such that, for all $v_1, v_2, v_3, v_4 \in H_K$,*

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \langle \pi(g)v_1, v_2\rangle \overline{\langle \pi(g)v_3, v_4\rangle}\, dg = \frac{1}{\boldsymbol{f}(\pi)} \langle v_1, v_3\rangle \overline{\langle v_2, v_4\rangle}.$$

Remark 1.4. It is well known that an irreducible, tempered representation as in Theorem 1.3 is unitarisable. We have chosen the somewhat redundant formulation above to emphasise that the unitarity of the representation plays a crucial role in the following. From now on, if (π, H) is a tempered, irreducible representation we will implicitly assume that π acts unitarily with respect to the inner product on H.

We need to modify the strategy above to account for the fact that the space of K-finite vectors of a unitary representation (π, H) of a real semisimple group does not afford a representation of G. It is, however, an admissible $(\mathfrak{g}, K)$-module.

Our approach relies crucially on the admissibility of irreducible, unitary representations of reductive Lie groups, a foundational theorem proved by Harish-Chandra. The theory of admissible $(\mathfrak{g}, K)$-modules then provides us with the appropriate version of the Schur lemma for $(\mathfrak{g}, K)$-invariant forms (Definition 2.11).

Hence, we are reduced to verifying that $D(\cdot, v_2, \cdot, v_4)$ and $D(v_1, \cdot, v_3, \cdot)$ are, indeed, $(\mathfrak{g}, K)$-invariant. Having established this, to conclude the proof of Theorem 1.3, we can argue as in [10, Section 4].

From now on, to make the notation look more compact, given a unitary representation (π, H) of G and vectors $v, w \in H$, we set

$$\phi_{v,w}(g) := \langle \pi(g)v, w\rangle.$$

For connected, semisimple Lie groups with finite centre, K-invariance is a consequence of $\mathfrak{g}$-invariance (Proposition 2.14). Therefore, the problem is establishing the $\mathfrak{g}$-invariance. Explicitly, we prove the following (Proposition 4.2).

Proposition 1.5. *Let G be a connected, semisimple Lie group with finite centre and let (π, H) be a tempered, irreducible, unitary representation of G. Then, for all $X \in \mathfrak{g}$, and for all $v_1, v_2, v_3, v_4 \in H_K$, we have*

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \phi_{\dot{\pi}(X)v_1,v_2}(g)\overline{\phi_{v_3,v_4}(g)}\, dg = -\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,v_2}(g)\overline{\phi_{\dot{\pi}(X)v_3,v_4}(g)}\, dg$$

and

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,\dot{\pi}(X)v_2}(g)\overline{\phi_{v_3,v_4}(g)}\, dg = -\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,v_2}(g)\overline{\phi_{v_3,\dot{\pi}(X)v_4}(g)}\, dg.$$

The key observation is that, by exploiting the theory of asymptotic expansions of matrix coefficients of tempered representations both with respect to a minimal parabolic subgroup $P = MAN$ and with respect to the standard (for P) parabolic subgroups of G, the expression

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \phi_{\dot{\pi}(X)v_1,v_2}(g)\overline{\phi_{v_3,v_4}(g)}\, dg$$

reduces, roughly, to a sum of finitely many terms of the form

$$\int_K \big\langle \Gamma_{\lambda,l}(m_\lambda, \pi(k)\dot{\pi}(X)v_1, w_2), \Gamma_{\mu,m}(m_\lambda, \pi(k)v_3, w_4)\big\rangle_{L^2(M_\lambda)}\, dk.$$

Here, M_λ comes from a standard parabolic subgroup $P_\lambda = M_\lambda A_{\lambda_0} N_{\lambda_0}$ of G. We denote by $\mathfrak{m}_\lambda$, $\mathfrak{a}_{\lambda_0}$, $\mathfrak{n}_{\lambda_0}$ the Lie algebras of M_λ, A_{λ_0}, N_{λ_0}, respectively. The pairs (λ, l) and (μ, m) will be introduced precisely in Theorem 3.3; we can think of λ, μ as n-tuples of complex numbers and of l, m as n-tuples of integers. The functions $\Gamma_{\lambda,l}$, $\Gamma_{\mu,m}$ are defined in (3.6). As functions of m_λ, they are analytic and square-integrable and they arise from the asymptotic expansion of the matrix coefficients $\phi_{\dot{\pi}(X)v_1,v_2}$ and ϕ_{v_3,v_4}, respectively, relative to P_λ (see Theorem 3.3). The subscript in P_λ is meant to indicate that the parabolic subgroup is obtained, in an appropriate sense, from the datum of λ. Moreover, (λ, l) and (μ, m) are related in a precise way (see the discussion after Theorem 3.1 and the proof of Proposition 4.2 after (4.4)).

We shall elaborate on these points later on. For the moment, let us point out that we reduced the initial problem to showing that, for every $X \in \mathfrak{g}$, and for all relevant pairs (λ, l) and (μ, m), the integral

$$\int_K \big\langle \Gamma_{\lambda,l}(m_\lambda, \pi(k)\dot{\pi}(X)v_1, w_2), \Gamma_{\mu,m}(m_\lambda, \pi(k)v_3, w_4)\big\rangle_{L^2(M_\lambda)}\, dk$$

equals

$$-\int_K \big\langle \Gamma_{\lambda,l}(m_\lambda, \pi(k)v_1, w_2), \Gamma_{\mu,m}(m_\lambda, \pi(k)\dot{\pi}(X)v_3, w_4)\big\rangle_{L^2(M_\lambda)}\, dk.$$

We will prove that, if (λ, l) and (μ, m) satisfy a certain condition (to be explained below), the functions $\Gamma_{\lambda,l}(\,\cdot\,, v_1, w_2)$ and $\Gamma_{\mu,m}(\,\cdot\,, v_3, w_4)$ are, in fact, $Z(\mathfrak{g}_{\mathbb{C}})$-finite, with $Z(\mathfrak{g}_{\mathbb{C}})$ denoting the centre of the universal enveloping algebra of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$, and $K \cap M_\lambda$-finite. It will then follow from a theorem of Harish-Chandra (Theorem 2.17) that they are smooth vectors in the right-regular representation $(R, L^2(M_\lambda))$ of M_λ.

The idea is to combine this observation with an appropriate form of the Frobenius reciprocity (Theorem 2.27), due to Casselman, to construct $(\mathfrak{g}, K)$-invariant maps

$$T_{w_2} : H_K \to \mathrm{Ind}_{P_\lambda, K_\lambda}(H_\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}), \quad T_{w_2}(v)(k)(m_\lambda) := \Gamma_{\lambda,l}(m_\lambda, \pi(k)v, w_2)$$

and

$$T_{w_2} : H_K \to \mathrm{Ind}_{P_\lambda, K_\lambda}(H_\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}), \quad T_{w_4}(v')(k)(m_\lambda) := \Gamma_{\lambda,l}(m_\lambda, \pi(k)v', w_4).$$

Here, the subgroup $\overline{P_\lambda}$ is the parabolic subgroup opposite to P_λ. The notation $\mathrm{Ind}_{\overline{P_\lambda}, K}(H_\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})$ stands for the space of K-finite vectors in the representation induced from the $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K \cap M_\lambda)$-module

$$H_\sigma \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}},$$

where (σ, H_σ) is an appropriately chosen admissible, unitary, subrepresentation of $(R, L^2(M_\lambda))$.

To apply the required form of the Frobenius reciprocity, we need to show that the maps

$$S_{w_2} : H_K \to H_\sigma \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_{w_2}(v)(m_\lambda) := \Gamma_{\lambda,l}(m_\lambda, v, w_2),$$

and

$$S_{w_2} : H_K \to H_\sigma \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_{w_4}(v')(m_\lambda) := \Gamma_{\lambda,l}(m_\lambda, v', w_4),$$

descend to $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-equivariant maps on $H_K/\mathfrak{n}_{\lambda_0} H_K$. Establishing this result is the technical heart of the article.

Assuming it, the integral

$$\int_K \big\langle \Gamma_{\lambda,l}(m_\lambda, \pi(k)\dot{\pi}(X)v_1, w_2), \Gamma_{\mu,m}(m_\lambda, \pi(k)v_3, w_4)\big\rangle_{L^2(M_\lambda)}\, dk$$

is nothing but

$$\big\langle \dot{\operatorname{Ind}}_{\overline{P_\lambda}}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})(X)\Gamma_{\lambda,l}(m_\lambda, v_1, w_2), \Gamma_{\mu,m}(m_\lambda, v_3, w_4)\big\rangle_{\operatorname{Ind}_{\overline{P_\lambda}}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})},$$

where

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\operatorname{Ind}_{\overline{P_\lambda}}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})}$$

is the inner product on $\operatorname{Ind}_{\overline{P_\lambda}}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})$. We will see that this makes sense since the inducing data ensure unitarity. The sought equality will then follow from the skew-invariance of the inner product on a unitary representation with respect to the action of the Lie algebra.

To explain how the functions $\Gamma_{\lambda,l}(\,\cdot\,, v_1, v_2)$ and $\Gamma_{\mu,m}(\,\cdot\,, v_3, v_4)$ arise, we need to recall the main features of the asymptotic expansions of K-finite matrix coefficients of tempered representations. If $\phi_{v,w}$ is such a matrix coefficient, then its restriction to a certain region of the subgroup A of a minimal parabolic subgroup $P = MAN$ of G admits an asymptotic expansion which can be thought of as a sum indexed by a countable collection

$$\Lambda := \{(\lambda, l)\}_{\lambda\in\mathcal{E},\, l\in\mathbb{Z}^n_{\geq 0}:|l|\leq l_0}.$$

The set $\mathcal{E}$ is a collection of complex-valued real-linear functionals on $\operatorname{Lie}(A)$ depending on (π, H) and not on the particular choice of $v, w \in H_K$. It is the set of *exponents* of (π, H). The number n is the rank of G and l_0, too, depends on (π, H) only.

The term indexed by (λ, l) is multiplied by a complex coefficient $c_{\lambda,l}(v, w)$. The choice of $v, w \in H_K$ determines the pairs in $\mathcal{C}$ for which $c_{\lambda,l}(v, w) \neq 0$. If $\lambda \in \mathcal{E}$, there exists at least a pair of $v, w \in H_K$ such that, for some $l \in \mathbb{Z}^n_{\geq 0}$ with $|l| \leq l_0$, we have $c_{\lambda,l}(v, w) \neq 0$.

For any standard (for P) parabolic subgroup $P' = M'A'N'$ of G, the restriction of the matrix coefficient $\phi_{v,w}$ to an appropriate region of A' admits a similar asymptotic expansion. It can be thought of as a sum indexed by a countable collection

$$\Lambda' := \{(\nu, q)\}_{\nu\in\mathcal{E}',\, q\in\mathbb{Z}^r_{\geq 0}:|q|\leq q_0}.$$

Here, $r \leq n$ is the dimension of A', the set $\mathcal{E}'$ consists of complex-valued real-linear functionals on $\operatorname{Lie}(A')$. On regions on which both the expansion relative to P and the expansion relative to P' are meaningful, by comparing the two it turns out that the element in $\mathcal{E}'$ are precisely the restrictions to $\operatorname{Lie}(A')$ of the elements in $\mathcal{E}$ and, making the appropriate identifications following from $A' \subset A$, each q is the projection to $\mathbb{Z}^r_{\geq 0}$ of an l appearing in the expansion relative to P.

While in the expansion relative to P the term indexed by (λ, l) is multiplied by the complex coefficient $c_{\lambda,l}(v, w)$, the term indexed by (ν, q) in the expansion relative to P' is multiplied by a real-analytic function

$$c^{P'}_{\nu,q}(\,\cdot\,, v, w) : M' \to \mathbb{C}. \tag{1.6}$$

We require one more piece of information to explain how $\Gamma_{\lambda,l}(\,\cdot\,, v_1, v_2)$ and $\Gamma_{\mu,m}(\,\cdot\,, v_3, v_4)$ arise: the construction of $\boldsymbol{d}(\pi)$ in [10]. The idea is as follows. We can think of $\lambda \in \mathcal{E}$ as an n-tuple of complex numbers $(\lambda_1, \dots, \lambda_n)$. It can be shown that there exist a finite subcollection $\mathcal{E}_0 \subset \mathcal{E}$ such that, for every $\lambda \in \mathcal{E}$, there exists $\hat{\lambda} \in \mathcal{E}_0$ such that

$$\hat{\lambda} - \lambda \in \mathbb{Z}^n_{\geq 0}.$$

Moreover, any two distinct elements in $\mathcal{E}_0$ are integrally inequivalent: their difference does not belong to $\mathbb{Z}^n$. By a result of Casselman (Theorem 3.2), for every $\hat{\lambda} \in \mathcal{E}_0$ and for every $i \in \{1, \dots, n\}$, we have

$$\operatorname{Re} \hat{\lambda}_i \leq 0,$$

and it is clear that this holds for every $\lambda \in \mathcal{E}$.

For $(\lambda, l) \in \Lambda$, we introduce the set $I_\lambda := \{i \in \{1, \dots, n\} \mid \operatorname{Re} \lambda_i < 0\}$, we define

$$\boldsymbol{d}_P(\lambda, l) := |I^c_\lambda| + \sum_{i \in I^c_\lambda} 2l_i, \tag{1.7}$$

and we take the maximum, $\boldsymbol{d}_P$, as (λ, l) ranges over all the pairs with $\lambda \in \mathcal{E}_0$.

We can proceed analogously for every standard parabolic P' and obtain a non-negative integer $\boldsymbol{d}_{P'}$. The maximum over all P' is $\boldsymbol{d}(\pi)$.

Now, given $\lambda \in \mathcal{E}_0$, identifying I_λ with a subset of the simple roots determined by an order on the root system attached to the pair $(\mathfrak{g}, \mathfrak{a})$, we can construct a standard (for P) parabolic subgroup $P_\lambda = M_\lambda A_{\lambda_0} N_{\lambda_0}$ associated to I_λ. We will show that if $(\lambda, l) \in \Lambda$ satisfies $\lambda \in \mathcal{E}_0$ and $\boldsymbol{d}_P(\lambda, l) = \boldsymbol{d}(\pi)$, then $\Gamma_{\lambda,l}(\,\cdot\,, v_1, v_2)$ is precisely the function $c^{P'}_{\nu,q}(\,\cdot\,, v_1, v_2)$ with $\nu := \lambda|_{\mathfrak{a}_{\lambda_0}}$, where $\mathfrak{a}_{\lambda_0} := \operatorname{Lie}(A_{\lambda_0})$, and q equal to the projection of l to $\mathbb{Z}^{I^c_\lambda}_{\geq 0}$.

Finally, we mentioned that in the integral

$$\int_K \big\langle \Gamma_{\lambda,l}(m_\lambda, \pi(k)v_1, w_2), \Gamma_{\mu,m}(m_\lambda, \pi(k)v_3, w_4) \big\rangle_{L^2(M_\lambda)} \, dk$$

the pairs (λ, l) and (μ, m) must be related in a precise way. First of all, $(\mu, m) \in \Lambda$ satisfies $\mu \in \mathcal{E}_0$ and $\boldsymbol{d}_P(\mu, m) = \boldsymbol{d}(\pi)$. Also, we must have $I_\lambda = I_\mu$ (so that $P_\lambda = P_\mu$) and $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$. The last condition, together with the unitarity of the representation (σ, H_σ) introduced above, is precisely what ensures that $\operatorname{Ind}_{P_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})$ is unitary.

Implementing the strategy sketched above requires gathering a number of intermediate results. Several are inspired from the chapter in [11] on the Langlands classification of tempered representations. Here is a more detailed outline of the article.

Section 2: The first part includes a discussion of the $(\mathfrak{g}, K)$-module version of the Schur lemma (Corollary 2.13). In the second part, we recall the result of

Harish-Chandra establishing that smooth, $Z(\mathfrak{g}_{\mathbb{C}})$-finite, K-finite, square-integrable functions on reductive groups are smooth vectors in the right-regular representation (Theorem 2.17). As a consequence, we prove that, on such a function, the action of $\mathfrak{g}$ through differentiation is the same as the action of the Lie algebra through the right-regular representation (Proposition 2.20). After stating the basic facts on parabolically induced representations that we need, we discuss Casselman's version of the Frobenius reciprocity (Theorem 2.27).

Section 3: In the first part, we recall the theory of asymptotic expansions of matrix coefficients of tempered representations both with respect to a minimal parabolic subgroup and with respect to standard parabolic subgroups. We then explain in detail how the functions $\Gamma_{\lambda,l}(\,\cdot\,, v_1, v_2)$, $\Gamma_{\mu,m}(\,\cdot\,, v_3, v_4)$ arise. We begin by introducing an equivalence relation on the data indexing the asymptotic expansion relative to P of the K-finite matrix coefficients of a tempered, irreducible, representation (π, H). This equivalence relation is motivated by the construction of $\boldsymbol{d}(\pi)$ in [10] and it is meant to exploit the criteria for the computation of asymptotic integrals in [10, Appendix A]. Imposing the conditions on (λ, l) and (μ, m) that we discussed above, we identify the functions $\Gamma_{\lambda,l}(\,\cdot\,, v_1, v_2)$ and $\Gamma_{\mu,m}(\,\cdot\,, v_3, v_4)$ with the coefficient functions in the asymptotic expansion relative to P_λ of ϕ_{v_1,v_2} and ϕ_{v_3,v_4} (Proposition 3.12). We then prove that they are smooth vectors in $(R, L^2(M_\lambda))$ (Proposition 3.14). Combining Proposition 3.14 with the technical Lemmas 3.15 and 3.16, we can construct unitary, admissible, finitely generated representations (σ_1, H_{σ_1}) and (σ_2, H_{σ_2}) whose direct sum is the unitary, admissible, finitely generated representation (σ, H_σ) introduced above (Proposition 3.19).

Section 4: Having gathered the results we need, we are able to prove Proposition 1.5 (Proposition 4.2). This consists in an application of the considerations in [10, Appendix A] to show that the integral

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,v_2}(g)\overline{\phi_{v_3,v_4}(g)}\, dg$$

can be computed in terms of a sum of integrals of the form

$$\int_K \big\langle \Gamma_{\lambda,l}(m_\lambda, \pi(k)v_1, w_2), \Gamma_{\mu,m}(m_\lambda, \pi(k)v_3, w_4)\big\rangle\, dk$$

with the pairs (λ, l) and (μ, m) both belonging to Λ with $\lambda, \mu \in \mathcal{E}_0$, $I_\lambda = I_\mu$, $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$ and

$$\boldsymbol{d}_P(\lambda, l) = \boldsymbol{d}_P(\mu, m) = \boldsymbol{d}(\pi).$$

At this point, the representation-theoretic arguments explained in the Introduction and proved in Section 3 conclude the proof of Proposition 1.5.

Finally, we proceed as explained in the first part of the Introduction to prove Theorem 1.3 (Theorem 4.6).

2. Recollections on representation theory

Our presentation of the theory of $(\mathfrak{g}, K)$-modules follows [18]. To discuss its basic features, we need to gather some results on unitary representations of compact groups. We begin by recalling the basic notions in the study of representations of topological groups, which we always assume to be Hausdorff.

First, following [18, Section 1.1], let G denote a second-countable, locally compact group, equipped with a left Haar measure dg, and let V denote a complex topological vector space. We denote by $\mathrm{GL}(V)$ the group of invertible continuous endomorphisms of V. A *representation* of G on V is a strongly continuous homomorphism $\pi : G \to \mathrm{GL}(V)$. Let (π, V) denote the datum of a representation of G. A subspace of V which is stable under the action of G through π is called an *invariant subspace*. A representation (π, V), with $V \neq 0$, is said to be *irreducible* if the only closed invariant subspaces are the trivial subspace and V itself.

If $(H, \langle \cdot, \cdot \rangle)$ is a separable Hilbert space, a representation π of G on H is termed a *Hilbert representation*. If, in addition, G acts by unitary operators through π, the representation is said to be *unitary*.

Next, following [14, Section 10], we introduce the basic features of the theory of vector-valued integration.

Let (X, dx) be a Radon measure space, let H be a Hilbert space and assume that

$$f : X \to H$$

is measurable. The function f is *integrable* if it satisfies these two conditions:

(1) For all $v \in H$,
$$\int_X |\langle f(x), v\rangle| \, dx < \infty.$$

(2) The map
$$v \mapsto \int_X \langle f(x), v\rangle \, dx$$
is a bounded conjugate-linear functional.

If $f : X \to H$ is integrable, then, by the Riesz representation theorem, there exists a unique element in H, denoted by

$$\int_X f(x) \, dx,$$

such that, for all $v \in H$, we have

$$\left\langle \int_X f(x) \, dx, v \right\rangle = \int_X \langle f(x), v\rangle \, dx.$$

Proposition 2.1. *Let (X, dx) be as above. Let H, E be Hilbert spaces, $f : X \to H$ a measurable function and $T : H \to E$ a bounded linear operator.*

(1) *If*

$$\int_X \| f(x) \| \, dx < \infty,$$

then $f : X \to H$ is integrable.

(2) *If $f : X \to H$ is integrable, then so is $Tf : X \to E$. Moreover,*

$$T\left(\int_X f(x) \, dx \right) = \int_X Tf(x) \, dx.$$

Proof. See [14, Propositions 10.8 and 10.9]. □

Now, let (π, H) be a unitary representation of G. Let $v \in H$ and $f : G \to H$ be such that the map

$$g \mapsto f(g)\pi(g)v$$

is integrable. Let $\pi(f)v$ denote the unique element in H such that, for all $w \in H$,

$$\langle \pi(f)v, w \rangle = \int_G f(g) \langle \pi(g)v, w \rangle \, dg.$$

Proposition 2.2. *Let (π, H) be as above. If $f \in L^1(G)$, then, for all $v \in H$, the map $g \mapsto f(g)\pi(g)v$ is integrable and the prescription*

$$\pi(f) : H \to H, \quad v \mapsto \pi(f)v,$$

defines a bounded linear operator.

Proof. See [14, Proposition 10.20]. □

With the integral operators introduced in Proposition 2.2 at our disposal, we have all the tools needed to state the main results on the unitary representations of compact groups.

Let K be a compact group. Let $\widehat{K}$ denote the set of equivalence classes of irreducible unitary representations of K. If (π, H) is a unitary representation, for each $[\gamma] \in \widehat{K}$ let $H(\gamma)$ denote the closure of the sum of all the closed invariant subspaces of H in the equivalence class of γ. We refer to $H(\gamma)$ as the γ-isotypic component of H. This notion is independent of the choice of representative for the equivalence class.

Proposition 2.3. *Let K be a compact group. Let (π, H) be an irreducible unitary representation of K. Then H is finite-dimensional.*

Proof. See [18, Proposition 1.4.2]. □

Given Proposition 2.3, we can associate, to each irreducible representation γ of K, the function

$$\chi_\gamma : K \to \mathbb{C}, \quad \chi_\gamma(g) := \operatorname{tr} \gamma(g),$$

the *character* of γ. A standard argument proves that equivalent representations have the same character.

Recall that if $\{(\pi_i, H_i) \mid i \in I\}$ is a countable family of unitary representations of a topological group G, we can construct a new unitary representation of G, the *direct sum*, on the Hilbert space completion of the algebraic direct sum of the H_i's. We refer the reader to [18, Section 1.4.1], for the details of this construction. We let

$$\bigoplus_{i \in I} H_i$$

denote the direct sum of the family $\{(\pi_i, H_i) \mid i \in I\}$, dropping explicit reference to the π_i's.

Proposition 2.4. *Let K be a compact group. Let (π, H) be a unitary representation of K. Then (π, H) is the direct sum representation of its K-isotypic components*:

$$H = \bigoplus_{[\gamma] \in \widehat{K}} H(\gamma).$$

Moreover, let α_γ denote the function

$$\alpha_\gamma(k) := \dim(\gamma) \overline{\chi_\gamma(k)}.$$

Then

$$H(\gamma) = \pi(\alpha_\gamma) H.$$

Proof. See [18, Lemma 1.4.7]. □

Proposition 2.5. *Let K be a compact group. If (π, H) is a Hilbert space representation of K, then there exists an inner product on H that induces the original topology on H and for which K acts unitarily through π.*

Proof. See [18, Lemma 1.4.8]. □

We are finally ready to introduce $(\mathfrak{g}, K)$-modules.

Definition 2.6. Let G be a connected, semisimple Lie group with finite centre. Let $\mathfrak{g}$ denote its Lie algebra. Let K be a maximal compact subgroup of G, which we fix from now on, with Lie algebra $\mathfrak{k}$. A vector space V, equipped with the structure of $\mathfrak{g}$-module and K-module, is called a $(\mathfrak{g}, K)$-*module* if the following conditions hold:

(1) For all $v \in V$, for all $X \in \mathfrak{g}$, for all $k \in K$,

$$kXv = \operatorname{Ad}(k)Xkv.$$

(2) For all $v \in V$, the span of the set

$$Kv := \{kv \mid k \in K\}$$

is a finite-dimensional subspace of V, on which the action of K is continuous.

(3) For all $v \in V$, for all $Y \in \mathfrak{k}$,

$$\frac{d}{dt}\exp(tY)v|_{t=0} = Yv.$$

We remark that (3) implicitly uses the smoothness of the action of K on the span of Kv. This follows from the fact that a continuous group homomorphism between Lie groups is automatically smooth.

Let V and W be $(\mathfrak{g}, K)$-modules and let $\mathrm{Hom}_{\mathfrak{g},K}(V, W)$ denote the space of $\mathfrak{g}$-morphisms that are also K-equivariant. Then V and W are said to be *equivalent* if $\mathrm{Hom}_{\mathfrak{g},K}(V, W)$ contains an invertible element.

A $(\mathfrak{g}, K)$-module V is called *irreducible* if the only subspaces that are invariant under the actions of $\mathfrak{g}$ and K are the trivial subspace and V itself. In this case, we have the following theorem:

Theorem 2.7. *Let V be an irreducible $(\mathfrak{g}, K)$-module. Then $\mathrm{Hom}_{\mathfrak{g},K}(V, V)$ is 1-dimensional.*

Proof. This is the result actually proved in [18, Lemma 3.3.2], although the statement there says $\mathrm{Hom}_{\mathfrak{g},K}(V, W)$, for an unspecified W. We believe it is a typo. □

Let V be a $(\mathfrak{g}, K)$-module. Since, given each $v \in V$, the span of Kv, say W_v, is a finite-dimensional continuous representation of K, we can use Proposition 2.5 and then apply Proposition 2.4, thus decomposing W_v into a finite sum of finite-dimensional K-invariant subspaces of V. For $\gamma \in \widehat{K}$, we let $V(\gamma)$ denote the sum of all the K-invariant finite-dimensional subspaces in the equivalence class of γ. Then the discussion above implies that

$$V = \bigoplus_{\gamma \in \widehat{K}} V(\gamma)$$

as a K-module, with the direct sum indicating the algebraic direct sum. A $(\mathfrak{g}, K)$-module V is called *admissible* if, for all $\gamma \in \widehat{K}$, $V(\gamma)$ is finite-dimensional.

Given a unitary representation (π, H), there exists a $(\mathfrak{g}, K)$-module naturally associated to it. To define it, recall that a vector $v \in H$ is called *smooth* if the map

$$g \mapsto \pi(g)v$$

is smooth. Let H^∞ denote the subspace of smooth vectors of H. It is a standard fact that the prescription

$$\dot{\pi}(X) := \frac{d}{dt}\pi(\exp(tX))v|_{t=0},$$

for $v \in H^\infty$ and $X \in \mathfrak{g}$, defines an action of $\mathfrak{g}$ on H^∞. Recall that a vector $v \in H$ is *K-finite* if the span of the set

$$\pi(K)v := \{\pi(k)v \mid k \in K\}$$

is finite-dimensional. Let H_K denote the subspace of K-finite vectors of H. By [18, Lemma 3.3.5], with the action of $\mathfrak{g}$ so defined and with the action of K through π, the space $H_K \cap H^\infty$ is a $(\mathfrak{g}, K)$-module. The representation (π, H) is said to be *admissible* if $H_K \cap H^\infty$ is admissible as a $(\mathfrak{g}, K)$-module and (π, H) is called *infinitesimally irreducible* if $H_K \cap H^\infty$ is irreducible as a $(\mathfrak{g}, K)$-module. It is in general not true that a K-finite vector is smooth. However, if (π, H) is admissible, we have the following result:

Theorem 2.8. *Let G be a connected, semisimple Lie group with finite centre. Let (π, H) be an admissible representation of G. Then every K-finite vector is smooth.*

Proof. See the proof [18, Theorem 3.4.10]. □

In light of the following fundamental result of Harish-Chandra, Theorem 2.8 will play an important role in this article.

Theorem 2.9. *Let G be a connected, semisimple Lie group with finite centre. Let (π, H) be an irreducible, Hilbert representation of G. Then (π, H) is admissible.*

Proof. See [13, Theorem 7.204]. □

In the following, given a unitary representation (π, H), we will write H_K for the $(\mathfrak{g}, K)$-module $H_K \cap H^\infty$ even if (π, H) is not admissible. We believe it will not cause any confusion.

We are now in position to prove the version of the Schur lemma for sesquilinear forms that we will use in Section 3. It is given as Corollary 2.13. First, we need:

Theorem 2.10. *Let G be a connected, semisimple Lie group with finite centre. Let (π, H) be an admissible Hilbert representation of G. Then (π, H) is irreducible if and only if it is infinitesimally irreducible.*

Proof. See [18, Theorem 3.4.11]. □

Definition 2.11. Let V and W be $(\mathfrak{g}, K)$-modules. A sesquilinear form

$$B(\,\cdot\,,\cdot\,) : V \times W \to \mathbb{C}$$

is $(\mathfrak{g}, K)$-invariant if it satisfies the following two conditions:

(i) For all $k_1, k_2 \in K$ and all $v, w \in V$ we have

$$B(k_1 v, k_2 w) = B(v, w).$$

(ii) For all $X \in \mathfrak{g}$ and all $v, w \in V$ we have

$$B(Xv, w) = -B(v, Xw).$$

Theorem 2.12. *Let G be a connected, semisimple Lie group with finite centre. Let V be an admissible $(\mathfrak{g}, K)$-module. Suppose that there exist a $(\mathfrak{g}, K)$-module W and a nondegenerate $(\mathfrak{g}, K)$-invariant sesquilinear form*

$$B(\,\cdot\,,\cdot\,) : V \times W \to \mathbb{C}.$$

Then W is $(\mathfrak{g}, K)$-isomorphic to $\overline{V}$.

Proof. This is [18, Lemma 4.5.1], except for the fact that our form is sesquilinear. To account for it, we modify the definition of the map T in the reference by setting, for a given $w \in W$, $T(w)(v) = B(w, v)$ for all $v \in V$. This defines a map from W to $\overline{V}$ obtained by sending w to $T(w)$ which, by the argument in the reference, is a $(\mathfrak{g}, K)$-isomorphism. □

The next corollary is proved by adapting to our case the argument in [4, Proposition 8.5.12] and using the beginning of the proof of [11, Proposition 9.1].

Corollary 2.13. *Let G be a connected, semisimple Lie group with finite centre. Let (π, H) be an irreducible, Hilbert representation of G. Then, up to a constant, there exists at most one nonzero $(\mathfrak{g}, K)$-invariant sesquilinear form on H_K. In particular, if (π, H) is irreducible unitary, then every such form is a constant multiple of $\langle\,\cdot\,,\cdot\,\rangle$.*

Proof. The irreducibility of (π, H) implies that of H_K, by Theorems 2.10 and 2.8. Let $B(\,\cdot\,,\cdot\,)$ be a $(\mathfrak{g}, K)$-invariant sesquilinear form. Consider the linear subspace V_0 of H_K defined as

$$V_0 := \{v \in H_K \mid B(v, w) = 0 \text{ for all } w \in H_K\}.$$

Since $B(\,\cdot\,,\cdot\,)$ is nonzero, V_0 is a proper subspace of H_K. Since $B(\,\cdot\,,\cdot\,)$ is moreover $(\mathfrak{g}, K)$-invariant, it follows that V_0 is a $(\mathfrak{g}, K)$-invariant subspace of H_K, and hence, by the irreducibility of H_K, it must be zero. Analogous considerations for the subspace

$$V^0 := \{w \in H_K \mid B(v, w) = 0 \text{ for all } v \in H_K\}$$

imply that $B(\,\cdot\,,\cdot\,)$ is nondegenerate. By Theorem 2.12, the map $v \mapsto T(v)$, $T(v)(\,\cdot\,) := B(v, \cdot\,)$, is a $(\mathfrak{g}, K)$-isomorphism. Since H_K is irreducible, the space $\mathrm{Hom}_{\mathfrak{g},K}(H_K, H_K)$ is 1-dimensional by Theorem 2.7. Now, let $B'(\,\cdot\,,\cdot\,)$ be another such form, with associated isomorphism T'. Then $T(T')^{-1} = cI$, for some $c \in \mathbb{C}$. For the last statement, the unitarity of (π, H) implies that $\langle\,\cdot\,,\cdot\,\rangle$ is a $(\mathfrak{g}, K)$-invariant nondegenerate sesquilinear form and Theorem 2.9, with the discussion above, implies the result. □

Since we are assuming that G is connected, proving $(\mathfrak{g}, K)$-invariance reduces to proving $\mathfrak{g}$-invariance. Indeed, by [9, Theorem 2.2, p. 256], any maximal compact subgroup K of G is connected. Therefore, by [12, Corollary 4.48], the exponential map

$$\exp : \mathfrak{k} \to K$$

is surjective.

Proposition 2.14. *Let G be a connected, semisimple Lie group with finite centre. Let V be a $(\mathfrak{g}, K)$-module, let*

$$B(\,\cdot\,,\,\cdot\,) : V \times V \to \mathbb{C}$$

be a $\mathfrak{g}$-invariant sesquilinear form. Then $B(\,\cdot\,,\,\cdot\,)$ is K-invariant.

Proof. Given any pair of vectors $v, w \in V$, we can find a finite-dimensional subspace of V, say W, which contains both and on which K acts continuously through a representation π. The restriction of the bilinear form $B(\,\cdot\,,\,\cdot\,)$ to W is continuous. To prove that $B(\pi(k)v, \pi(k)w) = B(v, w)$ for all $k \in K$, it suffices to prove that $B(\pi(k)v, w) = B(v, \pi(k^{-1})w)$ for all $k \in K$. Given $k \in K$, let $X \in \mathfrak{k}$ be such that $k = \exp X$. We begin by writing

$$B(\pi(k)v, w) = B(\pi(\exp X)v, w).$$

Since $\pi(\exp X) = \exp \dot{\pi}(X)v$, we obtain

$$B(\pi(\exp X)v, w) = B(\exp \dot{\pi}(X)v, w).$$

The continuity of $B(\,\cdot\,,\,\cdot\,)$ on V gives

$$B(\exp \dot{\pi}(X)v, w) = \exp B(\dot{\pi}(X)v, w).$$

By the $\mathfrak{g}$-invariance of $B(\,\cdot\,,\,\cdot\,)$, we have

$$\exp B(\dot{\pi}(X)v, w) = \exp B(v, \dot{\pi}(-X)w)$$

and, finally,

$$\exp B(v, \dot{\pi}(-X)w) = B(v, \pi(\exp(-X))w).$$

This is precisely

$$B(\pi(k)v, w) = B(v, \pi(k^{-1})w).$$ □

Let us recall that any locally compact Hausdorff group G acts on the Hilbert space $L^2(G)$ by the prescription

$$R(g)f(x) := f(xg).$$

The representation so obtained is unitary and if G is a Lie group the notion of smooth vectors in $L^2(G)$ makes sense. In the next section, we will need a criterion to establish that certain functions are smooth vectors in $L^2(G)$. We will make use of the following notion:

Definition 2.15. Let G be a Lie group and let (π, H) be a Hilbert representation of G. The *Gårding subspace* of H is the vector subspace of H spanned by the set

$$\{\pi(f)v \mid v \in H,\ f \in C_c^\infty(G)\}.$$

Proposition 2.16. *Let G be a Lie group with finitely many connected components, let (π, H) be a Hilbert representation of G. Then every vector in the Gårding subspace of H is a smooth vector in H.*

Proof. See [18, Lemma 1.6.1]. □

Recall that $f \in C^\infty(G)$ is called $\mathrm{Z}(\mathfrak{g}_\mathbb{C})$*-finite* if it is annihilated by an ideal of $\mathrm{Z}(\mathfrak{g}_\mathbb{C})$ of finite codimension. The criterion we need is the following result of Harish-Chandra:

Theorem 2.17. *Let G be a group in the class $\mathcal{H}$ as in [17, p. 192]. Let $f \in C^\infty(G)$ be K-finite and $\mathrm{Z}(\mathfrak{g}_\mathbb{C})$-finite. Then there exists a function $h \in C_c^\infty(G)$ which satisfies $h(kgk^{-1}) = h(g)$ for all $k \in K$ and for all $g \in G$ and such that $f * h = f$. If $f \in C^\infty(G)$, in addition, is square-integrable, then f is a smooth vector in $L^2(G)$.*

Proof. The first statement is [17, Proposition 14, p. 352]. The second conclusion follows from the observation found at the beginning of the proof of [11, Corollary 8.42] that f is in the Gårding subspace of $L^2(G)$ and it is therefore smooth by Proposition 2.16. That f is indeed in the Gårding subspace of $L^2(G)$ follows from the standard fact that

$$R(\tilde{\psi})f = f * \psi, \tag{2.18}$$

for every $\psi \in C_c^\infty(G)$. Here, $\tilde{\psi}(x) := \psi(x^{-1})$. The first statement then gives

$$R(\tilde{h})f = f * h = f. \tag{2.19}$$ □

Proposition 2.20. *Let G be a group in the class $\mathcal{H}$. Let $f \in C^\infty(G)$ be K-finite, $\mathrm{Z}(\mathfrak{g}_\mathbb{C})$-finite and square-integrable. Then, for every $X \in \mathfrak{g}$, we have*

$$Xf = \dot{R}(X)f,$$

where $Xf : G \to \mathbb{C}$ is defined as

$$Xf(g) := \frac{d}{dt}\big[f(g\exp(tX))\big]|_{t=0}. \tag{2.21}$$

Proof. By Theorem 2.17, there exists $h \in C_c^\infty(G)$ such that

$$f = f * h.$$

From

$$Xf = X(f * h) = f * Xh \quad \text{and} \quad f * Xh = \dot{R}(\widetilde{Xh})f,$$

the latter being an application of (2.18), we obtain

$$Xf = \dot{R}(\widetilde{Xh})f.$$

Since

$$\dot{R}(\widetilde{Xh})f = \dot{R}(X)R(\tilde{h})f \quad \text{and} \quad R(\tilde{h})f = f * h = f,$$

we conclude

$$Xf = \dot{R}(X)f. \qquad \square$$

We will apply Proposition 2.20 to the group M in the Langlands decomposition of a parabolic subgroup $P = MAN$ of a connected semisimple Lie group with finite centre. A group M of this form will not be connected, semisimple in general. However, it belongs to the class $\mathcal{H}$ by [5, Lemma 9, p. 108].

We briefly recall the construction of parabolically induced representations. We refer the reader to [13, Chapter XI], for a more thorough account.

Let G be a connected, semisimple Lie group with finite centre and let $P = MAN$ be a parabolic subgroup of G. The group $K_M := K \cap M$ is a maximal compact subgroup of M. Let λ be a complex-valued real-linear functional on $\mathfrak{a}$ and let (σ, H_σ) be a Hilbert representation of M. We define an action of G on the space of functions

$$\{f \in C(K, H_\sigma) \mid f(mk) = \sigma(m)f(k) \text{ for all } m \in K_M \text{ and all } k \in K\}$$

by declaring

$$\mathrm{Ind}_P(\sigma, \lambda, g)f(k) := e^{(\lambda+\rho)(\boldsymbol{h}(kg))}\sigma(\boldsymbol{m}(kg))f(\boldsymbol{k}(kg)),$$

where, if $g = kman$ for some $k \in K$, $m \in M$, $a \in A$, $n \in N$, we set $\boldsymbol{k}(g) := k$, $\boldsymbol{m}(g) := m$, $\boldsymbol{h}(g) := \log(a)$, $\boldsymbol{n}(g) := n$. The symbol ρ denotes half of the sum of the positive restricted roots determined by $\mathfrak{a}$ counted with multiplicities. On this space of functions, we introduce the norm

$$\|f\|_{\mathrm{Ind}_P(\sigma,\lambda)} := \left(\int_K \|f(k)\|_\sigma^2 \, dk \right)^{1/2}$$

and, upon completing, we obtain a Hilbert representation of G which we denote by $\mathrm{Ind}_P(\sigma, \lambda)$. We will denote by $\mathrm{Ind}_{P,K_M}(\sigma, \lambda)$ the space of K_M-finite vectors in $\mathrm{Ind}_P(\sigma, \lambda)$.

Proposition 2.22. *Let G be a connected, semisimple Lie group with finite centre and let $P = MAN$ be a parabolic subgroup of G. Let λ be a complex-valued, real-linear, totally imaginary functional on $\mathfrak{a}$ and let (σ, H_σ) be a unitary representation of M. Then $\mathrm{Ind}_P(\sigma, \lambda)$ is a unitary representation of G.*

Proof. See [13, Corollary 11.39]. $\square$

Corollary 2.23. *Let G be a connected, semisimple Lie group with finite centre and let $P = MAN$ be a parabolic subgroup of G. Let λ be a complex-valued, real-linear, totally imaginary functional on $\mathfrak{a}$ and let (σ, H_σ) be a unitary representation of M. Then, for every $f_1, f_2 \in \operatorname{Ind}_{P,K_M}(\sigma, \lambda)$ and for every $X \in \mathfrak{g}$, we have*

$$\langle \dot{\operatorname{Ind}}_P(\sigma, \lambda, X) f_1, f_2 \rangle_{\operatorname{Ind}_P(\sigma,\lambda)} = -\langle f_1, \dot{\operatorname{Ind}}_P(\sigma, \lambda, X) f_2 \rangle_{\operatorname{Ind}_P(\sigma,\lambda)}.$$

Proof. This is a consequence of Proposition 2.22 and the skew-invariance of the inner product on a unitary representation with respect to the action of the Lie algebra on the space of smooth vectors [19, p. 266]. □

Next, we recall a form of the Frobenius reciprocity originally observed by Casselman. We first need some preparation.

First of all, we record the following.

Lemma 2.24. *Let G be a connected, semisimple Lie group with finite centre and let $P = MAN$ be a parabolic subgroup of G. If V is a $(\mathfrak{g}, K)$-module, then the $(\mathfrak{g}, K)$-module structure on V induces a structure of $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$-module on $V \backslash \mathfrak{n} V$ in such a way that the quotient map*

$$q : V \to V/\mathfrak{n}V$$

is $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$-equivariant.

Proof. It suffices to show that if $v \in V$ is of the form

$$v = Xw$$

for some $w \in V$ and $X \in \mathfrak{n}$, then, for all $\xi \in K_M$, we have

$$\xi v \in \mathfrak{n}V,$$

and, for all $Y \in \mathfrak{m} \oplus \mathfrak{a}$, we have

$$Yv \in \mathfrak{n}V.$$

Let $\xi \in K_M$. We have

$$\xi v = \xi X w = \operatorname{Ad}(\xi) X \xi w$$

and, since K_M, being contained in M, normalises $\mathfrak{n}$ by [12, Proposition 7.83], it follows that $\operatorname{Ad}(\xi) X \in \mathfrak{n}$.

Let $Y \in \mathfrak{m} \oplus \mathfrak{a}$. We have

$$Yv = YXw = [Y, X]w + XYw.$$

The second term in the right-hand side belongs to $\mathfrak{n}V$ because $X \in \mathfrak{n}$ and the first belongs to $\mathfrak{n}V$ because $\mathfrak{n}$ is an ideal in $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$ by [12, Proposition 7.78]. □

Let us recall that a $(\mathfrak{g}, K)$-module is *finitely generated* if it is a finitely generated $U(\mathfrak{g}_{\mathbb{C}})$-module. We say that a Hilbert representation (π, H) of G is finitely generated if H_K is finitely generated. We record the following result of Casselman.

Theorem 2.25. *Let G be a connected, semisimple Lie group with finite centre and let $P = MAN$ be a parabolic subgroup of G. Let V be an admissible, finitely generated $(\mathfrak{g}, K)$-module. Then $V/\mathfrak{n}V$ is an admissible, finitely generated $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$-module.*

Proof. See [18, Lemma 4.3.1]. □

If V is an irreducible $(\mathfrak{g}, K)$-module, we say that V admits an infinitesimal character if the centre $Z(\mathfrak{g}_{\mathbb{C}})$ of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ acts on V by a character, that is, for every $Z \in Z(\mathfrak{g}_{\mathbb{C}})$ and for every $v \in V$, we have

$$Zv = \chi(Z)v,$$

where $\chi : Z(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C}$ is a morphism of complex, unital algebras. The action of $Z(\mathfrak{g}_{\mathbb{C}})$ on V in question is the one obtained by first extending the action of $\mathfrak{g}$ to an action of $\mathfrak{g}_{\mathbb{C}}$ and then to an action of $U(\mathfrak{g}_{\mathbb{C}})$ using the PBW theorem.

Corollary 2.26. *Let G be a connected, semisimple Lie group with finite centre and let $P = MAN$ be a parabolic subgroup of G. If V is an irreducible $(\mathfrak{g}, K)$-module admitting an infinitesimal character, then $V/\mathfrak{n}V$ is an admissible, finitely generated $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$-module.*

Proof. By [9, Theorem 2.2, p. 256], K is connected. By [13, Theorem 7.204], V is admissible. Combining [13, Example 1, p. 442] and [13, Corollary 7.207], it follows that V is finitely generated. The result now follows from Theorem 2.25. □

Let $\mathfrak{p}$, $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{n}$ denote the Lie algebras of P, M, A and N, respectively.

Let (σ, H_σ) be an admissible and finitely generated Hilbert representation of M which is unitary when restricted to K_M. Let λ be a complex-valued real-linear functional on $\mathfrak{a}$. Consider the $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$-module H^λ_{σ,K_M} defined as

$$H^\lambda_{\sigma,K_M} := H_{\sigma,K_M} \otimes \mathbb{C}_{\lambda+\rho},$$

where the pair $(\mathfrak{m}, K_M)$ acts on H_{σ,K_M} and $\mathfrak{a}$ acts on $\mathbb{C}_{\lambda+\rho}$ via the functional $\lambda + \rho$.

If V is a $(\mathfrak{g}, K)$-module and $T \in \mathrm{Hom}_{\mathfrak{g},K}(V, \mathrm{Ind}_{P,K_M}(\sigma, \lambda))$, then we can define an element $\hat{T} \in \mathrm{Hom}_{\mathfrak{m}\oplus\mathfrak{a},K_M}(V/\mathfrak{n}V, H^\lambda_{\sigma,K_M})$ by setting

$$\hat{T}(v) := T(v)(1).$$

Theorem 2.27. *Let G be a connected, semisimple Lie group with finite centre. Let V be a $(\mathfrak{g}, K)$-module. Let (σ, H_σ) be an admissible and finitely generated Hilbert* *representation of M which is unitary when restricted to K_M and let λ be a complex-valued real-linear functional on $\mathfrak{a}$. Consider the $(\mathfrak{m} \oplus \mathfrak{a}, K_M)$-module $H^{\lambda}_{\sigma, K_M}$. Then the map*

$$\mathrm{Hom}_{\mathfrak{g},K}(V, \mathrm{Ind}_{P,K_M}(\sigma, \lambda)) \to \mathrm{Hom}_{\mathfrak{m}\oplus\mathfrak{a},K_M}(V/\mathfrak{n}V, H^{\lambda}_{\sigma,K_M}), \quad T \mapsto \hat{T},$$

is a bijection.

Proof. See [18, Lemma 5.2.3] and the discussion preceding it. □

For clarity, we point out that the formulation in [18] seems to contain some typos and so we modified it following [8, Theorem 4.9].

The inverse of the map $T \mapsto \hat{T}$ is constructed as follows (see [18, Lemmas 5.2.3 and 3.8.2] or, alternatively, [8, Theorem 4.9]). Let $S \in \mathrm{Hom}_{\mathfrak{m}\oplus\mathfrak{a},K_M}(V/\mathfrak{n}V, H^{\lambda}_{\sigma,K_M})$. Then we obtain an element

$$\tilde{S} \in \mathrm{Hom}_{\mathfrak{g},K}(V, \mathrm{Ind}_{P,K_M}(\sigma, \lambda))$$

by setting

$$\tilde{S}(v)(k) := S(q(kv)),$$

where $q : V \to V/\mathfrak{n}V$ denotes the quotient map. Then the inverse of $T \mapsto \hat{T}$ is given by the map

$$\mathrm{Hom}_{\mathfrak{m}\oplus\mathfrak{a},K_M}(V/\mathfrak{n}V, H^{\lambda}_{\sigma,K_M}) \to \mathrm{Hom}_{\mathfrak{g},K}(V, \mathrm{Ind}_{P,K_M}(\sigma, \lambda)), \quad S \mapsto \tilde{S}.$$

3. Asymptotic behaviour of representations

3.1. ***Asymptotic expansions of matrix coefficients.*** We begin by collecting the fundamental facts concerning asymptotic expansions of matrix coefficients of tempered representations. We refer the reader to [11, Chapter VIII] for a more thorough exposition of the topic.

Let G be a connected, semisimple Lie group with finite centre, let K be a fixed maximal compact subgroup of G and let $\mathfrak{k}$ be its Lie algebra. Let $P = MAN$ denote the minimal parabolic subgroup of G with Lie algebra $\mathfrak{p}$. Given a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$, we call A the corresponding subgroup of P and M the centraliser of A in K. We fix a system Δ of simple roots of the restricted root system attached to $(\mathfrak{g}, \mathfrak{a})$, and we use Δ^+ to denote the corresponding set of positive roots.

Let $\mathfrak{a}^+$ denote the set $\{H \in \mathfrak{a} \mid \alpha(H) > 0 \text{ for all } \alpha \in \Delta\}$. Then the subset of regular elements G^{reg} of G admits a decomposition as $G^{\mathrm{reg}} = K \exp(\mathfrak{a}^+) K$ and G itself admits a decomposition $G = K\overline{\exp(\mathfrak{a}^+)}K$.

We write $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ and we identify it with the ordered set $\{1, \ldots, n\}$ in the obvious way. We adopt the following notation to simplify the appearance of the expansions we are going to work with.

For $H \in \mathfrak{a}$ and $l \in \mathbb{Z}^n_{\geq 0}$, we set $\alpha(H)^l := \prod_{i=1}^n \alpha_i(H)^{l_i}$.

If λ is a real-linear complex-valued functional on $\mathfrak{a}$, since, for every $H \in \mathfrak{a}$,

$$\lambda(H) = \sum_{i=1}^{n} \lambda_i \alpha_i(H)$$

for some $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$, we will often identify λ with the n-tuple $(\lambda_1, \ldots, \lambda_n)$.

The next result is concerned with the expansion of K-finite matrix coefficients relative to P.

Theorem 3.1. *Let G be a connected, semisimple Lie group with finite centre and let (π, H) be an irreducible, Hilbert representation of G. Then there exist a nonnegative integer l_0 and a finite set of real-linear complex-valued functionals on $\mathfrak{a}$, denoted by $\mathcal{E}_0$, such that, for every $v, w \in H_K$, the restriction to* $\exp(\mathfrak{a}^+)$ *of the matrix coefficient*

$$\phi_{v,w}(g) = \langle \pi(g)v, w \rangle$$

admits a uniformly and absolutely convergent expansion as

$$\phi_{v,w}(\exp H) = e^{-\rho(H)} \sum_{\lambda \in \mathcal{E}_0} \sum_{\substack{l \in \mathbb{Z}^n_{\geq 0} \\ |l| \leq l_0}} \sum_{k \in \mathbb{Z}^n_{\geq 0}} \alpha(H)^l e^{(\lambda - k)(H)} \langle c_{\lambda-k,l}(v), w \rangle,$$

where each $c_{\lambda-k,l} : H_K \to H_K$ is a complex-linear map and $\rho_{\mathfrak{p}}$ denotes half of the sum of the elements in Δ^+ counted with multiplicities. The maps $c_{\lambda-k,l}$ are completely determined by the representation (π, H).

Proof. By Theorem 2.9, (π, H) is admissible and therefore has an infinitesimal character. By [11, Theorem 8.32], we have the stated expansion for any τ-spherical function (in the sense of [11, p. 215]) F on G of the form

$$F(g) = E_2 \pi(g) E_1,$$

where τ_1 and τ_2 are subrepresentations of

$$\pi|_K \cong \bigoplus_{\gamma \in \tilde{K}} n_\gamma \gamma$$

of the form

$$\tau_1 := \bigoplus_{\gamma \in \Theta_1} n_\gamma \gamma \quad \text{and} \quad \tau_2 := \bigoplus_{\gamma \in \Theta_2} n_\gamma \gamma$$

for finite collections $\Theta_1, \Theta_2 \in \hat{K}$, and E_1, E_2 are the orthogonal projections to τ_1, τ_2, respectively. In this expansion, the set $\mathcal{E}_0$, the maps $c_{\lambda-k,l}$ and the number l_0 depend on $\tau = (\tau_1, \tau_2)$ and we can expand $\phi_{v,w}$ provided that $v \in \tau_1$ and $w \in \tau_2$. To obtain an expansion valid for every $v, w \in H_K$ and with $\mathcal{E}_0$, l_0 and the $c_{\lambda-k,l}$ independent of τ, we appeal to the theory developed in [2, Section 8], which we can apply since (π, H) is finitely generated by [13, Corollary 7.207] (for clarity, we should mention that the setup in [11] is different, but entirely equivalent to that in [2], translating between the two is just bookkeeping). First, by [2, Theorem 8.7], the representation (π, H) has a unique matrix coefficient map in the sense of [2, p. 907]. The required expansion on the region $\exp(\mathfrak{a}^+)$ of the matrix coefficient map is given by [2, Theorem 8.8]. For completeness, the relation between the τ-dependent expansion and the expansion in our statement is given in [2, Lemma 8.3]. □

We recall that if ν, ν' are real-linear complex-valued functionals on $\mathfrak{a}$ such that $\nu - \nu'$ is an integral linear combination of the simple roots, then we say that ν and ν' are *integrally equivalent*.

The set $\mathcal{E}_0$ has the property that if $\lambda, \lambda' \in \mathcal{E}_0$ with $\lambda \neq \lambda'$, then λ and λ' are not integrally equivalent.

If ν and ν' are integrally equivalent and $\nu - \nu'$ is a nonnegative integral combination of the simple roots, we write $\nu \geq \nu'$, thus introducing an order relation among integrally equivalent functionals on $\mathfrak{a}$.

If $k \in \mathbb{Z}^n_{\geq 0}$ is such that the term

$$\alpha(H)^l e^{(\lambda-k)(H)} \langle c_{\lambda-k,l}(v), w\rangle$$

is nonzero for some $\lambda \in \mathcal{E}_0$ and for some $v, w \in H_K$, then we say that $\nu := \lambda - k$ is an *exponent* and we let $\mathcal{E}$ denote the set of exponents. The exponents which are maximal with respect to the order relation introduced above are called *leading exponents*: $\mathcal{E}_0$ is precisely the set of leading exponents.

The following result is used crucially in [10] and in the following.

Theorem 3.2. *Let (π, H) be an irreducible, tempered, Hilbert representation of G. Then every $\lambda \in \mathcal{E}_0$ satisfies*

$$\operatorname{Re} \lambda_i \leq 0$$

for every $i \in \{1, \dots, n\}$.

Proof. See [11, Theorem 8.53]. Strictly speaking, in [loc. cit.] the theorem is formulated under some restrictions on G, but it is a convenient reference since we are adopting the same normalisation of the exponents. See [1, Proposition 3.7, p. 83] or [2, Corollary 8.12], for proofs for more general groups. □

We now turn to asymptotic expansions of matrix coefficients of (π, H) relative to standard (for P) parabolic subgroups of G. We follow [11, Chapter VIII, Section 12].

Given a subset $I \subset \{1, \dots, n\}$, and recalling that we identified Δ with $\{1, \dots, n\}$, we can associate to it a parabolic subgroup

$$P_I = M_I A_{I^c} N_{I^c}$$

of G containing P in such a way that the restricted root space $\mathfrak{g}_{-\alpha}$ satisfies $\mathfrak{g}_{-\alpha} \subset \mathfrak{m}_I$ if and only if $\alpha \in I$ (with $\mathfrak{m}_I$ denoting the Lie algebra of M_I). For the details, we refer the reader to [11, Proposition 5.23; 12, Chapter VII].

First, we introduce the basis $\{H_1, \dots, H_n\}$ of $\mathfrak{a}$ dual to Δ. We define the Lie algebra $\mathfrak{a}_I$ as

$$\mathfrak{a}_I := \sum_{i\in I} \mathbb{R}H_i$$

and the group A_I as

$$A_I := \exp\Bigl(\sum_{i\in I} \mathbb{R}\alpha_i\Bigr).$$

We can then write

$$\mathfrak{a} = \mathfrak{a}_I \oplus \mathfrak{a}_{I^c} \quad \text{and} \quad A = A_I A_{I^c}.$$

The groups N_I and N_{I^c} are the analytic subgroups of G corresponding to the Lie algebras

$$\mathfrak{n}_I := \sum_{\substack{\beta\in\Delta^+\\ \beta|_{\mathfrak{a}_{I^c}}=0}} \mathfrak{g}_\beta \quad \text{and} \quad \mathfrak{n}_{I^c} := \sum_{\substack{\beta\in\Delta^+\\ \beta|_{\mathfrak{a}_{I^c}}\neq 0}} \mathfrak{g}_\beta.$$

We have

$$\rho = \rho_I + \rho_{I^c}$$

with

$$\rho_I := \frac{1}{2} \sum_{\substack{\beta\in\Delta^+\\ \beta|_{\mathfrak{a}_{I^c}}=0}} (\dim \mathfrak{g}_\beta)\beta$$

and analogously for ρ_{I^c}. Denoting by $M_{0,I}$ the analytic subgroup of G corresponding to the Lie algebra

$$\mathfrak{m}_I = \mathfrak{m} \oplus \mathfrak{a}_I \oplus \mathfrak{n}_I \oplus \overline{\mathfrak{n}_I},$$

the group M_I is then given as

$$M_I := Z_K(\mathfrak{a}_{I^c})M_{0,I}.$$

Finally, $K_I := K \cap M_I$ is a maximal compact subgroup of M_I and MA_IN_I is a minimal parabolic subgroup of M_I.

Theorem 3.3. *Let G be a connected, semisimple Lie group with finite centre and let (π, H) be an irreducible, Hilbert representation of G. Let C be a compact subset of M_I satisfying $K_ICK_I = C$. Then there exists a positive real number R depending on C such that, for every $m \in C$ and for every $a = \exp H \in A_{I^c}$ which satisfies $\alpha_i(H) > \log R$ for every $i \in I^c$, we have*

$$\phi_{v,w}(m \exp H) = e^{-\rho_{I^c}(H)} \sum_{\nu\in\mathcal{E}_I} \sum_{\substack{q\in\mathbb{Z}^{I^c}_{\geq 0}\\ |q|\leq q_0}} \alpha(H)^q e^{\nu(H)} c^{P_I}_{\nu,q}(m, v, w)$$

for every $v, w \in H_K$. Here, $\mathcal{E}_I$ is a countable set of real-linear complex-valued functionals on $\mathfrak{a}_{I^c}$, each $c^{P_I}_{\nu,q}$ extends to a real-analytic function on M_I and satisfies

$$c^{P_I}_{\nu,q}(\xi_2 m \xi_1, v, w) = c^{P_I}_{\nu,q}(m, \pi(\xi_1)v, \pi(\xi_2^{-1})w)$$

for every $\xi_1, \xi_2 \in K_I$. Moreover, for every $m \in M_I$ and $w \in H_K$, the map

$$H_K \to \mathbb{C}, \quad v \mapsto c^{P_I}_{\nu,q}(m, v, w),$$

is complex-linear and, for every $m \in M_I$ and $v \in H_K$, the map

$$H_K \to \mathbb{C}, \quad w \mapsto c^{P_I}_{\nu,q}(m, v, w),$$

is conjugate-linear. The maps $c^{P_I}_{\nu,q} : M_I \times H_K \times H_K \to \mathbb{C}$ are completely determined by the representation (π, H).

Proof. For a τ-spherical function F as in the proof of Theorem 3.1, the result follows from [11, Theorem 8.45]. To obtain an expansion independent of τ, it suffices to prove that each $F_{\nu-\rho_{I^c_\lambda}}$ is independent of τ. Let $m \in M_I$ and write $m = \xi_2 a_I \xi_2$ for some $a_I \in \overline{A_I^+}$, where A_I^+ is the positive Weyl chamber, and some $\xi_1, \xi_2 \in K_I$. Since

$$F_{\nu-\rho_{I^c}}(ma, v, w) = F_{\nu-\rho_{I^c}}(a_I a, \pi(\xi_1)v, \pi(\xi_2^{-1})w),$$

relabelling things, it suffices to prove that $F_{\nu-\rho_{I^c}}(\,\cdot\,, v, w)$ is independent of τ as a function on $\overline{A_I^+} A_{I^c}$. By [11, Corollary 8.46], the functional $\nu \in \mathcal{E}_{\mathcal{I}}$ is the restriction of an element in the set of exponents $\mathcal{E}$ in the expansion relative to P and this set is independent of τ by [2, Theorem 8.8]. Therefore, it remains to prove that each $c^{P_\lambda}_{\nu,q}$ is independent of τ. Since $c^{P_\lambda}_{\nu,q}$ is analytic on M_I, it suffices to prove that $c^{P_\lambda}_{\nu,q}(\,\cdot\,, v, w)$ as a function on A_I^+ is independent of τ. Given $a_I \in A_I^+$, we can find a compact subset C of M_I containing and a_I such that $K_I C K_I = C$, and a positive R depending on C, such that for every $H \in \mathfrak{a}_{I^c}$ satisfying $\alpha_i(H) > \log R$ for every $i \in I^c$, the expansion of $\phi_{v,w}(a_I a)$ relative to P and the expansion relative to P_I are both valid. Comparing them as in [11, p. 251], it follows that expansion relative to P_I is completely determined by the expansion relative to P and the latter is independent of τ by Theorem 3.1. □

For every $\nu \in \mathcal{E}_I$, the term

$$\alpha(H)^q e^{(\nu-\rho_{I^c})(H)} c^{P_I}_{\nu,q}(m, v, w)$$

is nonzero for some $v, w \in H_K$ and some $m \in M$. The set $\mathcal{E}_I$ is the set of *exponents relative to P_I*.

To define the functions of the form $\Gamma_{\lambda,l}$ discussed in the Introduction, the first step consists in associating a standard (for P) parabolic subgroup of G to each $\lambda \in \mathcal{E}_0$.

Let (π, H) be an irreducible, tempered, Hilbert representation of G and let $\lambda \in \mathcal{E}_0$. We set $I_\lambda := \{i \in \{1, \dots, n\} \mid \operatorname{Re} \lambda_i < 0\}$ which we identify with the subset Δ_λ of Δ defined as

$$\Delta_\lambda := \{\alpha_i \in \Delta \mid i \in I_\lambda\}.$$

The construction of standard parabolic subgroups from the datum of a subset of Δ assigns to I_λ the standard parabolic subgroup P_λ defined as

$$P_\lambda := P_{I_\lambda}.$$

It admits a decomposition

$$P_\lambda = M_\lambda A_{\lambda_0} N_{\lambda_0},$$

where

$$A_{\lambda_0} := A_{I_\lambda^c}.$$

The subgroup M admits a decomposition

$$M_\lambda = K_\lambda A_\lambda K_\lambda,$$

where

$$A_\lambda := A_{I_\lambda} \quad \text{and} \quad K_\lambda := K \cap M_\lambda.$$

The group A decomposes as $A = A_\lambda A_{\lambda_0}$. We write $\mathfrak{a}_\lambda$ and $\mathfrak{a}_{\lambda_0}$ for $\mathfrak{a}_{I_\lambda}$ and $\mathfrak{a}_{I_\lambda^c}$, respectively. Similarly, we write ρ_λ and ρ_{λ_0} for ρ_{I_λ} and $\rho_{I_\lambda^c}$, respectively.

Remark 3.4. The theory recalled so far is sufficient to prove that tempered, irreducible, Hilbert representations are unitarisable. From now on, given a tempered, irreducible, Hilbert representation (π, H), we will implicitly assume that it is unitary and we will refer to it simply as a tempered, irreducible representation.

3.2. ***The functions*** $\boldsymbol{\Gamma_{\lambda,l}}$**.** We are going to introduce an equivalence relation on the data indexing the expansion of $\phi_{v,w}$ relative to P. The definition is motivated by the construction of $\boldsymbol{d}(\pi)$ in [10]. Let $v, w \in H_K$. We have

$$\phi_{v,w}(\exp H) = e^{-\rho(H)} \sum_{\lambda \in \mathcal{E}_0} \sum_{\substack{l \in \mathbb{Z}^n_{\geq 0} \\ |l| \leq l_0}} \alpha(H)^l e^{\lambda(H)} \Phi^{v,w}_{\lambda,l}(H),$$

where

$$\Phi^{v,w}_{\lambda,l}(H) := \sum_{k \in \mathbb{Z}^n_{\geq 0}} e^{-k(H)} \langle c_{\lambda-k,l}(v_1), v_2 \rangle.$$

The terms in this expansion are indexed by the finite set

$$\mathcal{C} := \{(\lambda, l)\}_{\substack{\lambda \in \mathcal{E}_0 \\ l \in \mathbb{Z}^n_{\geq 0} \\ |l| \leq l_0}}.$$

We introduce a relation on $\mathcal{C}$ by declaring that $(\lambda, l) \sim (\mu, m)$ if $I_\lambda = I_\mu$, $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$ and $\operatorname{res}_{I_\lambda^c} l = \operatorname{res}_{I_\mu^c} m$. To define this relation we have implicitly used the identification of I_λ with the subset Δ_λ of Δ at the end of the previous subsection.

It is clear that $\sim$ is an equivalence relation. We denote by $[\lambda, l]$ the equivalence class containing (λ, l).

We can therefore regroup the expansion of $\phi_{v,w}$ as

$$\phi_{v,w}(\exp H) = e^{-\rho(H)} \sum_{[\lambda,l]\in\mathcal{C}/\sim} \alpha(H_{\lambda_0})^{l_{\lambda_0}} e^{\lambda|_{\mathfrak{a}_{\lambda_0}}(H_{\lambda_0})} \sum_{(\lambda',l')\in[\lambda,l]} \alpha(H_\lambda)^{l'_\lambda} e^{\lambda'|_{\mathfrak{a}_\lambda}(H_\lambda)} \Phi^{v,w}_{\lambda',l'}(H),$$

where

$$l_{\lambda_0} := \operatorname{res}_{I_\lambda^c} l, \quad \alpha(H_{\lambda_0})^{l_{\lambda_0}} := \prod_{i\in I_\lambda^c} \alpha_i(H_{\lambda_0})^{l_i}, \quad l'_\lambda := \operatorname{res}_{I_\lambda} l', \quad \alpha(H_\lambda)^{l'_\lambda} := \prod_{i\in I_\lambda} \alpha(H_\lambda)^{l'_i}$$

and $H = H_{\lambda_0} + H_\lambda$ corresponds to the decomposition

$$\mathfrak{a}^+ = \mathfrak{a}^+_{\lambda_0} \oplus \mathfrak{a}^+_\lambda.$$

We are also implicitly using the fact that $\alpha(H)^l = \alpha(H_\lambda)^{l_\lambda} \alpha(H_{\lambda_0})^{l_{\lambda_0}}$ which follows from writing H with respect to the basis dual to Δ.

To proceed, we need to isolate certain equivalence classes in $\mathcal{C}/\sim$. First, we recall from the Introduction how the quantity $\boldsymbol{d}_P(\lambda, l)$, for $(\lambda, l) \in \mathcal{C}$ and P a fixed minimal parabolic subgroup of G, and the quantity $\boldsymbol{d}(\pi)$ are defined.

For $(\lambda, l) \in \mathcal{C}$, we set

$$\boldsymbol{d}_P(\lambda, l) := |I_\lambda^c| + \sum_{i\in I_\lambda^c} 2l_i$$

and we observe that this number only depends on the equivalence class of (λ, l). Then we take the maximum, $\boldsymbol{d}_P$, as (λ, l) ranges over $\mathcal{C}$. We can proceed analogously for every standard (for P) parabolic subgroup of P' of G to obtain a nonnegative integer $\boldsymbol{d}_{P'}$. Then $\boldsymbol{d}(\pi)$ is defined to be the maximum over all P' of the quantities $\boldsymbol{d}_{P'}$.

Definition 3.5. Let $[\lambda, l] \in \mathcal{C}/\sim$. We say that $[\lambda, l]$ is *relevant* if it satisfies

$$\boldsymbol{d}_P(\lambda, l) = \boldsymbol{d}(\pi),$$

where $\boldsymbol{d}_P(\lambda, l)$ is defined by (1.7).

Let $[\lambda, l] \in \mathcal{C}/\sim$ be a relevant equivalence class. For $H_\lambda \in \mathfrak{a}^+_\lambda$, we set

$$\Gamma_{\lambda,l}(\exp H_\lambda, v, w) := e^{-\rho(H)} \sum_{(\lambda',l')\in[\lambda,l]} \alpha(H_\lambda)^{l'_\lambda} e^{\lambda'|_{\mathfrak{a}_{\lambda_0}}(H_\lambda)} \Phi^{v,w}_{\lambda',l'}(H_\lambda). \tag{3.6}$$

Before establishing the properties of $\Gamma_{\lambda,l}$, let us pause to explain the motivation behind the definitions above. The discussion that follows will be used only in

Section 4. The reader who prefers to do so can skip to Proposition 3.12 without any loss of continuity.

Let $v_1, v_2, v_3, v_4 \in H_K$. We will be considering integrals of the form

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{\mathfrak{a}^+_{<r}} \phi_{v_1,v_2}(\exp H)\overline{\phi_{v_3,v_4}(\exp H)} \prod_{\beta\in\Delta^+} (e^{\beta(H)} - e^{-\beta(H)})^{\dim \mathfrak{g}_\beta}\, dH,$$

where

$$\mathfrak{a}^+_{<r} := \mathfrak{a}^+ \cap \{H \in \mathfrak{a} \mid \beta(H) < r \text{ for all } \beta \in \Delta^+\}. \tag{3.7}$$

Treating these is the content of [10, Appendix A]. We remark that our region of integration is defined as to exclude the subset of $\overline{\mathfrak{a}^+}$ where at least one of the simple roots vanishes. It is a set of measure zero.

We want to interpret [10, Lemma A.5] in group-theoretic terms.

Let us consider the matrix coefficients ϕ_{v_1,v_2} and ϕ_{v_3,v_4}. On $A^+ := \exp(\mathfrak{a}^+)$, they can be expanded as

$$\phi_{v_1,v_2}(\exp H) = e^{-\rho(H)} \sum_{[\lambda,l]\in\mathcal{C}/\sim} \alpha(H_{\lambda_0})^{l_{\lambda_0}} e^{\lambda|_{\mathfrak{a}_{\lambda_0}}(H_{\lambda_0})} \sum_{(\lambda',l')\in[\lambda,l]} \Psi^{v_1,v_2}_{\lambda',l'}(H)$$

and

$$\phi_{v_3,v_4}(\exp H) = e^{-\rho(H)} \sum_{[\mu,m]\in\mathcal{C}/\sim} \alpha(H_{\mu_0})^{m_{\mu_0}} e^{\mu|_{\mathfrak{a}_{\mu_0}}(H_{\mu_0})} \sum_{(\mu',m')\in[\mu,m]} \Psi^{v_3,v_4}_{\mu',m'}(H),$$

where, for $(\lambda', l') \in [\lambda, l]$, we set

$$\Psi^{v_1,v_2}_{\lambda',l'}(H) := \alpha(H_\lambda)^{l'_\lambda} e^{\lambda'|_{\mathfrak{a}_\lambda}(H_\lambda)} \Phi^{v_1,v_2}_{\lambda',l'}(H)$$

and similarly for $(\mu', m') \in [\mu, m]$. Let $[\lambda, l] \in \mathcal{C}/\sim$ and $[\mu, m] \in \mathcal{C}/\sim$ be such that $I_\lambda = I_\mu$, $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$ and

$$\boldsymbol{d}(\pi) = |I_\lambda| + \sum_{i\in I_\lambda} (l_i + m_i).$$

In view of the first condition, the third is equivalent to the requirement

$$\boldsymbol{d}_P(\lambda, l) = \boldsymbol{d}(\pi) \quad \text{and} \quad \boldsymbol{d}_P(\mu, m) = \boldsymbol{d}(\pi).$$

Consider the summand

$$e^{-2\rho(H)} \alpha(H)^{l'+m'} e^{(\lambda'+\overline{\mu'})(H)} \Phi^{v_1,v_2}_{\lambda',l'} \overline{\Phi^{v_3,v_4}_{\mu',m'}}(H)$$

in the expansion of the product $\phi_{v_1,v_2}\overline{\phi_{v_3,v_4}}$ on A^+. Taking into account the factor $e^{-2\rho(H)}$ and the fact that the term

$$\Omega(H) := \prod_{\beta\in\Delta^+} (e^{\beta(H)} - e^{-\beta(H)})^{\dim \mathfrak{g}_\beta} \tag{3.8}$$

is incorporated in the function ϕ in [10, Lemma A.5] (compare with Section 4.7 in [loc. cit.]), this lemma shows that, as $r \to \infty$, the integral

$$\frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{\mathfrak{a}^+_{<r}} e^{-2\rho(H)} \alpha(H)^{l'+m'} e^{(\lambda'+\overline{\mu'})(H)} \Phi^{v_1,v_2}_{\lambda',l'} \overline{\Phi^{v_3,v_4}_{\mu',m'}}(H)\Omega(H)\, dH$$

tends to

$$C(\lambda,l,m) \int_{\mathfrak{a}^+_\lambda} e^{-2\rho_\lambda(H_\lambda)} [\Psi^{v_1,v_2}_{\lambda',l'} \overline{\Psi^{v_3,v_4}_{\mu',m'}}]|_{\mathfrak{a}_\lambda}(H_\lambda)\Omega_\lambda(H_\lambda)\, dH_\lambda,$$

where

$$\Omega_\lambda(H_\lambda) := \prod_{\beta\in\Delta^+_\lambda} (e^{\beta(H_\lambda)} - e^{-\beta(H_\lambda)})^{\dim \mathfrak{g}_\beta}, \tag{3.9}$$

with

$$\Delta^+_\lambda := \{\beta \in \Delta^+ \mid \beta|_{\mathfrak{a}_{\lambda_0}} = 0\},$$

and the quantity $C(\lambda, l, m)$ is given by

$$C(\lambda,l,m) := \int_{\{H\in\mathfrak{a}_{\lambda_0} | \mathrm{ext}^{I^c_\lambda}(H)\in\mathfrak{a}^+_{<1}\}} \alpha(H_{\lambda_0})^{l_{\lambda_0}+m_{\mu_0}}\, dH_{\lambda_0}. \tag{3.10}$$

Now, summing over all $(\lambda', l') \in [\lambda, l]$ and over all $(\mu', m') \in [\mu, m]$, we obtain that the integral over $\mathfrak{a}^+_{<r}$ of

$$e^{-2\rho(H)} \sum_{(\lambda',l')\in[\lambda,l]} \sum_{(\mu',m')\in[\mu,m]} \alpha(H)^{l'+m'} e^{(\lambda'+\overline{\mu'})(H)} \Phi^{v_1,v_2}_{\lambda',l'} \overline{\Phi^{v_3,v_4}_{\mu',m}}(H)\Omega(H),$$

upon multiplying by $1/r^{\boldsymbol{d}(\pi)}$ and letting $r \to \infty$, equals

$$C(\lambda,l,m) \int_{\mathfrak{a}^+_\lambda} e^{-2\rho_\lambda(H_\lambda)} \sum_{(\lambda',l')\in[\lambda,l]} \sum_{(\mu',m')\in[\mu,m]} [\Psi^{v_1,v_2}_{\lambda',l'} \overline{\Psi^{v_3,v_4}_{\mu',m'}}]|_{\mathfrak{a}_\lambda}(H_\lambda)\Omega_\lambda(H_\lambda)\, dH_\lambda.$$

Finally, since

$$\Phi^{v_1,v_2}_{\lambda',l'}|_{\mathfrak{a}_\lambda}(H_\lambda) = \sum_{k\in\mathbb{Z}^{I_\lambda}_{\geq 0}} e^{-k(H_\lambda)} \langle c_{\lambda'-k,l'}(v_1), v_2\rangle,$$

and similarly for $\Phi^{v_3,v_4}_{\mu',m,}$, the integral above equals

$$C(\lambda,l,m) \int_{\mathfrak{a}^+_\lambda} \Gamma_{\lambda,l}(\exp H_\lambda, v, w) \overline{\Gamma_{\mu,m}(\exp H_\lambda, v, w)} \Omega_\lambda(H_\lambda)\, dH_\lambda. \tag{3.11}$$

If $[\lambda, l], [\mu, m] \in \mathcal{C}/{\sim}$ fail to satisfy any of the three conditions $I_\lambda = I_\mu$, $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_\lambda}$ and

$$\boldsymbol{d}_P(\lambda, l) = \boldsymbol{d}(\pi) = \boldsymbol{d}_P(\mu, m),$$

then, for every $(\lambda', l') \in [\lambda, l]$ and for every $(\mu', m') \in [\mu, m]$, by the considerations in the proof of Claim A.6 and Lemma A.5 in [10], the integral

$$\frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{A^+_{<r}} e^{-2\rho(H)} \alpha(H)^{l'+m'} e^{(\lambda'+\overline{\mu'})(H)} \Phi^{v_1,v_2}_{\lambda',l'} \overline{\Phi^{v_3,v_4}_{\mu',m'}}(H)\Omega(H)\, dH$$

vanishes as $r \to \infty$.

Therefore, the relevant equivalence classes $[\lambda, l] \in \mathcal{C}/\sim$, those for which the functions of the form $\Gamma_{\lambda,l}$ are defined, are precisely the ones that may contribute a nonzero term to the expression

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{\mathfrak{a}^+_{<r}} \phi_{v_1,v_2}(\exp H)\overline{\phi_{v_3,v_4}(\exp H)}\Omega(H)\, dH.$$

Throughout the rest of this section, we fix a tempered, irreducible representation of a connected, semisimple Lie group G with finite centre.

3.3. ***Some properties of the functions*** $\boldsymbol{\Gamma_{\lambda,l}}$**.** To study the properties of $\Gamma_{\lambda,l}$, we begin by showing that it is equal to a function of the form $c^{P_\lambda}_{\nu,q}$. More precisely, we have:

Proposition 3.12. *Let $v, w \in H_K$. Let $[\lambda, l] \in \mathcal{C}/\sim$ be a relevant equivalence class. Set $\nu := \lambda|_{\mathfrak{a}_{\lambda_0}}$ and $q := l_{\lambda_0}$. Then, for every $H_\lambda \in \mathfrak{a}^+_\lambda$, we have*

$$\Gamma_{\lambda,l}(\exp H_\lambda, v, w) = c^{P_\lambda}_{\nu,q}(\exp H_\lambda, v, w).$$

Proof. For every $H_\lambda \in \mathfrak{a}^+_\lambda$, we can find a compact subset C of M_λ such that $K_\lambda C K_\lambda = C$ and which contains H_λ, and a positive real $R > 0$ such that if $H_{\lambda_0} \in \mathfrak{a}^+_{\lambda_0}$ satisfies $\alpha_i(H_{\lambda_0}) > \log R$ for every $i \in I^c_\lambda$, then the expansion of $\phi_{v,w}$ with respect to P and the expansion with respect to P_λ are both valid at $H = H_\lambda + H_{\lambda_0}$. Comparing them as in [11, p. 251], we see that

$$c^{P_\lambda}_{\nu,q}(\exp H_\lambda, v, w) = \sum_{\substack{\lambda' \in \mathcal{E}_0 \\ \lambda'|_{\mathfrak{a}_{\lambda_0}} = \nu}} \sum_{\substack{l' \\ |l'| \le l_0 \\ l'_{\lambda_0} = q}} e^{-\rho_\lambda(H_\lambda)} \Psi^{v,w}_{\lambda',l'}(H_\lambda).$$

Since, by definition of $\Gamma_{\lambda,l}(\,\cdot\,, v, w)$, we have

$$\Gamma_{\lambda,l}(\exp H_\lambda, v, w) = e^{-\rho(H_\lambda)} \sum_{(\lambda',l') \in [\lambda,l]} \Psi^{v,w}_{\lambda',l'}(H_\lambda),$$

recalling the definition of the equivalence relation that we imposed on $\mathcal{C}$, we only need to show that the set

$$\{\lambda' \in \mathcal{E}_0 \mid \lambda'|_{\mathfrak{a}_{\lambda_0}} = \nu\}$$

is equal to the set

$$\{\lambda \in \mathcal{E}_0 \mid I_{\lambda'} = I_\lambda \text{ and } \lambda'|_{\mathfrak{a}_{\lambda_0}} = \lambda|_{\mathfrak{a}_{\lambda_0}}\}.$$

Because of the assumption on $[\lambda, l]$, for every $\lambda' \in \mathcal{E}_0$ such that $\lambda'|_{\mathfrak{a}_{\lambda_0}} = \nu$, we have $\operatorname{Re}\lambda'_j \neq 0$ for every $j \in I_\lambda$. Indeed, if there existed a $j \in I_\lambda$ for which $\operatorname{Re}\lambda'_j = 0$, we would have

$$|I^c_{\lambda'}| \geq 1 + |I^c_\lambda|$$

and, since $l'_{\lambda_0} = l_{\lambda_0}$, this would imply

$$\boldsymbol{d}_P(\lambda', l') > |I^c_\lambda| + \sum_{i \in I^c_{\lambda'}} 2l'_i \geq \boldsymbol{d}_P(\lambda, l) = \boldsymbol{d}(\pi),$$

contradicting the maximality of $\boldsymbol{d}(\pi)$. Since, by Theorem 3.2, we have $\operatorname{Re}\lambda'_i \leq 0$ for every $i \in \{1, \dots, n\}$, this concludes the proof. □

Theorem 8.45 in [11] and the discussion at the beginning of p. 251 in [loc. cit.] now show that $\Gamma_{\lambda,l}(\,\cdot\,, v, w)$, being equal to $c^{P_\lambda}_{v,q}$, extends to an analytic function on M_λ, which we denote again by $\Gamma_{\lambda,l}(\,\cdot\,, v, w)$. If we decompose M_λ as

$$M_\lambda = K_\lambda \exp(\overline{\mathfrak{a}^+_\lambda}) K_\lambda,$$

and if we write $m \in M_\lambda$ as $m = \xi_2 \exp H_\lambda \xi_1$ for some $\xi_1, \xi_2 \in K_\lambda$ and some $H_\lambda \in \overline{\mathfrak{a}^+_\lambda}$, then we have

$$\Gamma_{\lambda,l}(m, v, w) = \Gamma_{\lambda,l}(\exp H_\lambda, \pi(\xi_1)v, \pi(\xi_2)^{-1}w)$$

because $c^{P_\lambda}_{v,q}(\,\cdot\,, v, w)$ exhibits the same behaviour.

We want to prove that $\Gamma_{\lambda,l}(\,\cdot\,, v, w)$ belongs to $L^2(M_\lambda)$ and it is $Z(\mathfrak{m}_{\lambda\mathbb{C}})$-finite. An application of Theorem 2.17 will imply that $\Gamma_{\lambda,l}(\,\cdot\,, v, w)$ is a smooth vector in $L^2(M_\lambda)$. Similar ideas appear in [11, Chapter VIII; 15].

Proposition 3.13. *Let $v, w \in H_K$. Let $[\lambda, l] \in \mathcal{C}/\!\sim$ be a relevant equivalence class. Then $\Gamma_{\lambda,l}(\,\cdot\,, v, w)$ belongs to $L^2(M_\lambda)$.*

Proof. We argue as in the proof of [15, Lemma 4.10]. By the proof of Proposition 3.12, we have $\operatorname{Re}\lambda'_i < 0$ for every λ' appearing in the expansion of $\Gamma_{\lambda,l}(\,\cdot\,, v, w)$ on A^+_λ and for every $i \in I_\lambda$. Since $\Gamma_{\lambda,l}(\,\cdot\,, v, w)$ is analytic on $\overline{A^+_\lambda}$, we can apply [7, Theorem 4] and then argue as in [2, Theorem 7.5] to establish the desired square-integrability on $\overline{A^+_\lambda}$. The square-integrability on M_λ follows from combining the decomposition of M_λ as $M_\lambda = K_\lambda \overline{A^+_\lambda} K_\lambda$, the corresponding integral formula and the fact that if $m = \xi_2 \exp H_\lambda \xi_2$, for some $H_\lambda \in \overline{\mathfrak{a}^+_\lambda}$ and some $\xi_1, \xi_2 \in K_\lambda$, then

$$\Gamma_{\lambda,l}(m, v, w) = \Gamma_{\lambda,l}(\exp H_\lambda, \pi(\xi_1)v, \pi(\xi_2)^{-1}w). \qquad \square$$

We recall that there exists an injective algebra homomorphism

$$\mu_{P_\lambda} : \mathrm{Z}(\mathfrak{g}_\mathbb{C}) \to Z((\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0})_\mathbb{C}) \cong Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}}),$$

which turns $Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}})$ into a free module of finite rank over $\mu_{P_\lambda}(\mathrm{Z}(\mathfrak{g}_\mathbb{C}))$ by [6, Lemma 21].

Proposition 3.14. *Let $v, w \in H_K$. Let $[\lambda, l] \in \mathcal{C}/\sim$ be a relevant equivalence class. Then, for every $X \in U(\mathfrak{m}_{\lambda\mathbb{C}})$ and for every $m \in M_\lambda$, we have*

$$X\Gamma_{\lambda,l}(m, v, w) = \Gamma_{\lambda,l}(m, \dot{\pi}(X)v, w).$$

Moreover, the function $\Gamma_{\lambda,l}(\,\cdot\,, v, w)$ is a smooth vector in the right-regular representation $(R, L^2(M_\lambda))$ of M_λ.

Proof. For a given $X \in U(\mathfrak{m}_{\lambda\mathbb{C}})$ and every $g \in G$, we have

$$X\phi_{v,w}(g) = \phi_{\dot{\pi}(X)v,w}(g).$$

Therefore, the restriction of $X\phi_{v,w}(\,\cdot\,)$ to $M_\lambda A_{\lambda_0}$ satisfies

$$X\phi_{v,w}(ma) = \phi_{\dot{\pi}(X)v,w}(ma).$$

Given $m \in M_\lambda$ we can find a compact subset C of M_λ containing m such that $K_\lambda C K_\lambda = C$ and a positive R depending on C such that if $a = \exp H \in A^+_{\lambda_0}$ satisfies $\alpha_i(H) > \log R$ for every $i \in I^c_\lambda$, then $\phi_{\dot{\pi}(X)v,w}(ma)$ may be expanded with respect to P_λ. Since $X \in U(\mathfrak{m}_{\lambda\mathbb{C}})$, the restriction of $X\phi_{v,w}(\,\cdot\,)$ to $M_\lambda A_{\lambda_0}$ can also be computed as the action of the differential operator X on the restriction of $\phi_{v,w}(\,\cdot\,)$ to $M_\lambda A_{\lambda_0}$. For $m \in M_\lambda$ and $a \in A^+_{\lambda_0}$ as above, we expand the function so obtained with respect to P_λ and, as in the proof of (4.8) in [15], because of the convergence of the series, we can apply the differential operator term by term. By comparing the resulting expansion with the expansion of $\phi_{\dot{\pi}(X)v,w}(ma)$, and invoking [11, Corollary B.26], we obtain

$$Xc^{P_\lambda}_{\nu,q}(m, v, w) = c^{P_\lambda}_{\nu,q}(m, \dot{\pi}(X)v, w)$$

for every $\nu \in \mathcal{E}_I$ and every $q \in \mathbb{Z}^{I^c_\lambda}_{\geq 0}$. The first statement now follows from choosing ν and q as in Proposition 3.12.

For the last statement, we need to show that $\Gamma_{\lambda,l}(\,\cdot\,, v, w)$ is annihilated by an ideal of finite codimension in $Z(\mathfrak{m}_{\lambda\mathbb{C}})$; the result will then follow from Theorem 2.17. Let J be the kernel of the infinitesimal character of (π, H). Then J is an ideal of finite codimension in $\mathrm{Z}(\mathfrak{g}_\mathbb{C})$. As observed in [5, p. 182], the inverse image $J_{\mathfrak{m}_\lambda}$ along the inclusion

$$Z(\mathfrak{m}_{\lambda\mathbb{C}}) \to Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}}), \quad X \mapsto X \otimes 1,$$

of the ideal generated by $\mu_{P_\lambda}(J)$ in $Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}})$ is an ideal of finite codimension in $Z(\mathfrak{m}_{\lambda\mathbb{C}})$. This follows from the fact that the ideal generated by $\mu_{P_\lambda}(J)$ is of finite codimension in $Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}})$, since $Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}})$ is a free module of finite type over $\mu_{P_\lambda}(\mathrm{Z}(\mathfrak{g}_\mathbb{C}))$ by [6, Lemma 21]. Denoting by $\mu_{P_\lambda}(J)^e$ the ideal generated by $\mu_{P_\lambda}(J)$, we see that $J_{\mathfrak{m}_\lambda}$ is precisely the kernel of the homomorphism

$$Z(\mathfrak{m}_{\lambda\mathbb{C}}) \to (Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}}))/\mu_{P_\lambda}(J)^e, \quad X \mapsto (X \otimes 1) + \mu_{P_\lambda}(J)^e.$$

This exhibits $J_{\mathfrak{m}_\lambda}$ as an ideal of finite codimension in $Z(\mathfrak{m}_{\lambda\mathbb{C}})$. Now, if $X \in J_{\mathfrak{m}_\lambda}$, then $X \otimes 1$ belongs to $\mu_{P_\lambda}(J)^e$. Hence $X \otimes 1$ can be written as

$$X \otimes 1 = \sum_{i=1}^{r} Y_i \mu_{P_\lambda}(Z_i),$$

with $Y_i \in Z(\mathfrak{m}_{\lambda\mathbb{C}}) \otimes U(\mathfrak{a}_{\lambda_0\mathbb{C}})$ and $Z_i \in J$. For every $i \in \{1, \dots, r\}$, by (8.68) in [11, p. 251], the differential operator $\mu_{P_\lambda}(Z_i)$ annihilates the function

$$F_{\nu-\rho_{\lambda_0}}(ma, v, w) := \sum_{q:|q|\leq q_0} c_{\nu,q}^{P_\lambda}(m, v, w)\alpha(H)^q e^{(\nu-\rho_{\lambda_0})(H)}.$$

Therefore, $X \otimes 1$ annihilates it, as well. On the other hand, by the first part of the proof, we have

$$(X \otimes 1)F_{\nu-\rho_{\lambda_0}}(ma, v, w) = \sum_{q:|q|\leq q_0} c_{\nu,q}^{P_\lambda}(m, \dot{\pi}(X)v, w)\alpha(H)^q e^{(\nu-\rho_{\lambda_0})(H)}.$$

Since the left-hand side vanishes identically on $M_\lambda A_{\lambda_0}$, it follows that

$$c_{\nu,q}^{P_\lambda}(m, \dot{\pi}(X)v, w) = 0$$

for every $m \in M_\lambda$. Choosing ν and q as in Proposition 3.12, we find that $\Gamma_{\lambda,l}(\,\cdot\,, v, w)$ is annihilated by $J_{\mathfrak{m}_\lambda}$. □

3.4. ***The functions $\Gamma_{\lambda,l}$ as intertwining operators.*** Let $w \in H_K$. The following two technical lemmata, together with Proposition 3.14, will be used to prove the $(\mathfrak{m}_\lambda \oplus \mathfrak{a}, K_\lambda)$-equivariance of the map

$$S_w : H_K \to L^2(M_\lambda) \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_w(v)(m) := \Gamma_{\lambda,l}(m, v, w).$$

We are not claiming that for every $w \in H_K$ this map is nonzero: the only thing we need to know is that, whenever $w \in H_K$ is such that S_w is not identically zero, then S_w is $(\mathfrak{m}_\lambda \oplus \mathfrak{a}, K_\lambda)$-equivariant. In the final part of this subsection, we show the existence of an admissible, finitely generated, unitary representation of M_λ which will allow us to apply Theorem 2.27 in the way we explained in the Introduction.

Lemma 3.15. *Let $v, w \in H_K$. Let $[\lambda, l] \in \mathcal{C}/\sim$ be a relevant equivalence class. Then, for every $X \in \mathfrak{a}_{\lambda_0}$ and every $m \in M_\lambda$, we have*

$$\Gamma_{\lambda,l}(m, \dot{\pi}(X)v, w) = (\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(X)\Gamma_{\lambda,l}(m, v, w).$$

Proof. We write $m \in M_\lambda$ as $m = \xi_2 a_\lambda \xi_2$ for some $\xi_1, \xi_2 \in K_\lambda$ and some $a_\lambda \in \overline{A_\lambda^+}$. Then we have

$$\Gamma_{\lambda,l}(m, \dot{\pi}(X)v, w) = \Gamma_{\lambda,l}(a_\lambda, \pi(\xi_1)\dot{\pi}(X)v, \pi(\xi_2^{-1})w).$$

Recalling that

$$\pi(\xi_1)\dot{\pi}(X)v = \dot{\pi}(\mathrm{Ad}(\xi_1)X)\pi(\xi_1)v,$$

since M_λ centralises $\mathfrak{a}_{\lambda_0}$ [12, Proposition 7.82], and K_λ is contained in M_λ, we have

$$\Gamma_{\lambda,l}(a_\lambda, \pi(\xi_1)\dot{\pi}(X)v, \pi(\xi_2^{-1})w) = \Gamma_{\lambda,l}(a_\lambda, \dot{\pi}(X)\pi(\xi_1)v, \pi(\xi_2^{-1})w).$$

Therefore, relabelling things, it suffices to prove that for every $X \in \mathfrak{a}_{\lambda_0}$ and for every $a_\lambda \in \overline{A_\lambda^+}$, we have

$$\Gamma_{\lambda,l}(a_\lambda, \dot{\pi}(X)v, w) = (\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(X)\Gamma_{\lambda,l}(a_\lambda, v, w).$$

Moreover, since $\Gamma_{\lambda,l}(\,\cdot\,, v, w)$ is analytic, it suffices to prove the identity for every $a_\lambda \in A_\lambda^+$. Let $a_\lambda = \exp H_\lambda \in A_\lambda^+$. Then there exist a compact subset C of M_λ containing a_λ and such that $K_\lambda C K_\lambda = C$, and a positive R depending on C such that, for all $H_{\lambda_0} \in \mathfrak{a}_{\lambda_0}^+$ satisfying $\alpha_i(H_{\lambda_0}) > \log R$ for every $i \in I_\lambda^c$, the expansion of $\phi_{\dot{\pi}(X)v,w}(a_\lambda \exp H_{\lambda_0})$ relative to P (Theorem 3.1) and the expansion of $\phi_{\dot{\pi}(X)v,w}(a_\lambda \exp H_{\lambda_0})$ relative to P_λ (Theorem 3.3) are both valid. Setting $H := H_\lambda + H_{\lambda_0}$ for H_{λ_0} as above, the expansion in Theorem 3.1 gives

$$\phi_{\dot{\pi}(X)v,w}(H) = \sum_{\tilde{\lambda}\in\mathcal{E}} \sum_{\substack{\tilde{l}\in\mathbb{Z}^n_{\geq 0}\\ |\tilde{l}|\leq l_0}} \alpha(H)^{\tilde{l}} e^{(\tilde{\lambda}-\rho)(H)} \langle c_{\tilde{\lambda},\tilde{l}}(\dot{\pi}(X)v), w\rangle$$

By linearity we can assume that $X = H_i$ for some $i \in I_\lambda^c$, where H_i, we recall, is the element in $\mathfrak{a}_{\lambda_0}$ dual to the simple root α_i.

Differentiating term by term and taking into account the computation

$$H_i[\alpha(H)^{\tilde{l}} e^{(\tilde{\lambda}-\rho)(H)}] = \tilde{l}_i \alpha(H)^{\tilde{l}-e_i} e^{(\tilde{\lambda}-\rho)(H)} + (\tilde{\lambda}|_{\mathfrak{a}_{\lambda_0}} - \rho)(H_i)\alpha(H)^{\tilde{l}} e^{(\tilde{\lambda}-\rho)(H)},$$

where e_i is the element in $\mathbb{Z}^n_{\geq 0}$ having 1 as its i-th coordinate and 0 as every other coordinate, we observe that the only terms in the expansion

$$\phi_{v,w}(H) = \sum_{\tilde{\lambda}\in\mathcal{E}} \sum_{\substack{\tilde{l}\in\mathbb{Z}^n_{\geq 0}\\ |\tilde{l}|\leq l_0}} \alpha(H)^{\tilde{l}} e^{(\tilde{\lambda}-\rho)(H)} \langle c_{\tilde{\lambda},\tilde{l}}(v), w\rangle$$

that after differentiation by $H_i \in \mathfrak{a}_{\lambda_0}$ can contribute a term of the form

$$c\alpha(H)^{\tilde{l}} e^{(\tilde{\lambda}-\rho)(H)} \langle c_{\tilde{\lambda},\tilde{l}}(v), w\rangle,$$

with $c \in \mathbb{C}$, to the expansion of $\phi_{\dot{\pi}(X)v,w}(H)$, is precisely

$$\alpha(H)^{\tilde{l}} e^{(\tilde{\lambda}-\rho)(H)} \langle c_{\tilde{\lambda},\tilde{l}}(v), w\rangle.$$

This reasoning shows that in the expansion

$$\phi_{\dot\pi(H_i)v,w}(a_\lambda \exp H_{\lambda_0}) = \sum_{\nu\in\mathcal{E}_I} \sum_{\substack{q\in\mathbb{Z}_{\geq 0}^{I_\lambda^c} \\ |q|\leq q_0}} \alpha(H_{\lambda_0})^q e^{(\nu-\rho_{\lambda_0})(H_{\lambda_0})} c_{\nu,q}^{P_\lambda}(a_\lambda, \dot\pi(H_i)v, w)$$

relative to P_λ, the term indexed by (ν, q) with $\nu = \lambda|_{\mathfrak{a}_{\lambda_0}}$ and $q = l_{\lambda_0}$ satisfies

$$c_{\nu,q}^{P_\lambda}(a_\lambda, \dot\pi(H_i)v, w) = (\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(H_i) c_{\nu,q}^{P_\lambda}(a_\lambda, v, w).$$

Indeed, the comparison in [11, p. 251] shows that

$$\alpha(H_{\lambda_0})^q e^{(\nu-\rho_{\lambda_0})(H_{\lambda_0})} c_{\nu,q}^{P_\lambda}(a_\lambda, \dot\pi(H_i)v, w)$$

is the sum of all the terms in the expansion of $\phi_{\dot\pi(H_i)v,w}(H)$ relative to P which are indexed by couples $(\tilde\lambda, \tilde l)$ satisfying

$$\tilde\lambda|_{\mathfrak{a}_{\lambda_0}} = \lambda|_{\mathfrak{a}_{\lambda_0}} \quad \text{and} \quad \tilde l_{\lambda_0} = l_{\lambda_0}$$

and, as we saw, these are the terms of the form

$$(\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(H_i)\alpha(H)^{\tilde l} e^{(\tilde\lambda-\rho)(H)} \langle c_{\tilde\lambda,\tilde l}(v), w\rangle.$$

Finally, since

$$\Gamma_{\lambda,l}(a_\lambda, v, w) = c_{\nu,q}^{P_\lambda}(a_\lambda, v, w)$$

by Proposition 3.12, we obtain

$$\Gamma_{\lambda,l}(a_\lambda, v, w) = (\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(H_i)\Gamma_{\lambda,l}(a_\lambda, \dot\pi(H_i)v, w). \qquad \square$$

Lemma 3.16. *Let $v, w \in H_K$. Let $[\lambda, l] \in \mathcal{C}/\sim$ be a relevant equivalence class. Then, for every $X \in \mathfrak{n}_{\lambda_0}$ and every $m \in M_\lambda$, we have*

$$\Gamma_{\lambda,l}(m, \dot\pi(X)v, w) = 0.$$

Proof. We write $m \in M_\lambda$ as $m = \xi_2 a_\lambda \xi_2$ for some $\xi_1, \xi_2 \in K_\lambda$ and some $a_\lambda \in \overline{A_\lambda^+}$. Then we have

$$\Gamma_{\lambda,l}(m, \dot\pi(X)v, w) = \Gamma_{\lambda,l}(a_\lambda, \pi(\xi_1)\dot\pi(X)v, \pi(\xi_2^{-1})w).$$

Recalling that

$$\pi(\xi_1)\dot\pi(X)v = \dot\pi(\mathrm{Ad}(\xi_1)X)\pi(\xi_1)v,$$

since M_λ normalises $\mathfrak{n}_{\lambda_0}$ [12, Proposition 7.83], and K_λ is contained in M_λ, we have

$$\Gamma_{\lambda,l}(a_\lambda, \pi(\xi_1)\dot\pi(X)v, \pi(\xi_2^{-1})w) = \Gamma_{\lambda,l}(a_\lambda, \dot\pi(X')\pi(\xi_1)v, \pi(\xi_2^{-1})w)$$

for some $X' \in \mathfrak{n}_{\lambda_0}$. Therefore, relabelling things, it suffices to prove that for every $X \in \mathfrak{a}_{\lambda_0}$ and for every $a_\lambda \in \overline{A_\lambda^+}$, we have

$$\Gamma_{\lambda,l}(a_\lambda, \dot{\pi}(X)v, w) = 0.$$

Since $\Gamma_{\lambda,l}(\,\cdot\,, v, w)$ is analytic, it suffices to prove the identity for every $a_\lambda \in A_\lambda^+$.

As in the previous proof, we set $H := H_\lambda + H_{\lambda_0}$ for H_{λ_0} in an appropriate region and the expansion in Theorem 3.1 gives

$$\phi_{\dot{\pi}(X)v,w}(H) = \sum_{\tilde{\lambda}\in\mathcal{E}} \sum_{\substack{\tilde{l}\in\mathbb{Z}^n_{\geq 0}\\ |\tilde{l}|\leq l_0}} \alpha(H)^{\tilde{l}} e^{(\tilde{\lambda}-\rho)(H)} \langle c_{\tilde{\lambda},\tilde{l}}(\dot{\pi}(X)v), w\rangle.$$

The expansion in Theorem 3.3 gives

$$\phi_{\dot{\pi}(X)v,w}(a_\lambda \exp H_{\lambda_0}) = \sum_{\nu\in\mathcal{E}_I} \sum_{\substack{q\in\mathbb{Z}^{I_\lambda^c}_{\geq 0}\\ |q|\leq q_0}} \alpha(H_{\lambda_0})^q e^{(\nu-\rho_{\lambda_0})(H_{\lambda_0})} c^{P_\lambda}_{\nu,q}(a_\lambda, \dot{\pi}(X)v, w).$$

By [11, Corollary 8.46], each $\nu - \rho_{\lambda_0}$ in the second expansion is of the form $\tilde{\lambda}|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}$ for some exponent $\tilde{\lambda}$ in the first expansion. Therefore, it suffices to prove that if $\lambda \in \mathcal{E}_0$ and $l \in \mathbb{Z}^n_{\geq 0}$ with $|l| \leq l_0$ satisfy

$$\boldsymbol{d}_P(\lambda, l) = \boldsymbol{d}(\pi),$$

then no term with exponent $\tilde{\lambda} - \rho$ for which $\tilde{\lambda}|_{\mathfrak{a}_{\lambda_0}} = \lambda|_{\mathfrak{a}_{\lambda_0}}$ appears in the first expansion. Indeed, if we can show this, since by the comparison in [11, p. 251], the term

$$\alpha(H_{\lambda_0})^q e^{(\nu-\rho_{\lambda_0})(H_{\lambda_0})} c^{P_\lambda}_{\nu,q}(a_\lambda, \dot{\pi}(X)v, w),$$

for $\nu = \lambda|_{\mathfrak{a}_{\lambda_0}}$ and $q_{\lambda_0} = l_{\lambda_0}$ is the sum of all the terms in the expansion of $\phi_{\dot{\pi}(X)v,w}(H)$ relative to P which are indexed by couples $(\tilde{\lambda}, \tilde{l})$ satisfying

$$\tilde{\lambda}|_{\mathfrak{a}_{\lambda_0}} = \lambda|_{\mathfrak{a}_{\lambda_0}} \quad \text{and} \quad \tilde{l}_{\lambda_0} = l_{\lambda_0},$$

it would follow that

$$c^{P_\lambda}_{\nu,q}(a_\lambda, \dot{\pi}(X)v, w) = 0,$$

and therefore

$$\Gamma_{\lambda,l}(a_\lambda, \dot{\pi}(X)v, w) = 0.$$

By linearity we can assume that $X \in \mathfrak{g}_{-\alpha_i}$ for some $i \in I_\lambda^c$ [11, Proposition 5.23].

Computing as in [2, Lemma 8.16], we have

$$\phi_{\dot{\pi}(X)v,w}(a) = \big\langle \dot{\pi}(\mathrm{Ad}(a)X)\pi(a)v, w\big\rangle = -e^{-\alpha_i(H)}\phi_{v,\dot{\pi}(X)w}(a).$$

Hence every exponent in the expansion of $\phi_{\dot\pi(X)v,w}(a)$ relative to P is of the form $\tilde\lambda = \lambda' - e_i$ for some $\lambda' \in \mathcal{E}$. Now, if there existed $\lambda' \in \mathcal{E}$ with

$$(\lambda' - e_i)|_{\mathfrak{a}_{\lambda_0}} = \lambda|_{\mathfrak{a}_{\lambda_0}},$$

we would have

$$\operatorname{Re}(\lambda' - e_i)_i = \operatorname{Re}\lambda_i = 0$$

since $i \in I_\lambda^c$. This means that $\operatorname{Re}\lambda_i' > 0$, a contradiction. Indeed, since (π, H) is tempered, the real part of every coordinate of each leading exponent is at most zero by Theorem 3.2 and it follows that the same property holds for every element in $\mathcal{E}$. □

Lemma 3.17. *Let $w \in H_K$. Let $[\lambda, l] \in \mathcal{C}/\!\sim$ be a relevant equivalence class. Then the prescription*

$$S_w : H_K \to L^2(M_\lambda), \quad S_w(v)(m) := \Gamma_{\lambda,l}(m, v, w),$$

is a well-defined, $(\mathfrak{m}_\lambda, K_\lambda)$-equivariant map with image contained in $L^2(M_\lambda)_{K_\lambda}$.

Proof. The map S_w is well defined by Proposition 3.14. For every $\xi \in K_\lambda$ and every $m \in M_\lambda$, we have

$$S_w(\pi(\xi)v)(m) = \Gamma_{\lambda,l}(m, \pi(\xi)v, w) = \Gamma_{\lambda,l}(m\xi, v, w) = R(\xi)S_w(v)(m).$$

By Proposition 3.14, for all $X \in \mathfrak{m}_\lambda$ and for all $m \in M_\lambda$, we have

$$S_w(\dot\pi(X)v)(m) = X\Gamma_{\lambda,l}(m, v, w)$$

and, by Proposition 2.20, we have

$$X\Gamma_{\lambda,l}(m, v, w) = \dot R(X)\Gamma_{\lambda,l}(m, v, w).$$

Therefore

$$S_w(\dot\pi(X)v)(m) = \dot R(X)S_w(v)(m)$$

and this concludes the proof that S_w is $(\mathfrak{m}_\lambda, K_\lambda)$-equivariant. To prove that the image of S_w is contained in $L^2(M_\lambda)_{K_\lambda}$, we observe that, for every $v \in H_K$, the K_λ-finiteness of v implies the existence of finitely many $v_1, \dots, v_r \in H_K$ such that

$$R(K_\lambda)\Gamma_{\lambda,l}(\,\cdot\,, v, w) \in \operatorname{span}\{\Gamma_{\lambda,l}(\,\cdot\,, v_i, w) \mid i \in \{1, \dots, r\}\}.$$

Hence, $\Gamma_{\lambda,l}(\,\cdot\,, v, w)$ is K_λ-finite and, since it is a smooth vector in $(R, L^2(M_\lambda))$ by Proposition 3.14, it belongs to $L^2(M_\lambda)_{K_\lambda}$. □

We now construct a subrepresentation (Θ, H_Θ) of $(R, L^2(M_\lambda))$ which, as we will show in the next two results, has precisely those properties that we need to proceed with the strategy outlined in the Introduction. We will show that (Θ, H_Θ) is an admissible, finitely generated, unitary (this follows since it is a subrepresentation of $L^2(M_\lambda)$) representation of M_λ such that the image of the $(\mathfrak{m}_\lambda, K_\lambda)$-equivariant map S_w is precisely the $(\mathfrak{m}_\lambda, K_\lambda)$-module H_{Θ,K_λ} and such that the map

$$S_w : H_K \to H_{\Theta,K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_w(v)(m) := \Gamma_{\lambda,l}(m, v, w),$$

is $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-equivariant.

The representation (Θ, H_Θ) depends on the choice of $w \in H_K$ and of a relevant $[\lambda, l] \in \mathcal{C}/\sim$. However, and this is the important point, the construction can be formed for every choice of $\tilde{w} \in H_k$ and for every choice of relevant $[\tilde{\lambda}, \tilde{l}] \in \mathcal{C}/\sim$. In Proposition 3.21 below, we will use this construction to define the representation (σ, H_σ) discussed in the Introduction.

We adopt the notation of the previous lemma. In the proof of Proposition 3.14, we showed that, for each $v \in H_K$, the function $\Gamma_{\lambda,l}(\cdot, v, w)$ is a $Z(\mathfrak{m}_\lambda)$-finite function in $L^2(M_\lambda)$. By [11, Corollary 8.42], there exist finitely many orthogonal irreducible subrepresentations of $(R, L^2(M_\lambda))$ such that $\Gamma_{\lambda,l}(\cdot, v, w)$ is contained in their direct sum. It follows that there exists a (not necessarily finite) collection $\{(\theta, H_\theta)\}_{\theta\in\Theta}$ of orthogonal irreducible subrepresentations of $(R, L^2(M_\lambda))$ such that $S_w(H_K)$ is contained in their direct sum. Let (Θ, H_Θ) denote the direct sum of the subrepresentations in this collection.

Lemma 3.18. *The* $(\mathfrak{m}_\lambda, K_\lambda)$*-module* H_{Θ,K_λ} *is precisely the image of the* $(\mathfrak{m}_\lambda, K_\lambda)$*-equivariant map*

$$S_w : H_K \to L^2(M_\lambda), \quad S_w(v)(m) := \Gamma_{\lambda,l}(m, v, w).$$

Proof. By Lemma 3.17, $S_w(H_K) \subset H_\Theta \cap L^2(M_\lambda)_{K_\lambda} = H_{\Theta,K_\lambda}$. For the reverse inclusion, the irreducibility of each (θ, H_θ) implies that $S_w(H_K) \cap H_{\theta,K_\lambda} = H_{\theta,K_\lambda}$. Therefore H_{Θ,K_λ} is contained in the image of S_w, completing the proof. □

Proposition 3.19. *The representation* (Θ, H_Θ) *of* M_λ *is admissible, finitely generated and unitary. Moreover, the map*

$$S_w : H_K \to H_{\Theta,K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_w(v)(m) := \Gamma_{\lambda,l}(m, v, w),$$

is $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$*-equivariant.*

Proof. By Lemma 3.18 we have $S_w(H_K) = H_{\Theta,K_\lambda}$. By Lemma 3.15, for all $X \in \mathfrak{a}_{\lambda_0}$ and for all $m \in M_\lambda$, we have

$$S_w(\dot{\pi}(X)v)(m) = (\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(X)\Gamma_{\lambda,l}(m, v, w) = (\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0})(X)S_w(v).$$

By Lemma 3.16, for all $X \in \mathfrak{n}_{\lambda_0}$ and for all $m \in M_\lambda$, we have

$$S_w(\dot{\pi}(X)v)(m) = \Gamma_{\lambda,l}(m, \dot{\pi}(X)v, w) = 0.$$

We thus obtained an $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-equivariant map

$$S_w : H_K \to H_{\Theta, K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_w(v)(m) := \Gamma_{\lambda,l}(m, v, w),$$

which factors through the quotient map

$$q : H_K \to H_K / \mathfrak{n}_{\lambda_0} H_K,$$

which is $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-equivariant by Lemma 2.24.

Since H_K, being irreducible (and hence admissible by Theorem 2.9), has an infinitesimal character, by Corollary 2.26 the $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-module $H_K/\mathfrak{n}_{\lambda_0} H_K$ is admissible and finitely generated. It follows that

$$S_w(H_K) = H_{\Theta, K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}$$

is an admissible and finitely generated $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-module. The fact that $\mathfrak{a}_{\lambda_0}$ acts by scalars implies that H_{Θ, K_λ} itself is finitely generated (as $U(\mathfrak{m}_{\lambda\mathbb{C}})$-module) and admissible. □

In the next corollary, we apply Casselman's version of the Frobenius reciprocity to construct $(\mathfrak{g}, K)$-intertwining operators from the functions $\Gamma_{\lambda,l}$. We recall that $\overline{P_\lambda}$ denotes the parabolic subgroup opposite to P_λ and that the half-sum of positive roots determined by $\overline{P_\lambda}$ is precisely $-\rho_{\lambda_0}$.

Corollary 3.20. *The map*

$$T_w : H_K \to \operatorname{Ind}_{\overline{P_\lambda}, K_\lambda}(\Theta, \lambda|_{\mathfrak{a}_{\lambda_0}}), \quad T_w(v)(k)(m) := \Gamma_{\lambda,l}(m, \pi(k)v, w),$$

is $(\mathfrak{g}, K)$*-equivariant.*

Proof. The equivariance follows from Proposition 3.19, in combination with Theorem 2.27 and the discussion following it. More precisely, we have $T_w = \tilde{S}_w$ in the notation of the discussion following Theorem 2.27. □

The next proposition is the core of the article: it allows us to prove an identity of certain integrals using representation-theoretic methods. In the final section, it will be shown that the identity in question implies Proposition 1.5.

Proposition 3.21. *Let* $[\lambda, l], [\mu, m] \in \mathcal{C}/\sim$ *be relevant equivalence classes such that* $I_\lambda = I_\mu$, $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$ *and* $\boldsymbol{d}_P(\lambda, l) = \boldsymbol{d}_P(\mu, m)$. *Then, for all* $X \in \mathfrak{g}$, *for all* $k \in K$, *and for all* $v_1, v_2, v_3, v_4 \in H_K$, *the integral*

$$\int_K \big\langle \Gamma_{\lambda,l}(m_\lambda, \pi(k)\dot{\pi}(X)v_1, v_2), \Gamma_{\mu,m}(m_\lambda, \pi(k)v_3, v_4)\big\rangle_{L^2(M_\lambda)}\, dk$$

is equal to the integral

$$-\int_K \big\langle \Gamma_{\lambda,l}(m_\lambda, \pi(k)v_1, v_2), \Gamma_{\mu,m}(m_\lambda, \pi(k)\dot{\pi}(X)v_3, v_4)\big\rangle_{L^2(M_\lambda)}\, dk.$$

Proof. By Proposition 3.19 and the discussion before Lemma 3.17, we can construct a representation (σ_1, H_{σ_1}) of M_λ that is finitely generated, unitary and such that the image of the $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-equivariant map

$$S_{v_2} : H_K \to L^2(M_\lambda)_{K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_{v_2}(v)(m) := \Gamma_{\lambda,l}(m, v, v_2),$$

is precisely $H_{\sigma_1,K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}$. Similarly, we can construct an admissible, finitely generated, unitary representation (σ_2, H_{σ_2}) such that the image of the $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-equivariant map

$$S_{v_4} : H_K \to L^2(M_\lambda)_{K_\lambda} \otimes \mathbb{C}_{\mu|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_{v_4}(v)(m) := \Gamma_{\mu,m}(m, v, v_4),$$

is precisely $H_{\sigma_2,K_\lambda} \otimes \mathbb{C}_{\mu|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}$. Let (σ, H_σ) denote the direct sum of (σ_1, H_{σ_1}) and (σ_2, H_{σ_2}). It is an admissible, finitely generated, unitary representation which restricts to a unitary representation of K_λ. Since $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$, by the same computations as in Lemma 3.17 and Proposition 3.19 we obtain $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-equivariant maps

$$S_{v_2} : H_K \to H_{\sigma,K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_{v_2}(v)(m) := \Gamma_{\lambda,l}(m, v, v_2),$$

and

$$S_{v_4} : H_K \to H_{\sigma,K_\lambda} \otimes \mathbb{C}_{\lambda|_{\mathfrak{a}_{\lambda_0}} - \rho_{\lambda_0}}, \quad S_{v_4}(v)(m) := \Gamma_{\mu,m}(m, v, v_4),$$

factoring through the $(\mathfrak{m}_\lambda \oplus \mathfrak{a}_{\lambda_0}, K_\lambda)$-equivariant quotient map

$$q : H_K \to H_K/\mathfrak{n}_{\lambda_0} H_K.$$

From Corollary 3.20, we obtain $(\mathfrak{g}, K)$-equivariant maps

$$T_{v_2} : H_K \to \mathrm{Ind}_{\overline{P_\lambda},K_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}), \quad T_{v_2}(v)(k)(m) := \Gamma_{\lambda,l}(m, \pi(k)v, v_2),$$

and

$$T_{v_4} : H_K \to \mathrm{Ind}_{\overline{P_\lambda},K_\lambda}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}), \quad T_{v_4}(v)(k)(m) := \Gamma_{\mu,m}(m, \pi(k)v, v_4).$$

By definition of the inner product on $\mathrm{Ind}_{\overline{P_\lambda}}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}})$, we see that proving the sought identity is equivalent to proving that

$$\big\langle T_{v_2}(\dot{\pi}(X)v_1), T_{v_4}(v_3)\big\rangle_{\mathrm{Ind}_{\overline{P_\lambda}}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})} = -\big\langle T_{v_2}(v_1), T_{v_4}(\dot{\pi}(X)v_3)\big\rangle_{\mathrm{Ind}_{\overline{P_\lambda}}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})}.$$

By the $(\mathfrak{g}, K)$-equivariance of T_{v_2}, we have

$$\big\langle T_{v_2}(\dot{\pi}(X)v_1), T_{v_4}(v_3)\big\rangle_{\mathrm{Ind}_{\overline{P_\lambda}}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})} = \big\langle \dot{\mathrm{Ind}}_{\overline{P_\lambda}}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}, X) T_{v_2}(v_1), T_{v_4}(v_3)\big\rangle_{\mathrm{Ind}_{\overline{P_\lambda}}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})}$$

and, since $\lambda|_{\mathfrak{a}_{\lambda_0}}$ is totally imaginary, from Corollary 2.23 we deduce

$$\begin{aligned}\big\langle \dot{\mathrm{Ind}}_{\overline{P_\lambda}}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}, X) T_{v_2}(v_1), T_{v_4}(v_3)\big\rangle_{\mathrm{Ind}_{\overline{P_\lambda}}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})}& \\ = -\big\langle T_{v_2}(v_1), \dot{\mathrm{Ind}}_{\overline{P_\lambda}}(\sigma, \lambda|_{\mathfrak{a}_{\lambda_0}}, X) T_{v_4}(v_3)\big\rangle_{\mathrm{Ind}_{\overline{P_\lambda}}(\sigma,\lambda|_{\mathfrak{a}_{\lambda_0}})}&.\end{aligned}$$

The result follows from the $(\mathfrak{g}, K)$-equivariance of T_{v_4}. □

4. Asymptotic orthogonality

For a tempered, irreducible representation (π, H) of G, for $v, w \in H$, let

$$\phi_{v,w}(g) := \langle \pi(g)v, w\rangle$$

denote the associated matrix coefficient. By (2) of Theorem 1.2, there exists $\boldsymbol{d}(\pi) \in \mathbb{Z}_{\geq 0}$ such that

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} |\phi_{v,w}(g)|^2 \, dg < \infty$$

for all $v, w \in H_K$.

As in [10, Section 4.1], by the polarisation identity and by (2) of Theorem 1.2, the prescription

$$D(v_1, v_2, v_3, v_4) := \lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,v_2}(g)\overline{\phi_{v_3,v_4}(g)} \, dg$$

is a well-defined form on H_K that is linear in the first and fourth variable and conjugate-linear in the second and the third.

We explained in the Introduction that the crucial point is the proof of Proposition 1.5. We begin with the following reduction.

Lemma 4.1. *Let G be a connected, semisimple Lie group with finite centre and let (π, H) be a tempered, irreducible representation of G. If for all $X \in \mathfrak{g}$ and for all $v_1, v_2, v_3, v_4 \in H_K$ we have*

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \phi_{\dot\pi(X)v_1,v_2}(g)\overline{\phi_{v_3,v_4}(g)} \, dg = -\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,v_2}(g)\overline{\phi_{\dot\pi(X)v_3,v_4}(g)} \, dg,$$

then

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,\dot\pi(X)v_2}(g)\overline{\phi_{v_3,v_4}(g)} \, dg = -\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,v_2}(g)\overline{\phi_{v_3,\dot\pi v_4}(g)} \, dg$$

holds for every $X \in \mathfrak{g}$ and for every $v_1, v_2, v_3, v_4 \in H_K$.

Proof. We write

$$\phi_{v_1,\dot\pi(X)v_2}(g)\overline{\phi_{v_3,v_4}(g)} = \langle v_1, \pi(g^{-1})\dot\pi(X)v_2\rangle \overline{\langle v_3, \pi(g^{-1})v_4\rangle}$$

and since $\langle\cdot,\cdot\rangle$ is Hermitian we have

$$\langle v_1, \pi(g^{-1})\dot\pi(X)v_2\rangle \overline{\langle v_3, \pi(g^{-1})v_4\rangle} = \phi_{v_4,v_3}(g^{-1})\overline{\phi_{\dot\pi(X)v_2,v_1}(g^{-1})}.$$

Now, since $G_{<r}$ is invariant under $\iota(g) = g^{-1}$ and G is unimodular, we have

$$\int_{G_{<r}} \phi_{v_4,v_3}(g^{-1})\overline{\phi_{\dot\pi(X)v_2,v_1}(g^{-1})} \, dg = \int_{G_{<r}} \phi_{v_4,v_3}(g)\overline{\phi_{\dot\pi(X)v_2,v_1}(g)} \, dg$$

and therefore

$$\int_{G_{<r}} \phi_{v_1,\dot\pi(X)v_2}(g)\overline{\phi_{v_3,v_4}(g)}\,dg = \int_{G_{<r}} \phi_{v_4,v_3}(g)\overline{\phi_{\dot\pi(X)v_2,v_1}(g)}\,dg.$$

Applying complex conjugation, we obtain

$$\overline{\int_{G_{<r}} \phi_{v_1,\dot\pi(X)v_2}(g)\overline{\phi_{v_3,v_4}(g)}\,dg} = \int_{G_{<r}} \phi_{\dot\pi(X)v_2,v_1}(g)\overline{\phi_{v_4,v_3}(g)}\,dg.$$

Assuming the validity of the first identity in the statement, we can write

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \phi_{\dot\pi(X)v_2,v_1}(g)\overline{\phi_{v_4,v_3}(g)}\,dg = -\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \phi_{v_2,v_1}(g)\overline{\phi_{\dot\pi(X)v_4,v_3}(g)}\,dg.$$

Now, since

$$\int_{G_{<r}} \phi_{\dot\pi(X)v_2,v_1}(g)\overline{\phi_{v_4,v_3}(g)}\,dg = \overline{\int_{G_{<r}} \phi_{v_1,\dot\pi(X)v_2}(g)\overline{\phi_{v_3,v_4}(g)}\,dg},$$

it follows that

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,\dot\pi(X)v_2}(g)\overline{\phi_{v_3,v_4}(g)}\,dg = -\lim_{r\to\infty} \overline{\int_{G_{<r}} \phi_{v_2,v_1}(g)\overline{\phi_{\dot\pi(X)v_4,v_3}(g)}\,dg}.$$

Observing that

$$\overline{\int_{G_{<r}} \phi_{v_2,v_1}(g)\overline{\phi_{\dot\pi(X)v_4,v_3}(g)}\,dg} = \int_{G_{<r}} \phi_{\dot\pi(X)v_4,v_3}(g)\overline{\phi_{v_2,v_1}(g)}\,dg$$

and that, using the invariance of $G_{<r}$ under $\iota(g) = g^{-1}$ and the unimodularity of G,

$$\int_{G_{<r}} \phi_{\dot\pi(X)v_4,v_3}(g)\overline{\phi_{v_2,v_1}(g)}\,dg = \int_{G_{<r}} \phi_{v_1,v_2}(g)\overline{\phi_{v_3,\dot\pi(X)v_4}(g)}\,dg,$$

we finally obtain

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,\dot\pi(X)v_2}(g)\overline{\phi_{v_3,v_4}(g)}\,dg = -\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,v_2}(g)\overline{\phi_{v_3,\dot\pi(X)v_4}(g)}\,dg.$$

This completes the proof. □

Proposition 4.2. *Let G be a connected, semisimple Lie group with finite centre and let (π, H) be a tempered, irreducible representation of G. Then, for all $X \in \mathfrak{g}$ and for all $v_1, v_2, v_3, v_4 \in H_K$, we have*

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \phi_{\dot\pi(X)v_1,v_2}(g)\overline{\phi_{v_3,v_4}(g)}\,dg = -\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \phi_{v_1,v_2}(g)\overline{\phi_{\dot\pi(X)v_3,v_4}(g)}\,dg.$$

Remark 4.3. Some of the integral manipulations in the proof require careful justification. We decided to provide this in Lemma 4.5 after the proof of Proposition 4.2.

Proof. The integral formula for the Cartan decomposition, taking into account the fact that, except for a set of measure zero, every $g \in G_{<r}$ can be written as $g = k_2 \exp H\, k_1$, for some $k_1, k_2 \in K$ and some $H \in \mathfrak{a}^+_{<r}$, with $\mathfrak{a}^+_{<r}$ as in (3.7), gives

$$\int_{G_{<r}} \phi_{\dot\pi(X)v_1,v_2}(g)\overline{\phi_{v_3,v_4}(g)}\,dg = \int_K \int_{\mathfrak{a}^+_{<r}} \int_K \phi_{\dot\pi(X)v_1,v_2}(k_2 \exp H\, k_1)\overline{\phi_{v_3,v_4}(k_2 \exp H\, k_1)}\Omega(H)\,dk_1\,dH\,dk_2,$$

with $\Omega(H)$ defined in (3.8).

Arguing as in [10, p. 258], we can interchange the two innermost integrals in the right-hand side and, upon multiplying both sides by $1/r^{\boldsymbol{d}(\pi)}$ and taking the limit as $r \to \infty$, the right-hand side can be computed as the integral over $K \times K$ of

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{\mathfrak{a}^+_{<r}} \phi_{\dot\pi(X)v_1,v_2}(k_2 \exp H\, k_1)\overline{\phi_{v_3,v_4}(k_2 \exp H\, k_1)}\Omega(H)\,dH.$$

We expand ϕ_{v_1,v_2} and ϕ_{v_3,v_4} as

$$\phi_{v_1,v_2}(k_2 \exp H\, k_1) = e^{-\rho_{\mathfrak{p}}(H)} \sum_{[\lambda,l]\in\mathcal{C}/\sim} \alpha(H_{\lambda_0})^{l_{\lambda_0}} e^{\lambda|_{\mathfrak{a}_{\lambda_0}}(H_{\lambda_0})} \sum_{(\lambda',l')\in[\lambda,l]} \Psi^{\pi(k_1)v_1,\pi(k_2^{-1})v_2}_{\lambda',l'}(H)$$

and

$$\phi_{v_3,v_4}(k_2 \exp H\, k_1) = e^{-\rho_{\mathfrak{p}}(H)} \sum_{[\mu,m]\in\mathcal{C}/\sim} \alpha(H_{\mu_0})^{m_{\mu_0}} e^{\mu|_{\mathfrak{a}_{\mu_0}}(H_{\mu_0})} \sum_{(\mu',m')\in[\mu,m]} \Psi^{\pi(k_1)v_1,\pi(k_2^{-1})v_2}_{\mu',m'}(H).$$

By [10, Lemma A.5 and Claim A.6], the only nonzero contributions to

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{\mathfrak{a}^+_{<r}} \phi_{\dot\pi(X)v_1,v_2}(k_2 \exp H\, k_1)\overline{\phi_{v_3,v_4}(k_2 \exp H\, k_1)}\Omega(H)\,dH \tag{4.4}$$

may come from those $[\lambda, l] \in \mathcal{C}/\sim$ and those $[\mu, m] \in \mathcal{C}/\sim$ for which

$$I_\lambda = I_\mu, \quad \lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}, \quad \boldsymbol{d}(\pi) = |I_\lambda| + \sum_{i\in I_\lambda}(l_i + m_i).$$

In view of the first condition, the third is equivalent to requiring that

$$\boldsymbol{d}(\pi) = \boldsymbol{d}_P(\lambda, l) = \boldsymbol{d}_P(\mu, m),$$

where $\boldsymbol{d}_P(\lambda, l)$ and $\boldsymbol{d}_P(\mu, m)$ are defined by (1.7).

By the discussion in Section 3 and by Proposition 3.12,

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{\mathfrak{a}^+_{<r}} \phi_{\dot\pi(X)v_1,v_2}(k_2 \exp H\, k_1)\overline{\phi_{v_3,v_4}(k_2 \exp H\, k_1)}\Omega(H)\,dH$$

is equal to a finite sum of terms of the form

$$C(\lambda, l, m) \int_{\mathfrak{a}_\lambda^+} \Gamma_{\lambda,l}(\exp H_\lambda, \pi(k_1)\dot{\pi}(X)v_1, \pi(k_2^{-1})v_2) \overline{\Gamma_{\mu,m}(\exp H_\lambda, \pi(k_1)v_3, \pi(k_2^{-1})v_4)}\Omega_\lambda(H_\lambda)\, dH_\lambda,$$

with $C(\lambda, l, m)$ as in (3.10), the functions $\Gamma_{\lambda,l}$ and $\Gamma_{\mu,m}$ defined as in (3.6) and $\Omega_\lambda(H_\lambda)$ defined as in (3.9).

Taking into account the integration over $K \times K$, we proved that

$$\lim_{r\to\infty} \frac{1}{r^{d(\pi)}} \int_K \int_{\mathfrak{a}_{<r}^+} \int_K \phi_{\dot{\pi}(X)v_1,v_2}(k_2 \exp H\, k_1)\overline{\phi_{v_3,v_4}(k_2 \exp H\, k_1)}\Omega(H)\, dk_1\, dH\, dk_2$$

is equal to a finite sum of terms of the form

$$C(\lambda, l, m) \int_K \int_K \int_{\mathfrak{a}_\lambda^+} \Gamma_{\lambda,l}(\exp H_\lambda, \pi(k_1)\dot{\pi}(X)v_1, \pi(k_2^{-1})v_2) \cdot \overline{\Gamma_{\mu,m}(\exp H_\lambda, \pi(k_1)v_3, \pi(k_2^{-1})v_4)}\Omega_\lambda(H_\lambda)\, dH_\lambda\, dk_1\, dk_2.$$

By (1) of Lemma 4.5 and applying the Fubini–Tonelli theorem, we can interchange the two innermost integral and we therefore need to prove that

$$\int_K \int_{\mathfrak{a}_\lambda^+} \int_K \Gamma_{\lambda,l}(\exp H_\lambda, \pi(k_1)\dot{\pi}(X)v_1, \pi(k_2^{-1})v_2) \cdot \overline{\Gamma_{\mu,m}(\exp H_\lambda, \pi(k_1)v_3, \pi(k_2^{-1})v_4)}\Omega_\lambda(H_\lambda)\, dk_1\, dH_\lambda\, dk_2$$

is equal to

$$-\int_K \int_{\mathfrak{a}_\lambda^+} \int_K \Gamma_{\lambda,l}(\exp H_\lambda, \pi(k_1)v_1, \pi(k_2^{-1})v_2) \cdot \overline{\Gamma_{\mu,m}(\exp H_\lambda, \pi(k_1)\dot{\pi}(X)v_3, \pi(k_2^{-1})v_4)}\Omega_\lambda(H_\lambda)\, dk_1\, dH_\lambda\, dk_2.$$

Set

$$\mathcal{I}(\exp H_\lambda, k_1, k_2^{-1}) := \Gamma_{\lambda,l}(\exp_\lambda, \pi(k_1)\dot{\pi}(X)v_1, \pi(k_2^{-1})v_2)\overline{\Gamma_{\mu,m}(\exp H_\lambda, \pi(k_1)v_3, \pi(k_2^{-1})v_4)}.$$

We apply the quotient integral formula [3, Theorem 2.51], to write the integral

$$\int_K \int_{\mathfrak{a}_\lambda^+} \int_K \mathcal{I}(\exp H_\lambda, k_1, k_2^{-1})\Omega(H_\lambda)\, dk_1\, dH_\lambda\, dk_2$$

as

$$\int_K \int_{\mathfrak{a}_\lambda^+} \int_{K_\lambda\backslash K} \int_{K_\lambda} \mathcal{I}(\exp H_\lambda, \xi_1 k_1, k_2^{-1})\Omega_\lambda(H_\lambda)\, d\xi_1\, d\dot{k}_1\, dH_\lambda\, dk_2$$

and again to write it as

$$\int_{K/K_\lambda}\int_{K_\lambda}\int_{\mathfrak{a}_\lambda^+}\int_{K_\lambda\backslash K}\int_{K_\lambda}\mathcal{I}(\exp H_\lambda,\xi_1k_1,\xi_2^{-1}k_2^{-1})\Omega_\lambda(H_\lambda)\,d\xi_1\,d\dot{k}_1\,dH_\lambda\,d\xi_2\,d\dot{k}_2.$$

By (3) of Lemma 4.5, we can appeal to the Fubini–Tonelli theorem to interchange the two innermost integrals and to obtain

$$\int_{K/K_\lambda}\int_{K_\lambda}\int_{\mathfrak{a}_\lambda^+}\int_{K_\lambda}\int_{K_\lambda\backslash K}\mathcal{I}(\exp H_\lambda,\xi_1k_1,\xi_2^{-1}k_2^{-1})\Omega_\lambda(H_\lambda)\,d\dot{k}_1\,d\xi_1\,dH_\lambda\,d\xi_2\,d\dot{k}_2.$$

Now, combining the fact that $M_\lambda^{\mathrm{reg}}=K_\lambda A_\lambda^+K_\lambda$, the relevant integral formula and the fact that the complement of M^{reg} has measure zero in M, it follows that the integral

$$\int_{K_\lambda}\int_{\mathfrak{a}_\lambda^+}\int_{K_\lambda}\int_{K_\lambda\backslash K}\mathcal{I}(\exp H_\lambda,\xi_1k_1,\xi_2^{-1}k_2^{-1})\Omega_\lambda(H_\lambda)\,d\dot{k}_1\,d\xi_1\,dH_\lambda\,d\xi_2$$

is equal to

$$\int_{M_\lambda}\int_{K_\lambda\backslash K}\Gamma_{\lambda,l}(m_\lambda,\pi(k_1)\dot{\pi}(X)v_1,\pi(k_2^{-1})v_2)\overline{\Gamma_{\mu,m}(m_\lambda,\pi(k_1)v_3,\pi(k_2^{-1})v_4)}\,d\dot{k}_1\,dm_\lambda.$$

For $k_1\in K$, we define

$$f(k_1):=\big\langle\Gamma_{\lambda,l}(m_\lambda,\pi(k_1)\dot{\pi}(X)v_1,\pi(k_2^{-1})v_2),\Gamma_{\mu,m}(m_\lambda,\pi(k_1)v_3,\pi(k_2^{-1})v_4)\big\rangle_{L^2(M_\lambda)}.$$

The function f is invariant under left-multiplication by K_λ. Indeed, if $\xi\in K_\lambda$, then

$$\Gamma_{\lambda,l}(m_\lambda,\pi(\xi k_1)\dot{\pi}(X)v_1,\pi(k_2)v_2)=\Gamma_{\lambda,l}(m_\lambda\xi,\pi(k_1)\dot{\pi}(X)v_1,\pi(k_2)v_2)$$

and similarly for the $\Gamma_{\mu,m}$-term. Since the right-regular representation of M_λ is unitary, we have

$$\big\langle\Gamma_{\lambda,l}(m_\lambda\xi,\pi(k_1)\dot{\pi}(X)v_1,\pi(k_2^{-1})v_2),\Gamma_{\mu,m}(m_\lambda\xi,\pi(k_1)v_3,\pi(k_2^{-1})v_4)\big\rangle_{L^2(M_\lambda)}=f(k).$$

An application of the quotient integral formula [3, Theorem 2.51] gives

$$\int_K f(k_1)\,dk_1=\int_{K_\lambda\backslash K}\int_K f(\xi k_1)\,d\xi\,d\dot{k}_1=\mathrm{vol}(K_\lambda)\int_{K_\lambda\backslash K}f(k_1)\,d\dot{k}_1.$$

By (2) in Lemma 4.5 and appealing again to the Fubini–Tonelli theorem, we interchange the integrals over M_λ and $K_\lambda\backslash K$ to obtain that

$$\int_{K/K_\lambda}\int_{M_\lambda}\int_{K_\lambda\backslash K}\Gamma_{\lambda,l}(m_\lambda,\pi(k_1)\dot{\pi}(X)v_1,\pi(k_2^{-1})v_2)\cdot\overline{\Gamma_{\mu,m}(m_\lambda,\pi(k_1)v_3,\pi(k_2^{-1})v_4)}\,d\xi_1\,dm_\lambda\,d\xi_2$$

equals

$$\frac{1}{\mathrm{vol}(K_\lambda)}\int_{K/K_\lambda}\int_K f(k_1)\,dk_1\,d\dot{k}_2,$$

which, in turn, equals

$$\frac{1}{\operatorname{vol}(K_\lambda)}\int_{K/K_\lambda}\int_K \big\langle \Gamma_{\lambda,l}(m_\lambda, \pi(k_1)\dot{\pi}(X)v_1, \pi(k_2^{-1})v_2), \\ \Gamma_{\mu,m}(m_\lambda, \pi(k_1)v_3, \pi(k_2^{-1})v_4)\big\rangle_{L^2(M_\lambda)}\, dk_1\, d\dot{k}_2.$$

For fixed $k_2 \in K$, set $w_2 := \pi(k_2^{-1})v_2$ and $w_4 := \pi(k_2^{-1})v_4$. We reduced the problem to proving that

$$\int_K \big\langle \Gamma_{\lambda,l}(m_\lambda, \pi(k_1)\dot{\pi}(X)v_1, w_2), \Gamma_{\mu,m}(m_\lambda, \pi(k_1)v_3, w_4)\big\rangle_{L^2(M_\lambda)}\, dk_1$$

equals

$$-\int_K \big\langle \Gamma_{\lambda,l}(m_\lambda, \pi(k_1)v_1, w_2), \Gamma_{\mu,m}(m_\lambda, \pi(k_1)\dot{\pi}(X)v_3, w_4)\big\rangle_{L^2(M_\lambda)}\, dk_1.$$

The result is therefore a consequence of Proposition 3.21. □

Lemma 4.5. *Let $v_1, w_2, v_3, w_4 \in H_K$. Let $[\lambda, l], [\mu, m] \in \mathcal{C}/\!\sim$ be such that $I_\lambda = I_\mu$, $\lambda|_{\mathfrak{a}_{\lambda_0}} = \mu|_{\mathfrak{a}_{\lambda_0}}$ and $\boldsymbol{d}(\pi) = |I_\lambda| + \sum_{i \in I_\lambda}(l_i + m_i)$. Then the following holds*:

(1) $\displaystyle\int_K \int_{\mathfrak{a}_\lambda^+} \big|\Gamma_{\lambda,l}(\exp H_\lambda, \pi(k_1)v_1, w_2)\overline{\Gamma_{\mu,m}(\exp H_\lambda, \pi(k_1)v_3, w_4)}\big|\, dH_\lambda\, dk_1 < \infty.$

(2) $\displaystyle\int_{K_\lambda\backslash K}\int_{M_\lambda} \big|\Gamma_{\lambda,l}(m_\lambda, \pi(k)v_1, w_2)\overline{\Gamma_{\mu,m}(m_\lambda, \pi(k)v_3, w_4)}\big|\, dm_\lambda\, d\dot{k} < \infty.$

(3) *For any fixed $H_\lambda \in \mathfrak{a}_\lambda^+$, we have*

$$\int_{K_\lambda\backslash K}\int_{K_\lambda} \big|\Gamma_{\lambda,l}(\exp H_\lambda, \pi(\xi k)v_1, w_2)\overline{\Gamma_{\mu,m}(\exp H_\lambda, \pi(\xi k)v_3, w_4)}\big|\, d\xi\, d\dot{k} < \infty.$$

Proof. To prove (1), we begin by observing that, for a fixed element k of K, the functions $\Gamma_{\lambda,l}(\exp H_\lambda, \pi(k)v_1, v_2)$ and $\Gamma_{\mu,m}(\exp H_\lambda, \pi(k)v_3, v_4)$ are square-integrable on $\mathfrak{a}_\lambda^+$ by Proposition 3.13. Therefore, we have

$$\int_{\mathfrak{a}_\lambda^+} \big|\Gamma_{\lambda,l}(\exp H_\lambda, \pi(k)v_1, w_2)\overline{\Gamma_{\mu,m}(\exp H_\lambda, \pi(k)v_3, w_4)}\big|\, dm_\lambda < \infty.$$

Hence, we can define the function

$$h : K \to \mathbb{R}_{\geq 0}, \quad h(k) = \int_{\mathfrak{a}_\lambda^+} \big|\Gamma_{\lambda,l}(\exp H_\lambda, \pi(k)v_1, w_2)\overline{\Gamma_{\mu,m}(\exp H_\lambda, \pi(k)v_3, w_4)}\big|\, dH_\lambda,$$

and the result will follow if we establish the continuity of h. The K-finiteness of v_1 and v_3 implies the existence of finitely many K-finite vectors $v_1^{(1)}, \dots, v_1^{(p)}$ and finitely many K-finite vectors $v_3^{(1)}, \dots, v_3^{(q)}$ such that

$$\pi(k)v_1 = \sum_{i=1}^{p} a_i(k)v_1^{(i)} \quad \text{and} \quad \pi(k)v_3 = \sum_{j=1}^{q} b_j(k)v_3^{(j)}$$

for continuous complex-valued functions a_i and b_j. Let $k_0 \in K$. Then

$$|h(k) - h(k_0)|$$

is majorised by the integral over $\mathfrak{a}_\lambda^+$ of

$$\begin{aligned}\Big|\big|\Gamma_{\lambda,l}(\exp H_\lambda, \pi(k)v_1, w_2)\overline{\Gamma_{\mu,m}(\exp H_\lambda, \pi(k)v_3, w_4)}\big| \\ - \big|\Gamma_{\lambda,l}(\exp H_\lambda, \pi(k_0)v_1, w_2)\overline{\Gamma_{\mu,m}(\exp H_\lambda, \pi(k_0)v_3, w_4)}\big|\Big|.\end{aligned}$$

By reverse triangle inequality, the integrand is majorised by

$$\begin{aligned}\big|\Gamma_{\lambda,l}(\exp_\lambda, \pi(k)v_1, w_2)\overline{\Gamma_{\mu,m}(\exp H_\lambda, \pi(k)v_3, w_4)} \\ - \Gamma_{\lambda,l}(\exp H_\lambda, \pi(k_0)v_1, w_2)\overline{\Gamma_{\mu,m}(\exp H_\lambda, \pi(k_0)v_3, w_4)}\big|,\end{aligned}$$

which, in turn, is less than or equal to

$$\sum_{i=1}^{p}\sum_{j=1}^{q} |a_i(k)\overline{b_j(k)} - a_i(k_0)\overline{b_j(k_0)}|\, |\Gamma_{\lambda,l}(\exp H_\lambda, v_1^{(i)}, w_2)\overline{\Gamma_{\mu,m}(\exp H_\lambda, v_3^{(j)}, w_4)}|.$$

We obtained

$$\begin{aligned}|h(k) - h(k_0)| \le \sum_{i=1}^{p}\sum_{j=1}^{q} |a_i(k)\overline{b_j(k)} - a_i(k_0)\overline{b_j(k_0)}| \\ \cdot \int_{\mathfrak{a}_\lambda^+} |\Gamma_{\lambda,l}(\exp H_\lambda, v_1^{(i)}, w_2)\overline{\Gamma_{\mu,m}(\exp H_\lambda, v_3^{(j)}, w_4)}|\, dH_\lambda,\end{aligned}$$

and the continuity follows from the continuity of the a_i's and b_j's.

For (2), we first observe that for fixed $k \in K$, the functions $\Gamma_{\lambda,l}(m_\lambda, \pi(k)v_1, w_2)$ and $\Gamma_{\mu,m}(m_\lambda, \pi(k)v_3, w_4)$ are square-integrable on M_λ by Proposition 3.13. Therefore, we have

$$\int_{M_\lambda} \big|\Gamma_{\lambda,l}(m_\lambda, \pi(k)v_1, w_2)\overline{\Gamma_{\mu,m}(m_\lambda, \pi(k)v_3, w_4)}\big|\, dm_\lambda < \infty.$$

Hence, we can define the function

$$h : K \to \mathbb{R}_{\ge 0}, \quad h(k) = \int_{M_\lambda} \big|\Gamma_{\lambda,l}(m_\lambda, \pi(k)v_1, w_2)\overline{\Gamma_{\mu,m}(m_\lambda, \pi(k)v_3, w_4)}\big|\, dm_\lambda.$$

Arguing as for (1), we obtain that h is continuous.

By the right-invariance of the Haar measure on M_λ and since

$$\Gamma_{\lambda,l}(m_\lambda, \pi(\xi k)v_1, w_2) = \Gamma_{\lambda,l}(m_\lambda \xi, \pi(k)v_1, w_2)$$

for every $\xi \in K_\lambda$ (and similarly for the $\Gamma_{\mu,m}$-term), the function h is invariant under multiplication on the left by elements in K_λ and it therefore descends to a continuous function on $K_\lambda \backslash K$, concluding the proof of (2).

For (3), given a fixed $H_\lambda \in \mathfrak{a}_\lambda^+$ the function

$$K_\lambda \to \mathbb{C}, \quad \xi \mapsto \Gamma_{\lambda,l}(\exp H_\lambda, \pi(\xi k)v_1, w_2),$$

is continuous. Indeed, let $\xi_0 \in K_\lambda$. Since $\pi(k)v_1$ is K-finite, it is in particular K_λ-finite. Hence, there exist finitely many K_λ-finite vectors $v_1^{(1)}, \ldots, v_1^{(r)}$ such that

$$\pi(\xi)\pi(k)v = \sum_{i=1}^{r} c_i(\xi) v_1^{(i)},$$

where each c_i is a complex-valued continuous function on K_λ. Therefore,

$$\left|\Gamma_{\lambda,l}(\exp H_\lambda, \pi(\xi k)v_1, w_2) - \Gamma_{\lambda,l}(\exp H_\lambda, \pi(\xi_0 k)v_1, w_2)\right|$$

is bounded by

$$\sum_{i=1}^{r} |c_i(\xi) - c_i(\xi_0)|\, |\Gamma_{\lambda,l}(\exp H_\lambda, v_1^{(i)}, w_2)|$$

and the claim follows from the continuity of the c_i's.

The same argument shows that, for fixed $H_\lambda \in \mathfrak{a}_\lambda^+$, the function

$$K_\lambda \to \mathbb{C}, \quad \xi \mapsto \Gamma_{\mu,m}(\exp H_\lambda, \pi(\xi)v_3, w_4),$$

is continuous and it follows that

$$\int_{K_\lambda} \left|\Gamma_{\lambda,l}(\exp H_\lambda, \pi(\xi k)v_1, w_2)\overline{\Gamma_{\mu,m}(\exp H_\lambda, \pi(\xi k)v_3, w_4)}\right| d\xi < \infty.$$

Hence, we can define the function

$$f : K \to \mathbb{R}_{\geq 0},\ f(k) = \int_{K_\lambda} \left|\Gamma_{\lambda,l}(\exp H_\lambda, \pi(\xi k)v_1, w_2)\overline{\Gamma_{\mu,m}(\exp H_\lambda, \pi(\xi k)v_3, w_4)}\right| d\xi$$

and argue as in the proof of (2). □

We now complete the strategy outlined in the Introduction. For fixed $v_2, v_4 \in H_K$, we define

$$A_{v_2,v_4} := D(\,\cdot\,, v_2, \,\cdot\,, v_4),$$

which is linear in the first variable and conjugate linear in the second. For fixed $v_1, v_3 \in H_K$, we define

$$B_{v_1,v_3} := D(v_1, \,\cdot\,, v_3, \,\cdot\,),$$

which is conjugate-linear in the first variable and linear in the second.

Theorem 4.6. *Let G be a connected, semisimple Lie group with finite centre. Let (π, H) be a tempered, irreducible representation of G. Then there exists $\boldsymbol{f}(\pi) \in \mathbb{R}_{>0}$ such that, for all $v_1, v_2, v_3, v_4 \in H_K$, we have*

$$\lim_{r\to\infty} \frac{1}{r^{\boldsymbol{d}(\pi)}} \int_{G_{<r}} \langle \pi(g)v_1, v_2\rangle \overline{\langle \pi(g)v_3, v_4\rangle}\, dg = \frac{1}{\boldsymbol{f}(\pi)} \langle v_1, v_3\rangle \overline{\langle v_2, v_4\rangle}.$$

Proof. Fix $v_2, v_4 \in H_K$. By Proposition 4.2, we can apply Corollary 2.13 to the form A_{v_2,v_4}. Hence there exists $c_{v_2,v_4} \in \mathbb{C}$ such that for all $v_1, v_3 \in H_K$ we have

$$A_{v_2,v_4}(v_1, v_3) = c_{v_2,v_4} \langle v_1, v_3\rangle.$$

Similarly, fixing $v_1, v_3 \in H_K$, by Proposition 4.2 and Lemma 4.1 there exists a $d_{v_1,v_3} \in \mathbb{C}$ such that

$$\overline{B_{v_3,v_1}(v_4, v_2)} = d_{v_1,v_3} \langle v_4, v_2\rangle,$$

since the left-hand side is conjugate-linear in the first variable. Hence, since

$$\overline{B_{v_3,v_1}(v_4, v_2)} = B_{v_1,v_3}(v_2, v_4),$$

we obtain

$$B_{v_1,v_3}(v_2, v_4) = d_{v_1,v_3} \overline{\langle v_2, v_4\rangle}.$$

By definition, we have

$$D(v_1, v_2, v_3, v_4) = A_{v_2,v_4}(v_1, v_3) = B_{v_1,v_3}(v_2, v_4),$$

so, for a vector $v_0 \in H_K$ of norm 1, using (2) of Theorem 1.2, we obtain a real number $C(v_0, v_0) > 0$ such that

$$D(v_0, v_0, v_0, v_0) = C(v_0, v_0) = c_{v_0,v_0} = d_{v_0,v_0}.$$

Computing $D(v_1, v_0, v_3, v_0)$, we have

$$d_{v_1,v_3} = c_{v_0,v_0} \langle v_1, v_3\rangle.$$

Therefore, we obtained

$$D(v_1, v_2, v_3, v_4) = c_{v_0,v_0} \langle v_1, v_3\rangle \overline{\langle v_2, v_4\rangle},$$

showing that $\boldsymbol{f}(\pi) := \frac{1}{C(v_0,v_0)}$ does not depend on the choice of v_0, as required. □

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References

[1] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, 2nd ed., Mathematical Surveys and Monographs **67**, Amer. Math. Soc., Providence, RI, 2000. MR

[2] W. Casselman and D. Miličić, "Asymptotic behavior of matrix coefficients of admissible representations", *Duke Math. J.* **49**:4 (1982), 869–930. MR

[3] G. B. Folland, *A course in abstract harmonic analysis*, 2nd ed., CRC Press, Boca Raton, FL, 2015. MR

[4] D. Goldfeld and J. Hundley, *Automorphic representations and L-functions for the general linear group, I*, Cambridge Studies in Advanced Mathematics **129**, Cambridge Univ. Press, 2011. MR

[5] Harish-Chandra, "Harmonic analysis on real reductive groups, I: The theory of the constant term", in *Collected Papers*, *IV*, Springer, Berlin, 2014.

[6] Harish-Chandra, "Invariant eigendistributions on a semisimple Lie group", in *Collected Papers*, *III*, edited by V. S. Varadarajan, Springer, Berlin, 2014.

[7] Harish-Chandra, "Supplement to 'Some results on differential equations' ", in *Collected papers*, *III*, edited by V. S. Varadarajan, Springer, Berlin, 2014.

[8] H. Hecht and W. Schmid, "Characters, asymptotics and $\mathfrak{n}$-homology of Harish-Chandra modules", *Acta Math.* **151**:1-2 (1983), 49–151. MR

[9] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics **34**, Amer. Math. Soc., Providence, RI, 2001. MR

[10] D. Kazhdan and A. Yom Din, "On tempered representations", *J. Reine Angew. Math.* **788** (2022), 239–280. MR

[11] A. W. Knapp, *Representation theory of semisimple groups: an overview based on examples*, Princeton Mathematical Series **36**, Princeton Univ. Press, 1986. MR

[12] A. W. Knapp, *Lie groups beyond an introduction*, 2nd ed., Progress in Mathematics **140**, Birkhäuser, Boston, MA, 2002. MR

[13] A. W. Knapp and D. A. Vogan, Jr., *Cohomological induction and unitary representations*, Princeton Mathematical Series **45**, Princeton Univ. Press, 1995. MR

[14] A. Knightly and C. Li, *Traces of Hecke operators*, Mathematical Surveys and Monographs **133**, Amer. Math. Soc., Providence, RI, 2006. MR

[15] R. P. Langlands, "On the classification of irreducible representations of real algebraic groups", pp. 101–170 in *Representation theory and harmonic analysis on semisimple Lie groups*, edited by J. Sally, Paul J. and J. Vogan, David A., Math. Surveys Monogr. **31**, Amer. Math. Soc., Providence, RI, 1989. MR

[16] D. Renard, *Représentations des groupes réductifs p-adiques*, Cours Spécialisés **17**, Soc. Math. France, Paris, 2010. MR

[17] V. S. Varadarajan, *Harmonic analysis on real reductive groups*, Lecture Notes in Mathematics **576**, Springer, 1977. MR

[18] N. R. Wallach, *Real reductive groups*, *I*, Pure and Applied Mathematics **132**, Academic Press, Boston, MA, 1988. MR

[19] G. Warner, *Harmonic analysis on semi-simple Lie groups, I*, Grundl. Math. Wissen. **188**, Springer, 1972. MR

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A FUNCTORIALITY PROPERTY FOR SUPERCUSIPIDAL L-PACKETS

ADÈLE BOURGEOIS AND PAUL MEZO

Kaletha constructed L-packets for supercuspidal L-parameters of tame p-adic groups. These L-packets consist entirely of supercuspidal representations, which are explicitly described. In the setting of quasisplit reductive groups, we show that Kaletha's L-packets satisfy a functoriality property for homomorphisms with central kernel and abelian cokernel.

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1. Introduction

Let G be a connected reductive algebraic group defined over a nonarchimedean local field F. The local Langlands correspondence (LLC) is a conjectural map

$$\varphi \mapsto \Pi_\varphi$$

from *L-parameters* to *L-packets* [Borel 1979, Chapter III]. The latter are finite sets of (equivalence classes of) irreducible representations of the group of F-points, $G(F)$. The LLC is expected to satisfy numerous additional properties which give it content. We focus on only two properties. The first property concerns *supercuspidal* representations. A large class of supercuspidal representations have been grouped into L-packets by Kaletha [2019; 2021]. We shall *exclusively* be dealing with these supercuspidal representations. The second property concerns *functoriality* for homomorphisms with central kernel and abelian cokernel. This form of functoriality was introduced by Borel [1979, Desideratum 10.3(5)] and was later refined by Solleveld [2020, Corollary 2].

To describe the expected properties for supercuspidal representations, we recall that an L-parameter is an L-homomorphism

$$\varphi : W_F \times \mathrm{SL}_2 \to {}^L G$$

from the Weil–Deligne group into the L-group of G [Borel 1979, Section 8.2]. Following [Kaletha 2021, Section 4.1], the L-parameter φ is defined to be *supercuspidal* if it is trivial on SL_2, i.e.,

$$\varphi : W_F \to {}^L G,$$

and its image is not contained in a proper parabolic subgroup of ${}^L G$ [Borel 1979, Section 3.3]. As observed in [Kaletha 2021, Section 4.1], "compound" L-packets (or L-packets when G is quasisplit) consisting entirely of supercuspidal representations are conjectured to correspond precisely to supercuspidal L-parameters [DeBacker and Reeder 2009, Section 3.5; Aubert et al. 2018]. Kaletha [2019; 2021] provided an explicit construction for these conjectured L-packets, under the additional assumptions that G splits over a tamely ramified extension, and that the residual characteristic p of F does not divide the order of the Weyl group of G. He further proved that the L-packets satisfy some important properties (e.g., stability).

The first goal of this paper is to show that these L-packets satisfy the desired functorial property [Borel 1979, Desideratum 10.3(5)]. For this reason, and from now on, we work under the assumptions on G and the residual characteristic of F given in the previous paragraph. For the sake of simplicity, we also assume that G is quasisplit over F (see the discussion surrounding (1)). Let $\Phi_{\mathrm{sc}}(G)$ denote the set (of conjugacy classes) of supercuspidal L-parameters of G. Given $\varphi \in \Phi_{\mathrm{sc}}(G)$, we let Π_φ denote the associated supercuspidal L-packet obtained via Kaletha's construction.

Theorem (Theorem 4.1). *Suppose G is quasisplit and splits over a tamely ramified extension. Suppose further that the residual characteristic p of F does not divide the order of the Weyl group of G. Let $\eta : G \to \underline{G}$ be an F-morphism of connected reductive F-groups such that*

(i) *the kernel of $d\eta : \mathrm{Lie}(G) \to \mathrm{Lie}(\underline{G})$ is central,*

(ii) *the cokernel of η is an abelian F-group.*

Let $\underline{\varphi} \in \Phi_{\mathrm{sc}}(\underline{G})$ and set $\varphi = {}^L\eta \circ \underline{\varphi}$. Then for all $\underline{\pi} \in \Pi_{\underline{\varphi}}$, $\underline{\pi} \circ \eta$ is the direct sum of finitely many irreducible supercuspidal representations belonging to Π_{φ}.

Here, the map ${}^L\eta : {}^L\underline{G} \to {}^LG$ takes the form ${}^L\eta(g, w) = (\hat{\eta}(g), w)$ for all $g \in \widehat{\underline{G}}$, $w \in W_F$. The map $\hat{\eta} : \widehat{\underline{G}} \to \widehat{G}$ on the Langlands dual groups is recalled in Section 4.1.

The above theorem is a modified version of [Borel 1979, Desideratum 10.3(5)], in which η is required to have abelian kernel and cokernel. The hypothesis on η is precisely [Solleveld 2020, Condition 1], and is stronger [Solleveld 2020, Lemma 5.1] than that of [Borel 1979, Desideratum 10.3(5)]. It ensures that the root systems of G and $\underline{G}$ are identified through η in arbitrary characteristic (see [SGA 3_{III} 1970, Sections 6.8, 7.5]).

In addition to proving Theorem 4.1, we provide a description of the components of $\underline{\pi} \circ \eta$. The supercuspidal representations that make up the L-packets of Theorem 4.1 are constructed from *tame F-nonsingular elliptic pairs*, which consist of a particular kind of torus and a character thereof [Kaletha 2021, Definition 3.4.1]. Given such a pair $(\underline{S}, \underline{\theta})$ of $\underline{G}$, we let $\pi_{(\underline{S},\underline{\theta})}$ denote the attached supercuspidal representation of $\underline{G}(F)$, which is obtained from the *Kaletha–Yu construction*. This construction consists of applying the *J.-K. Yu construction* [2001] after unfolding $(\underline{S}, \underline{\theta})$ into an appropriate *$\underline{G}$-datum* [Kaletha 2019; 2021]. The representation $\pi_{(\underline{S},\underline{\theta})}$ may be reducible, and its irreducible components form part of an L-packet. The first big result of this paper is writing a decomposition formula for $\pi_{(\underline{S},\underline{\theta})} \circ \eta$.

Theorem (Theorem 3.1). *Let $(\underline{S}, \underline{\theta})$ and (S, θ) be tame F-nonsingular elliptic pairs for $\underline{G}$ and G, respectively. Assume that $\eta(S) \subset \underline{S}$ and $\theta = \underline{\theta} \circ \eta$. Then*

$$\pi_{(\underline{S},\underline{\theta})} \circ \eta \simeq \bigoplus_{\underline{c} \in \underline{C}} \pi_{(S,\theta)} \circ \underline{\mathrm{Ad}}(\underline{c}^{-1}),$$

where $\underline{C}$ is a set of coset representatives of $\eta(G(F))\backslash\underline{G}(F)/\underline{S}(F)$ and $\underline{\mathrm{Ad}}$ is the $\underline{G}(F)$-action on $G(F)$ described in Section 2.3.

The following three paragraphs sketch the main ideas required to prove this theorem, and its complete proof is given in Section 4.2.

The composition $\pi_{(\underline{S},\underline{\theta})} \circ \eta$ can be viewed as the restriction of $\pi_{(\underline{S},\underline{\theta})}$ to $\eta(G(F))$. Having abelian cokernel implies that $\eta(G)$ is a subgroup of $\underline{G}$ which contains the derived subgroup $[\underline{G}, \underline{G}]$. The kernel of η, which we denote by Z, is a central

$$(\pi_{(\underline{S},\underline{\theta})}, \underline{G}(F)) \xrightarrow{\text{[Bourgeois 2021]}} (\pi_{(\underline{S},\underline{\theta})}|_{G_Z(F)}, G_Z(F)) \xrightarrow{\text{Theorem 3.17}} (\pi_{(\underline{S},\underline{\theta})} \circ \eta, G(F))$$

Figure 1. Illustration of the two-step process required to compute $\pi_{(\underline{S},\underline{\theta})} \circ \eta$.

subgroup by [Solleveld 2020, Lemma 5.1]. We will write G_Z for $\eta(G)$ and use the Z in subscript for all objects attached to G_Z. In this notation, $G_Z \simeq G/Z$ and $\underline{G} = G_Z Z(\underline{G})$, where $Z(\underline{G})$ denotes the center of $\underline{G}$. We compute the restriction of $\pi_{(\underline{S},\underline{\theta})}$ to $\eta(G(F)) \simeq G(F)/Z(F)$ in two steps (as illustrated in Figure 1). First, by restricting to $\eta(G)(F) = G_Z(F)$, and second, by further restricting to $G(F)/Z(F)$. The group $\eta(G(F))$ is a normal subgroup of $G_Z(F)$ (Corollary 2.7) and the quotient is parameterized by a subgroup of a Galois cohomology group $H^1(F, Z)$ [Springer 2009, Proposition 12.3.4], which may be nontrivial. The restriction of supercuspidal representations to algebraic subgroups that contain the derived subgroup was extensively studied in [Bourgeois 2021]. We can apply the results therein to obtain a description for $\pi_{(\underline{S},\underline{\theta})}|_{G_Z(F)}$ (Theorem 3.19). The second restriction (Theorem 3.17) can be computed via Mackey theory, as the quotient $G_Z(F)/(G(F)/Z(F))$ is compact and abelian [Silberger 1979].

In order to describe the supercuspidal representations in the L-packets Π_φ and $\Pi_{\underline{\varphi}}$, one must know which tame F-nonsingular elliptic pairs to use. These pairs are provided by *supercuspidal L-packet data* [Kaletha 2021, Definition 4.1.4]. The supercuspidal L-packet data for φ and $\underline{\varphi}$ consist of tuples $(S, \hat{j}, \chi_0, \theta)$ and $(\underline{S}, \hat{\underline{j}}, \underline{\chi}_0, \underline{\theta})$, respectively. Unlike the previous paragraph, S and $\underline{S}$ are not subtori of G and $\underline{G}$. Rather, they are embedded into subtori of the respective groups. The elements $\hat{j}$ and $\hat{\underline{j}}$ specify families of *admissible embeddings* $S(F) \to G(F)$ and $\underline{S}(F) \to \underline{G}(F)$, denoted by $\mathcal{J}_F$ and $\underline{\mathcal{J}}_F$, respectively. Each embedding $j \in \mathcal{J}_F$ ($\underline{j} \in \underline{\mathcal{J}}_F$) is used to generate a tame F-nonsingular elliptic pair $(jS, j\theta)$ $((\underline{j}\underline{S}, \underline{j}\underline{\theta}))$. We let the components of $\pi_{(jS,j\theta)}$ $(\pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})})$ be elements of Π_φ $(\Pi_{\underline{\varphi}})$.

In order to apply our decomposition formula for $\pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})} \circ \eta$ and relate it to representations in Π_φ, we must first establish an appropriate relationship between the supercuspidal L-packet data and the admissible embeddings. This is given to us by Theorem 4.2, another key result of this paper, in which we show that for all $\underline{j} \in \underline{\mathcal{J}}_F$, there exists $j \in \mathcal{J}_F$ such that $\eta(jS) \subset \underline{j}\underline{S}$ and $j\theta = \underline{j}\underline{\theta} \circ \eta$. As such, we obtain a decomposition formula for $\pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})} \circ \eta$ in terms of certain conjugates of $\pi_{(jS,j\theta)}$. This completes the proof of Theorem 4.1.

The second goal of this paper is to provide a more detailed description of the decomposition of $\underline{\pi} \circ \eta$ in Theorem 4.1 in the special case that both φ and $\underline{\varphi}$ are *regular* supercuspidal L-parameters [Kaletha 2019, Definition 5.2.3]. The regularity assumptions on the L-parameters have several pleasant consequences. We list them for φ, with the understanding that their analogs hold for $\underline{\varphi}$. First, the representations $\pi_{(jS,j\theta)}$ are irreducible for all $j \in \mathcal{J}_F$. From this it follows that the set $\mathcal{J}_F$ parameterizes the representations in Π_{φ}. Second, the set $\mathcal{J}_F$ is in bijection with characters of the usual component group that one is accustomed to seeing in Langlands correspondences. More precisely, the component group of the centralizer of the image of φ is in bijection with the Galois-fixed subgroup of some torus $\widehat{S}$ [Kaletha 2019, Lemma 5.3.4], and certain characters of this component group are in bijection with $\mathcal{J}_F$ (see (17)). Since $\widehat{S}$ is abelian, so too is the component group.

The regularity assumptions consequently allow us to write

$$\underline{\pi} = \pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})} = \pi_{(\underline{\varphi},\underline{\varrho})},$$

where $\underline{j} \in \underline{\mathcal{J}}_F$ corresponds to a character $\underline{\varrho}$ of the component group for $\underline{\varphi}$. The map $\hat{\eta}$ sends the component group for $\underline{\varphi}$ to the one for φ. The precise description of $\underline{\pi} \circ \eta$ is given in Proposition 5.12. We summarize it here as follows.

Theorem 1.1. *Let $\eta : G \to \underline{G}$, $\underline{\varphi}$ and $\varphi = {}^L\eta \circ \underline{\varphi}$ be as in Theorem 4.1. Assume that φ and $\underline{\varphi}$ are regular. Then*

$$\pi_{(\underline{\varphi},\underline{\varrho})} \circ \eta \simeq \bigoplus_{\varrho} \operatorname{Hom}(\underline{\varrho}, (\varrho \circ \hat{\eta})) \otimes \pi_{(\varphi,\varrho)},$$

where $\underline{\varrho}$ and ϱ are characters of the Langlands component groups.

Theorem 1.1 is the proof of a conjecture of Solleveld for regular supercuspidal L-parameters [2020, Conjecture 2]. Solleveld proved his conjecture in a variety of cases [2020, Theorem 3]. The only overlap of Theorem 1.1 with these cases is when G and $\underline{G}$ are inner forms of GL_n, SL_n or PGL_n.

One might hope that the regularity of $\varphi = {}^L\eta \circ \underline{\varphi}$ in Theorem 1.1 would follow from the regularity of $\underline{\varphi}$. While this is not true in general, as illustrated with a counterexample at the end of Section 5.1, the converse implication holds (Corollary 5.3). Furthermore, as explained after [Kaletha 2019, Definition 3.7.3], regular L-parameters are typical among all supercuspidal L-parameters.

Let us discuss how one might extend Theorem 1.1 to nonregular supercuspidal L-parameters. In this case, $\underline{\mathcal{J}}_F$ is no longer a parameterizing set for $\Pi_{\underline{\varphi}}$ since the representations $\pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})}$, $\underline{j} \in \underline{\mathcal{J}}_F$, may be reducible. For each $\underline{j} \in \underline{\mathcal{J}}_F$, the irreducible components of $\pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})}$ are parameterized by certain representations of $N(\underline{j}\underline{S}, \underline{G})(F)_{\underline{j}\underline{\theta}}$, the stabilizer of the pair $(\underline{j}\underline{S}, \underline{j}\underline{\theta})$ in $N(\underline{j}\underline{S}, \underline{G})(F)$ [Kaletha 2021, Corollary 3.4.7]. It appears that [Kaletha 2021, Proposition 4.3.2] serves as a bridge between $\{N(\underline{j}\underline{S}, \underline{G})(F)_{\underline{j}\underline{\theta}} : \underline{j} \in \underline{\mathcal{J}}_F\}$ and the component group of the centralizer of the image of $\underline{\varphi}$. Another key step in the proof of Theorem 1.1 is the decomposition formula for $\pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})} \circ \eta$. Removing the regularity hypothesis means one would need to derive the decomposition formula of $\underline{\pi} \circ \eta$, where $\underline{\pi}$ is an irreducible component of $\pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})}$. This would require a deeper study of the results in [Kaletha 2021, Section 3]. Once one has such a decomposition formula, we believe that similar arguments as the ones in Section 5.2 could be applied.

Let us briefly indicate what is required to extend Theorems 4.1 and 1.1 to nonquasisplit groups. Every connected reductive algebraic F-group G' is an inner form of a quasisplit form G. When char $F = 0$, the group G' may be assigned to a class of *rigid inner twists* for G [Kaletha 2016, Corollary 3.8 and Section 5.1]. This class is an element in a set of the form $H^1(u \to W, Z' \to G)$ which we shall not describe. For any $j \in \mathcal{J}_F$, there is a natural surjection

$$H^1(u \to W, Z' \to jS) \to H^1(u \to W, Z' \to G), \tag{1}$$

where jS is a maximal torus of G. The elements in a supercuspidal L-packet of G' are indexed by the fiber in $H^1(u \to W, Z' \to jS)$ over the class in $H^1(u \to W, Z' \to G)$ corresponding to G' [Kaletha 2019, Section 5.3]. If one is only interested in the quasisplit form, that is, $G' = G$, the classes of rigid inner twists may be chosen to equal the usual Galois cohomology sets (which we recall more precisely below), and a supercuspidal L-packet is indexed by the fiber of the more familiar map

$$H^1(F, jS) \to H^1(F, G) \tag{2}$$

over the trivial class. This fiber is in bijection with the set of admissible embeddings $\mathcal{J}_F$ above. In general, the fiber of (1) corresponding to G' is in bijection with the set $\mathcal{J}'_F$ of admissible embeddings into (the rigid inner twist for) $G'(F)$. When char $F \neq 0$, a parallel picture is given in [Dillery 2023]. The constructions and results of Sections 3 and 4 apply to $G'(F)$ and $\mathcal{J}'_F$ in exactly the same manner as they do to $G(F)$ and $\mathcal{J}_F$. More work is required to accommodate $G'(F)$ and $\mathcal{J}'_F$ in the constructions of Section 5. Rather than working with characters of $\pi_0(\widehat{S}^\Gamma / Z(\widehat{G})^\Gamma)$, one works with the characters of the larger group $\pi_0([\widehat{\bar{S}}]^+)$ which appear in [Kaletha 2019, Lemma 5.3.4; 2016, Corollary 5.4]. The admissible embeddings $\mathcal{J}'_F$ correspond to certain characters of $\pi_0([\widehat{\bar{S}}]^+)$ and these characters correspond to the representations in the L-packet for G'. A discussion of such matters may be found in [Kaletha 2016, Section 5.4]. In view of the length of this paper, which deals only with quasisplit groups, it seems prudent to leave the treatment of nonquasisplit groups to some future work.

The paper is organized as follows. Section 2 contains preliminaries, beginning with an outline of the notation and conventions used throughout this paper (Section 2.1). We also present results concerning the structure theory of G, G_Z and $\underline{G}$ (Sections 2.2 and 2.3), and provide summaries for the Kaletha–Yu construction of supercuspidal representations (Section 2.4) as well as Kaletha's construction of supercuspidal L-packets (Section 2.5). In Section 3, we prove Theorem 3.1, which describes the decomposition of $\pi_{(\underline{S},\underline{\theta})} \circ \eta$. Most of the section (Sections 3.1 and 3.2) focuses on proving the second part of the two-step restriction illustrated in Figure 1, that is, describing restrictions from $G_Z(F)$ to $G(F)/Z(F)$. In particular, a deep analysis of the Kaletha–Yu construction is required, and we show that the J.-K. Yu construction essentially commutes with η. In Section 4, we prove the functoriality of Kaletha's supercuspidal L-packets (Theorem 4.1). We first establish the relationship between the supercuspidal L-packet data associated to φ and $\underline{\varphi}$ (Section 4.1) and end with the proof of Theorem 4.1 (Section 4.2). In Section 5, we start by describing the regular supercuspidal L-parameters and their corresponding L-packet structure, as well as a discussion on when one might expect both parameters φ and $\underline{\varphi}$ to be regular (Section 5.1). We then proceed to reparameterize the L-packets in terms of characters of their corresponding component groups and proving Theorem 1.1 (Section 5.2).

2. Preliminaries

We set up results concerning the structure theory of G, G_Z and $\underline{G}$, and summarize the constructions that will be needed in this paper. We begin with notation and conventions in Section 2.1, after which we discuss how the structure theory of G_Z and that of G relate in Section 2.2. In Section 2.3, we describe an action of $\underline{G}(F)$ on representations of $G(F)$. In Section 2.4, we summarize the Kaletha–Yu construction, which produces supercuspidal representations from F-nonsingular elliptic pairs. We summarize Kaletha's construction of supercuspidal L-packets in Section 2.5.

2.1. ***Notation and conventions.*** Given the nonarchimedean local field F, we denote by $\mathcal{O}_F$ its ring of integers, $\mathfrak{p}_F$ the unique maximal ideal of $\mathcal{O}_F$ and $\mathfrak{f}$ its residue field of prime characteristic p. Let F^{un} be a maximal unramified extension of F. The residue field of F^{un} is an algebraic closure of $\mathfrak{f}$, so we denote it by $\bar{\mathfrak{f}}$. The Galois group $\mathrm{Gal}(F^{\mathrm{un}}/F)$ is canonically isomorphic to $\mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$, and we denote the Frobenius automorphism by Fr. Let $\Gamma = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ denote the Galois groups of F, where F^{sep} is a separable closure of F. We use the notation I_F and P_F for the inertia subgroup and wild inertia subgroup of the Weil group W_F, respectively. We also let E denote the tamely ramified extension of F over which G splits.

In this paper, we will encounter different types of cohomology groups. Given an algebraic group G' that is defined over a field F', we take $G'(F')$ to be the set of F'-points in the sense of [Springer 2009, Section 2.1]. For G' defined over F we write $H^1(F, G')$ for $H^1(\Gamma, G'(F^{\mathrm{sep}}))$. Similarly, given an algebraic group $\mathcal{G}'$ that is defined over $\mathfrak{f}$, we write $H^1(\mathfrak{f}, \mathcal{G}')$ for $H^1(\mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f}), \mathcal{G}'(\bar{\mathfrak{f}}))$. Furthermore, given a group $\widetilde{G}$ with $\mathrm{Gal}(F^{\mathrm{un}}/F)$-action, we write $H^1(\mathrm{Fr}, \widetilde{G})$ for $H^1(\mathrm{Gal}(F^{\mathrm{un}}/F), \widetilde{G})$ and $\widetilde{G}^{\mathrm{Fr}}$ for $\widetilde{G}^{\mathrm{Gal}(F^{\mathrm{un}}/F)}$.

Given a maximal torus T of G, we let $R(G, T)$ denote the root system of G with respect to T. Given $\alpha \in R(G, T)$, we denote the associated root subgroup by U_α. Letting $\underline{T}$ denote the maximal torus of $\underline{G}$ such that $\eta(T) = \underline{T} \cap \eta(G)$ (given by [Bourgeois 2021, Theorem 2.2]), the root systems $R(G, T)$ and $R(\underline{G}, \underline{T})$ are identified by η, and the Weyl groups of G and $\underline{G}$ coincide. We use $\mathfrak{g}$ and $\underline{\mathfrak{g}}$ for the Lie algebras of G and $\underline{G}$, respectively.

Furthermore, given groups $H_1 \subset H_2$, $h \in H_2$ and a representation γ of H_1, we let ${}^hH_1 := \mathrm{Ad}(h)(H_1) = hH_1h^{-1}$ and ${}^h\gamma := \gamma \circ \mathrm{Ad}(h^{-1})$.

The reader is assumed to be familiar with the structure theory of p-adic groups. Following the notation from [Kaletha 2019], we write $\mathcal{B}(G, F)$ for the reduced building of G over F and $\mathcal{A}(G, T, F)$ for the apartment associated to any maximal torus T of G which is maximally split. For each $x \in \mathcal{B}(G, F)$, we set $G(F)_x$ to be the stabilizer of x in $G(F)$. Furthermore, for $r > 0$, $G(F)_{x,r}$ denotes the Moy–Prasad filtration subgroup of the parahoric subgroup $G(F)_{x,0}$. We will be using Kaletha and Prasad's definitions [2023, Definition 13.2.1], which coincide with the ones of Moy and Prasad given our tameness assumption [2023, p. XXV]. In particular, we have $E_r^\times = 1 + \mathfrak{p}_E^{\lceil er \rceil}$, where e denotes the ramification degree of E/F. We also set $G(F)_{x,r^+} = \bigcup_{t>r} G(F)_{x,t}$. We use colons to abbreviate quotients, that is $G(F)_{x,r:t} = G(F)_{x,r}/G(F)_{x,t}$ for $t > r$. We have analogous filtrations of $\mathcal{O}_F$-submodules at the level of the Lie algebra.

For all $r > 0$, the quotient $G(F)_{x,r:r^+}$ is an abelian group and is isomorphic to its Lie algebra analog $\mathfrak{g}(F)_{x,r:r^+}$ via Adler's mock exponential map [1998]. The quotient $G(F)_{x,0:0^+}$ is also very important, as it results in the $\mathfrak{f}$-points of a reductive group $\mathcal{G}$, which we refer to as the *reductive subquotient of G at x*.

The construction of $\mathcal{G}$ is summarized in [Kaletha and Prasad 2023, Section 8.4.2]. One starts with the relative identity component $\mathcal{G}_x$ of a $\mathcal{O}_F$-group scheme associated to x, whose existence is guaranteed by [Kaletha and Prasad 2023, Proposition 8.3.1 and Section 9.2.5]. One then takes the special fiber $\overline{\mathcal{G}}_x$ of $\mathcal{G}_x$, and defines $\mathcal{G}$ to be the quotient by its unipotent radical, $\mathcal{G} := \overline{\mathcal{G}}_x / R_u(\overline{\mathcal{G}}_x)$. By [Kaletha and Prasad 2023, Theorem 8.3.13], $G(F^{\mathrm{un}})_{x,0} = \mathcal{G}_x(\mathcal{O}_{F^{\mathrm{un}}})$. The projection map

$$G(F^{\mathrm{un}})_{x,0} = \mathcal{G}_x(\mathcal{O}_{F^{\mathrm{un}}}) \to \overline{\mathcal{G}}_x(\bar{\mathfrak{f}}) \tag{3}$$

is surjective, and the preimage of $R_u(\overline{\mathcal{G}}_x)(\bar{\mathfrak{f}})$ under this map is equal to $G(F^{\mathrm{un}})_{x,0^+}$ [Kaletha and Prasad 2023, Corollary 8.4.12], whence $\mathcal{G}(\bar{\mathfrak{f}}) \simeq G(F^{\mathrm{un}})_{x,0:0^+}$.

There is a natural action of $\mathrm{Gal}(F^{\mathrm{un}}/F)$ on $\mathcal{G}_x(\mathcal{O}_{F^{\mathrm{un}}})$ and a natural action of $\mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$ on $\overline{\mathcal{G}}_x(\bar{\mathfrak{f}})$ and the map (3) is Galois-equivariant with respect to these actions. The Galois-equivariance passes to the isomorphism $G(F^{\mathrm{un}})_{x,0:0^+} \simeq \mathcal{G}(\bar{\mathfrak{f}})$.

All of the objects described above have their analogs in G_Z and $\underline{G}$, which we will denote using the subscript Z and the underline, respectively.

2.2. ***Structure theory of G_Z in relation to G.*** In this section, η is the map on G, as stated in the introduction (Theorem 4.1) and $G_Z = \eta(G) \simeq G/Z$. We let (S, θ) and (S_Z, θ_Z) be tame F-nonsingular elliptic pairs of G and G_Z, respectively, which satisfy $\eta(S) = S_Z$ and $\theta = \theta_Z \circ \eta$. The goal of this section is to compare certain subgroups of $G(F)$ to their analogs in $G_Z(F)$. In particular, we focus on Moy–Prasad filtration subgroups and reductive subquotients of G and G_Z, respectively.

These results will be useful when computing a decomposition formula for $\pi_{(S,\theta)} \circ \eta$ in Section 3.

First, we note that $\eta : G \to G_Z$ induces an equivariant isomorphism $\eta_{\mathcal{B}} : \mathcal{B}(G, F) \to \mathcal{B}(G_Z, F)$ by [Kaletha and Prasad 2023, Axiom 4.1.1]. In particular, this map satisfies $\eta_{\mathcal{B}}(gx) = \eta(g) \cdot \eta_{\mathcal{B}}(x)$ for all $g \in G(F)$, $x \in \mathcal{B}(G, F)$, where $\cdot$ refers to the action of $G(F)$ on $\mathcal{B}(G, F)$. For the rest of this section, let us fix $y \in \mathcal{B}(G, F)$ and set $y_Z = \eta_{\mathcal{B}}(y)$. We begin with a result stating a relationship between $G_Z(F)_{y_z}$ and $G(F)_y$.

Lemma 2.1. *We have $\eta(G(F)_y) = G_Z(F)_{y_Z} \cap \eta(G(F))$. Furthermore, $\eta(G(F)_y)$ is normal in $G_Z(F)_{y_Z}$.*

Proof. We start by showing that $\eta(G(F)_y)$ is a normal subgroup of $G_Z(F)_{y_Z}$. Let $g \in G(F)_y$. Then $g \cdot y = y$ by definition. It follows that $\eta_{\mathcal{B}}(g \cdot y) = \eta_{\mathcal{B}}(y)$, or equivalently $\eta(g) \cdot y_Z = y_Z$. Thus $\eta(G(F)_y) \subset G_Z(F)_{y_Z}$. For normality, take $g \in G(F)_y$ and $g_Z \in G_Z(F)_{y_Z}$. Since $\eta(G(F))$ is a normal subgroup of $G_Z(F)$, it follows that $g_Z \eta(g) g_Z^{-1} = \eta(g')$ for some $g' \in G(F)$. By what precedes, we also have $\eta(g') \in G_Z(F)_{y_Z}$, or equivalently $\eta(g') \cdot y_Z = y_Z$. Since the map $\eta_{\mathcal{B}}$ is a bijection, it follows that $g' \cdot y = y$. Thus, $g' \in G(F)_y$ and $g_Z \eta(g) g_Z^{-1} \in \eta(G(F)_y)$. For the intersection, it is clear that $G_Z(F)_{y_Z} \cap \eta(G(F)) \supset \eta(G(F)_y)$. Conversely, take $g_Z \in G_Z(F)_{y_Z} \cap \eta(G(F))$. Then $g_Z = \eta(g')$ for some $g' \in G(F)$ and $\eta(g') \cdot y_Z = y_Z$. Using the bijectivity of $\eta_{\mathcal{B}}$, it follows that $g' \in G(F)_y$, and thus $g_Z \in \eta(G(F)_y)$. □

Next, we describe the relationship between the Moy–Prasad filtration subgroups.

Lemma 2.2. *For all $r > 0$ we have $\eta(G(F)_{y,r}) = G_Z(F)_{y_Z,r}$.*

Proof. Let $r > 0$. Following the proof of [Kaletha 2019, Lemma 3.3.2], use [Bruhat and Tits 1972, Lemma 6.4.48] to write $G(F)_{y,r}$ as the direct product of (topological spaces) $T(F)_r$ and the appropriate affine root subgroups. Here T is a maximally unramified maximally split maximal torus, whose existence is guaranteed by [Bruhat and Tits 1984, Corollary 5.1.12]. Since η induces an isomorphism on the affine root subgroups, it suffices to show that $\eta(T(F)_r) = T_Z(F)_r$, where $T_Z = \eta(T) \simeq T/Z$. To do so, let Z° denote the identity component of Z. The map η factors as follows:

$$\begin{array}{ccc} T & \xrightarrow{\quad\quad\eta\quad\quad} & T_Z \simeq (T/Z^\circ)/(Z/Z^\circ) \\ & {\scriptstyle\eta^\circ}\searrow \qquad \nearrow{\scriptstyle\bar{\eta}} & \\ & T/Z^\circ & \end{array}$$

We have that Z° is a torus by [Humphreys 1975, Theorem 16.2] as it is a closed and connected subgroup of the torus $Z(G)^\circ$. By [Kaletha 2019, Lemma 3.1.3], we have an exact sequence

$$1 \to Z^\circ(F)_r \to T(F)_r \to (T/Z^\circ)(F)_r \to 1,$$

implying that $(T/Z^\circ)(F)_r \simeq T(F)_r/Z^\circ(F)_r \simeq \eta^\circ(T(F)_r)$. Furthermore, since $\bar{\eta}: T/Z^\circ \to T_Z$ is an isogeny, [Kaletha 2019, Lemma 3.1.3] tells us that

$$\bar{\eta}((T/Z^\circ)(F)_r) = T_Z(F)_r.$$

Combining these two equations allows us to conclude that $\eta(T(F)_r) = T_Z(F)_r$. □

Remark 2.3. For all $r > 0$, the map η induces a surjection $G(F)_{y,r:r^+} \to G_Z(F)_{y_Z,r:r^+}$, $0 \le i \le d$. At the depth-zero level, we can only guarantee an inclusion. In other words, $\eta(G(F))_{y,0} \subset G_Z(F)_{y_Z,0}$. This induces a homomorphism $G(F)_{y,0:0^+} \to G_Z(F)_{y_Z,0:0^+}$.

Now, let $\mathcal{G}$ and $\mathcal{G}_Z$ denote the reductive subquotients of G and G_Z at y and y_Z, respectively (see discussion surrounding (3)). In light of Remark 2.3, the map η induces a map between $\mathcal{G}$ and $\mathcal{G}_Z$, which we describe as follows.

Since η induces a homomorphism $G(F^{\text{un}})_{y,0} \to G_Z(F^{\text{un}})_{y_Z,0}$, we can compose with the quotient map to obtain a homomorphism

$$\begin{aligned} G(F^{\text{un}})_{y,0} &\to G_Z(F^{\text{un}})_{y_Z,0:0^+}, \\ g &\mapsto \eta(g)G_Z(F^{\text{un}})_{y,0^+}. \end{aligned}$$

The kernel of this homomorphism is $(Z \cap G(F^{\text{un}})_{y,0})G(F^{\text{un}})_{y,0^+}$, resulting in an embedding

$$G(F^{\text{un}})_{y,0}/(Z \cap G(F^{\text{un}})_{y,0})G(F^{\text{un}})_{y,0^+} \hookrightarrow G_Z(F^{\text{un}})_{y_Z,0:0^+}.$$

By the third isomorphism theorem, the domain of the previous embedding is isomorphic to

$$G(F^{\text{un}})_{y,0:0^+}/\big((Z \cap G(F^{\text{un}})_{y,0})G(F^{\text{un}})_{y,0^+}/G(F^{\text{un}})_{y,0^+}\big).$$

Given that $Z \subset Z(G)$, it follows that $(Z \cap G(F^{\text{un}})_{y,0})G(F^{\text{un}})_{y,0^+}/G(F^{\text{un}})_{y,0^+}$ is a closed central subgroup of $G(F^{\text{un}})_{y,0:0^+}$. As such, it corresponds to a closed central subgroup of $\mathcal{G}(\bar{\mathfrak{f}})$, which we denote by $\mathcal{Z}(\bar{\mathfrak{f}})$. Given that we identify reductive groups with their $\bar{\mathfrak{f}}$-points, what we have just described is an embedding

$$\bar{\eta}: \mathcal{G}/\mathcal{Z} \hookrightarrow \mathcal{G}_Z.$$

We expect this embedding to be surjective when the groups are split. Even though we do not have a counterexample, it is not clear to us whether it is surjective in general. Note that the groups of the embedding are defined over $\mathfrak{f}$. The embedding is $\text{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$-equivariant, as η is $\text{Gal}(F^{\text{un}}/F)$-equivariant. Furthermore, $\mathcal{Z}$ is defined over $\mathfrak{f}$ by [Springer 2009, Corollary 12.1.3]. Thus, the map between $\mathcal{G}$ and $\mathcal{G}_Z$

(which is defined over $\mathfrak{f}$) is given by

$$
\begin{array}{ccccc}
\mathcal{G} & \xrightarrow{\quad q \quad} & \mathcal{G}/\mathcal{Z} & \xhookrightarrow{\quad \bar{\eta} \quad} & \mathcal{G}_Z \\
\updownarrow \simeq & & \updownarrow \simeq & & \updownarrow \simeq \\
G(F^{\mathrm{un}})_{y,0:0^+} & \longrightarrow & G(F^{\mathrm{un}})_{y,0}/(Z \cap G(F^{\mathrm{un}})_{y,0})G(F^{\mathrm{un}})_{y,0^+} & \hookrightarrow & G_Z(F^{\mathrm{un}})_{y_Z,0:0^+} \\
gG(F^{\mathrm{un}})_{y,0^+} & & \longmapsto & & \eta(g)G_Z(F^{\mathrm{un}})_{y_Z,0^+}
\end{array} \tag{4}
$$

where q is the obvious quotient map. To alleviate notation, we will keep the isomorphisms implicit and say that an element of $\mathcal{G}(\bar{\mathfrak{f}})$ (or $\mathcal{G}(\mathfrak{f})$) is of the form $gG(F^{\mathrm{un}})_{y,0^+}$ for some $g \in G(F^{\mathrm{un}})_{y,0}$ (or $gG(F)_{y,0^+}$ for some $g \in G(F)_{y,0}$).

Remark 2.4. The map $\bar{\eta} \circ q$ as defined above corresponds to the restriction of the map

$$
\begin{gathered}
G(F^{\mathrm{un}})_y/G(F^{\mathrm{un}})_{y,0^+} \to G_Z(F^{\mathrm{un}})_{y_Z}/G_Z(F^{\mathrm{un}})_{y_Z,0^+}, \\
gG(F^{\mathrm{un}})_{y,0^+} \mapsto \eta(g)G_Z(F^{\mathrm{un}})_{y_Z,0^+}.
\end{gathered}
$$

We will abuse notation and also denote this map by $\bar{\eta} \circ q$ when called upon.

We now prove two elementary results involving the maps $\bar{\eta}$ and q.

Lemma 2.5. *Let $\bar{\eta}$ and $\mathcal{Z}$ be as above. Then $\bar{\eta}(\mathcal{G}/\mathcal{Z}) \supseteq [\mathcal{G}_Z, \mathcal{G}_Z]$.*

Proof. We identify the reductive groups with their $\bar{\mathfrak{f}}$-points. Based on the definitions above, we have

$$\bar{\eta}(\mathcal{G}(\bar{\mathfrak{f}})/\mathcal{Z}(\bar{\mathfrak{f}})) = \eta(G(F^{\mathrm{un}})_{y,0})G_Z(F^{\mathrm{un}})_{y_Z,0^+}/G_Z(F^{\mathrm{un}})_{y_Z,0^+}.$$

Let S'_Z be a maximal F^{un}-split torus of G_Z and let T_Z be its centralizer in G_Z. By definition (see, for example, [Fintzen 2021, Section 2.4] or [Kaletha and Prasad 2023, Definition 13.2.1]),

$$G_Z(F^{\mathrm{un}})_{y,0} = \langle T_Z(F^{\mathrm{un}})_0, U_\alpha(F^{\mathrm{un}}) : \alpha \in R^{\mathrm{aff}}_{F^{\mathrm{un}}}, \langle \alpha,\ y_Z \rangle \geq 0 \rangle,$$

where $R^{\mathrm{aff}}_{F^{\mathrm{un}}} = \{\lambda + k : \lambda \in R(G_Z, T_Z) \text{ such that } \lambda|_{S'_Z} \neq 1,\ k \in \mathbb{R}\}$, and $U_\alpha(F^{\mathrm{un}})$ is the affine root subgroup associated to the affine root α. Given that root subgroups are normalized by toral elements, it follows that $[G_Z(F^{\mathrm{un}})_{y_Z,0}, G_Z(F^{\mathrm{un}})_{y_Z,0}]$ consists only of products of root subgroup elements. Since η induces an isomorphism on the affine root subgroups of G and G_Z, we conclude that $[G_Z(F^{\mathrm{un}})_{y_Z,0}, G_Z(F^{\mathrm{un}})_{y_Z,0}] \subseteq \eta(G(F^{\mathrm{un}})_{y,0})$. It follows that

$$
\begin{aligned}
[\mathcal{G}_Z(\bar{\mathfrak{f}}), \mathcal{G}_Z(\bar{\mathfrak{f}})] &= [G_Z(F^{\mathrm{un}})_{y_Z,0}, G_Z(F^{\mathrm{un}})_{y_Z,0}]G_Z(F^{\mathrm{un}})_{y_Z,0^+}/G_Z(F^{\mathrm{un}})_{y_Z,0^+} \\
&\subseteq \bar{\eta}(\mathcal{G}_Z(\bar{\mathfrak{f}})/\mathcal{Z}(\bar{\mathfrak{f}})).
\end{aligned}
$$

□

2.3. ***The action of $\underline{G}(F)$ on $G(F)$.*** The decomposition formula for $\pi_{(\underline{S},\underline{\theta})} \circ \eta$ involves an action of $\underline{G}(F)$ on representations of $G(F)$. The purpose of this section is to describe this action.

Let $\underline{g} \in \underline{G}(F)$. Using $\underline{G} = G_Z Z(\underline{G})$, write $\underline{g} = g_Z z$ for some $g_Z \in G_Z$, $z \in Z(\underline{G})$. Since G_Z is the image of η, there exists $g \in G$ such that $g_Z = \eta(g)$. It follows that $\mathrm{Ad}(\underline{g}) = \mathrm{Ad}(\eta(g))$. We also set $\underline{\mathrm{Ad}}(\underline{g}) := \mathrm{Ad}(g)$, an automorphism of G.

Lemma 2.6. *For all $\underline{g} \in \underline{G}(F)$, $\underline{\mathrm{Ad}}(\underline{g}) \in \mathrm{Aut}(G(F))$ is defined over F. Furthermore,*

$$\underline{\mathrm{Ad}} : \underline{G}(F) \to \mathrm{Aut}(G(F)),$$
$$\underline{g} \mapsto \underline{\mathrm{Ad}}(\underline{g}),$$

is a well-defined homomorphism.

Proof. It is clear that $\underline{\mathrm{Ad}}(\underline{g})$ maps G to G. We first prove that $\underline{\mathrm{Ad}}(\underline{g})$ is defined over F^{sep}. According to [Springer 2009, 12.3.3], a quotient map carries the F^{sep}-points of its domain surjectively onto the F^{sep}-points of its image. Now, the group $\underline{G}$ is the quotient of $G_Z \times Z(\underline{G})$ by $G_Z \cap Z(\underline{G})$. Therefore $\underline{g} = g_Z z$ where $g_Z \in G_Z(F^{\mathrm{sep}})$ and $z \in Z(\underline{G})(F^{\mathrm{sep}})$. Our map η is also a quotient map so g_Z can be written as $\eta(g)$ where $g \in G(F^{\mathrm{sep}})$. Consequently $\mathrm{Ad}(g) = \underline{\mathrm{Ad}}(\underline{g})$ is defined over F^{sep}.

To conclude that $\underline{\mathrm{Ad}}(\underline{g})$ is defined over F (and therefore maps $G(F)$ to $G(F)$), we show that $\underline{\mathrm{Ad}}(\underline{g}) \circ \sigma = \sigma \circ \underline{\mathrm{Ad}}(\underline{g})$ for all $\sigma \in \Gamma$. Recall that $\underline{\mathrm{Ad}}(\underline{g}) = \mathrm{Ad}(g)$, where $g \in G$ is such that $\underline{g} = \eta(g)z$ for some $z \in Z(\underline{G})$. Since η is defined over F and $\underline{g} \in \underline{G}(F)$, we have

$$\begin{aligned} \eta \circ \sigma \circ \underline{\mathrm{Ad}}(\underline{g}) &= \sigma \circ \mathrm{Ad}(\eta(g)) \circ \eta \\ &= \sigma \circ \mathrm{Ad}(\underline{g}) \circ \eta \\ &= \mathrm{Ad}(\underline{g}) \circ \eta \circ \sigma \\ &= \mathrm{Ad}(\eta(g)) \circ \eta \circ \sigma \\ &= \eta \circ \underline{\mathrm{Ad}}(\underline{g}) \circ \sigma. \end{aligned}$$

Given $x \in G$, the previous equality implies $(\sigma \circ \underline{\mathrm{Ad}}(\underline{g}))(x) = (\underline{\mathrm{Ad}}(\underline{g}) \circ \sigma)(x) z_x$ for some $z_x \in Z$. Define the map

$$f : G \to Z,$$
$$x \mapsto z_x.$$

This is a homomorphism, and is trivial on $Z(G)$. Furthermore, because Z is abelian, f is also trivial on $[G, G]$. Thus, f is trivial on $G = [G, G]Z(G)$, and $z_x = 1$ for all $x \in G$. We conclude that $\underline{\mathrm{Ad}}(\underline{g}) \circ \sigma = \sigma \circ \underline{\mathrm{Ad}}(\underline{g})$, as desired. To show that the map $\underline{\mathrm{Ad}}$ is well defined, assume $\underline{g} = \eta(g_1)z_1 = \eta(g_2)z_2$, where $g_1, g_2 \in G$, $z_1, z_2 \in Z(\underline{G})$. It follows that $\eta(g_1 g_2^{-1}) = z_1^{-1} z_2 \in Z(\underline{G}) \cap G_Z \subset Z(G_Z)$, and therefore $g_1 g_2^{-1} \in Z(G)$. We conclude that $\mathrm{Ad}(g_1) = \mathrm{Ad}(g_2)$, and thus $\underline{\mathrm{Ad}}(\underline{g})$ is well defined. It is straightforward to show that $\underline{\mathrm{Ad}}$ is a homomorphism. □

Corollary 2.7. *The group $\eta(G(F))$ is normal in $\underline{G}(F)$.*

Proof. Let $h \in G(F)$ and $\underline{g} \in \underline{G}(F)$. We show that $\mathrm{Ad}(\underline{g})(\eta(h)) \in \eta(G(F))$. Following the notation above, we have that $\mathrm{Ad}(\underline{g}) = \mathrm{Ad}(\eta(g))$ for some $g \in G$. It follows that

$$\mathrm{Ad}(\underline{g})(\eta(h)) = (\eta \circ \mathrm{Ad}(g))(h) = (\eta \circ \underline{\mathrm{Ad}}(\underline{g}))(h).$$

By the previous lemma, $\underline{\mathrm{Ad}}(\underline{g})$ is defined over F, which implies $\underline{\mathrm{Ad}}(\underline{g})(h) \in G(F)$. Thus, we conclude that

$$\mathrm{Ad}(\underline{g})(\eta(h)) \in \eta(G(F)). \qquad \square$$

The following lemma will also be useful in proving the main statements of this section.

Lemma 2.8. *Let π_Z be a representation of $G_Z(F)$ and $\underline{g} \in \underline{G}(F)$. Then*

$$^{\underline{g}}\pi_Z \circ \eta = \pi_Z \circ \eta \circ \underline{\mathrm{Ad}}(\underline{g}^{-1}).$$

Proof. We have $\underline{\mathrm{Ad}}(\underline{g}) = \mathrm{Ad}(g)$, where $g \in G$ satisfies $\underline{g} = \eta(g)z$ for some $z \in Z(\underline{G})$. For all $h \in G(F)$,

$$\begin{aligned}(\pi_Z \circ \eta \circ \underline{\mathrm{Ad}}(\underline{g}^{-1}))(h) &= (\pi_Z \circ \eta)(g^{-1}hg) \\ &= \pi_Z(\eta(g)^{-1}\eta(h)\eta(g)) \\ &= ({}^{\eta(g)}\pi_Z \circ \eta)(h) \\ &= ({}^{\underline{g}}\pi_Z \circ \eta)(h). \end{aligned} \qquad \square$$

2.4. ***Summary of the Kaletha–Yu construction.*** Let us recall the construction of supercuspidal representations from tame F-nonsingular elliptic pairs as per [Kaletha 2019; 2021], which we refer to as the Kaletha–Yu construction. For simplicity of notation, we will describe the construction over G, though it is also applied to $G_Z \simeq G/Z$ and $\underline{G}$.

The construction of the supercuspidal representation $\pi_{(S,\theta)}$ of G starts from a tame F-nonsingular elliptic pair (S, θ) in the sense of [Kaletha 2021, Definition 3.4.1]. The representation $\pi_{(S,\theta)}$ is obtained in two steps. One starts by unfolding the pair (S, θ) into a G-datum $\Psi_{(S,\theta)} = (\vec{G}, y, \vec{r}, \rho, \vec{\phi})$ in the sense of [Yu 2001, Section 3]. We will refer to $\Psi_{(S,\theta)}$ as the *corresponding G-datum* of (S, θ). The properties of S and θ provided by [Kaletha 2021, Definition 3.4.1] allow us to go to the reductive subquotient and use the theory of Deligne–Lusztig cuspidal representations in order to construct ρ, the so-called *depth-zero* part of the datum $\Psi_{(S,\theta)}$. The second step consists of applying the J.-K. Yu construction [2001] on the obtained G-datum. The unfolding of the tame F-nonsingular elliptic pair into a G-datum is given as follows.

2.4.1. *The twisted Levi sequence $\vec{G}$ and the sequence $\vec{r}$.* We recall how to construct a Levi sequence from S as per [Kaletha 2019, Section 3.6]. We consider the set $R_r := \{\alpha \in R(G, S) : \theta(N_{E/F}(\check{\alpha}(E_r^\times))) = 1\}$,where $\check{\alpha}$ is the coroot associated to α and $N_{E/F}$ is the norm of E/F, and set $R_{r^+} = \bigcap_{s>r} R_s$. There will be breaks in this filtration, $r_{d-1} > r_{d-2} > \cdots > r_0 > 0$. We set $r_{-1} = 0$ and $r_d = \operatorname{depth}(\theta)$. For all $0 \le i \le d$, $G^i := \langle S, U_\alpha : \alpha \in R_{r_{i-1}^+} \rangle$ is a tamely ramified twisted Levi subgroup of G [Kaletha 2019, Lemma 3.6.1]. These twisted Levi subgroups are what we use to form the twisted Levi sequence $\vec{G} = (G^0, \dots, G^d)$. We also set $G^{-1} = S$ and $\vec{r} = (r_0, \dots, r_d)$.

2.4.2. *The character sequence $\vec{\phi}$.* By [Kaletha 2019, Proposition 3.6.7], given the character θ of $S(F)$, there exists a Howe factorization, that is, a sequence of characters $\phi^i : G^i(F) \to \mathbb{C}^\times$ for $i = -1, \dots, d$ such that:

(1) $\theta = \prod_{i=-1}^{d} \phi^i|_{S(F)}$.

(2) For all $0 \le i \le d$, ϕ^i is trivial on the simply connected cover of G^i.

(3) For all $0 \le i < d$, ϕ^i is a $G^{i+1}(F)$-generic character of depth r_i in the sense of [Hakim and Murnaghan 2008, Definition 3.9]. For $i = d$, ϕ^d is trivial if $r_d = r_{d-1}$ and has depth r_d otherwise. For $i = -1$, ϕ^{-1} is trivial if $G^0 = S$ and otherwise satisfies $\phi^{-1}|_{S(F)_{0^+}} = 1$.

Given such a factorization, we set $\vec{\phi} = (\phi^0, \dots, \phi^d)$.

2.4.3. *The point y.* Since (S, θ) is a tame F-nonsingular elliptic pair, the torus S is a maximally unramified elliptic maximal F-torus of G^0 in the sense of [Kaletha 2019, Definition 3.4.2]. As such, we can associate to it a vertex y of $\mathcal{B}(G^0, F) \subset \mathcal{B}(G, F)$ [Kaletha 2019, Lemma 3.4.3], which is the unique $\operatorname{Gal}(F^{\mathrm{un}}/F)$-fixed point of $\mathcal{A}(G^0, S, F^{\mathrm{un}})$ [Kaletha and Prasad 2023, Section 17.8].

2.4.4. *The representation ρ.* Let $\mathcal{G}^0$ denote the reductive subquotient of G^0 at y, that is, the connected reductive $\mathfrak{f}$-group such that $\mathcal{G}^0(\bar{\mathfrak{f}}) \simeq G^0(F^{\mathrm{un}})_{y,0:0^+}$ and $\mathcal{G}^0(\mathfrak{f}) \simeq G^0(F)_{y,0:0^+}$, as recalled at the end of Section 2.1.

By [Kaletha 2019, Lemma 3.4.4], there exists an elliptic maximal $\mathfrak{f}$-torus $\mathcal{S}$ of $\mathcal{G}^0$ such that for every unramified extension F' of F, the image of $S(F')_0$ in $G(F')_{y,0:0^+}$ is isomorphic to $\mathcal{S}(\mathfrak{f}')$. For every character $\bar{\chi}$ of $\mathcal{S}(\mathfrak{f})$, one can construct a virtual character of $\mathcal{G}(\mathfrak{f})$ as per [Deligne and Lusztig 1976], which we denote by $R_{\mathcal{S},\bar{\chi}}$. When $\bar{\chi}$ is nonsingular in the sense of [Deligne and Lusztig 1976, Definition 5.15], $\pm R_{\mathcal{S},\bar{\chi}}$ is a Deligne–Lusztig cuspidal representation of $\mathcal{G}(\mathfrak{f})$ [1976, Proposition 7.4, Theorem 8.3]. The sign $\pm$ refers to $(-1)^{r_{\mathfrak{f}}(\mathcal{G}^0) - r_{\mathfrak{f}}(\mathcal{S})}$, where $r_{\mathfrak{f}}(\mathcal{G}^0)$ and $r_{\mathfrak{f}}(\mathcal{S})$ denote the $\mathfrak{f}$-split ranks of $\mathcal{G}^0$ and $\mathcal{S}$, respectively. The character ϕ^{-1} factors through to a character $\bar{\phi}^{-1}$ of $S(F)/S(F)_{0^+}$, which restricts to a character of $\mathcal{S}(\mathfrak{f})$. By [Kaletha 2019, Lemma 3.4.14], this character of $\mathcal{S}(\mathfrak{f})$ is nonsingular, meaning that the virtual character $\pm R_{\mathcal{S},\bar{\phi}^{-1}}$ is a (possibly reducible) Deligne–Lusztig cuspidal

$$(\pm R_{\mathcal{S},\bar{\phi}^{-1}}, \mathcal{G}^0(\mathfrak{f})) \xmapsto{\text{pullback, extend}} (\kappa_{(S,\phi^{-1})}, S(F)G^0(F)_{y,0}) \xmapsto{\text{induce}} (\rho, G^0(F)_y)$$

Figure 2. Summary of the construction of ρ.

representation of $\mathcal{G}^0(\mathfrak{f})$. The pullback of $\pm R_{\mathcal{S},\bar{\phi}^{-1}}$ to $G^0(F)_{y,0}$ then gets extended to a representation $\kappa_{(S,\phi^{-1})}$ of $S(F)G^0(F)_{y,0}$. This extension process is explained in [Kaletha 2019, Section 3.4.4] as well as [Kaletha 2021, Remark 2.6.5, p. 35], and will be recalled when needed in Corollary 3.12. We then define

$$\rho := \operatorname{Ind}_{S(F)G^0(F)_{y,0}}^{G^0(F)_y} \kappa_{(S,\phi^{-1})}.$$

The construction of ρ is summarized in Figure 2. Note that we are following the notation from [Kaletha 2019] in the paragraph above. What we have denoted by ρ is denoted by $\kappa_{(S,\phi^{-1})}$ in [Kaletha 2021, Section 3.3].

Once we have the G-datum $\Psi_{(S,\theta)} = (\vec{G}, y, \vec{r}, \rho, \vec{\phi})$, we apply the J.-K. Yu construction to obtain the supercuspidal representation $\pi_{(S,\theta)}$. We do not recall all the details of this construction, but provide a summary in the form of a diagram (Figure 3). We invite the reader to consult [Bourgeois 2021, Section 3] for a brief description of the steps involved. We point out that it is sometimes convenient to write $\kappa_G(\Psi_{(S,\theta)})$ for $\kappa_{(S,\theta)}$ and $\pi_G(\Psi_{(S,\theta)})$ for $\pi_{(S,\theta)}$ to indicate that we are applying the J.-K. Yu construction on the G-datum $\Psi_{(S,\theta)}$.

The representation ρ above may be reducible, and its irreducible components are given by [Kaletha 2021, Theorem 2.7.7]. While the definition of a G-datum in [Yu 2001] requires ρ to be irreducible, we may still apply the steps of the J.-K. Yu construction on $(\vec{G}, y, \vec{r}, \rho, \vec{\phi})$ to obtain $\pi_{(S,\theta)}$, which is a completely reducible supercuspidal representation independent of the chosen Howe factorization [Kaletha 2021, Corollary 3.4.7]. We write $[\pi_{(S,\theta)}]$ for the set of irreducible components of $\pi_{(S,\theta)}$.

$$\begin{array}{cccccc}
 & (\phi^0, K^0) & \cdots & (\phi^{d-2}, K^{d-2}) & (\phi^{d-1}, K^{d-1}) & (\phi^d, K^d) \\
 & \downarrow \text{enlarge} & & \downarrow \text{enlarge} & \downarrow \text{enlarge} & \downarrow = \\
(\rho, K^0) & (\phi^{0\prime}, K^1) & \cdots & (\phi^{d-2\prime}, K^{d-1}) & (\phi^{d-1\prime}, K^d) & (\phi^{d\prime}, K^d) \\
\downarrow \text{inflate} & \downarrow \text{inflate} & & \downarrow \text{inflate} & \downarrow = & \downarrow = \\
(\kappa^{-1}, K^d) & (\kappa^0, K^d) & \cdots & (\kappa^{d-2}, K^d) & (\kappa^{d-1}, K^d) & (\kappa^d, K^d)
\end{array}$$

$$\pi_{(S,\theta)} := \operatorname{Ind}_{K^d}^G \kappa_{(S,\theta)}, \text{ where}$$

$$\kappa_{(S,\theta)} = \kappa^{-1} \otimes \kappa^0 \otimes \kappa^1 \otimes \kappa^2 \cdots \otimes \kappa^{d-2} \otimes \kappa^{d-1} \otimes \kappa^d$$

Figure 3. Summary of the J.-K. Yu construction for $\pi_{(S,\theta)}$, where $K^0 = G^0(F)_y$ and $K^{i+1} = K^0 G^1(F)_{y,r_0/2} \cdots G^{i+1}(F)_{y,r_i/2}$, $0 \le i \le d-1$.

From the pair (S, θ), one may also perform what we call a *twisted* J.-K. Yu construction. Indeed, following [Fintzen et al. 2023, Section 4.1 page 2259], let $\epsilon = \prod_{i=1}^{d} \epsilon^{G^i/G^{i-1}}$, where $\epsilon^{G^i/G^{i-1}}$ is the quadratic character of K^d that is trivial on $G^1(F)_{y,r_0/2} \cdots G^d(F)_{y,r_{d-1}/2}$ and whose restriction to K^0 is given by $\epsilon_y^{G^i/G^{i-1}}$ defined in [Fintzen et al. 2023, Definition 4.1.10]. The so-called twisted representation then refers to $\mathrm{Ind}_{K^d}^{G}(\kappa_{(S,\theta)} \cdot \epsilon)$, which is equivalent to constructing $\pi_{(S,\theta\cdot\epsilon)}$ via the above steps. Since one obtains the Levi sequence $\vec{G}$ from S, we simply say that ϵ is constructed from S.

2.5. ***The construction of supercuspidal L-packets.*** Recall that $\Phi_{\mathrm{sc}}(G)$ denotes the set (of conjugacy classes) of supercuspidal L-parameters of G. Given our hypothesis on p, every $\varphi \in \Phi_{\mathrm{sc}}(G)$ has the property that $\varphi(P_F)$ is contained in a maximal torus of $\widehat{G}$ [Kaletha 2021, Lemma 4.1.3]. Such parameters are called *torally wild* in [Kaletha 2021]. Since all supercuspidal parameters we consider in this paper are torally wild, we will omit these adjectives.

Given $\varphi \in \Phi_{\mathrm{sc}}(G)$, we let Π_φ denote the associated L-packet of [Kaletha 2021]. Kaletha provides an explicit parameterization for Π_φ, and elements therein consist entirely of supercuspidal representations obtained from the construction outlined in Section 2.4. Thus, when $\varphi \in \Phi_{\mathrm{sc}}(G)$, we shall refer to Π_φ as a *supercuspidal* L-packet.

In order to describe Kaletha's construction of supercuspidal L-packets, we must first familiarize ourselves with his notion of a supercuspidal L-packet datum. We start this section by recalling the definition below.

Definition 2.9 [Kaletha 2021, Definition 4.1.4]. A supercuspidal L-packet datum of G is a tuple $(S, \hat{j}, \chi_0, \theta)$, where

(1) S is a torus of dimension equal to the absolute rank of G, defined over F and split over a tame extension of F;

(2) $\hat{j} : \widehat{S} \to \widehat{G}$ is an embedding of complex reductive groups whose $\widehat{G}$-conjugacy class is Γ-stable;

(3) $\chi_0 = (\chi_{\alpha_0})_{\alpha_0}$ is tamely ramified χ-data for $R(G, S^0)$, where S^0 is a particular subtorus of S defined from R_{0+} as explained in [Kaletha 2021, p. 41];

(4) and $\theta : S(F) \to \mathbb{C}^\times$ is a character;

subject to the condition that (S, θ) is a tame F-nonsingular elliptic pair in the sense of [Kaletha 2021, Definition 3.4.1].

Despite appearances, the torus S does not actually live inside G. It is an abstract torus that will be embedded into G below.

The notion of χ-data was introduced in [Langlands and Shelstad 1987] and is recalled in [Kaletha 2019, Section 4.6]. It is not necessary for the reader to be familiar with χ-data in what follows. For our purposes, one can think of χ-data for $R(G, S^0)$ simply as a set of characters of unit groups of finite extensions of F which are indexed by roots.

By [Kaletha 2021, Proposition 4.1.8], there is a one-to-one correspondence between the $\widehat{G}$-conjugacy classes of supercuspidal L-parameters for G and isomorphism classes of supercuspidal L-packet data. Following the proof of [Kaletha 2021, Proposition 4.1.8], given $\varphi \in \Phi_{\mathrm{sc}}(G)$, one constructs a representative $(S, \hat{j}, \chi_0, \theta)$ of the corresponding isomorphism class of supercuspidal L-packet data as follows:

- S: Let $\widehat{M} = \mathrm{Cent}(\varphi(P_F), \widehat{G})^\circ$, $\widehat{C} = \mathrm{Cent}(\varphi(I_F), \widehat{G})^\circ$ and $\widehat{S} = \mathrm{Cent}(\widehat{C}, \widehat{M})$. By [Kaletha 2019, Lemma 5.2.2; 2021, Lemma 4.1.3], $\widehat{M}$ is Levi subgroup of $\widehat{G}$, $\widehat{C}$ is a torus of $\widehat{G}$ and $\widehat{S}$ is a maximal torus of $\widehat{G}$. The action of W_F (which extends to Γ) on $\widehat{S}$ is defined as $\mathrm{Ad}(\varphi(-))$. The torus S is then the torus dual to $\widehat{S}$.
- $\hat{j}$: One simply takes $\hat{j}$ to be the set inclusion $\widehat{S} \hookrightarrow \widehat{G}$.
- χ_0: One chooses tame χ-data χ_0 for S^0, which extends to χ-data for S by [Kaletha 2021, Remark 4.1.5].
- θ: Following [Langlands and Shelstad 1987, Section 2.6], the χ-data allow one to extend $\hat{j}$ to an embedding ${}^L j : {}^L S \to {}^L G$. The image of ${}^L j$ contains the image of φ so that we may write $\varphi = {}^L j \circ \varphi_S$ for some L-parameter φ_S of S. We let θ be the corresponding character of $S(F)$ via the LLC for tori.

We will say that $(S, \hat{j}, \chi_0, \theta)$ is the *supercuspidal L-packet datum associated to $\varphi \in \Phi_{\mathrm{sc}}(G)$*. The embedding $\hat{j}$ belongs to a Γ-stable $\widehat{G}$-conjugacy class $\widehat{\mathcal{J}}$ of embeddings $\widehat{S} \to \widehat{G}$, and from $\widehat{\mathcal{J}}$ we obtain a Γ-stable $G(F^{\mathrm{sep}})$-conjugacy class $\mathcal{J}$ of embeddings $S \to G$ (called *admissible embeddings*) as per [Kaletha 2019, Section 5.1; Dillery 2023, Sections 6.1 and 7.1]. We denote by $\mathcal{J}_F$ the set of $G(F)$-conjugacy classes of elements of $\mathcal{J}$ which are defined over F. For each $j \in \mathcal{J}_F$, we consider the torus $jS = j(S)$ and let $j\theta = \theta \circ j^{-1} \cdot \epsilon_j$, where ϵ_j is the specific character from [Fintzen et al. 2023, Section 4.1] constructed from jS, described at the end of Section 2.4. Each pair $(jS, j\theta)$ is a tame F-nonsingular elliptic pair from which we can construct a supercuspidal representation $\pi_{(jS, j\theta)}$ as described in Section 2.4. The supercuspidal L-packet Π_φ is then defined as

$$\Pi_\varphi := \{[\pi_{(jS, j\theta)}] : j \in \mathcal{J}_F\},$$

where j is identified with its $G(F)$-conjugacy class and $\pi_{(jS, j\theta)}$ is identified with its equivalence class. Similarly, given $\underline{\varphi} \in \Phi_{\mathrm{sc}}(\underline{G})$, we denote the associated supercuspidal L-packet datum by $(\underline{S}, \underline{\hat{j}}, \underline{\chi}_0, \underline{\theta})$, and let $\underline{\mathcal{J}}_F$ be the set of $\underline{G}(F)$-conjugacy classes of admissible embeddings obtained from the Γ-stable $\widehat{\underline{G}}$-conjugacy class of $\underline{\hat{j}}$ which are defined over F, so that

$$\Pi_{\underline{\varphi}} = \{[\pi_{(\underline{j}\underline{S}, \underline{j}\underline{\theta})}] : \underline{j} \in \underline{\mathcal{J}}_F\}.$$

What we have denoted by Π_φ is what Kaletha [2021] denotes as $\Pi_\varphi(G)$. Kaletha assigns the notation Π_φ to his "compound L-packet" which encompasses rigid inner forms of G.

3. The decomposition of $\pi_{(\underline{S},\underline{\theta})} \circ \eta$

Kaletha [2019; 2021] described a way to construct a supercuspidal representation $\pi_{(\underline{S},\underline{\theta})}$ of $\underline{G}$ from a tame F-nonsingular elliptic pair $(\underline{S},\underline{\theta})$ [2021, Definition 3.4.1] (recalled above in Section 2.4). Here $\underline{S}$ is a maximally unramified elliptic maximal torus and $\underline{\theta}$ is a character of $\underline{S}(F)$ satisfying a certain *nonsingularity* condition. As seen in Section 2.5, the irreducible components of these representations are what make up the supercuspidal L-packets. As such, finding a decomposition formula for $\pi_{(\underline{S},\underline{\theta})} \circ \eta$ is crucial in proving Theorem 4.1.

The main result of this section is precisely this decomposition formula, and is given by the following theorem.

Theorem 3.1. *Let $(\underline{S},\underline{\theta})$ and (S,θ) be tame F-nonsingular elliptic pairs for $\underline{G}$ and G, respectively. Assume that $\eta(S) \subset \underline{S}$ and $\theta = \underline{\theta} \circ \eta$. Then*

$$\pi_{(\underline{S},\underline{\theta})} \circ \eta \simeq \bigoplus_{\underline{c} \in \underline{C}} \pi_{(S,\theta)} \circ \underline{\mathrm{Ad}}(\underline{c}^{-1}),$$

where $\underline{C}$ is a set of coset representatives of $\eta(G(F))\backslash\underline{G}(F)/\underline{S}(F)$ and $\underline{\mathrm{Ad}}$ is the $\underline{G}(F)$-action on $G(F)$ described in Section 2.3.

The proof of Theorem 3.1 is done in two steps (as illustrated in Figure 1). Indeed, by noting that

$$\pi_{(\underline{S},\underline{\theta})} \circ \eta = \big(\mathrm{Res}^{\underline{G}(F)}_{G_Z(F)} \pi_{(\underline{S},\underline{\theta})}\big) \circ \eta,$$

we first seek a decomposition formula for $\mathrm{Res}^{\underline{G}(F)}_{G_Z(F)} \pi_{(\underline{S},\underline{\theta})}$. The results from [Bourgeois 2021] grant us such a formula, as G_Z is a normal subgroup of $\underline{G}$ that contains $[\underline{G},\underline{G}]$. This is stated in Theorem 3.19. The second step is describing the composition of a supercuspidal representation of $G_Z(F)$ with η. More specifically, we prove the following theorem.

Theorem (Theorem 3.17). *Let (S,θ) and (S_Z,θ_Z) be tame F-nonsingular elliptic pairs of G and G_Z, respectively. Assume that $\eta(S) = S_Z$ and $\theta = \theta_Z \circ \eta$. Then*

$$\pi_{(S_Z,\theta_Z)} \circ \eta \simeq \bigoplus_{d_Z \in D_Z} \pi_{(S,\theta)} \circ \underline{\mathrm{Ad}}(d_Z^{-1}),$$

where D_Z is a set of coset representatives of $\eta(G(F))\backslash G_Z(F)/S_Z(F)$ and $\underline{\mathrm{Ad}}$ is the $\underline{G}(F)$-action on $G(F)$ described in Section 2.3.

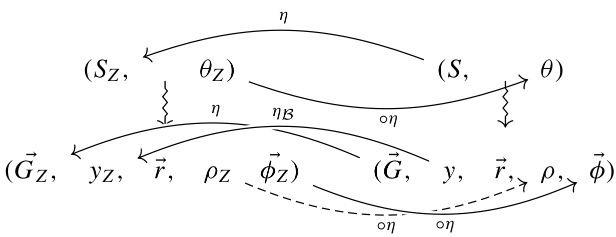

Figure 4. Relationship between the corresponding J.-K. Yu data given the relationship between the tame F-nonsingular elliptic pairs.

The strategy that we use to prove Theorem 3.17 is similar to the one from [Bourgeois 2021] in the sense that we go through the constructions of $\pi_{(S,\theta)}$ and $\pi_{(S_Z,\theta_Z)}$ step-by-step to make comparisons along the way.

Recall that the Kaletha–Yu construction involves two steps: producing a datum (in the sense of [Yu 2001, Section 3]) from the F-nonsingular elliptic pair, and then applying the J.-K. Yu construction to this datum. As such, the first thing we do is establish the relationship between the G-datum $\Psi_{(S,\theta)}$ and G_Z-datum $\Psi_{(S_Z,\theta_Z)}$ produced from (S,θ) and (S_Z,θ_Z), respectively. In particular, these data are related via the map η, a statement we will make precise with Theorem 3.2 and illustrate in Figure 4.

Once the relationship between $\Psi_{(S,\theta)}$ and $\Psi_{(S_Z,\theta_Z)}$ is established, we start comparing the representations at each step of the J.-K. Yu construction. What we show is that the steps of the construction commute with the composition with η (Propositions 3.14 and 3.15), as illustrated in Figure 5.

Establishing these relationships between the data and representations is a lengthy process. However, once complete, the proof of Theorem 3.17 reduces to an application of Mackey theory.

The rest of this section is organized as follows. Most of its contents are dedicated to the proof of Theorem 3.17. In Section 3.1, we establish the relationship between $\Psi_{(S,\theta)}$ and $\Psi_{(S_Z,\theta_Z)}$. In Section 3.2, we go through the steps of the J.-K. Yu

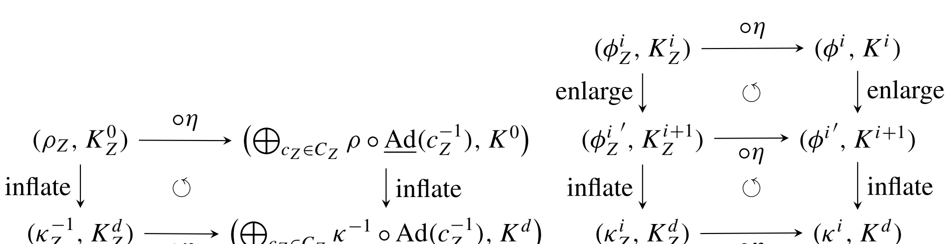

Figure 5. Commutativity of the composition with η with the steps of the J.-K. Yu construction. Here, C_Z denotes a set of coset representatives of $\eta(K^0)\backslash K^0_Z/S_Z(F)$.

construction to show that it commutes with the composition with η, and prove the statement of Theorem 3.17. Finally, in Section 3.3, we restate the results from [Bourgeois 2021] in the context of the Kaletha–Yu construction and provide the proof of Theorem 3.1.

3.1. ***Relationship between the G-datum and the G_Z-datum.*** In this section, η is the map on G, as stated in the introduction (Theorem 4.1) and $G_Z = \eta(G) \simeq G/Z$. We let (S, θ) and (S_Z, θ_Z) be tame F-nonsingular elliptic pairs of G and G_Z, respectively, which satisfy $\eta(S) = S_Z$ and $\theta = \theta_Z \circ \eta$. The goal of this section is to show that the corresponding G-datum and G_Z-datum are also related via the map η, a statement we illustrated in Figure 4 and will make precise with Theorem 3.2. First, we note that $\eta : G \to G_Z$ induces an equivariant isomorphism $\eta_{\mathcal{B}} : \mathcal{B}(G, F) \to \mathcal{B}(G_Z, F)$ by [Kaletha and Prasad 2023, Axiom 4.1.1]. In particular, this map satisfies $\eta_{\mathcal{B}}(gx) = \eta(g) \cdot \eta_{\mathcal{B}}(x)$ for all $g \in G(F)$, $x \in \mathcal{B}(G, F)$, where $\cdot$ refers to the action of $G(F)$ on $\mathcal{B}(G, F)$.

Theorem 3.2. *Let (S, θ) and (S_Z, θ_Z) be tame F-nonsingular elliptic pairs of G and G_Z, respectively, such that $\eta(S) = S_Z$ and $\theta = \theta_Z \circ \eta$. Let $\Psi_{(S,\theta)} = (\vec{G}, y, \vec{r}, \rho, \vec{\phi})$ and $\Psi_{(S_Z,\theta_Z)} = (\vec{G}_Z, y_Z, \vec{r}_Z, \rho_Z, \vec{\phi}_Z)$ be the corresponding J.-K. Yu data as described in Section 2.4. Then,*

(a) $\vec{r} = \vec{r}_Z$ *and* $\eta(\vec{G}) = \vec{G}_Z$,

(b) $y_Z = \eta_{\mathcal{B}}(y)$,

(c) $\vec{\phi} = \vec{\phi}_Z \circ \eta$, *and*

(d) $\rho_Z \circ \eta \simeq \bigoplus_{c_Z \in C_Z} \rho \circ \underline{\mathrm{Ad}}(c_Z^{-1})$, *where C_Z is a set of coset representatives of $\eta(K^0)\backslash K_Z^0/S_Z(F)$.*

The proof of this theorem will be divided into four parts. Lemma 3.3 shows that $\eta(\vec{G}) = \vec{G}_Z$ and $\vec{r} = \vec{r}_Z$. Lemma 3.4 gives us $y_Z = \eta_{\mathcal{B}}(y)$. Proposition 3.8 allows us to set $\vec{\phi} = \vec{\phi}_Z \circ \eta$. Finally, we obtain the decomposition formula of $\rho_Z \circ \eta$ from Proposition 3.13. We note that proving part (d) is itself a multistep process, having to work over the reductive subquotients and call on Deligne–Lusztig theory.

Lemma 3.3 (Theorem 3.2(a)). *Let $\vec{G} = (G^0, \dots, G^d)$ and $\vec{G}_Z = (G_Z^0, \dots, G_Z^{d_Z})$ be the twisted Levi sequences obtained from S and S_Z, respectively, as per Section 2.4. Then $d = d_Z$ and $\eta(G^i) = G_Z^i$ for all $0 \le i \le d$.*

The previous lemma easily follows from the fact that the root systems $R(G, S)$ and $R(G_Z, S_Z)$ are identified by η. Furthermore, the induced sequence of numbers $\vec{r}$ is the same for both S and S_Z.

Lemma 3.4 (Theorem 3.2(b)). *Let y be the vertex of $\mathcal{B}(G, F)$ associated to S and y_Z be the vertex of $\mathcal{B}(G_Z, F)$ associated to S_Z as per [Kaletha 2019, Lemma 3.4.3]. Then $y_Z = \eta_{\mathcal{B}}(y)$.*

Proof. The bijection $\eta_{\mathcal{B}}$ maps $\mathcal{A}(G, S, F^{\mathrm{un}})$ to $\mathcal{A}(G_Z, S_Z, F^{\mathrm{un}})$. Recall that y is fixed by $\mathrm{Gal}(F^{\mathrm{un}}/F)$ by definition. Since $\eta_{\mathcal{B}}$ is also defined over F, $\eta_{\mathcal{B}}(y)$ is a $\mathrm{Gal}(F^{\mathrm{un}}/F)$-fixed point. Since S_Z is elliptic, this fixed point is unique [Kaletha and Prasad 2023, Section 17.8], and thus $y_Z = \eta_{\mathcal{B}}(y)$. □

Remark 3.5. Given that $\eta(G^i) = G^i_Z$ and that Z is a central subgroup of G^i for all $0 \le i \le d$, the results of Lemma 2.2 and Remark 2.3 apply to G and G_Z replaced by G^i and G^i_Z. In particular, $\eta(G^i(F)_{y,r_i}) = G^i_Z(F)_{y_Z,r_i}$, which induces a surjection $G^i(F)_{y,r_i:r_i^+} \to G^i_Z(F)_{y_Z,r_i:r_i^+}$, $0 \le i \le d$. Using a similar argument, we also have $\eta(J^{i+1}) = J^{i+1}_Z$ and $\eta(J^{i+1}_+) = J^{i+1}_{Z+}$ for all $0 \le i \le d-1$, where $J^{i+1} = (G^i, G^{i+1})(F)_{y,(r_i,r_i/2)}$ and $J^{i+1}_+ = (G^i, G^{i+1})(F)_{y,(r_i,r_i/2^+)}$ as per [Yu 2001, Section 1], and J^{i+1}_Z and J^{i+1}_{Z+} are defined analogously. Furthermore, one sees from [Yu 2001, Section 1] that $J^{i+1} \cap Z = J^{i+1}_+ \cap Z$ so that we have an isomorphism $J^{i+1}/J^{i+1}_+ \simeq J^{i+1}_Z/J^{i+1}_{Z+}$ for all $0 \le i \le d-1$.

Lemma 3.6. *Let* $K^0_Z = G^0_Z(F)_{y_Z}$ *and* $K^i_Z = K^0_Z G^1_Z(F)_{y_Z,r_0/2} \cdots G^i_Z(F)_{y_Z,r_{i-1}/2}$ *for all* $0 \le i \le d-1$. *Then* $\eta(K^i) = K^i_Z \cap \eta(G(F))$ *for all* $0 \le i \le d$. *Furthermore,* $\eta(K^i)$ *is normalized by* K^0_Z *for all* $0 \le i \le d$.

Proof. The case of $i = 0$ is given by Lemma 2.1, in which we replace G by G^0.

For $0 < i \le d$, we proceed as follows. Let $J = G^1(F)_{y,r_0/2} \cdots G^i(F)_{y,r_{i-1}/2}$ and $J_Z = G^1_Z(F)_{y_Z,r_0/2} \cdots G^i_Z(F)_{y_Z,r_{i-1}/2}$. By Remark 3.5, we have $\eta(J) = J_Z$. It follows that $K^i_Z = K^0_Z J_Z = K^0_Z \eta(J)$, and therefore

$$K^i_Z \cap \eta(G(F)) = \big(K^0_Z \cap \eta(G(F))\big)\eta(J) = \eta(K^0)\eta(J) = \eta(K^i).$$

Using the fact that ${}^g G^j_Z(F)_{y_Z,r} = G^j_Z(F)_{g\cdot y_Z,r}$ for all $g \in G^j_Z(F)$, one sees that $\eta(J) = J_Z$ is normalized by K^0_Z. We conclude, with what precedes for $i = 0$, that $\eta(K^i)$ is normalized by K^0_Z. □

Corollary 3.7. *Let* $0 \le i \le d$. *Then* $\underline{\mathrm{Ad}}(k_Z)(K^i) = K^i$ *for all* $k_Z \in K^0_Z$.

Proof. By the previous lemma, we have $k_Z\eta(K^i)k_Z^{-1} = \eta(K^i)$ for all $k_Z \in K^0_Z$. Given $k_Z \in K^0_Z \subset G^0_Z(F)$, $k_Z = \eta(g)$ for some $g \in G^0(F^{\mathrm{sep}})$ such that $\sigma(g)g^{-1} \in Z$ for all $\sigma \in \Gamma$. It follows that $\eta(gK^ig^{-1}) = \eta(K^i)$, with $gK^ig^{-1} \subset G(F)$ (or equivalently, $\sigma(gK^ig^{-1}) = gK^ig^{-1}$ for all $\sigma \in \Gamma$). As a map on $G(F)$, the kernel of η is $Z(F)$, which implies $gK^iZ(F)g^{-1} = K^iZ(F)$. By [Yu 2001, Lemma 3.3], one has $K^0 = N_{G^0(F)}(G^0(F)_{y,0})$, and hence $Z(F) \subset K^0 \subset K^i$. Thus, we conclude that $gK^ig^{-1} = K^i$, or equivalently, $\underline{\mathrm{Ad}}(k_Z)(K^i) = K^i$. □

Proposition 3.8 (Theorem 3.2(c)). *Let* $(\phi^{-1}_Z, \phi^0_Z, \dots, \phi^d_Z)$ *be a Howe factorization for* θ_Z. *For each* $-1 \le i \le d$, *set* $\phi^i = \phi^i_Z \circ \eta$. *Then* $(\phi^{-1}, \phi^0, \dots, \phi^d)$ *is a Howe factorization for* θ.

Proof. One sees that $(\phi^{-1}, \phi^0, \dots, \phi^d)$ satisfies the two first axioms to be a Howe factorization of θ, so it remains to verify the third axiom. Let $0 \le i < d$. By [Kaletha 2019, Lemma 3.6.8], proving genericity is equivalent to showing that

$$\phi^i\big(N_{E/F}(\check{\alpha}(E_r^\times))\big) \neq 1$$

for all $\alpha \in R(G^{i+1}, S)\backslash R(G^i, S)$. Since the character ϕ_Z^i is generic, we have that

$$\phi_Z^i\big(N_{E/F}(\check{\alpha_Z}(E_r^\times))\big) \neq 1$$

for all $\alpha_Z \in R(G_Z^{i+1}, S)\backslash R(G_Z^i, S)$. Since η is defined over F, we have that

$$\phi^i\big(N_{E/F}(\check{\alpha}(E_r^\times))\big) = (\phi_Z^i \circ \eta)\big(N_{E/F}(\check{\alpha}(E_r^\times))\big) = \phi_Z^i\big(N_{E/F}((\eta \circ \check{\alpha})(E_r^\times))\big).$$

Since the root systems of G and G_Z are identified by η, we conclude from the genericity of ϕ_Z^i that ϕ^i is generic.

For $i = d$, we see that ϕ^d is trivial whenever ϕ_Z^d is. When $\phi_Z^d \neq 1$, ϕ^d must be of the same depth, as a consequence of the surjection $G(F)_{y,r_d:r_d^+} \to G_Z(F)_{y_Z,r_d:r_d^+}$.

Finally, for $i = -1$, it is clear that ϕ^{-1} is trivial in the case where ϕ_Z^{-1} is trivial, and that $\phi^{-1}|_{S(F)_{0+}} = 1$ whenever $\phi_Z^{-1}|_{S_Z(F)_{0+}} = 1$ as $\eta(S(F)_{0^+}) \subset S_Z(F)_{0^+}$. ☐

It now remains to prove part (d) of Theorem 3.2. Recall from Figure 2 that $\rho = \mathrm{Ind}_{S(F)G^0(F)_{y,0}}^{G^0(F)_y} \kappa_{(S,\phi^{-1})}$, where $\kappa_{(S,\phi^{-1})}$ is Kaletha's extension of the Deligne–Lusztig cuspidal representation $\pm R_{\mathcal{S},\bar{\phi}^{-1}}$ of $\mathcal{G}^0(\mathfrak{f})$ with $\mathcal{S}$ a maximal torus of $\mathcal{G}^0$ which satisfies $\mathcal{S}(\mathfrak{f}) \simeq S(F)_{0:0^+}$. Adopting similar notation for G_Z^0, we have that

$$\rho_Z = \mathrm{Ind}_{G_Z^0(F)_{y_Z,0}}^{G_Z^0(F)_{y_Z}} \kappa_{(S_Z,\phi_Z^{-1})},$$

where $\kappa_{(S_Z,\phi_Z^{-1})}$ is Kaletha's extension of the Deligne–Lusztig cuspidal representation $\pm R_{\mathcal{S}_Z,\bar{\phi}_Z^{-1}}$ of $\mathcal{G}_Z^0(\mathfrak{f})$, the reductive subquotient of G_Z^0 at y_Z, with $\mathcal{S}_Z$ a maximal torus of $\mathcal{G}_Z^0$ which satisfies $\mathcal{S}_Z(\mathfrak{f}) \simeq S_Z(F)_{0:0^+}$. To understand the relationship between ρ and ρ_Z, we start by studying the relationship between $\pm R_{\mathcal{S},\bar{\phi}^{-1}}$ and $\pm R_{\mathcal{S}_Z,\bar{\phi}_Z^{-1}}$.

Given that $\eta(G^0) = G_Z^0$ and that Z is a central subgroup of G^0 (Lemma 3.3), the map η induces a map

$$\mathcal{G}^0 \xrightarrow{q} \mathcal{G}^0/\mathcal{Z}^0 \xrightarrow{\bar{\eta}} \mathcal{G}_Z^0,$$

where $\mathcal{Z}^0$ is such that $\mathcal{Z}^0(\bar{\mathfrak{f}}) = (Z \cap G^0(F^{\mathrm{un}})_{y,0})G^0(F^{\mathrm{un}})_{y,0^+}$. This is precisely the map illustrated in (4), with G and G_Z replaced by G^0 and G_Z^0, respectively. The tori $\mathcal{S}$ and $\mathcal{S}_Z$ are related via this map, as per the following lemma.

Lemma 3.9. *One has* $(\bar{\eta} \circ q)(\mathcal{S}) = \mathcal{S}_Z \cap \bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)$.

Proof. We identify the reductive groups with their $\bar{\mathfrak{f}}$-points. Combining the lower map of (4) with the definitions above, we have

$$\begin{aligned}(\bar{\eta}\circ q)(\mathcal{S}(\bar{\mathfrak{f}})) &= (\bar{\eta}\circ q)\big(S(F^{\mathrm{un}})_0 G^0(F^{\mathrm{un}})_{y,0^+}/G^0(F^{\mathrm{un}})_{y,0^+}\big)\\ &= \eta(S(F^{\mathrm{un}})_0)G^0_Z(F^{\mathrm{un}})_{y_Z,0^+}/G^0_Z(F^{\mathrm{un}})_{y_Z,0^+}\\ &\subseteq \big(S_Z(F^{\mathrm{un}})_0 G^0_Z(F^{\mathrm{un}})_{y_Z,0^+}/G^0_Z(F^{\mathrm{un}})_{y_Z,0^+}\big)\\ &\qquad\qquad \cap\big(\eta(G^0(F^{\mathrm{un}})_{y,0})G^0_Z(F^{\mathrm{un}})_{y_Z,0^+}/G^0_Z(F^{\mathrm{un}})_{y_Z,0^+}\big)\\ &= \mathcal{S}_Z(\bar{\mathfrak{f}})\cap\bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)(\bar{\mathfrak{f}}).\end{aligned}$$

Given that both tori are maximal, the equality follows. □

Lemma 3.10. *We have* $G^0_Z(F)_{y_Z,0} = S_Z(F)_0\,\eta(G^0(F)_{y,0})$.

Proof. Using Lemma 2.5, we have that $\mathcal{G}^0_Z = \mathcal{S}_Z\,\bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)$. Furthermore, the intersection of $\mathcal{S}_Z$ with $\bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)$ is a maximal torus of $\bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)$. As a consequence of Lang's theorem, $H^1(\mathfrak{f}, \mathcal{S}_Z\cap\bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)) = 1$. Combining this with the usual Galois cohomology sequence,

$$\mathcal{G}^0_Z(\mathfrak{f}) = \mathcal{S}_Z(\mathfrak{f})\bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)(\mathfrak{f}) = \mathcal{S}_Z(\mathfrak{f})\bar{\eta}((\mathcal{G}^0/\mathcal{Z}^0)(\mathfrak{f})). \tag{5}$$

We have that $(\mathcal{G}^0/\mathcal{Z}^0)(\mathfrak{f}) = C(\mathcal{G}^0(\mathfrak{f})/\mathcal{Z}^0(\mathfrak{f}))$, where C is a set of coset representatives of $(\mathcal{G}^0/\mathcal{Z}^0)(\mathfrak{f})/(\mathcal{G}^0(\mathfrak{f})/\mathcal{Z}^0(\mathfrak{f}))$.

Without loss of generality, we may assume that $C\subseteq q(\mathcal{S})(\mathfrak{f})$. To see this, consider the exact sequences

$$1\to\mathcal{Z}^0\to\mathcal{G}^0\to\mathcal{G}^0/\mathcal{Z}^0\to 1$$

and

$$1\to\mathcal{Z}^0\to\mathcal{S}\to q(\mathcal{S})\to 1.$$

Given that $\mathcal{G}^0$ and $\mathcal{S}$ are connected, Lang's theorem implies $H^1(\mathfrak{f},\mathcal{G}^0) = 1 = H^1(\mathfrak{f},\mathcal{S})$, giving us exact cohomology sequences [Springer 2009, Theorem 12.3.4]

$$1\to\mathcal{Z}^0(\mathfrak{f})\to\mathcal{G}^0(\mathfrak{f})\to(\mathcal{G}^0/\mathcal{Z}^0)(\mathfrak{f})\to H^1(\mathfrak{f},\mathcal{Z}^0)\to 1$$

and

$$1\to\mathcal{Z}^0(\mathfrak{f})\to\mathcal{S}(\mathfrak{f})\to q(\mathcal{S})(\mathfrak{f})\to H^1(\mathfrak{f},\mathcal{Z}^0)\to 1.$$

The exactness of the sequences implies

$$(\mathcal{G}^0/\mathcal{Z}^0)(\mathfrak{f})/(\mathcal{G}^0(\mathfrak{f})/\mathcal{Z}^0(\mathfrak{f}))\simeq H^1(\mathfrak{f},\mathcal{Z}^0)\simeq q(\mathcal{S})(\mathfrak{f})/(\mathcal{S}(\mathfrak{f})/\mathcal{Z}^0(\mathfrak{f})).$$

The definitions of the connecting homomorphisms $(\mathcal{G}^0/\mathcal{Z}^0)(\mathfrak{f})\to H^1(\mathfrak{f},\mathcal{Z}^0)$ and $q(\mathcal{S})(\mathfrak{f})\to H^1(\mathfrak{f},\mathcal{Z}^0)$ from [Springer 2009, Section 12.3.3] allow us to conclude that we may choose $C\subset q(\mathcal{S})(\mathfrak{f})$.

Having $C\subseteq q(\mathcal{S})(\mathfrak{f})$ allows us to rewrite (5) as

$$\mathcal{G}^0_Z(\mathfrak{f}) = \mathcal{S}_Z(\mathfrak{f})\bar{\eta}(C)\bar{\eta}(\mathcal{G}^0(\mathfrak{f})/\mathcal{Z}^0(\mathfrak{f})) = \mathcal{S}_Z(\mathfrak{f})\bar{\eta}(\mathcal{G}^0(\mathfrak{f})/\mathcal{Z}^0(\mathfrak{f})). \tag{6}$$

Given that $\mathcal{G}^0(\mathfrak{f}) = G^0(F)_{y,0:0^+}$, it follows that

$$\mathcal{Z}^0(\mathfrak{f}) = (Z \cap G^0(F)_{y,0})G^0(F)_{y,0^+}/G^0(F)_{y,0^+}.$$

Therefore,

$$\bar{\eta}(\mathcal{G}^0(\mathfrak{f})/\mathcal{Z}^0(\mathfrak{f})) = \eta(G^0(F)_{y,0})G_Z^0(F)_{y_Z,0^+}/G_Z^0(F)_{y_Z,0^+}.$$

Using $\mathcal{G}_Z^0(\mathfrak{f}) \simeq G_Z^0(F)_{y_Z,0:0^+}$ and $\mathcal{S}_Z(\mathfrak{f}) \simeq S_Z(F)_0 G_Z^0(F)_{y_Z,0^+}/G_Z^0(F)_{y_Z,0^+}$, we rewrite (6) as

$$G_Z^0(F)_{y_Z,0:0^+} = \big(S_Z(F)_0 G_Z^0(F)_{y_Z,0^+}/G_Z^0(F)_{y_Z,0^+}\big)\big(\eta(G^0(F)_{y,0})G_Z^0(F)_{y_Z,0^+}/G_Z^0(F)_{y_Z,0^+}\big),$$

from which we conclude that

$$G_Z^0(F)_{y_Z,0} = S_Z(F)_0\,\eta(G^0(F)_{y,0})G_Z^0(F)_{y_Z,0^+}.$$

Finally, we have that $G_Z^0(F)_{y_Z,0^+} = \eta(G^0(F)_{y,0^+})$. Indeed, using Lemma 2.2,

$$\begin{aligned} G_Z^0(F)_{y_Z,0^+} &= \bigcup_{r>0} G_Z^0(F)_{y_Z,r} = \bigcup_{r>0} \eta(G^0(F)_{y,0}) \\ &= \eta\Big(\bigcup_{r>0} G^0(F)_{y,0}\Big) = \eta(G^0(F)_{y,0^+}). \end{aligned}$$

Thus, we conclude that

$$G_Z^0(F)_{y_Z,0} = S_Z(F)_0\,\eta(G^0(F)_{y,0}). \qquad \square$$

Using the map $\bar{\eta} \circ q$, and Lemma 2.5, we can establish the following relationship between the virtual characters $\pm R_{\mathcal{S},\bar{\phi}^{-1}}$ and $\pm R_{\mathcal{S}_Z,\bar{\phi}_Z^{-1}}$.

Proposition 3.11. *Given the above notation, one has* $\pm R_{\mathcal{S},\bar{\phi}^{-1}} = \pm R_{\mathcal{S}_Z,\bar{\phi}_Z^{-1}} \circ (\bar{\eta} \circ q)$.

Proof. Let us recall the construction of $\pm R_{\mathcal{S},\bar{\phi}^{-1}}$. Following the notation of [Kaletha 2019, Section 3.4.4; 2021, Section 2.4], let $\mathcal{U}$ be the unipotent radical of a Borel subgroup $\mathcal{B}$ of $\mathcal{G}^0$ which contains $\mathcal{S}$ and define $Y_{\mathcal{U}}^{\mathcal{G}^0} = \{g \in \mathcal{G}^0/\mathcal{U} : g^{-1}\operatorname{Fr}(g) \in \mathcal{U}\operatorname{Fr}(\mathcal{U})\}$, where Fr is a generator of $\mathrm{Gal}(\bar{\mathfrak{f}}/\mathfrak{f})$. Let $d\mathcal{U}$ denote the number of hyperplanes separating the Weyl chambers of $\mathcal{U}$ and $\operatorname{Fr}(\mathcal{U})$, and consider the ℓ-adic cohomology group $H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)$. The virtual character $\pm R_{\mathcal{S},\bar{\phi}^{-1}}$ is then defined to be the action of $\mathcal{G}^0(\mathfrak{f})$ on $H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\bar{\phi}^{-1}}$, the $\bar{\phi}^{-1}$-isotypic component of $H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)$. Similarly, $\pm R_{\mathcal{S}_Z,\bar{\phi}_Z^{-1}}$ is the action of $\mathcal{G}_Z^0(\mathfrak{f})$ on $H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0}, \overline{\mathbb{Q}}_\ell)_{\bar{\phi}_Z^{-1}}$, where $\mathcal{U}_Z$ is the unipotent radical of a Borel subgroup $\mathcal{B}_Z$ of $\mathcal{G}_Z^0$ containing $\mathcal{S}_Z$.

Using Lemma 3.9, we have that $\bar{\eta} \circ q(\mathcal{S}) = \mathcal{S}_Z \cap \bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)$. Since the virtual characters do not depend on the choice of Borel subgroup (see, e.g., [Deligne and Lusztig 1976, Corollary 4.3; Carter 1993, Proposition 7.3.6; Kaletha 2021, Section 2.5]), we may assume without loss of generality that $\bar{\eta} \circ q(\mathcal{B}) = \mathcal{B}_Z \cap \bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)$, and therefore $\bar{\eta} \circ q(\mathcal{U}) = \mathcal{U}_Z \cap \bar{\eta}(\mathcal{G}^0/\mathcal{Z}^0)$. Thus, $\bar{\eta} \circ q$ induces a map $Y_{\mathcal{U}}^{\mathcal{G}^0} \to Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0}$.

Furthermore, since we are setting $\phi_Z^{-1} \circ \eta = \phi^{-1}$ (Proposition 4.5), we have that $\overline{\phi}_Z^{-1} \circ (\bar{\eta} \circ q) = \overline{\phi}^{-1}$. Indeed, for all $g \in G^0(F)_{y,0}$, we obtain

$$\begin{aligned}\overline{\phi}_Z^{-1} \circ (\bar{\eta} \circ q)(gG^0(F)_{y,0^+}) = \overline{\phi}_Z^{-1}(\eta(g)G_Z^0(F)_{y,0^+}) &= \phi_Z^{-1}(\eta(g)) \\ &= \phi^{-1}(g) = \overline{\phi}^{-1}(gG^0(F)_{y,0^+}).\end{aligned}$$

By [Kaletha 2021, D.4], $\bar{\eta} \circ q$ induces an isomorphism

$$(\bar{\eta} \circ q)^* : H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}_Z^{-1}} \to H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}}.$$

Now, for all $g \in \mathcal{G}^0(\mathfrak{f})$, letting $L(g) : Y_{\mathcal{U}}^{\mathcal{G}^0} \to Y_{\mathcal{U}}^{\mathcal{G}^0}$ be the map corresponding to left multiplication by g, one sees that the following diagram commutes:

$$\begin{array}{ccc} Y_{\mathcal{U}}^{\mathcal{G}^0} & \xrightarrow{\bar{\eta}\circ q} & Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0} \\ {\scriptstyle L(g)}\downarrow & & \downarrow{\scriptstyle L_Z((\bar{\eta}\circ q)(g))} \\ Y_{\mathcal{U}}^{\mathcal{G}^0} & \xrightarrow{\bar{\eta}\circ q} & Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0} \end{array}$$

Given that ℓ-adic cohomology is functorial, we have that the following diagram also commutes for all $g \in \mathcal{G}^0(\mathfrak{f})$:

$$\begin{array}{ccc} H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}} & \xleftarrow{(\bar{\eta}\circ q)^*} & H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}_Z^{-1}} \\ {\scriptstyle \pm R_{\mathcal{S},\overline{\phi}^{-1}}(g)}\uparrow & & \uparrow{\scriptstyle \pm R_{\mathcal{S}_Z,\overline{\phi}_Z^{-1}}((\bar{\eta}\circ q)(g))} \\ H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}} & \xleftarrow[(\bar{\eta}\circ q)^*]{} & H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}_Z^{-1}} \end{array} \tag{7}$$

Thus, we conclude that $\pm R_{\mathcal{S},\overline{\phi}^{-1}} = \pm R_{\mathcal{S}_Z,\overline{\phi}_Z^{-1}} \circ (\bar{\eta} \circ q)$. □

Corollary 3.12. *Let $\kappa_{(S,\phi^{-1})}$ and $\kappa_{(S_Z,\phi_Z^{-1})}$ be the representations of $S(F)G^0(F)_{y,0}$ and $S_Z(F)G_Z^0(F)_{y_Z,0}$ (as in* [Kaletha 2019, Section 3.4.4]*) which extend the pullbacks of $\pm R_{\mathcal{S},\overline{\phi}^{-1}}$ and $\pm R_{\mathcal{S}_Z,\overline{\phi}_Z^{-1}}$, respectively. Then $\kappa_{(S,\phi^{-1})} \simeq \kappa_{(S_Z,\phi_Z^{-1})} \circ \eta$.*

Proof. Since we are building up from Proposition 3.11, let us follow the notation within its proof.

As in [Kaletha 2021, Section 3], we have an $\mathfrak{f}$-group scheme $\mathcal{S}'$, which satisfies $\mathcal{S}'(\mathfrak{f}) = S(F)/S(F)_{0^+}$. Every $s' \in \mathcal{S}'(\mathfrak{f})$ acts on $Y_{\mathcal{U}}^{\mathcal{G}^0}$ by conjugation, and induces an action on $H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}}$, denoted by $\mathrm{Ad}(s')$. As explained in [Kaletha 2019, Section 3.4.4; 2021, Remark 2.6.5], this allows us to define an action of $\mathcal{S}'(\mathfrak{f})\mathcal{G}^0(\mathfrak{f})$ on $H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}}$, denoted by κ, as

$$\kappa(s'g)(v) = \overline{\phi}^{-1}(s') \cdot (\pm R_{\mathcal{S},\overline{\phi}^{-1}}(g) \circ \mathrm{Ad}(s'))(v)$$

for all $s' \in \mathcal{S}'(\mathfrak{f})$, $g \in \mathcal{G}^0(\mathfrak{f})$, $v \in H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}}$. The action pulls back to an action of $S(F)G^0(F)_{y,0}$ on $H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}}$, which is the representation $\kappa_{(S,\phi^{-1})}$.

Similarly, $\kappa_{(S_Z,\phi_Z^{-1})}$ is the action of $S_Z(F)G_Z^0(F)_{y_Z,0}$ on $H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}_Z^{-1}}$, obtained by pulling back the action

$$\kappa_Z(s_Z' g_Z)(v_Z) = \overline{\phi}_Z^{-1}(s_Z') \cdot (\pm R_{\mathcal{S}_Z,\overline{\phi}_Z^{-1}}(g_Z) \circ \mathrm{Ad}(s_Z'))(v_Z)$$

for all $s_Z' \in \mathcal{S}_Z'(\mathfrak{f}) = S_Z(F)/S_Z(F)_{0^+}$, $g_Z \in \mathcal{G}_Z^0(\mathfrak{f})$, $v_Z \in H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}_Z^{-1}}$.

Using Remark 2.4, one sees that the following diagram commutes for all $s' \in \mathcal{S}'(\mathfrak{f})$:

$$\begin{array}{ccc} Y_{\mathcal{U}}^{\mathcal{G}^0} & \xrightarrow{\bar{\eta}\circ q} & Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0} \\ {\scriptstyle \mathrm{Ad}(s')}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}((\bar{\eta}\circ q)(s'))} \\ Y_{\mathcal{U}}^{\mathcal{G}^0} & \xrightarrow{\bar{\eta}\circ q} & Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0} \end{array}$$

Given that ℓ-adic cohomology is functorial, we have that the following diagram also commutes for all $s' \in \mathcal{S}'(\mathfrak{f})$:

$$\begin{array}{ccc} H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}} & \xleftarrow{(\bar{\eta}\circ q)^*} & H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}_Z^{-1}} \\ {\scriptstyle \mathrm{Ad}(s')}\uparrow & & \uparrow{\scriptstyle \mathrm{Ad}((\bar{\eta}\circ q)(s'))} \\ H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}} & \xleftarrow[(\bar{\eta}\circ q)^*]{} & H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}_Z^{-1}} \end{array} \tag{8}$$

where $(\bar{\eta} \circ q)^*$ is an isomorphism.

Combining the diagrams (7) and (8), we obtain a final commutative diagram

$$\begin{array}{ccc} H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}} & \xleftarrow{(\bar{\eta}\circ q)^*} & H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}_Z^{-1}} \\ {\scriptstyle \kappa(s'g)}\uparrow & & \uparrow{\scriptstyle (\kappa_Z\circ(\bar{\eta}\circ q))(s'g)} \\ H_c^{d\mathcal{U}}(Y_{\mathcal{U}}^{\mathcal{G}^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}^{-1}} & \xleftarrow[(\bar{\eta}\circ q)^*]{} & H_c^{d\mathcal{U}_Z}(Y_{\mathcal{U}_Z}^{\mathcal{G}_Z^0}, \overline{\mathbb{Q}}_\ell)_{\overline{\phi}_Z^{-1}} \end{array}$$

for all $s' \in \mathcal{S}'(\mathfrak{f})$, $g \in \mathcal{G}^0(\mathfrak{f})$.

As such, we conclude that $\kappa \simeq \kappa_Z \circ (\bar{\eta} \circ q)$, and thus $\kappa_{(S,\phi^{-1})} \simeq \kappa_{(S_Z,\phi_Z^{-1})} \circ \eta$. □

Proposition 3.13 (Theorem 3.2(d)). *Let ρ and ρ_Z be the representations of $G^0(F)_y$ and $G_Z^0(F)_{y_Z}$ constructed from (S, θ) and (S_Z, θ_Z), respectively, as per Section 2.4. Then*

$$\rho_Z \circ \eta = \bigoplus_{c_Z \in C_Z} \rho \circ \underline{\mathrm{Ad}}(c_Z^{-1}),$$

where C_Z is a set of coset representatives of $\eta(K^0)\backslash K_Z^0/S_Z(F)$.

By Corollary 3.7, we have that $\underline{\mathrm{Ad}}(c_Z^{-1})(K^0) = K^0$ for all $c_Z \in K_Z^0$. Therefore, the direct sum decomposition above makes sense as a representation of K^0.

Proof of Proposition 3.13. From the Mackey decomposition, we have

$$\begin{aligned}\rho_Z\circ\eta &= \big(\mathrm{Ind}_{S_Z(F)G_Z^0(F)_{y_Z,0}}^{K_Z^0}\kappa_{(S_Z,\phi_Z^{-1})}\big)\circ\eta\\ &= \big(\mathrm{Res}_{\eta(K^0)}^{K_Z^0}\mathrm{Ind}_{S_Z(F)G_Z^0(F)_{y_Z,0}}^{K_Z^0}\kappa_{(S_Z,\phi_Z^{-1})}\big)\circ\eta\\ &= \bigoplus_{c_Z\in C_Z}\big[\big(\mathrm{Ind}_{\eta(K^0)\cap{}^{c_Z}(S_Z(F)G_Z^0(F)_{y_Z,0})}^{\eta(K^0)}\mathrm{Res}_{\eta(K^0)\cap{}^{c_Z}(S_Z(F)G_Z^0(F)_{y_Z,0})}^{{}^{c_Z}(S_Z(F)G_Z^0(F)_{y_Z,0})}{}^{c_Z}\kappa_{(S_Z,\phi_Z^{-1})}\big)\circ\eta\big],\end{aligned}$$

where C_Z is a set of coset representatives of $\eta(K^0)\backslash K_Z^0/S_Z(F)G_Z^0(F)_{y_Z,0}$, which is equal to $\eta(K^0)\backslash K_Z^0/S_Z(F)$ by Lemma 3.10. Given that $\eta(K^0)$ is a normal subgroup of K_Z^0 (Lemma 3.6) and that $\eta(K^0)\cap S_Z(F)G_Z^0(F)_{y_Z,0}=\eta(S(F)G^0(F)_{y,0})$ (as a consequence of Lemma 3.10), it follows that

$$\begin{aligned}\eta(K^0)\cap{}^{c_Z}(S_Z(F)G_Z^0(F)_{y_Z,0}) &= {}^{c_Z}(\eta(K^0)\cap S_Z(F)G_Z^0(F)_{y_Z,0})\\ &= {}^{c_Z}\eta(S(F)G^0(F)_{y,0})\\ &= \eta(\underline{\mathrm{Ad}}(c_Z)(S(F)G^0(F)_{y,0})).\end{aligned}$$

Thus, we may simplify the above expression and apply Proposition A.2 to obtain

$$\begin{aligned}\rho_Z\circ\eta &= \bigoplus_{c_Z\in C_Z}\big[\big(\mathrm{Ind}_{\eta(\underline{\mathrm{Ad}}(c_Z)(S(F)G^0(F)_{y,0}))}^{\eta(K^0)}\mathrm{Res}_{\eta(\underline{\mathrm{Ad}}(c_Z)(S(F)G^0(F)_{y,0}))}^{{}^{c_Z}(S_Z(F)G_Z^0(F)_{y_Z,0})}{}^{c_Z}\kappa_{(S_Z,\phi_Z^{-1})}\big)\circ\eta\big]\\ &\simeq \bigoplus_{c_Z\in C_Z}\mathrm{Ind}_{\underline{\mathrm{Ad}}(c_Z)(S(F)G^0(F)_{y,0})}^{\underline{\mathrm{Ad}}(c_Z)(K^0)}({}^{c_Z}\kappa_{(S_Z,\phi_Z^{-1})}\circ\eta).\end{aligned}$$

Applying Lemma 2.8, followed by Corollary 3.12 on the previous expression,

$$\begin{aligned}\rho_Z\circ\eta &\simeq \bigoplus_{c_Z\in C_Z}\mathrm{Ind}_{\underline{\mathrm{Ad}}(c_Z)(S(F)G^0(F)_{y,0})}^{\underline{\mathrm{Ad}}(c_Z)(K^0)}[(\kappa_{(S_Z,\phi_Z^{-1})}\circ\eta)\circ\underline{\mathrm{Ad}}(c_Z^{-1})]\\ &= \bigoplus_{c_Z\in C_Z}\mathrm{Ind}_{\underline{\mathrm{Ad}}(c_Z)(S(F)G^0(F)_{y,0})}^{\underline{\mathrm{Ad}}(c_Z)(K^0)}(\kappa_{(S,\phi^{-1})}\circ\underline{\mathrm{Ad}}(c_Z^{-1})).\end{aligned}$$

Finally, we apply Proposition A.1 to extract the $\underline{\mathrm{Ad}}$ map from the induction and get

$$\begin{aligned}\rho_Z\circ\eta &\simeq \bigoplus_{c_Z\in C_Z}\big(\mathrm{Ind}_{S(F)G^0(F)_{y,0}}^{K^0}\kappa_{(S,\phi^{-1})}\big)\circ\underline{\mathrm{Ad}}(c_Z^{-1})\\ &= \bigoplus_{c_Z\in C_Z}\rho\circ\underline{\mathrm{Ad}}(c_Z^{-1}).\end{aligned}$$ □

Proposition 3.13 completes the proof of Theorem 3.2.

3.2. ***Going through the steps of the J.-K. Yu construction.*** Let (S,θ) and (S_Z,θ_Z) be tame F-nonsingular elliptic pairs of G and G_Z, respectively, such that $\eta(S)=S_Z$ and $\theta=\theta_Z\circ\eta$. In the previous section, we have established the relationship between the corresponding J.-K. Yu data, $(\vec{G}, y, \vec{r}, \rho, \vec{\phi})$ and $(\vec{G}_Z, y_Z, \vec{r}, \rho_Z, \vec{\phi}_Z)$, respectively. It is from these data that we construct the representations $\pi_{(S,\theta)}$ and $\pi_{(S_Z,\theta_Z)}$ following the steps of the J.-K. Yu construction as outlined in Figure 3. To be consistent with notation, we will keep using subscript Z to differentiate between the construction over G_Z from that of G. Since we have that $\phi^i = \phi^i_Z \circ \eta$ for all $0 \le i \le d$ and $\rho_Z \circ \eta = \bigoplus_{c_Z \in C_Z} \rho \circ \underline{\mathrm{Ad}}(c_Z^{-1})$ with C_Z a set of coset representatives of $\eta(K^0)\backslash K^0_Z/S_Z(F)$, it is natural to expect that we also have $\phi^{i'} = \phi^{i\,'}_Z \circ \eta$, $\kappa_Z^{-1} \circ \eta = \bigoplus_{c_Z \in C_Z} \kappa^{-1} \circ \underline{\mathrm{Ad}}(c_Z^{-1})$ and $\kappa^i = \kappa^i_Z \circ \eta$ for all $0 \le i \le d$, as illustrated in Figure 5. Indeed, we prove these equalities and inclusion with Propositions 3.14 and 3.15. In particular, one can say that the J.-K. Yu construction commutes with the map η. The above results allow us to complete the proof of Theorem 3.17 at the end of this section.

In order to define the representation $\phi^{i'}$, we require the groups J^{i+1} and J^{i+1}_+, which were previously mentioned in Remark 3.5. The construction of $\phi^{i'}$ from ϕ^i is divided into two steps: the first step consists of extending ϕ^i to a character $\widehat{\phi^i}$ of $K^i G^{i+1}(F)_{y,s_i^+}$, where $s_i = r_i/2$. The character $\widehat{\phi^i}$ is the unique character of $K^i G^{i+1}(F)_{y,s_i^+}$ that agrees with ϕ^i on K^i and is trivial on $(G^i, G^{i+1})(F)_{y,(r_i^+,s_i^+)}$ [Hakim and Murnaghan 2008, Section 3.1]. When $J^{i+1} \ne J^{i+1}_+$, a second step is required to enlarge the character $\widehat{\phi^i}$ a little further to a representation of K^{i+1} by means of a Heisenberg–Weil lift. We adopt analogous notation to describe the construction of $\phi^{i\,'}_Z$ from ϕ^i_Z. We note that $J^{i+1} = J^{i+1}_+$ if and only if $J^{i+1}_Z = J^{i+1}_{Z+}$ (as a consequence of Remark 3.5), which ensures that the construction of $\phi^{i'}$ requires a Heisenberg–Weil lift if and only if that of $\phi^{i\,'}_Z$ does.

Proposition 3.14. *For all* $0 \le i \le d$ *we have* $\phi^{i'} = \phi^{i\,'}_Z \circ \eta$.

Proof. We have that $\widehat{\phi^i} = \widehat{\phi^i_Z} \circ \eta$. Indeed, given that

$$\eta(K^i) \subset K^i_Z \quad \text{and} \quad \eta((G^i, G^{i+1})(F)_{y,(r_i^+,s_i^+)}) = (G^i_Z, G^{i+1}_Z)(F)_{y_Z,(r_i^+,s_i^+)}$$

(Remark 3.5), one sees that $\widehat{\phi^i_Z} \circ \eta$ agrees with ϕ^i on K^i and that it is trivial on $(G^i, G^{i+1})(F)_{y,(r_i^+,s_i^+)}$.

If $J^{i+1} = J^{i+1}_+$, we have $\phi^{i'} = \widehat{\phi^i}$ and $\phi^{i\,'}_Z = \widehat{\phi^i_Z}$ and we are done. If $J^{i+1} \ne J^{i+1}_+$, we have that $\phi^{i'}$ is constructed using a Heisenberg–Weil lift ω^i, which is a representation of $K^i \ltimes \mathcal{H}^i$, where $\mathcal{H}^i = J^{i+1}/\ker(\xi^i)$ and $\xi^i = \widehat{\phi^i}|_{J^{i+1}_+}$. We then have $\phi^{i'}(kj) = \widehat{\phi^i}(k)\omega^i(k, j\ker(\xi^i))$ for all $k \in K^i$, $j \in J^{i+1}$. Since $J^{i+1} \ne J^{i+1}_+$ if and only if $J^{i+1}_Z \ne J^{i+1}_{Z+}$ (Remark 3.5), we also require a Heisenberg–Weil lift ω^i_Z, which is a representation of $K^i_Z \ltimes \mathcal{H}^i_Z$, where $\mathcal{H}^i_Z = J^{i+1}_Z/\ker(\xi^i_Z)$ and $\xi^i_Z = \widehat{\phi^i_Z}|_{J^{i+1}_{Z+}}$, and have that

$$\phi^{i\,'}_Z(k_Z j_Z) = \widehat{\phi^i_Z}(k)\omega^i_Z(k_Z, j_Z\ker(\xi^i_Z))$$

for all $k_Z \in K^i_Z$, $j_Z \in J^{i+1}_Z$.

Since we already know that $\widehat{\phi^i} = \widehat{\phi^i_Z} \circ \eta$, it then suffices to show that $\omega^i = \omega^i_Z \circ \eta$. The map η induces isomorphisms $\mathcal{H}^i \simeq \mathcal{H}^i_Z$ and $W^i \simeq W^i_Z$ (Remark 3.5), where $W^i = J^{i+1}/J^{i+1}_+$ and $W^i_Z = J^{i+1}_Z/J^{i+1}_{Z+}$. We then obtain that $\omega^i = \omega^i_Z \circ \eta$ as an application of [Nevins 2015, Proposition 3.2], in which we set $H_1 = \mathcal{H}^i$, $H_2 = \mathcal{H}^i_Z$, $W_1 = W^i$, $W_2 = W^i_Z$, $T_1 = K^i$, $T_2 = K^i_Z$, $\alpha = \delta = \eta$, ν_1 and ν_2 the corresponding special isomorphisms from [Hakim and Murnaghan 2008, Lemma 2.35], and f_1 and f_2 the homomorphisms coming from the actions by conjugation of K^i and K^i_Z on J^{i+1} and J^{i+1}_Z, respectively. □

Proposition 3.15. *For all* $0 \le i \le d$ *we have* $\kappa^i = \kappa^i_Z \circ \eta$. *Furthermore,*

$$\kappa_Z^{-1} \circ \eta = \bigoplus_{c_Z \in C_Z} \kappa^{-1} \circ \underline{\mathrm{Ad}}(c_Z^{-1}),$$

where C_Z *is a set of coset representatives of* $\eta(K^0)\backslash K^0_Z/S_Z(F)$.

Proof. Let $0 \le i \le d-1$. Let us briefly recall the process of inflation. We have that $K^d = K^{i+1}J$, where $J = J^{i+2} \cdots J^d$. Then, for all $k \in K^{i+1}$, $j \in J$, $\kappa^i(kj) = \phi^{i\prime}(k)$. Similarly, we have $K^d_Z = K^{i+1}_Z J_Z$, where $J_Z = J^{i+2}_Z \cdots J^d_Z$, and $\kappa^i_Z(k_Z j_Z) = {\phi^i_Z}'(k_Z)$ for all $k_Z \in K^{i+1}_Z$, $j_Z \in J_Z$.

Using these definitions, for all $k \in K^{i+1}$, $j \in J$, we have

$$\kappa^i(kj) = \phi^{i\prime}(k) = {\phi^i_Z}'(\eta(k)).$$

By Remark 3.5, we have that $\eta(k) \in K^{i+1}_Z$ and $\eta(j) \in J_Z$. Therefore,

$${\phi^i_Z}'(\eta(k)) = \kappa^i_Z(\eta(k)\eta(j)) = \kappa^i_Z \circ \eta(kj).$$

Thus, we conclude that $\kappa^i = \kappa^i_Z \circ \eta$.

By a similar argument, we have that

$$\kappa_Z^{-1} \circ \eta = \bigoplus_{c_Z \in C_Z} \kappa^{-1} \circ \underline{\mathrm{Ad}}(c_Z^{-1})$$

as a consequence of having (Proposition 3.13)

$$\rho_Z \circ \eta = \bigoplus_{c_Z \in C_Z} \rho \circ \underline{\mathrm{Ad}}(c_Z^{-1}),$$

where C_Z is a set of coset representatives of $\eta(K^0)\backslash K^0_Z/S_Z(F)$. □

Combining the previous proposition with the Mackey decomposition formula, we obtain the following relationship between $\kappa_{(S,\theta)}$ and $\kappa_{(S_Z,\theta_Z)}$.

Proposition 3.16. *Let* (S,θ) *and* (S_Z,θ_Z) *be tame* F*-nonsingular elliptic pairs of* G *and* G_Z*, respectively. Assume that* $\eta(S) = S_Z$ *and* $\theta = \theta_Z \circ \eta$. *Then,* $\kappa_{(S_Z,\theta_Z)} \circ \eta = \bigoplus_{l_Z \in L_Z} \kappa_{(S,\theta)} \circ \underline{\mathrm{Ad}}(l_Z^{-1})$, *where* L_Z *is a set of coset representatives of* $\eta(K^0)\backslash K^0_Z/S_Z(F)$.

By Corollary 3.7, we have that $\underline{\mathrm{Ad}}(l_Z^{-1})(K^d) = K^d$ for all $l_Z \in L_Z \subseteq K_Z^0$. Therefore, the direct sum decomposition above makes sense as a representation of K^d.

Proof of Proposition 3.16. For all $l_Z \in L_Z$, we have that ${}^{l_Z}\phi_Z^i = \phi_Z^i$, as $K_Z^0 \subset G^i_Z(F)$ for all i. It follows that ${}^{l_Z}\kappa_Z^i \simeq \kappa_Z^i$, which implies ${}^{l_Z}\kappa_Z^i \circ \eta \simeq \kappa_Z^i \circ \eta$. Using Proposition 3.15, we conclude that $\kappa^i \circ \underline{\mathrm{Ad}}(l_Z^{-1}) \simeq \kappa_i$ for all $l_Z \in L_Z$. Furthermore, Proposition 3.15 also tells us that $\kappa_Z^{-1} \circ \eta = \bigoplus_{l_Z \in L_Z} \kappa^{-1} \circ \underline{\mathrm{Ad}}(l_Z^{-1})$. We thus obtain

$$\begin{aligned}
\kappa_{(S_Z,\theta_Z)} \circ \eta &= \Big[\kappa_Z^{-1} \otimes \Big(\bigotimes_{i=0}^{d} \kappa_Z^i\Big)\Big] \circ \eta \\
&= \kappa_Z^{-1} \circ \eta \otimes \Big(\bigotimes_{i=0}^{d} \kappa_Z^i \circ \eta\Big) \\
&= \Big(\bigoplus_{l_Z \in L_Z} \kappa^{-1} \circ \underline{\mathrm{Ad}}(l_Z^{-1})\Big) \otimes \bigotimes_{i=0}^{d} \kappa^i \\
&= \bigoplus_{l_Z \in L_Z} \Big(\kappa^{-1} \circ \underline{\mathrm{Ad}}(l_Z^{-1}) \otimes \bigotimes_{i=0}^{d} \kappa^i\Big) \\
&\simeq \bigoplus_{l_Z \in L_Z} \Big(\kappa^{-1} \otimes \bigotimes_{i=0}^{d} \kappa^i\Big) \circ \underline{\mathrm{Ad}}(l_Z^{-1}) \\
&= \bigoplus_{l_Z \in L_Z} \kappa_{(S,\theta)} \circ \underline{\mathrm{Ad}}(l_Z^{-1}).
\end{aligned}$$

□

We are now in a position to complete the proof of our main theorem.

Theorem 3.17. *Let (S,θ) and (S_Z,θ_Z) be tame F-nonsingular elliptic pairs of G and G_Z, respectively. Assume that $\eta(S) = S_Z$ and $\theta = \theta_Z \circ \eta$. Then*

$$\pi_{(S_Z,\theta_Z)} \circ \eta \simeq \bigoplus_{d_Z \in D_Z} \pi_{(S,\theta)} \circ \underline{\mathrm{Ad}}(d_Z^{-1}),$$

where D_Z is a set of coset representatives of $\eta(G(F))\backslash G_Z(F)/S_Z(F)$.

Proof. From the Mackey decomposition, we have

$$\begin{aligned}
\pi_{(S_Z,\theta_Z)} \circ \eta &= \big(\mathrm{Ind}_{K_Z^d}^{G_Z(F)} \kappa_{(S_Z,\theta_Z)}\big) \circ \eta \\
&= \big(\mathrm{Res}^{G_Z(F)}_{\eta(G(F))} \mathrm{Ind}_{K_Z^d}^{G_Z(F)} \kappa_{(S_Z,\theta_Z)}\big) \circ \eta \\
&= \bigoplus_{\ell_Z \in \mathcal{L}_Z} \big(\mathrm{Ind}^{\eta(G(F))}_{\eta(G(F)) \cap {}^{\ell_Z}K_Z^d} \mathrm{Res}^{{}^{\ell_Z}K_Z^d}_{\eta(G(F)) \cap {}^{\ell_Z}K_Z^d} {}^{\ell_Z}\kappa_{(S_Z,\theta_Z)}\big) \circ \eta,
\end{aligned}$$

where $\mathcal{L}_Z$ is a set of coset representatives of $\eta(G(F))\backslash G_Z(F)/K_Z^d$. Given that $\eta(G(F))$ is a normal subgroup of $G_Z(F)$ and $\eta(G(F)) \cap K_Z^d = \eta(K^d)$ (Lemma 3.6),

$$\eta(G(F)) \cap {}^{\ell_Z}K_Z^d = {}^{\ell_Z}(\eta(G(F)) \cap K_Z^d) = {}^{\ell_Z}\eta(K^d) = \eta(\underline{\mathrm{Ad}}(\ell_Z)(K^d)).$$

As a result, we may simplify the above expression and apply Proposition A.2 to obtain

$$\begin{aligned}\pi_{(S_Z,\theta_Z)}\circ\eta &= \bigoplus_{\ell_Z\in\mathcal{L}_Z}\bigl(\mathrm{Ind}_{\eta(\underline{\mathrm{Ad}}(\ell_Z)(K^d))}^{\ell_Z\eta(G(F))}\,\mathrm{Res}_{\eta(\underline{\mathrm{Ad}}(\ell_Z)(K^d))}^{\ell_Z K_Z^d}\,{}^{\ell_Z}\kappa_{(S_Z,\theta_Z)}\bigr)\circ\eta\\ &\simeq \bigoplus_{\ell_Z\in\mathcal{L}_Z}\mathrm{Ind}_{\underline{\mathrm{Ad}}(\ell_Z)(K^d)}^{\underline{\mathrm{Ad}}(\ell_Z)(G(F))}\bigl({}^{\ell_Z}\kappa_{(S_Z,\theta_Z)}\circ\eta\bigr).\end{aligned}$$

We then apply Lemma 2.8, followed by Proposition A.1 on the previous expression and get

$$\begin{aligned}\pi_{(S_Z,\theta_Z)}\circ\eta &\simeq \bigoplus_{\ell_Z\in\mathcal{L}_Z}\mathrm{Ind}_{\underline{\mathrm{Ad}}(\ell_Z)(K^d)}^{\underline{\mathrm{Ad}}(\ell_Z)(G(F))}[(\kappa_{(S_Z,\theta_Z)}\circ\eta)\circ\underline{\mathrm{Ad}}(\ell_Z^{-1})]\\ &\simeq \bigoplus_{\ell_Z\in\mathcal{L}_Z}\bigl(\mathrm{Ind}_{K^d}^{G(F)}\,\kappa_{(S_Z,\theta_Z)}\circ\eta\bigr)\circ\underline{\mathrm{Ad}}(\ell_Z^{-1}).\end{aligned}$$

Replacing $\kappa_{(S_Z,\theta_Z)}\circ\eta$ by its equivalent direct sum decomposition in Proposition 3.16, with L_Z a set of coset representatives of $\eta(K^0)\backslash K_Z^0/S_Z(F)$, it follows that

$$\begin{aligned}\pi_{(S_Z,\theta_Z)}\circ\eta &= \bigoplus_{\ell_Z\in\mathcal{L}_Z}\mathrm{Ind}_{K^d}^{G(F)}\Bigl(\bigoplus_{l_Z\in L_Z}\kappa_{(S,\theta)}\circ\underline{\mathrm{Ad}}(l_Z^{-1})\Bigr)\circ\underline{\mathrm{Ad}}(\ell_Z^{-1})\\ &= \bigoplus_{\ell_Z\in\mathcal{L}_Z}\bigoplus_{l_Z\in L_Z}\mathrm{Ind}_{\underline{\mathrm{Ad}}(l_Z)(K^d)}^{\underline{\mathrm{Ad}}(l_Z)(G(F))}(\kappa_{(S,\theta)}\circ\underline{\mathrm{Ad}}(l_Z^{-1}))\circ\underline{\mathrm{Ad}}(\ell_Z^{-1})\\ &\simeq \bigoplus_{\ell_Z\in\mathcal{L}_Z}\bigoplus_{l_Z\in L_Z}\bigl(\mathrm{Ind}_{K^d}^{G(F)}\,\kappa_{(S,\theta)}\bigr)\circ\underline{\mathrm{Ad}}(l_Z^{-1})\circ\underline{\mathrm{Ad}}(\ell_Z^{-1})\\ &= \bigoplus_{\ell_Z\in\mathcal{L}_Z}\bigoplus_{l_Z\in L_Z}\pi_{(S,\theta)}\circ\underline{\mathrm{Ad}}((\ell_Z l_Z)^{-1}).\end{aligned}$$

Finally, we claim that $\{\ell_Z l_Z : \ell_Z\in\mathcal{L}_Z,\ l_Z\in L_Z\}$ is a set of coset representatives of $\eta(G(F))\backslash G_Z(F)/S_Z(F)$, which we denote by D_Z, allowing us to write

$$\pi_{(S_Z,\theta_Z)}\circ\eta = \bigoplus_{d_Z\in D_Z}\pi_{(S,\theta)}\circ\underline{\mathrm{Ad}}(d_Z^{-1}).$$

To prove this last claim, we set $N=\eta(G(F))$, $A=G_Z(F)$, $B=K_Z^d$, $\overline{N}=\eta(K^0)$, $\overline{A}=K_Z^0$ and $\overline{B}=S_Z(F)$ and show that N, A, B, $\overline{N}$, $\overline{A}$, $\overline{B}$ satisfy the hypotheses of Lemma A.4(1). It is clear that N and $\overline{N}$ are normal subgroups of A and $\overline{A}$, respectively, and that $\overline{A}\subseteq B$ and $N\cap\overline{A}\subseteq\overline{N}$. It remains to show that $B/(N\cap B)\overline{B}\simeq\overline{A}/\overline{N}\overline{B}$.

Recall that $K_Z^d=K_Z^0 J_Z$ and $\eta(K^d)=\eta(K^0 J)=\eta(K^0)J_Z$, where J and J_Z are as in the proof of Lemma 3.6. Therefore,

$$B/(N\cap B)\overline{B} = K_Z^d/\eta(K^d)\overline{B} = K_Z^0 J_Z/\eta(K^0)J_Z\overline{B}.$$

Given that $\bar{B}$ is in the stabilizer of y_Z, it follows that $\bar{B}$ normalizes J_Z and thus $J_Z\bar{B} = \bar{B}J_Z$. This, in combination with Lemma A.3, allows us to obtain

$$B/(N\cap B)\bar{B} \simeq K_Z^0/\eta(K^0)\bar{B}(K_Z^0\cap J_Z) = K_Z^0/\eta(K^0)\bar{B},$$

where the last equality follows from the fact that $K_Z^0\cap J_Z = K_Z^0\cap\eta(J)\subset\eta(K^0)$ (Lemma 3.6). Thus, we conclude from Lemma A.4(1) that $\{\ell_Z l_Z : \ell_Z\in\mathcal{L}_Z,\ l_Z\in L_Z\}$ is a set of coset representatives of $\eta(G(F))\backslash G_Z(F)/S_Z(F)$. □

3.3. ***The proof of Theorem 3.1.*** As mentioned earlier in this section, and illustrated in Figure 1, computing the decomposition of $\pi_{(\underline{S},\underline{\theta})}\circ\eta$ is done in two steps. So far, this section has focused on the second step. The first step consists of applying the results of [Bourgeois 2021] to compute the restriction of $\pi_{(\underline{S},\underline{\theta})}$ to $G_Z(F)$. We begin this section with a lemma on parahoric subgroups. We then summarize the results of [Bourgeois 2021] in the context of the Kaletha–Yu construction, and end with a proof of Theorem 3.1.

Lemma 3.18. *Let G be a reductive F-group, and H be an F-subgroup that contains $[G,G]$. Let S be a maximally unramified elliptic maximal torus of G, and let $S_H = S\cap H$, so that S_H is a maximally unramified elliptic maximal torus of H. Denote by y the point of the reduced building associated to S (and S_H) via* [Kaletha 2019, Lemma 3.4.3]. *Let G^0 and G_H^0 denote the smallest Levi subgroups of the Levi sequences obtained from S and S_H, respectively, and recall that $G_H^0 = G^0\cap H$. Then*

$$G^0(F)_{y,0} = S(F)_0\,H^0(F)_{y,0}.$$

Proof. By [Kaletha 2019, Lemma 3.4.2], we have that S is the centralizer of a maximal F^{un}-split torus S' of G^0. It follows by definition (see, for example, [Fintzen 2021, Section 2.4] or [Kaletha and Prasad 2023, Definition 13.2.1]) that

$$G^0(F^{\mathrm{un}})_{y,0} = \langle S(F^{\mathrm{un}})_0,\ U_\alpha(F^{\mathrm{un}}) : \alpha\in R^{\mathrm{aff}}_{F^{\mathrm{un}}},\ \langle\alpha, y\rangle\ge 0\rangle,$$

where $R^{\mathrm{aff}}_{F^{\mathrm{un}}} = \{\lambda + k : \lambda\in R(G^0, S) \text{ such that } \lambda|_{S'}\ne 1,\ k\in\mathbb{R}\}$, and $U_\alpha(F^{\mathrm{un}})$ is the affine root subgroup associated to the affine root α. The affine root subgroups are normalized by $S(F^{\mathrm{un}})_0$, allowing us to write

$$\begin{aligned} G^0(F^{\mathrm{un}})_{y,0} &= S(F^{\mathrm{un}})_0\,\langle S_H(F^{\mathrm{un}})_0,\ U_\alpha(F^{\mathrm{un}}) : \alpha\in R^{\mathrm{aff}}_{F^{\mathrm{un}}},\ \langle\alpha, y\rangle\ge 0\rangle \\ &= S(F^{\mathrm{un}})_0\,H^0(F^{\mathrm{un}})_{y,0}, \end{aligned}$$

and

$$G^0(F)_{y,0} = (G^0(F^{\mathrm{un}})_{y,0})^{\mathrm{Fr}} = (S(F^{\mathrm{un}})_0\,H^0(F^{\mathrm{un}})_{y,0})^{\mathrm{Fr}}.$$

Using [Kaletha 2019, Lemma 3.4.6] and the definition of S_H, we have that

$$S(F^{\mathrm{un}})_0\cap H^0(F^{\mathrm{un}})_{y,0} = S_H(F^{\mathrm{un}})_0.$$

Furthermore, $H^1(\mathrm{Fr}, S_H(F^{\mathrm{un}})_0)$ is trivial [Kaletha and Prasad 2023, Lemma 8.1.4]. It follows from the usual sequence of Galois cohomology [Springer 2009, Proposition 12.3.4] that

$$(S(F^{\mathrm{un}})_0 H^0(F^{\mathrm{un}})_{y,0})^{\mathrm{Fr}} = S(F^{\mathrm{un}})_0^{\mathrm{Fr}} (H^0(F^{\mathrm{un}})_{y,0})^{\mathrm{Fr}} = S(F)_0 H^0(F)_{y,0},$$

and therefore

$$G^0(F)_{y,0} = S(F)_0 H^0(F)_{y,0}. \qquad \square$$

Theorem 3.19. *Let (S,θ) be a tame F-nonsingular elliptic pair of G and let y be the vertex of $\mathcal{B}(G,F)$ associated to S. Let H be a closed connected F-subgroup of G that contains $[G,G]$. Set $S_H = S \cap H$ and $\theta_H = \theta|_{S_H}$. Then (S_H,θ_H) is a tame F-nonsingular elliptic pair of H and*

$$\pi_{(S,\theta)}|_{H(F)} = \bigoplus_{d\in D} {}^d\pi_{(S_H,\theta_H)},$$

where D is a set of coset representatives of $H(F)\backslash G(F)/S(F)$.

Proof. Let $\Psi_{(S,\theta)} = (\vec{G}, y, \vec{r}, \rho, \vec{\phi})$ be the G-datum obtained from the pair (S,θ) as in Section 2.4. Recall that we may write $\pi_G(\Psi_{(S,\theta)})$ for $\pi_{(S,\theta)}$ and $\kappa_G(\Psi_{(S,\theta)})$ for $\kappa_{(S,\theta)}$ to indicate that we are applying the J.-K. Yu construction to $\Psi_{(S,\theta)}$. Set $K_H^i = K^i \cap H$ for all $0 \le i \le d$ and $\Psi_{(S,\theta)}^H = (\vec{H}, y, \vec{\tilde{r}}, \rho|_{K_H^0}, \vec{\phi}_H)$, where $\vec{H}$, $\vec{\tilde{r}}$ and $\vec{\phi}_H$ are as per [Bourgeois 2021, Theorem 4.1]. Then, it follows from [Bourgeois 2021, Theorems 5.7 and 5.8] that

$$\pi_{(S,\theta)}|_{H(F)} = \pi_G(\Psi_{(S,\theta)})|_{H(F)} \simeq \bigoplus_{l\in L} {}^l\pi_H(\Psi_{(S,\theta)}^H),$$

where L is a set of coset representatives of $H(F)\backslash G(F)/K^d$.

We have that the elements $\vec{H}$, y, $\vec{\tilde{r}}$ and $\vec{\phi}_H$ from the datum $\Psi_{(S,\theta)}^H$ also appear in the datum $\Psi_{(S_H,\theta_H)}$. Indeed, $\vec{H}$ is the twisted Levi sequence associated to S_H by [Bourgeois 2021, Theorem 2.3] and the discussion preceding it, and the point y is the vertex of $\mathcal{B}(H,F)$ associated to S_H by [Bourgeois 2021, Lemma 7.1]. The character sequence $\vec{\phi}_H$ clearly satisfies the first two axioms to be a Howe factorization of θ_H, and genericity is given by [Bourgeois 2021, Proposition 4.7]. Therefore, assembling these pieces along with the construction from Figure 2, we have

$$\Psi_{(S_H,\theta_H)} = \big(\vec{H}, y, \vec{\tilde{r}}, \mathrm{Ind}_{S_H(F)H^0(F)_{y,0}}^{K_H^0} \kappa_{(S_H,\theta_H)}, \vec{\phi}_H\big).$$

Applying the Mackey decomposition as in the proof of [Bourgeois 2021, Proposition 7.5], we have

$$\rho|_{K_H^0} \simeq \bigoplus_{\ell\in\mathcal{L}} {}^\ell\mathrm{Ind}_{S_H(F)H^0(F)_{y,0}}^{K_H^0} \kappa_{(S_H,\theta_H)},$$

where $\mathcal{L}$ is a set of coset representatives of $K_H^0 \backslash K^0 / S(F) G^0(F)_{y,0}$. Therefore,

$$\pi_H(\Psi_{(S,\theta)}^H) \simeq \bigoplus_{\ell \in \mathcal{L}} {}^{\ell}\pi_H(\Psi_{(S_H,\theta_H)}) = \bigoplus_{\ell \in \mathcal{L}} {}^{\ell}\pi_{(S_H,\theta_H)},$$

which implies

$$\pi_{(S,\theta)}|_{H(F)} \simeq \bigoplus_{l \in L} \bigoplus_{\ell \in \mathcal{L}} {}^{l\ell}\pi_{(S_H,\theta_H)}.$$

Using Lemma 3.18, one rewrites $\mathcal{L}$ as $K_H^0 \backslash K^0 / S(F)$. We claim that $L\mathcal{L} = \{l\ell : l \in L,\ \ell \in \mathcal{L}\}$ is a set of coset representatives of $H(F) \backslash G(F) / S(F)$, which we denote by D, allowing us to write

$$\pi_{(S,\theta)} \simeq \bigoplus_{d \in D} {}^{d}\pi_{(S_H,\theta_H)}.$$

To prove this last claim, we set $N = H(F)$, $A = G(F)$, $B = K^d$, $\bar{N} = K_H^0$, $\bar{A} = K^0$ and $\bar{B} = S(F)$, and show that N, A, B, $\bar{N}$, $\bar{A}$, $\bar{B}$ satisfy the hypotheses of Lemma A.4(1). It is clear that N and $\bar{N}$ are normal subgroups of A and $\bar{A}$, respectively, and that $\bar{A} \subseteq B$ and $N \cap \bar{A} = \bar{N}$. It remains to show that $B/(N \cap B)\bar{B} \simeq \bar{A}/\bar{N}\bar{B}$.

Setting $J_H = H^1(F)_{y,r_0/2} \cdots H^d(F)_{y,r_{d-1}/2}$, we have $K_H^d = K_H^0 J_H$ by definition, and $K^d = K^0 J_H$ as per [Bourgeois 2021, Proof of Proposition 5.1]. It follows that

$$B/(N \cap B)\bar{B} = K^d / K_H^d S(F) = K^0 J_H / K_H^0 J_H S(F).$$

Given that $S(F)$ is in the stabilizer of y, we have that ${}^sH^i(F)_{y,r} = H^i(F)_{s \cdot y,r} = H^i(F)_{y,r}$ for all $s \in S(F)$, $r \geq 0$, $0 \leq i \leq d$. Therefore, $S(F)$ normalizes J_H and $J_H S(F) = S(F) J_H$, allowing us to write

$$B/(N \cap B)\bar{B} = K^0 J_H / (K_H^0 S(F)) J_H.$$

Applying Lemma A.3, we obtain

$$B/(N \cap B)\bar{B} \simeq K^0 / K_H^0 S(F)(K^0 \cap J_H) = K^0 / K_H^0 S(F),$$

where the last equality follows from the fact that $K^0 \cap J_H \subset K_H^0$. Thus,

$$B/(N \cap B)\bar{B} \simeq \bar{A}/\bar{N}\bar{B}. \qquad \square$$

Proof of Theorem 3.1. Setting $G = \underline{G}$ and $H = G_Z$ in Theorem 3.19, we have

$$\pi_{(\underline{S},\underline{\theta})} \circ \eta = \left(\mathrm{Res}_{G_Z(F)}^{\underline{G}(F)} \pi_{(\underline{S},\underline{\theta})}\right) \circ \eta \simeq \left(\bigoplus_{\underline{c} \in \underline{C}} {}^{\underline{c}}\pi_{(S_Z,\theta_Z)} \right) \circ \eta,$$

where $S_Z = \underline{S} \cap G_Z$, $\theta_Z = \theta|_{S_Z}$ and $\underline{C}$ is a set consisting of coset representatives of $G_Z(F) \backslash \underline{G}(F) / \underline{S}(F)$.

By Lemma 2.8, ${}^{\underline{c}}\pi_{(S_Z,\theta_Z)} \circ \eta = \pi_{(S_Z,\theta_Z)} \circ \eta \circ \underline{\mathrm{Ad}}(\underline{c}^{-1})$. Using this last equality, and applying Theorem 3.17, it follows that

$$\begin{aligned} \pi_{(\underline{S},\underline{\theta})} \circ \eta &\simeq \bigoplus_{\underline{c}\in\underline{C}} (\pi_{(S_Z,\theta_Z)} \circ \eta) \circ \underline{\mathrm{Ad}}(\underline{c}^{-1}) \\ &\simeq \bigoplus_{\underline{c}\in\underline{C}} \Big(\bigoplus_{d_Z \in D_Z} \pi_{(S,\theta)} \circ \underline{\mathrm{Ad}}(d_Z^{-1}) \Big) \circ \underline{\mathrm{Ad}}(\underline{c}^{-1}) \\ &= \bigoplus_{\underline{c}\in\underline{C}} \bigoplus_{d_Z \in D_Z} \pi_{(S,\theta)} \circ \underline{\mathrm{Ad}}((\underline{c} d_Z)^{-1}), \end{aligned}$$

where D_Z is a set of coset representatives of $\eta(G(F))\backslash G_Z(F)/S_Z(F)$. Setting $N = G_Z(F)$, $A = \underline{G}(F)$, $B = \underline{S}(F)$, $\bar{N} = \eta(G(F))$, $\bar{A} = G_Z(F)$, $\bar{B} = S_Z(F)$, one sees from Lemma A.4(2) that $\{\underline{c} d_Z : \underline{c} \in \underline{C},\ d_Z \in D_Z\}$ is a set of coset representatives of $\eta(G(F))\backslash \underline{G}(F)/\underline{S}(F)$. The decomposition formula thus follows. □

4. Functoriality for supercuspidal L-packets

One of the main aims of this paper is to show that Kaletha's supercuspidal L-packets satisfy a certain functoriality property. More specifically, the goal of this section is to prove the following theorem.

Theorem 4.1. *Suppose G is quasisplit and splits over a tamely ramified extension. Suppose further that the residual characteristic p of F does not divide the order of the Weyl group of G. Let $\eta : G \to \underline{G}$ be an F-morphism of connected reductive F-groups such that*

(i) *the kernel of $d\eta : \mathrm{Lie}(G) \to \mathrm{Lie}(\underline{G})$ is central,*

(ii) *the cokernel of η is an abelian F-group.*

Let $\underline{\varphi} \in \Phi_{\mathrm{sc}}(\underline{G})$ and set $\varphi = {}^L\eta \circ \underline{\varphi}$. Then for all $\underline{\pi} \in \Pi_{\underline{\varphi}}$, $\underline{\pi} \circ \eta$ is the direct sum of finitely many irreducible supercuspidal representations belonging to Π_{φ}.

Recall from Section 2.5 that the packet $\Pi_{\underline{\varphi}}$ is constructed from $\underline{\varphi}$ by first taking its associated supercuspidal L-packet datum $(\underline{S}, \hat{\underline{j}}, \underline{\chi}_0, \underline{\theta})$, and then taking the irreducible components of $\pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})}$ as $\underline{j}$ varies over $\underline{\mathcal{J}}_F$. Similarly, we let (S, j, χ_0, θ) be the supercuspidal L-packet datum associated to φ so that Π_{φ} consists of the irreducible components of $\pi_{(jS,j\theta)}$ as j varies over $\mathcal{J}_F$

The strategy to prove Theorem 4.1 is to apply the decomposition formula from Section 3 on the representation $\pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})} \circ \eta$. However, to do so, we must first find an F-nonsingular elliptic pair of G that relates to $(\underline{j}\underline{S}, \underline{j}\underline{\theta})$ in the sense of Theorem 3.1. To achieve this, we establish the relationship between the supercuspidal L-packet data $(\underline{S}, \hat{\underline{j}}, \underline{\chi}_0, \underline{\theta})$ and (S, j, χ_0, θ). In particular, the relationship between these data is induced by the map ${}^L\eta$, a statement we will make precise with Theorem 4.2 and illustrate in Figure 6.

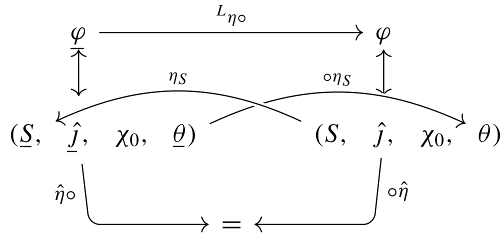

Figure 6. Summary of the relationship between the supercuspidal L-packet data associated to $\underline{\varphi}$ and φ. The map $^L\eta$ is constructed from a map $\hat{\eta}$, dual to η, which we recall in Section 4.1. The map η_S is the map dual to $\hat{\eta}|_{\widehat{\underline{S}}}$.

In Section 4.1, we prove the relationship between the L-packet data, and then prove Theorem 3.1 in Section 4.2.

4.1. ***Matching the supercuspidal L-packet data.*** Let $\underline{\varphi} \in \Phi_{\mathrm{sc}}(\underline{G})$ and $\varphi = {}^L\eta \circ \underline{\varphi}$. The goal of this section is to match the corresponding parameterizing data for $\underline{\varphi}$ and φ through $^L\eta$. This matching is crucial for establishing the relationship between $\Pi_{\underline{\varphi}}$ and Π_{φ}.

The map $^L\eta$ is defined by $^L\eta(g, w) = (\hat{\eta}(\underline{g}), w)$ for all $\underline{g} \in \widehat{\underline{G}}$ and $w \in W_F$, and so we begin with a review of the map $\hat{\eta} : \widehat{\underline{G}} \to \widehat{G}$ [Springer 1979, Sections 1 and 2]. Recall that we are assuming that G is quasisplit. We may therefore fix a Borel subgroup $B \subset G$ defined over F, and fix a maximally F-split maximal torus T of G with $T \subset B$. This choice of Borel pair (B, T) is equivalent to fixing a based root datum for G. The Γ-equivariant map η carries T to a maximally F-split torus $\eta(T)$ of $\eta(G) \supset [G, G]$. Set $\underline{T} = \eta(T)Z(\underline{G})$ and $\underline{B} = \eta(B)Z(\underline{G})$. Then $(\underline{B}, \underline{T})$ is a Borel pair for $\underline{G}$ which is defined over F [Springer 2009, Corollary 8.1.6]. Thus, we have fixed based root data for both G and $\underline{G}$, and η induces a homomorphism between them, in the sense of [Jantzen 2003, II.1.13]. This homomorphism of root data determines a homomorphism from the dual based root datum of $\underline{G}$ to the dual based root datum of G. These two dual based data may be identified with the based root data arising from Γ-invariant Borel pairs $(\widehat{\underline{B}}, \widehat{\underline{T}})$ and $(\widehat{B}, \widehat{T})$ which are implicit in the definition of $^L\underline{G}$ and ^LG [Borel 1979, 2.3]. Under this identification and the assumptions of Theorem 4.1, the homomorphism of dual based root data produces the homomorphism $\hat{\eta} : \widehat{\underline{G}} \to \widehat{G}$ of algebraic groups with abelian kernel and cokernel [Jantzen 2003, Proposition II.1.14]. It is only unique up to conjugation by $\widehat{T}$.

A change in the choice of based root data above has the effect of conjugating $\hat{\eta}$ by an element of $\widehat{G}$. This change has no effect on the equivalence classes of the objects under consideration. However, even for fixed based root data and a fixed choice of $\hat{\eta}$, there remains an ambiguity between η and $\hat{\eta}$. Indeed, conjugating η by an element $\underline{t} \in \underline{T}$ such that $\underline{t}Z(\underline{G}) \in (\underline{T}/Z(\underline{G}))(F)$ produces another F-homomorphism whose dual is identical to $\hat{\eta}$. As observed in [Solleveld 2020, Section 3], this ambiguity is consequential in matching the parameters of the L-packets under consideration.

In order to be more precise about these matters, we fix some pinnings. We extend the Γ-invariant Borel pair (B, T) to a Γ-invariant pinning $(T, B, \{X_\alpha\})$ of G. The application of η to this pinning fixes a Γ-invariant pinning $\big(\underline{T}, \underline{B}, \{\eta(X_\alpha)\}\big)$ of $\underline{G}$. It follows from [Solleveld 2020, Theorem 3.2] that $\eta : G \to \underline{G}$ is the unique F-homomorphism in its $(\underline{G}/Z(\underline{G}))(F)$-conjugacy class which carries the former pinning to the latter (see the discussion preceding [Solleveld 2020, Theorem 6.2]). Underlying the definition of ${}^LG$ and ${}^L\underline{G}$ are respective pinnings $(\widehat{T}, \widehat{B}, \{Y_{\hat{\alpha}}\})$ of $\widehat{G}$ and $(\widehat{\underline{T}}, \widehat{\underline{B}}, \{\underline{Y}_{\hat{\alpha}}\})$ of $\widehat{\underline{G}}$. The fixed pinnings determine $\hat{\eta}$ uniquely by the requirement that $\{\hat{\eta}(Y_{\hat{\alpha}})\} = \{\underline{Y}_{\hat{\alpha}}\}$. Furthermore, if $\eta' : G \to \underline{G}$ is another F-homomorphism satisfying the assumptions of Theorem 4.1 and $\widehat{\eta'} = \hat{\eta}$, then there is a unique element $\underline{t}' Z(\underline{G}) \in (\underline{T}/Z(\underline{G}))(F)$ such that $\eta' = \mathrm{Ad}(\underline{t}') \circ \eta$ [Solleveld 2020, Proposition 3.4].

Having set the foundation for the comparison between G and $\underline{G}$, let us return to the matching of the apposite parameters. They are to be matched as in Figure 6.

Theorem 4.2. *Let $\underline{\varphi} \in \Phi_{\mathrm{sc}}(\underline{G})$, $\varphi = {}^L\eta \circ \underline{\varphi}$ and $(\underline{S}, \hat{\underline{j}}, \underline{\chi}_0, \underline{\theta})$ and $(S, \hat{j}, \chi_0, \theta)$ be the associated supercuspidal L-packet data. Let $\mathcal{J}_F$ and $\underline{\mathcal{J}}_F$ be the sets of embeddings which parameterize Π_φ and $\Pi_{\underline{\varphi}}$, respectively. Then*

(a) $\hat{\eta}(\widehat{\underline{S}}) \subset \widehat{S}$, $\underline{\chi}_0 = \chi_0$ *and* $\hat{\eta} \circ \hat{\underline{j}} = \hat{j} \circ \hat{\eta}$,

(b) $\theta = \underline{\theta} \circ \eta_S$, *where η_S is the dual map of $\hat{\eta}|_{\widehat{\underline{S}}} : \widehat{\underline{S}} \to \widehat{S}$, and*

(c) *for all $\underline{j} \in \underline{\mathcal{J}}_F$, there exists $j \in \mathcal{J}_F$ such that $\eta(jS) \subset \underline{j}\underline{S}$ and $j\theta = \underline{j}\underline{\theta} \circ \eta$.*

The proof of this theorem will be divided into three parts. Proposition 4.4 will give the relationship between the tori, and consequently the embeddings and χ-data. Proposition 4.5 will give the relationship between the characters and Proposition 4.7 will provide the statement regarding the sets $\mathcal{J}_F$ and $\underline{\mathcal{J}}_F$.

Before proceeding to the statements of these propositions, we begin with a useful result, which can be shown using [Humphreys 1995, Theorem 2.2] and the presentation of reductive groups in terms of generators from [Humphreys 1975, Theorem 26.3].

Proposition 4.3. *Let T be a maximal torus of a connected reductive group G', and assume H is a subtorus of T. Then $T = \mathrm{Cent}(H, G')$ if and only if for every $\alpha \in R(G', T)$ there exists $h_\alpha \in H$ such that $\alpha(h_\alpha) \neq 1$.*

Proposition 4.4 (Theorem 4.2(a)). *Let $\underline{S}$ and S be as in Theorem 4.2. Then $\hat{\eta}(\widehat{\underline{S}}) \subset \widehat{S}$.*

Proof. Recall that $\widehat{S} = \mathrm{Cent}(\widehat{C}, \widehat{M})$, where $\widehat{M} = \mathrm{Cent}(\varphi(P_F), \widehat{G})^\circ$ and $\widehat{C} = \mathrm{Cent}(\varphi(I_F), \widehat{G})^\circ$. Similarly, $\widehat{\underline{S}} = \mathrm{Cent}(\widehat{\underline{C}}, \widehat{\underline{M}})$, where $\widehat{\underline{M}} = \mathrm{Cent}(\underline{\varphi}(P_F), \widehat{\underline{G}})^\circ$ and $\widehat{\underline{C}} = \mathrm{Cent}(\underline{\varphi}(I_F), \widehat{\underline{G}})^\circ$. We start by showing that

$$\hat{\eta}(\widehat{\underline{M}}) = \widehat{M} \cap \hat{\eta}(\widehat{\underline{G}}).$$

We have that $\underline{\varphi}(P_F)$ is contained in some maximal torus $\widehat{\underline{\mathcal{T}}}$ of $\widehat{\underline{G}}$ [Kaletha 2021, Lemma 4.1.3], and therefore $\varphi(P_F)$ is contained in the maximal torus $\hat{\eta}(\widehat{\underline{\mathcal{T}}})$ of $\hat{\eta}(\widehat{\underline{G}})$. Since $\hat{\eta}(\widehat{\underline{G}})$ contains $[\widehat{G}, \widehat{G}]$, we have that $\hat{\eta}(\widehat{\underline{\mathcal{T}}}) = \widehat{\mathcal{T}} \cap \hat{\eta}(\widehat{\underline{G}})$ for some maximal torus $\widehat{\mathcal{T}}$ of $\widehat{G}$ [Bourgeois 2021, Theorem 2.2]. By definition, we have

$$\widehat{\underline{M}} = \mathrm{Cent}(\underline{\varphi}(P_F), \widehat{\underline{G}})^\circ = \Bigl(\bigcap_{s \in \underline{\varphi}(P_F)} \mathrm{Cent}(s, \widehat{\underline{G}})\Bigr)^\circ.$$

Using the description from [Humphreys 1995, Theorem 2.2] for each set $\mathrm{Cent}(s, \widehat{\underline{G}})$, $s \in \underline{\varphi}(P_F)$, it follows that

$$\widehat{\underline{M}} = \langle \widehat{\underline{\mathcal{T}}}, U_\beta : \beta(s) = 1 \text{ for all } s \in \underline{\varphi}(P_F) \rangle,$$

where U_α denotes the root subgroup of $\widehat{\underline{G}}$ associated to the root $\alpha \in R(\widehat{\underline{G}}, \widehat{\underline{\mathcal{T}}})$.

Using a similar argument, and given that the root systems of $\widehat{\underline{G}}$ and $\widehat{G}$ are identified, we have that

$$\widehat{M} = \langle \widehat{\mathcal{T}}, \hat{\eta}(U_\beta) : \beta(s) = 1 \text{ for all } s \in \varphi(P_F) \rangle.$$

We then deduce from [Bourgeois 2021, Section 2B] that $\hat{\eta}(\widehat{\underline{M}}) = \widehat{M} \cap \hat{\eta}(\widehat{\underline{G}})$. Analogously, one has $\hat{\eta}(\widehat{\underline{C}}) = \widehat{C} \cap \hat{\eta}(\widehat{\underline{G}})$.

Since $\widehat{\underline{S}} = \mathrm{Cent}(\widehat{\underline{C}}, \widehat{\underline{M}})$ it follows from Proposition 4.3 that for all $\alpha \in R(\widehat{\underline{M}}, \widehat{\underline{S}})$ there exists $c_\alpha \in \widehat{\underline{C}}$ such that $\alpha(c_\alpha) \neq 1$. Applying $\hat{\eta}$, we have that for all $\alpha \in R(\hat{\eta}(\widehat{\underline{M}}), \hat{\eta}(\widehat{\underline{S}}))$, there exists $\hat{\eta}(c_\alpha) \in \hat{\eta}(\widehat{\underline{C}})$ such that $\alpha(\hat{\eta}(c_\alpha)) \neq 1$. Reapplying Proposition 4.3, we obtain $\hat{\eta}(\widehat{\underline{S}}) = \mathrm{Cent}(\hat{\eta}(\widehat{\underline{C}}), \hat{\eta}(\widehat{\underline{M}}))$. It follows that

$$\widehat{S} \cap \hat{\eta}(\widehat{\underline{G}}) = \mathrm{Cent}(\widehat{C}, \widehat{M}) \cap \hat{\eta}(\widehat{\underline{G}}) = \mathrm{Cent}(\widehat{C}, \hat{\eta}(\widehat{\underline{M}})) \subset \mathrm{Cent}(\hat{\eta}(\widehat{\underline{C}}), \hat{\eta}(\widehat{\underline{M}})) = \hat{\eta}(\widehat{\underline{S}}).$$

Since both $\widehat{S} \cap \hat{\eta}(\widehat{\underline{G}})$ and $\hat{\eta}(\widehat{\underline{S}})$ are maximal tori of $\hat{\eta}(\widehat{\underline{G}})$, we conclude that they are equal, and therefore $\hat{\eta}(\widehat{\underline{S}}) \subset \widehat{S}$. □

Having $\hat{\eta}(\widehat{\underline{S}}) \subset \widehat{S}$ implies that the root systems $R(\widehat{\underline{G}}, \widehat{\underline{S}})$ and $R(\widehat{G}, \widehat{S})$, together with their Γ-actions, are identified, which allows us to choose $\chi_0 = \underline{\chi}_0$, as the χ-data are parameterized by roots. Also, $\underline{\hat{j}} : \widehat{\underline{S}} \to \widehat{\underline{G}}$ and $\hat{j} : \widehat{S} \to \widehat{G}$ are simply inclusions. This means we have the commutative diagram

$$\begin{array}{ccc} \widehat{\underline{S}} & \xrightarrow{\ \underline{\hat{j}}\ } & \widehat{\underline{G}} \\ \downarrow{\scriptstyle \hat{\eta}} & & \downarrow{\scriptstyle \hat{\eta}} \\ \widehat{S} & \xrightarrow{\ \hat{j}\ } & \widehat{G} \end{array}$$

Proposition 4.5 (Theorem 4.2(b)). *Let $\theta, \underline{\theta}$ and η_S be as in Theorem 4.2. Then,* $\theta = \underline{\theta} \circ \eta_S$.

Proof. Using the χ-data as in [Langlands and Shelstad 1987, Section 2.6], the above diagram extends into the commutative diagram

$$\begin{array}{ccc} \widehat{\underline{S}} \rtimes W_F & \xrightarrow{{}^L\underline{j}} & \widehat{\underline{G}} \rtimes W_F \\ \big\downarrow {}^L\eta & & \big\downarrow {}^L\eta \\ \widehat{S} \rtimes W_F & \xrightarrow{{}^Lj} & \widehat{G} \rtimes W_F \end{array}$$

where ${}^L\eta(g, w) = (\hat{\eta}(g), w)$ for all $g \in \widehat{\underline{G}}$, $w \in W_F$.

Following [Kaletha 2021, Proposition 4.1.8], $\mathrm{Im}(\underline{\varphi}) \subset \mathrm{Im}({}^L\underline{j})$ and $\mathrm{Im}(\phi) \subset \mathrm{Im}({}^Lj)$, meaning that $\underline{\varphi} = {}^L\underline{j} \circ \varphi_{\underline{S}}$ and $\varphi = {}^Lj \circ \varphi_S$ for some L-parameters $\varphi_{\underline{S}}$ and φ_S of $\underline{S}$ and S, respectively. We claim that $\varphi_S = {}^L\eta \circ \varphi_{\underline{S}}$. Indeed, by definition, $\varphi = {}^L\eta \circ \underline{\varphi}$, which implies ${}^Lj \circ \varphi_S = {}^L\eta \circ {}^L\underline{j} \circ \varphi_{\underline{S}}$. Using the commutative diagram above, it follows that ${}^Lj \circ \varphi_S = {}^Lj \circ {}^L\eta \circ \varphi_{\underline{S}}$. Given that ${}^Lj$ is an embedding, it is injective by definition, which implies that $\varphi_S = {}^L\eta \circ \varphi_{\underline{S}}$ as claimed.

By definition, $\underline{\theta}$ and θ are the characters which correspond to $\varphi_{\underline{S}}$ and φ_S, respectively, under the LLC for tori. Since L-packets of tori are singletons, we apply the functoriality property for the LLC of tori to conclude that $\theta = \underline{\theta} \circ \eta_S$. □

We now arrive to the final statement of Theorem 4.2 which matches the embeddings in $\mathcal{J}_F$ and $\underline{\mathcal{J}}_F$. The description of these embeddings depends on our fixed pinnings. We will require the following lemma for our proof.

Lemma 4.6. *Let (S, θ) and $(\underline{S}, \underline{\theta})$ be tame F-nonsingular elliptic pairs of G and $\underline{G}$, respectively, which satisfy $\eta(S) \subset \underline{S}$ and $\theta = \underline{\theta} \circ \eta$. Let ϵ and $\underline{\epsilon}$ be the characters from* [Fintzen et al. 2023, Section 4.1] *(recalled at the end of Section 2.4) constructed from S and $\underline{S}$, respectively. Then $\epsilon = \underline{\epsilon} \circ \eta$.*

Proof. Let $(G^0, \dots, G^d)$ and $(\underline{G}^0, \dots, \underline{G}^d)$ be the twisted Levi sequences obtained from S and $\underline{S}$, respectively. As defined in [Fintzen et al. 2023, p.2259], we have that $\epsilon = \prod_{i=1}^d \epsilon^{G^i/G^{i-1}}$, where $\epsilon^{G^i/G^{i-1}}$ is the quadratic character of K^d that is trivial on $G^1(F)_{y,r_0/2} \cdots G^d(F)_{y,r_d/2}$ and whose restriction to K^0 is given by $\epsilon_y^{G^i/G^{i-1}}$ defined in [Fintzen et al. 2023, Definition 4.1.10]. The character $\epsilon_y^{G^i/G^{i-1}}$ is essentially just a composition of a sign character constructed from the adjoint groups of G^i and G^{i-1}, and the adjoint map of G^i. The character $\underline{\epsilon}$ is defined similarly. Given that $\eta(G^i) = \underline{G}^i \cap \eta(G)$ (Lemma 3.3 and Theorem 3.19), it follows that G^i and $\underline{G}^i$ have the same adjoint group and that the adjoint map of G^i is the composition of the adjoint map of $\underline{G}^i$ with η. It follows that $\epsilon_y^{G^i/G^{i-1}} = \underline{\epsilon}_{\underline{y}}^{\underline{G}^i/\underline{G}^{i-1}} \circ \eta$ for all $1 \le i \le d$ and therefore $\epsilon = \underline{\epsilon} \circ \eta$. □

Proposition 4.7 (Theorem 4.2(c)). *For all $\underline{j} \in \underline{\mathcal{J}}_F$, there exists $j \in \mathcal{J}_F$ such that $\eta(jS) \subset \underline{j}\underline{S}$ and $j\theta = \underline{j}\underline{\theta} \circ \eta$.*

Proof. Our fixed Γ-invariant pinnings satisfy $\eta(T)=\underline{T}\cap\eta(G)$ and $\hat{\eta}(\widehat{\underline{T}})=\widehat{T}\cap\hat{\eta}(\widehat{\underline{G}})$. Following [Kaletha 2019, Section 5.1], we may describe $\underline{\mathcal{J}}$ and $\mathcal{J}$ as follows. Choose $\hat{\underline{\imath}}$ in $\widehat{\underline{\mathcal{J}}}$ such that $\hat{\underline{\imath}}(\widehat{\underline{S}})=\widehat{\underline{T}}$ and define $\underline{i}$ to be the inverse of the isomorphism $\underline{T}\to\underline{S}$ induced by $\hat{\underline{\imath}}$. We have that $\hat{\underline{\imath}}=\mathrm{Ad}(\hat{g})\circ\hat{\underline{\jmath}}$ for some $\hat{g}\in\widehat{\underline{G}}$. Let $\hat{\imath}\in\widehat{\mathcal{J}}$ be defined by $\hat{\imath}=\mathrm{Ad}(\hat{\eta}(\hat{g}))\circ\hat{\jmath}$. Since $\hat{\eta}\circ\hat{\underline{\jmath}}=\hat{\jmath}\circ\hat{\eta}$, we have the commutative diagram

$$\begin{array}{ccc} \widehat{\underline{S}} & \xrightarrow{\ \hat{\underline{\imath}}\ } & \widehat{\underline{T}} \\ \big\downarrow{\scriptstyle \hat{\eta}} & & \big\downarrow{\scriptstyle \hat{\eta}} \\ \widehat{S} & \xrightarrow{\ \hat{\imath}\ } & \widehat{T} \end{array} \tag{9}$$

It follows that

$$\widehat{T}\cap\hat{\eta}(\widehat{\underline{G}})=\hat{\eta}(\widehat{\underline{T}})=\hat{\eta}(\hat{\underline{\imath}}(\widehat{\underline{S}}))=\hat{\imath}(\hat{\eta}(\widehat{\underline{S}}))\subset\hat{\imath}(\widehat{S}).$$

Since we know $\hat{\imath}(\widehat{S})$ has to be a maximal torus of $\widehat{G}$, we conclude from [Bourgeois 2021, Theorem 2.2] that $\hat{\imath}(\widehat{S})=\widehat{T}$. Therefore, $\underline{\mathcal{J}}$ corresponds to the $\underline{G}(F^{\mathrm{sep}})$-conjugacy class of $\underline{i}$, and $\mathcal{J}$ corresponds to the $G(F^{\mathrm{sep}})$-conjugacy class of i.

Now, given $\underline{j}\in\underline{\mathcal{J}}_F$, we have that $\underline{j}=\mathrm{Ad}(\underline{g})\circ\underline{i}$ for some $\underline{g}\in\underline{G}(F^{\mathrm{sep}})$. Using the fact that $\underline{G}=Z(\underline{G})G_Z$, we may assume without loss of generality that $\underline{g}\in G_Z(F^{\mathrm{sep}})$. Let g be any preimage of $\underline{g}$ in $G(F^{\mathrm{sep}})$ by η and set $j=\mathrm{Ad}(g)\circ i$. By taking the dual of diagram (9), we have

$$\begin{array}{ccc} \underline{S} & \xrightarrow{\ \underline{i}\ } & \underline{T} \\ {\scriptstyle \eta_S}\big\uparrow & & {\scriptstyle \eta}\big\uparrow \\ S & \xrightarrow{\ i\ } & T \end{array} \tag{10}$$

Here, η_S is the dual map of $\hat{\eta}|_{\widehat{\underline{S}}}:\widehat{\underline{S}}\to\widehat{S}$. It follows that

$$\eta(jS)=\eta(giSg^{-1})=\eta(g)\eta(iS)\eta(g)^{-1}=\underline{g}\underline{i}(\eta_S(S))\underline{g}^{-1}=\underline{j}(\eta_S(S))\subset\underline{j}\underline{S}.$$

Since $\underline{j}$ and η are defined over F, we have that $j\in\mathcal{J}_F$. Indeed, by [Dillery 2023, Lemma 6.2], which generalizes [Kottwitz 1982, Corollary 2.2] to arbitrary local fields, there exists $h\in G(F^{\mathrm{sep}})$ such that $\mathrm{Ad}(h)\circ j$ is defined over F. Then $\mathrm{Ad}(\eta(h))\circ\underline{j}$ is also defined over F, implying $\sigma\big(\mathrm{Ad}(\eta(h))\circ\underline{j}\big)\sigma^{-1}=\mathrm{Ad}(\eta(h))\circ\underline{j}$ for all $\sigma\in\Gamma$. Equivalently, $\eta(h)^{-1}\sigma(\eta(h))=\eta(h^{-1}\sigma(h))\in\underline{j}\underline{S}$ for all $\sigma\in\Gamma$. This implies $h^{-1}\sigma(h)\in jS$, and therefore $\mathrm{Ad}(h)\circ j=\mathrm{Ad}(\sigma(h))\circ j$ for all $\sigma\in\Gamma$. Using the fact that $\mathrm{Ad}(h)\circ j$ is defined over F, we rewrite this last equality as $\sigma(\mathrm{Ad}(h)\circ j)\sigma^{-1}=\mathrm{Ad}(\sigma(h))\circ j$ for all $\sigma\in\Gamma$. It follows that $\mathrm{Ad}(\sigma(h))\circ\sigma j\sigma^{-1}=\mathrm{Ad}(\sigma(h))\circ j$, and therefore $\sigma j\sigma^{-1}=j$ for all $\sigma\in\Gamma$.

It remains to show that $j\theta=\underline{j}\underline{\theta}\circ\eta$. We have that $j\theta=\theta\circ j^{-1}\cdot\epsilon_j$ and $\underline{j}\underline{\theta}=\underline{\theta}\circ\underline{j}^{-1}\cdot\epsilon_{\underline{j}}$, where ϵ_j and $\epsilon_{\underline{j}}$ are the characters from [Fintzen et al. 2023, Section 4.1] constructed from jS and $\underline{j}\underline{S}$, respectively, which we briefly recalled at the end of Section 2.4. By what precedes, we have that $\eta(jS) \subset \underline{j}\underline{S}$ and $\underline{\theta} \circ \underline{j}^{-1} \circ \eta = \underline{\theta} \circ \eta_S \circ j^{-1} = \theta \circ j^{-1}$. Using Lemma 4.6, we have $\epsilon_j = \epsilon_{\underline{j}} \circ \eta$, and thus $j\theta = \underline{j}\underline{\theta} \circ \eta$. □

4.2. *The proof of Theorem 4.1.* In this section, we begin with the statement of a lemma, after which we will combine the results of the previous section with the decomposition formula from Section 3 to prove Theorem 4.1.

Lemma 4.8. *Let (S, θ) be a tame F-nonsingular elliptic pair of G, and let $c \in G$ be such that* $\mathrm{Ad}(c)$ *is defined over F. Then, the following statements hold.*

(1) ${}^c\pi_{(S,\theta)} \simeq \pi_{({}^cS, {}^c\theta)}$.

(2) *If ϵ is the character from* [Fintzen et al. 2023, Section 4.1] *constructed from S, then ${}^c\epsilon$ is the character from* [Fintzen et al. 2023, Section 4.1] *constructed from ${}^cS$.*

Proof. We have that $\mathrm{Ad}(c) : G \to G$ is a map which satisfies hypotheses (i) and (ii) of Theorem 4.1, having trivial kernel and trivial cokernel.

For (1), we apply Theorem 3.17, in which we set $\eta = \mathrm{Ad}(c)$, $S_Z = \mathrm{Ad}(c)(S) = {}^cS$ and $\theta_Z = \theta \circ \mathrm{Ad}(c^{-1}) = {}^c\theta$. We obtain as a result $\pi_{({}^cS, {}^c\theta)} \circ \mathrm{Ad}(c) \simeq \pi_{(S,\theta)}$, or equivalently, $\pi_{({}^cS, {}^c\theta)} \simeq {}^c\pi_{(S,\theta)}$.

For (2), we apply Lemma 4.6, in which we set $\eta = \mathrm{Ad}(c)$, $\underline{S} = {}^cS$ and $\underline{\theta} = {}^c\theta$. □

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $(\underline{S}, \hat{\underline{j}}, \chi_0, \underline{\theta})$ and $(S, \hat{j}, \chi_0, \theta)$ be the supercuspidal L-packet data associated to $\underline{\varphi}$ and φ, respectively. By construction of $\Pi_{\underline{\varphi}}$, we have that $\underline{\pi} \subset \pi_{(\underline{j}\underline{S}, \underline{j}\underline{\theta})}$ for some $\underline{j} \in \underline{\mathcal{J}}_F$. By Theorem 4.2, there exists $j \in \mathcal{J}_F$ such that $\eta(jS) \subset \underline{j}\underline{S}$ and $j\theta = \underline{j}\underline{\theta} \circ \eta$. By Theorem 3.1, it follows that

$$\underline{\pi} \circ \eta \subset \pi_{(\underline{j}\underline{S}, \underline{j}\underline{\theta})} \circ \eta = \bigoplus_{\underline{c} \in \underline{C}} \pi_{(jS, j\theta)} \circ \underline{\mathrm{Ad}}(\underline{c}^{-1}), \tag{11}$$

where $\underline{C}$ is a set of coset representatives of $\eta(G(F)) \backslash \underline{G}(F) / \underline{j}\underline{S}(F)$. Recall from Section 2.3 that $\underline{\mathrm{Ad}}(\underline{c}^{-1}) = \mathrm{Ad}(c^{-1})$, where $c \in G$ satisfies $\underline{c} = \eta(c)z$ for some $z \in Z(\underline{G})$. Using Lemma 4.8(1) one sees that

$$\pi_{(jS, j\theta)} \circ \underline{\mathrm{Ad}}(\underline{c}^{-1}) = {}^c\pi_{(jS, j\theta)} \simeq \pi_{({}^cjS, {}^cj\theta)},$$

where ${}^cjS = (\mathrm{Ad}(c) \circ j)S$ and

$${}^cj\theta = j\theta \circ \mathrm{Ad}(c^{-1}) = ((\theta \circ j^{-1}) \cdot \epsilon_j) \circ \mathrm{Ad}(c^{-1}) = (\theta \circ j^{-1} \circ \mathrm{Ad}(c^{-1})) \cdot (\epsilon_j \circ \mathrm{Ad}(c^{-1})).$$

By Lemma 4.8(2), $\epsilon_j \circ \mathrm{Ad}(c^{-1}) = \epsilon_{\mathrm{Ad}(c) \circ j}$ so that

$${}^cj\theta = \left(\theta \circ (\mathrm{Ad}(c) \circ j)^{-1}\right) \cdot \epsilon_{\mathrm{Ad}(c) \circ j} = (\mathrm{Ad}(c) \circ j)\theta.$$

Since $\mathrm{Ad}(c)$ is defined over F (Lemma 2.6), $\mathrm{Ad}(c) \circ j \in \mathcal{J}_F$, and therefore

$$[\pi_{(jS, j\theta)} \circ \underline{\mathrm{Ad}}(\underline{c}^{-1})] = [\pi_{((\mathrm{Ad}(c) \circ j)S, (\mathrm{Ad}(c) \circ j)\theta)}] \subset \Pi_\varphi$$

for all $\underline{c} \in \underline{C}$. Thus, all irreducible components of $\underline{\pi} \circ \eta$ belong to Π_φ. □

5. Specializing to regular supercuspidal parameters

Part of the local Langlands conjectures is a correspondence between the irreducible representations in an L-packet Π_φ and the irreducible representations of the component group of $\mathrm{Cent}(\varphi(W_F), \widehat{G})$ [Vogan 1993, Conjecture 1.15]. In this section, we review this correspondence for *regular* supercuspidal L-parameters.

The regular supercuspidal L-parameters [Kaletha 2019, Definition 5.2.3] are an important subclass of the supercuspidal L-parameters. They are also easier to study, as their corresponding L-packets are simpler to describe (see (12) below), and their corresponding component groups are abelian [Kaletha 2019, Lemma 5.3.4].

Under the assumption that φ and $\underline{\varphi}$ are both regular, we show how to reparameterize the elements of Π_φ and $\Pi_{\underline{\varphi}}$ in terms of characters of their respective component groups. From this reparameterization, we obtain an alternate formulation for the decomposition formula for $\underline{\pi} \circ \eta$, $\underline{\pi} \in \Pi_{\underline{\varphi}}$, obtained from Theorem 3.1. This reformulation amounts to a proof of [Solleveld 2020, Conjecture 2] for regular supercuspidal L-packets of quasisplit groups (Theorem 1.1, Proposition 5.12).

First we explicitly describe the regular supercuspidal L-parameters, their corresponding L-packet structure, and the relationship between the regularity of φ and $\underline{\varphi}$.

5.1. ***Regular L-Packets and conditions for regularity of φ and $\underline{\varphi}$.*** One way to describe the regular supercuspidal L-parameters is via the notion of regular supercuspidal L-packet data [Kaletha 2019, Definition 5.2.4]. A regular supercuspidal L-packet datum of G is a supercuspidal L-packet datum $(S, \hat{j}, \chi_0, \theta)$ (see Definition 2.9), with the stronger condition that (S, θ) is an extra regular elliptic pair in the sense of [Kaletha 2019, Definition 3.7.5]. In particular, this means that the character $\theta|_{S(F)_0}$ has trivial stabilizer for the action of $\Omega(S, G)(F) := (N_G(S)/S)(F)$.

By [Kaletha 2019, Proposition 5.2.7], there is a one-to-one correspondence between the $\widehat{G}$-conjugacy classes of regular supercuspidal L-parameters for G and isomorphism classes of regular supercuspidal L-packet data. Given a regular supercuspidal L-parameter φ of G with associated regular supercuspidal L-packet datum $(S, \hat{j}, \chi_0, \theta)$, the representations $\pi_{(jS, j\theta)}$ are irreducible for all $j \in \mathcal{J}_F$ [Kaletha 2019, Lemma 3.4.20]. Thus, the corresponding L-packet is

$$\Pi_\varphi = \{\pi_{(jS, j\theta)} : j \in \mathcal{J}_F\}, \tag{12}$$

where j is identified with its $G(F)$-conjugacy class and $\pi_{(jS, j\theta)}$ is identified with its equivalence class. Furthermore, as stated in [Kaletha 2019, Section 5.3; 2021, Section 4.2], the elements of Π_φ are in one-to-one correspondence with the elements of $\mathcal{J}_F$. The following lemma is a proof of this statement.

Lemma 5.1. *Let φ be a regular supercuspidal L-parameter of G with associated regular L-packet datum $(S, \hat{j}, \chi_0, \theta)$. Then the map $j \mapsto \pi_{(jS, j\theta)}$ induces a bijection $\mathcal{J}_F \to \Pi_\varphi$.*

Proof. We prove an equivalent statement: $\pi_{(j_1 S, j_1\theta)} \simeq \pi_{(j_2 S, j_2\theta)}$ if and only if j_1 and j_2 are $G(F)$-conjugate.

Assume $\pi_{(j_1 S, j_1\theta)} \simeq \pi_{(j_2 S, j_2\theta)}$. Then, by [Kaletha 2019, Corollary 3.7.10], there exists $g \in G(F)$ such that $j_1 S = \mathrm{Ad}(g) j_2 S$ and $j_1\theta = {}^g j_2\theta$. Using [Kaletha 2019, Lemmas 3.4.10 and 3.4.12], there exists $j' \in \mathcal{J}_F$ such that $(j'S, j'\theta)$ is extra regular in the sense of [Kaletha 2019, Definition 3.7.5]. As in the proof of Proposition 4.7, $j_1 = \mathrm{Ad}(h_1) \circ j'$ and $j_2 = \mathrm{Ad}(h_2) \circ j'$ for some $h_1, h_2 \in G(F^{\mathrm{sep}})$ such that $\mathrm{Ad}(h_1)$ and $\mathrm{Ad}(h_2)$ (as maps of $j'S$) are defined over F. Thus, $j_1 S = \mathrm{Ad}(g) j_2 S$ and $j_1\theta = {}^g j_2\theta$ if and only if $h_1^{-1} g h_2 \in N_G(j'S)$ and $j'\theta = {}^{h_1^{-1} g h_2} j'\theta$. Because $\mathrm{Ad}(h_1^{-1} g h_2)$ is defined over F, it is an easy exercise to show that $\sigma(h_1^{-1} g h_2)^{-1}(h_1^{-1} g h_2) \in C_G(j'S) = j'S$ for all $\sigma \in \Gamma$. It follows that $(h_1^{-1} g h_2) j'S \in \Omega(j'S, G)(F) = (N_G(j'S)/j'S)(F)$. By the extra regularity of $j'\theta$, we conclude that $h_1^{-1} g h_2 \in j'S$, and therefore $\mathrm{Ad}(h_1^{-1} g h_2) \circ j' = j'$. Thus, $\mathrm{Ad}(g) \circ j_2 = j_1$.

The converse is a direct consequence of [Kaletha 2019, Corollary 3.7.10], and is built into the definition of Π_φ. □

Suppose as usual that $\underline{\varphi} \in \Phi_{\mathrm{sc}}(\underline{G})$ and $\varphi = {}^L\eta \circ \underline{\varphi} \in \Phi_{\mathrm{sc}}(G)$. It is natural to ask under what conditions φ and $\underline{\varphi}$ are both regular. The following lemma and corollary address this question from the perspective of regular supercuspidal L-packet data.

Lemma 5.2. *Let (S, θ) and $(\underline{S}, \underline{\theta})$ be F-nonsingular elliptic pairs of G and $\underline{G}$, respectively, satisfying $\eta(S) \subset \underline{S}$ and $\theta = \underline{\theta} \circ \eta$. If (S, θ) is extra regular, then $(\underline{S}, \underline{\theta})$ is also extra regular.*

Corollary 5.3. *Let $\underline{\varphi}$ be a supercuspidal L-parameter of $\underline{G}$ and let $\varphi = {}^L\eta \circ \underline{\varphi}$. If φ is regular, then $\underline{\varphi}$ is also regular.*

Before proving Lemma 5.2, recall from Section 2.5 that S is not a subtorus of G. Rather, as in [Kaletha 2019, p. 1145], we have a structure on S that is given to us by $\hat{j}$. This means that the action of $\Omega(S, G)$ on S corresponds to the action of $\Omega(T, G)$ twisted by $i : S \to T$. More precisely, given $w \in \Omega(S, G) = \Omega(T, G)$, ${}^w s = i^{-1}({}^w i(s))$ for all $s \in S$. Here, T is the maximal torus from our fixed Γ-invariant pinning, and i is as per the proof of Proposition 4.7. The same is also true of $\underline{S}$, for which we adopt analogous notation. Furthermore, since $\eta(T) = \eta(G) \cap \underline{T}$ and $\eta(G) \supset [\underline{G}, \underline{G}]$, one sees from [Bourgeois 2020, Proposition 2.1.24] that η induces a Γ-equivariant isomorphism

$$\eta_\Omega : \Omega(T, G) \xrightarrow{\eta} \Omega(\eta(T), \eta(G)) \to \Omega(\underline{T}, \underline{G}),$$

which sends gT to $\eta(g)\underline{T}$ for all $g \in N_G(T)$.

Proof of Lemma 5.2. It is clear that the first two conditions in the definition of extra regularity [Kaletha 2019, Definition 3.7.5] are satisfied for $(\underline{S}, \underline{\theta})$ if and only if they are satisfied for (S, θ). We focus our attention on the third and final condition.

That is, we assume that $\theta|_{S(F)_0}$ has trivial stabilizer for the action of $\Omega(S, G)(F)$ and show that $\underline{\theta}|_{\underline{S}(F)_0}$ has trivial stabilizer for the action of $\Omega(\underline{S}, \underline{G})(F)$.

Recall from Proposition 4.5 and (10) that $\theta = \tilde{\theta} \circ \eta_S$ and $\underline{i} \circ \eta_S = \eta \circ i$. Using these equalities in combination with the definitions of the actions of $\Omega(S, G)$ and $\Omega(\underline{S}, \underline{G})$, we obtain

$$ {}^{w}\theta = {}^{\eta_\Omega(w)}\tilde{\theta} \circ \eta_S \quad \text{for all } w \in \Omega(S, G)(F). \tag{13} $$

Assume that ${}^{\underline{w}}\underline{\theta}|_{\underline{S}(F)_0} = \underline{\theta}|_{\underline{S}(F)_0}$ for some $\underline{w} \in \Omega(\underline{S}, \underline{G})(F)$. By the above discussion, $\underline{w} = \eta_\Omega(w)$ for some $w \in \Omega(S, \mathrm{G})(F)$. Using (13), it follows that

$$ \theta|_{S(F)_0} = (\underline{\theta} \circ \eta_S)|_{S(F)_0} = ({}^{\underline{w}}\underline{\theta} \circ \eta_S)|_{S(F)_0} = {}^{w}\theta|_{S(F)_0}. $$

Given the assumption on θ, we conclude that $w = 1$. Thus $\underline{w} = 1$ and $\underline{\theta}|_{\underline{S}(F)_0}$ has trivial stabilizer for the action of $\Omega(\underline{S}, \underline{G})(F)$. □

The converse of Lemma 5.2 and Corollary 5.3 is not true in general. Consider the case of $G = \mathrm{SL}_2$ and $\underline{G} = \mathrm{GL}_2$, with η being the inclusion map. Then, all irreducible supercuspidal representations of $\underline{G}(F)$ are extra regular [Kaletha 2019, Lemma 3.7.7], whereas there exist irreducible supercuspidal representations of $G(F)$ which are not regular (e.g., the four *exceptional* supercuspidal representations from [Adler et al. 2011]). Given one such representation of $G(F)$, say π, the irreducible components of $\mathrm{Ind}_{G(F)}^{\underline{G}(F)} \pi$ are all extra regular, and thus correspond to extra regular elliptic pairs of $\underline{G}(F)$. We claim that the restrictions of these extra regular elliptic pairs to $G(F)$ can not be extra regular (or even regular). Indeed, given $\underline{\pi} = \pi_{(\underline{S}, \underline{\theta})} \subset \mathrm{Ind}_{G(F)}^{\underline{G}(F)} \pi$, with $(\underline{S}, \underline{\theta})$ extra regular, Theorem 3.1 says that $\underline{\pi} \circ \eta$ is a sum of conjugates of $\pi_{(S,\theta)}$, where $S = \underline{S} \cap G$ and $\theta = \underline{\theta} \circ \eta$. Assuming (S, θ) is extra regular (or even regular) contradicts the nonregularity of $\pi \subset \underline{\pi} \circ \eta$, and thus (S, θ) cannot be extra regular.

It is worth pointing out that the instances for which the converse holds are not extremely rare. Indeed, assume that $\underline{\theta}$ is extra regular and that ${}^{w}\theta|_{S(F)_0} = \theta|_{S(F)_0}$ for some $w \in \Omega(S, G^0)(F)$. Then (13) tells us ${}^{\eta_\Omega(w)}\underline{\theta} \circ \eta_S|_{S(F)_0} = \underline{\theta} \circ \eta_S|_{S(F)_0}$, or equivalently, ${}^{\eta_\Omega(w)}\underline{\theta}|_{\eta_S(S(F)_0)} = \underline{\theta}|_{\eta_S(S(F)_0)}$. In order to conclude that $w = 1$ (i.e., θ is extra regular), we must have ${}^{\eta_\Omega(w)}\underline{\theta}|_{\underline{S}(F)_0} = \underline{\theta}|_{\underline{S}(F)_0}$. If the difference between $\underline{S}(F)_0$ and $\eta_S(S(F)_0)$ is centralized by $\Omega(\underline{S}, \underline{G})(F)$, then we can conclude that $w = 1$, and so θ is extra regular when $\underline{\theta}$ is extra regular. This happens, for example, if the equality $\underline{T} = Z(\underline{G})\eta(T)$ (or equivalently, $\underline{S} = \underline{i}^{-1}(Z(\underline{G}))\eta_S(S)$) remains true at the level of the F-points.

5.2. ***Functoriality and the characters of the dual component group.*** Let $\varphi \in \Phi_{\mathrm{sc}}(G)$ be a regular supercuspidal L-parameter. Up until now in this section, the elements of Π_φ have been parameterized by regular elliptic pairs. However, in Solleveld's conjecture [2020, Conjecture 2], the elements of the L-packets are parameterized by φ and characters of the associated component group. Thus, the first step in proving Theorem 1.1 is to reconcile the two parameterizations.

As discussed in Section 5.1, the L-parameter φ corresponds to a regular supercuspidal L-packet datum $(S, \hat{j}, \chi_0, \theta)$, and the (equivalence classes of) irreducible representations in Π_φ correspond bijectively to the ($G(F)$-conjugacy classes of) admissible embeddings in $\mathcal{J}_F$

(14)
$$\Pi_\varphi \longleftrightarrow \mathcal{J}_F.$$

Thus far, there has been no need to specify a particular irreducible representation or admissible embedding in (14) as being special. However, it shall be necessary to specify a particular representation and embedding to correspond to the identity character of the dual component group. Kaletha [2019, Sections 5.3, 6.2; 2021, Lemma 4.2.1] discusses what the particular choice of representation and embedding should be upon fixing a Whittaker datum, and when F is of characteristic zero the particular representation and embedding is fixed on [Fintzen et al. 2023, page 2273]. For F of characteristic zero we may fix a Whittaker datum for G and thereby an embedding $j \in \mathcal{J}_F$ as on [Fintzen et al. 2023, page 2273]. At the time of writing, a preferred choice of embedding does not appear to be available for regular supercuspidal L-packets in positive characteristic. In this case, we arbitrarily fix $j \in \mathcal{J}_F$ and thereby its corresponding irreducible representation $\pi_{(jS, j\theta)} \in \Pi_\varphi$. The bijection in (14) is now an isomorphism of pointed sets.

The dual group attached to $\varphi \in \Phi_{\mathrm{sc}}(G)$ is $\mathrm{Cent}(\varphi(W_F), \widehat{G})$, and according to [Kaletha 2019, Lemma 5.3.4], it is naturally isomorphic to the fixed-point subgroup $\widehat{S}^\Gamma$. We denote the finite abelian component group $\widehat{S}^\Gamma/(\widehat{S}^\Gamma)^\circ$ by $\pi_0(\widehat{S}^\Gamma)$, and denote its group of characters by $\pi_0(\widehat{S}^\Gamma)^D$. What we wish to do here is supplement (14) with an inclusion

(15)
$$\Pi_\varphi \longleftrightarrow \mathcal{J}_F \hookrightarrow \pi_0(\widehat{S}^\Gamma)^D$$

and to describe the image of this inclusion.

Recall from (2) that $\mathcal{J}_F$ is in bijection with $\ker(H^1(F, jS) \to H^1(F, G))$. An arbitrary element of $\ker(H^1(F, jS) \to H^1(F, G))$ is represented by a cocycle

$$z_g(\sigma) = g^{-1}\sigma(g), \quad \sigma \in \Gamma,$$

for some $g \in G(F^{\mathrm{sep}})$. By fixing $j \in \mathcal{J}_F$ as we have above, we fix a bijection from $\ker(H^1(F, jS) \to H^1(F, G))$ to $\mathcal{J}_F$ given by

(16)
$$z_g \mapsto \mathrm{Ad}(g) \circ j.$$

The desired inclusion of (15) is given through this fixed bijection and the commutative diagram

(17)
$$\begin{array}{ccc} H^1(F, jS) & \longrightarrow & \pi_0(\widehat{S}^\Gamma)^D \\ \downarrow & & \downarrow \\ H^1(F, G) & \longrightarrow & \pi_0(Z(\widehat{G})^\Gamma)^D \end{array}$$

of [Kottwitz 1986, Theorem 1.2; Thang 2011, Theorem 2.1]. In this diagram the upper and lower maps are bijections which arise from perfect pairings in Tate–Nakayama duality. The map on the left is given by the inclusion $jS \subset G$, and the map on the right is given by restriction to $Z(\widehat{G})^\Gamma \subset \widehat{S}^\Gamma$. Since (17) is commutative, $\ker(H^1(F, jS) \to H^1(F, G))$ is in bijection with the kernel of the restriction map on the right of the diagram. The kernel of the restriction map is isomorphic to $\pi_0(\widehat{S}^\Gamma/Z(\widehat{G})^\Gamma)^D$.

Combining these observations with (16), we obtain a bijection between $\mathcal{J}_F$ and $\pi_0(\widehat{S}^\Gamma/Z(\widehat{G})^\Gamma)^D$ given by the map

$$\mathrm{Ad}(g) \circ j \mapsto \tau_g \tag{18}$$

in which $\mathrm{Ad}(g) \circ j$ is a representative of a $G(F)$-conjugacy class in $\mathcal{J}_F$ and $\tau_g \in \pi_0(\widehat{S}^\Gamma/Z(\widehat{G})^\Gamma)^D$ is obtained through z_g and Tate–Nakayama duality.

In summary, the desired arrangement of (15) takes the shape of three bijections

$$\Pi_\varphi \longleftrightarrow \mathcal{J}_F \longleftrightarrow \big(\ker(H^1(F, jS) \to H^1(F, G))\big) \longleftrightarrow \pi_0(\widehat{S}^\Gamma/Z(\widehat{G})^\Gamma)^D. \tag{19}$$

On the level of elements, the bijections have the form

$$\pi_{({}^g jS, {}^g j\theta)} \longleftrightarrow \mathrm{Ad}(g) \circ j \longleftrightarrow z_g \longleftrightarrow \tau_g,$$

where $g \in G(F^{\mathrm{sep}})$ and $z_g \in Z^1(F, jS)$. We can go one step further and obtain an alternative description for $\ker(H^1(F, jS) \to H^1(F, G))$ as follows.

Given a maximal torus S' of G which is defined over F, we have in particular that S' is a closed subgroup of G. The quotient G/S' is therefore a variety defined over F. An element $gS' \in (G/S')(F^{\mathrm{sep}})$ belongs to $(G/S')(F)$ if and only if $gS' = \sigma(g)S'$ for all $\sigma \in \Gamma$. The group $G(F)$ acts on $(G/S')(F)$ by left multiplication. Let $G(F)\backslash(G/S')(F)$ denote the set of $G(F)$-orbits. The following lemma is a special case of [Serre 2002, I.5.4 Corollary 1].

Lemma 5.4. *Let $gS' \in (G/S')(F)$. Then the map $z_g : \Gamma \to S'$ defined by*

$$z_g(\sigma) = g^{-1}\sigma(g), \quad \sigma \in \Gamma,$$

is a cocycle in $Z^1(F, S')$. In addition, the map $g \mapsto z_g$ induces a bijection from $G(F)\backslash(G/S')(F)$ to $\ker(H^1(F, S') \to H^1(F, G))$.

We may now rewrite (19) using the bijection of Lemma 5.4 with $S' = jS$:

$$\begin{array}{ccccccc} \Pi_\varphi & \longleftrightarrow & \mathcal{J}_F & \longleftrightarrow & G(F)\backslash(G/jS)(F) & \longleftrightarrow & \pi_0(\widehat{S}^\Gamma/Z(\widehat{G})^\Gamma)^D, \\ \pi_{({}^{g^{-1}}jS, {}^{g^{-1}}j\theta)} & \longleftrightarrow & \mathrm{Ad}(g) \circ j & \longleftrightarrow & gjS & \longleftrightarrow & \tau_g. \end{array} \tag{20}$$

Assume now that we have two regular supercuspidal L-parameters $\underline{\varphi} \in \Phi_{\mathrm{sc}}(\underline{G})$ and $\varphi = {}^{L}\eta \circ \underline{\varphi}$. In light of the discussion after Corollary 5.3, it is not sufficient to assume that only $\underline{\varphi}$ is regular. Our next objective is to describe a commutative diagram

$$\begin{array}{ccccccc}
\Pi_{\underline{\varphi}} & \longleftrightarrow & \underline{\mathcal{J}}_F & \xleftrightarrow{\mathrm{Ad}(\,\cdot\,)\circ \underline{j}} & \underline{G}(F)\backslash(\underline{G}/\underline{j}\underline{S})(F) & \xleftrightarrow{\underline{\tau}} & \pi_0(\widehat{\underline{S}}^{\Gamma}/Z(\widehat{\underline{G}})^{\Gamma})^D \\
\uparrow & & \uparrow{\scriptstyle \bar{\eta}} & & \uparrow{\scriptstyle \bar{\eta}} & & \uparrow{\scriptstyle \circ\hat{\eta}} \\
\Pi_{\varphi} & \longleftrightarrow & \mathcal{J}_F & \xleftrightarrow[\mathrm{Ad}(\,\cdot\,)\circ j]{} & G(F)\backslash(G/jS)(F) & \xleftrightarrow[\tau]{} & \pi_0(\widehat{S}^{\Gamma}/Z(\widehat{G})^{\Gamma})^D
\end{array} \tag{21}$$

Once this diagram is in place, the decomposition formula for $\underline{\pi}_{(\underline{j}\underline{S},\underline{j}\underline{\theta})} \circ \eta$ that one would obtain by applying Theorem 3.1 may be transferred to the right-hand square of the diagram, which is key for proving Theorem 1.1.

The starting point is the top row of (21). Replacing G with $\underline{G}$, we fix (the $\underline{G}(F)$-conjugacy class of) an admissible embedding $\underline{j} \in \underline{\mathcal{J}}_F$ relative to a fixed Whittaker datum for $\underline{G}$ when char $F = 0$. When char $F \neq 0$ we fix $\underline{j} \in \underline{\mathcal{J}}_F$ arbitrarily. The top row is then the sequence of bijections in (20), in which S and j are replaced by $\underline{S}$ and $\underline{j}$. By Proposition 4.7, there exists $j \in \mathcal{J}_F$ such that $\eta(jS) \subset \underline{j}\underline{S}$ and $j\theta = \underline{j}\underline{\theta} \circ \eta$. The embedding j in Proposition 4.7 is specified by the alignment of the Γ-invariant pinnings of $\underline{G}$ and G through η. This alignment preserves the simple root spaces and therefore transfers a fixed Whittaker datum of $\underline{G}$ to a Whittaker datum for G. Consequently, when char $F = 0$, the embedding j may be chosen as on [Fintzen et al. 2023, page 2273] relative to the latter Whittaker datum. Otherwise, we fix j as in Proposition 4.7 arbitrarily. In any case, the bottom row is now given by (20).

We continue by describing the middle two vertical maps of (21). Since $\eta(jS) \subset \underline{j}\underline{S}$ the map $\bar{\eta}$ sending $gjS \in G/jS$ to $\eta(g)\underline{j}\underline{S} \in \underline{G}/\underline{j}\underline{S}$ is well defined. Furthermore, η is defined over F so that, for $g \in G(F^{\mathrm{sep}})$,

$$\sigma(\bar{\eta}(gjS)) = \eta(\sigma(g))\underline{j}\underline{S} = \bar{\eta}(\sigma(gjS)), \quad \sigma \in \Gamma.$$

This means that the restriction of $\bar{\eta}$ to $G(F^{\mathrm{sep}})/jS(F^{\mathrm{sep}})$ is defined over F, and passes to a map $G(F)\backslash(G/jS)(F) \to \underline{G}(F)\backslash(\underline{G}/\underline{j}\underline{S})(F)$. This defines the second map from the right in (21). We define the map $\mathcal{J}_F \to \underline{\mathcal{J}}_F$ to its left as the map which takes $\mathrm{Ad}(g) \circ j$ to $\mathrm{Ad}(\eta(g)) \circ \underline{j}$. In this way the middle square in (21) commutes. The arguments in the proof of Proposition 4.7 imply the surjectivity of the two vertical maps in the middle square.

The vertical map $\circ\,\hat{\eta}$ on the right of (21) is defined by composition with $\hat{\eta}$ (see Proposition 4.4). The commutativity of the right-hand square of (21) may be explained as follows. The functoriality of [Kottwitz 1986, Theorem 1.2; Thang

2011, Theorem 2.1] imply that the following diagram commutes:

$$\begin{array}{ccc} H^1(F, \underline{j}\underline{S}) & \longrightarrow & \pi_0(\widehat{\underline{S}}^\Gamma)^D \\ \eta\circ\uparrow & & \uparrow\circ\hat{\eta} \\ H^1(F, jS) & \longrightarrow & \pi_0(\widehat{S}^\Gamma)^D \end{array}$$

Therefore, the restriction

$$\begin{array}{ccc} \ker(H^1(F, \underline{j}\underline{S}) \to H^1(F, \underline{G})) & \longrightarrow & \pi_0(\widehat{\underline{S}}^\Gamma / Z(\widehat{\underline{G}})^\Gamma)^D \\ \eta\circ\uparrow & & \uparrow\circ\hat{\eta} \\ \ker(H^1(F, jS) \to H^1(F, G)) & \longrightarrow & \pi_0(\widehat{S}^\Gamma / Z(\widehat{G})^\Gamma)^D \end{array}$$

also commutes. Applying Lemma 5.4 then gives us the right-hand square of (21) as a commutative diagram. As a consequence, the surjectivity of $\circ\,\hat{\eta}$ follows from the surjectivity of $\bar{\eta}$.

The leftmost vertical arrow of (21) is defined as the unique map which makes the leftmost square of (21) commute. It is defined by

$$\pi_{({}^g jS, {}^g j\theta)} \mapsto \pi_{({}^{\eta(g)}\underline{j}\underline{S}, {}^{\eta(g)}\underline{j}\underline{\theta})}$$

for all $\mathrm{Ad}(g)\circ j \in \mathcal{J}_F$. This map is surjective, as $\bar{\eta}$ is surjective. Given $\mathrm{Ad}(\underline{g})\circ\underline{j} \in \underline{\mathcal{J}}_F$ the preimage of $\pi_{((\mathrm{Ad}(\underline{g})\circ\underline{j})\underline{S}, (\mathrm{Ad}(\underline{g})\circ\underline{j})\underline{\theta})} \in \Pi_{\underline{\varphi}}$ under this map is

$$\{\pi_{({}^g jS, {}^g j\theta)} : \mathrm{Ad}(g)\circ j \in \bar{\eta}^{-1}(\mathrm{Ad}(\underline{g})\circ\underline{j})\}. \tag{22}$$

5.3. ***The proof of Theorem 1.1.*** The commutative diagram (21) from the previous section will allow us to rewrite the decomposition formula provided in the proof of Theorem 4.1 with respect to the parameterization in terms of characters of the component groups.

Let φ, $\underline{\varphi}$, $\underline{j}$, j be as in the previous section. In the proof of Theorem 4.1, equation (11) takes the form

$$\pi_{(\underline{j}\underline{S}, \underline{j}\underline{\theta})} \circ \eta \simeq \bigoplus_{\underline{c}\in\underline{C}} \pi_{((\mathrm{Ad}(c^{-1})\circ j)S, (\mathrm{Ad}(c^{-1})\circ j)\theta)}, \tag{23}$$

where $\underline{C}$ is a set of coset representatives of $\eta(G(F))\backslash\underline{G}(F)/\underline{j}\underline{S}(F)$ and $c \in G(F^{\mathrm{sep}})$ is such that $\underline{c} = \eta(c)z$ for some $z \in Z(\underline{G})(F^{\mathrm{sep}})$. Note that, by construction, $\mathrm{Ad}(c^{-1})\circ j \in \bar{\eta}^{-1}(\mathrm{Ad}(\underline{c}^{-1})\circ\underline{j})$ and $\mathrm{Ad}(\underline{c}^{-1})\circ\underline{j}$ and $\underline{j}$ belong to the same $\underline{G}(F)$-equivalence class.

The following proposition tells us that the set of representations (22) coincides with the irreducible components of the decomposition formula (23).

Proposition 5.5. *Suppose* $\underline{g}\,\underline{j}\,\underline{S} \in (\underline{G}/\underline{j}\,\underline{S})(F)$. *Then*

$$\pi_{({}^{\underline{g}}\underline{j}\,\underline{S},\,{}^{\underline{g}}\underline{j}\,\underline{\theta})} \circ \eta \cong \bigoplus_{gjS \in \bar{\eta}^{-1}(\underline{g}\,\underline{j}\,\underline{S})} \pi_{({}^{g}jS,\,{}^{g}j\theta)}. \tag{24}$$

Equivalently, an irreducible representation π *of* $G(F)$ *is a subrepresentation of* $\pi_{({}^{\underline{g}}\underline{j}\,\underline{S},\,{}^{\underline{g}}\underline{j}\,\underline{\theta})} \circ \eta$ *if and only if* $\pi \cong \pi_{({}^{g}jS,\,{}^{g}j\theta)}$ *for* $gjS \in \bar{\eta}^{-1}(\underline{g}\,\underline{j}\,\underline{S})$ (*i.e.*, $\mathrm{Ad}(g) \circ j$ *in the fiber of* $\bar{\eta}$ *over* $\mathrm{Ad}(\underline{g}) \circ \underline{j}$).

The key to proving (24) is to identify the fiber over $\underline{j}$ with $\eta(G(F))\backslash\underline{G}(F)/\underline{j}\,\underline{S}(F)$ in (23), which is done via the following lemma.

Lemma 5.6. *Let* $\bar{\eta}^{-1}(\underline{j})$ *be the fiber of* $\mathcal{J}_F \xrightarrow{\bar{\eta}} \underline{\mathcal{J}}_F$ *over* $\underline{j}$. *The map of Lemma 5.4 and the map* η *induce horizontal maps in the commutative diagram*

$$\begin{array}{ccccc}
\mathcal{J}_F & \longrightarrow & G(F)\backslash(G/jS)(F) & \longrightarrow & \eta(G(F))\backslash(\underline{G}/\underline{j}\,\underline{S})(F) \\
\uparrow & & \uparrow & & \uparrow \\
\bar{\eta}^{-1}(\underline{j}) & \longrightarrow & G(F)\backslash\bar{\eta}^{-1}(\{\underline{g}\,\underline{j}\,\underline{S} : \underline{g} \in \underline{G}(F)\}) & \longrightarrow & \eta(G(F))\backslash\underline{G}(F)/\underline{j}\,\underline{S}
\end{array}$$

In addition, the horizontal maps are bijections.

Proof. We first prove the assertion for the square on the left. The upper map is bijective by Lemma 5.4 and the vertical maps are inclusions. For the lower horizontal map, suppose $\mathrm{Ad}(g) \circ j \in \mathcal{J}$ is a representative of some $G(F)$-orbit in $\mathcal{J}_F$ where $gjS \in (G/jS)(F)$, and $\bar{\eta}(\mathrm{Ad}(g) \circ j) = \mathrm{Ad}(\underline{g}) \circ \underline{j}$ for some $\underline{g} \in \underline{G}(F)$. The bijectivity of the lower horizontal map follows from the equivalences

$$\begin{aligned}
\mathrm{Ad}(\underline{g}^{-1}\eta(g)) \circ \underline{j} = \underline{j} \iff{}& \underline{g}^{-1}\eta(g) \in \underline{j}\,\underline{S} \\
\iff{}& \eta(g)\underline{j}\,\underline{S} = \underline{g}\,\underline{j}\,\underline{S} \iff gjS \in \bar{\eta}^{-1}(\mathrm{Ad}(\underline{G}(F))\underline{j}).
\end{aligned}$$

We continue by examining the upper horizontal map in the square on the right. This map may be described as

$$G(F)gjS \mapsto \eta(G(F))\tilde{\eta}(g\underline{j}\,\underline{S}), \quad \text{where } \tilde{\eta}(gjS) = \eta(g)\underline{j}\,\underline{S},\ gjS \in G/jS.$$

Clearly, the upper horizontal map is bijective if $\tilde{\eta} : G/jS \to \underline{G}/\underline{j}\,\underline{S}$ yields an isomorphism on F-points. Let us prove that $\tilde{\eta}$ is an isomorphism. The injectivity of $\tilde{\eta}$ follows from $jS = \eta^{-1}(\underline{j}\,\underline{S})$. Suppose $\underline{g}\,\underline{j}\,\underline{S} \in \underline{G}/\underline{j}\,\underline{S}$. We may assume that $\underline{g} \in [\underline{G}, \underline{G}]$ [Springer 2009, Corollary 8.1.6]. Since $\eta(G) \supset [\underline{G}, \underline{G}]$, there exists $g \in G$ such that $\eta(g) = \underline{g}$ and $\tilde{\eta}(gjS) = \underline{g}\,\underline{j}\,\underline{S}$. This proves the surjectivity of $\tilde{\eta}$. It also proves that η induces a transitive G-action on $\underline{G}/\underline{j}\,\underline{S}$. In other words, $\underline{G}/\underline{j}\,\underline{S}$ is a homogeneous space for G. Thus, $\tilde{\eta}$ is a bijective G-equivariant morphism of homogeneous spaces. According to [Springer 2009, Theorem 5.3.2 (ii)], the bijective morphism $\tilde{\eta}$ is an isomorphism if and only if its differential $d\tilde{\eta} : \mathfrak{g}/dj\mathfrak{s} \to \underline{\mathfrak{g}}/d\underline{j}\underline{\mathfrak{s}}$ is an isomorphism. We observe that

$$d\tilde{\eta}(X + dj\mathfrak{s}) = d\eta(X) + d\underline{j}\underline{\mathfrak{s}}, \quad X \in \mathfrak{g}.$$

By hypothesis (Theorem 4.1(i)), the kernel of $d\eta$ is central in $\mathfrak{g}$. The one-dimensional root space $\mathfrak{g}_\alpha$, $\alpha \in R(G, S)$, is not central in $\mathfrak{g}$. Consequently $d\eta$ carries $\mathfrak{g}_\alpha$ onto the root space $\underline{\mathfrak{g}}_\alpha$ of $\underline{\mathfrak{g}}$. As the diagram

$$\begin{array}{ccc} \mathfrak{g}/dj\mathfrak{s} & \xrightarrow{d\tilde{\eta}} & \underline{\mathfrak{g}}/d\underline{j}\underline{\mathfrak{s}} \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \bigoplus_\alpha \mathfrak{g}_\alpha & \xrightarrow[d\eta]{\cong} & \bigoplus_\alpha \underline{\mathfrak{g}}_\alpha \end{array}$$

commutes, we see in turn that $d\tilde{\eta}$ is an isomorphism and that $\tilde{\eta}$ is an isomorphism. Since η is defined over F, so is $\tilde{\eta}$, and it follows that $\tilde{\eta}$ yields an isomorphism $(G/jS)(F) \cong (\underline{G}/\underline{j}\underline{S})(F)$. This proves the bijectivity of the upper horizontal map.

Finally, the lower horizontal map on the right of the main diagram is defined by restricting the upper right horizontal map. This yields a bijection

$$G(F)\backslash\bar{\eta}^{-1}(\{\underline{g}\underline{j}\underline{S} : \underline{g} \in \underline{G}(F)\}) \to \eta(G(F))\backslash\underline{G}(F)\underline{j}\underline{S}/\underline{j}\underline{S},$$

and the set on the right is $\eta(G(F))\backslash\underline{G}(F)/\underline{j}\underline{S}(F)$. □

The following lemma is an immediate consequence of Lemma 5.6 and decomposition formula (23).

Lemma 5.7. *An irreducible representation π of $G(F)$ is a subrepresentation of $\pi_{(\underline{j}\underline{S},\underline{j}\underline{\theta})} \circ \eta$ if and only if $\pi \cong \pi_{({}^g jS,{}^g j\theta)}$ for $gjS \in \bar{\eta}^{-1}(\underline{j}\underline{S})$ $(\mathrm{Ad}(g) \circ j$ is in the fiber of $\bar{\eta}$ over $\underline{j})$. In particular, (24) holds for $\underline{g} = 1$.*

We are now ready to prove Proposition 5.5

Proof of Proposition 5.5. Let $\underline{j}' = \mathrm{Ad}(\underline{g}) \circ \underline{j}$. Arguing as in the proof of Proposition 4.7 there exists $g'S \in (G/S)(F)$ such that $j' = \mathrm{Ad}(g') \circ j$ is sent to $\underline{j}'$ under $\bar{\eta}$. We may replace $\underline{j}$ with $\underline{j}'$, and j with j' in the earlier results. Lemma 5.7 then tells us that the irreducible subrepresentations of

$$\pi_{({}^{\underline{g}}\underline{j}\underline{S},{}^{\underline{g}}\underline{j}\underline{\theta})} \circ \eta = \pi_{(\underline{j}'\underline{S},\underline{j}'\underline{\theta})} \circ \eta$$

are $\pi_{({}^h j'S,{}^h j'\theta)} = \pi_{({}^{hg}jS,{}^{hg}j\theta)}$, where $hj'S \in \bar{\eta}^{-1}(\underline{j}'\underline{S})$. The corollary now follows from

$$\begin{aligned} hj'S \in \bar{\eta}^{-1}(\underline{j}'\underline{S}) &\iff \eta(hg'jS(g')^{-1}) \subset \underline{g}\underline{j}\underline{S}\underline{g}^{-1} \\ &\iff \eta(hg')\underline{j}\underline{S} = \underline{g}\underline{j}\underline{S} \iff (hg')jS \in \bar{\eta}^{-1}(\underline{g}\underline{j}\underline{S}) \end{aligned}$$

and setting $g = hg'$. □

A simple consequence of Proposition 5.5 and diagram (21) is the following corollary, analogous to [Solleveld 2020, Corollary 5.8].

Corollary 5.8. *The L-packet Π_φ consists of the irreducible representations appearing on the right of (24). More precisely,*

$$\Pi_\varphi = \coprod_{\mathrm{Ad}(\underline{g})\circ\underline{j}\in\underline{\mathcal{J}}_F} \{\pi_{({}^g jS, {}^g j\theta)} : \mathrm{Ad}(g)\circ j \in \bar{\eta}^{-1}(\mathrm{Ad}(\underline{g})\circ\underline{j})\},$$

or equivalently, $\Pi_\varphi = \{[\underline{\pi}\circ\eta] : \underline{\pi}\in\Pi_{\underline{\varphi}}\}$.

Proof. The corollary follows from the commutativity of (21) and the partition

$$\mathcal{J}_F = \coprod_{\mathrm{Ad}(\underline{g})\circ\underline{j}\in\underline{\mathcal{J}}_F} \bar{\eta}^{-1}(\mathrm{Ad}(\underline{g})\circ\underline{j}). \qquad \square$$

We continue by expressing Proposition 5.5, which concerns the left-hand side of diagram (21), in terms of the characters of the dual groups, which appear on the right of the diagram. Recall that the characters τ and $\underline{\tau}$ on the right of the diagram are defined in (18).

Corollary 5.9. *Suppose $\underline{g}\,\underline{j}\,\underline{S}\in(\underline{G}/\underline{j}\,\underline{S})(F)$ and $gjS\in(G/jS)(F)$. Then $\pi_{({}^g jS, {}^g j\theta)}$ is a subrepresentation of $\pi_{({}^{\underline{g}}\underline{j}\,\underline{S}, {}^{\underline{g}}\underline{j}\,\underline{\theta})}\circ\eta$ if and only if $\tau_g\circ\hat{\eta} = \underline{\tau}_{\underline{g}}$. In addition,*

$$\pi_{({}^{\underline{g}}\underline{j}\,\underline{S}, {}^{\underline{g}}\underline{j}\,\underline{\theta})}\circ\eta \cong \bigoplus_{g\in G(F)\backslash(G/jS)(F)} \mathrm{Hom}(\underline{\tau}_{\underline{g}}, \tau_g\circ\hat{\eta})\otimes\pi_{({}^g jS, {}^g j\theta)}.$$

Remark 5.10. Since $\underline{\tau}_{\underline{g}}$ and τ_g are characters, $\dim\mathrm{Hom}(\underline{\tau}_{\underline{g}}, \tau_g\circ\hat{\eta})$ is either equal to 1 or 0. As such, $\pi_{({}^{\underline{g}}\underline{j}\,\underline{S}, {}^{\underline{g}}\underline{j}\,\underline{\theta})}\circ\eta$ is multiplicity free for all $\underline{g}\,\underline{j}\,\underline{S}\in(\underline{G}/\underline{j}\,\underline{S})(F)$. One can also prove that the decomposition is multiplicity free using tools directly from the classification theory of supercuspidal representations such as [Hakim and Murnaghan 2008; Murnaghan 2011] (as done, for instance, in [Bourgeois 2021, Section 6]).

Yet another manner of expressing the decompositions of the corollaries is to set $\underline{\varrho} = \underline{\tau}_{\underline{g}}$ and set $\pi(\underline{\varphi}, \underline{\varrho}) = \pi_{({}^{\underline{g}}\underline{j}\,\underline{S}, {}^{\underline{g}}\underline{j}\,\underline{\theta})}$. Then the decomposition of Corollary 5.9 reads as

$$\pi(\underline{\varphi}, \underline{\varrho})\circ\eta \cong \bigoplus_{\varrho\in\pi_0(\widehat{S}^\Gamma/Z(\widehat{G})^\Gamma)^D} \mathrm{Hom}(\underline{\varrho}, \varrho\circ\hat{\eta})\otimes\pi(\varphi, \varrho), \tag{25}$$

thus completing the proof of Theorem 1.1.

This form of the decomposition is the one proposed by Solleveld [2020, Conjecture 2] when η is chosen to preserve fixed pinnings of G and $\underline{G}$ as done in Section 4.1.

In Section 4.1 we also remarked that dropping the requirement of preserving the pinnings, but keeping the dual homomorphism $\hat{\eta}$ fixed, allows one to replace η with

$\eta' = \mathrm{Ad}(\underline{t}') \circ \eta$ where $\underline{t}' Z(\underline{G}) \in (\underline{T}/Z(\underline{G}))(F)$. It is convenient to write $\underline{t} = (\underline{t}')^{-1}$, for in this arrangement

$$\pi_{({}^{\underline{g}}j\underline{S}, {}^{\underline{g}}j\underline{\theta})} \circ \eta' = \pi_{({}^{\underline{g}}j\underline{S}, {}^{\underline{g}}j\underline{\theta})} \circ \mathrm{Ad}(\underline{t}') \circ \eta = \pi_{({}^{\underline{tg}}j\underline{S}, {}^{\underline{tg}}j\underline{\theta})} \circ \eta$$

and Corollary 5.9 yields

$$\pi_{({}^{\underline{g}}j\underline{S}, {}^{\underline{g}}j\underline{\theta})} \circ \eta' \cong \bigoplus_{g \in G(F)\backslash(G/jS)(F)} \mathrm{Hom}(\underline{\tau}_{\underline{tg}}, \tau_g \circ \hat{\eta}) \otimes \pi_{({}^{g}jS, {}^{g}j\theta)}. \tag{26}$$

This decomposition can be rephrased in terms of characters on the dual group as in (25). The introduction of $\mathrm{Ad}(\underline{t}')$ on the left of (25) introduces the character $\underline{\tau}_{\underline{t}'}$ on the right, as one sees in the following corollary.

Corollary 5.11. *Suppose* $\underline{g} \in (\underline{G}/j\underline{S})(F)$ *and* $\eta' = \mathrm{Ad}(\underline{t}') \circ \eta$, *where* $\underline{t}' Z(\underline{G}) \in (\underline{T}/Z(\underline{G}))(F)$. *Set* $\underline{\varrho} = \underline{\tau}_{\underline{g}}$, $\pi(\underline{\varphi}, \underline{\varrho}) = \pi_{({}^{\underline{g}}j\underline{S}, {}^{\underline{g}}j\underline{\theta})}$. *Then*

$$\pi(\underline{\varphi}, \underline{\varrho}) \circ \eta' \cong \bigoplus_{\varrho \in \pi_0(\widehat{S}^{\Gamma}/Z(\widehat{G})^{\Gamma})^D} \mathrm{Hom}(\underline{\varrho}, (\varrho \circ \hat{\eta}) \otimes \underline{\tau}_{\underline{t}'}) \otimes \pi(\varphi, \varrho).$$

Proof. The character $\underline{\tau}_{\underline{tg}}$ appearing in (26) corresponds to the element $\underline{tg} \in (\underline{G}/j\underline{S})$. According to Lemma 5.4 (applied to $\underline{G}$ and $j\underline{S}$), the element $\underline{tg}$ represents the cocycle $z_{\underline{tg}}$ defined by

$$z_{\underline{tg}}(\sigma) = (\underline{tg})^{-1}\sigma(\underline{tg}), \quad \sigma \in \Gamma.$$

Since $\underline{t}Z(\underline{G}) \in (\underline{T}/Z(\underline{G}))(F)$, we have in turn that $\underline{t}^{-1}\sigma(\underline{t}) \in Z(\underline{G})$ and

$$(\underline{tg})^{-1}\sigma(\underline{tg}) = \underline{g}^{-1}(\underline{t}^{-1}\sigma(\underline{t}))\sigma(\underline{g}) = \underline{t}^{-1}\sigma(\underline{t})\underline{g}^{-1}\sigma(\underline{g}).$$

Therefore $z_{\underline{tg}} = z_{\underline{t}} z_{\underline{g}}$ in the group $Z^1(F, jS)$. Applying $\underline{\tau}$, we obtain $\underline{\tau}_{\underline{tg}} = \underline{\tau}_{\underline{t}} \otimes \underline{\tau}_{\underline{g}}$. A similar argument leads to $1 = \underline{\tau}_{\underline{t}} \otimes \underline{\tau}_{\underline{t}^{-1}}$, and so $\underline{\tau}_{\underline{t}}^{-1} = \underline{\tau}_{\underline{t}'}$. By setting $\varrho = \tau_g$ in (26) we see that

$$\mathrm{Hom}(\underline{\tau}_{\underline{tg}}, \tau_g \circ \hat{\eta}) = \mathrm{Hom}(\underline{\tau}_{\underline{t}}\underline{\tau}_{\underline{g}}, \varrho \circ \hat{\eta}) = \mathrm{Hom}(\underline{\varrho}, (\varrho \circ \hat{\eta}) \otimes \underline{\tau}_{\underline{t}'}).$$

The corollary now follows from (26) and the commutativity of (21). □

The decomposition of Corollary 5.11 resembles the one appearing in Solleveld's conjecture [2020, Conjecture 2]. The only difference is that in place of the term $(\varrho \circ \hat{\eta}) \otimes \underline{\tau}_{\underline{t}'}$ in Corollary 5.11, Solleveld [2020, (5.4)] has a term ${}^{S}\eta'^{*}(\varrho)$. Translated into our setting, Solleveld's term is expressed as

$${}^{S}\eta'^{*}(\varrho) = (\varrho \circ \hat{\eta}') \otimes \tau_{\underline{\varphi}}(t') = (\varrho \circ \hat{\eta}) \otimes \tau_{\underline{\varphi}}(t'),$$

where $\tau_{\underline{\varphi}} : \underline{G}(F)\backslash(\underline{G}/Z(\underline{G}))(F) \to \pi_0(\widehat{\underline{S}}^{\Gamma}/Z(\widehat{\underline{G}})^{\Gamma})^D$ is a homomorphism defined in [Solleveld 2020, (2.12), Lemma 2.1].

Proposition 5.12. *Under the regularity assumptions on φ and $\underline{\varphi}$, Corollary 5.11 coincides with* [Solleveld 2020, Conjecture 2]. *That is,* $(\varrho \circ \hat{\eta}) \otimes \underline{\tau}_{\underline{t}'} = {}^{S}\eta'^{*}(\varrho)$, *and therefore,*

$$\pi(\underline{\varphi}, \underline{\varrho}) \circ \eta' \cong \bigoplus_{\varrho \in \pi_0(\widehat{S}^{\Gamma}/Z(\widehat{G})^{\Gamma})^D} \operatorname{Hom}(\underline{\varrho}, (\varrho \circ \hat{\eta}) \otimes \underline{\tau}_{\underline{t}'}) \otimes \pi(\varphi, \varrho)$$
$$= \bigoplus_{\varrho \in \pi_0(\widehat{S}^{\Gamma}/Z(\widehat{G})^{\Gamma})^D} \operatorname{Hom}(\underline{\varrho}, {}^{S}\eta'^{*}(\varrho)) \otimes \pi(\varphi, \varrho).$$

Proof. All we need to show is

$$\tau_{\underline{\varphi}}(\underline{t}') = \underline{\tau}_{\underline{t}'}. \tag{27}$$

Under the assumption that the characteristic of F is zero, this identity is given in the proof of [2013, Lemma 4.2] in which Kaletha writes $\tau_{\underline{\varphi}}(\underline{t}')$ as $(\mathfrak{w}, \mathfrak{w}')$ for a pair of Whittaker data conjugate under $\underline{t}'$ [2013, pp. 2454–2455], and $\underline{\tau}_{\underline{t}'}$ is expressed in terms of the cocycle $z = z_{\underline{t}'}$ and the Tate–Nakayama pairing (Lemma 5.4). His argument references the work in [Kottwitz and Shelstad 1999, Appendix A] on the hypercohomology of complexes of tori of length two. The argument is the same in positive characteristic, and to convince the reader that nothing runs awry we offer a sketch.

We write a complex of F-tori of length two simply as $\mathcal{T} \to \mathcal{S}$, concentrated in degrees 0 and 1. Let $\mathcal{T}$ be a maximal torus in $\underline{G}$ which is defined over F. Let $\underline{Z} = Z(\underline{G})$ and $\mathcal{T}_{\text{ad}} = \mathcal{T}/\underline{Z}$. Then $\mathcal{T}_{\text{ad}}$ is a maximal torus in $\underline{G}_{\text{ad}} = \underline{G}/\underline{Z}$, and its Langlands dual is $(\widehat{\mathcal{T}})_{\text{sc}}$, a maximal torus in the simply connected dual group $(\widehat{\underline{G}})_{\text{sc}} = \widehat{\underline{G}_{\text{ad}}}$. The map $(\widehat{\underline{G}})_{\text{sc}} \to [\widehat{\underline{G}}, \widehat{\underline{G}}] \to \widehat{G}$ induces a map $(\widehat{\mathcal{T}})_{\text{sc}} \to \widehat{\mathcal{T}}$ with kernel denoted by $\widehat{\underline{Z}}$.

The sequence

$$0 \to (0 \to \mathcal{T}_{\text{ad}}) \xrightarrow{(0,id)} (\mathcal{T} \to \mathcal{T}_{\text{ad}}) \xrightarrow{(id,0)} (\mathcal{T} \to 0) \to 0$$

is a short exact sequence, and therefore gives rise to a long exact sequence of Galois hypercohomology. The first hypercohomology portion of this long exact sequence appears in the second row of the diagram

$$\begin{array}{ccccc}
\mathcal{T}_{\text{ad}}(F) & \longrightarrow & H^1(F, \underline{Z}) & \longrightarrow & H^1(F, \mathcal{T}) \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
H^1(F, 0 \to \mathcal{T}_{\text{ad}}) & \longrightarrow & H^1(F, \mathcal{T} \to \mathcal{T}_{\text{ad}}) & \longrightarrow & H^1(F, \mathcal{T} \to 0) \\
\downarrow & & \downarrow & & \downarrow \\
H^1(W_F, (\widehat{\mathcal{T}})_{\text{sc}} \to 0)^D & \longrightarrow & H^1(W_F, (\widehat{\mathcal{T}})_{\text{sc}} \to \widehat{\mathcal{T}})^D & \longrightarrow & H^1(W_F, 0 \to \widehat{\mathcal{T}})^D \\
\cong\downarrow & & \cong\downarrow & & \cong\downarrow \\
H^1(W_F, (\widehat{\mathcal{T}})_{\text{sc}})^D & \longrightarrow & H^1(W_F, \widehat{\underline{Z}})^D & \longrightarrow & (\widehat{\mathcal{T}}^{W_F})^D
\end{array} \tag{28}$$

The third row of the diagram is the Pontryagin dual of the analogous sequence of the dual tori together with the action of the Weil group on the dual tori. The vertical arrows between the second and third rows are given by the pairing [Kottwitz and Shelstad 1999, (A.3.15)] (see [Dillery 2022, Proposition A.4]).

The first row of the diagram is defined as follows. Any element in $\mathcal{T}_{\mathrm{ad}}(F)$ may be written as $t'\underline{Z}$ where $t' \in \mathcal{T}$. The first horizontal map sends $t'\underline{Z}$ to the class of the 1-cocycle $z_{t'}$ defined by

$$z_{t'}(\sigma) = (t')^{-1}\sigma(t'), \quad \sigma \in \Gamma. \tag{29}$$

The second horizontal map in the top row carries $z_{t'}$ to itself. The vertical isomorphisms between the first and second rows are canonical and left as exercises (see [Kottwitz and Shelstad 1999, A.1]).

The maps of the fourth row and the isomorphisms with the third row follow just as the ones for the first and second rows. Starting with $h \in \widehat{\mathcal{T}}^{W_F}$, we choose $h' \in (\widehat{\mathcal{T}})_{\mathrm{sc}}$ so that $h = h'\widehat{\underline{Z}}$. We then define an element $c_{h'}$ in $H^1(W_F, \widehat{\underline{Z}})$ or $H^1(W_F, (\widehat{\mathcal{T}})_{\mathrm{sc}})$ by imitating (29) (see [Solleveld 2020, (2.8)]). The maps in the fourth row are the ones dual to those just defined.

Diagram (28) is commutative, due to the functoriality of the vertical morphisms (see [Kottwitz and Shelstad 1999, (A.3.5)]). By making a comparison with the cohomology crossed modules, one can also see that the map $H^1(F, \underline{Z}) \to H^1(W_F, \widehat{\underline{Z}})^D$ in the middle of (28) is independent of the choice of maximal torus $\mathcal{T}$ (see the proof of [Kaletha 2015, Proposition 5.19]). Combining these facts with $\mathcal{T} = \underline{T}$ and $\mathcal{T} = j\underline{S}$, we obtain the commutative diagram

$$\begin{array}{ccccc} \underline{T}_{\mathrm{ad}}(F) & \longrightarrow & H^1(F, \underline{Z}) & \longrightarrow & H^1(F, j\underline{S}) \\ \downarrow & & \downarrow & & \downarrow \\ H^1(W_F, (\widehat{\underline{T}})_{\mathrm{sc}})^D & \longrightarrow & H^1(W_F, \widehat{\underline{Z}})^D & \longrightarrow & (\widehat{\underline{S}}^{\varphi(W_F)})^D \end{array}$$

Consider the element $t'\underline{Z} \in \underline{T}_{\mathrm{ad}}(F)$ in the top left of this diagram. If we trace $t' \in \underline{T}$ through the vertical map on the left and the lower horizontal maps, we arrive at $\tau_{\underline{\varphi}}(t')$ [Kottwitz and Shelstad 1999, (A.3.13); Solleveld 2020, (2.11)]. Alternatively, tracing $t' \in \underline{T}$ through the upper horizontal maps followed by the vertical map on the right we arrive at $\tau_{t'}$ [Kottwitz and Shelstad 1999, (A.3.14)]. The commutativity of the diagram yields the desired identity (27). □

Appendix: Intertwining maps and coset identities

This appendix groups together some important results that were cited in the main text, but whose proofs did not necessarily fit with the flow of the main narrative.

Proposition A.1. *Let G and $\underline{G}$ be as per Theorem 4.1, and let $G_1 \subseteq G_2 \subseteq G(F)$. Let $\underline{g} \in \underline{G}(F)$ and let $\underline{\mathrm{Ad}}(\underline{g})$ be the automorphism of $G(F)$ as defined in Section 2.3.*

Then, given a representation π of G_1,

$$\mathrm{Ind}_{\underline{\mathrm{Ad}}(\underline{g})(G_1)}^{\underline{\mathrm{Ad}}(\underline{g})(G_2)}(\pi \circ \underline{\mathrm{Ad}}(\underline{g}^{-1})) \simeq \mathrm{Ind}_{G_1}^{G_2} \pi \circ \underline{\mathrm{Ad}}(\underline{g}^{-1}).$$

Proof. Let V denote the vector space on which the representations π and $\pi \circ \underline{\mathrm{Ad}}(\underline{g}^{-1})$ act. Then, $\mathrm{Ind}_{\underline{\mathrm{Ad}}(\underline{g})(G_1)}^{\underline{\mathrm{Ad}}(\underline{g})(G_2)}(\pi \circ \underline{\mathrm{Ad}}(\underline{g}^{-1}))$ and $\mathrm{Ind}_{G_1}^{G_2} \pi \circ \underline{\mathrm{Ad}}(\underline{g}^{-1})$ act on the vector spaces

$$\underline{W} = \{\underline{f} : \underline{\mathrm{Ad}}(\underline{g})(G_2) \to V \text{ locally constant} \mid \underline{f}(gh) = (\pi \circ \underline{\mathrm{Ad}}(\underline{g}))(h)^{-1} \underline{f}(g) \quad \text{for all } h \in \underline{\mathrm{Ad}}(\underline{g})(G_1)\}$$

and

$$W = \{f : G_2 \to V \text{ locally constant} \mid f(gh) = \pi(h)^{-1} f(g) \text{ for all } h \in G_1\},$$

respectively. Define a linear map $\mathcal{F}$ as

$$\mathcal{F} : W \to \underline{W}, \quad f \mapsto f \circ \underline{\mathrm{Ad}}(\underline{g}^{-1}).$$

The map $\mathcal{F}$ is bijective as a consequence of $\underline{\mathrm{Ad}}(\underline{g}^{-1}) : \underline{\mathrm{Ad}}(\underline{g})(G_2) \to G_2$ being bijective. It is then an easy computation to verify that $\mathcal{F}$ intertwines the representations $\mathrm{Ind}_{\underline{\mathrm{Ad}}(\underline{g})(G_1)}^{\underline{\mathrm{Ad}}(\underline{g})(G_2)}(\pi \circ \underline{\mathrm{Ad}}(\underline{g}^{-1}))$ and $\mathrm{Ind}_{G_1}^{G_2} \pi \circ \underline{\mathrm{Ad}}(\underline{g}^{-1})$. □

Proposition A.2. *Let $\mu : H_2' \to H_2$ be a morphism of locally profinite groups, $H_1 \subset H_2$ and $H_1' \subset H_2'$ subgroups such that $\mu(H_2') \cap H_1 = \mu(H_1')$ and $\ker(\mu) \subset H_1'$. Let π be a representation of H_1. Then,*

$$\mathrm{Ind}_{H_1'}^{H_2'}(\pi \circ \mu) \simeq \left(\mathrm{Ind}_{\mu(H_1')}^{\mu(H_2')} \mathrm{Res}_{\mu(H_1')}^{H_1} \pi\right) \circ \mu.$$

Proof. Let V denote the vector space on which the representations π and $\pi \circ \mu$ act. Then, the representations $\left(\mathrm{Ind}_{\mu(H_1')}^{\mu(H_2')} \mathrm{Res}_{\mu(H_1')}^{H_1} \pi\right) \circ \mu$ and $\mathrm{Ind}_{H_1'}^{H_2'}(\pi \circ \mu)$ act on the vector spaces

$$W_\mu = \{f_\mu : \mu(H_2') \to V \text{ locally constant} \mid f_\mu(gh) = \pi(h)^{-1} f_\mu(g) \text{ for all } h \in \mu(H_1')\}$$

and

$$W = \{f : H_2' \to V \text{ locally constant} \mid f(gh) = (\pi \circ \mu)(h)^{-1} f(g) \text{ for all } h \in H_1'\},$$

respectively. Define a linear map $\mathcal{F}$ as

$$\mathcal{F} : W_\mu \to W, \quad f_\mu \mapsto f_\mu \circ \mu.$$

One sees that the map $\mathcal{F}$ is injective, as $\mu : H_2' \to \mu(H_2')$ is surjective.

Next, we show that $\mathcal{F}$ is surjective. Given $f \in W$, we have that f is constant on the coset $h \ker \mu$ for all $h \in H_2'$. Indeed, for all $z \in \ker \mu \subset H_1'$, we have $f(hz) = (\pi \circ \mu)(z)^{-1} f(h) = f(h)$. This allows us to define a map

$$f_\mu : H_2' / \ker \mu \to V, \quad h \ker \mu \mapsto f(h),$$

which we view as an element of W_μ under the isomorphism $\mu(H_2') \simeq H_2'/\ker\mu$. By construction we have $\mathcal{F}(f_\mu) = f$, which proves surjectivity.

It is then an easy computation to verify that $\mathcal{F}$ intertwines the representations $\big(\mathrm{Ind}_{\mu(H_1')}^{\mu(H_2')} \mathrm{Res}_{\mu(H_1')}^{H_1} \pi\big) \circ \mu$ and $\mathrm{Ind}_{H_1'}^{H_2'}(\pi \circ \mu)$. □

Lemma A.3. *Let A, B, C be subgroups of a fixed group such that $B \subseteq A$, and both A and B normalize C. Then there is a natural bijection*

$$AC/BC \simeq A/B(A \cap C).$$

Proof. Note that $B \subseteq A$ normalizes both A and C, so that B normalizes $A \cap C$. Consequently, $B(A \cap C)$ is a group, and $A/B(A \cap C)$ is a set of cosets. Consider the map $A/B(A \cap C) \to AC/BC$ of cosets defined by

$$aB(A \cap C) \mapsto aBC, \quad a \in A.$$

This map is clearly well defined and surjective. Furthermore, if $a_1BC = a_2BC$ for $a_1, a_2 \in A$, then $a_1 = a_2bc$, where $b \in B$ and $c = b^{-1}a_2^{-1}a_1 \in A \cap C$. This proves that the map is injective. □

Lemma A.4. *Let N, A, B, $\bar{N}$, $\bar{A}$, $\bar{B}$ be groups that satisfy the conditions*

$$\begin{array}{ccccccc} N & \lhd & A & & \bar{A} & \rhd & \bar{N} \\ & & \text{\rotatebox{90}{$\subseteq$}} & & \text{\rotatebox{90}{$\subseteq$}} & & \\ & & B & \supseteq & \bar{B} & & \end{array}$$

Let L be a set of coset representatives of $N\backslash A/B$, $\mathcal{L}$ be a set of coset representatives of $\bar{N}\backslash\bar{A}/\bar{B}$, and suppose that both L and $\mathcal{L}$ are finite. Write $L\mathcal{L} = \{l\ell : l \in L,\ \ell \in \mathcal{L}\}$.

1. *Suppose $\bar{A} \subseteq B$, $N \cap \bar{A} \subseteq \bar{N}$, and $B/(N \cap B)\bar{B} \simeq \bar{A}/\bar{N}\bar{B}$. Then $L\mathcal{L}$ is a set of coset representatives of $N\backslash A/\bar{B}$.*
2. *If $\bar{N} \subseteq N$, $N \cap B = \bar{B}$, $N = \bar{A}$ and $\bar{N}$ is a normal subgroup of A, then $L\mathcal{L}$ is a set of coset representatives of $\bar{N}\backslash A/B$.*

Proof. Since N is normal in A, and $\bar{N}$ is normal in $\bar{A}$, we note that $N\backslash A/B = A/NB$, $N\backslash A/\bar{B} = A/N\bar{B}$ and $\bar{N}\backslash\bar{A}/\bar{B} = \bar{A}/\bar{N}\bar{B}$.

To prove (1), we use the normality of N, the inclusion $\bar{B} \subseteq B$ and the isomorphism $B/(N \cap B)\bar{B} \simeq \bar{A}/\bar{N}\bar{B}$, along with the second and third isomorphism theorems to show that

$$A/NB \simeq A/N\bar{B}/\bar{A}/\bar{N}\bar{B}.$$

As both L and $\mathcal{L}$ are finite it follows that

$$|A/NB| \cdot |\bar{A}/\bar{N}\bar{B}| = |A/N\bar{B}|.$$

We then use the inclusions $\bar{A} \subseteq B$ and $N \cap \bar{A} \subseteq \bar{N}$ to show that the map

$$\mu : L \times \mathcal{L} \to A/N\bar{B}, \quad (l, \ell) \mapsto l\ell N\bar{B},$$

is injective.

The proof of (2) follows the exact same strategy as the proof of (1). □

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References

[Adler 1998] J. D. Adler, "Refined anisotropic K-types and supercuspidal representations", *Pacific J. Math.* **185**:1 (1998), 1–32. MR

[Adler et al. 2011] J. D. Adler, S. DeBacker, P. J. Sally, Jr., and L. Spice, "Supercuspidal characters of SL_2 over a p-adic field", pp. 19–69 in *Harmonic analysis on reductive, p-adic groups*, edited by R. S. Doran et al., Contemp. Math. **543**, Amer. Math. Soc., Providence, RI, 2011. MR

[Aubert et al. 2018] A.-M. Aubert, A. Moussaoui, and M. Solleveld, "Generalizations of the Springer correspondence and cuspidal Langlands parameters", *Manuscripta Math.* **157**:1-2 (2018), 121–192. MR

[Borel 1979] A. Borel, "Automorphic L-functions", pp. 27–61 in *Automorphic forms, representations and L-functions, part* 2 (Corvallis, OR, 1977), edited by A. Borel and W. Casselman, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, RI, 1979. MR

[Bourgeois 2020] A. Bourgeois, *On the restriction of supercuspidal representations: an in-depth exploration of the data*, Ph.D. thesis, University of Ottawa, 2020, available at http://hdl.handle.net/10393/40901.

[Bourgeois 2021] A. Bourgeois, "Restricting supercuspidal representations via a restriction of data", *Pacific J. Math.* **312**:1 (2021), 1–39. MR

[Bruhat and Tits 1972] F. Bruhat and J. Tits, "Groupes réductifs sur un corps local", *Inst. Hautes Études Sci. Publ. Math.* 41 (1972), 5–251. MR

[Bruhat and Tits 1984] F. Bruhat and J. Tits, "Groupes réductifs sur un corps local, II: Schémas en groupes; existence d'une donnée radicielle valuée", *Inst. Hautes Études Sci. Publ. Math.* 60 (1984), 5–184. MR

[Carter 1993] R. W. Carter, *Finite groups of Lie type: conjugacy classes and complex characters*, Wiley, Chichester, 1993. MR

[DeBacker and Reeder 2009] S. DeBacker and M. Reeder, "Depth-zero supercuspidal L-packets and their stability", *Ann. of Math.* (2) **169**:3 (2009), 795–901. MR

[Deligne and Lusztig 1976] P. Deligne and G. Lusztig, "Representations of reductive groups over finite fields", *Ann. of Math.* (2) **103**:1 (1976), 103–161. MR

[Dillery 2022] P. E. Dillery, *Rigid inner forms over function fields*, Ph.D. thesis, University of Michigan, Ann Arbor, MI, 2022, available at https://www.proquest.com/docview/2723382422. MR

[Dillery 2023] P. Dillery, "Rigid inner forms over local function fields", *Adv. Math.* **430** (2023), art. id. 109204. MR

[Fintzen 2021] J. Fintzen, "On the Moy–Prasad filtration", *J. Eur. Math. Soc.* (*JEMS*) **23**:12 (2021), 4009–4063. MR

[Fintzen et al. 2023] J. Fintzen, T. Kaletha, and L. Spice, "A twisted Yu construction, Harish-Chandra characters, and endoscopy", *Duke Math. J.* **172**:12 (2023), 2241–2301. MR

[Hakim and Murnaghan 2008] J. Hakim and F. Murnaghan, "Distinguished tame supercuspidal representations", *Int. Math. Res. Pap. IMRP* 2 (2008), art. id. rpn005. MR

[Humphreys 1975] J. E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics **21**, Springer, 1975. MR

[Humphreys 1995] J. E. Humphreys, *Conjugacy classes in semisimple algebraic groups*, Mathematical Surveys and Monographs **43**, Amer. Math. Soc., Providence, RI, 1995. MR

[Jantzen 2003] J. C. Jantzen, *Representations of algebraic groups*, 2nd ed., Mathematical Surveys and Monographs **107**, Amer. Math. Soc., Providence, RI, 2003. MR

[Kaletha 2013] T. Kaletha, "Genericity and contragredience in the local Langlands correspondence", *Algebra Number Theory* **7**:10 (2013), 2447–2474. MR

[Kaletha 2015] T. Kaletha, "Epipelagic L-packets and rectifying characters", *Invent. Math.* **202**:1 (2015), 1–89. MR

[Kaletha 2016] T. Kaletha, "Rigid inner forms of real and p-adic groups", *Ann. of Math.* (2) **184**:2 (2016), 559–632. MR

[Kaletha 2019] T. Kaletha, "Regular supercuspidal representations", *J. Amer. Math. Soc.* **32**:4 (2019), 1071–1170. MR

[Kaletha 2021] T. Kaletha, "Supercuspidal L-packets", preprint, 2021. arXiv 1912.03274

[Kaletha and Prasad 2023] T. Kaletha and G. Prasad, *Bruhat–Tits theory — a new approach*, New Mathematical Monographs **44**, Cambridge Univ. Press, 2023. MR

[Kottwitz 1982] R. E. Kottwitz, "Rational conjugacy classes in reductive groups", *Duke Math. J.* **49**:4 (1982), 785–806. MR

[Kottwitz 1986] R. E. Kottwitz, "Stable trace formula: elliptic singular terms", *Math. Ann.* **275**:3 (1986), 365–399. MR

[Kottwitz and Shelstad 1999] R. E. Kottwitz and D. Shelstad, *Foundations of twisted endoscopy*, Astérisque **255**, Soc. Math. de France, 1999. MR

[Langlands and Shelstad 1987] R. P. Langlands and D. Shelstad, "On the definition of transfer factors", *Math. Ann.* **278**:1-4 (1987), 219–271. MR

[Murnaghan 2011] F. Murnaghan, "Parametrization of tame supercuspidal representations", pp. 439–469 in *On certain L-functions*, edited by J. Arthur et al., Clay Math. Proc. **13**, Amer. Math. Soc., Providence, RI, 2011. MR

[Nevins 2015] M. Nevins, "Restricting toral supercuspidal representations to the derived group, and applications", *J. Pure Appl. Algebra* **219**:8 (2015), 3337–3354. MR

[Serre 2002] J.-P. Serre, *Galois cohomology*, Springer, 2002. MR

[SGA 3_{III} 1970] M. Demazure and A. Grothendieck, *Schémas en groupes, Tome III: Structure des schémas en groupes réductifs, Exposés XIX–XXVI* (Séminaire de Géométrie Algébrique du Bois Marie 1962–1964), Lecture Notes in Math. **153**, Springer, 1970. MR

[Silberger 1979] A. J. Silberger, "Isogeny restrictions of irreducible admissible representations are finite direct sums of irreducible admissible representations", *Proc. Amer. Math. Soc.* **73**:2 (1979), 263–264. MR

[Solleveld 2020] M. Solleveld, "Langlands parameters, functoriality and Hecke algebras", *Pacific J. Math.* **304**:1 (2020), 209–302. MR

[Springer 1979] T. A. Springer, "Reductive groups", pp. 3–27 in *Automorphic forms, representations and L-functions, part* 1 (Corvallis, OR, 1977)), edited by A. Borel and W. Casselman, Proc. Sympos. Pure Math. **33**, Amer. Math. Soc., Providence, RI, 1979. MR

[Springer 2009] T. A. Springer, *Linear algebraic groups*, 2nd ed., Birkhäuser, Boston, MA, 2009. MR

[Thang 2011] N. Q. Thang, "On Galois cohomology and weak approximation of connected reductive groups over fields of positive characteristic", *Proc. Japan Acad. Ser. A Math. Sci.* **87**:10 (2011), 203–208. MR

[Vogan 1993] D. A. Vogan, Jr., "The local Langlands conjecture", pp. 305–379 in *Representation theory of groups and algebras*, edited by J. Adams et al., Contemp. Math. **145**, Amer. Math. Soc., Providence, RI, 1993. MR

[Yu 2001] J.-K. Yu, "Construction of tame supercuspidal representations", *J. Amer. Math. Soc.* **14**:3 (2001), 579–622. MR

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THE LOCAL GROSS–PRASAD CONJECTURE OVER ARCHIMEDEAN LOCAL FIELDS

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Following the approach of C. Mœglin and J.-L. Waldspurger, this article proves the local Gross–Prasad conjecture over $\mathbb{R}$ and $\mathbb{C}$ based on the tempered cases of Luo and the author.

1. Introduction

In [19; 20], B. Gross and D. Prasad formulated a conjecture on the local multiplicity for Bessel models of special orthogonal groups over a local field of characteristic 0, known as the *local Gross–Prasad conjecture*. When the local field is non-Archimedean, the conjecture was proved in [29] based on the tempered cases proved in [36; 37; 38; 39; 40]. This paper proves the local Gross–Prasad conjecture over Archimedean local fields. The proof over the real field follows Mœglin and Waldspurger's approach and is based on the tempered cases proved in [28; 10].

There are some recent applications of the local Gross–Prasad conjecture. The paper [22] takes it as an input to prove one direction of the global Gross–Prasad

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conjecture, and the paper [24] uses the local Gross–Prasad conjecture to develop the theory of arithmetic wavefront sets for irreducible admissible representations of classical groups. We refer to the ICM report of R. Beuzart-Plessis [6] for a general discussion of the significance of the local Gross–Prasad conjecture in arithmetic.

The local Gross–Prasad conjecture is set up as follows: Let F be a local field of characteristic 0, and (W, V) be a pair of nondegenerate quadratic spaces over F such that the orthogonal complement $W^\perp$ of W in V is odd-dimensional and split over F. We let G be the algebraic group $\mathsf{SO}(W) \times \mathsf{SO}(V)$ over F and take its subgroup $\mathsf{H} = \Delta\mathsf{SO}(W) \ltimes \mathsf{N}$, where $\Delta\mathsf{SO}(W)$ is the image of the diagonal embedding $\mathsf{SO}(W) \hookrightarrow \mathsf{SO}(W) \times \mathsf{SO}(V)$ and N is the unipotent part of a parabolic subgroup stabilizing a full totally isotropic flag on $W^\perp$. We fix a generic character ξ_N of $N = \mathsf{N}(F)$ that uniquely extends to a character ξ of $H = \mathsf{H}(F)$. For every irreducible admissible representation π of $G = \mathsf{G}(F)$ (we require the representation to be Casselman–Wallach when F is Archimedean), we define the multiplicity

$$m(\pi) := \dim \operatorname{Hom}_H(\pi|_H, \xi).$$

It was proved in [1; 16; 40] over non-Archimedean fields and in [34; 23] over Archimedean fields that

$$m(\pi) \leqslant 1.$$

This result is known as the *multiplicity-one theorem*. The local Gross–Prasad conjecture is a refinement of the multiplicity-one theorem that takes representations of pure inner forms of G into consideration.

For every $\alpha \in H^1(F, \mathsf{H}) \hookrightarrow H^1(F, \mathsf{G})$, the inner twists of G, H by α give pure inner forms G_α, H_α, respectively. Then $\mathsf{G}_\alpha = \mathsf{SO}(W_\alpha) \times \mathsf{SO}(V_\alpha)$ and $\mathsf{H}_\alpha = \Delta\mathsf{SO}(W_\alpha) \ltimes \mathsf{N}$, where W_α is the inner twist of W by $\alpha \in H^1(F, \mathsf{H}) = H^1(F, \mathsf{SO}(W_\alpha))$ and $V_\alpha = W_\alpha \perp S$. Let ξ_α be the character of $H_\alpha = \mathsf{H}_\alpha(F)$ obtained by the extension of ξ_N. For every irreducible admissible representation π of $G_\alpha = \mathsf{G}_\alpha(F)$ (we require the representation to be Casselman–Wallach when F is Archimedean), we extend the definition of multiplicity by setting

$$m(\pi) := \dim \operatorname{Hom}_{H_\alpha}(\pi|_{H_\alpha}, \xi_\alpha).$$

For every local L-parameter $\phi : \mathcal{W}_F \to {}^L\mathsf{G}$, we denote by $\Pi_{F,\phi}(\mathsf{G})$ the corresponding L-packet, which consists of finitely many irreducible admissible representations of $\mathsf{G}(F)$, which are Casselman–Wallach when F is Archimedean. For every $\alpha \in H^1(F, \mathsf{G})$. the Langlands dual group ${}^L\mathsf{G}_\alpha$ of G_α is isomorphic to that of G, so ϕ also represents a local L-parameter of G_α. Following D. Vogan [35], we can define the Vogan L-packet associated to ϕ as

$$\Pi^{\text{Vogan}}_{F,\phi} := \bigsqcup_{\alpha \in H^1(F,\mathsf{G})} \Pi_\phi(\mathsf{G}_\alpha).$$

The L-parameter ϕ is called *tempered* if $\mathrm{Im}(\phi)$ is bounded. The L-parameter ϕ is called *generic* if there is a generic representation in $\Pi^{\mathrm{Vogan}}_{F,\phi}$. In particular, tempered parameters are generic.

When ϕ is generic, it was conjectured by Vogan and known over Archimedean local fields [35, Theorem 6.3], that, fixing a Whittaker datum of $\{\mathsf{G}_\alpha\}_{\alpha\in H^1(F,\mathsf{G})}$, there is a bijection

$$\pi \in \Pi^{\mathrm{Vogan}}_{F,\phi} \longleftrightarrow \eta_\pi \in \widehat{\mathcal{S}_\phi}.$$

Here $\widehat{\mathcal{S}_\phi}$ is the set of (complex) characters of component group

$$\mathcal{S}_\phi := \pi_0(\mathrm{Cent}_{\widehat{G}}(\mathrm{Im}(\phi))),$$

where $\mathrm{Cent}_{\widehat{G}}(\mathrm{Im}(\phi))$ is the centralizer of the image $\mathrm{Im}(\phi)$ in the dual group $\widehat{\mathsf{G}}$. Gross and Prasad suggested that one may consider the *relevant Vogan packet*, defined as

$$\Pi^{\mathrm{Vogan}}_{F,\phi,\mathrm{rel}} := \bigsqcup_{\alpha\in H^1(F,\mathsf{H})} \Pi_{F,\phi}(\mathsf{G}_\alpha) \subset \Pi^{\mathrm{Vogan}}_{F,\phi}.$$

In particular, the multiplicity $m(\pi)$ is well-defined for representations in $\Pi^{\mathrm{Vogan}}_{F,\phi,\mathrm{rel}}$.

Conjecture 1 [19; 20]. With the notions above, the following two statements hold.

(1) (multiplicity one) For every generic parameter ϕ of G, we have

$$\sum_{\pi\in\Pi^{\mathrm{Vogan}}_{F,\phi,\mathrm{rel}}} m(\pi) = 1.$$

This implies that there is an unique representation $\pi \in \Pi^{\mathrm{Vogan}}_{F,\phi,\mathrm{rel}}$ such that $m(\pi) = 1$.

(2) (epsilon dichotomy) Fix the Whittaker datum of $\{\mathsf{G}_\alpha\}_{\alpha\in H^1(F,\mathsf{G})}$ as [20, (6.3)]. The unique representation $\pi \in \Pi^{\mathrm{Vogan}}_{F,\phi,\mathrm{rel}}$ such that $m(\pi) = 1$ can be characterized as

$$\eta_\pi = \eta_\phi,$$

where η_ϕ is defined in (2.3.2).

When F is non-Archimedean and ϕ is tempered, Waldspurger proved the conjecture in [36; 37; 38; 39; 40]. Mœglin and Waldspurger completed the proof of Conjecture 1 for generic parameters based on the results in the tempered cases.

When $F = \mathbb{R}$ and the parameter ϕ is tempered, Z. Luo proved the multiplicity-one part of Conjecture 1 in [28] following the work of R. Beuzart-Plessis in [5]. The author and Luo proved the epsilon-dichotomy part of Conjecture 1 in [10] by a simplification of Waldspurger's approach.

The main result of the paper is the following.

Theorem 1.0.1. *When $F = \mathbb{R}$ or $\mathbb{C}$, Conjecture 1 holds for generic parameters.*

The proof over $\mathbb{C}$ is done by construction based on results in [18] and the proof over $\mathbb{R}$ follows the strategy in [29]. The proof consists of a structure theorem (Proposition 4.0.5) for representations in generic packets and a multiplicity formula (Theorem 5.0.1). With these results, we can reduce all situations of the conjecture into the tempered cases.

In Section 4, we prove the structure theorem using the standard module conjecture. The proof of the multiplicity formula, however, is more intricate. Following [29], this requires a formula for reduction to basic cases and two multiplicity formulas that establish inequalities needed to prove the basic cases.

In the basic case, one inequality of the multiplicity formula is proved using orbit analysis (Section 5.3). The proof of the other inequality is expected to be completed using harmonic analysis in Section 5.4. The formula for reduction to the basic cases, which is an equality, can be established by proving two inequalities in a manner similar to the inequalities in the basic case. The non-Archimedean counterpart is discussed in [29, Section 2], [29, Sections 1.4 1.6], and [29, Sections 1.7 1.8].

There is a parallel conjecture for unitary groups, formulated by W. Gan, Gross, and Prasad. Over non-Archimedean local fields, the conjecture for tempered parameters was treated by Beuzart-Plessis in [3; 4]; Based on the tempered cases, Gan and A. Ichino proved the conjecture for generic parameters in [15]. Over Archimedean local fields, Beuzart-Plessis proved the multiplicity-one part of the conjecture in [5] for tempered parameters using local trace formula and endoscopy. Xue completed the proof for tempered cases in [43] using theta correspondence and proved the generic cases in [42].

Although it is not necessary for the proof for the local Gan–Gross–Prasad conjecture, the multiplicity formula (Theorem 5.0.1) also works for reducible representations obtained from parabolic induction. This result can be applied to the study of local descents in my joint work with D. Jiang, D. Liu, L. Zhang [12].

Organization. In Section 2, we recall the statement of the local Gross–Prasad conjecture following [19; 20]. In Section 3, we work over the complex field $\mathbb{C}$. We follow the observation in [19, §11] and prove the conjecture by constructing an explicit functional of the representation $\pi_V \boxtimes \pi_W$ using the results in [18].

In Sections 4 5, we work over the real field $\mathbb{R}$. Section 4 provides a structure theorem for representations in generic packets, using a sufficient condition for irreducibility. In Section 5, we reduce the conjecture to the tempered cases by employing a multiplicity formula, following the approach in [29].

For the basic case of the multiplicity formula, we prove one inequality using representation theory and orbit analysis (Section 5.3) and the other using harmonic analysis (Section 5.4). Additionally, in Sections 5.3 5.4, we establish a formula that reduces the multiplicity to the basic cases.

2. Local Gross–Prasad Conjecture

In this section, we review the local Gross–Prasad conjecture over Archimedean local fields following [19] and [20].

2.1. ***Gross–Prasad triples.*** Let $F=\mathbb{R}$ or $\mathbb{C}$ and (W, V) be a pair of nondegenerate quadratic spaces over F. The pair (W, V) is called *relevant* if and only if there exists an anisotropic line D and a nondegenerate even-dimensional split quadratic space Z over F such that

$$V = W \perp D \perp Z.$$

We set $r=\dfrac{\dim Z}{2}$. There exists a basis $\{z_i\}_{i=\pm 1}^{\pm r}$ of Z such that

$$q(z_i, z_j) = \delta_{i,-j}, \quad \forall i, j \in \{\pm 1, \dots, \pm r\},$$

where q is the quadratic form on V. We denote by P_V the parabolic subgroup of the special orthogonal group $\mathsf{SO}(V)$ stabilizing the totally isotropic flag

$$\langle z_r\rangle \subset \langle z_r, z_{r-1}\rangle \subset \cdots \subset \langle z_r, \dots, z_1\rangle. \tag{2.1.1}$$

We take $\mathsf{P}_V=\mathsf{M}_V\cdot\mathsf{N}$ to be its Levi decomposition. In particular, the Levi subgroup $\mathsf{M}_V \simeq \mathsf{SO}(W\oplus D)\times \mathrm{GL}_1^r$.

Let $\mathsf{G}=\mathsf{SO}(W)\times\mathsf{SO}(V)$. We identify N as a subgroup of G via the embedding $\mathsf{SO}(V)\hookrightarrow 1\times\mathsf{SO}(V)$. We set $\Delta\mathsf{SO}(W)$ as the image of the diagonal embedding $\mathsf{SO}(W)\hookrightarrow\mathsf{G}$. Then $\Delta\mathsf{SO}(W)$ acts on N by adjoint action of $\mathsf{SO}(W)\subset\mathsf{M}_V$. We set

$$\mathsf{H}=\Delta\mathsf{SO}(W)\ltimes\mathsf{N}.$$

We define a morphism $\lambda:\mathsf{N}\to\mathbb{G}_a$ by

$$\lambda(n)=\sum_{i=0}^{r-1} q(z_{-i-1}, n z_i), \quad n\in\mathsf{N}.$$

Then λ is $\Delta\mathsf{SO}(W)$-conjugation invariant and hence λ admits an unique extension to H that is trivial on $\Delta\mathsf{SO}(W)$. We still denote this character by λ. Let $\lambda_F:\mathsf{H}(F)\to F$ be the induced morphism on F-rational points. We define an unitary character of $H=\mathsf{H}(F)$ by

$$\xi(h)=\lambda_F(h), \quad h\in H,$$

where ψ is a fixed additive (unitary) character ψ of F. The triple $(\mathsf{G},\mathsf{H},\xi)$ is called the *Gross–Prasad triple* associated with the relevant pair (W, V).

2.2. ***Vogan L-packets.*** We now recall the notion of Vogan L-packets for special orthogonal groups over Archimedean local fields following [35] and review the definition of the relevant Vogan L-packet following [19; 16].

For any reductive algebraic group G over a local field F, we denote by $\widehat{\mathsf{G}}$ the dual group of G and by ${}^L\mathsf{G}$ the Langlands dual group of G. It was established by Langlands in [27] that every local L-parameter $\phi : \mathcal{L}_F \to {}^L\mathsf{G}$ gives a local L-packet $\Pi_{F,\phi}(\mathsf{G})$, which consists of a finite set of irreducible admissible representations of $G = \mathsf{G}(F)$. In particular, when F is Archimedean, the representations in the packet are Casselman–Wallach [7; 41], which means that they are smooth Fréchet representations of moderate growth and the associated Harish-Chandra modules are admissible.

A *pure inner form* G_α is an inner twist of G by $\alpha \in H^1(F, \mathsf{G})$. Since pure inner forms of G share the same dual group, every local L-parameter $\phi : \mathcal{L}_F \to {}^L\mathsf{G}$ of G can be viewed as an L-parameter for any pure inner form G_α. Hence, one can define the *Vogan L-packet* as

$$\Pi_{F,\phi}^{\text{Vogan}} := \bigsqcup_{\alpha \in H^1(F,\mathsf{G})} \Pi_{F,\phi}(\mathsf{G}').$$

Now we consider reductive group G with a quasisplit pure inner form. A *Whittaker datum* $\mathfrak{w}$ for G is a triple $(\mathsf{G}', \mathsf{B}', \psi')$ where G' is a quasisplit pure inner form of G, B' is a Borel subgroup of G', and ψ' is a generic character of the unipotent radical $N' = \mathsf{N}'(F)$ of $\mathsf{B}'(F)$. A representation π' of $\mathsf{G}'(F)$ is called $\mathfrak{w}$-generic if $\mathrm{Hom}_{N'}(\pi'|_{N'}, \xi') \neq 0$. An L-parameter ϕ is called ($\mathfrak{w}$-)*generic* if the Vogan L-packet contains a generic representation. As argued in [16, §18], the genericity of an L-parameter is independent of the choice of the Whittaker datum.

From [35], when F is Archimedean, fixing a generic L-parameter ϕ and a Whittaker datum $\mathfrak{w}$ of G, there is a bijection

$$\pi \in \Pi_{F,\phi}^{\text{Vogan}} \mapsto \eta_\pi \in \Pi(\mathcal{S}_\phi), \tag{2.2.1}$$

where $\Pi(\mathcal{S}_\phi)$ is the set of characters of the *component group*

$$\mathcal{S}_\phi := \pi_0(\mathrm{Cent}_{\widehat{\mathsf{G}}}(\mathrm{Im}(\phi))).$$

Therefore, we can parametrize representations in Vogan packets with characters $\eta : \mathcal{S}_\phi \to \{\pm 1\}$.

Now we return to the setting in Section 2.1. For $\alpha \in H^1(F, \mathsf{H}) = H^1(F, \mathsf{SO}(W))$, we denote by W_α the inner twist of W by α and set $V_\alpha = W_\alpha \perp D \perp Z$. Then the inner twists of G and H by $\alpha \in H^1(F, \mathsf{H}) \subset H^1(F, \mathsf{G})$ are

$$\mathsf{G}_\alpha = \mathsf{SO}(V_\alpha) \times \mathsf{SO}(W_\alpha) \text{ and } \mathsf{H}_\alpha = \Delta\mathsf{SO}(W_\alpha) \ltimes \mathsf{N}.$$

Together with the character $\xi_\alpha : \mathsf{N}(F) \to \mathbb{C}$ obtained by the extension of ξ_N, we obtain the Gross–Prasad triple associated to the relevant pair (W_α, V_α). The *relevant*

Vogan packet is defined by

$$\Pi^{\mathrm{Vogan}}_{F,\phi,\mathrm{rel}} := \bigsqcup_{\alpha\in H^1(F,\mathsf{H})} \Pi_\phi(\mathsf{G}_\alpha). \tag{2.2.2}$$

It is a subset of $\Pi^{\mathrm{Vogan}}_{F,\phi}$ and thus can be parametrized with a subset of $\Pi(\mathcal{S}_\phi)$ via (2.2.1).

2.3. ***The conjecture.*** In this subsection, we review the statement of the local Gross–Prasad conjecture formulated in [19; 20].

Let (W, V) be a relevant pair over an Archimedean local field F and $(\mathsf{G},\mathsf{H},\xi)$ be the Gross–Prasad triple associate to it. For an irreducible Casselman–Wallach representation π of $G=\mathsf{G}(F)$, we set $H=\mathsf{H}(F)$ and define the multiplicity

$$m(\pi) := \dim \mathrm{Hom}_H(\pi,\xi). \tag{2.3.1}$$

From the multiplicity-one theorem established in [34; 23], we have

$$m(\pi)\leqslant 1.$$

The local Gross–Prasad conjecture (Conjecture 1) studies the refinement behavior of the multiplicity $m(\pi)$ in a relevant Vogan L-packet, which shows that there is exactly one representation π_ϕ in $\Pi^{\mathrm{Vogan}}_{F,\mathrm{rel},\phi}$ with multiplicity equal to 1 and the character $\eta_{\pi_\phi}:\mathcal{S}_\phi\to\{\pm1\}$ attached to π_ϕ is equal to an explicit character η_ϕ.

For a generic character $\phi=\phi_V\times\phi_W$ of G, the character

$$\eta_\phi=\eta^W_{\phi_V}\times\eta^V_{\phi_W}:\mathcal{S}_{\phi_V}\times\mathcal{S}_{\phi_W}\to\{\pm1\}$$

was constructed explicitly in [19, §10]. For every element $s\in\mathcal{S}_{\phi_W}\times\mathcal{S}_{\phi_V}$, set

$$\begin{aligned}\eta^W_{\phi_V}(s_V)&=\det(\mathrm{M}_V^{s_V=-1})(-1)^{\frac{\dim \mathrm{M}_W}{2}}\det(\mathrm{M}_W)(-1)^{\frac{\dim \mathrm{M}_V^{s_V=-1}}{2}}\varepsilon\big(\tfrac12,\mathrm{M}_V^{s_V=-1}\otimes\mathrm{M}_W,\psi\big),\\ \eta^V_{\phi_W}(s_W)&=\det(\mathrm{M}_W^{s_W=-1})(-1)^{\frac{\dim \mathrm{M}_V}{2}}\det(\mathrm{M}_V)(-1)^{\frac{\dim \mathrm{M}_W^{s_W=-1}}{2}}\varepsilon\big(\tfrac12,\mathrm{M}_W^{s_W=-1}\otimes\mathrm{M}_V,\psi\big).\end{aligned} \tag{2.3.2}$$

Here M_V and M_W are the spaces of the standard representation of ${}^L\mathrm{SO}(V)$ and ${}^L\mathsf{SO}(W)$, respectively. The notion $\det(\cdot)$ makes a finite-dimensional representation into a character and the $\det(\cdot)(-1)$ means its value at $-1\in\mathcal{W}^{ab}_{\mathbb{R}}\cong\mathbb{R}^\times$, equivalently, $\det(\cdot)(j)$ for $j\in\mathcal{W}_{\mathbb{R}}$. The space $\mathrm{M}_V^{s_V=-1}$ denotes the $s_V=(-1)$-eigenspace of M_V and $\varepsilon(\dots)$ is the local root number defined by the Rankin–Selberg integral [21].

When $F=\mathbb{C}$, the relevant Vogan L-packet $\Pi^{\mathrm{Vogan}}_{F,\phi,\mathrm{rel}}$ contains only one element. Hence, part (1) of the conjecture implies part (2) of the conjecture. We will prove the following theorem by constructing a nonzero element in $\mathrm{Hom}_H(\pi,\xi)$ in Section 3.

Theorem 2.3.1. *When $F=\mathbb{C}$, Conjecture 1 holds.*

When $F=\mathbb{R}$, in [28], following the work of Waldspurger [36; 38] and Beuzart-Plessis [5], Luo proved part (1) of Conjecture 1 when the parameter ϕ is tempered. In [10], by simplifying Waldspurger's approach [36; 37; 38; 39; 40], the author and Luo proved part (2) of Conjecture 1 when the parameter ϕ is tempered. The main result in Section 5 is to prove Theorem 5.0.1 that implies the following theorem based on the Conjecture 1 for tempered parameters.

Theorem 2.3.2. *When $F=\mathbb{R}$, Conjecture 1 holds.*

3. Integral method and the proof for the complex case

One of the main tools for proving Conjecture 1 is the integral method. In particular, this is the only tool we would apply to prove Conjecture 1 when $F=\mathbb{C}$. When $F=\mathbb{C}$ and $\dim V=\dim W+1$, Conjecture 1 was proved by J. Möllers in [14] using an equivalent method. In Section 3, we use some computation in [14] and present the proof using the integral method following [18].

Let $F=\mathbb{R},\mathbb{C}$. Let G be a quasisplit group over F and H be a closed subgroup of G such that G/H is absolutely spherical. Suppose there is a Borel subgroup B of G such that

$$\mathsf{B}\cap\mathsf{H}=1.$$

Let T be the Levi component of B. We set

$$G=\mathsf{G}(F),\quad H=\mathsf{H}(F),\quad B=\mathsf{B}(F),\quad T=\mathsf{T}(F).$$

Fix a unitary character ψ of F. For an algebraic character $\lambda:\mathsf{H}\to\mathbb{G}_a$, we set $\xi=\psi\circ\lambda_F$, which is a unitary character of H.

As a consequence of the integral method in [18], we have the following theorem.

Theorem 3.0.1. *Let G,H,B,T as above. For every character σ of T, we have*

$$\dim\operatorname{Hom}_H(\operatorname{Ind}_B^G(\sigma),\xi)\geqslant 1.$$

First, we construct a measure μ on $B\cdot H\subset G$ by setting $\mu=f(bh)dbdh$ where

$$f(bh):=\delta_B^{-1/2}(b)\sigma^{-1}(b)\xi(h),\quad b\in B,\ h\in H.$$

We can express the function f in the form of

$$f(bh)=t^{\mu_1}\bar{t}^{\mu_2}e^{is_1\mathrm{Re}(\lambda(h))+s_2\mathrm{Im}(\lambda(h))}\quad\forall b=t\cdot n\in B=T\cdot N,\ h\in H$$

for certain $s_1,s_2\in\mathbb{R}$ and $\mu_1,\mu_2\in\operatorname{Hom}(\mathsf{T},\mathbb{G}_m)$. Hence, for every differential operator D on $B\times H$, the growth of $|Df|$ can be controlled by a polynomial. Therefore, μ is a tempered measure on $B\cdot H$, which is left-$(B,\delta_B^{1/2}\sigma)$-equivariant and right-$(\mathsf{H}(F),\xi)$-equivariant. Because B is solvable, from [18, Theorem B], one can construct a left-$(B, \delta_{\mathsf{B}}^{1/2}\sigma)$-equivariant and right-$(H, \xi)$-equivariant distribution on G.

From [13] and the compactness of $B\backslash G$, there is a one-to-one correspondence between $\mathrm{Hom}(\mathrm{Ind}_B^G(\sigma), \xi)$ and the space of left-$(B, \delta_B^{1/2}\sigma)$-equivariant and right-$(H, \xi)$-equivariant distributions on G.

Now we return to the Gross–Prasad conjecture over $F = \mathbb{C}$. As argued in [19, §11], since there is exactly one representation in the relevant Vogan L-packet and this representation is a principal series, it suffices to verify that $m(\pi) \geqslant 1$ for every principal series representation $\pi = \mathrm{Ind}_B^G(\sigma)$. For this purpose, we verify $\mathsf{B} \cap \mathsf{H} = 1$ when $(\mathsf{G}, \mathsf{H}, \xi)$ is the Gross–Prasad triple associated to a relevant pair (W, V).

Set $\mathsf{P}_V = \mathsf{M}_V \cdot \mathsf{N}$ be the parabolic subgroup stabilizing the totally isotropic flag (2.1.1) and the Levi subgroup M_V can be decomposed as $\mathsf{M}_V = \prod_{i=1}^r \mathrm{GL}(\mathbb{C} \cdot z_i) \times \mathrm{SO}(V \oplus D)$. Let $\overline{\mathsf{P}}_V = \mathsf{M}_V \cdot \overline{\mathsf{N}}$ be the opposite parabolic subgroup of P_V.

Let $(\mathsf{G}', \mathsf{H}', \xi')$ be the Gross–Prasad triple associated to the relevant pair $(W, W \oplus D)$. From [14, §6.2.4], there exists a Borel subgroup B' of $\mathsf{G}' = \mathrm{SO}(W \oplus D) \times \mathrm{SO}(W)$ such that $\mathsf{B}' \cap \mathsf{H}' = 1$. We set $\mathsf{B} = \mathsf{B}' \cdot \prod_{i=1}^r \mathrm{GL}(\mathbb{C} \cdot z_i) \cdot \mathsf{B}' \cdot (\bar{\mathsf{N}} \times 1)$. Consider the parabolic subgroup $\mathsf{P} = \mathsf{P}_V \times \mathrm{SO}(W) = \mathsf{M} \cdot (\mathsf{N} \times 1)$ of G. Since $\prod_{i=1}^r \mathrm{GL}(\mathbb{C} \cdot z_i)\mathsf{B}'$ and H' are subgroups of $\mathsf{M} = \mathsf{M}_V \times \mathrm{SO}(W)$ such that

$$\prod_{i=1}^r \mathrm{GL}(\mathbb{C} \cdot z_i)\mathsf{B}' \cap \mathsf{H}' = 1,$$

we have

$$\mathsf{B} \cap \mathsf{H} = \overline{\mathsf{N}} \cdot \prod_{i=1}^r \mathrm{GL}(\mathbb{C} \cdot z_i)\mathsf{B}' \cap \mathsf{H}' \cdot \mathsf{N} = \mathrm{GL}(\mathbb{C} \cdot z_i)\mathsf{B}' \cap \mathsf{H}' = 1.$$

This completes the proof for Theorem 2.3.1.

4. Representations in generic packets

In this section, we prove that, for every parameter ϕ of a special orthogonal group over $\mathbb{R}$, there is a tempered L-parameter ϕ_0 of a smaller special orthogonal group with decomposition $\phi = \phi^{\mathrm{GL}} \oplus \phi_0 \oplus (\phi^{\mathrm{GL}})^\vee$, such that the parabolic induction

$$\pi_0 \mapsto \sigma \ltimes \pi_0$$

induces isomorphism before $\Pi_{\phi_0}^{\mathrm{Vogan}}$ and $\Pi_{\phi}^{\mathrm{Vogan}}$, where σ is the unique representation in the packet $\Pi_{\phi^{\mathrm{GL}}}$.

Let V be a nondegenerate quadratic space over $\mathbb{R}$. It is well-known that an L-parameter ϕ_V of $\mathrm{SO}(V)$ is generic if and only if the adjoint L-function $L(s, \phi_V, \mathrm{Ad})$ is holomorphic at $s = 1$ (see [19, Conjecture 2.6] and the remark after it). Based on this property, we first compute an equivalent condition for ϕ_V to be generic.

Definition 4.0.1. Given a generic L-parameter $\phi_V : \mathcal{W}_{\mathbb{R}} \to {}^L\mathrm{SO}(V)$, we denote by ϕ_V^{ss} the *semisimplification* of ϕ_V, that is, the semisimple representation on M_V

defined by the composition ϕ_V with the standard representation $\mathrm{std}_V : {}^L\mathrm{SO}(V) \to \mathrm{GL}(M_V)$.

Given an L-parameter ϕ_V, its semisimplification ϕ_V^{ss} can be decomposed as

$$\phi_V^{\mathrm{ss}} = \bigoplus |\cdot|^{s^1_{V,i}} \phi^1_{l_{V,i}} + \bigoplus |\cdot|^{s^2_{V,i}} \phi^2_{m_{V,i}}. \tag{4.0.1}$$

Here $\phi^1_{l_{V,i}}$ ($l_{V,i} \in \mathbb{Z}$) is a one-dimensional representation of $\mathcal{W}_{\mathbb{R}} = \mathbb{C} \cup \mathbb{C} \cdot j$ ($j^2 = -1$) defined by

$$\phi^1_{l_{V,i}}(z) = 1, \quad \phi^2_{l_{V,i}}(z \cdot j) = (-1)^{l_{V,i}}, \quad z \in \mathbb{C},$$

and $\phi^2_{m_{V,i}}$ ($m_{V,i} \in \mathbb{N}$) is the two-dimensional representation of $\mathcal{W}_{\mathbb{R}}$ with basis u, v satisfying

$$\phi^2_{m_{V,i}}(z)u = u, \quad \phi^2_{m_{V,i}}(z \cdot j)u = (-1)^{m_{V,i}} v,$$
$$\phi^2_{m_{V,i}}(z)v = v, \quad \phi^2_{m_{V,i}}(z \cdot j)v = u.$$

The adjoint L-function $L(s, \phi_V, \mathrm{Ad}) = L(s, \phi_V^{\mathrm{ss}} \otimes \phi_V^{\mathrm{ss},\vee})$ is a product of factors

$$L(s, \phi_V, |\cdot|^{s^1_{V,i}} \phi^1_{l_{V,i}} \otimes (|\cdot|^{s^1_{V,j}} \phi^1_{l_{V,j}})^\vee), \quad L(s, \phi_V, |\cdot|^{s^1_{V,i}} \phi^1_{l_{V,i}} \otimes (|\cdot|^{s^2_{V,j}} \phi^2_{m_{V,j}})^\vee),$$
$$L(s, \phi_V, |\cdot|^{s^2_{V,i}} \phi^2_{m_{V,i}} \otimes (|\cdot|^{s^1_{V,j}} \phi^1_{l_{V,j}})^\vee), \quad L(s, \phi_V, |\cdot|^{s^2_{V,i}} \phi^2_{l_{V,i}} \otimes (|\cdot|^{s^2_{V,j}} \phi^2_{m_{V,j}})^\vee).$$

From [25], we can compute the value of these L-functions and obtain that:

(1) $L(s, \phi_V, |\cdot|^{s^1_{V,i}} \phi^1_{l_{V,i}} \otimes (|\cdot|^{s^1_{V,j}} \phi^1_{l_{V,j}})^\vee)$ has a pole at $s = 1$ if and only if $\frac{1}{2}(1 + s^1_{V,i} - s^1_{V,j} + (1 - (-1)^{l_{V,i}+l_{V,j}})/2)$ is a nonpositive integer.

(2) $L(s, \phi_V, |\cdot|^{s^1_{V,i}} \phi^1_{m_{V,i}} \otimes (|\cdot|^{s^2_{V,j}} \phi^2_{m_{V,j}})^\vee)$ has a pole at $s = 1$ if and only if $1 + s^1_{V,i} - s^2_{V,j} + \frac{1}{2} m_{V,j}$ is a nonpositive integer.

(3) $L(s, \phi_V, |\cdot|^{s^2_{V,i}} \phi^2_{m_{V,i}} \otimes (|\cdot|^{s^1_{V,j}} \phi^1_{l_{V,j}})^\vee)$ has a pole at $s = 1$ if and only if $1 + s^2_{V,i} - s^1_{V,j} + \frac{1}{2} m_{V,i}$ is a nonpositive integer.

(4) $L(s, \phi_V, |\cdot|^{s^2_{V,i}} \phi^2_{m_{V,i}} \otimes (|\cdot|^{s^2_{V,j}} \phi^2_{m_{V,j}})^\vee)$ has a pole at $s = 1$ if and only if $1 + s^2_{V,i} - s^2_{V,j} + \frac{1}{2}(m^2_{V,i} + m_{V,j})$ or $1 + s^2_{V,i} - s^2_{V,j} + \frac{1}{2}(|m_{V,i} - m_{V,j}|)$ is a nonpositive integer.

Lemma 4.0.2. *A parameter ϕ_V with semisimplification ϕ_V^{ss} in (4.0.1) is generic if and only if none of*

$$\tfrac{1}{2}(1 + s^1_{V,i} - s^1_{V,j} + (1 - (-1)^{l_{V,i}+l_{V,j}})/2), \ 1 + s^1_{V,i} - s^2_{V,j} + \tfrac{1}{2} m_{V,j},$$
$$1 + s^2_{V,i} - s^1_{V,j} + \tfrac{1}{2} l_{V,i}, \ 1 + s^2_{V,i} - s^2_{V,j} + \tfrac{1}{2}(|m_{V,i} - m_{V,j}|)$$

is a nonpositive integer.

Irreducibility criteria. B. Speh and D. Vogan gave a sufficient condition for the irreducibility of limits of generalized principal series representations in [32, Theorem 6.19]. We apply this result to prove the irreducibility of standard models for representations in generic packets.

Definition 4.0.3. Given $\sigma_1 \in \Pi(\mathrm{GL}_{n_1}), \ldots, \sigma_r \in \Pi(\mathrm{GL}_{n_r})$ and $\pi_{V_0} \in \Pi(\mathrm{SO}(p,q))$. We denote by

$$\sigma_1 \times \cdots \times \sigma_r \ltimes \pi_{V_0}$$

the normalized parabolic induction

$$\mathrm{I}_{P_{n_1,\cdot,n_r,p+q}}^{\mathrm{SO}(p+n,q+n)}(\sigma_1 \otimes \cdots \otimes \sigma_r \otimes \pi_{V_0}) \in \Pi(\mathrm{SO}(p+n,q+n)), \quad n = n_1 + \cdots + n_r.$$

Lemma 4.0.4. *Fix a generic parameter $\phi_V = \phi_V^{\mathrm{GL}} \oplus \phi_{V_0} \oplus (\phi_V^{\mathrm{GL}})^\vee$ of $\mathrm{SO}(p,q)$ $(p > q)$. For $\sigma \in \Pi_{\phi_V^{\mathrm{GL}}}$ and $\pi_{V_0} \in \Pi_{\phi_{V_0}}^{\mathrm{Vogan}}$, the representation $\sigma \ltimes \pi_{V_0}$ is irreducible.*

Proof. From [25, Theorem 14.2], we may write the tempered representation π_{V_0} as a parabolic induction from a limit of discrete series representations. Then we can express $\sigma \ltimes \pi_{V_0}$ as

$$\sigma_1 \times \cdots \times \sigma_l \ltimes \pi_{V_0'} \quad \sigma_i \in \Pi(\mathrm{GL}_{n_{V,i}}) \tag{4.0.2}$$

where $\pi'_{V_0} \in \Pi(\mathrm{SO}(V_0'))$ is a limit of discrete series representation and

$$\sigma_i = |\cdot|^{s_{V,i}^1}\mathrm{sgn}^i \text{ or } \sigma_i = |\det|^{s_{V,i}^2} D_{m_{V,i}}.$$

Following [32, Theorem 6.19], it suffices to check the following conditions:

(4.0.3) For every root α such that

$$n_\alpha = \langle \alpha, \nu\rangle / \langle \alpha, \alpha\rangle \in \mathbb{Z},$$

(1) if α is a complex root ($\alpha \neq -\theta\alpha$), then $\langle \alpha, \nu\rangle\langle \theta\alpha, \nu\rangle \geqslant 0$;

(2) if α is a real root ($\alpha = -\theta\alpha$), then

$$(-1)^{n_\alpha+1} = \epsilon_\alpha \cdot \lambda(m_\alpha)$$

Here λ is the central character of σ, m_α is the image of $\rho_\alpha(-I_2)$ in G for the embedding $\rho_\alpha : \mathrm{SL}_2(\mathbb{R}) \to G(\mathbb{R})$ determined by α and $\epsilon_\alpha = -1$.

Then we check them using Lemma 4.0.2.

(1) For every complex root α such that $n_\alpha \in \mathbb{Z}$,

(a) if α is a root of $\mathrm{SO}(p-q)$, then $\langle \alpha, \nu\rangle\langle \theta\alpha, \nu\rangle = 0$;

(b) otherwise, $\theta\alpha = \alpha$, and then $\langle \alpha, \nu\rangle\langle \theta\alpha, \nu\rangle = \langle \alpha, \nu\rangle^2 \geqslant 0$.

(2) For every real root $\beta_{ab} = e_a - e_b$ such that $n_{\beta_{ab}} \in \mathbb{Z}$.

(a) If E_{aa} is in the GL_1-block $\mathrm{GL}_{n_{V,i}}$ and E_{bb} is the in a GL_1-block $\mathrm{GL}_{n_{V,j}}$ (in the inducing datum in (4.0.2)), then $n_{\beta_{ab}} = \frac{1}{2}(s^1_{V,i} - s^1_{V,j})$ is an integer, and both

$\frac{1}{2}(1+s^1_{V,i}-s^1_{V,j}+(1-(-1)^{l_{V,i}+l_{V,j}})/2)$ and $\frac{1}{2}(1+s^1_{V,j}-s^1_{V,i}+(1-(-1)^{l_{V,i}+l_{V,j}})/2)$

are not nonpositive integers. If $l_{V,i}+l_{V,j}$ is odd, the sum is equal to 2, then $s^1_{V,i} = s^1_{V,j}$ or $s^1_{V,i} - s^1_{V,j}$ is odd. If $l_{V,i}+l_{V,j}$ is even, the sum is equal to 3/2, then $s^1_{V,i} - s^1_{V,j}$ is even.

(b) If E_{aa} is in the GL_1-block $\mathrm{GL}_{n_{V,i}}$ and E_{bb} is the in a GL_2-block $\mathrm{GL}_{n_{V,j}}$, Lemma 4.0.2 implies

$$s^2_{V,j} - \tfrac{1}{2}l_{V,j} \leqslant s^1_{V,i} \leqslant s^2_{V,j} + \tfrac{1}{2}l_{V,j}.$$

(c) If E_{aa} is in the GL_2-block $\mathrm{GL}_{n_{V,i}}$ and E_{bb} is the in a GL_2-block $\mathrm{GL}_{n_{V,j}}$, we may assume $l_{V,j} \geqslant l_{V,i}$, Lemma 4.0.2 implies

$$s^2_{V,j} - \tfrac{1}{2}l_{V,j} \leqslant s^2_{V,i} - \tfrac{1}{2}l_{V,i} \leqslant s^2_{V,i} + \tfrac{1}{2}l_{V,i} \leqslant s^2_{V,j} + \tfrac{1}{2}l_{V,j}.$$

Therefore, we have checked cases (b) and (c) following an understanding of the parity condition in [30, Theorem 2]. For case (a), parity holds unless $l_{V,i}+l_{V,j}$ is odd and $s^1_{V,i} = s^1_{V,j}$. In this situation

$$|\cdot|^{s^1_{V,i}}\mathrm{sgn}^{l_{V,i}} \times |\cdot|^{s^1_{V,j}}\mathrm{sgn}^{l_{V,j}} = |\cdot|^{s^1_{V,i}}\mathrm{sgn}^{l_{V,i}}(1 \times \mathrm{sgn})$$

And $1 \times \mathrm{sgn}$ is the limit of a discrete series representation with parameter ϕ^2_0, which can be treated as in cases (b) and (c). □

Representations in generic packets. The classification of representations of $\mathcal{W}_{\mathbb{R}}$ [25] shows the following factorization into irreducible representations:

$$\phi^{\mathrm{ss}}_V = \phi^{\mathrm{GL}}_V \oplus \phi_{V_0} \oplus (\phi^{\mathrm{GL}}_V)^\vee, \tag{4.0.4}$$

where ϕ_{V_0} is tempered and

$$\phi^{\mathrm{GL}}_V = \bigoplus_{i=1}^{l_V} |\cdot|^{s_i}\phi^{\mathrm{GL}}_{V,i} \quad \text{where } \mathrm{Re}(s_i) > 0 \text{ for } 1 \leqslant i \leqslant l_V$$

for discrete series parameter $\phi_{V,i}$ (i.e., the image of $\phi_{V,i}$ is bounded and does not lie in any proper Levi).

It is straightforward that ϕ^{GL}_V is unpaired. Let $n_{V,i} = \dim \phi^{\mathrm{GL}}_{V,i}$, $n_V = \dim \phi^{\mathrm{GL}}_V$ and $\sigma_{V,i}$ be the unique representation of GL_n in the L-packet $\Pi_{\phi^{\mathrm{GL}}_{V,i}}(\mathrm{GL}_{n_{V,i}})$, then

$$\Pi_{\phi^{\mathrm{GL}}_V}(\mathrm{GL}_{n_V}) = \{\sigma_V\} \quad \text{where } \sigma_V = |\det|^{s_1}\sigma_{V,1} \times \cdots \times |\det|^{s_{l_V}}\sigma_{V,l_V} \tag{4.0.5}$$

By Lemma 4.0.4, there is an injective map

$$\Pi^{\mathrm{Vogan}}_{\phi_{V_0}} \to \Pi^{\mathrm{Vogan}}_{\phi_V}, \quad \pi_{V_0} \mapsto \sigma_V \ltimes \pi_{V_0}. \tag{4.0.6}$$

Since ϕ_V^{GL} is unpaired, $|\mathcal{S}_{\phi_{V_0}}| = |\mathcal{S}_{\phi_V}|$ and thus $|\Pi^{\mathrm{Vogan}}_{\phi_{V_0}}| = |\Pi^{\mathrm{Vogan}}_{\phi_V}|$. This implies that the above map is an isomorphism and we have the following result.

Proposition 4.0.5. *For a generic L-parameter $\phi_V = \phi_V^{\mathrm{GL}} \oplus \phi_{V_0} \oplus (\phi_V^{\mathrm{GL}})^\vee$, every representation π_V in $\Pi^{\mathrm{Vogan}}_{\phi_V}$ can be expressed as $\pi_V = \sigma_V \ltimes \pi_{V_0}$ where $\pi_{V_0} \in \Pi^{\mathrm{Vogan}}_{\phi_{V_0}}$ and σ_V given in (4.0.5).*

This result shows that representations in the generic packets are in the form

$$\pi_V = \sigma_V \ltimes \pi_{V_0}, \quad \sigma_V = |\det|^{s_{V,1}}\sigma_{V,1} \times \cdots \times |\det|^{s_{V,r}}\sigma_{V,r}, , \tag{4.0.7}$$

where $\mathrm{Re}(s_{V,1}) \geqslant \mathrm{Re}(s_{V,2}) \geqslant \cdots \geqslant \mathrm{Re}(s_{V,r}) > 0$, and tempered $\pi_0 \in \mathrm{Irr}(\mathrm{SO}(V_0))$. And $\sigma_{V,i} = \mathrm{sgn}^{l_{V,i}}$ for $l_{V,i} = 0, 1$ or $\sigma_{V_i} = D_{m_{V,i}}$ for $m_i \in \mathbb{N}_+$.

For π_V in the form of (4.0.7), we define the following notions.

Definition 4.0.6. We parametrize the infinitesimal character of π_V with the *Harish-Chandra parameter* for π_V in (4.0.7) is defined as

$$\nu = (\nu_1, \ldots, \nu_r, \nu_{\pi_{V_0}})$$

where $\nu_{\pi_{V_0}}$ is the Harish-Chandra parameter of the tempered representation π_{V_0}, $\nu_i = s_i$ when $\rho_{V,i} = \mathrm{sgn}^{l_{V,i}}$, and $\nu_i = \left(s_{V,i} + \frac{1}{2}m_{V,i}, s_{V,i} - \frac{1}{2}m_{V,i}\right)$ when $\rho_{V,i} = D_{m_{V,i}}$.

Definition 4.0.7. We define the *leading index* of π_V as the largest number among $\mathrm{Re}(s_{V,i})$. We denote it by $\mathrm{LI}(\pi_V)$.

5. Proof for the real case

We now complete the proof of the local Gross–Prasad conjecture (Conjecture 1) over the real field based on the tempered cases. More specifically, following the approach in [29], we prove a multiplicity formula for the reduction to the tempered cases and conclude the conjecture with the tempered cases proved in [10].

The proof uses the idea of Mackey's theory. Let G be a reductive group over $\mathbb{R}$, H is a closed subgroup of G and P is a parabolic subgroup of G with Levi decomposition $\mathsf{P} = \mathsf{MN}$. We denote by $G = \mathsf{G}(\mathbb{R})$, $H = \mathsf{H}(\mathbb{R})$ and $P = \mathsf{P}(\mathbb{R})$. For a representation σ of $M = \mathsf{M}(\mathbb{R})$, we study the space $\mathrm{Hom}_H(\mathrm{Ind}_P^G(\sigma), 1_H)$ by analyzing the double coset $P\backslash G/H$. Since $P\backslash G$ is compact, the smooth induction $\mathrm{Ind}_P^G(\sigma)$ is equal to the Schwartz induction in the sense of [13]. In order to use the analytic tools established in [13] and [11], we work within the category of almost linear Nash groups [33, Definition 1.1] and consider the category of Nash manifolds [33, Definition 2.1], with the possible action of certain almost linear Nash groups.

In particular, for a linear algebraic group G over $\mathbb{R}$, $\mathsf{G}(\mathbb{R})$ can be treated as an almost linear Nash group.

Let G be an almost linear Nash group. We denote by $\mathcal{SF}(G)$ the categories of smooth Fréchet G-representations of moderate growth. We denote by $\mathcal{CW}(G)$ the subcategory of $\mathcal{SF}(G)$ consisting of representations with admissible $(\mathfrak{g}_{\mathbb{C}}, K)$-modules, that is, the category of Casselman–Wallach representations of G. We use $\mathrm{Irr}(G)$ to denote the set of irreducible Casselman–Wallach representations of G.

Our main result in this section is the following theorem.

Theorem 5.0.1 (multiplicity formula). *Let V, W be quadratic spaces with decompositions $V = V_0 \perp (X_V + X_V^\vee)$, $W = W_0 \perp (X_W + X_W^\vee)$. Let $\pi_{V_0} \in \mathrm{Irr}(\mathrm{SO}(V_0))$, $\pi_{W_0} \in \mathrm{Irr}(\mathrm{SO}(W_0))$ be tempered representations and $\sigma_V \in \mathcal{CW}(\mathrm{SO}(V))$, $\sigma_W \in \mathcal{CW}(\mathrm{SO}(W))$ such that*

$$\sigma_V = |\det|^{s_{V,1}} \sigma_{V,1} \times \cdots \times |\det|^{s_{V,r_V}} \sigma_{V,r_V},$$
$$\sigma_W = |\det|^{s_{W,1}} \sigma_{W,1} \times \cdots \times |\det|^{s_{W,r_W}} \sigma_{W,r_W}, \tag{5.0.1}$$

for $\mathrm{Re}(s_{V,i}), \mathrm{Re}(s_{W,i}) > 0$ and tempered representations $\sigma_{V,i} \in \mathrm{Irr}(\mathrm{GL}_{n_{V,i}}(F))$ $(i = 1, \ldots, r_V)$, $\sigma_{W,i} \in \mathrm{Irr}(\mathrm{GL}_{n_{W,i}}(F))$ $(j = 1, \ldots, r_W)$; here $n_{V,i}$, $n_{W,i}$ are integers such that $\sum_{i=1}^{r_V} n_{V,i} = \dim X_V$ and $\sum_{i=1}^{r_W} n_{W,i} = \dim X_W$. Then we have

$$m((\sigma_V \ltimes \pi_{V_0}) \boxtimes (\sigma_W \ltimes \pi_{W_0})) = m(\pi_{V_0} \boxtimes \pi_{W_0}).$$

Note that in the theorem, the representations $\sigma_V \ltimes \pi_{V_0}$ and $\sigma_W \ltimes \pi_{W_0}$ can be reducible. The reducible case of the multiplicity formula is actually necessary when it is applied in [12]. In this article, to complete the proof for the real case of the Gross–Prasad conjecture, we only use the formula when both $\sigma_V \ltimes \pi_{V_0}$ and $\sigma_W \ltimes \pi_{W_0}$ are irreducible.

Proof of Theorem 2.3.2 given Theorem 5.0.1. Given generic parameters ϕ_V, ϕ_W, from Proposition 4.0.5, we can express the parameters as

$$\phi_V = \phi_V^{\mathrm{GL}} + \phi_{V_0} + (\phi_V^{\mathrm{GL}})^\vee, \quad \phi_W = \phi_W^{\mathrm{GL}} + \phi_{W_0} + (\phi_W^{\mathrm{GL}})^\vee \tag{5.0.2}$$

such that ϕ_V^{GL} has no self-dual subrepresentation.

Let σ_V be the unique representation in $\Pi^{\mathrm{Vogan}}_{\phi_V^{\mathrm{GL}}}$ and σ_W be the unique representation in $\Pi^{\mathrm{Vogan}}_{\phi_W^{\mathrm{GL}}}$. For every $\pi_V \boxtimes \pi_W \in \Pi^{\mathrm{Vogan}}_{\phi_V \times \phi_W}$, there exists $\pi_{V_0} \boxtimes \pi_{W_0} \in \Pi^{\mathrm{Vogan}}_{\phi_V \times \phi_W}$ such that

$$\pi_V = \sigma_V \ltimes \pi_{V_0}, \quad \pi_W = \sigma_W \ltimes \pi_{W_0}.$$

Therefore, the maps

$$\Pi^{\mathrm{Vogan}}_{\phi_{V_0}} \to \Pi^{\mathrm{Vogan}}_{\phi_V}, \quad \pi_{V_0} \mapsto \sigma_V \ltimes \pi_{V_0}, \quad \text{and}$$
$$\Pi^{\mathrm{Vogan}}_{\phi_{W_0}} \to \Pi^{\mathrm{Vogan}}_{\phi_W}, \quad \pi_{W_0} \mapsto \sigma_W \ltimes \pi_{W_0}$$

are isomorphisms. Hence, we can identify the component group $\mathcal{S}_{\phi_{V_0}\times\phi_{W_0}}$ with $\mathcal{S}_{\phi_V\times\phi_W}$. Under this identification, it can be easily verified that for ϕ_V, ϕ_W, ϕ_{V_0}, ϕ_{W_0}, we have

$$\eta_{\phi_{V_0}\times\phi_{W_0}} = \eta_{\phi_V\times\phi_W}.$$

Theorem 5.0.1 reduces Conjecture 1 for ϕ_V, ϕ_W to that for ϕ_{V_0}, ϕ_{W_0}, which is a tempered case proved in [28; 10]. □

Following [29], there are three steps in our proof for Theorem 5.0.1: reduction to basic cases, the first inequalities, and the second inequalities.

A relevant pair (W, V) is called *basic* if $\dim V = \dim W + 1$. For a general relevant pair (W, V) with decomposition $V = W \perp Z \perp D$, we let D^+ be the anisotropic line with the opposite signature to D. We set $Z^+ = Z \perp (D + D^+)$ and set $(V, W^+) = (V, Z^+ \oplus W)$ and we call (V, W^+) the *basic relevant pair associate to* (W, V).

Definition 5.0.2. Let $s_1, s_2, \dots, s_{r+1}$ be complex numbers. We say the $(r+1)$-tuple $\underline{s} = (s_1, \dots, s_{r+1})$ are *in general position*, if $\underline{s} \in \mathbb{C}^{t+1}$ does not lie in the set of zeros of countably many polynomial functions on $\mathbb{C}^{t+1}$.

For the $(r+1)$-tuple $\underline{s} = (s_1, \dots, s_{r+1})$, we denote by $\sigma_{\underline{s}}$ the spherical principal series representation $|\cdot|^{s_1} \times \cdots \times |\cdot|^{s_{r+1}}$.

Lemma 5.0.3 (reduction to basic cases). *For every* $\pi_V \in \mathrm{Irr}(\mathrm{SO}(V))$ *and* $\pi_W \in \mathrm{Irr}(\mathrm{SO}(W))$, *we have*

$$m(\pi_V \boxtimes \pi_W) = m((\sigma_{\underline{s}} \ltimes \pi_W) \boxtimes \pi_V)$$

for $\underline{s} = (s_1, \dots, s_{r+1}) \in \mathbb{C}^{r+1}$ *in general position.*

With this, we find such a spherical principal series $\sigma_{\underline{s}}$ and reduce Theorem 5.0.1 to the case for a relevant pair $(V, W \oplus Z^+)$ and representations $\sigma_{\underline{s}} \ltimes \pi_W$, π_V that can be expressed in the parabolic induction form as in (4.0.7), which is a basic case.

Proposition 5.0.4 (basic case of the multiplicity formula). *Given a basic relevant pair* (W, V), *let* $\pi_V \in \mathcal{CW}(\mathrm{SO}(V))$ *and* $\pi_W \in \mathcal{CW}(\mathrm{SO}(W))$ *as in Theorem 5.0.1, we have*

$$m(\pi_V \boxtimes \pi_W) = m(\pi_{V_0} \boxtimes \pi_{W_0}).$$

The inequalities $m(\pi_V \boxtimes \pi_W) \geqslant m(\pi_{V_0} \boxtimes \pi_{W_0})$ and $m(\pi_V \boxtimes \pi_W) \leqslant m(\pi_{V_0} \boxtimes \pi_{W_0})$ are called “the first inequality” and “the second inequality” in [29]. Using a similar approach as [29], we prove the first inequality using mathematical induction with the following lemma as the building block (Section 5.3).

Lemma 5.0.5. *Let* π_V *be a representation in a generic packet and* $\pi_W \in \mathrm{Irr}(\mathrm{SO}(W))$.

(1) *When* $\dim V = \dim W + 1$ *and* $\mathrm{Re}(s) \geqslant \mathrm{LI}(\pi_V)$, *we have*

$$m(\pi_V \boxtimes \pi_W) \geqslant m((|\cdot|^s \mathrm{sgn}^m \ltimes \pi_W) \boxtimes \pi_V).$$

(2) *When* $\dim V = \dim W + 3$ *and* $\mathrm{Re}(s) \geqslant \mathrm{LI}(\pi_V)$, *we have*

$$m(\pi_V \boxtimes (|\cdot|^{s+\frac{m}{2}} \mathrm{sgn}^{m+1} \ltimes \pi_W)) \geqslant m((|\det|^s D_m \ltimes \pi_W) \boxtimes \pi_V),$$

where D_m *is the Langlands quotient of the induction* $|\cdot|^{-\frac{m}{2}} \times |\cdot|^{\frac{m}{2}} \mathrm{sgn}^{m+1}$.

The second inequality holds in a more general setup.

Lemma 5.0.6. *For* $\pi_V \in \mathcal{CW}(\mathrm{SO}(V))$, $\pi_W \in \mathcal{CW}(\mathrm{SO}(W))$ *and* σ_{X^+} *is a generic representation in* $\mathcal{CW}(\mathrm{GL}(X^+))$, *we have*

$$m(\pi_V \boxtimes \pi_W) \leqslant m((\sigma_{X^+} \ltimes \pi_W) \boxtimes \pi_V)$$

We prove one inequality of Lemma 5.0.3 and Lemma 5.0.5 in Section 5.3 and prove the other inequality of Lemma 5.0.3 and Lemma 5.0.6 in Section 5.4. It is worth mentioning that Lemma 5.0.3 can also be proved with Schwartz homology as in [43].

5.1. ***Some functors and vanishing theorems.*** In this section, we review some analytic tools established in [13] and [11] to study certain Fréchet spaces of moderate growth.

Schwartz induction. Let G be an almost linear Nash group.

Proposition 5.1.1. *For* $\pi \in \mathcal{CW}(G)$, *the projective tensor product* $\cdot \widehat{\otimes} \pi$ *is an exact functor in* $\mathcal{SF}(G)$.

Proof. From [2], the underlying Fréchet space of π is nuclear and the proposition follows from [8, Lemma A.3]. □

Let H be a Nash subgroup of G and $\pi_H \in \mathcal{SF}(H)$. We denote by $H\backslash(G \times \pi_H)$ the vector bundle over $H\backslash G$ obtained by $G \times \pi_H$ quotient by left H-action

$$h.(g, v) = (h \cdot g, \pi_H(h).v) \quad \text{for } h \in H, \quad g \in G \text{ and } v \in \pi_H. \tag{5.1.1}$$

This vector bundle is tempered. We define the *Schwartz induction* as the functor

$$\mathrm{Ind}_P^{\mathcal{S},G} : \mathcal{SF}(H) \to \mathcal{SF}(G), \quad \pi_H \mapsto \Gamma^{\mathcal{S}}(H\backslash G, \pi_H),$$

where $\Gamma^{\mathcal{S}}(H\backslash G, \pi_H)$ stands for the space of Schwartz sections over the tempered vector bundle $H\backslash(G \times \pi_H)$. In particular, when G is reductive and $P \subset G$ is a parabolic subgroup of it, $P\backslash G$ is compact, so the Schwartz induction $\mathrm{Ind}_P^{\mathcal{S},G}$ coincides with the smooth induction, and we denote by I_P^G the normalized induction $\mathrm{Ind}_P^{\mathcal{S},G}(\delta_P^{1/2} \cdot)$, where δ_P is the modular characters of P. We will use the following properties of Schwartz inductions.

Proposition 5.1.2. (1) [11, Propositon 7.1] $\mathrm{Ind}_H^{\mathcal{S},G}$ *is an exact functor* $\mathcal{SF}(H) \to \mathcal{SF}(G)$.

(2) [11, Proposition 7.2] *For a closed subgroup* H' *of* H, *we have*

$$\mathrm{Ind}_H^{\mathcal{S},G} \circ \mathrm{Ind}_{H'}^{\mathcal{S},H} = \mathrm{Ind}_{H'}^{\mathcal{S},G}.$$

(3) [11; 2, Proposition 7.4] *For* $\pi_G \in \mathcal{CW}(G)$ *and* $\pi_H \in \mathcal{SF}(H)$, *then*

$$\mathrm{Ind}_H^{\mathcal{S},G}(\pi_H \widehat{\otimes} \pi_G|_H) = \mathrm{Ind}_H^{\mathcal{S},G}(\pi_H) \widehat{\otimes} \pi_G.$$

The* Hom*-functor. For any category $\mathcal{C}$ and object M, it is well-known that the functor $\mathrm{Hom}(-, M)$ is left exact and invariant under projective limit. We first apply this result to the category $\mathcal{SF}(G)$ and obtain the following result.

Lemma 5.1.3. (1) *For an exact sequence* $0 \to \pi_1 \to \pi_2 \to \pi_3 \to 0$ *in* $\mathcal{SF}(G)$, *suppose* $\mathrm{Hom}_G(\pi_1, 1_G) = \mathrm{Hom}_G(\pi_3, 1_G) = 0$. *Then*

$$\mathrm{Hom}_G(\pi_2, 1_G) = 0.$$

(2) *For a directed set* I *and projective system* $(\pi_\alpha, f_{\alpha\beta})_{\alpha,\beta\in I}$ *in* $\mathcal{SF}(G)$, *and for* $I' \subset I$, *suppose* $\mathrm{Hom}_G(\pi_\alpha, 1_G) = 0$ *for all* $\alpha \in I'$. *Then*

$$\mathrm{Hom}(\varprojlim_{i\in I} \pi_\alpha, 1_G) = 0.$$

Definition 5.1.4. (1) For a countable directed set I and a Fréchet space V, a set $\{V_k\}_{k\in I}$ of subspaces of V is called a *complete decreasing filtration* of π if

(a) $V_j \subset V_i$ for $i < j$, and, denoting by f_{ji} the injection maps,
(b) $\{V_i, f_{ji}\}_{i<j\in I}$ is a complete projective system, that is,

$$\varprojlim_{i\in I} V/V_i = V.$$

(2) The *composition factors* of a complete decreasing filtration are

$$V_\alpha/V_{\alpha+}, \quad \alpha \in I,$$

where $\alpha+$ is the successor of α in I.

Corollary 5.1.5. *For an almost linear Nash group* G, $\pi \in \mathcal{SF}(G)$ *and a complete decreasing filtration* $\{\pi_k\}_{k\in I}$ *of* π, *suppose* $\mathrm{Hom}_G(V_\alpha/V_{\alpha+}, 1_G) = 0$ *for all* $\alpha \in I$. *Then we have*

$$\mathrm{Hom}_G(\pi, 1_G) = 0.$$

Proof. This can be obtained from Lemma 5.1.3 with the arguments in [42]. □

Propositions 8.2 and 8.3 of [11] provide a complete decreasing filtration that is helpful for distributional analysis.

Theorem 5.1.6. *Let $\mathcal{X}$ be a Nash manifold, $\mathcal{Z}$ be a closed Nash manifold of $\mathcal{X}$ and $\mathcal{U} = \mathcal{X} - \mathcal{Z}$. There is a decreasing complete filtration on $\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}, \mathcal{E})$, denoted by $\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}, \mathcal{E})_k$, whose composition factors are isomorphic to*

$$\Gamma^{\mathcal{S}}(\mathcal{Z}, \mathrm{Sym}^k \mathcal{N}^{\vee}_{\mathcal{Z}/\mathcal{X}} \otimes \mathcal{E}|_{\mathcal{Z}}), \quad k = 0, 1, \dots, \tag{5.1.2}$$

where $\mathcal{N}^{\vee}_{\mathcal{Z}/\mathcal{X}}$ is the conormal bundle over $\mathcal{Z}$ (see [11, Section 6.1].)

Vanishing by infinitesimal characters.

Definition 5.1.7. For an infinitesimal character $\chi : \mathcal{Z}(\mathcal{U}(\mathfrak{g}_{\mathbb{C}})) \to \mathbb{C}$, we denote by $\chi^{\vee}$ the infinitesimal character generated by the relation

$$\chi^{\vee}(X) = \chi(-X), \quad X \in \mathfrak{g}_{\mathbb{C}}.$$

Theorem 5.1.8. *For representations π_1, π_2 of G with infinitesimal characters $\chi_{\pi_1}, \chi_{\pi_2}$, satisfying $\chi_{\pi_1} \neq \chi^{\vee}_{\pi_2}$, we have*

$$\mathrm{Hom}_G(\pi_1 \widehat{\otimes} \pi_2, 1_G) = 0.$$

Proof. The existence of elements in $\mathrm{Hom}_G(\pi_1 \widehat{\otimes} \pi_2, 1_G)$ implies the existence of a homomorphism on $(\mathfrak{g}_{\mathbb{C}}, K)$-modules. This contradicts the relation of infinitesimal characters. □

We apply the above theorem in the following setup:

Corollary 5.1.9. *Suppose $\pi_{V_0} \in \mathcal{SF}(\mathrm{SO}(V_0))$ and $\pi_V \in \mathrm{Irr}(\mathrm{SO}(V))$.*

$$(\sigma_{\underline{s}} \ltimes \pi_{V_0}) \widehat{\otimes} \pi_V$$

for $\sigma_{\underline{s}} = |\cdot|^{s_1} \times \cdots \times |\cdot|^{s_r}$ and $\underline{s} = (s_1, \dots, s_r)$ in general positions.

Vanishing by leading index.

Definition 5.1.10. By the Langlands classification, for every $\pi_V \in \mathrm{Irr}(\mathrm{SO}(V))$, we can express π_V as the Langlands quotient of a certain induction

$$|\det|^{s_1}\rho_1 \times \cdots \times |\det|^{s_r}\rho_r \ltimes \pi_{V_0} \tag{5.1.3}$$

for $\mathrm{Re}(s_1) \geqslant \cdots \geqslant \mathrm{Re}(a_r) > 0$ and tempered representations $\rho_1, \dots, \rho_r, \pi_{V_0}$. We define the *leading index for Langlands quotient* as $\mathrm{LI}(\pi_V) = \mathrm{Re}(s_1)$. This definition is compatible with Definition 4.0.7 when the standard module (5.1.3) is irreducible. In particular, the definitions are compatible when π_V is in a generic packet.

Theorem 5.1.11 [9, Theorem A.1.1]. *If $\mathrm{Re}(s) > \mathrm{LI}(\pi_V)$, then*

$$\mathrm{Hom}_{\Delta\mathrm{SO}(V)}((|\det|^s \rho \ltimes \pi_{V_0}) \boxtimes \pi_V, 1_{\Delta\mathrm{SO}(V)}) = 0$$

for $\pi_{V_0} \in \mathcal{SF}(\mathrm{SO}(V_0))$ and $\pi_V \in \mathrm{Irr}(\mathrm{SO}(V))$.

5.2. *The restriction of principal series to mirabolic subgroups.* We now turn to the graded structure of the restriction of certain principal series of GL_n to the mirabolic subgroup $R_{n-1,1}$ as in [42, §5], that is, the subgroup of GL_n leaving V_n/V_{n-1} invariant, where V_n is the space of the standard representation of GL_n and V_{n-1} is an $(n-1)$-dimensional subspace of V_n. These results will be used in the distributional analysis of the open orbit in Section 5.3.

Graded structure of $|\cdot|^{-\frac{m}{2}} \times |\cdot|^{\frac{m}{2}}\mathrm{sgn}^{m+1}$. By definition, the discrete series D_m of $\mathrm{GL}_2(\mathbb{R})$ is the unique quotient of the induction $\pi_I = |\cdot|^{-\frac{m}{2}} \times |\cdot|^{\frac{m}{2}}\mathrm{sgn}^{m+1}$. We denote π_F the unique subrepresentation of this induction π_I, then π_m is an m-dimensional irreducible representation of $\mathrm{GL}_2(\mathbb{R})$.

- Let B_2 be the (upper-triangular) Borel subgroup of GL_2 with Levi decomposition $\mathsf{B}_2 = \mathsf{T}_2\mathsf{N}_2$. Let $K = \mathrm{SO}_2(\mathbb{R})$, $B_2 = \mathsf{B}_2(\mathbb{R})$, $T_2 = \mathsf{T}_2(\mathbb{R})$, $N_2 = \mathsf{N}_2(\mathbb{R})$ and $R_{1,1} = \mathsf{R}_{1,1}(\mathbb{R})$.
- We write
$$n_x = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \quad k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \quad w_2 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix},$$
then $N_2 = \{n_x : x \in \mathbb{R}\}$ and $K = \{k_\theta : \theta \in [0, 2\pi)\}$.
- We write $\mathcal{X}_2 = B_2\backslash\mathrm{GL}_2(\mathbb{R})$, $\mathcal{U}_2 = B_2\backslash B_2 w_2 B_2 \subset \mathcal{X}_2$ and $\mathcal{Z}_2 = B_2\backslash B_2$.
- By definition,
$$\pi_I = \mathrm{Ind}_{B_2}^{\mathcal{S},\mathrm{GL}_2(\mathbb{R})}(|\cdot|^{\frac{m+1}{2}} \otimes |\cdot|^{\frac{m-1}{2}}\mathrm{sgn}^{m+1}).$$
We write $\chi_1 = |\cdot|^{-m+1}\mathrm{sgn}^{m+1}$ and $\chi_2 = |\cdot|^{\frac{m-1}{2}}\mathrm{sgn}^{m+1}$. Then
$$\pi_I = \mathrm{Ind}_{B_2}^{\mathcal{S},\mathrm{GL}_2(\mathbb{R})}(\chi_1\chi_2 \otimes \chi_2).$$

Lemma 5.2.1. (1) *The representation π_F is isomorphic to the n-dimensional* $\mathrm{GL}_2(\mathbb{R})$*-representation*
$$\chi_1\chi_2(\det(\cdot))\mathrm{Sym}^{n-1}(\mathbb{C}^2),$$
where $\mathbb{C}^2$ is the standard representation of $\mathrm{GL}_2(\mathbb{R})$.

(2) *The restriction $\pi_F|_{R_{1,1}}$ has irreducible components*
$$|\det(\cdot)|^k\mathrm{sgn}^k(\det(\cdot)),\ \textit{for}\ k = 0, 1, \dots, m-1.$$

Proof. Part (1) follows directly from [17, §2.3]. Part (2) follows from direct computation based on (1). □

Using the left quotient in the sense of (5.1.1), we define
$$\mathcal{E}_2 := B_2\backslash(\mathrm{GL}_2(\mathbb{R}) \times \chi_1\chi_2 \otimes \chi_2).$$

Extension by zero gives a natural embedding of $R_{1,1}$-representations

$$i_{UX} : \Gamma^{\mathcal{S}}(\mathcal{U}_2, \mathcal{E}_2) \to \Gamma^{\mathcal{S}}(\mathcal{X}_2, \mathcal{E}_2). \tag{5.2.1}$$

Lemma 5.2.2. *There is a complete decreasing filtration* $\{\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}_2, \mathcal{E}_2)_i\}_{i\in\mathbb{N}}$ *of submodules of* $\Gamma^{\mathcal{S}}(\mathcal{X}_2, \mathcal{E}_2)/\,\Gamma^{\mathcal{S}}(\mathcal{U}_2, \mathcal{E}_2)$ *such that the composition factors are* $R_{1,1}$-*isomorphic to*

$$\chi_1\chi_2(\det(\cdot))\mathrm{sgn}^k(\det(\cdot))|\det(\cdot)|^k\Big|_{R_{1,1}}, \quad k \in \mathbb{N}.$$

Proof. This lemma follows from [11, Propositions 8.2, 8.3]. □

We identify $\Gamma^{\mathcal{S}}(\mathcal{U}_2, \mathcal{E}_2)$ as $\mathrm{Ind}^{\mathcal{S},R_{1,1}}_{\mathbb{R}^\times\times 1}(\chi_2)$ using the equation

$$\begin{aligned}\Gamma^{\mathcal{S}}(\mathcal{U}_2, \mathcal{E}_2) &= \Gamma^{\mathcal{S}}(B_2\backslash B_2 w_2 B_2, \mathcal{E}_2)\\ &= \Gamma^{\mathcal{S}}(T_2\backslash B_2, \chi_2\otimes\chi_1\chi_2)\\ &= \Gamma^{\mathcal{S}}(\mathbb{R}^\times\times 1\backslash R_{1,1}, \chi_2) = \mathrm{Ind}^{\mathcal{S},R_{1,1}}_{\mathbb{R}^\times\times 1}(\chi_2),\end{aligned}$$

and then define an $R_{1,1}$-homomorphism

$$T_d : \mathrm{Ind}^{\mathcal{S},R_{1,1}}_{\mathbb{R}^\times\times 1}(\chi_2) \to \pi_D$$

by composing the embedding (5.2.1) and the quotient map π_I to π_F:

$$T_d : \mathrm{Ind}^{\mathcal{S},R_{1,1}}_{\mathbb{R}^\times\times 1}(\chi_2) = \Gamma^{\mathcal{S}}(\mathcal{U}_2, \mathcal{E}_2) \hookrightarrow \Gamma^{\mathcal{S}}(\mathcal{X}_2, \mathcal{E}_2) = \pi_I \to \pi_I/\pi_F = \pi_D.$$

Lemma 5.2.3. *The homomorphism T_d is injective.*

Proof. Suppose T_d is not injective. Then there exist $\tilde f \in \Gamma^{\mathcal{S}}(\mathcal{U}, \chi_1\chi_2 \otimes \chi_2)$ whose extension by zero $\tilde f_G$ in π_I is contained in π_F.

On the one hand, $f(x) = \tilde f(w_2 n_x)$ is a Schwartz function. For $\theta \in (0, \pi)$, we can compute $\tilde f$ with the decomposition

$$k_\theta = \begin{pmatrix} 1/\sin\theta & \cos\theta \\ & \sin\theta \end{pmatrix} w_2 \begin{pmatrix} 1 & -\cot\theta \\ & 1 \end{pmatrix}.$$

Then we have

$$\tilde f_G(k_\theta) = \tilde f(k_\theta) = \chi_1\chi_2(1/\sin\theta)\chi_2(\sin\theta) f(-\cot\theta) = o(\theta^l), \qquad \text{for every } l > 0.$$

Then $\left(\frac{d}{d\theta}\right)^l \tilde f_G(k_\theta)|_{\theta=0} = 0$ for every positive integer l.

On the other hand, from [17, Section 2.3], π_F is generated by the functions

$$\phi_{-m+1}, \phi_{-m+3}, \ldots, \phi_{m-1},$$

where $\phi_l\,(n_x \cdot t(a, b) \cdot k_\theta) = \chi_1\chi_2(a)\chi_2(b)e^{il\theta}$.

Then $\tilde{f}_G \in \pi_F$ is a linear combination of ϕ_k, that is, there is a nonzero n-tuple $(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ such that $\tilde{f}_G = \sum_{k=1}^{n} \lambda_k \phi_{2k-n-1}$. Then we have

$$\left(\frac{d}{d\theta}\right)^l \tilde{f}_G(k_\theta)\Big|_{\theta=0} = \sum_{k=0}^{n-1} \lambda_k((2k-n-1)i)^l,$$

Hence, there exists l such that $\left(\frac{d}{d\theta}\right)^l (\tilde{f}_G(k_\theta))|_{\theta=0} \neq 0$, which leads to a contradiction. Therefore, the $R_{1,1}$-homomorphism T_d is injective. □

Proposition 5.2.4. Coker(T_d) *has a decreasing complete filtration* $\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}_2, \mathcal{E}_2)_k$ *with composition factors isomorphic to*

$$|\det(\,\cdot\,)|^{k+\frac{m-1}{2}} \operatorname{sgn}(\,\cdot\,)^k\big|_{R_{1,1}}, \text{ for } k = 1, 2, \dots. \tag{5.2.2}$$

Proof. From Lemma 5.2.2, $\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}_2, \mathcal{E}_2) = \pi_I / \Gamma^{\mathcal{S}}(\mathcal{U}_2, \mathcal{E}_2)$ has a decreasing complete filtration $\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}_2, \mathcal{E}_2)_k$ with composition factors isomorphic to

$$|\det(\,\cdot\,)|^k \operatorname{sgn}(\,\cdot\,)^k \chi_1\chi_2(\det(\,\cdot\,))\big|_{R_{1,1}}, \text{ for } k = 0, 1, \dots. \tag{5.2.3}$$

From Lemma 5.2.1, the finite-dimensional representation π_F in π_I has $R_{1,1}$-composition factors with irreducible pieces

$$|\det(\,\cdot\,)|^k \operatorname{sgn}^k(\det(\,\cdot\,))\chi_1\chi_2(\det(\,\cdot\,))\big|_{R_{1,1}}, \text{ for } k = 0, 1, \dots, m-1.$$

Then the projection $\pi_I \to \pi_I / i_{UX}(\Gamma^{\mathcal{S}}(\mathcal{U}_2, \mathcal{E}_2))$ gives an isomorphism between π_F and $\overline{\pi}_F = \Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}_2, \mathcal{E}_2) / \Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}_2, \mathcal{E}_2)_n$, implying that

$$\Gamma^{\mathcal{S}}_{\mathcal{Z}_2}(\mathcal{X}_2, \mathcal{E}_2) = \pi_F \oplus \Gamma^{\mathcal{S}}_{\mathcal{Z}_2}(\mathcal{X}_2, \mathcal{E}_2)_m.$$

Therefore,

$$\operatorname{Coker}(T_d) = \pi_D / i_{UX}(\Gamma^{\mathcal{S}}(\mathcal{U}_2, \mathcal{E}_2)) = (\pi_I / \Gamma^{\mathcal{S}}(\mathcal{U}_2, \mathcal{E}_2))/\pi_F = \Gamma^{\mathcal{S}}_{\mathcal{Z}_2}(\mathcal{X}_2, \mathcal{E}_2)_m,$$

and thus Coker(T_d) has a decreasing complete filtration with composition factors isomorphic to

$$\sigma_k = |\det(\,\cdot\,)|^k \operatorname{sgn}^k(\det(\,\cdot\,))\chi_2(\det(\,\cdot\,))\big|_{R_{1,1}} = |\det(\,\cdot\,)|^{k+\frac{m-1}{2}} \operatorname{sgn}(\,\cdot\,)^k\big|_{R_{1,1}},$$

for $k = 1, 2, \dots$. □

Graded structure of spherical principal series. Let $(s_1, \dots, s_{r+1}) \in \mathbb{C}^{r+1}$, and set $\sigma_{X^+} = |\cdot|^{s_1} \times \cdots \times |\cdot|^{s_{r+1}}$, which is a spherical principal series. The computation in [42, Section 5.1] for the restriction of spherical principal series representations to the mirabolic subgroup $R_{r,1}$ can be generalized over the real field verbatim and we can obtain a proposition parallel to [42, Proposition 5.1].

Following [42, §5], we denote by $Q_{a,b,c}$ the intersection of the parabolic subgroup $P_{a,b,c}$ associated to the partition (a, b, c) in GL_{a+b+c} and the mirabolic subgroup

$R_{a+b+c-1}$. We let the "Levi part" $L_{a,b,c}$ of $Q_{a,b,c}$ to be the image of $\mathrm{GL}_a \times \mathrm{GL}_b \times R_{c-1,1}$ diagonally embedded into $\mathrm{GL}_{a,b,c}$. Then $Q_{a,b,c} = L_{a,b,c}U_{a,b,c}$ for the unipotent group associated to the partition (a, b, c).

Proposition 5.2.5. *When restricted to $R_{r,1}$, the representation σ_{X^+} has a subrepresentation $\mathrm{Ind}_{N_{r+1}}^{\mathcal{S},R_{r,1}}(\psi_{r+1}^{-1})$. Moreover, the quotient $\sigma_{X^+}/\mathrm{Ind}_{N_{r+1}}^{\mathcal{S},\mathrm{GL}_{r+1}}(\psi_{r+1}^{-1})$ admits an $R_{r,1}$-stable complete filtration whose composition factors have the shape*

$$\mathrm{Ind}_{Q_{a,b,c}}^{\mathcal{S},R_{r,1}}(\tau_a \boxtimes \tau_b \boxtimes \tau_c)$$

where $a+b+c = t+1$, $a+b \neq 0$ and the tensor $\tau_a \boxtimes \tau_b \boxtimes \tau_c$ is regarded as a $Q_{a,b,c}$ representation by trivial extension on $N_{a,b,c}$.

(1) $\tau_a = \mathrm{Ind}_{B_a}^{\mathcal{S},\mathrm{GL}_a(\mathbb{R})}(\mathrm{sgn}^{m_1}|\cdot|^{s_{i_1}+k_1} \boxtimes \cdots \boxtimes \mathrm{sgn}^{m_a}|\cdot|^{s_{i_a}+k_a})$ *where* $1 \leqslant i_1, \dots, i_a \leqslant t+1$ *are integers*, $l_1, \dots, l_a \in \mathbb{Z}$ *and* $k_1, \dots, k_a \in \frac{1}{2}\mathbb{Z}$;

(2) $\tau_b = \tau_b' \otimes \rho$ *where τ_b' is a representation of the same form as τ_a and ρ is a finite-dimensional representation of* $\mathrm{GL}_b(\mathbb{R})$;

(3) $\tau_c = \mathrm{Ind}_{N_c}^{\mathcal{S},R_{c-1,1}}(\psi_c^{-1})$.

5.3. *Multiplicity formula: first inequality.* In this section, we prove Lemma 5.0.5 and one inequality of Lemma 5.0.3. More precisely, in the setting of Theorem 5.0.1, we prove the inequality

$$m(\pi_V \boxtimes \pi_W) \geqslant m((|\det|^s \sigma_{X^+} \ltimes \pi_W) \boxtimes \pi_V)$$

for a basic relevant pair (W^+, V) when

(1) $\sigma_{X^+} = \mathrm{sgn}^l$ and $s \geqslant \mathrm{LI}(\pi_V)$, or

(2) $\sigma_{X^+} = \sigma_{\underline{s}}$ for $\underline{s}$ in general positions.

With a similar approach, we show that

$$m(\pi_V \boxtimes (|\cdot|^{s+\frac{m}{2}}\mathrm{sgn}^{m+1} \ltimes \pi_W)) \geqslant m((|\det|^s \sigma_{X^+} \ltimes \pi_W) \boxtimes \pi_V)$$

when $\sigma_{X^+} = D_m$ and $s \geqslant \mathrm{LI}(\pi_V)$.

For a relevant pair (W, V) and we let (V, W^+) be the associated basic relevant pair with the decomposition $W^+ = W \perp (X^+ \oplus Y^+)$. We denote by $(\mathsf{G}^+, \mathsf{H}^+, \xi^+)$ the Gross–Prasad triple associated to (V, W^+).

Let P_{X^+} be the parabolic subgroup of $\mathrm{SO}(W^+)$ stabilizing X^+. For $\sigma_{X^+} \in \mathcal{SF}(\mathrm{GL}(X^+))$ and $\pi_W \in \mathcal{SF}(\mathrm{SO}(W))$, from Definition 4.0.3,

$$\sigma_{X^+} \ltimes \pi_W = \mathrm{Ind}_{P_{X^+}}^{\mathcal{S},G}(|\det|^s \sigma_{X^+} \ltimes \pi_W) = \Gamma^S(P_{X^+}\backslash \mathrm{SO}(W^+), \mathcal{E})$$

where

$$\mathcal{E} = \mathcal{E}_{\sigma_{X^+},\pi_W} = P_{X^+}\backslash\big(\mathrm{SO}(W^+) \times (\delta_{P_{X^+}}^{1/2} |\det|^s \sigma_{X^+} \boxtimes \pi_W)\big). \tag{5.3.1}$$

We first study the structure of the right-$\mathrm{SO}(V)$-orbits of $\mathcal{X} = P_{W^+}\backslash \mathrm{SO}(W^+)$.

(1) When $\dim W^+ > 2(r+1)$, $\mathcal{X}$ consists of all k-dimensional totally isotropic subspaces of V. When $\dim W^+ = 2(r+1)$, there are exactly two maximal totally isotropic spaces and $\mathcal{X}$ is exactly one of them.

(2) When $\dim W^+ > 2(r+1)$, there is an open $\mathrm{SO}(V)$-orbit $\mathcal{U}$ consisting of $(r+1)$-dimensional totally isotropic spaces that are not contained in V. Its complement $\mathcal{Z}$ is the space of $(r+1)$-dimensional totally isotropic spaces contained in V. When $\dim V = 2(r+1)$ and $X^+.g_0 \subset V$ for some $g_0 \in \mathrm{SO}(W^+)$, $\mathcal{Z}$ has two orbits and both of them are singletons, more precisely, $[X^+.g_0]$ and $[X^+.g_0 g]$ for any $g \in \mathrm{O}(V)\backslash \mathrm{SO}(V)$; when $\dim V = 2(r+1)$ and if $X^+.g_0 \not\subseteq V$ for all $g_0 \in \mathrm{SO}(W^+)$, $\mathcal{Z}$ is empty; otherwise, $\mathcal{Z}$ has just one orbit.

We can draw the following conclusion:

Lemma 5.3.1. (1) *$\mathcal{Z}$ is empty when* $\dim W^+ = 2(r+1)$ *or* $\dim V = 2(r+1)$ *and* $X^+.g_0 \not\subseteq V$ *for all* $g_0 \in \mathrm{SO}(W^+)$.

(2) *$\mathcal{Z}$ has two* $\mathrm{SO}(V)$*-orbits*, *when* $\dim V \neq 2(r+1)$.

(3) *$\mathcal{Z}$ has a single* $\mathrm{SO}(V)$*-orbit*, *when* $\dim V = 2(r+1)$ *and* $X^+.g_0 \subseteq V$ *for some* $g_0 \in \mathrm{SO}(W^+)$.

Let $\Gamma^S_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}) = \Gamma^S(\mathcal{X}, \mathcal{E})/\,\Gamma^S(\mathcal{U}, \mathcal{E})$. From Proposition 5.1.1, there is a short exact sequence

$$0 \to \Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V \to \Gamma^S(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V \to \Gamma^S_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V \to 0. \tag{5.3.2}$$

Hence, we have the short exact sequence

$$\begin{aligned}0 \to \mathrm{Hom}_{H^+}(\Gamma^S_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}) \to \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+})\\ \to \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}).\end{aligned} \tag{5.3.3}$$

When $\mathrm{Hom}_{H^+}(\Gamma^S_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}) = 0$, we have

$$m((\sigma_{X^+} \ltimes \pi_W) \boxtimes \pi_V) \leqslant \dim \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}).$$

We first analyze the closed orbits on $\mathcal{Z}$ to prove

$$\mathrm{Hom}_{H^+}(\Gamma^S_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}) = 0$$

and then analyze the open orbit $\mathcal{U}$ to prove

$$\dim \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}) \leqslant m(\pi_V \boxtimes \pi_W),$$

under the given conditions.

Closed orbits. Suppose $\mathcal{Z}$ is nonempty. Let $\gamma \in \mathrm{SO}(W^+)$ be a representative of an orbit of $\mathcal{Z}$ such that $X^+.\gamma = X'$ where X' is a totally isotropic subspace of V satisfying $\dim X^+ = \dim X'$. Then the stabilizer group S_γ at $[X]$ is equal to $\gamma^{-1} P_{W^+}\gamma \cap \mathrm{SO}(V)$, which a parabolic subgroup of $\mathrm{SO}(V)$ with Levi decomposition $S_\gamma = M_\gamma N_\gamma$ and the Levi subgroup $M_\gamma = \mathrm{GL}(X') \times \mathrm{SO}(V_0)$. The cotangent bundles and their fibers at $[X']$ are

$$T^*_{\mathcal{Z}} = \mathrm{SO}(V) \times_{S_\gamma} S_\gamma^\perp, \qquad \mathrm{Fib}_{[X']}(T^*_{\mathcal{Z}}) = S_\gamma^\perp$$
$$T^*_{\mathcal{X}} = \mathrm{SO}(W^+) \times_{P_{W^+}} P_{W^+}^\perp, \quad \mathrm{Fib}_{[X']}(T^*_{\mathcal{X}}) = P_{W^+}^\perp$$

and S_γ acts by adjoint action. Then the fiber of the conormal bundle at $[X']$

$$\mathrm{Fib}_{[X']}(\mathcal{N}^\vee_{\mathcal{Z}/\mathcal{X}}) = \mathrm{Fib}_{[X']}(T^*_{\mathcal{X}})/\mathrm{Fib}_{[X']}(T^*_{\mathcal{Z}}) = P_{W^+}^\perp/S_\gamma^\perp,$$

which is $\dim(X')$-dimensional. The $\mathrm{SO}(V_0)$ and N_γ act trivially and $\mathrm{GL}(X')$ acts as the standard representations. Then

$$\begin{aligned}\Gamma^{\mathcal{S}}(\mathrm{SO}(V).[X], \mathrm{Sym}^k \mathcal{N}^\vee_{\mathcal{Z}/\mathcal{X}} \otimes \mathcal{E}|_{\mathcal{Z}}) \\ &= \mathrm{Ind}_{S_\gamma}^{\mathcal{S},\mathrm{SO}(V)}(\mathrm{Fib}_{[X]}(\mathrm{Sym}^k \mathcal{N}_{\mathcal{Z}/\mathcal{X}} \otimes \mathcal{E}|_{\mathcal{Z}})) \\ &= \mathrm{I}_{S_\gamma}^{\mathrm{SO}(V)}((|\det(\cdot)|^{s+\frac{1}{2}}\sigma_{X^+} \otimes \mathrm{Sym}^k \rho_{X'}^{\mathrm{std}}) \boxtimes ({}^\gamma\pi_W|_{\mathrm{SO}(V_0)}))\end{aligned}$$

Therefore,

$$\begin{aligned}(5.3.4)\quad \Gamma^{\mathcal{S}}(\mathcal{Z}, \mathrm{Sym}^k \mathcal{N}^\vee_{\mathcal{Z}/\mathcal{X}} \otimes \mathcal{E}|_{\mathcal{Z}}) \\ &= (\mathrm{I}_{S_\gamma}^{\mathrm{SO}(V)}((|\det(\cdot)|^{s+\frac{1}{2}}\sigma_{X^+} \otimes \mathrm{Sym}^k \rho_{X'}^{\mathrm{std}}) \boxtimes ({}^\gamma\pi_W|_{\mathrm{SO}(V_0)})))^{\oplus c}\end{aligned}$$

where $\rho_{X'}^{\mathrm{std}}$ is the standard representation of $\mathrm{GL}(X')$ and c is the number of $\mathrm{SO}(V)$-orbits in $\mathcal{Z}$.

Proposition 5.3.2. *We have*

$$\mathrm{Hom}_{H^+}(\Gamma^{\mathcal{S}}_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}) = 0$$

under any of the following conditions:

(1) $\sigma_{X^+} = \mathrm{sgn}^l$ $(l = 0, 1)$ *or* $\sigma_{X^+} = D_m$ $(m \in \mathbb{N}_+)$, *and* $s \geqslant \mathrm{LI}(\pi_V)$, *or*

(2) $\sigma_{X^+} = \sigma_{\underline{s}} \in \mathbb{C}^r$ *and* $\underline{s}$ *is in general position.*

Proof. By (5.3.4), we have

$$\begin{aligned}\Gamma^{\mathcal{S}}(\mathcal{Z}, \mathrm{Sym}^k \mathcal{N}^\vee_{\mathcal{Z}/\mathcal{X}} \otimes \mathcal{E}|_{\mathcal{Z}}) \boxtimes \pi_V \\ &= (\mathrm{I}_{S_\gamma}^{\mathrm{SO}(V)}(|\det(\cdot)|^{s+\frac{1}{2}}(\sigma_{X^+} \otimes \mathrm{Sym}^k \rho_{X'}^{\mathrm{std}}) \boxtimes ({}^\gamma\pi_W|_{\mathrm{SO}(V_0)})))^{\oplus c} \boxtimes \pi_V.\end{aligned}$$

• When $\sigma_{X^+} = \mathrm{sgn}^m$, we have $\sigma_{X^+} \otimes \mathrm{Sym}^k \rho = |\det|^k \mathrm{sgn}^m$. When $\mathrm{Re}(s) \geqslant \mathrm{LI}(\pi_V)$, we have $s + \frac{1}{2} + k > \mathrm{LI}(\pi_V)$, from Theorem 5.1.11, we have

$$\mathrm{Hom}_{H^+}(\Gamma^{\mathcal{S}}(\mathcal{Z}, \mathrm{Sym}^k \mathcal{N}^\vee_{\mathcal{Z}/\mathcal{X}} \otimes \mathcal{E}|_{\mathcal{Z}}) \boxtimes \pi_V, 1_{H^+}) = 0.$$

• When $\sigma_{X^+} = D_m$, by computation with the base of D_{m+2a} in [17, §2.3], we have

$$\sigma_{X^+} \otimes \mathrm{Sym}^k \rho = \bigoplus_{a=0}^{k} D_{m+2a}.$$

When $\mathrm{Re}(s) \geqslant \mathrm{LI}(\pi_V)$, we have $s + \frac{1}{2} > \mathrm{LI}(\pi_V)$, from Theorem 5.1.11, we have

$$\mathrm{Hom}_{H^+}(\Gamma^{\mathcal{S}}(\mathcal{Z}, \mathrm{Sym}^k \mathcal{N}^\vee_{\mathcal{Z}/\mathcal{X}} \otimes \mathcal{E}|_{\mathcal{Z}}) \boxtimes \pi_V, 1_{H^+}) = 0.$$

• When $\sigma_{X^+} = |\cdot|^{s_1} \times \cdots \times |\cdot|^{s_r}$, from [26, Corollary 5.6], the Harish-Chandra parameter of the infinitesimal character of $\sigma_{X^+} \otimes \mathrm{Sym}^k \rho$ is

$$[(s_1 + a_1, \ldots, s_{r+1} + a_{r+1})],$$

where the a_i are nonnegative integers. From Corollary 5.1.9, we have

$$\mathrm{Hom}_{H^+}(\Gamma^{\mathcal{S}}(\mathcal{Z}, \mathrm{Sym}^k \mathcal{N}^\vee_{\mathcal{Z}/\mathcal{X}} \otimes \mathcal{E}|_{\mathcal{Z}}) \boxtimes \pi_V, 1_{H^+}) = 0$$

for $\underline{s} \in \mathbb{C}^{r+1}$ in general positions.

From Corollary 5.1.5, we can conclude that, under the conditions given in the proposition, we have

$$\mathrm{Hom}_{H^+}(\Gamma^{S}_{\mathcal{Z}}(\mathcal{X}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}) = 0.$$

Hence, from (5.3.3), we have

$$\dim \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}) \leqslant \dim \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+}). \quad \square$$

The open orbit. We study $\Gamma^S(\mathcal{U}, \mathcal{E})$ and show that $\dim \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U}, \mathcal{E}) \boxtimes \pi_V, 1_{H^+})$ is less than or equal to $m(\pi_V \boxtimes \pi_W)$ under the given conditions.

We introduce the following notations just for this section:

• Let $d = \dim V$, $r = \frac{1}{2}(\dim V - \dim W - 1)$. We can compute the modular character

$$\delta_{P_{X^+}}((m \times g_W) \ltimes n) = |\det(m)|^{d-1-r}, \quad m \in \mathrm{GL}(X^+),\ g_W \in \mathrm{SO}(W),\ n \in N.$$

• Let N_{r+1} be the unipotent subgroup of $\mathrm{GL}_{r+1}(\mathbb{R})$ consisting of upper-triangular unipotent matrices, and let $R_{r,1}$ be the mirabolic subgroup of GL_{r+1}. We denote by $N_{r,1}$ the unipotent radical of $R_{r,1}$.

• We define a generic character π_{r+1} of N_{r+1} by letting

$$\psi_{r+1}(n) = \psi\Big(\sum_{i=1}^{r+1} n_{i,i+1}\Big),$$

where $n_{i,j}$ is the entry of matrix n at i-th row and j-th column.

Recall the decomposition $V = W \perp D \perp Z$ in Section 2.1. Let $X = X^+ \cap Z$ and we have X is totally isotropic and $\dim X = \dim X^+ - 1$. Let N be the unipotent radical of the parabolic subgroup P_X of $\mathrm{SO}(V)$ stabilizing X. We define N'_V the subgroup of N stabilizing D, then $H = (N_{r+1} \times \Delta\mathrm{SO}(W)) \ltimes N'_V$.

From Frobenius reciprocity, we have

$$\mathrm{Hom}_H(\xi^{-1} \otimes (\pi_V \boxtimes \pi_W), 1_H) = \mathrm{Hom}_{H^+}\big(\mathrm{Ind}_H^{\mathcal{S},H^+}(\xi^{-1} \otimes (\pi_V \boxtimes \pi_W)), 1_{H^+}\big). \tag{5.3.5}$$

By definition, the dimension of the left-hand side of (5.3.5) is equal to $m(\pi_V \boxtimes \pi_W)$. The right-hand side of (5.3.5) can be expressed as

$$\begin{aligned}\mathrm{Ind}_H^{\mathcal{S},H^+}&(\xi^{-1} \otimes (\pi_V \boxtimes \pi_W)) \\ &= \mathrm{Ind}^{\mathcal{S},H^+}_{(N_{r+1}\times\Delta\mathrm{SO}(W))\ltimes N'_V}(\xi^{-1} \otimes (\pi_V \boxtimes \pi_W)) \\ &= \mathrm{Ind}^{\mathcal{S},H^+}_{(R_{r,1}\times\Delta\mathrm{SO}(W))\ltimes N'_V}(\mathrm{Ind}^{\mathcal{S},R_{r,1}}_{N_{r+1}}(\psi_{r+1}^{-1})|_{R_{r,1}} \boxtimes \pi_W \boxtimes \pi_V).\end{aligned} \tag{5.3.6}$$

Recall that the open orbit $\mathcal{U} = P_{W^+}\backslash P_{W^+}\mathrm{SO}(V)$ equals $(P_{W^+} \cap \mathrm{SO}(V))\backslash\mathrm{SO}(V)$ and the stabilizer group can be decomposed as

$$P_{W^+} \cap \mathrm{SO}(V) = (\mathrm{GL}(X) \times 1 \times \mathrm{SO}(W)) \ltimes N = \mathrm{SO}(W) \ltimes (R_{r,1} \ltimes N'_V). \tag{5.3.7}$$

By definition, we have

$$\begin{aligned}\Gamma^{S}_{\mathcal{Z}}(\mathcal{U},\mathcal{E}) \boxtimes \pi_V &= \mathrm{Ind}^{\mathcal{S},\mathrm{SO}(V)}_{P_{W^+}\cap\mathrm{SO}(V)}(|\det|^{\frac{d-1-r}{2}}\sigma_{X^+} \otimes \pi_W|_{P_{W^+}\cap\mathrm{SO}(V)}) \boxtimes \pi_V \\ &= \mathrm{Ind}^{\mathcal{S},\mathrm{SO}(V)}_{P_{W^+}\cap\mathrm{SO}(V)}(|\det|^{\frac{d-1-r}{2}}\sigma_{X^+}|_{R_{r,1}} \boxtimes \pi_W) \boxtimes \pi_V \\ &= \mathrm{Ind}^{\mathcal{S},H^+}_{(R_{r,1}\times\Delta\mathrm{SO}(W))\ltimes N'_V}(|\det|^{\frac{d-1-r}{2}}\sigma_{X^+}|_{R_{r,1}} \boxtimes \pi_W \boxtimes \pi_V)\end{aligned} \tag{5.3.8}$$

- When $r = 0$ and $\sigma_{X^+} = \mathrm{sgn}^l$, we have

$$\mathrm{Ind}^{\mathcal{S},R_{r,1}}_{N_{r+1}}(\psi_{r+1}^{-1})|_{R_{r,1}} = |\det|^{\frac{d-1-r}{2}}\sigma_{X^+}|_{R_{r,1}},$$

so the right sides of (5.3.8) and (5.3.6) are the same. Hence, we have

$$m(\pi_V \boxtimes \pi_W) = \dim \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U},\mathcal{E}) \boxtimes \pi_V, 1_{H^+}).$$

- When $\sigma_{X^+} = |\cdot|^{s_1} \times \cdots \times |\cdot|^{s_{r+1}}$ for $(s_1, \ldots, s_{r+1}) \in \mathbb{C}^n$, from Proposition 5.2.5, there is an $R_{r,1}$-equivariant embedding

$$\mathrm{Ind}^{\mathcal{S},R_{r,1}}_{N_{r+1}}(\psi_{r+1}^{-1}) \hookrightarrow |\det|^{\frac{d-1-r}{2}}\sigma_{X^+}. \tag{5.3.9}$$

Applying the quotient of (5.3.8) and (5.3.6), we obtain

$$\Gamma^{S}_{\mathcal{Z}}(\mathcal{U},\mathcal{E}) \boxtimes \pi_V / \mathrm{Ind}_H^{\mathcal{S},H^+}(\xi^{-1} \otimes (\pi_V \boxtimes \pi_W)) = \mathfrak{Q}, \tag{5.3.10}$$

where

$$\mathfrak{Q} = \mathrm{Ind}^{\mathcal{S}, H^+}_{(R_{r,1}\times \Delta \mathrm{SO}(W))\ltimes N'_V}\big((|\det|^{\frac{d-1-r}{2}}\sigma_{X^+}|_{R_{r,1}}/\mathrm{Ind}^{\mathcal{S}, R_{r,1}}_{N_{r+1}}\psi^{-1}_{r+1})\boxtimes \pi_W \boxtimes \pi_V\big).$$

Therefore, to conclude that $\dim \mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U}, \mathcal{E})\boxtimes \pi_V, 1_{H^+}) \leqslant m(\pi_V \boxtimes \pi_W)$, it suffices to prove that

$$\mathrm{Hom}_{H^+}(\mathfrak{Q}, 1_{H^+}) = 0. \tag{5.3.11}$$

Using Proposition 5.2.5, from the exactness of Schwartz induction (Proposition 5.1.2) and projective tensor product (Proposition 5.1.1), we obtain that the quotient $\mathfrak{Q}$ has composition factors

$$\mathrm{Ind}^{\mathcal{S}, H^+}_{(R_{r,1}\times \Delta \mathrm{SO}(W))\ltimes N'_V}\big(\mathrm{Ind}^{\mathcal{S}, R_{r,1}}_{Q_{a,b,c}}(\tau_a\boxtimes\tau_b\boxtimes\tau_c)\boxtimes \pi_W\big), \tag{5.3.12}$$

where $Q_{a,b,c} = P_{a,b,c}\cap R_{r,1}$ and τ_a, τ_b, τ_c are defined in Proposition 5.2.5. Since (5.3.12) can be expressed as the parabolic induction

$$\big(|\det|^{-\frac{d-1-r+c}{2}}(\tau_a\boxtimes\tau_b)\big)\ltimes \mathrm{Ind}^{\mathcal{S}, \mathrm{SO}(W\oplus D\oplus X_c)}_{(R_{c-1,1}\times \mathrm{SO}(W))\ltimes N_{W^+,c}}(\xi_c^{-1}\otimes \pi_W),$$

based on Corollary 5.1.5 and the fact that $a+b\geqslant 1$, the Hom-space in (5.3.12) vanishes for $(s_1, \ldots, s_{r+1}) \in \mathbb{C}^n$ in general position.

- When $r=1$ and $\sigma_{X^+} = D_l$, instead of (5.3.6), we use the equality

$$\begin{aligned}\mathrm{Ind}^{\mathcal{S}, H^+}_{\Delta\mathrm{SO}(W\oplus\mathbb{R})}&((|\cdot|^{s+\frac{m}{2}}\mathrm{sgn}^{m+1}\ltimes \pi_W)\boxtimes\pi_V)\\ &= \mathrm{Ind}^{\mathcal{S}, H^+}_{(R_{1,1}\times \Delta \mathrm{SO}(W))\ltimes N'_V}(\mathrm{Ind}^{\mathcal{S}, R_{1,1}}_{\mathbb{R}^\times\times 1}(\chi_2)\boxtimes\pi_W\boxtimes\pi_V).\end{aligned} \tag{5.3.13}$$

From Section 5.2, there is an injection $T_d : \mathrm{Ind}^{\mathcal{S}, R_{1,1}}_{\mathbb{R}^\times\times 1}(\chi_2)\hookrightarrow D_m$, and it induces an injection

$$\mathrm{Ind}^{\mathcal{S}, R_{1,1}}_{\mathbb{R}^\times\times 1}(|\cdot|^s\chi_2)\hookrightarrow |\det|^s D_m.$$

Applying the quotient of (5.3.8) and (5.3.13), we obtain

$$\begin{aligned}\Gamma^S_{\mathcal{Z}}(\mathcal{U}, \mathcal{E})\boxtimes\pi_V/&\mathrm{Ind}^{\mathcal{S}, H^+}_{\Delta\mathrm{SO}(W\oplus\mathbb{R})}((|\cdot|^{s+\frac{m}{2}}\mathrm{sgn}^{m+1}\ltimes \pi_W)\boxtimes\pi_V)\\ &= \mathrm{Ind}^{\mathcal{S}, H^+}_{(R_{r,1}\times \Delta \mathrm{SO}(W))\ltimes N'_V}\big((|\det|^{\frac{d-2}{2}}\sigma_{X^+}|_{R_{1,1}}/\mathrm{Ind}^{\mathcal{S}, R_{1,1}}_{N_2}(\psi_2^{-1}))\boxtimes\pi_W\boxtimes\pi_V\big).\end{aligned} \tag{5.3.14}$$

From Proposition 5.2.4, the quotient $|\det|^s\sigma_{X^+}|_{R_{1,1}}/\mathrm{Ind}^{\mathcal{S}, R_{1,1}}_{\mathbb{R}^\times\times 1}(|\cdot|^s\chi_2)|_{R_{1,1}}$ has composition factors

$$\sigma_k := |\det(\cdot)|^{s+k+\frac{m-1}{2}}\mathrm{sgn}(\cdot)^k|_{R_{1,1}}, \quad k = 1, 2, \ldots$$

From the exactness of Schwartz induction (Proposition 5.1.2) and projective tensor product (Proposition 5.1.1), there is a decreasing complete filtration of

$$\mathrm{Ind}^{\mathcal{S}, H^+}_{(R_{r,1}\times \Delta \mathrm{SO}(W))\ltimes N'_V}\big((|\det|^{s+\frac{d-2}{2}}\sigma_{X^+}|_{R_{1,1}}/\mathrm{Ind}^{\mathcal{S}, R_{1,1}}_{N_2}(\psi_2^{-1})|_{R_{1,1}})\boxtimes\pi_W\boxtimes\pi_V\big)$$

with composition factors

$$\mathrm{Ind}^{\mathcal{S},H^+}_{(R_{1,1}\times\Delta\mathrm{SO}(W))\ltimes N'_V}(\sigma_k\boxtimes\pi_W\boxtimes\pi_V).$$

Notice that

$$\mathrm{Ind}^{\mathcal{S},H^+}_{(R_{1,1}\times\Delta\mathrm{SO}(W))\ltimes N'_V}(\sigma_k\boxtimes\pi_W\boxtimes\pi_V)=(|\cdot|^{s+\frac{m}{2}+k}\mathrm{sgn}^m\ltimes\mathrm{Ind}^{\mathcal{S},\mathrm{SO}(W\oplus\mathbb{R})}_{\mathrm{SO}(W)}(\pi_W))\boxtimes\pi_V.$$

Since we have assumed that $\mathrm{Re}(s)\geqslant\mathrm{LI}(\pi_V)$ and k is a positive integer, we have

$$s+\frac{m}{2}+k>\mathrm{LI}(\pi_V).$$

Then, from Theorem 5.1.11, we have

$$\mathrm{Hom}_{H^+}\big((|\cdot|\mathrm{sgn}^m\ltimes\mathrm{Ind}^{\mathcal{S},\mathrm{SO}(W\oplus\mathbb{R})}_{\mathrm{SO}(W)}(\pi_W))\boxtimes\pi_V,1_{H^+}\big)=0,\quad k=1,2,\ldots$$

From Corollary 5.1.5, this implies

$$\mathrm{Hom}_{H^+}\big(\Gamma^S_{\mathcal{Z}}(\mathcal{U},\mathcal{E})\boxtimes\pi_V/\mathrm{Ind}^{\mathcal{S},H^+}_{\Delta\mathrm{SO}(W\oplus\mathbb{R})}((|\cdot|^{s+\frac{m}{2}}\mathrm{sgn}^{m+1}\ltimes\pi_W)\boxtimes\pi_V),1_{H^+}\big)=0.$$

Hence, Lemma 5.1.3, we have

$$m(\pi_V\boxtimes(|\cdot|^{s+\frac{m}{2}}\mathrm{sgn}^{m+1}\ltimes\pi_W))\geqslant\dim\mathrm{Hom}_{H^+}(\Gamma^S(\mathcal{U},\mathcal{E})\boxtimes\pi_V,1_{H^+}).$$

Proof of the "first inequality". We now make use of Lemma 5.0.5 to prove one side of the equality in Proposition 5.0.4, namely

$$m(\pi_V\boxtimes\pi_W)\leqslant m(\pi_{V_0}\boxtimes\pi_{W_0}).\tag{5.3.15}$$

We express $\pi_V=\sigma_V\ltimes\pi_{V_0}$, $\pi_W=\sigma_W\ltimes\pi_{W_0}$ in the form of (4.0.7) and prove the inequality by induction on

$$N(\sigma_V,\sigma_W)=\sum_{\mathrm{Re}(s_{V,i})\neq0}n_{V,i}+\sum_{\mathrm{Re}(s_{W,i})\neq0}n_{W,i},$$

where $s_{V,i},s_{W,i},n_{V,i},n_{W,i}$ are defined as in (4.0.7).

If $N(\sigma_V,\sigma_W)=0$, both π_V and π_W are tempered; then the inequality follows from Conjecture 1 for tempered parameters, which was proved in [28; 10].

In other cases, we may assume

$$\mathrm{Re}(s_{V,1})\geqslant\mathrm{Re}(s_{V,2})\geqslant\cdots\geqslant\mathrm{Re}(s_{V,l})>0,$$
$$\mathrm{Re}(s_{W,1})\geqslant\mathrm{Re}(s_{W,2})\geqslant\cdots\geqslant\mathrm{Re}(s_{W,l})>0.$$

Suppose the proposition holds when $N(\sigma_V,\sigma_W)\leqslant k$, then when $N(\sigma_V,\sigma_W)=k+1$, we consider the following cases.

Case 1: If $l_V\neq0$ and $\mathrm{Re}(s_{V,1})\geqslant\mathrm{Re}(s_{W,1})$, then let $\widetilde{\sigma}_V=|\det(\cdot)|^{s_{V,2}}\sigma_{V,2}\times\cdots\times|\det(\cdot)|^{s_{V,l}}\sigma_{V,l}$.

(1) If $n_{V,1} = 1$, from Lemma 5.0.5(1) we have

$$m((\sigma_V \ltimes \pi_{V_0}) \boxtimes (\sigma_W \ltimes \pi_{W_0})) \leqslant m((\sigma_W \ltimes \pi_{W_0}) \boxtimes (\widetilde{\sigma}_V \ltimes \pi_{V_0})).$$

(2) If $n_{V,1} = 2$, let $\widehat{\sigma}_V = |\cdot|^{s_{V,1}+\frac{m_{V,1}}{2}} \mathrm{sgn}^{m_{V,1}+1} \ltimes \widetilde{\sigma}_V$. From Lemma 5.0.5(2), we have

$$m((\sigma_V \ltimes \pi_{V_0}) \boxtimes (\sigma_W \ltimes \pi_{W_0})) \leqslant m((\sigma_W \ltimes \pi_{W_0}) \boxtimes (\widehat{\sigma}_V \ltimes \pi_{V_0})).$$

Since $N(\widetilde{\sigma}_V, \sigma_W), N(\widehat{\sigma}_V, \sigma_W) \leqslant N(\sigma_V, \sigma_W) - 1 = k$, we have

$$m((\sigma_W \ltimes \pi_{W_0}) \boxtimes (\widetilde{\sigma}_V \ltimes \pi_{V_0})) \leqslant m(\pi_{V_0} \boxtimes \pi_{W_0}),$$
$$m((\sigma_W \ltimes \pi_{W_0}) \boxtimes (\widehat{\sigma}_V \ltimes \pi_{V_0})) \leqslant m(\pi_{V_0} \boxtimes \pi_{W_0}).$$

Therefore, we have

$$m((\sigma_V \ltimes \pi_{V_0}) \boxtimes (\sigma_W \ltimes \pi_{W_0})) \leqslant m(\pi_{V_0} \boxtimes \pi_{W_0}),$$

Case 2: If $l_V = 0$ or $\mathrm{Re}(s_{V,1}) < \mathrm{Re}(s_{W,1})$, we switch the order of V, W to reduce to Case 1. More explicitly, we take $\sigma_W^{(s')} = |\cdot|^{s'} \times \sigma_{W_0}$. There is an $s' \in i\mathbb{R}$ such that

$$m((\sigma_V \ltimes \pi_{V_0}) \boxtimes (\sigma_W \ltimes \pi_{W_0})) = m((\sigma_W^{(s')} \ltimes \pi_{W_0}) \boxtimes (\sigma_V \ltimes \pi_{V_0}))$$

From [32, Theorem 1.1] and Langlands classification, we may assume $\sigma_W^{(s')} \ltimes \pi_{W_0}$ is irreducible. Then the pair $(\sigma_W^{(s')}, \sigma_V)$ belongs to Case 1 and $N(\sigma_W^{(s')}, \sigma_V) = N(\sigma_V, \sigma_W) = k+1$, so

$$m((\sigma_W^{(s')} \ltimes \pi_{W_0}) \boxtimes (\sigma_V \ltimes \pi_{V_0})) \leqslant m(\pi_{V_0} \boxtimes \pi_{W_0}).$$

Therefore, we have

$$m((\sigma_V \ltimes \pi_{V_0}) \boxtimes (\sigma_W \ltimes \pi_{W_0})) \leqslant m(\pi_{V_0} \boxtimes \pi_{W_0}).$$

The proposition now follows by induction on $N(\sigma_V, \sigma_W)$. □

5.4. ***Multiplicity formula: the second inequality.*** In this section, we complete the proof for the "second inequality" of Proposition 5.0.4.

A construction. We prove Lemma 5.0.6 by construction. Recall that, for a relevant pair (W, V), we can construct a basic relevant pair (V, W^+) by taking $W^+ = W \perp (X^+ \oplus Y^+)$ for certain totally isotropic spaces X^+ and Y^+. Let $\mathsf{G}^+ = \mathrm{SO}(W^+) \times \mathrm{SO}(V)$, $\mathsf{H}^+ = \Delta\mathrm{SO}(V)$, P^+ is the parabolic subgroup $\mathsf{P}_{X^+} \times \mathrm{SO}(V)$, where P_{X^+} is the parabolic subgroup of $\mathrm{SO}(W^+)$ stabilizing X^+. We note

$$G^+ = \mathsf{G}^+(\mathbb{R}), \quad H^+ = \mathsf{H}^+(\mathbb{R}), \quad P^+ = \mathsf{P}^+(\mathbb{R}).$$

From the multiplicity-one theorem [34], we have $m(\pi_V \boxtimes \pi_W) \leqslant 1$, so it suffices to prove the following proposition.

Proposition 5.4.1. *When* $m(\pi_V \boxtimes \pi_W) \neq 0$ *and* σ_{X^+} *is a generic representation of* $\mathrm{GL}(X^+)$, *then one can construct a nonzero element in*

$$\mathrm{Hom}_{H^+}((\sigma_{X^+} \ltimes \pi_W) \boxtimes \pi_V, 1_{H^+}).$$

The main idea for proving this proposition is from the following theorem.

Theorem 5.4.2 [18, Proposition 4.9]. *For a Casselman–Wallach representation* σ^+ *of* P^+, *suppose*:

(1) *The complement* $G^+ - P^+H^+$ *is the zero set of a polynomial* f^+ *on* G^+ *that is left-*H^+*-invariant and right-*(P^+, ψ_{P^+})*-equivariant for an algebraic character* ψ_{P^+} *of* P^+.

(2) H^+ *has finitely many orbits on the flag of a minimal parabolic subgroup of* G^+

(3) σ^+ *admits a nonzero* $(P^+ \cap H^+, \delta_{P^+\cap H^+}\delta_{H^+}^{-1})$*-equivariant continuous linear functional, where* $\delta_{P^+\cap H^+}, \delta_{H^+}$ *are the modular characters of* $P^+ \cap H^+$ *and* H^+ *respectively.*

Then $\mathrm{Ind}_{P^+}^{\mathcal{S},G^+}(\sigma^+)$ *admits a nonzero* H^+*-invariant functional.*

We first verify (1) and (2) in the setup of Proposition 5.4.1.

(1) Fix a basis $v_1, \dots, v_n$ of V and a basis $v_1^+, \dots, v_{r+1}^+$ of X^+. For every $(g_{W^+}, g_V) \in G^+$, $g \in G^+ - P^+H^+$ if and only if $Xg_{W^+} \subset V$, equivalently, the $(n+1) \times (n+1+r)$-matrix

$$A_g = \left[v_1 g_V, \dots, v_n g_V, v_1^X g_{W^+}^{-1}, \dots, v_{r+1}^X g_{W^+}^{-1}\right]$$

is of rank n. We let

$$f(g) = \det(A_g A_g^t); \tag{5.4.1}$$

then f is left-(P^+, ψ_{P^+})-equivariant and right-H^+-invariant, where

$$\psi_{P^+}(p_{X^+}, g_V) = \det(g_{X^+})^2 \quad \text{for } p_{X^+} = (g_{X^+}, g_W)\cdot n_{X^+} \in P_{W^+} \text{ and } g_V \in \mathrm{SO}(V).$$

(2) Since G^+/H^+ is an absolutely spherical variety (Section 3), the Borel subgroup has finitely many orbits, so the complexification of the minimal parabolic also has finitely many orbits. Then condition (2) is a direct consequence of the finiteness of the first Galois cohomology for groups over local fields.

Therefore, to complete the proof for Proposition 5.4.1, it suffices to construct a nonzero $(P^+ \cap H^+, \delta_{P^+\cap H^+}\delta_{H^+}^{-1})$-equivariant continuous linear functional.

As computed in Section 5.3, we have

$$H\backslash P^+ \cap H^+ = N_{r+1}\backslash R_{r,1},$$

where N_{r+1} and $R_{r,1}$ are the unipotent group and mirabolic group defined in Section 5.3. Hence, from [31], the Rankin–Selberg integral

$$F_s(v_{\pi_V}, v_{\pi_W}, v_{\sigma_{X^+}}) := \int_{P^+\cap H^+} \mu(\pi_V(p_{X^+})v_{\pi_V}, v_{\pi_W})\lambda(\sigma_{X^+}(p_{X^+})v_{\sigma_{X^+}})|\det(g_{X^+})|^s d(p_{X^+}, p_{X^+})$$

is absolutely convergent when $\mathrm{Re}(s)$ is large enough and extends to a meromorphic family in

$$F_s \in \mathrm{Hom}_{P^+\cap H^+}(\pi_V \boxtimes \pi_W \boxtimes \sigma_{X^+}, |\det(g_X)|^{s-s_0}),$$

where $s_0 = \dim W - \dim X^+$, which is the real number satisfying $\delta_{P^+}(p_{X^+}) = |\det(g_{X^+})|^{s_0}$. From [18], we know

$$\frac{F_s}{(s-s_0)^{n_{s_0}}}\Big|_{s=s_0}$$

is a nonzero element

$$\mathrm{Hom}_{P^+\cap H^+}(\pi_V \boxtimes \pi_W \boxtimes \sigma_{X^+}, 1_{P^+\cap H^+}),$$

where n_{s_0} is the order of poles of F_s at $s = s_0$. This completes the proof for Proposition 5.4.1.

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References

[1] A. Aizenbud, D. Gourevitch, S. Rallis, and G. Schiffmann, "Multiplicity one theorems", *Ann. of Math.* (2) **172**:2 (2010), 1407–1434. MR

[2] J. Bernstein and B. Krötz, "Smooth Fréchet globalizations of Harish-Chandra modules", *Israel J. Math.* **199**:1 (2014), 45–111. MR

[3] R. Beuzart-Plessis, "Expression d'un facteur epsilon de paire par une formule intégrale", *Canad. J. Math.* **66**:5 (2014), 993–1049. MR

[4] R. Beuzart-Plessis, *La conjecture locale de Gross–Prasad pour les représentations tempérées des groupes unitaires*, Mém. Soc. Math. Fr. (N.S.) **149**, 2016. MR

[5] R. Beuzart-Plessis, *A local trace formula for the Gan–Gross–Prasad conjecture for unitary groups: the Archimedean case*, Astérisque **418**, 2020. MR

[6] R. Beuzart-Plessis, "Relative trace formulae and the Gan–Gross–Prasad conjectures", pp. 1712–1743 in *Proceedings of the International Congress of Mathematicians*, vol. 3, sections 1–4, EMS Press, Berlin, 2023. MR

[7] W. Casselman, "Canonical extensions of Harish-Chandra modules to representations of G", *Canad. J. Math.* **41**:3 (1989), 385–438. MR

[8] W. Casselman, H. Hecht, and D. Miličić, "Bruhat filtrations and Whittaker vectors for real groups", pp. 151–190 in *The mathematical legacy of Harish-Chandra* (Baltimore, MD, 1998), Proc. Sympos. Pure Math. **68**, Amer. Math. Soc., Providence, RI, 2000. MR

[9] C. Chen, "Multiplicity formula for induced representations: Bessel and Fourier–Jacobi models over Archimedean local fields", preprint, 2023. arXiv 2308.02912

[10] C. Chen and Z. Luo, "The local Gross–Prasad conjecture over $\mathbb{R}$: epsilon dichotomy", 2022. arXiv 2204.01212

[11] Y. Chen and B. Sun, "Schwartz homologies of representations of almost linear Nash groups", *J. Funct. Anal.* **280**:7 (2021), art. id. 108817, 50 pp. MR

[12] C. Chen, D. Jiang, D. Liu, and L. Zhang, "Arithmetic branching law and generic L-packets", *Represent. Theory* **28** (2024), 328–365. MR

[13] F. du Cloux, "Sur les représentations différentiables des groupes de Lie algébriques", *Ann. Sci. École Norm. Sup.* (4) **24**:3 (1991), 257–318. MR

[14] J. Frahm, "Symmetry breaking operators for strongly spherical reductive pairs", preprint, 2017. arXiv 1705.06109

[15] W. T. Gan and A. Ichino, "The Gross–Prasad conjecture and local theta correspondence", *Invent. Math.* **206**:3 (2016), 705–799. MR

[16] W. T. Gan, B. H. Gross, and D. Prasad, "Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups", pp. 1–109 in *Sur les conjectures de Gross et Prasad*, vol. I, Astérisque **346**, 2012. MR

[17] R. Godement, "Notes on Jacquet–Langlands theory", *Matematika* **18**:2 (1974), 28–78.

[18] D. Gourevitch, S. Sahi, and E. Sayag, "Analytic continuation of equivariant distributions", *Int. Math. Res. Not.* **2019**:23 (2019), 7160–7192. MR

[19] B. H. Gross and D. Prasad, "On the decomposition of a representation of SO_n when restricted to SO_{n-1}", *Canad. J. Math.* **44**:5 (1992), 974–1002. MR

[20] B. H. Gross and D. Prasad, "On irreducible representations of $\mathrm{SO}_{2n+1} \times \mathrm{SO}_{2m}$", *Canad. J. Math.* **46**:5 (1994), 930–950. MR

[21] H. Jacquet, "Archimedean Rankin–Selberg integrals", pp. 57–172 in *Automorphic forms and L-functions, II: Local aspects*, Contemp. Math. **489**, Amer. Math. Soc., Providence, RI, 2009. MR

[22] D. Jiang and L. Zhang, "Arthur parameters and cuspidal automorphic modules of classical groups", *Ann. of Math.* (2) **191**:3 (2020), 739–827. MR

[23] D. Jiang, B. Sun, and C.-B. Zhu, "Uniqueness of Bessel models: the Archimedean case", *Geom. Funct. Anal.* **20**:3 (2010), 690–709. MR

[24] D. Jiang, D. Liu, and L. Zhang, "Arithmetic Wavefront Sets and Generic L-packets", 2022. arXiv 2207.04700

[25] A. W. Knapp and G. J. Zuckerman, "Classification of irreducible tempered representations of semisimple groups", *Ann. of Math.* (2) **116**:2 (1982), 389–455. MR

[26] B. Kostant, "On the tensor product of a finite and an infinite dimensional representation", *J. Functional Analysis* **20**:4 (1975), 257–285. MR

[27] R. P. Langlands, "On the classification of irreducible representations of real algebraic groups", preprint, Institute for Advance Study, Princeton, 1973.

[28] Z. Luo, *A local trace formula for the local Gross–Prasad conjecture for special orthogonal groups*, Ph.D. thesis, University of Minnesota, 2021, available at https://www.proquest.com/docview/2594696789. MR

[29] C. Mœglin and J.-L. Waldspurger, "La conjecture locale de Gross–Prasad pour les groupes spéciaux orthogonaux: le cas général", pp. 167–216 in *Sur les conjectures de Gross et Prasad*, vol. II, Astérisque **347**, 2012. MR

[30] D. Prasad, "Reducible principal series representations, and Langlands parameters for real groups", preprint, 2017. arXiv 1705.01445

[31] D. Soudry, *Rankin–Selberg convolutions for* $\mathrm{SO}_{2l+1} \times \mathrm{GL}_n$*: local theory*, Mem. Amer. Math. Soc. **500**, 1993. MR

[32] B. Speh and D. A. Vogan, Jr., "Reducibility of generalized principal series representations", *Acta Math.* **145**:3-4 (1980), 227–299. MR

[33] B. Sun, "Almost linear Nash groups", *Chinese Ann. Math. Ser. B* **36**:3 (2015), 355–400. MR

[34] B. Sun and C.-B. Zhu, "Multiplicity one theorems: the Archimedean case", *Ann. of Math.* (2) **175**:1 (2012), 23–44. MR

[35] D. A. Vogan, Jr., "The local Langlands conjecture", pp. 305–379 in *Representation theory of groups and algebras*, Contemp. Math. **145**, Amer. Math. Soc., Providence, RI, 1993. MR

[36] J.-L. Waldspurger, "Une formule intégrale reliée à la conjecture locale de Gross–Prasad", *Compos. Math.* **146**:5 (2010), 1180–1290. MR

[37] J.-L. Waldspurger, "Calcul d'une valeur d'un facteur ϵ par une formule intégrale", pp. 1–102 in *Sur les conjectures de Gross et Prasad*, vol. II, Astérisque **347**, 2012. MR

[38] J.-L. Waldspurger, "La conjecture locale de Gross–Prasad pour les représentations tempérées des groupes spéciaux orthogonaux", pp. 103–165 in *Sur les conjectures de Gross et Prasad*, vol. II, Astérisque **347**, 2012. MR

[39] J.-L. Waldspurger, "Une formule intégrale reliée à la conjecture locale de Gross–Prasad, 2$^{\mathrm{e}}$ partie: extension aux représentations tempérées", pp. 171–312 in *Sur les conjectures de Gross et Prasad*, vol. I, Astérisque **346**, 2012. MR

[40] J.-L. Waldspurger, "Une variante d'un résultat de Aizenbud, Gourevitch, Rallis et Schiffmann", pp. 313–318 Astérisque **346**, 2012. MR

[41] N. R. Wallach, *Real reductive groups*, *II*, Pure and Applied Mathematics **132-II**, Academic Press, Boston, 1992. MR

[42] H. Xue, “Bessel models for real unitary groups and Schwartz homology”, preprint, 2020.

[43] H. Xue, “Bessel models for real unitary groups: the tempered case”, *Duke Math. J.* **172**:5 (2023), 995–1031. MR

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VOLUME BOUNDS FOR HYPERBOLIC ROD COMPLEMENTS IN THE 3-TORUS

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The study of rod complements is motivated by rod packing structures in crystallography. We view them as complements of links comprised of Euclidean geodesics in the 3-torus. Recently, Hui classified when such rod complements admit hyperbolic structures, but their geometric properties are yet to be well understood. In this paper, we provide upper and lower bounds for the volumes of all hyperbolic rod complements in terms of rod parameters, and show that these bounds may be loose in general. We introduce better and asymptotically sharp volume bounds for a family of rod complements. The bounds depend only on the lengths of the continued fractions formed from the rod parameters.

1. Introduction

The present work is motivated by the notion of rod packing structures in crystallography. In 1977, O'Keeffe and Andersson observed that many crystal structures can be described as a packing of uniform cylinders, representing linear or zigzag chains of atoms or connected polyhedra [24]. In 2001, O'Keeffe et al. classified some of the simplest so-called rod packings in terms of arrangements in Euclidean space [25]. Rod packings have also appeared in the biological science and materials science literature [9; 11; 23; 27].

A rod packing structure exhibits translational symmetry along each dimension in a three-dimensional Euclidean space; it is thus natural to view a rod packing structure as a geodesic link in the 3-torus, whose covering space is the three-dimensional Euclidean space. In this paper, we use tools from 3-manifold geometry and topology to study the complements of these geodesic links, called *rod complements*. In particular, Thurston's geometrisation theorem implies that each rod complement can be decomposed into geometric pieces. Indeed, each rod complement with three

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or more linearly independent rods is hyperbolic or has a unique hyperbolic rod complement component in its JSJ decomposition [5; 20; 21]. This means that geometric invariants such as the hyperbolic volume could be used to classify and distinguish rod complements. However, we need to know how geometry relates to the vector descriptions of rods in the 3-torus, which aligns with the descriptions of rods in crystallography. This is still unknown in general.

Previously, Hui and Purcell used vector description and the theory of links in the 3-sphere to identify an infinite family of rod complements that admit complete hyperbolic structures [21]. Following this, Hui provided a complete classification of the geometric structures on rod complements in the 3-torus [20]. As a consequence, checking the hyperbolicity of a rod complement reduces to a linear algebra problem. While [20; 21] provide a convenient characterisation of when a rod complement is hyperbolic, they do not give further information on the metric. In this paper, we provide more information on the hyperbolic structures of rod complements via the study of their volumes.

The Mostow–Prasad rigidity theorem states that a complete hyperbolic metric on a finite-volume hyperbolic 3-manifold is unique, so hyperbolic volume is a topological invariant. In the crystallographic setting, the uniqueness of hyperbolic structures allows us to associate each rod packing structure with a real number, namely the volume. When volumes are distinct, this provides a simple way to distinguish rod packing structures and avoid the more complicated symmetry descriptions that are often used in chemistry; see [25] for examples.

For a rod complement in the 3-torus, each rod has an associated direction in the unit cube fundamental region of the 3-torus. We encode the direction of each rod by integer vector coordinates, which we call *rod parameters*. Our most general result provides upper and lower volume bounds in terms of the number of rods and their rod parameters.

Theorem 3.2. *Let $R_1, R_2, \dots, R_n$ be disjoint rods in the 3-torus whose complement is a hyperbolic 3-manifold M. After applying a linear homeomorphism and renumbering, if necessary, we may assume that there is a positive integer $k < n$ such that $R_{k+1}, R_{k+2}, \dots, R_n$ are exactly the $(0,0,1)$-rods. Suppose that R_i has direction vector (p_i, q_i, z_i), for $i = 1, 2, \dots, n$. Then*

$$n v_{\text{tet}} < \operatorname{Vol}(M) \le 8 v_{\text{tet}} \Big(\sum_{1 \le i < j \le k} |p_i q_j - p_j q_i| + \sum_{1 \le i \le k} (\gcd(p_i, q_i) - 1) \Big),$$

where $v_{\text{tet}} \approx 1.01494$ is the volume of the regular ideal tetrahedron.

The lower bound is due to a result proved by Adams, which applies to any cusped hyperbolic 3-manifold [1]. Such a bound can be loose in general; indeed, we find families of rod complements for which the number of rods is fixed at $n = 3$, but for which the volumes approach infinity.

The upper bound uses more recent results of Cremaschi and Rodríguez-Migueles [6, Theorem 1.5], which can be applied to many complements of geodesic links in Seifert fibred spaces; see also [7]. Such a bound can be loose, even when restricted to rod complements; there are families of rod complements for which the volumes are bounded but for which the right side of the inequality above grows to infinity.

Thus, while Theorem 3.2 provides reasonable initial bounds that may be strong in certain cases, they are somewhat unsatisfying in general. It would be desirable to have upper and lower volume bounds that depend linearly on the same quantity. For example, hyperbolic volumes of 2-bridge knots [18], alternating knots [22], and highly twisted knots [16] are known to be bounded above and below by linear functions of the number of twist regions. For all of these knot complements, the upper bound is asymptotically sharp. The lower bound is asymptotically sharp in the 2-bridge case [18], and sharp, realised by the Borromean rings, in the alternating case [2, Theorem 2.2]. Similarly, there are upper and lower volume bounds for adequate knots in terms of coefficients of coloured Jones polynomials [8; 16; 17]. There are also upper and lower volume bounds for fibred 3-manifolds in terms of a quantity related to the action of the monodromy map [4], with analogous results for cusp volumes [15]. One would like to obtain such results for rod complements.

While we have not obtained coarse volume bounds of this form in general, we do find improved, asymptotically sharp volume bounds for infinite families of rod complements in terms of the lengths of the continued fractions formed from their rod parameters. These lengths of continued fractions can remain the same when rod parameters increase significantly.

Theorem 5.7. *Let $R_1, R_2, \dots, R_n$ be disjoint rods in the 3-torus whose complement is M, where $n \geq 3$. Suppose that R_n has direction vector $(0, 0, 1)$ and for $i < n$, R_i has direction vector $(p_i, q_i, 0)$, with $(p_i, q_i) \neq (p_{i+1}, q_{i+1})$ for $i = 1, 2, \dots, n-2$ and $(p_{n-1}, q_{n-1}) \neq (p_1, q_1)$. Suppose that $R_1, R_2, \dots, R_{n-1}$ are positioned from top to bottom in the unit cube representation of the 3-torus. Let $[c_{i1}; c_{i2}, \dots, c_{im_i}]$ be a continued fraction expansion for p_i/q_i. Then M is hyperbolic and its volume satisfies the asymptotically sharp upper bound*

$$\mathrm{Vol}(M) \leq 2v_{\mathrm{oct}} \sum_{i=1}^{n-1} m_i.$$

Suppose in addition that

$$C := \min_{\substack{1 \leq i \leq n-1 \\ j \geq 2}} \{|c_{ij}|, |c_{i1} - c_{(i-1)1}|\} \geq 6,$$

where c_{01} is interpreted as $c_{(n-1)1}$. Then the volume satisfies the lower bound

$$\mathrm{Vol}(M) \geq \left(1 - \frac{4\pi^2}{C^2 + 4}\right)^{3/2} 2v_{\mathrm{oct}} \sum_{i=1}^{n-1} m_i.$$

Theorem 5.7 leads to the following consequences.

Corollary 5.9. *There exists a sequence of hyperbolic rod complements with bounded volume, but for which the upper bound of Theorem 3.2 grows to infinity.*

Corollary 5.10. *There exists a sequence of hyperbolic rod complements, each with three rods, whose volumes grow to infinity.*

Other works related to geometry and periodic links include [12], in which Evans, Robins, and Hyde studied 3-periodic links using energy functions. In [10], Evans and Schröder-Turk used two-dimensional hyperbolic geometry to study triply periodic links embedded in the three-dimensional Euclidean space.

The structure of the paper is as follows.

- In Section 2, we introduce some terminology, notation and foundational results that are used throughout the paper. These pertain to rod complements, continued fractions and homeomorphisms from the n-dimensional torus to itself.
- In Section 3, we provide general volume bounds for all hyperbolic rod complements in the 3-torus (Theorem 3.2). The upper bound is in terms of the rod parameters, while the lower bound is only in terms of the number of rods.
- In Section 4, we introduce the notion of nested annular Dehn filling in the 3-torus.
- In Section 5, we use the notion of nested annular Dehn filling to provide more refined volume bounds for a particular class of rod complements (Theorem 5.7). This is sufficient to exhibit a family of rod complements with bounded volumes for which the upper bound of Theorem 3.2 grows to infinity (Corollary 5.9) and another family with bounded number of rods whose volumes grow to infinity (Corollary 5.10).
- In Section 6, we conclude with brief discussion of open questions that are motivated by the present work.

2. Preliminaries

2.1. ***Rod complements.*** We consider the 3-torus $\mathbb{T}^3$ as the cube $[0, 1]\times[0, 1]\times[0, 1]$ in three-dimensional Euclidean space, with opposite faces glued identically, as in [20; 21]. Its universal cover is $\mathbb{R}^3$ and it inherits the Euclidean metric from $\mathbb{R}^3$.

A *rod* is the projection of a Euclidean straight line with rational slope in $\mathbb{R}^3$ to $\mathbb{T}^3$ under the covering map.

For n a positive integer, an *n-rod complement* is the complement of n disjoint rods in the 3-torus. When n is unspecified, we refer to such a manifold simply as a *rod complement*.

Let p, q, z be integers, not all zero, with $\gcd(p, q, z) = 1$. A (p, q, z)-rod is a geodesic in $\mathbb{T}^3$ that has (p, q, z) as a tangent vector. We call (p, q, z) a *direction vector* of the rod, where we consider (p, q, z) only up to a change of sign. We call the integers p, q, z the *rod parameters* of the rod. A *standard rod* is a $(1, 0, 0)$-rod, a $(0, 1, 0)$-rod, or a $(0, 0, 1)$-rod.

A rod complement is said to be *hyperbolic* if it admits a complete hyperbolic structure; for further details on hyperbolic geometry, see, for example, [26]. Previously, Hui classified exactly when rod complements are hyperbolic, Seifert fibred or toroidal.

Theorem 2.1 (Hui, [20]). *Let $R_1, R_2, \dots, R_n$ be disjoint rods in $\mathbb{T}^3$. The rod complement $\mathbb{T}^3 \setminus (R_1 \cup R_2 \cup \cdots \cup R_n)$ is*

(1) *hyperbolic if and only if $\{R_1, R_2, \dots, R_n\}$ contains three linearly independent rods and each pair of disjoint parallel rods are not linearly isotopic in the complement of the other rods*;

(2) *Seifert fibred if and only if all rods have the same direction vector*; *and*

(3) *toroidal if*

(a) *the direction vectors of the rods all lie in the same plane*; *or*

(b) *there exist two distinct rods that are linearly isotopic in the complement of the other rods.*

In case (3)(b), suppose without loss of generality that R_{n-1} and R_n are linearly isotopic in the complement of the other rods. Then an essential torus encircling the linearly isotopic rods will cut the rod complement into a solid torus containing R_{n-1} and R_n, and a new rod complement with rods $R_1, R_2, \dots, R_{n-1}$. So if there were three linearly independent rods to begin with, there would be a unique hyperbolic rod complement appearing as a component of the JSJ decomposition; see [5, Theorem 19]. The upshot of this discussion is that rod complements are very commonly hyperbolic, in a certain sense.

Observe that in a hyperbolic rod complement, there may be several rods with the same direction vector, provided that for any two such rods, at least one other rod intersects the linear annuli bound by them. Two or more rods with the same direction vector are said to be *parallel.*

2.2. ***Continued fractions.*** Let p, q be nonzero relatively prime integers. Without loss of generality, we may assume $q > 0$ unless otherwise specified. The rational number p/q can be expressed as a finite continued fraction

$$\frac{p}{q} = [c_1; c_2, \dots, c_m] \coloneqq c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{1}{\ddots + \cfrac{1}{c_m}}}},$$

where c_1 is an integer and $c_2, \ldots, c_m$ are nonzero integers. The integers $c_1, c_2, \ldots, c_m$ are called *coefficients* or *terms* of the continued fraction and the number m is called the *length* of the continued fraction.

Observe that a continued fraction expansion for a given rational number is not unique. For example, the rational number $\frac{7}{4}$ can be expressed in several ways, including $[1; 1, 3]$, $[1; 1, 2, 1]$ and $[2; -4]$. The upper bound of Theorem 5.7 is strengthened by using continued fraction expansions that have minimal length. In particular, if $m \geq 2$, we do not allow $c_m = 1$ in the continued fraction expansion above.

The length of the continued fraction $[0] = \frac{0}{1}$ is one. For convenience, we define the "empty" continued fraction $[\ ] = \frac{1}{0}$ and consider its length to be zero.

The rational numbers whose continued fraction expansions we consider arise as slopes on the 2-torus. We consider the 2-torus $\mathbb{T}^2$ as the unit square $[0, 1] \times [0, 1]$ in two-dimensional Euclidean space, with opposite faces glued identically. Its universal cover is $\mathbb{R}^2$ and it inherits the Euclidean metric from $\mathbb{R}^2$.

Let p and q be integers, not both zero, with $\gcd(p, q) = 1$. A simple closed geodesic on $\mathbb{T}^2$ is said to have *slope* p/q or to be a (p, q)*-curve* if it is isotopic to the projection of a line in $\mathbb{R}^2$ with slope q/p. Observe that our definition of slope on the torus is the reciprocal of the corresponding slope on the plane. We defined slope of simple closed geodesics in this way because of our choices of notation in Section 4.

2.3. ***Homeomorphisms of the n-torus.*** The following are useful results concerning homeomorphisms of the n-dimensional torus $\mathbb{T}^n$. The statements are well known, but short proofs have been provided for completeness.

Lemma 2.2. *For $n \geq 2$, an element $A \in \mathrm{GL}(n, \mathbb{Z})$ induces a homeomorphism from $\mathbb{T}^n$ to itself.*

Proof. The element $A \in \mathrm{GL}(n, \mathbb{Z})$ gives rise to a homeomorphism from $\mathbb{R}^n$ to itself that sends the integer lattice $\mathbb{Z}^n$ to itself. In particular, it takes the standard basis of $\mathbb{R}^n$ to a basis formed by the columns of A, whose coordinates are integers. This produces a new fundamental domain for the torus. The induced homeomorphism simply maps the standard fundamental domain of the torus to this new fundamental domain via A. □

In fact, it is known that when $n = 2$ or $n = 3$, $\mathrm{GL}(n, \mathbb{Z})$ is the mapping class group of $\mathbb{T}^n$. (The result for $n = 2$ appears in [13, Theorem 2.5] while the result for $n = 3$ follows from work of Hatcher [19].)

Remark 2.3. Given a rod complement in the 3-torus that contains an (a, b, c)-rod R, there exists an element of $\mathrm{GL}(3, \mathbb{Z})$ that sends (a, b, c) to $(0, 0, 1)$. By Lemma 2.2, we may change the fundamental region of the 3-torus to ensure that R is a $(0, 0, 1)$-rod. In the rest of the paper, we often assume without loss of generality that one of the rods in a rod complement has direction vector $(0, 0, 1)$.

Lemma 2.4 (Bézout's lemma in n-dimensions). *Let $n \geq 2$ be an integer. Suppose that $\boldsymbol{a}_n = (a_{1n}, a_{2n}, \dots, a_{nn})^\intercal$ is a nonzero vector in $\mathbb{Z}^n \subset \mathbb{R}^n$ such that* $\gcd(a_{1n}, a_{2n}, \dots, a_{nn}) = 1$. *Then there exist vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{n-1}$ in $\mathbb{Z}^n$ such that* $\det(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n) = 1$.

Proof. We prove the result by induction on n. Suppose that $\boldsymbol{a}_2 = (a_{12}, a_{22})^\intercal$ is a nonzero vector in $\mathbb{Z}^2$ with $\gcd(a_{12}, a_{22}) = 1$. By Bézout's lemma, there exist integers a_{11}, a_{21} such that $a_{11}a_{22} - a_{21}a_{12} = 1$. So defining $\boldsymbol{a}_1 = (a_{11}, a_{21})^\intercal$ leads to $\det(\boldsymbol{a}_1, \boldsymbol{a}_2) = 1$. This proves the base case $n = 2$.

Now let $n \geq 3$ be an integer. Suppose that $\boldsymbol{a}_n = (a_{1n}, a_{2n}, \dots, a_{nn})^\intercal$ is a nonzero vector in $\mathbb{Z}^n$ with $\gcd(a_{1n}, a_{2n}, \dots, a_{nn}) = 1$. Without loss of generality, suppose that $a_{nn} \neq 0$ so that the vector $\tilde{\boldsymbol{a}}_n := (a_{2n}, a_{3n}, \dots, a_{nn})^\intercal$ is nonzero. Let

$$d := \gcd(a_{2n}, a_{3n}, \dots, a_{nn}).$$

Since $\gcd(a_{1n}, d) = \gcd(a_{1n}, a_{2n}, \dots, a_{nn}) = 1$, by Bézout's lemma, there exist integers s and t such that $sd - ta_{1n} = 1$.

Set $a_{11} = s$ and

$$(a_{21}, a_{31}, \dots, a_{n1}) := \frac{t}{d}\tilde{\boldsymbol{a}}_n^\intercal = \frac{t}{d}(a_{2n}, a_{3n}, \dots, a_{nn}).$$

Since $\frac{1}{d}\tilde{\boldsymbol{a}}_n \in \mathbb{Z}^{n-1}$ and $\gcd\big(\frac{1}{d}a_{2n}, \frac{1}{d}a_{3n}, \dots, \frac{1}{d}a_{nn}\big) = 1$, by induction there exist $\tilde{\boldsymbol{a}}_2, \tilde{\boldsymbol{a}}_3, \dots, \tilde{\boldsymbol{a}}_{n-1}$ in $\mathbb{Z}^{n-1}$ such that $\det\big(\tilde{\boldsymbol{a}}_2, \tilde{\boldsymbol{a}}_3, \dots, \tilde{\boldsymbol{a}}_{n-1}, \frac{1}{d}\tilde{\boldsymbol{a}}_n\big) = 1$.

Now define $\boldsymbol{a}_1 := \big(s, \frac{t}{d}\tilde{\boldsymbol{a}}_n^\intercal\big)^\intercal, \boldsymbol{a}_2 := (0, \tilde{\boldsymbol{a}}_2^\intercal)^\intercal, \dots, \boldsymbol{a}_{n-1} := (0, \tilde{\boldsymbol{a}}_{n-1}^\intercal)^\intercal$. Then by expanding along the first row, we find that

$$\begin{aligned}
&\det(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{n-1}, \boldsymbol{a}_n)\\
&\quad = a_{11}\det(\tilde{\boldsymbol{a}}_2, \dots, \tilde{\boldsymbol{a}}_{n-1}, \tilde{\boldsymbol{a}}_n) + (-1)^{1+n}a_{1n}\det\left(\frac{t}{d}\tilde{\boldsymbol{a}}_n, \tilde{\boldsymbol{a}}_2, \dots, \tilde{\boldsymbol{a}}_{n-1}\right)\\
&\quad = sd\det\left(\tilde{\boldsymbol{a}}_2, \dots, \tilde{\boldsymbol{a}}_{n-1}, \frac{1}{d}\tilde{\boldsymbol{a}}_n\right) + (-1)^{(1+n)+(n-2)}a_{1n}t\det\left(\tilde{\boldsymbol{a}}_2, \dots, \tilde{\boldsymbol{a}}_{n-1}, \frac{1}{d}\tilde{\boldsymbol{a}}_n\right)\\
&\quad = sd - a_{1n}t\\
&\quad = 1.
\end{aligned}$$

□

Proposition 2.5. *For fixed $n \geq 2$, all 1-rod complements in the n-torus are homeomorphic.*

Proof. Let R be a rod in the n-torus whose fundamental region is $[0, 1]^n$. Suppose $\boldsymbol{a}_n = (a_{1n}, a_{2n}, \dots, a_{nn})^\intercal$ is the direction vector of R. We may translate the rod R so that it intersects the origin. As R is a simple closed curve, we must have $\gcd(a_{1n}, a_{2n}, \dots, a_{nn}) = 1$. By Lemma 2.4, there exist vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{n-1}$ in $\mathbb{Z}^n \subset \mathbb{R}^n$ such that

$$\det(\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_n) = 1.$$

Hence, the matrix $(\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n)$ lies in $\mathrm{GL}(n, \mathbb{Z})$ and by Lemma 2.2, it induces a homeomorphism that maps the $(0, 0, \ldots, 0, 1)$-rod to the $\boldsymbol{a}_n$-rod. Therefore, any 1-rod complement $\mathbb{T}^n \setminus R$ is homeomorphic to $\mathbb{T}^n \setminus R_z$, where R_z represents a standard $(0, 0, \ldots, 0, 1)$-rod. □

3. Volume bounds for all rod complements

In this section, we obtain upper and lower bounds on the volumes of all hyperbolic rod complements.

Proposition 3.1. *An n-rod complement in the 3-torus with $k \geq 1$ parallel rods is an $(n-k)$-rod complement in the Seifert fibred space $\mathbb{T}^2_k \times \mathbb{S}^1$, where $\mathbb{T}^2_k$ is a torus with k punctures.*

Proof. Let M be an n-rod complement in the 3-torus with parallel rods $R_1, R_2, \ldots, R_k$. Suppose that these parallel rods have direction vector (a, b, c), where a, b, c are integers such that $\gcd(a, b, c) = 1$. By Lemma 2.4, there exist integers f, g, h, p, q, r such that

$$\det\begin{pmatrix} a & f & p \\ b & g & q \\ c & h & r \end{pmatrix} = 1 \quad \Rightarrow \quad \begin{pmatrix} a & f & p \\ b & g & q \\ c & h & r \end{pmatrix} \in \mathrm{GL}(3, \mathbb{Z}).$$

By Lemma 2.2, such a matrix represents an orientation-preserving homeomorphism of $\mathbb{T}^3$ sending the rods with direction vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ to rods with direction vectors (a, b, c), (f, g, h), (p, q, r), respectively.

Define $T \subset \mathbb{T}^3$ to be a 2-torus spanned by the vectors (f, g, h) and (p, q, r). Note that $T \setminus (R_1 \cup R_2 \cup \cdots \cup R_k)$ is a k-punctured torus. As the homeomorphism represented by the above matrix sends the standard fundamental region of the 3-torus to the fundamental region spanned by the vectors (a, b, c), (f, g, h), and (p, q, r), M is homeomorphic to an $(n-k)$-rod complement in the Seifert fibred space $T \setminus (R_1 \cup R_2 \cup \cdots \cup R_k) \times \mathbb{S}^1$. □

Theorem 3.2. *Let $R_1, R_2, \ldots, R_n$ be disjoint rods in the 3-torus whose complement is a hyperbolic 3-manifold M. After applying a linear homeomorphism and renumbering, if necessary, we may assume that there is a positive integer $k < n$ such that $R_{k+1}, R_{k+2}, \ldots, R_n$ are exactly the $(0, 0, 1)$-rods. Suppose that R_i has direction vector (p_i, q_i, z_i), for $i = 1, 2, \ldots, n$. Then*

$$n v_{\mathrm{tet}} < \mathrm{Vol}(M) \leq 8 v_{\mathrm{tet}} \Bigg(\sum_{1 \leq i < j \leq k} |p_i q_j - p_j q_i| + \sum_{1 \leq i \leq k} (\gcd(p_i, q_i) - 1) \Bigg),$$

where $v_{\mathrm{tet}} \approx 1.01494$ is the volume of the regular ideal tetrahedron.

Proof. From Theorem 2.1, we deduce that $n \geq 3$. Adams proved that an n-cusped hyperbolic 3-manifold M with $n \geq 3$ satisfies $\mathrm{Vol}(M) > n v_{\mathrm{tet}}$, which is the desired lower bound [1, Theorem 3.4].

We obtain the upper bound using a result of Cremaschi and Rodríguez-Migueles [6, Theorem 1.5]. They proved that for a link $\overline{\mathcal{L}}$ in an orientable Seifert fibred space N over a hyperbolic 2-orbifold O in which $\mathcal{L}$ projects injectively to a filling geodesic multicurve $\mathcal{L} \subseteq O$, one has the volume bound

$$\mathrm{Vol}(N \setminus \overline{\mathcal{L}}) < 8 v_{\mathrm{tet}} i(\mathcal{L}, \mathcal{L}).$$

Here, $i(\mathcal{L}, \mathcal{L})$ denotes the geometric self-intersection number of $\mathcal{L}$.

In our particular setting, Proposition 3.1 asserts that M is homeomorphic to a k-rod complement in the Seifert fibred space

$$N = \mathbb{T}^2_{n-k} \times \mathbb{S}^1,$$

where $\mathbb{T}^2_{n-k} := (T \setminus (R_{k+1} \cup R_{k+2} \cup \cdots \cup R_n))$ with $T \subseteq \mathbb{T}^3$ being the 2-torus with fundamental region equal to the unit square $[0, 1]^2$ in the xy-plane. Observe that T is a 2-torus such that the intersection number between R_n and T is 1. Denote by $\overline{\mathcal{L}}$ the k-component link $R_1 \cup R_2 \cup \cdots \cup R_k$ in N. Here, R_i is a (p_i, q_i, z_i)-rod with $(p_i, q_i) \neq (0, 0)$ for $i = 1, 2, \ldots, k$.

Let $\mathcal{P} : N \to \mathbb{T}^2_{n-k}$ be the bundle projection map. Note that the link $\overline{\mathcal{L}}$ projects to $\mathcal{L}$, a union of k rods in the base space $\mathbb{T}^2_{n-k}$, which is a 2-torus in $\mathbb{T}^3$ with $n-k$ punctures. The rod R_i projects to a (p_i, q_i)-curve on this punctured torus.

After a small deformation of the rods, we may ensure that their projections intersect transversely, with at most two arcs meeting at each intersection point. Since the number of rods is finite, we can also ensure, up to small deformation, that

(1) any pair of projections $\mathcal{P}(R_i)$ and $\mathcal{P}(R_j)$ intersect exactly $|p_i q_j - p_j q_i|$ times; see, for example, [13, Section 1.2.3]; and

(2) the (p_i, q_i)-curve $\mathcal{P}(R_i)$ intersects itself exactly $\gcd(p_i, q_i) - 1$ times.

Item (2) above can be seen as follows. Consider a (p_i, q_i)-curve γ with $d_i := \gcd(p_i, q_i) > 1$. Up to homeomorphism of the torus, γ is equivalent to the $(d_i, 0)$-curve, for which a representative consists of $d_i - 1$ horizontal arcs connected by a single arc running across, meeting $d_i - 1$ strands.

Hence, the total geometric intersection number of $\mathcal{P}(\mathcal{L})$ is

$$\sum_{1 \leq i < j \leq k} |p_i q_j - p_j q_i| + \sum_{1 \leq i \leq k} (\gcd(p_i, q_i) - 1).$$

Thus, applying the result of Cremaschi and Rodríguez-Migueles leads to the upper bound. □

Remark 3.3. The upper volume bound in Theorem 3.2 depends on the choice of rod that is sent to the (0, 0, 1)-rod via a homeomorphism of $\mathbb{T}^3$. For example, if we consider four rods R_1, R_2, R_3, R_4 with direction vectors (2, 4, 3), (5, 7, 1), (9, 8, 6), (0, 0, 1), respectively, Theorem 3.2 will give us an upper volume bound $8v_{\text{tet}} \times 50$. Using the constructive proof of Lemma 2.4, we obtain the following matrices in $\mathrm{GL}(3, \mathbb{Z})$ that map (0, 0, 1) to R_1, R_2, R_3, respectively:

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 4 \\ 0 & -1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -4 & 0 & 9 \\ -4 & -1 & 8 \\ -3 & -1 & 6 \end{pmatrix}.$$

By taking the inverses of these matrices and computing the new rod parameters, we now obtain upper volume bounds of $8v_{\text{tet}} \times 116$, $8v_{\text{tet}} \times 114$, and $8v_{\text{tet}} \times 132$, respectively. The minimum among all such choices naturally provides an upper bound.

4. Nested annular Dehn filling in the 3-torus

We will show that neither the upper nor lower bound of Theorem 3.2 can be part of a two-sided coarse volume bound in terms of the given parameters. That is, we exhibit a family of rod complements with fixed number of cusps whose volumes grow to infinity as well as a family of rod complements with bounded volume for which the intersection number in the upper bound of Theorem 3.2 grows to infinity. For both of these results, we use the machinery of annular Dehn filling.

Definition 4.1 (annular Dehn filling). Let A be an annulus embedded in a 3-manifold M, with boundary curves L^+ and L^-. Let $\mu^\pm$ denote a meridian of $\partial N(L^\pm)$ and let $\lambda^\pm$ denote a longitude of $\partial N(L^\pm)$ that is parallel to ∂A.

For an integer n, define $(1/n)$*-annular Dehn surgery* to be the process of drilling $N(L^+)$ and $N(L^-)$ from M, performing $(+1/n)$-Dehn filling on $\partial N(L^+)$ and performing $(-1/n)$-Dehn filling on $\partial N(L^-)$.

The surgery can be realised by cutting along A, performing n Dehn twists along the core of A in the anticlockwise direction (where the induced orientation puts L^+ on the right of the core of A), and then regluing; see, for example, [3, Section 2.3].

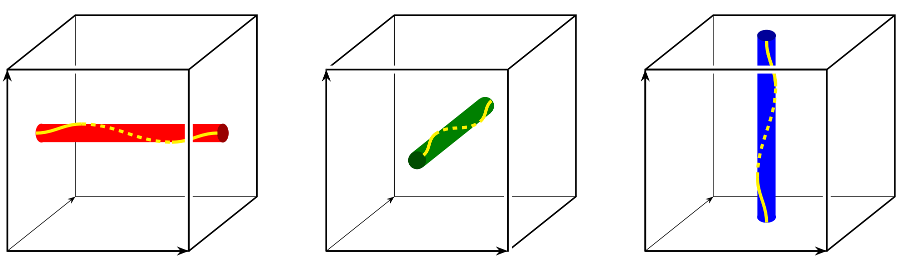

Figure 1. A (1, 1)-curve on the standard rods with direction vectors (1, 0, 0), (0, 1, 0) and (0, 0, 1), respectively.

If the curves $L^{\pm}$ are already drilled, such as in the case of a link complement, define $(1/n)$-*annular Dehn filling along* A to be the process of performing $(+1/n)$-Dehn filling on L^{+} and performing $(-1/n)$-Dehn filling on L^{-}, where the framing on the link components is as above.

In our case, we perform annular Dehn filling on an annulus bounded by a pair of parallel rods in the 3-torus. This can be done by applying Dehn fillings along slopes $\pm 1/n$ on the parallel rods. For example, the slopes $1/1$ on standard rods are shown in Figure 1. Note that parallel rods bound many annuli in $\mathbb{T}^3$. The following result confirms that the resulting link is well defined, regardless of our choice of annulus.

Lemma 4.2. *Let R^{+} and R^{-} be parallel rods in $\mathbb{T}^3$ that form the boundary of two nonisotopic annuli A^{+} and A^{-} with disjoint interiors. Suppose that A^{+} is the annulus oriented with R^{+} on the right of the core, under the induced orientation from $\mathbb{T}^3$. Then $(1/n)$-annular Dehn filling on A^{+} and $(-1/n)$-annular Dehn filling on A^{-} result in homeomorphic manifolds.*

More generally, suppose that A_1 and A_2 are disjoint annuli with A_1 cobounded by rods R_0 and R_1, with R_1 to the right, and A_2 cobounded by R_1 and a rod R_2, with R_1 to the left. Let M be the result of performing $(1/n)$-annular Dehn filling on A_1 followed by $(1/m)$-annular Dehn filling on A_2. Then M is also the result of performing $(-1/n)$-Dehn filling on R_0, followed by $(1/(n-m))$-Dehn filling on R_1, followed by $(1/m)$-Dehn filling on R_2, when $R_0 \neq R_2$. If $R_0 = R_2$, then the Dehn filling coefficient on $R_0 = R_2$ is $1/(m-n)$.

Proof. Let N^{+} be the manifold obtained by $(1/n)$-annular Dehn filling A^{+} and let N^{-} be the manifold obtained by $(-1/n)$-annular Dehn filling A^{-}. The fact that N^{+} and N^{-} are homeomorphic follows from the fact that the link complements have the same Dehn surgery coefficients. Thus, the results of the Dehn fillings must be homeomorphic.

To prove the more general statement, we again consider the Dehn surgery coefficients. Annular Dehn filling first along A_1 gives surgery slope $\mu + n\lambda$ on R_1 and $\mu - n\lambda$ on R_0, where μ denotes a meridian and λ is parallel to ∂A_1. Then performing $(1/m)$-annular Dehn filling along A_2 adjusts the surgery slope on R_1 by subtracting m longitudes, giving $\mu + (n-m)\lambda$. It gives a surgery slope of $\mu + m\lambda$ on R_2 when $R_0 \neq R_2$. When $R_0 = R_2$, the slopes combine as on R_1 to give $\mu - (n-m)\lambda$. $\square$

Definition 4.3. Let m be an even positive integer. Consider a unit cube fundamental region of $\mathbb{T}^3$. For each $i = 1, 2, \ldots, m/2$, let $(R^{+}_{2i-1}, R^{-}_{2i-1})$ be a pair of $(1,0,0)$-rods bounding a vertical xz-plane annulus within the unit cube, with R^{+}_{2i-1} above and R^{-}_{2i-1} below. Let (R^{-}_{2i}, R^{+}_{2i}) be a pair of $(0,1,0)$-rods bounding a vertical yz-plane annulus with R^{-}_{2i} above and R^{+}_{2i} below. A rod R is said to be *sandwiched along the xy-plane by nested pairs of rods with order* $(R^{+}_1, R^{-}_2, \ldots, R^{+}_{m-1}, R^{-}_m)$ if

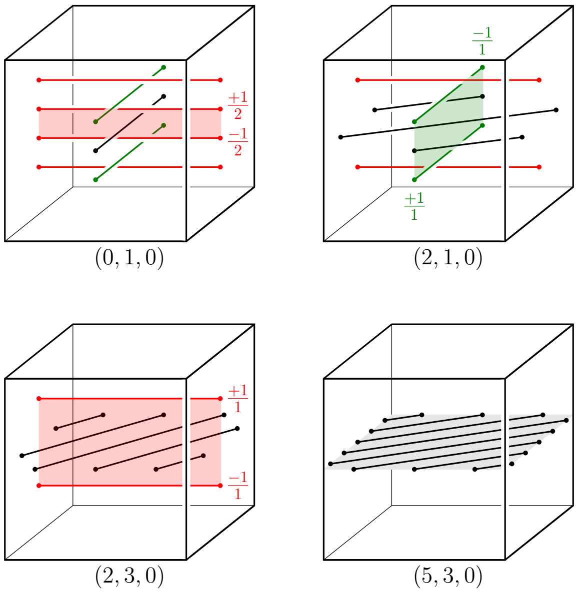

Figure 2. The (5, 3)-nested annular Dehn filling on the (0, 1, 0)-core rod. The vector under each 3-torus is the direction vector of the corresponding black rod (up to ambient isotopy).

and only if R lies in an xy-plane and the rods are positioned from top to bottom in the unit cube in the order

$$(R_1^+, R_2^-, \dots, R_{m-1}^+, R_m^-, R, R_m^+, R_{m-1}^-, \dots, R_2^+, R_1^-).$$

Similarly, for m an odd positive integer, we can say R is *sandwiched along the xy-plane by nested pairs of rods with order* $(R_1^+, R_2^-, \dots, R_{m-1}^-, R_m^+)$ if and only if R lies in an xy-plane and the rods are positioned from top to bottom in the unit cube in the order

$$(R_1^+, R_2^-, \dots, R_{m-1}^-, R_m^+, R, R_m^-, R_{m-1}^+, \dots, R_2^+, R_1^-).$$

See the top-right picture of Figure 2 for an example of a rod (black) sandwiched by nested pairs of rods with $m = 2$.

Lemma 4.4. *Let p and q be integers with* $\gcd(p, q) = 1$. *Suppose that* $[c_1; c_2, \dots, c_m]$ *is a continued fraction expansion of p/q. If m is even, consider a* $(1, 0, 0)$*-rod R_x sandwiched along the xy-plane by nested pairs of rods with order*

$$(R_1^+, R_2^-, \dots, R_{m-1}^+, R_m^-).$$

If m is odd, consider a $(0, 1, 0)$*-rod R_y sandwiched along the xy-plane by nested pairs of rods with order*

$$(R_1^+, R_2^-, \dots, R_{m-1}^-, R_m^+).$$

Sequentially apply $(1/c_i)$*-annular Dehn filling to the pair* (R_i^+, R_i^-) *of rods, starting with* $i = m$ *and ending with* $i = 1$. *Then the rod* R_x *for* m *is even (respectively,* R_y *for* m *odd) is transformed to a* $(p, q, 0)$*-rod.*

Proof. We will focus on the case when the length m of the continued fraction is odd. The argument for m even follows similarly.

Starting with the $(0, 1, 0)$-rod R_y and applying $(1/c_m)$-annular Dehn filling to (R_m^+, R_m^-) transforms the $(0, 1, 0)$-rod R_y to a $(c_m, 1, 0)$-rod $R^{(1)}$. See the top row of Figure 2 for an example.

The $(c_m, 1, 0)$-rod $R^{(1)}$ intersects the annulus bounded by R_{m-1}^- and R_{m-1}^+ a total of c_m times. Applying $(1/c_{m-1})$-annular Dehn filling to (R_{m-1}^+, R_{m-1}^-) transforms the $(c_m, 1, 0)$-rod $R^{(1)}$ into a $(c_m, 1+c_m c_{m-1}, 0)$-rod $R^{(2)}$. See the transition from the top right to the bottom left of Figure 2 for an example. Observe that the ratio of the rod parameters satisfies

$$\frac{1 + c_m c_{m-1}}{c_m} = c_{m-1} + \frac{1}{c_m}.$$

The $(c_m, 1+c_m c_{m-1}, 0)$-rod $R^{(2)}$ intersects the annulus bounded by R_{m-2}^+ and R_{m-2}^- a total of $1 + c_m c_{m-1}$ times. Applying $(1/c_{m-2})$-annular Dehn filling to (R_{m-2}^+, R_{m-2}^-) transforms $R^{(2)}$ into a $(c_m+(1+c_m c_{m-1})c_{m-2}, 1+c_m c_{m-1}, 0)$-rod $R^{(3)}$. See the bottom row of Figure 2 for an example. Now observe that the ratio of the rod parameters satisfies

$$\frac{c_m + (1 + c_m c_{m-1})c_{m-2}}{1 + c_m c_{m-1}} = c_{m-2} + \frac{c_m}{1 + c_m c_{m-1}} = c_{m-2} + \cfrac{1}{c_{m-1} + \cfrac{1}{c_m}}.$$

Continuing in this way, we apply $(1/c_{m-3})$-annular Dehn filling, $(1/c_{m-4})$-annular Dehn filling, and so on, until we finally apply $(1/c_1)$-annular Dehn filling. Each successive annular Dehn filling prepends a term to the continued fraction expansion for the ratio of the rod parameters. Hence, the final rod $R^{(m)}$ has direction vector $(p, q, 0)$, where $p/q = [c_1; c_2, \dots, c_m]$. □

Lemma 4.4 holds for any continued fraction expansion of p/q, without any restriction on the signs of the terms.

Definition 4.5. Let p and q be integers such that $\gcd(p, q) = 1$. Suppose that $[c_1; c_2, \dots, c_m]$ is a continued fraction expansion of p/q. Define (p, q)*-nested annular Dehn filling* to be the process of performing the sequence of $(1/c_i)$-annular Dehn fillings from $i = m$ to $i = 1$ on the rod R_x or R_y, as described in Lemma 4.4. The rod R_x or R_y is called the *core rod* of the nested annular Dehn filling. The rods R_i^+ and R_i^- for $i = 1, 2, \dots, m$ are called the *filling rods* of the nested annular Dehn filling.

For example, consider (p, q)-nested annular Dehn filling with $(p, q) = (5, 3)$, using the continued fraction expansion $p/q = 5/3 = [1; 1, 2]$. Since the number of terms is odd, we start with a $(0, 1, 0)$-rod R_y sandwiched along the xy-plane by nested pairs of rods with order (R_1^+, R_2^-, R_3^+), as shown in the top-left picture of Figure 2. After applying $(1/2)$-annular Dehn filling to the pair of innermost red rods (R_3^+, R_3^-), we obtain the rod complement shown in the top-right picture of Figure 2. Then after applying $(1/1)$-annular Dehn filling to the pair of green rods (R_2^+, R_2^-), we obtain the rod complement shown in the bottom-left picture of Figure 2. Finally, after applying a $(1/1)$-annular Dehn filling to the outermost pair of red rods (R_1^+, R_1^-), we obtain the rod complement shown in the bottom-right picture of Figure 2. The result is a single rod with direction vector $(5, 3, 0)$.

Remark 4.6. Any rod that does not intersect the annulus used in annular Dehn filling is unaffected by the filling. In particular, such rods maintain their direction vectors. This straightforward observation is crucial for our use of annular Dehn fillings below.

5. Asymptotically sharp volume bounds

With nested annular Dehn filling introduced in the last section, we can now proceed to show some asymptotically sharp volume bounds for a family of rod complements.

Lemma 5.1. *Let $R_1, R_2, \ldots, R_n$ be disjoint rods in $\mathbb{T}^3$ with $n \geq 3$. Suppose that R_n has direction vector $(0, 0, 1)$ while each of the other rods R_i has direction vector of the form $(p_i, q_i, 0)$. If any two neighbouring rods, ordered by z-coordinate, are not parallel, then the rod complement $\mathbb{T}^3 \setminus (R_1 \cup R_2 \cup \cdots \cup R_n)$ is hyperbolic.*

Proof. The direction vectors of rods R_1, R_2, R_n are linearly independent, since R_1 and R_2 are not parallel, and R_n is orthogonal to the plane spanned by the direction vectors of R_1 and R_2. Since no two neighbouring rods are parallel, each pair of disjoint parallel rods are not linearly isotopic in the complement of the other rods. Thus, the result follows from Theorem 2.1. □

Definition 5.2. A *standard rod complement* is the complement of a finite number of rods in $\mathbb{T}^3$, each with direction vector $(1, 0, 0)$, $(0, 1, 0)$ or $(0, 0, 1)$.

A *standard parent manifold* of a rod complement $\mathbb{T}^3 \setminus (R_1 \cup R_2 \cup \cdots \cup R_n)$ is a standard rod complement from which $\mathbb{T}^3 \setminus (R_1 \cup R_2 \cup \cdots \cup R_n)$ can be obtained after a finite sequence of Dehn fillings.

Proposition 5.3 (standard parent manifolds exist). *Let $R_1, R_2, \ldots, R_n$ be disjoint rods in $\mathbb{T}^3$ with $n \geq 3$. Suppose that R_n has direction vector $(0, 0, 1)$ while each of the other rods R_i has direction vector of the form $(p_i, q_i, 0)$. Suppose that p_i/q_i has a continued fraction expansion with m_i terms. Let E denote the number of $(p_i, q_i, 0)$-rods with even m_i and let O denote the number of $(p_i, q_i, 0)$-rods*

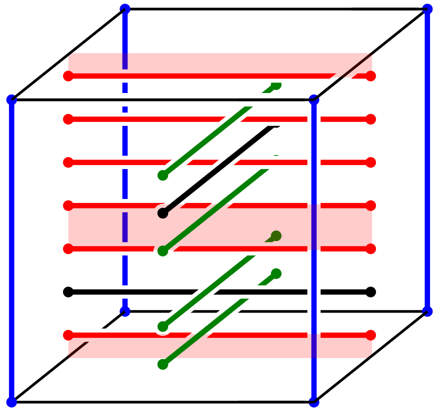

Figure 3. An example of a standard parent manifold with essential annuli.

with odd m_i. *Then there exists a standard rod complement* M *with* E $(1,0,0)$-*core rods and* O $(0,1,0)$-*core rods together with* $2\sum_{i=1}^{n-1} m_i$ *filling rods such that* $\mathbb{T}^3 \setminus (R_1 \cup R_2 \cup \cdots \cup R_n)$ *can be obtained by applying* (p_i, q_i)-*nested annular Dehn filling to the core rods of* M *for* $i = 1, 2, \ldots, n-1$.

Proof. For $i = 1, 2, \ldots, n-1$, since $\gcd(p_i, q_i) = 1$, Lemma 4.4 and Definition 4.5 ensure that the $(p_i, q_i, 0)$-rod R_i can be obtained by applying a (p_i, q_i)-nested annular Dehn filling to one of the $E+O$ core rods. The $2m_i$ filling rods sandwiching the core rod will be removed in the process of Dehn filling. Observe that a (p_i, q_i)-nested annular Dehn filling does not affect the isotopy classes of rods disjoint from the associated annuli. Hence, after applying $n-1$ nested annular Dehn fillings on the $E+O = n-1$ core rods, we obtain a 3-manifold homeomorphic to $\mathbb{T}^3 \setminus (R_1 \cup R_2 \cup \cdots \cup R_n)$. □

Proposition 5.3 provides an explicit procedure to obtain a standard parent manifold of a rod complement with the particular form for which the result applies. The manifold M in Proposition 5.3 is a standard parent manifold of $\mathbb{T}^3 \setminus (R_1 \cup R_2 \cup \cdots \cup R_n)$. Note that for each sandwich of a nested annular Dehn filling, the outermost pair of filling rods are $(1,0,0)$-rods. Between each pair of adjacent (possibly the same) sandwiches, the bottom filling rod of the top sandwich is linearly isotopic to the top filling rod of the bottom sandwich, so there is a natural choice of essential plane annulus between these two filling rods. To obtain a hyperbolic standard parent manifold, we cut along any such essential plane annuli in M. Observe this merges two parallel rods into a single rod.

An example of a standard parent manifold with essential annuli and two core rods is shown in Figure 3. Figure 4 below shows a hyperbolic standard parent manifold. For that example, black and pale green rods are both core rods, and the outermost red rods correspond to the top and bottom filling rods of the sandwiches.

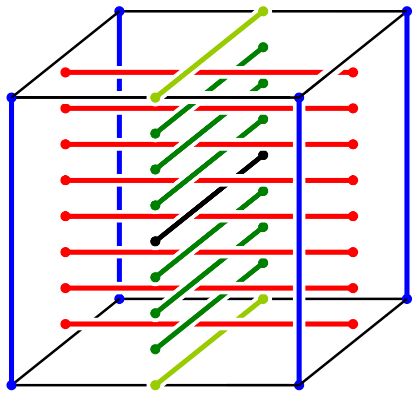

Figure 4. The standard parent manifold of $\mathbb{T}^3 \setminus (R_1^{(7)} \cup R_2 \cup R_3)$. The black rod is the core rod with direction vector $(0, 1, 0)$; the pale green $(0, 1, 0)$-rod that lies on the boundary (top and bottom) of the unit cube is R_2; the blue $(0, 0, 1)$-rod is R_3.

Lemma 5.4. *Consider a standard parent manifold M with exactly one* $(0, 0, 1)$*-rod and $m \geq 2$ additional rods, which alternate between* $(1, 0, 0)$*-rods and* $(0, 1, 0)$*-rods. Then M is hyperbolic and can be decomposed into m regular ideal octahedra. Thus, its volume is* $\mathrm{Vol}(M) = m v_{\mathrm{oct}}$, *where $v_{\mathrm{oct}} \approx 3.66386$ is the volume of the regular ideal octahedron.*

Proof. The fact that M is hyperbolic follows from Theorem 2.1. Alternatively, one can construct the hyperbolic structure directly as follows. Cut M along an xz-plane torus, a yz-plane torus, and all xy-plane tori that contain $(1, 0, 0)$-rods or $(0, 1, 0)$-rods. We obtain m three-dimensional balls, each with six arcs removed from the boundary. By shrinking these arcs, one obtains m ideal octahedra; see Figure 5.

We can assign a complete hyperbolic metric on M by setting each ideal octahedron to be regular. Such a polyhedron has dihedral angles equal to $\pi/2$. The gluing of the octahedra identifies four such dihedral angles around each edge and tiles each cusp by Euclidean squares, so one obtains a complete hyperbolic structure; see [26, Theorem 4.10]. The volume of M is then $m v_{\mathrm{oct}}$, the sum of the volumes of the octahedra. □

Lemma 5.5. *Let M be a hyperbolic standard parent manifold. The fundamental region of the torus cusp boundary corresponding to each filling rod of M is a Euclidean rectangle formed by gluing two squares corresponding to cusp neighbourhoods of ideal vertices of octahedra. The meridian forms one of the sides of the rectangle, running along one edge of each square. The longitude forms the other side of the rectangle, running along an edge of one of the squares. Finally, there exists a choice of horoball neighbourhoods with disjoint interiors for the rod complement such that the meridian has length* 2 *and the longitude has length* 1.

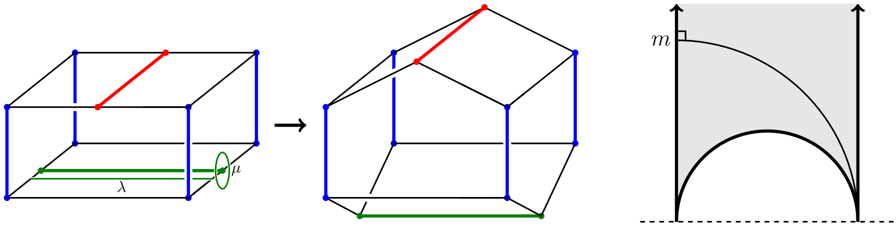

Figure 5. Left: The complement of rods in a standard parent manifold can be decomposed into ideal octahedra. Shown are the meridian μ and the longitude λ for the green rod on the bottom. Right: The midpoint m of an ideal edge in a hyperbolic ideal triangle.

Proof. Consider how the octahedra in the proof of Lemma 5.4 fit together. Since the rod complement can be decomposed into ideal octahedra, the cusps corresponding to the filling rods are tiled by Euclidean squares that are cusp neighbourhoods of the ideal vertices of the octahedra.

Each horizontal rod R meets exactly two octahedra: one above the xy-plane containing R, which we cut along to obtain the decomposition, and one below. The meridian μ runs once through each and can be isotoped to run through the xz- or yz-plane as in the left of Figure 5. Hence, it lies on faces of the two octahedra. Thus, the meridian forms a closed curve running along one edge in each of the two squares corresponding to the two octahedra.

The longitude may be isotoped to run through a single octahedron, say the one above the xy-plane containing R, as in the left of Figure 5. Thus, it forms one side of a cusp square. Finally, observe that the square is glued to itself by the identity, with one side glued to the opposite side. The cusp is a Euclidean rectangle, comprised of two squares, with the meridian running along the long edge of the rectangle and the longitude running along the short edge.

It remains to argue that the lengths of the meridian and the longitude are 2 and 1, respectively. To do so, we show that we can choose horoballs about the cusps of M with disjoint interiors such that when we intersect with the ideal octahedra, the boundary of the intersection is a collection of squares, each with side length 1. The horoball expansion we use is the same as that appearing in [26, Lemma 7.22] or [14, Lemma 3.7]. That is, each edge e of the octahedron borders two triangular faces. The midpoint of the edge e with respect to one of the triangles is the unique point on the edge e that lies on a perpendicular hyperbolic geodesic running from the opposite vertex to e; see the right of Figure 5. Since our ideal octahedron is regular, the midpoints obtained from either adjacent triangle agree. When the vertices of the ideal triangle are placed at 0, 1 and ∞, the midpoint has height 1. If we place a regular ideal octahedron containing a side with vertices at 0, 1, and ∞, the midpoints of each of the edges meeting infinity also have height 1. This remains true after applying a Möbius transformation taking any vertex to infinity. Thus, we may expand horoballs about each ideal vertex to the height of the midpoints of the four edges meeting that vertex. This gives a collection of horoballs that are tangent exactly at the midpoints of edges, with disjoint interiors. The boundary of each horoball meets the octahedron in a square of side length 1. Finally, since the octahedra are glued in such a way that cusp squares glue to cusp squares with the same side lengths, this gluing must preserve this choice of horoballs. Hence, these define horoball neighbourhoods with disjoint interiors and lengths as claimed. □

Lemma 5.6. *Let M be a hyperbolic standard parent manifold, with slope $1/n$ on one of the horizontal rods. Then in the horoball neighbourhood described in Lemma 5.5, the length of the slope is $\sqrt{n^2+4}$.*

Proof. The slope $1/n$ runs once along a meridian and n times along the longitude. In the universal cover of the cusp torus, it can be lifted to an arc with one endpoint at $(0,0)$ and the other at $(2,n)$. The meridian and longitude are orthogonal, with the meridian of length 2 and the longitude of length 1. Hence, length of the slope is $\sqrt{n^2+2^2}$. □

We are now ready to prove the coarse volume bound discussed in the introduction.

Theorem 5.7. *Let $R_1, R_2, \ldots, R_n$ be disjoint rods in the 3-torus whose complement is M, where $n \geq 3$. Suppose that R_n has direction vector $(0,0,1)$ and for $i<n$, R_i has direction vector $(p_i, q_i, 0)$, with $(p_i, q_i) \neq (p_{i+1}, q_{i+1})$ for $i = 1, 2, \ldots, n-2$ and $(p_{n-1}, q_{n-1}) \neq (p_1, q_1)$. Suppose that $R_1, R_2, \ldots, R_{n-1}$ are positioned from top to bottom in the unit cube representation of the 3-torus. Let $[c_{i1}; c_{i2}, \ldots, c_{im_i}]$ be a continued fraction expansion for p_i/q_i. Then M is hyperbolic and its volume satisfies the asymptotically sharp upper bound*

$$\operatorname{Vol}(M) \leq 2v_{\text{oct}} \sum_{i=1}^{n-1} m_i.$$

Suppose in addition that

$$C := \min_{\substack{1\leq i\leq n-1\\ j\geq 2}} \{|c_{ij}|, |c_{i1} - c_{(i-1)1}|\} \geq 6,$$

where c_{01} is interpreted as $c_{(n-1)1}$. Then the volume satisfies the lower bound

$$\operatorname{Vol}(M) \geq \left(1 - \frac{4\pi^2}{C^2+4}\right)^{3/2} 2v_{\text{oct}} \sum_{i=1}^{n-1} m_i.$$

Proof. By Lemma 5.1, the manifold M must be hyperbolic.

We construct standard parent manifolds with ideal octahedral decompositions. By Proposition 5.3, there exists a standard rod complement N with $n-1$ core rods and $\sum_{i=1}^{n-1} 2m_i$ filling rods such that M can be obtained by applying a (p_i, q_i)-nested annular Dehn filling to each of the core rods of N. Observe that the outermost pair of filling rods for each nested annular Dehn filling are $(1, 0, 0)$-rods. Each of the two outermost filling rods for each nested annular Dehn filling will be linearly isotopic to an outermost filling rod for another nested annular Dehn filling. By cutting along the essential annuli arising from all of these linear isotopies, we obtain a standard parent manifold N_{p} with exactly one $(0, 0, 1)$-rod, namely R_n, and alternating $(1, 0, 0)$-rods and $(0, 1, 0)$-rods.

By Lemma 5.4, N_{p} has a decomposition into $\sum_{i=1}^{n-1} 2m_i$ regular ideal octahedra and it admits a complete hyperbolic structure.

We obtain $M = \mathbb{T}^3 \setminus (R_1 \cup R_2 \cup \cdots \cup R_n)$ by Dehn filling the standard parent manifold N_{p}. Since Dehn filling decreases volume [29], we obtain the bound

$$\mathrm{Vol}(M) < \mathrm{Vol}(N_{\mathrm{p}}) = v_{\mathrm{oct}} \sum_{i=1}^{n-1} 2m_i.$$

Furthermore, this bound is asymptotically sharp. Taking larger and larger values for the coefficients c_{ij} of the continued fraction expansion while fixing the lengths m_i will produce Dehn fillings of the same parent manifold whose volumes converge to that of the parent manifold.

For the lower bound, we consider the slopes of the Dehn filling. These are of the form $1/c_{ij}$ for filling components with $2 \leq j \leq m_i$. For the outermost filling rods, the coefficient of the Dehn filling combines the $1/c_{i1}$ from one side with $-1/c_{(i-1)1}$ from the other side, as in Lemma 4.2. Thus, the slope is $1/(c_{i1} - c_{(i-1)1})$.

By Lemma 5.5, for any integer ℓ, the length of the slope $1/\ell$ on a filling rod is $\sqrt{\ell^2+4}$. So under the hypotheses required for the lower bound, the minimum length slope will be at least $\sqrt{6^2+4} > 2\pi$. We may now apply a theorem of Futer, Kalfagianni and Purcell, which states that if the minimum slope length is larger than 2π, then the volume change under Dehn filling is a multiple of the volume of the unfilled manifold [16, Theorem 1.1]. In our case, this leads to

$$\mathrm{Vol}(M) \geq \left(1 - \frac{4\pi^2}{C^2+4}\right)^{3/2} 2v_{\mathrm{oct}} \sum_{i=1}^{n-1} m_i. \qquad \square$$

Remark 5.8. The upper bound of Theorem 5.7 motivates one to seek an efficient expression for such rod complements, with the complexity measured by $\sum_{i=1}^{n-1} m_i$, the sum of the lengths of the continued fractions. One may simultaneously switch each $(p_i, q_i, 0)$-rod to a $(q_i, p_i, 0)$-rod, which may change $\sum_{i=1}^{n-1} m_i$. Recall that we allow negative terms in our continued fractions, as per the discussion in Section 2.2. Typically, one obtains shorter continued fractions this way than if one restricts to using positive integers as terms.

Corollary 5.9. *There exists a sequence of hyperbolic rod complements with bounded volume, but for which the upper bound of Theorem 3.2 grows to infinity.*

Proof. For n a positive integer, let $R_1^{(n)}$ be an $(n, 1, 0)$-rod, let R_2 be a $(0, 1, 0)$-rod, and let R_3 be a $(0, 0, 1)$-rod. These rods satisfy the hypotheses of the first part of Theorem 5.7. The continued fraction associated to the rod $R_1^{(n)}$ is $n/1 = [n]$. Thus, in the notation of Theorem 5.7, we have $m_1 = 1$ for any choice of n and we also have $m_2 = 1$. So the upper bound of Theorem 5.7 implies that

$$\mathrm{Vol}(\mathbb{T}^3 \setminus (R_1^{(n)} \cup R_2 \cup R_3)) \le 4v_{\mathrm{oct}}.$$

On the other hand, $(p_1, q_1) = (n, 1)$ and $(p_2, q_2) = (0, 1)$, so $|p_1q_2 - p_2q_1| = n$, which is unbounded as n grows to infinity. □

Corollary 5.10. *There exists a sequence of hyperbolic rod complements, each with three rods, whose volumes grow to infinity.*

Proof. Define the sequence of rational slopes

$$\frac{p_k}{q_k} = [\underbrace{k; k, k, \ldots, k}_{k \text{ terms}}].$$

for $k \ge 6$. For example, we have

$$\frac{p_6}{q_6} = [6; 6, 6, 6, 6, 6] = \frac{53353}{8658},$$
$$\frac{p_7}{q_7} = [7; 7, 7, 7, 7, 7, 7] = \frac{927843}{129949},$$
$$\frac{p_8}{q_8} = [8; 8, 8, 8, 8, 8, 8, 8] = \frac{18674305}{2298912}.$$

Let $R_1^{(k)}$ be a $(p_k, q_k, 0)$-rod, let R_2 be a $(0, 1, 0)$-rod, and let R_3 be a $(0, 0, 1)$-rod. Let $M_k = \mathbb{T}^3 \setminus (R_1^{(k)} \cup R_2 \cup R_3)$ be the associated rod complement. Using the notation of Theorem 5.7, we have $m_1 = k$, $m_2 = 1$, and $C = k \ge 6$. Figure 4 shows the standard parent manifold of $\mathbb{T}^3 \setminus (R_1^{(7)} \cup R_2 \cup R_3)$. So Theorem 5.7 implies that

$$\mathrm{Vol}(M_k) \ge \left(1 - \frac{4\pi^2}{k^2+4}\right)^{3/2} 2v_{\mathrm{oct}}(k+1) > \left(1 - \frac{4\pi^2}{6^2+4}\right)^{3/2} 2v_{\mathrm{oct}}k > 0.01091k.$$

Since the right side grows to infinity with k, the volume of M_k also grows to infinity. □

6. Further discussion

Our results on the volumes of rod complements suggest various natural questions worthy of further exploration, such as the following.

Question 6.1. Do there exist two-sided coarse volume bounds for all rod complements in terms of the rod parameters?

By Corollaries 5.9 and 5.10, such bounds cannot depend only on the number of rods nor on the number of intersections of the rods in a particular projection. It would be natural to wonder whether two rod complements with the same rod parameters have volumes with bounded ratio.

Question 6.2. Does hyperbolic volume distinguish rod complements up to homeomorphism?

It would be surprising if any two rod complements with the same hyperbolic volume were necessarily homeomorphic. It is well known that hyperbolic volume does not distinguish hyperbolic 3-manifolds in general. In particular, mutation of cusped hyperbolic 3-manifolds can change its homeomorphism class, but necessarily preserves the hyperbolicity and volume [28]. An example of mutation involves cutting along an essential embedded 4-punctured sphere bounding a tangle in a ball, rotating the ball via a certain involution, and then regluing. Mutation can also be performed with respect to surfaces of other topologies that possess a suitable involution. It is not immediately obvious whether rod complements contain such embedded essential surfaces along which mutation can be performed.

Question 6.3. Does there exist a rod complement with a nontrivial mutation?

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References

[1] C. C. Adams, "Volumes of N-cusped hyperbolic 3-manifolds", *J. London Math. Soc.* (2) **38**:3 (1988), 555–565. MR

[2] I. Agol, P. A. Storm, and W. P. Thurston, "Lower bounds on volumes of hyperbolic Haken 3-manifolds", *J. Amer. Math. Soc.* **20**:4 (2007), 1053–1077. MR

[3] K. L. Baker, "Surgery descriptions and volumes of Berge knots, I: Large volume Berge knots", *J. Knot Theory Ramifications* **17**:9 (2008), 1077–1097. MR

[4] J. F. Brock, "Weil–Petersson translation distance and volumes of mapping tori", *Comm. Anal. Geom.* **11**:5 (2003), 987–999. MR

[5] B. A. Burton, T. de Paiva, A. He, and C. O. Y. Hui, "Crushing surfaces of positive genus", preprint, 2024. arXiv 2403.11523

[6] T. Cremaschi and J. A. Rodríguez-Migueles, "Hyperbolicity of link complements in Seifert-fibered spaces", *Algebr. Geom. Topol.* **20**:7 (2020), 3561–3588. MR

[7] T. Cremaschi, J. A. Rodriguŕz-Migueles, and A. Yarmola, "On volumes and filling collections of multicurves", *J. Topol.* **15**:3 (2022), 1107–1153. MR

[8] O. T. Dasbach and X.-S. Lin, "A volumish theorem for the Jones polynomial of alternating knots", *Pacific J. Math.* **231**:2 (2007), 279–291. MR

[9] M. E. Evans and S. T. Hyde, "From three-dimensional weavings to swollen corneocytes", *J. R. Soc. Interface* **8**:62 (2011), 1274–1280.

[10] M. E. Evans and G. E. Schröder-Turk, "In a material world: hyperbolic geometry in biological materials", *Asia Pac. Math. Newsl.* **5**:2 (2015), 21–30.

[11] M. E. Evans, V. Robins, and S. T. Hyde, "Periodic entanglement, II: Weavings from hyperbolic line patterns", *Acta Crystallogr. Sect. A* **69**:3 (2013), 262–275. MR

[12] M. E. Evans, V. Robins, and S. T. Hyde, "Ideal geometry of periodic entanglements", *Proc. A.* **471**:2181 (2015), art. id. 20150254. MR

[13] B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton Mathematical Series **49**, Princeton Univ. Press, 2012. MR

[14] D. Futer and J. S. Purcell, "Links with no exceptional surgeries", *Comment. Math. Helv.* **82**:3 (2007), 629–664. MR

[15] D. Futer and S. Schleimer, "Cusp geometry of fibered 3-manifolds", *Amer. J. Math.* **136**:2 (2014), 309–356. MR

[16] D. Futer, E. Kalfagianni, and J. S. Purcell, "Dehn filling, volume, and the Jones polynomial", *J. Differential Geom.* **78**:3 (2008), 429–464. MR

[17] D. Futer, E. Kalfagianni, and J. Purcell, *Guts of surfaces and the colored Jones polynomial*, Lecture Notes in Mathematics **2069**, Springer, 2013. MR

[18] F. Guéritaud, "On canonical triangulations of once-punctured torus bundles and two-bridge link complements", *Geom. Topol.* **10** (2006), 1239–1284. MR

[19] A. Hatcher, "Homeomorphisms of sufficiently large P^2-irreducible 3-manifolds", *Topology* **15**:4 (1976), 343–347. MR

[20] C. O. Y. Hui, "A geometric classification of rod complements in the 3-torus", *Proc. Amer. Math. Soc.* **153**:1 (2025), 381–394. MR

[21] C. O. Y. Hui and J. S. Purcell, "On the geometry of rod packings in the 3-torus", *Bull. Lond. Math. Soc.* **56**:4 (2024), 1291–1309. MR

[22] M. Lackenby, "The volume of hyperbolic alternating link complements", *Proc. London Math. Soc.* (3) **88**:1 (2004), 204–224. MR

[23] L. Norlén and A. Al-Amoudi, "Stratum corneum keratin structure, function, and formation: the cubic rod-packing and membrane templating model", *Journal of Investigative Dermatology* **123**:4 (2004), 715–732.

[24] M. O'Keeffe and S. Andersson, "Rod packings and crystal chemistry", *Acta Cryst.* **33**:6 (1977), 914–923.

[25] M. O'Keeffe, J. Plévert, Y. Teshima, Y. Watanabe, and T. Ogama, "The invariant cubic rod (cylinder) packings: symmetries and coordinates", *Acta Cryst. Sect. A* **57**:1 (2001), 110–111. MR

[26] J. S. Purcell, *Hyperbolic knot theory*, Graduate Studies in Mathematics **209**, Amer. Math. Soc., Providence, RI, 2020. MR

[27] N. L. Rosi, J. Kim, M. Eddaoudi, B. Chen, M. O'Keeffe, and O. M. Yaghi, "Rod packings and metal-organic frameworks constructed from rod-shaped secondary building units", *J. Am. Chem. Soc.* **127**:5 (2005), 1504–1518.

[28] D. Ruberman, "Mutation and volumes of knots in S^3", *Invent. Math.* **90**:1 (1987), 189–215. MR

[29] W. P. Thurston, *The geometry and topology of three-manifolds*, *IV*, Amer. Math. Soc., Providence, RI, 2022. MR

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A REMARK ON THE LEWARK–ZIBROWIUS INVARIANT

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We prove a conjecture about the concordance invariant ϑ, defined in a recent paper by Lewark and Zibrowius. This result simplifies the relation between ϑ and Rasmussen's s-invariant. The proof relies on Bar-Natan's tangle version of Khovanov homology or, more precisely, on its distillation in the case of 4-ended tangles into the immersed curve theory of Kotelskiy, Watson and Zibrowius.

1. Introduction

Lewark and Zibrowius [2024] defined two new families of smooth concordance invariants,

$$\{\vartheta_c : \mathcal{C}_{\mathrm{sm}} \to \mathbb{Z}\} \quad \text{and} \quad \{\vartheta'_c : \mathcal{C}_{\mathrm{sm}} \to \mathbb{Z} \cup \{\infty\}\},$$

parametrized by a prime c. These invariants exploit the following linearity property of Rasmussen's invariant in characteristic c. Given a knot $K \subset S^3$ and a pattern $P \subset D^2 \times S^1$ of wrapping number 2, the function

$$t \mapsto s_c(P_t(K))$$

is the restriction to $\mathbb{Z}$ of a piecewise affine function $\mathbb{R} \to \mathbb{R}$ of slope 1 or 0 that has at most one jump discontinuity. If the winding number of P is ± 2 then the function has slope 1, otherwise the winding number and slope are 0 and, in this latter case, the function does have a jump discontinuity. In the case of winding number ± 2, the invariant $\vartheta'_c(K)$ is defined to be the value of t for which

$$s_c(P_{\vartheta'_c(K)}(K)) = s_c(P_{\vartheta'_c(K)-1}(K)),$$

if it exists. If no such value exists because the piecewise affine function is affine, then $\vartheta'_c(K) := \infty$. Not only do Lewark and Zibrowius prove that ϑ_c and ϑ'_c are concordance invariants and that ϑ_c is a homomorphism $\mathcal{C}_{\mathrm{sm}} \to \mathbb{Z}$, but they also show that ϑ_c is a genuinely new invariant, in that it is not simply a multiple of s_c, in contrast to the τ-invariant [2024, §2.2].

The knots K with $\vartheta'_c(K) \neq \infty$ are of particular interest, and they are called ϑ_c-rational. We establish here a conjecture on the expected simplicity of ϑ'_c:

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Theorem 1.1 [Lewark and Zibrowius 2024, Conjecture 2.24]. *If K is a ϑ_c-rational knot, then $\vartheta_c(K) = 0$.*

Since ϑ_c agrees with ϑ'_c on the class of ϑ_c-rational knots [Lewark and Zibrowius 2024, Theorem 2.23], it follows that the second family of invariants $\{\vartheta'_c\}$ contains no more information than a single $\mathbb{Z}/2\mathbb{Z}$-valued invariant. A consequence noted by Lewark and Zibrowius [2024, p. 250] is the following simplification of their Theorem 2.23:

Corollary 1.2. *Let $K \subset S^3$ be a ϑ_c-rational knot and let P be a pattern with wrapping number 2 and winding number ± 2. Then*

$$s_c(P(K)) = s_c(P_{-\vartheta_c(K)}(U)) = s_c(P_0(U)). \qquad \square$$

Our argument uses the immersed curve theory of 4-ended tangles, constructed in [Kotelskiy et al. 2019] as a specialization of the theory developed in [Bar-Natan 2005], and a property of Lee's homology [2005].

2. Background

Tangles are considered modulo isotopy fixing the endpoints. Let $(K, *)$ be a pointed oriented knot and let T_K be the 4-ended tangle obtained by taking a copy of the long knot $K \setminus \{*\}$ together with its Seifert push-off, as in Figure 1. We generally also orient our tangles and mark an endpoint, as required for the theory in [Kotelskiy et al. 2019].

To specify notation for the cut-and-paste procedures used, let $n \in \mathbb{Z} \cup \{\infty\}$. First, the rational n-tangle Q_n is the one in Figure 2 for $n > 0$. If $n < 0$, then $Q_n = mQ_{-n}$, where m denotes the mirror. And if $n = 0, \infty$, we set $Q_0 =$  and $Q_\infty =$ . Second, given two 4-ended tangles T_1 and T_2, the link $\mathcal{L}(T_1, T_2)$ is obtained by identifying endpoints as in Figure 2 below. Finally, let the n-closure $T(n)$ of a 4-ended tangle T be $\mathcal{L}(T, Q_{-n})$. By convention, diagrams for the tangle T_K are chosen so that their ∞-closure is the unknot, and the tangle is oriented compatibly with the 0-closure, as in Figure 1.

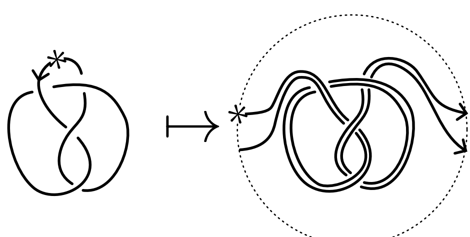


Figure 1. A pointed oriented knot $(K, *)$ and its associated double T_K.

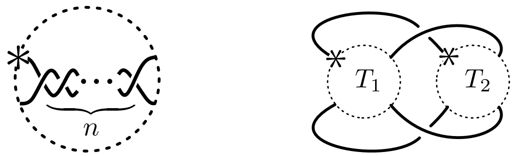


Figure 2. Left: The tangle Q_n. Right: The link $\mathcal{L}(T_1, T_2)$.

2.1. ***Bar-Natan homology.*** The Bar-Natan homology of a link is a version of Khovanov homology [2000] defined in [Bar-Natan 2005] with coefficients in the field with two elements $\mathbb{F}_2$, and later extended as a theory with coefficients in any prime field in [Mackaay et al. 2007]. It has been observed that varying the field characteristic results in interesting differences [Lewark and Zibrowius 2021], so let $\mathbb{F}_c$ be the prime field of characteristic c (in particular, $\mathbb{F}_0 = \mathbb{Q}$). We use the setup in [Kotelskiy et al. 2019, §3].

Given a link L, its Bar-Natan homology is a bigraded $\mathbb{F}_c[H]$-module $\mathrm{BN}(L; \mathbb{F}_c)$, where H is a formal variable that lowers the secondary (quantum) grading by 2. The shift operators for the homological and quantum gradings are denoted using square and curly brackets, respectively. For example,

$$\mathrm{BN}(L; \mathbb{F}_c)\{-1\}$$

is the Bar-Natan homology of L with coefficients in $\mathbb{F}_c$, but with quantum gradings formally reduced by 1.

If the link L is pointed, then there is a reduced theory $\widetilde{\mathrm{BN}}(L; \mathbb{F}_c)$, which is related to unreduced Bar-Natan homology by a short exact sequence of bigraded $\mathbb{F}_c[H]$-complexes:

$$0 \to \widetilde{\mathrm{CBN}}(D; \mathbb{F}_c)\{-1\} \to \mathrm{CBN}(D; \mathbb{F}_c) \to \widetilde{\mathrm{CBN}}(D; \mathbb{F}_c)\{1\} \to 0, \tag{1}$$

where D is a choice of diagram for L.

Notation. Free summands of the bigraded $\mathbb{F}_c[H]$-module $\widetilde{\mathrm{BN}}(L; \mathbb{F}_c)$ are called towers. The grading of a tower refers to the grading of a corresponding free generator.

2.2. ***Lee's deformation.*** Rasmussen [2010] used the work in [Lee 2005] to define the s-invariant of a knot. While the s-invariant can also be defined for links, as in [Beliakova and Wehrli 2008; Pardon 2012], this construction is not used as much, and Lewark and Zibrowius arranged so that their work only dealt with s-invariants of knots. This subsection recalls an aspect of the definition of the s-invariant for links in Lemma 2.1 below. This result is known to the experts and is the main observation needed to prove Theorem 1.1. See also [Lee 2005, Proposition 4.3].

Lemma 2.1. *Let L be an oriented 2-component pointed link. If* $\mathrm{lk}(L) \neq 0$, *then there is a unique tower* $\mathbb{F}_c[H] \hookrightarrow \widetilde{\mathrm{BN}}(L; \mathbb{F}_c)$ *in homological grading* 0. *Otherwise, if* $\mathrm{lk}(L) = 0$, *then both towers have homological grading* 0.

Proof. The idea is that, by setting $H = 1$ in the chain complex $\mathrm{CBN}(L; \mathbb{F}_c)$, we obtain a chain complex $\mathrm{fCBN}(L; \mathbb{F}_c)$ that is no longer bigraded, but rather homologically graded and quantum filtered. Courtesy of the filtration, there is an induced spectral sequence

$$\mathrm{fCBN}(L; \mathbb{F}_c) \rightrightarrows H_*(\mathrm{fCBN}(L; \mathbb{F}_c)).$$

Theorem 2.2 of [Lipshitz and Sarkar 2014] establishes that the vector space $H_*(\mathrm{fCBN}(L; \mathbb{F}_c))$ is 4-dimensional, and there is a canonical identification between the set of orientations on L and a set of generators of $H_*(\mathrm{fCBN}(L; \mathbb{F}_c))$. To understand this identification, note that each orientation on L determines an oriented resolution of a diagram for L. Lee's argument applies in this context to show that each generator of $H_*(\mathrm{fCBN}(L; \mathbb{F}_c))$ is the homology class of an algebra element assigned to an oriented resolution of L by the TQFT defining fCBN; see [Lee 2005, Theorem 4.2] or [Rasmussen 2010, §2.4] for the construction and [Lipshitz and Sarkar 2014, Theorem 2.2] for the applicability of Lee's work in this slightly different context.

Now, as explained in [Kotelskiy et al. 2019, Proposition 3.8], the components of the differential $\partial_{\mathrm{CBN}(L)}$ that are given by $1 \mapsto H^l$ induce differentials on the l-th page of the spectral sequence above, and this implies that

$$\mathrm{BN}(L; \mathbb{F}_c) \cong (\mathbb{F}_c[H])^{\oplus 4} \oplus \mathrm{Tors},$$

where the towers in $\mathrm{BN}(L; \mathbb{F}_c)$ correspond to the generators of $H_*(\mathrm{fCBN}(L; \mathbb{F}_c)$. Moreover it follows from the short exact sequence (1) that there is a 2-to-1 correspondence that preserves homological grading between the towers of $\mathrm{BN}(L)$ and the towers of $\widetilde{\mathrm{BN}}(L)$.

Finally, fix an oriented diagram $(D, \mathfrak{o}_0)$ for L, where $\mathfrak{o}_0$ is the orientation on D induced from L. Let $n_+(\mathfrak{o}_0)$ and $n_-(\mathfrak{o}_0)$ be the number of positive and negative crossings in $(D, \mathfrak{o}_0)$. Pick a component K of L and let $\mathfrak{o}_1$ be the orientation on D which is obtained by reversing the orientation on K. Then the number of negative crossings in $(D, \mathfrak{o}_1)$ is

$$n_-(\mathfrak{o}_1) = n_-(\mathfrak{o}_0) + 2\,\mathrm{lk}(L).$$

It follows that, while the oriented resolution of $(D, \mathfrak{o}_0)$ lies in homological grading 0, the $\mathfrak{o}_1$-oriented resolution $D^{\mathfrak{o}_1}$ lies in homological grading $2\,\mathrm{lk}(L)$. □

2.3. ***The immersed curve theory.*** In [Kotelskiy et al. 2019], two equivalent invariants of pointed 4-ended oriented tangles are defined:

$$T \mapsto \mathcal{D}(T; \mathbb{F}_c) \in \mathbf{Mod}^{\mathcal{B}} \quad \text{and} \quad T \mapsto \widetilde{\mathrm{BN}}(T; \mathbb{F}_c) \in \mathbf{Fuk}(S^2_{4,*}).$$

The first produces type-D structures over the Bar-Natan algebra $\mathcal{B}$, which we will describe in Section 4. The second lands in the (partially wrapped) Fukaya category of S^2, punctured at four points, one of which is marked $*$. In other words, $\widetilde{\mathrm{BN}}(T; \mathbb{F}_c)$ is an immersed curve in $S^2_{4,*}$, possibly carrying a nontrivial local system. This possibility does not occur for noncompact curves, which are the only curves of interest in what follows. Moreover, the invariants are bigraded in an appropriate sense. Our main tool is the following pairing theorem.

Theorem 2.2 [Kotelskiy et al. 2019, Theorem 7.2]. *Let T_1 and T_2 be two pointed 4-ended tangles, and let $L = \mathcal{L}(T_1, T_2)$. Then the Bar-Natan homology is isomorphic to the wrapped Lagrangian intersection Floer homology of the tangle invariants, as bigraded $\mathbb{F}_c[H]$-modules*:

$$\widetilde{\mathrm{BN}}(L; \mathbb{F}_c)\{-1\} \cong \mathrm{HF}(\widetilde{\mathrm{BN}}(mT_1; \mathbb{F}_c), \widetilde{\mathrm{BN}}(T_2; \mathbb{F}_c)).$$

3. The proof of Theorem 1.1

Suppose now that K is a ϑ_c-rational knot. Lewark and Zibrowius identified $\vartheta_c(K)$ with a certain slope of $\widetilde{\mathrm{BN}}(T_K; \mathbb{F}_c)$, and this allows us to reduce the proof to a simple statement that can be checked using Lemma 2.1. Let $\widetilde{\mathrm{BN}}_a(T; \mathbb{F}_c)$ consist of the noncompact component(s) of $\widetilde{\mathrm{BN}}(T; \mathbb{F}_c)$.

Proposition 3.1 [Lewark and Zibrowius 2024, Proposition 6.18]. *If K is ϑ_c-rational, then the immersed curve $\widetilde{\mathrm{BN}}_a(T_K; \mathbb{F}_c)$ is equal to the immersed curve of the rational tangle Q_n, for some choice of $n \in 2\mathbb{Z}$, up to some grading shift.*

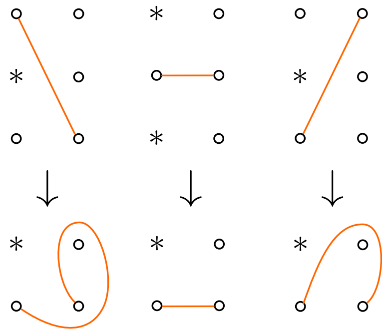
We have then $\widetilde{\mathrm{BN}}_a(T_K; \mathbb{F}_c) = \widetilde{\mathrm{BN}}(Q_n; \mathbb{F}_c)$, for some $n \in 2\mathbb{Z}$, up to grading shift. The immersed curve invariants $\widetilde{\mathrm{BN}}(Q_n; \mathbb{F}_c)$ are calculated in [Kotelskiy et al. 2019]. It turns out that they are independent of the coefficient field, so we may drop it from the notation. These invariants are best described in the following covering space of the 4-punctured sphere:

$$\mathbb{R}^2 \setminus \left(\tfrac{1}{2}\mathbb{Z}\right)^2 \xrightarrow{\alpha} T^2_{4,*} \xrightarrow{\beta} S^2_{4,*},$$

where β is the double cover given by hyperelliptic involution and α is the universal Abelian cover of the punctured torus. The puncture $*$ lifts to the integer lattice $\mathbb{Z}^2 \subset \frac{1}{2}\mathbb{Z}^2$. The lift of $\widetilde{\mathrm{BN}}(Q_n)$ is (isotopic to) a line of slope n, as depicted in Figure 3 in the cases $n = -2, 0, 2$.

Proposition 3.2 [Lewark and Zibrowius 2024, Corollary 6.14]. *Given a knot K in S^3, let σ_c be the slope of $\widetilde{\mathrm{BN}}_a(T_K; \mathbb{F}_c)$ near the bottom-right tangle end. Then $\vartheta_c(K) = \lceil \sigma_c \rceil$.*

Since the curve $\widetilde{\mathrm{BN}}(Q_n)$ lifts to a curve that is isotopic to a line of slope n, the above two propositions reduce the proof of Theorem 1.1 to proving that $\widetilde{\mathrm{BN}}_a(T_K; \mathbb{F}_c) = \widetilde{\mathrm{BN}}(Q_0)$, up to grading shift. Consider the Bar-Natan homology of the 0-closure $T_K(0)$. Since T_K is obtained by taking the union of a long knot with its



$\widetilde{\mathrm{BN}}(Q_{-2})$ $\widetilde{\mathrm{BN}}(Q_0)$ $\widetilde{\mathrm{BN}}(Q_2)$

Figure 3. Some immersed curve invariants of Q_n and their lifts to the covering space $\mathbb{R}^2 \setminus \mathbb{Z}^2$.

Seifert push-off, the closure $T_K(0)$ has linking number 0. Thus, by Lemma 2.1, the Bar-Natan homology $\widetilde{\mathrm{BN}}(T_K(0); \mathbb{F}_c)$ has both $\mathbb{F}_c[H]$ towers in grading 0. We may compute this homology using Theorem 2.2:

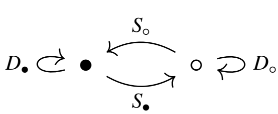
$$\begin{aligned}\widetilde{\mathrm{BN}}(T_K(0))\{-1\} &\cong \mathrm{HF}\Big(\widetilde{\mathrm{BN}}\big(m\asymp\big), \widetilde{\mathrm{BN}}(T_K)\Big)\\ &\cong \mathrm{HF}\Big(\widetilde{\mathrm{BN}}\big(m\asymp\big), \widetilde{\mathrm{BN}}_a(T_K)[h]\{q\}\Big) \oplus \mathrm{Tors}\\ &\cong \widetilde{\mathrm{BN}}(T(2,2n); \mathbb{F}_c)[h]\{q\} \oplus \mathrm{Tors},\end{aligned}$$

where Tors is a torsion $\mathbb{F}_c[H]$-module, $T(2,2n)$ is the $(2,2n)$-torus link and $[h]\{q\}$ is a possible bigrading shift. Clearly both towers of $\widetilde{\mathrm{BN}}(T_K(0))$ sit in a summand of the homology that is isomorphic to $\widetilde{\mathrm{BN}}(T(2,2n))$, up to a grading shift. But the homology of 2-strand torus links is well understood — indeed, we will indicate how to compute it in the next section. In particular, the only way for both towers of $\widetilde{\mathrm{BN}}(T(2,2n))$ to be in the same homological grading is if $n = 0$. □

4. Epilogue

Let us now indicate how to compute $\widetilde{\mathrm{BN}}(T(2,n); \mathbb{F}_c)$, using a technique that applies more generally and that is the honest source of the proof above. To that end, we will need to look under the hood of Theorem 2.2 and use the bigraded type-D structures $\mathcal{I}(Q_n; \mathbb{F}_c) \in \mathbf{Mod}^{\mathcal{B}}$. First, we will write $\Bbbk$ instead of $\mathbb{F}_c$ in what follows, since the characteristic does not matter and clutters the notation.

Definition 4.1. The Bar-Natan algebra $\mathcal{B}$ is the bigraded path algebra over $\Bbbk$ of the quiver



subject to the relations

$$D_\circ S_\bullet = S_\bullet D_\bullet = 0 \quad \text{and} \quad D_\bullet S_\circ = S_\circ D_\circ = 0,$$

and with bigrading given by

$$q(1_{◒}) = 0, \quad q(S_{◒}) = -1, \quad q(D_{◒}) = -2, \quad h(1_{◒}) = h(S_{◒}) = h(D_{◒}) = 0,$$

where $◒ \in \{\circ, \bullet\}$.

Remark. Alternatively, consider the quiver above as describing an additive category with two objects and with four nonidentity morphisms indicated, and suppose that the composites DS and SD vanish. Then the algebra $\mathcal{B}$ is the collection of all morphisms of this category, where the algebra operation corresponds to composition of morphisms, and we formally set the composite of noncomposable morphisms to 0. This is a bigraded category in the sense of [Bar-Natan 2005].

Remark. By definition, path algebras have idempotent elements $1_{◒}$: the constant paths at each vertex. These correspond to identity morphisms in the categorical perspective. The idempotents generate a subring $\mathcal{I} := \Bbbk\langle 1_\circ, 1_\bullet\rangle \cong \Bbbk^2$, giving $\mathcal{B}$ the additional structure of an $\mathcal{I}$-algebra.

Now a type-D structure over $\mathcal{B}$ is, by definition, an $\mathcal{I}$-module M together with a map $\delta : M \to M \otimes_{\mathcal{I}} \mathcal{B}$ subject to an appropriate "$d^2 = 0$" condition:

$$(\mathrm{Id}_M \otimes m) \circ (\delta \otimes \mathrm{Id}_{\mathcal{B}}) \circ \delta = 0.$$

Notation. Type-D structures are described as labeled directed graphs, with vertices labeled by $\bullet$ or $\circ$, and edges labeled with elements of $\mathcal{B}$. The vertices correspond to homogeneous generators (with respect to the action of $\mathcal{I}$) and the edges are the homogeneous components of the differential δ. To avoid heavy use of brackets, we denote homological and quantum shifts by subscripts and left-superscripts, respectively. For example, ${}^{q}\bullet_h$ is a type-D structure generator fixed by $1_\bullet$ and in (homological, quantum)-bigrading (h, q).

The Д-invariants of Q_n are explicitly computed as Example 4.27 of [Kotelskiy et al. 2019] (where Q_n is oriented compatibly with the 0-closure): $\text{Д}(Q_0) = {}^{0}\bullet_0$ and, more generally,

$$\text{Д}(Q_n; k) = \begin{cases} \underbrace{{}^{3n-1}\circ_n \xrightarrow{X} \cdots \xrightarrow{D} \circ \xrightarrow{SS} \circ \xrightarrow{D} \circ \xrightarrow{S} {}^{n}\bullet_0}_{-n+1} & \text{if } n < 0, \\ \underbrace{{}^{n}\bullet_0 \xrightarrow{S} \circ \xrightarrow{D} \circ \xrightarrow{SS} \circ \xrightarrow{D} \cdots \xrightarrow{X} {}^{3n-1}\circ_n}_{n+1} & \text{if } n > 0, \end{cases}$$

where the algebra element X is D if n is even and SS if n is odd.

Finally, the following element is defined in $\mathcal{B}$:

$$H := SS_\bullet - D_\bullet + SS_\circ - D_\circ.$$

This gives the Bar-Natan algebra the structure of a $\Bbbk[H]$-algebra, and, by design, this structure is compatible with the $\Bbbk[H]$-module structure of Bar-Natan homology:

Theorem 4.2 [Kotelskiy et al. 2019, Proposition 4.31]. *Let T_1 and T_2 be two pointed oriented 4-ended tangles. Then there is a homotopy*

$$\widetilde{\mathrm{CBN}}(\mathcal{L};\Bbbk)\{-1\} \simeq \mathrm{Mor}(\text{Д}(mT_1;\Bbbk), \text{Д}(T_2;\Bbbk)) \tag{2}$$

of bigraded chain complexes of $\Bbbk[H]$-modules, where m denotes the mirror, and the bifunctor $\mathrm{Mor}(-,-)$ *above is the internal Hom in the category of bigraded type-D structures.*

The type-D structure of $\mathrm{Mor}(\text{Д}_1, \text{Д}_2)$ is defined in [Kotelskiy et al. 2019, §2]. Briefly, $\mathrm{Mor}(\text{Д}_1, \text{Д}_2)$ consists of all morphisms $\text{Д}_1 \to \text{Д}_2$, not just the grading-preserving ones. Given generators $x_i \in \text{Д}_i$ the quantum and homological grading of a morphism is given by

$$\mathrm{gr}(x_1 \xrightarrow{f} x_2) = \mathrm{gr}(x_2) - \mathrm{gr}(x_1) + \mathrm{gr}(f).$$

Finally, a differential D on $\mathrm{Mor}(\text{Д}_1, \text{Д}_2)$ is given on morphisms between generators by pre- and post-composing with the δ_i differentials on Д_i:

$$D(x_1 \xrightarrow{f} x_2) = f \circ \delta_1 - \delta_2 \circ f.$$

For our purposes, note the computations

$$\mathrm{Mor}({}^i\bullet_j, {}^k\circ_l) = \Bbbk[H]\langle {}^i\bullet_j \xrightarrow{S_\bullet} {}^k\circ_l\rangle \cong {}^{k-i-1}(\Bbbk[H])_{l-j},$$
$$\mathrm{Mor}({}^i\bullet_j, {}^k\bullet_l) = \Bbbk[H]\langle {}^i\bullet_j \xrightarrow{1_\bullet} {}^k\bullet_l, {}^i\bullet_j \xrightarrow{D_\bullet} {}^k\bullet_l\rangle \cong {}^{k-i}(\Bbbk[H])_{l-j} \oplus {}^{k-i-2}(\Bbbk[H])_{l-j}.$$

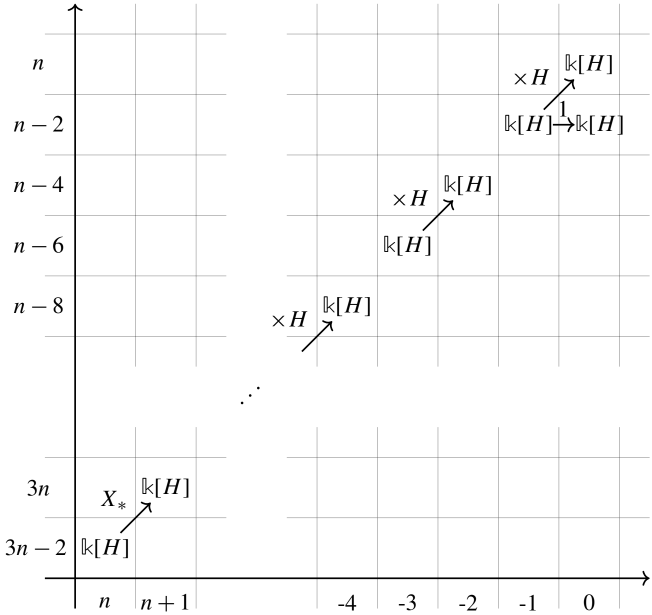
To give the simplest application of Theorem 4.2, the unknot U is $\mathcal{L}(\asymp,)()$. Thus

$$\widetilde{\mathrm{BN}}(U)\{-1\} \cong H_*[\mathrm{Mor}({}^0\bullet_0, {}^0\circ_0)] \cong {}^{-1}(\Bbbk[H])_0.$$

Now we can give a rapid computation of $\widetilde{\mathrm{BN}}(T(2,n)) = \widetilde{\mathrm{BN}}(\mathcal{L}(\asymp, Q_n))$. If $n < 0$, then

$$\begin{aligned}\widetilde{\mathrm{BN}}(T(2,n))\{-1\} &\cong H_*[\mathrm{Mor}({}^0\bullet_0, {}^{3n-1}\circ_n \xrightarrow{X} \circ \xrightarrow{D} \circ \xrightarrow{SS} \cdots \to {}^n\bullet_0)] \\ &\cong H_*\big[\mathrm{Mor}({}^0\bullet_0, {}^{3n-1}\circ_n) \xrightarrow{X_*} \mathrm{Mor}({}^0\bullet_0, \circ) \xrightarrow{D_*} \mathrm{Mor}({}^0\bullet_0, \circ) \\ &\qquad \xrightarrow{SS_*} \cdots \to \mathrm{Mor}({}^0\bullet_0, {}^n\bullet_0)\big],\end{aligned}$$

where the maps above are the ones induced by postcomposing with the components of the differential on $\text{Д}(Q_n)$. It is convenient to organize the above complex in a grid as follows:



Here, the horizontal and vertical axes measure the homological and quantum grading, respectively, and only the nonzero components of the differential are indicated. These components are easy to compute: every morphism group, except for the last one, is generated over $\mathbb{k}[H]$ by an $S_\bullet$, which D_* takes to 0 and SS_* takes to $SSS = HS$. The last morphism group is generated by $1_\bullet$ and $D_\bullet$ and the incoming differential is $S_\bullet \mapsto S_\circ S_\bullet = H1_\bullet + D_\bullet$.

Taking homology of the above bigraded complex of free $\mathbb{k}[H]$-modules yields $\widetilde{\mathrm{BN}}(T(2,n);\mathbb{k})$. In particular, when n is even, the two towers are in homological grading n and 0, in accordance with Lemma 2.1. The computation for $n \geq 0$ is analogous.

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References

[Bar-Natan 2005] D. Bar-Natan, "Khovanov's homology for tangles and cobordisms", *Geom. Topol.* **9** (2005), 1443–1499. MR

[Beliakova and Wehrli 2008] A. Beliakova and S. Wehrli, "Categorification of the colored Jones polynomial and Rasmussen invariant of links", *Canad. J. Math.* **60**:6 (2008), 1240–1266. MR

[Khovanov 2000] M. Khovanov, "A categorification of the Jones polynomial", *Duke Math. J.* **101**:3 (2000), 359–426. MR

[Kotelskiy et al. 2019] A. Kotelskiy, L. Watson, and C. Zibrowius, "Immersed curves in Khovanov homology", preprint, 2019. arXiv 1910.14584

[Lee 2005] E. S. Lee, "An endomorphism of the Khovanov invariant", *Adv. Math.* **197**:2 (2005), 554–586. MR

[Lewark and Zibrowius 2021] L. Lewark and C. Zibrowius, "Rasmussen invariants", *Mathematical Research Postcards* **1** (2021).

[Lewark and Zibrowius 2024] L. Lewark and C. Zibrowius, "Rasmussen invariants of Whitehead doubles and other satellites", *J. Reine Angew. Math.* **816** (2024), 241–296. MR

[Lipshitz and Sarkar 2014] R. Lipshitz and S. Sarkar, "A refinement of Rasmussen's S-invariant", *Duke Math. J.* **163**:5 (2014), 923–952. MR

[Mackaay et al. 2007] M. Mackaay, P. Turner, and P. Vaz, "A remark on Rasmussen's invariant of knots", *J. Knot Theory Ramifications* **16**:3 (2007), 333–344. MR

[Pardon 2012] J. Pardon, "The link concordance invariant from Lee homology", *Algebr. Geom. Topol.* **12**:2 (2012), 1081–1098. MR

[Rasmussen 2010] J. Rasmussen, "Khovanov homology and the slice genus", *Invent. Math.* **182**:2 (2010), 419–447. MR

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