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# STARTING THE STUDY OF OUTER LENGTH BILLIARDS

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**We focus on outer length billiard dynamics, acting on the exterior of a strictly convex planar domain. We first show that ellipses are totally integrable. We then provide an explicit representation of first order terms for the formal Taylor expansion of the corresponding Mather's $\beta$-function. Finally, we provide explicit Lazutkin coordinates up to order 4.**

## 1. Introduction

The aim of the present paper is starting an accurate study of outer length billiards, first described by P. Albers and S. Tabachnikov in 2024, see [2, Section 3.4]. These billiards are the counterpart of Birkhoff ones since the generating function is the outer length instead of the inner length. They are also called "fourth billiards". In fact, two billiards systems — Birkhoff and outer area billiards — have been extensively studied; we refer respectively to [22] and [21] for exhaustive surveys. Another type of billiards, namely symplectic billiards, whose generating function is the inner area, were introduced in 2018 by P. Albers and S. Tabachnikov [1] and their study started to become more intensive only recently. We refer to [4], [6] and [23] for integrability results and to [5] and [12] for area spectral rigidity results for symplectic billiards. Regarding outer length billiards, to the best of our knowledge, they were not studied yet. However, the seminal idea on the base of the definition of this dynamical system (detecting, in particular, circumscribed polygons to a strictly convex domain with minimal perimeter) can already be found in some former papers in convex planar geometry; see [11, Theorem 1] and [10, Section 2], for example.

We first give all the details to introduce this dynamical system, acting on the exterior of a strictly convex planar domain. We then prove, using elementary planar geometry, that ellipses are totally integrable, that is the phase space is fully foliated by continuous invariant curves which are not null-homotopic.

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*Keywords:* monotone twist maps, mathematical billiards, Mather beta function, geometric approximations.

We successively focus on the main topic of the paper, which is providing an explicit representation of first order terms for the formal Taylor expansion of Mather's $\beta$-function (or minimal average function) for outer length billiards. In particular, we write these coefficients (up to order 5) by means of the ordinary curvature and length of the boundary of the billiard table. As already noticed, for such a dynamical system, Mather's $\beta$-function is related to the minimal perimeter of polygons circumscribed to a strictly convex domain. These perimeters are special cases (i.e., for periodic trajectories of winding number $= 1$) of the corresponding marked length spectrum for outer length billiards.

Finally, by using the computations we made to obtain minimal average function's coefficients, we provide explicit Lazutkin coordinates up to order 4 and discuss straightforward facts regarding the existence/nonexistence of caustics for outer length billiards.

## 2. Twist maps and Mather's $\beta$-function

Let $\mathbb{T} \times (a, b)$ be the annulus, where $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1]/\sim$ identifying $0 \sim 1$ and (eventually) $a = -\infty$ and/or $b = +\infty$. Given a diffeomorphism $\Phi : \mathbb{S}^1 \times (a, b) \to \mathbb{S}^1 \times (a, b)$, we denote by

$$\phi : \mathbb{R} \times (a, b) \to \mathbb{R} \times (a, b), \quad (x_0, y_0) \mapsto (x_1, y_1)$$

a lift of $\Phi$ to the universal cover. Then $\phi$ is a diffeomorphism and $\phi(x + 1, y) = \phi(x, y) + (1, 0)$. In the case where $a$ (resp. $b$) is finite, we assume that $\phi$ extends continuously to $\mathbb{R} \times \{a\}$ (resp. $\mathbb{R} \times \{b\}$) by a rotation of fixed angle:

$$\phi(x_0, a) = (x_0 + \rho_a, a) \quad (\text{resp. } \phi(x_0, b) = (x_0 + \rho_b, b)). \tag{2-1}$$

Once fixed the lift, the numbers $\rho_a, \rho_b$ are unique. The choice of $\rho_a$ (resp. $\rho_b$) if $a = -\infty$ (resp. $b = +\infty$) depends on the dynamics at infinity. For example, in the case of outer length billiards, where $b = +\infty$, it is natural to set $\rho_b = \frac{1}{2}$. We refer to point 1 of Section 3 for details.

We recall for convenience the definition of a monotone twist map (see [18, page 2], for instance).

**Definition 1.** A monotone twist map $\phi : \mathbb{R} \times (a, b) \to \mathbb{R} \times (a, b)$, $(x_0, y_0) \mapsto (x_1, y_1)$ is a diffeomorphism satisfying

1. $\phi(x_0 + 1, y_0) = \phi(x_0, y_0) + (1, 0)$.
2. $\phi$ preserves orientations and the boundaries of $\mathbb{R} \times (a, b)$.
3. $\phi$ extends to the boundaries by rotation, as in (2-1).

4. $\phi$ satisfies a monotone twist condition, that is

$$\frac{\partial x_1}{\partial y_0} > 0. \tag{2-2}$$

5. $\phi$ is exact symplectic; this means that there exists a generating function $H \in C^2(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ for $\phi$ such that

$$y_1\, dx_1 - y_0\, dx_0 = dH(x_0, x_1). \tag{2-3}$$

Clearly, $H(x_0+1, x_1+1) = H(x_0, x_1)$ and, due to the twist condition, the domain of $H$ is the strip $\{(x_0, x_1) : \rho_a + x_0 < x_1 < x_0 + \rho_b\}$. Moreover, equality (2-3) reads

$$\begin{cases} y_1 = H_2(x_0, x_1), \\ y_0 = -H_1(x_0, x_1), \end{cases} \tag{2-4}$$

and the twist condition (2-2) becomes $H_{12} < 0$. As a consequence, $\{(x_i, y_i)\}_{i\in\mathbb{Z}}$ is an orbit of $\phi$ if and only if $H_2(x_{i-1}, x_i) = y_i = -H_1(x_i, x_{i+1})$ for all $i \in \mathbb{Z}$. Formally, this means that the corresponding bi-infinite sequence $x := \{x_i\}_{i\in\mathbb{Z}}$ is a so-called critical configuration of the action functional $\sum_{i\in\mathbb{Z}} H(x_i, x_{i+1})$. In such a setting, minimal orbits play a fundamental role. We recall that a critical configuration $x$ of $\phi$ is minimal if every finite segment of $x$ minimizes the action functional with fixed end points (we refer to [18, page 7] for details). Clearly, all these facts remain true if we consider a monotone twist map on $\{(x_0, x_1) :\ u_a(x_0) < x_1 < u_b(x_0)\}$, where $u_a, u_b : \mathbb{R} \to \mathbb{R}$ are two continuous 1-periodic functions such that $u_a < u_b$.

For a twist map $\phi$ generated by $H$, we finally introduce the rotation number and the Mather $\beta$-function (or minimal average action).

**Definition 2.** The rotation number of an orbit $\{(x_i, y_i)\}_{i\in\mathbb{Z}}$ of $\phi$ is

$$\rho := \lim_{i\to\pm\infty} \frac{x_i}{i}$$

if such a limit exists.

An important class of monotone twist maps are planar billiard maps. In this setting, the rotation number of a periodic trajectory is the rational number

$$\frac{m}{n} = \frac{\text{winding number}}{\text{number of reflections}} \in \left(0, \tfrac{1}{2}\right];$$

see [18, page 40] for details.

In view of the celebrated Aubry–Mather theory (see [3], for example), a monotone twist map possesses minimal orbits for every rotation number $\rho$ inside the so-called twist interval $(\rho_a, \rho_b)$. As a consequence, we can associate to each $\rho$ the average action of any minimal orbit having that rotation number.

**Definition 3.** The Mather $\beta$-function of $\phi$ is $\beta : (\rho_a, \rho_b) \to \mathbb{R}$ with

$$\beta(\rho) := \lim_{N\to\infty} \frac{1}{2N} \sum_{i=-N}^{N-1} H(x_i, x_{i+1})$$

where $\{x_i\}_{i\in\mathbb{Z}}$ is any minimal configuration of $\phi$ with rotation number $\rho$.

In the framework of Birkhoff billiards, A. Sorrentino in [19] gave an explicit representation of the coefficients of the (formal) Taylor expansion at zero of the corresponding Mather's $\beta$-function. More recently, J. Zhang in [24] got (locally) an explicit formula for this function via a Birkhoff normal form. Moreover, M. Bialy in [9] obtained an explicit formula for Mather's $\beta$-function for ellipses by using a nonstandard generating function, involving the support function. Regarding symplectic and outer billiards, the first two authors and A. Nardi in [7] computed explicitly the higher order terms of such an expansion, by using tools from affine differential geometry. As anticipated, one of the target of the present paper is writing explicitly these coefficients (up to order 5) in the case of forth billiards.

## 3. The dynamical system

Let $\Omega$ be a strictly convex planar domain with smooth boundary $\partial\Omega$. Assume that the perimeter of $\partial\Omega$ is $\ell = |\partial\Omega|$. Fixing the positive counterclockwise orientation, let $\gamma : \mathbb{T} \to \partial\Omega$ be the smooth arc-length parametrization of $\partial\Omega$. For every $s \in \mathbb{T}$, we denote by $s^* \in \mathbb{T}$ the (unique, by strict convexity) arc-length parameter such that $T_{\gamma(s)}\partial\Omega = T_{\gamma(s^*)}\partial\Omega$. We refer to

$$\mathcal{P} = \{(s, r) \in \mathbb{T} \times \mathbb{T} : s < r < s^*\}$$

as the (open, positive) phase space and we define the outer length billiard map as follows [2, Section 3.4].

Since $\Omega$ is strictly convex, to every point $P \in \mathbb{R}^2 \setminus \mathrm{cl}(\Omega)$ can be uniquely associated a pair $(s_0, s_1) \in \mathbb{T} \times \mathbb{T}$ with $s_0 < s_1$ and such that the lines $P\gamma(s_0)$ and $P\gamma(s_1)$ are the (negative and positive) tangents to $\partial\Omega$. Consider the circle in $\mathbb{R}^2 \setminus \Omega$ tangent to $\partial\Omega$ at $\gamma(s_1)$ and to the line $P\gamma(s_0)$. Then the image $P'$ of $P$ is defined as the intersection point between the lines $P\gamma(s_1)$ and the other common tangent line of the circle and $\partial\Omega$ (hence passing through $P'$ and $\gamma(s_2)$):

$$T : \mathcal{P} \to \mathcal{P}, \quad (s_0, s_1) \mapsto (s_1, s_2).$$

(We refer to Figure 1.) Setting $\varepsilon_0 = s_1 - s_0$ and

$$\hat{\mathcal{P}} = \{(s, \varepsilon) \in \mathbb{T} \times \mathbb{R} :\ 0 < \varepsilon < s^* - s\},$$

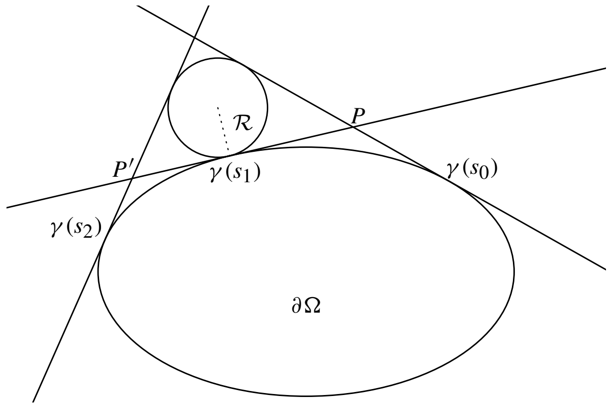


**Figure 1.** The outer-length billiard map around the domain $\Omega$ associates the point $P$ to the point $P'$.

the outer length billiard map can be equivalently defined as

$$T : \hat{\mathcal{P}} \to \hat{\mathcal{P}}, \quad (s_0, \varepsilon_0) \mapsto (s_1, \varepsilon_1).$$

Here are some properties of the outer length billiard map.

1. $T$ is continuous and can be continuously extended so that $T(s, s) = (s, s)$ and $T(s, s^*) = (s^*, s)$.

2. The function

$$H : \mathcal{P} \to \mathbb{R}, \quad H(s_0, s_1) := |P\gamma(s_0)| + |P\gamma(s_1)|,$$

generates $T$, that is

$$T(s_0, s_1) = (s_1, s_2) \iff H_2(s_0, s_1) + H_1(s_1, s_2) = 0. \tag{3-1}$$

See [2, Lemma 3.1] for the proof. In view of (3-1), we can equivalently refer to

$$\bar{H} : \mathcal{P} \to \mathbb{R}, \quad \bar{H}(s_0, s_1) := |P\gamma(s_0)| + |P\gamma(s_1)| - s_1 + s_0,$$

as a generating function, which is exactly the Lazutkin parameter of $\partial\Omega$, interpreted as convex caustic for a Birkhoff billiard.

3. $T$ is a twist map preserving the area form $-H_{12}(s_0, s_1)\, ds_0 \wedge ds_1$.

4. By introducing new variables

$$y_0 = -H_1(s_0, s_1), \quad y_1 = H_2(s_0, s_1),$$

$(s, y)$ are coordinates on $\mathcal{P}$ and the outer length billiard map results a (negative) twist map, since

$$\frac{\partial y_1}{\partial s_0} = H_{12}(s_0, s_1) = -\frac{k(s_0)k(s_1)H(s_0, s_1)}{2\sin^2(\varphi/2)} < 0,$$

where $\varphi$ is the angle between the tangent lines $P\gamma(s_0)$ and $P\gamma(s_1)$ (see also [2, page 11]). In these coordinates, the preserved area form is the standard one: $ds \wedge dy$.

5. The marked length spectrum for the outer length billiard is the map $\mathcal{ML}_o(\Omega) : \mathbb{Q} \cap \left(0, \frac{1}{2}\right) \to \mathbb{R}$ that associates to any $m/n$ in lowest terms the minimal perimeter of the periodic trajectories having rotation number $m/n$. We refer to [18, Sections 3.1 and 3.2] for a general treatment of the marked spectrum. Clearly, periodic outer length billiard minimal trajectories (with winding number $= 1$) correspond to convex polygons realizing the minimal (circumscribed) perimeter, so that

$$\beta\left(\tfrac{1}{n}\right) = \tfrac{1}{n}\mathcal{ML}_o(\Omega)\left(\tfrac{1}{n}\right). \tag{3-2}$$

**3.1.** ***Circles and ellipses.*** As expected, the outer length billiard on the circle (of center $O$) is totally integrable: the phase space is completely foliated by concentric invariant circles. By using as coordinates $(\alpha_0, \alpha_1) \in \mathbb{T} \times \mathbb{T}$, where $\alpha_0$ and $\alpha_1$ are respectively the angles of $O\gamma(s_0)$ and $O\gamma(s_1)$ with respect to the positive horizontal direction, the generating function in the case of disk of unit radius is

$$H(\alpha_0, \alpha_1) = 2\tan\frac{\alpha_1 - \alpha_0}{2}.$$

Equivalently, in terms of $(\alpha_0, y_0) = (\alpha_0, -H_1(\alpha_0, \alpha_1)) = \left(\alpha_0, 1 + \tan^2\frac{\alpha_1 - \alpha_0}{2}\right)$, we have

$$H(\alpha_0, y_0) = 2\sqrt{y_0 - 1}$$

and total integrability follows.

An unexpected fact — at least from the authors' point of view, since the billiard dynamics is not invariant by affine transformations — is that also the outer length billiard on the ellipse is totally integrable, as stated in the next proposition.

**Proposition 4.** *Let $\mathcal{E}$ and $\Gamma$ be two confocal nested ellipses, $\mathcal{E} \subset \Gamma$. Then $\Gamma$ is a caustic for the outer-length billiard dynamics outside $\mathcal{E}$.*

The proof of Proposition 4 relies on a lemma from elementary plane geometry:

**Lemma 5** [20, Lemma 2.4]. *Let $P_0, P_1 \in \Gamma$ two distinct points such that the line $P_0P_1$ is tangent to $\mathcal{E}$ at a point $Q$. Let $R$ be the intersection point of the tangent lines to $\Gamma$ at $P_0$ and $P_1$. Then the lines $P_0P_1$ and $RQ$ are orthogonal.* (See Figure 2.)

*Proof of Proposition 4.* Let a point $P_0$ on $\Gamma$. Consider the positive tangent line to $\mathcal{E}$ at a point $Q$ and passing through $P_0$. Let $P_1 \in \Gamma$ be the intersection point of the latter tangent line $P_0Q$ with $\Gamma$, see Figure 3. We need to show that $P_1$ is the image of $P_0$ under the outer-length billiard reflection outside $\mathcal{E}$. Consider the point $P$, such that $PP_0$ and $PP_1$ are the two tangent lines to $\mathcal{E}$ passing through $P$, see Figure 3. Since $\mathcal{E}$ and $\Gamma$ are confocal, $\mathcal{E}$ is a caustic for the classical billiard in $\Gamma$. In particular, the tangent line $T_{P_0}\Gamma$ is a bisector of the angle $\widehat{P_1P_0P}$. With

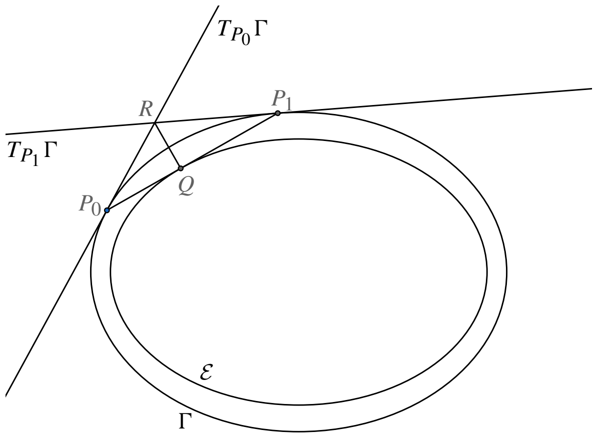


**Figure 2.** The line $RQ$ is orthogonal to the line $P_0P_1$.

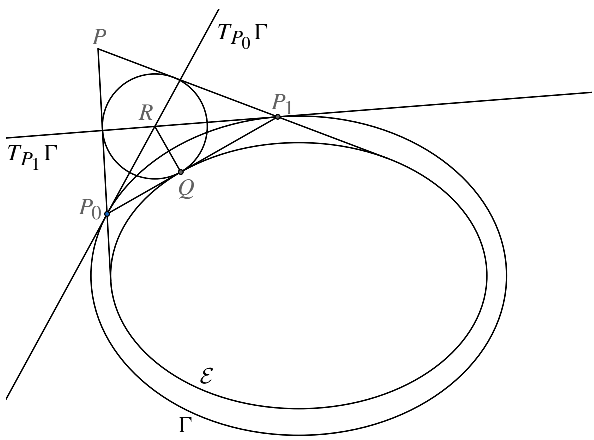


**Figure 3.** The point $P_0 \in \Gamma$ is reflected to the point $P_1 \in \Gamma$ by the outer-length billiard dynamics around $\mathcal{E}$.

the same argument the tangent line $T_{P_1}\Gamma$ is a bisector of the angle $\widehat{P_0P_1P}$. Hence $T_{P_0}\Gamma$ and $T_{P_1}\Gamma$ intersects at a point $R$ which is the center of the inscribed circle $\mathcal{D}$ to the triangle $P_0PP_1$. By Lemma 5, the lines $RQ$ and $P_0P_1$ are orthogonal. In particular $\mathcal{D}$ is tangent to the ellipse $\mathcal{E}$. This implies that $P_1$ is obtained from $P_0$ by the outer-length billiard law of reflection. $\square$

It would be interesting to investigate if these are the only cases. This fundamental problem (possibly to be studied by an integral inequality à la Bialy [8]) may present nontrivial difficulties, due to the infinite total area of the phase space.

## 4. Asymptotic expansions

S. Marvizi and R. Melrose's theory, first stated and proved for Birkhoff billiards [15, Theorem 3.2], can be applied to the general case of (strongly) billiard-like maps, see [13, Section 2.1]. As an outcome, the following expansion at $\rho = 0$ of the corresponding minimal average function holds:

$$\beta(\rho) \sim \beta_1 \rho + \beta_3 \rho^3 + \beta_5 \rho^5 + \cdots$$

in terms of odd powers of $\rho$. It is well-known (see, e.g., [15, Section 7] again) that for usual billiards the sequence $\{\beta_k\}$ can be interpreted as a spectrum of a differential operator, see also Remark 2.11 in [1]. The question is wide open for other types of billiards, included outer length billiards.

In this section, we gather all the technical results in order to prove the next theorem, providing the coefficient $\beta_5$ for the outer length billiard map. This result is a refinement of [17, Theorem 1(iii)]. In fact, in a genuine framework of convex planar geometry, D.E. Vitale and R.A. McClure computed $\beta_3$ by using as coordinate the support function and as parameter the angle with respect to a fixed direction.

**Theorem 6.** *Let $\Omega$ be a strictly convex planar domain with smooth boundary $\partial\Omega$. Suppose that $\partial\Omega$ has everywhere positive curvature. Denote by $k(s)$ the (ordinary) curvature of $\partial\Omega$ with arc-length parameter $s$. Let $\ell$ be the length of the boundary and*

$$L := \int_0^\ell k^{2/3}(s)\, ds.$$

*The formal Taylor expansion at $\rho = 0$ of Mather's $\beta$-function for the outer length billiard map has coefficients*

$$\beta_{2k} = 0 \text{ for all } k, \quad \beta_1 = \ell, \quad \beta_3 = \frac{L^3}{12},$$
$$\beta_5 = L^4 \int_0^\ell \left( \frac{k^{4/3}(s)}{120} + \frac{k^{-\frac{8}{3}}(s) k'^2(s)}{2160} \right) ds.$$

As expected, a straightforward consequence of the previous result is that, as for other billiards, also for outer length ones, the two coefficients $\beta_1$ and $\beta_3$ allow one to recognize a circle among all strictly convex planar domains.

**Corollary 7.** *The coefficients $\beta_1$ and $\beta_3$ recognize a circle. In particular,*

$$3\beta_3 + \pi^2 \beta_1 \le 0$$

*with equality if and only if $\partial\Omega$ is a circle.*

*Proof.* We apply Hölder's inequality with $p = 3/2$ and $q = 3$ to obtain

$$L = \int_0^\ell k^{2/3}(s)\,ds \le \Big(\int_0^\ell (k^{2/3}(s))^{3/2} ds\Big)^{2/3} \Big(\int_0^\ell 1^3 ds\Big)^{1/3} = (2\pi)^{2/3}\ell^{1/3}, \tag{4-1}$$

since $\int_0^\ell k(s)\,ds = 2\pi$. Using the expressions of $\beta_1$ and $\beta_3$ found in Theorem 6, we can write

$$3\beta_3 + \pi^2\beta_1 = \tfrac{1}{4}L^3 - \pi^2\ell \le \tfrac{1}{4}(2\pi)^2\ell - \pi^2\ell = 0.$$

In the case of equality, namely if $3\beta_3 + \pi^2\beta_1 = 0$, then $L = (2\pi)^{2/3}\ell^{1/3}$, and the case of equality is reached in (4-1). In that case, $k$ is constant. Hence $\Omega$ is a disk. □

**Remark 8.** Let $\mathcal{P}_n^c$ be the set of all convex polygons with at most $n$ vertices which are circumscribed to $\Omega$. We define

$$\delta(\Omega; \mathcal{P}_n^c) := \inf\{\ell(P_n) : P_n \in \mathcal{P}_n^c\},$$

where $\ell(P_n)$ is the perimeter length of $P_n$. Clearly, essentially in view of equality (3-2), Theorem 6 gives also the formal expansion of $\delta(\Omega; \mathcal{P}_n^c)$ at $n \to +\infty$.

Since we use the arc-length parametrization of $\partial\Omega$, it is useful to recall that

$$\begin{cases} \gamma'' = kJ\gamma', \quad \gamma''' = -k^2\gamma' + k'J\gamma', \\ \gamma^{(4)} = -3kk'\gamma' + (-k^3 + k'')J\gamma', \\ \gamma^{(5)} = (k^4 - 4kk'' - 3k'^2)\gamma' + (-6k^2k' + k''')J\gamma', \\ \gamma^{(6)} = (10k^3k' - 10k'k'' - 5kk''')\gamma' + (k^5 - 10k^2k'' - 15kk'^2 + k^{(4)})J\gamma', \end{cases} \tag{4-2}$$

where $J$ is the counterclockwise rotation of angle $\pi/2$.

**Proposition 9.** *For $0 \le r \le s \le \ell$, let $\delta := s - r$. Then*

$$H(r,s) = \delta + \frac{k^2(r)}{12}\delta^3 + \frac{k(r)k'(r)}{12}\delta^4 + \frac{2k^4(r) + 4k'^2(r) + 7k(r)k''(r)}{240}\delta^5 + O(\delta^6), \tag{4-3}$$

*uniformly as $\delta \to 0$.*

*Proof.* We start by writing separately the Taylor expansions of numerator and denominator of the generating function

$$H(r,s) = \frac{(\gamma(s) - \gamma(r)) \wedge (\gamma'(s) - \gamma'(r))}{\gamma'(r) \wedge \gamma'(s)}. \tag{4-4}$$

From now on, we omit the dependence on $r$ of $\gamma$, $k$ and their derivatives. We have

$$\gamma(s) - \gamma(r) = \gamma'\delta + \frac{\gamma''}{2}\delta^2 + \frac{\gamma'''}{6}\delta^3 + \frac{\gamma^{(4)}}{24}\delta^4 + \frac{\gamma^{(5)}}{5!}\delta^5 + O(\delta^6)$$

and likewise for $\gamma'(s) - \gamma'(r)$; thus the Taylor expansion of the numerator of (4-4) is

$$k\delta^2+\frac{k'}{2}\delta^3+\frac{1}{6}\frac{2k''-k^3}{2}\delta^4+\frac{k'''-3k^2k'}{24}\delta^5+\frac{2k^5-48kk'^2-29k^2k''+6k^{(4)}}{720}\delta^6+O(\delta^7),$$

where we have used (4-2). Similarly, the Taylor expansion of the denominator is

$$\begin{aligned}\gamma'(r)\wedge\gamma'(s)\\ &=\gamma'\wedge\Big(\gamma'+\gamma''\delta+\tfrac{\gamma'''}{2}\delta^2+\tfrac{\gamma^{(4)}}{6}\delta^3+\tfrac{\gamma^{(5)}}{24}\delta^4+\tfrac{\gamma^{(6)}}{5!}\delta^5+O(\delta^6)\Big)\\ &=k\delta+\tfrac{k'}{2}\delta^2+\tfrac{-k^3+k''}{6}\delta^3+\tfrac{-6k^2k'+k'''}{24}\delta^4+\tfrac{k^5-10k^2k''-15kk'^2+k^{(4)}}{5!}\delta^5+O(\delta^6)\\ &=k\delta\Big(1-\tfrac{k'}{2k}\delta+\tfrac{2k^4+3k'^2-2kk''}{12k^2}\delta^2-\tfrac{3k'^3-2k'(k^4+2kk'')+k^2k'''}{24k^3}\delta^3+D\delta^4+O(\delta^5)\Big)^{-1},\end{aligned}$$

where

$$D=\frac{45k'^4-90kk'^2k''+30k^2k'k'''+2k^2(7k^6+10k^3k''+10k''^2-3kk^{(4)})}{720k^4}.$$

Using the above expansions for numerator and denominator, we obtain (4-3). □

**Proposition 10.** *The outer length billiard map $T:(s_0,\varepsilon_0)\mapsto(s_1,\varepsilon_1)$ has the expansion*

$$\begin{cases}s_1=s_0+\varepsilon_0,\\ \varepsilon_1=\varepsilon_0+A(s_0)\varepsilon_0^2+B(s_0)\varepsilon_0^3+C(s_0)\varepsilon_0^4+O(\varepsilon_0^5),\end{cases}\tag{4-5}$$

*where*

$$\begin{aligned}A(s)&=-\frac{2k'(s)}{3k(s)},\qquad B(s)=\frac{10k'^2(s)}{9k^2(s)}-\frac{2k''(s)}{3k(s)},\\ C(s)&=\frac{-24k^4(s)k'(s)-1160k'^3(s)+1200k(s)k'(s)k''(s)-216k^2(s)k'''(s)}{540k^3(s)}.\end{aligned}\tag{4-6}$$

*Proof.* We start by writing separately the Taylor expansions of numerator and denominator of the radius $\mathcal{R}$ of the circle in $\mathbb{R}^2\setminus\Omega$ tangent to $\partial\Omega$ at $\gamma(s_1)$ and to the line $P\gamma(s_0)$; see Figure 1.

$$\mathcal{R}=\frac{(\gamma(s_1)-\gamma(s_0))\wedge\gamma'(s_1)}{1+\gamma'(s_1)\cdot\gamma'(s_0)}=\frac{(\gamma(s_2)-\gamma(s_1))\wedge\gamma'(s_2)}{1+\gamma'(s_2)\cdot\gamma'(s_1)}.$$

From now on, we indicate, by subscripting 1, the dependence on $s_1$ of $\gamma$, $k$ and their derivatives. Recall that $\varepsilon_1=s_2-s_1$. The Taylor expansion of the numerator is

$$\begin{aligned}&\Big(\gamma_1'\varepsilon_1+\tfrac{\gamma_1''}{2}\varepsilon_1^2+\tfrac{\gamma_1'''}{6}\varepsilon_1^3+\tfrac{\gamma_1^{(4)}}{24}\varepsilon_1^4+\tfrac{\gamma_1^{(5)}}{5!}\varepsilon_1^5+O(\varepsilon_1^6)\Big)\\ &\qquad\wedge\Big(\gamma_1'+\gamma_1''\varepsilon_1+\tfrac{\gamma_1'''}{2}\varepsilon_1^2+\tfrac{\gamma_1^{(4)}}{6}\varepsilon_1^3+\tfrac{\gamma_1^{(5)}}{24}\varepsilon_1^4+O(\varepsilon_1^5)\Big)\\ &\qquad\qquad=\tfrac{k_1}{2}\varepsilon_1^2+\tfrac{k_1'}{3}\varepsilon_1^3+\Big(\tfrac{-k_1^3+3k_1''}{24}\Big)\varepsilon_1^4+\Big(\tfrac{-9k_1^2k_1'+4k_1'''}{120}\Big)\varepsilon_1^5+O(\varepsilon_1^6),\end{aligned}$$

where we have used (4-2).

Similarly, the Taylor expansion of the denominator is

$$1+\Big(\gamma_1'+\gamma_1''\varepsilon_1+\tfrac{\gamma_1'''}{2}\varepsilon_1^2+\tfrac{\gamma_1^{(4)}}{6}\varepsilon_1^3+\tfrac{\gamma_1^{(5)}}{24}\varepsilon_1^4+O(\varepsilon_1^5)\Big)\cdot\gamma_1'$$
$$=2\Big(1-\tfrac{k_1^2}{4}\varepsilon_1^2-\tfrac{k_1k_1'}{4}\varepsilon_1^3+O(\varepsilon_1^4)\Big)=2\Big(1+\tfrac{k_1^2}{4}\varepsilon_1^2+\tfrac{k_1k_1'}{4}\varepsilon_1^3+O(\varepsilon_1^4)\Big)^{-1}.$$

Using the above expansions for numerator and denominator, we obtain

$$2\mathcal{R}=\frac{k_1}{2}\varepsilon_1^2+\frac{k_1'}{3}\varepsilon_1^3+\frac{2k_1^3+3k_1''}{24}\varepsilon_1^4+\frac{16k_1^2k_1'+4k_1'''}{120}\varepsilon_1^5+O(\varepsilon_1^6) \tag{4-7}$$

or, equivalently,

$$2\mathcal{R}=\frac{k_1}{2}\varepsilon_0^2-\frac{k_1'}{3}\varepsilon_0^3+A_4\varepsilon_0^4+A_5\varepsilon_0^5+O(\varepsilon_0^6), \tag{4-8}$$

with

$$A_4=\frac{2k_1^3+3k_1''}{24},\qquad A_5:=\frac{16k_1^2k_1'+4k_1'''}{120}$$

Substituting the powers of the expansion

$$\varepsilon_1=\varepsilon_0+\alpha(s_1)\varepsilon_0^2+\beta(s_1)\varepsilon_0^3+\gamma(s_1)\varepsilon_0^4+O(\varepsilon_0^5)$$

in (4-7), we obtain (omitting the dependence on $s_0$ in $\alpha$, $\beta$ and $\gamma$)

$$2\mathcal{R}=\frac{k_1}{2}\varepsilon_0^2+\left(k_1\alpha+\frac{k_1'}{3}\right)\varepsilon_0^3+\left(\frac{k_1}{2}(\alpha^2+2\beta)+k_1'\alpha+A_4\right)\varepsilon_0^4$$
$$+\big(k_1(\alpha\beta+\gamma)+k_1'(\alpha^2+\beta)+4\alpha A_4+A_5\big)\,\varepsilon_0^5+O(\varepsilon_0^6).$$

Equaling this to (4-8), we obtain

$$\alpha(s)=-\frac{2k'(s)}{3k(s)},\qquad \beta(s)=\frac{4k'^2(s)}{9k^2(s)},$$
$$\gamma(s)=\frac{-320k'^3(s)+3k'(s)(-8k^4(s)+60k(s)k''(s))-36k^2(s)k'''(s)}{540k^3(s)}.$$

Finally, from

$$\varepsilon_1=\varepsilon_0+\alpha(s_0)\varepsilon_0^2+(\alpha'(s_0)+\beta(s_0))\varepsilon_0^3+\left(\frac{\alpha''(s_0)}{2}+\beta'(s_0)+\gamma(s_0)\right)\varepsilon_0^4+O(\varepsilon_0^5),$$

we obtain the formulas (4-6). □

**Proposition 11.** *Let $q \geq 3$. The $q$-periodic orbits of rotation number $1/q$ for the outer length billiard map have the expansion*

$$\begin{cases} s_k = s_0^q + a_0(k/q) + \dfrac{a_1(k/q)}{q} + \dfrac{a_2(k/q)}{q^2} + O\left(\dfrac{1}{q^3}\right) \\ \varepsilon_k = \dfrac{b_1(k/q)}{q} + \dfrac{b_2(k/q)}{q^2} + \dfrac{b_3(k/q)}{q^3} + O\left(\dfrac{1}{q^4}\right) \end{cases} \tag{4-9}$$

*where $s_0^q \in \mathbb{R}$ converges to $0$ with $q$, $a_0 : \mathbb{R} \to \mathbb{R}$ is a map such that $a_0(x+1) = a_0(x) + \ell$ for any $x$ and $a_1, a_2, b_1, b_2, b_3 : \mathbb{R} \to \mathbb{R}$ are $1$-periodic maps which can be expressed as*

$$\begin{cases} a_0^{-1}(s) = \dfrac{1}{L}\displaystyle\int_0^s k^{2/3}(r)\,dr := x(s), \qquad L := \int_0^\ell k^{2/3}(r)\,dr, \\ a_1(x) = 0, \\ a_2(x) = k^{-\frac{2}{3}}(a_0(x)) \\ \qquad \times \left(\displaystyle\int_0^x L^3\Big(\frac{1}{810}\big(9k''k^{-\frac{7}{3}} - 12(k')^2k^{-\frac{10}{3}}\big) + \frac{k^{\frac{2}{3}}}{15}\Big)(a_0(t))\,dt + cx\right), \\ b_1(x) = a_0'(x) = Lk^{-2/3}(a_0(x)), \\ b_2(x) = \dfrac{a_0''(x)}{2} = -\dfrac{L^2k'(a_0(x))k^{-7/3}(a_0(x))}{3}, \\ b_3(x) = a_2' + \dfrac{a_0'''}{6}. \end{cases} \tag{4-10}$$

*The constant $c$ in the expression of $a_2$ is such that*

$$L^3\Big(\frac{1}{810}\big(9k''k^{-\frac{7}{3}} - 12(k')^2k^{-\frac{10}{3}}\big) + \frac{k^{\frac{2}{3}}}{15}\Big) + c$$

*has zero mean.*

*Proof.* Since the points in the orbits are equidistributed as $q \to +\infty$, for any $q$ we can choose the first point of the orbit $s_0^q$ such as $s_0^q \to 0$ for $q \to +\infty$. For simplicity, we omit the dependence of $a_i$ and $b_j$ on $k/q$.

Combining the expansions in (4-5), we have

$$\begin{aligned} \varepsilon_{k+1} - \varepsilon_k &= A(s_k)\varepsilon_k^2 + B(s_k)\varepsilon_k^3 + C(s_k)\varepsilon_k^4 + O(\varepsilon_k^5) \\ &= \frac{A(s_0^q + a_0)b_1^2}{q^2} + \frac{B(s_0^q + a_0)b_1^3 + A'(s_0^q + a_0)a_1b_1^2 + 2A(s_0^q + a_0)b_1b_2}{q^3} \\ &\qquad + \frac{F(a_i, b_j)}{q^4} + O\Big(\frac{1}{q^5}\Big), \end{aligned}$$

where

$$F(a_i, b_j) := A(s_0^q+a_0)b_2^2+2A(s_0^q+a_0)b_1b_3+2A'(s_0^q+a_0)a_1b_1b_2+A'(s_0^q+a_0)a_2b_1^2 + A''(s_0^q+a_0)a_1^2b_1^2/2+3B(s_0^q+a_0)b_1^2b_2+B'(s_0^q+a_0)a_1b_1^3+C(s_0^q+a_0)b_1^4.$$

Moreover, directly from the second expansion in (4-9), we have

$$\varepsilon_{k+1}-\varepsilon_k = \frac{b_1'}{q^2}+\frac{b_2'+b_1''/2}{q^4}+\frac{b_3'+b_2''/2+b_1'''/6}{q^5}+O\Big(\frac{1}{q^5}\Big).$$

Equaling these two expansions, we obtain that $a_i$ and $b_j$ solve

$$\text{(4-11)}\qquad \begin{cases} A(s_0^q+a_0)b_1^2 = b_1', \\ B(s_0^q+a_0)b_1^3+A'(s_0^q+a_0)a_1b_1^2+2A(s_0^q+a_0)b_1b_2 = b_2'+b_1''/2, \\ F(a_i,b_j) = b_3'+b_2''/2+b_1'''/6. \end{cases}$$

On the other hand, directly from the first expansion in (4-5), we conclude that

$$s_{k+1}-s_k = \frac{a_0'}{q}+\frac{a_1'+a_0''/2}{q^2}+\frac{a_2'+a_1''/2+a_0'''/6}{q^3}+O\left(\frac{1}{q^4}\right),$$

which, compared which the second expansion in (4-5), gives the system

$$\text{(4-12)}\qquad a_0' = b_1,\quad a_1'+a_0''/2 = b_2,\quad a_2'+a_1''/2+a_0'''/6 = b_3.$$

***Expressions of $a_0$ and $b_1$.*** To compute $a_0$ and $b_1$, we solve the system

$$\text{(4-13)}\qquad b_1 = a_0',\quad b_1' = A(s_0^q+a_0)b_1^2.$$

Replacing $b_1$ by $a_0'$ in the second equation, we get

$$\text{(4-14)}\qquad a_0'' = (a_0')^2A(s_0^q+a_0).$$

If we let $A_1(s) = -\frac{2}{3}\log k(s)$ be a primitive of $A$, it follows from (4-14) that

$$\big(a_0'e^{-A_1(s_0^q+a_0)}\big)' = 0.$$

Hence $a_0'e^{-A_1(s_0^q+a_0)}$ is constant. Consider now $A_2(s) = \int_0^s k^{2/3}(r)\,dr$, which is a primitive of $\exp(-A_1)$. We just proved that $A_2(s_0^q+a_0)$ has constant derivative, hence it must be of the form $A_2(s_0^q+a_0(x)) = ux+v$ for any $x\in\mathbb{R}$, where $u, v\in\mathbb{R}$. Since, by definition, $A_2(s_0^q+a_0(0)) = A_2(s_0^q) = v$, we have $v = A_2(s_0^q)$. The expression of $u$ is

$$u = A_2(s_0^q+a_0(1))-A_2(s_0^q) = A_2(s_0^q+\ell)-A_2(s_0^q) = \int_0^\ell k^{2/3}(r)\,dr.$$

Finally, $b_1$ follows from $b_1 = a_0'$.

***Expressions of $a_1$ and $b_2$.*** To compute $a_1$ and $b_2$, we solve the system

$$\begin{cases} b_2 = a_1' + a_0''/2, \\ b_2' + b_1''/2 = B(s_0^q + a_0)b_1^3 + A'(s_0^q + a_0)a_1 b_1^2 + 2A(s_0^q + a_0)b_1 b_2. \end{cases} \tag{4-15}$$

The terms containing $a_1$ nor $b_2$ can be computed using the expression of $a_0$ and $b_1$ we just obtained. Let us replace in the second equation of (4-15) $b_2$ by the expression given by the first equation: we obtain an equation for which we split the terms containing $a_1$ from the others. Namely,

$$\begin{aligned} a_1'' - 2A(s_0^q + a_0)b_1 a_1' - A'(s_0^q + a_0)a_1 b_1^2 \\ = A(s_0^q + a_0)b_1 a_0'' + B(s_0^q + a_0)b_1^3 - \tfrac{1}{2}b_1'' - \tfrac{1}{2}a_0^{(3)}. \end{aligned} \tag{4-16}$$

Replacing $a_0$ and $b_1$ by the expressions we just found, the left-hand side of (4-16) can be expressed as

$$a_1'' + \tfrac{4}{3}Lk^{-5/3}k'a_1' + \tfrac{2}{3}L^2\big(k^{-7/3}k'' - k^{-10/3}k'^2\big)a_1 = k^{-2/3}(a_1 k^{2/3})'',$$

where it is understood that $k$ and its derivatives are evaluated at $s_0^q + a_0$. The right-hand side of (4-16) vanishes. Hence (4-16) is equivalent to

$$k^{-2/3}(a_1 k^{2/3})'' = 0.$$

Since $a_1$ is periodic and vanishes at 0, we necessarily have $a_1 = 0$. The expression of $b_2$ comes from the first equation of (4-15), namely $b_2 = a_0''/2$.

***Expressions of $a_2$ and $b_3$.*** Although not used later, we derive an explicit expression for the coefficient $a_2$. By making use of (4-11) and (4-12), and taking into account that $a_1 = 0$, we obtain the system

$$\begin{cases} b_3 = a_2' + a_0'''/6, \\ A'(s_0^q + a_0)b_1^2 a_2 + A(s_0^q + a_0)(b_2^2 + 2b_1 b_3) + B(s_0^q + a_0)3b_1^2 b_2 + C(s_0^q + a_0)b_1^4 \\ \qquad = b_1'''/6 + b_2''/2 + b_3' \end{cases} \tag{4-17}$$

From the first equation of (4-17) we have $b_3' = a_2'' + a_0^{(4)}/6$, which in turn gives

$$b_3' = a_2'' + \frac{11k'k''k^{-\frac{10}{3}}}{27} - \frac{8(k')^3 k^{\frac{13}{3}}}{27} - \frac{k'''k^{-\frac{7}{3}}}{9}.$$

Replacing into the second of (4-17) and grouping all the terms with $a_2$, we get

$$(k^{\frac{2}{3}}a_2)'' = L^4\left(\frac{40(k')^3 - 45kk'k'' + 9k^2k''}{810k^5} + \frac{2k'}{45k}\right). \tag{4-18}$$

The right-hand side is the derivative of

$$L^3\left(\frac{1}{810}\left(9k''k^{-\frac{7}{3}} - 12(k')^2k^{-\frac{10}{3}}\right) + \frac{k^{\frac{2}{3}}}{15}\right) + c,$$

where $c$ is a constant such that this function has zero mean. At this point we can integrate again and get for $a_2(x)$ the value

$$k^{-\frac{2}{3}}(a_0(x))\left(\int_0^x L^3\left(\frac{1}{810}\left(9k''k^{-\frac{7}{3}} - 12(k')^2k^{-\frac{10}{3}}\right) + \frac{k^{\frac{2}{3}}}{15}\right)(a_0(t))\,dt + cx\right).$$

The value of $b_3$ can now be easily derived from the first one of (4-17). □

## 5. Proof of Theorem 6

This section is entirely devoted to the proof of Theorem 6, providing the coefficient $\beta_5$ for the outer length billiard map.

*Proof.* We start the computation of the beta function by writing its value at rational points of the form $\frac{1}{q}$, which (by the expansion (4-3) of the generating function $H$) is

$$\begin{aligned}\beta\left(\frac{1}{q}\right) &= \frac{1}{q}\sum_{n=0}^{q-1} H\left(s_n, s_{n+1}\right) \\ &= \frac{1}{q}\sum_{n=0}^{q-1} \varepsilon_n + \frac{k^2}{12}\varepsilon_n^3 + \frac{kk'}{12}\varepsilon_n^4 + \frac{2k^4 + 4k'^2 + 7kk''}{240}\varepsilon_n^5 + O(\varepsilon_n^6).\end{aligned} \tag{5-1}$$

Here, the curvature $k$ and its derivatives $k'$ and $k''$ are to be understood as evaluated in $s_n$.

Now, we substitute in the above formula $s_n$ and $\varepsilon_n$ with their corresponding Taylor expansions obtained in Proposition 11. We then proceed to group the various terms according to their order of magnitude $q^k$.

First, we observe that the summation of $\varepsilon_n$ is simply equal to the perimeter $\ell$ of $D$, so that $\beta_1 = \ell$.

By inspecting the formula even before performing the substitution, we see that there are no terms of order $q^{-2}$, so that $\beta_2 = 0$, as expected by Marvizi–Melrose theory.

The second term of the summation on the right-hand side of (5-1) becomes, after the substitution and after grouping the various powers of $q$,

$$\text{(5-2)}\quad \frac{1}{12}\sum_{n=0}^{q-1} k^2(s_n)\,\varepsilon_n^3 = \frac{1}{12}\sum_{n=0}^{q-1} k^2\left(a_0+\frac{a_2}{q^2}+O\left(\frac{1}{q^3}\right)\right)\left(\frac{b_1}{q}+\frac{b_2}{q^2}+\frac{b_3}{q^3}\right)^3 = \sum_{n=0}^{q-1}\frac{k^2b_1^3}{12}\frac{1}{q^3}+\frac{k^2b_1^2b_2}{4}\frac{1}{q^4}+\frac{2kk'b_1^3a_2+3k^2(b_1^2b_3+b_1b_2^2)}{12}\frac{1}{q^5}+O\Big(\frac{1}{q^6}\Big).$$

Similarly, we have

$$\text{(5-3)}\quad \frac{1}{12}\sum_{n=0}^{q-1} k(s_n)\,k'(s_n)\,\varepsilon_n^4 = \frac{1}{12}\sum_{n=0}^{q-1} kk'\left(\frac{b_1}{q}+\frac{b_2}{q^2}+O\Big(\frac{1}{q^3}\Big)\right)^4 = \sum_{n=0}^{q-1}\frac{kk'b_1^4}{12}\frac{1}{q^4}+\frac{kk'b_1^3b_2}{3}\frac{1}{q^5}+O\Big(\frac{1}{q^6}\Big).$$

Finally, the last term is

$$\text{(5-4)}\quad \sum_{n=0}^{q-1}\left(\frac{2k^4+4k'^2+7kk''}{240}\right)b_1^5\frac{1}{q^5}+O\Big(\frac{1}{q^6}\Big).$$

We recall that, in the last three formulas, it is implicitly understood that all functions $a_i$, $b_i$ are evaluated at $n/q$, and the curvature $k$ and its derivatives $k'$ and $k''$, where not explicitly specified, are computed at $s_0^q+a_0(n/q)$. To determine $\beta_3$, we compute $\lim_{q\to+\infty} q^3\big(\beta\big(\frac{1}{q}\big)-\frac{\ell}{q}\big)$.

From (5-2), (5-3), and (5-4), we obtain

$$\beta_3=\lim_{q\to+\infty}\frac{1}{12q}\sum_{n=0}^{q-1}\left(k^2b_1^3+O\Big(\frac{1}{q}\Big)\right)$$

By Proposition 11, we have $b_1=Lk^{-\frac{2}{3}}\Rightarrow k^2b_1^3=L^3$, so that

$$\beta_3=\frac{1}{12}L^3=\frac{1}{12}\left(\int_0^\ell k^{2/3}(r)dr\right)^3.$$

The leading part of this limit is constant, while the term denoted by $O(1/q)$ contains only higher-order terms. We will take this into account when analyzing $\beta-\ell/q-\beta_3/q^3$, considering only the terms present in $O(1/q)$.

For the terms of order 4, we obtain the expression

$$\sum_{n=0}^{q-1}\left(\frac{k^2b_1^2b_2}{4}+\frac{kk'b_1^4}{12}\right)\frac{1}{q^4}.$$

Since, by Proposition 11, we have $b_1=Lk^{-\frac{2}{3}}$ and $b_2=-\frac{1}{3}L^2k'k^{-\frac{7}{3}}$, we immediately conclude (again, as expected by Marvizi–Melrose theory), that $\beta_4=0$.

The terms of order 5 are

$$\sum_{n=0}^{q-1}\left[\frac{2k^4+4k'^2+7kk''}{240}b_1^5+\frac{1}{12}\left(2kk'b_1^3a_2+3k^2\left(b_3b_1^2+b_2^2b_1\right)+4kk'b_1^3b_2\right)\right]\frac{1}{q^5}$$
$$=S_1+S_2,$$

where

$$S_1:=\sum_{n=0}^{q-1}\frac{1}{12}\left(2kk'b_1^3a_2+3k^2b_1^2a_2'\right),$$
$$S_2:=\sum_{n=0}^{q-1}\left(\frac{2k^4+4k'^2+7kk''}{240}b_1^5+\frac{1}{12}\left(3k^2b_2^2b_1+4kk'b_1^3b_2+\frac{1}{2}k^2a_0'''b_1^2\right)\right);$$

where we substituted the value $b_3=a_2'+a_0'''/6$ from Proposition 11. We remark that the sum $S_1$ contains $a_2$ and the sum $S_2$ doesn't contain $a_2$. As established earlier, we have

$$\beta_5=\lim_{q\to+\infty}q^5\left(\beta\left(\frac{1}{q}\right)-\frac{\ell}{q}-\frac{\beta_3}{q^3}\right)=\lim_{q\to+\infty}\frac{1}{q}(S_1+S_2).$$

By studying $\lim_{q\to+\infty}\frac{1}{q}S_1$, we obtain

$$\text{(5-5)}\quad \frac{1}{12}\lim_{q\to+\infty}\frac{1}{q}\sum_{n=0}^{q-1}\left(2kk'k^{-2}a_2L^3+3k^2k^{-\frac{4}{3}}a_2'L^2\right)$$
$$=\frac{1}{12}\int_0^1\left(\frac{2k'}{k}\left(a_0(x)\right)a_2(x)L^3+3k^{\frac{2}{3}}\left(a_0(x)\right)a_2'(x)L^2\right)dx,$$

where once again in the summations we have used the convention that the functions $a_i, b_i$ are evaluated at $n/q$, while the functions $k, k'$ are evaluated at $a_0(n/q)$. Similarly, in the integral on the right-hand side, $a_i, b_i$ are evaluated at $x$, and $k, k'$ at $a_0(x)$. Integrating by parts the second term inside the integral, we have

$$\text{(5-6)}\quad \int_0^1 k^{\frac{2}{3}}\left(a_0(x)\right)a_2'(x)L^2dx$$
$$=k^{\frac{2}{3}}\left(a_0(x)\right)a_2(x)L^2\Big|_0^1-\int_0^1\frac{2k'}{3k}\left(a_0(x)\right)a_2(x)L^3dx.$$

By periodicity, the first term is 0. By substituting the remaining expression of (5-6) inside (5-5), we conclude that the first limit is 0.

Let us proceed with the calculation of $\lim_{q\to+\infty}\frac{1}{q}S_2$. Recalling from (4-10) that $a_0'(x)=Lk^{-\frac{2}{3}}(a_0(x))$, we have

$$a_0'''=L^3\left(-\frac{2k''}{3k^3}+\frac{14k'^2}{9k^4}\right).$$

Taking into account the expressions of $a_i, b_j$ given in (4-10) and by substituting the previous expression into $S_2$, we obtain

$$\begin{aligned}(5\text{-}7)\quad &\lim_{q\to+\infty}\frac{1}{q}S_2\\ &=\lim_{q\to+\infty}\frac{1}{q}\sum_{n=0}^{q-1}\frac{2k^4+4k'^2+7kk''}{240}b_1^5+\tfrac{1}{12}\big(3k^2b_2^2b_1+4kk'b_1^3b_2+\tfrac{1}{2}k^2a_0'''b_1^2\big)\\ &=\lim_{q\to+\infty}\frac{L^5}{q}\sum_{n=0}^{q-1}\left(\frac{k^{2/3}}{120}-\frac{k^{-\frac{10}{3}}k'^2}{540}+\frac{k^{-\frac{7}{3}}k''}{720}\right)\\ &=L^5\int_0^1\left(\frac{k^{2/3}}{120}-\frac{k^{-\frac{10}{3}}k'^2}{540}+\frac{k^{-\frac{7}{3}}k''}{720}\right)dx.\end{aligned}$$

We finally integrate by parts the last term of the integral, obtaining

$$\begin{aligned}\int_0^1 L^5k^{-\frac{7}{3}}k''dx=L^4\int_0^1 k^{-5/3}k''(k^{-\frac{2}{3}}L)\,dx=L^4\int_0^1 k^{-\frac{5}{3}}\left(k'\right)'dx\\ =\frac{5L^5}{3}\int_0^1 k^{-\frac{10}{3}}k'^2dx.\end{aligned}$$

Substituting this in (5-7), we conclude that

$$\lim_{q\to+\infty}\frac{1}{q}S_2=L^5\int_0^1\left(\frac{k^{2/3}}{120}+\frac{k^{-\frac{10}{3}}k'^2}{2160}\right)dx.$$

Finally, switching to arc length as the variable of integration, we obtain the desired result:

$$\beta_5=L^4\int_0^\ell\left(\frac{k^{4/3}(s)}{120}+\frac{k^{-\frac{8}{3}}(s)k'^2(s)}{2160}\right)ds.\qquad\square$$

## 6. Lazutkin coordinates and caustics

A consequence of Proposition 10 is that we can compute explicitly Lazutkin coordinates [14] for order 4 in the case of outer length billiards.

**Lemma 12** (Lazutkin for outer length billiards). *The coordinates*

$$x(s) = \frac{1}{L}\int_0^s k^{2/3}(r)\,dr, \quad L := \int_0^\ell k^{2/3}(r)\,dr,$$
$$y(s,\varepsilon) = x(s+\varepsilon) - x(s)$$

*are such that the outer length billiard dynamics is given by*

$$x \mapsto x + y, \quad y \mapsto y + O(y^4).$$

*Proof.* Let

$$(s,\varepsilon) \mapsto (x,y) := (f(s), f(s+\varepsilon) - f(s))$$

a change of coordinates so that $(x_k, y_k) \mapsto (x_k + y_k, y_{k+1})$. Then, by using the expansion of $\varepsilon_1$ given in 10, we have

$$\begin{aligned}
y_1 = x_2 - x_1 &= f(s_1+\varepsilon_1) - f(s_1) = f'(s_1)\varepsilon_1 + \frac{f''(s_1)}{2}\varepsilon_1^2 + \frac{f'''(s_1)}{6}\varepsilon_1^3 + O(\varepsilon_1^4)\\
&= \left(f'(s_0) + f''(s_0)\varepsilon_0 + \frac{f'''(s_0)}{2}\varepsilon_0^3\right)\left(\varepsilon_0 + A(s_0)\varepsilon_0^2 + B(s_0)\varepsilon_0^3\right)\\
&\quad + \left(f''(s_0) + f'''(s_0)\varepsilon_0\right)\frac{\left(\varepsilon_0 + A(s_0)\varepsilon_0^2 + B(s_0)\varepsilon_0^3\right)^2}{2} + \frac{f'''(s_0)}{6}\varepsilon_0^3 + O(\varepsilon_0^4)\\
&= \left(f'(s_0)\varepsilon_0 + \frac{f''(s_0)}{2}\varepsilon_0^2 + \frac{f'''(s_0)}{6}\varepsilon_0^3\right) + \left(f''(s_0) + f'(s_0)A(s_0)\right)\varepsilon_0^2\\
&\quad + \left(f'(s_0)B(s_0) + 2f''(s_0)A(s_0) + f'''(s_0)\right)\varepsilon_0^3 + O(\varepsilon_0^4)\\
&= y_0 + \left(f''(s_0) + f'(s_0)A(s_0)\right)\varepsilon_0^2 + \left(f'(s_0)B(s_0) + 2f''(s_0)A(s_0) + f'''(s_0)\right)\varepsilon_0^3\\
&\quad + O(\varepsilon_0^4).
\end{aligned}$$

Thus, to get rid of the $\varepsilon_0^2$ and $\varepsilon_0^3$ terms, we need to choose $f$ solving

$$\begin{cases} f''(s_0) + f'(s_0)A(s_0) = 0,\\ f'(s_0)B(s_0) + 2f''(s_0)A(s_0) + f'''(s_0) = 0. \end{cases}$$

Integrating the first equation, we immediately obtain the desired formula for $f$, giving, up to normalization,

$$x(s) = \frac{1}{L}\int_0^s k^{2/3}(r)\,dr, \quad L := \int_0^\ell k^{2/3}(r)\,dr.$$

Then, by direct computation, it is easy to check that such a function solves also the second equation. □

As a consequence, the outer length billiard map is a small perturbation of the integrable map

$$(x,y) \mapsto (x+y, y),$$

satisfying the assumptions of Lazutkin's theorem [14, Theorem 1]. Applying this theorem, the next corollary of Proposition 10 immediately follows.

**Theorem 13.** *Arbitrarily close to the boundary $\partial\Omega$, there exist smooth caustics for the outer length billiard map. The union of these caustics has positive measure.*

On the other hand, regarding the nonexistence of caustics, we underline that the following outer length billiard version of Mather s theorem still holds.

**Theorem 14.** *If the curvature of the boundary $\partial\Omega$ vanishes at some point, then the outer length billiard in $\partial\Omega$ has no caustics.*

*Proof.* We use Mather's necessary analytic condition for the existence of a caustic [16], that is

$$H_{22}(s_0, s_1) + H_{11}(s_1, s_2) < 0.$$

By using the general expression of the generating function (4-4), it is easily seen that

$$H_1(s_1, s_2) = -1 - \frac{(\gamma(s_2) - \gamma(s_1)) \wedge \gamma''(s_1)}{\gamma'(s_1) \wedge \gamma'(s_2)} - \frac{(\gamma(s_2) - \gamma(s_1)) \wedge (\gamma'(s_2) - \gamma'(s_1)) \cdot (\gamma''(s_1) \wedge \gamma'(s_2))}{(\gamma'(s_1) \wedge \gamma'(s_2))^2}.$$

Hence,

$$\begin{aligned} H_{11}(s_1, s_2) = {} & \frac{\gamma'(s_1)\wedge\gamma''(s_1)}{\gamma'(s_1)\wedge\gamma'(s_2)} - \frac{(\gamma(s_2)-\gamma(s_1))\wedge\gamma'''(s_1)}{\gamma'(s_1)\wedge\gamma'(s_2)} \\ & + 2\frac{(\gamma(s_2)-\gamma(s_1))\wedge\gamma''(s_1)}{(\gamma'(s_1)\wedge\gamma'(s_2))^2}(\gamma''(s_1)\wedge\gamma'(s_2)) + \frac{\gamma''(s_1)\wedge\gamma'(s_2)}{(\gamma'(s_1)\wedge\gamma'(s_2))^2} \\ & - \frac{(\gamma(s_2)-\gamma(s_1))\wedge(\gamma'(s_2)-\gamma'(s_1))}{(\gamma'(s_1)\wedge\gamma'(s_2))^2}(\gamma'''(s_1)\wedge\gamma'(s_2)) \\ & + 2\frac{(\gamma(s_2)-\gamma(s_1))\wedge(\gamma'(s_2)-\gamma'(s_1))}{(\gamma'(s_1)\wedge\gamma'(s_2))^3}(\gamma''(s_1)\wedge\gamma'(s_2))^2. \end{aligned}$$

Now assume that at a point on the boundary corresponding to the arc-length parameter value $s_1$, the curvature is zero, that is, $k(s_1) = 0$. Since the set is convex, this condition implies that also $k'(s_1) = 0$. From formulas (4-2), it follows that $\gamma''(s_1) = \gamma'''(s_1) = 0$. Substituting into the previous formula, we see that all the terms composing $H_{11}$ vanish, and a similar argument holds for $H_{22}$. As a consequence, we have $H_{11}(s_1, s_2) + H_{22}(s_0, s_1) = 0$ for every $s_0, s_2$, and therefore no topologically nontrivial invariant curve can exist. □

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