Pacific Journal of Mathematics

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Volume 339 No. 2 December 2025

PACIFIC JOURNAL OF MATHEMATICS
Vol. 339, No. 2, 2025
https://doi.org/10.2140/pjm.2025.339.223

MORI DREAM SPACES AND $\mathbb{Q}$-HOMOLOGY QUADRICS

PAOLO CASCINI, FABRIZIO CATANESE, YIFAN CHEN AND JONGHAE KEUM

We show that Shavel-type surfaces are fake $\mathbb{Q}$-homology quadrics of even type which are not Mori dream surfaces, yet there are infinitely many primes p such that the reduction modulo p is a Mori dream surface.

We investigate fake $\mathbb{Q}$-homology quadrics, first concerning the property of being a Mori dream surface, then trying to determine which families are of even type among the surfaces isogenous to a higher product which are fake $\mathbb{Q}$-homology quadrics.

1. Introduction

In [20] the authors considered complex surfaces of general type with $q = p_g = 0$ which are Mori dream surfaces, and asked in section 3.2 whether there are fake quadrics which are not Mori dream surfaces.

We produce here the first examples.

Recall that, for complex surfaces isogenous to a product of curves, it was established in [20] and [16] that they are Mori dream surfaces.

At the Hefei Conference in September 2024, the second author pointed out that such examples should be provided by the surfaces constructed by Shavel in 1978 [24]. Our first aim is therefore to give a complete proof of this assertion.

Our second aim is to discuss several problems related to minimal surfaces of general type which are $\mathbb{Q}$-homology quadrics: that is, smooth surfaces with $q = p_g = 0$, and with second Betti number $b_2(S) = 2$ (equivalently, with $K_S^2 = 8$). All known such examples have universal cover equal to the bidisk $\mathbb{H} \times \mathbb{H}$.

Indeed Hirzebruch ([18] Problem 25; see also pages 779–780 of [19]) was the first to ask the question whether there exists a surface of general type which is homeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and one can respectively ask the same question for a surface homeomorphic to the blow up $\mathbb{F}_1$ of $\mathbb{P}^2$ in one point. The answer is suspected to be negative.

MSC2020: 14J15, 14J25, 14J29, 14J80.

Keywords: surfaces of general type, Mori dream space, homology quadric, fake quadric, group action, surface isogenous to a product, foliation.

The two manifolds above are topologically distinguished by the property that the intersection form on the Severi group $\mathrm{Num}(S)$ is even in the first case, odd in the second.

We show here that many of our surfaces (surfaces of general type which are $\mathbb{Q}$-homology quadrics) have even intersection form. The existence of the case of odd intersection form was shown in the affirmative in [11] after the present paper was written.

The third question we consider is: what happens when, instead of complex surfaces, we consider surfaces defined over an algebraically closed field of positive characteristic? When are they Mori dream surfaces?

This is related to a beautiful conjecture by Ekedahl, Shepherd-Barron and Taylor [14] about algebraic integrability of foliations via reduction modulo primes p.

We can summarize our result in this regard as follows.

Theorem 1.1. *A Shavel-type surface S is an even $\mathbb{Q}$-homology fake quadric which is not a Mori dream surface.*

There are infinitely many primes p such that the reduction of S modulo p is a Mori dream surface.

We also give results stating when a fake $\mathbb{Q}$-homology quadric is a Mori dream space.

2. Definitions and basic properties

In these first sections we shall mostly work with projective smooth surfaces defined over the field $\mathbb{C}$, most definitions however make also sense if $\mathbb{C}$ is replaced by an algebraically closed field $\mathcal{K}$ of arbitrary characteristic.

Definition 2.1 ($\mathbb{Q}$-homology quadric). Let S be a smooth projective surface over $\mathbb{C}$.

The surface S is called a *$\mathbb{Q}$-homology quadric* if $q(S) = p_g(S) = 0$, $b_2(S) = 2$.

In turn, it will be called an *even homology quadric* if

(1) $q(S) = p_g(S) = 0$, $b_2(S) = 2$;

(2) the intersection form on $\mathrm{Num}(S)$ is even, that is, it is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We shall call it an *odd homology quadric* if instead the intersection form is odd, and hence diagonalizable with diagonal entries $(+1, -1)$.

Remarks 2.2. (1) A normal projective surface X is called a $\mathbb{Q}$-homology projective plane if $b_1(X) = b_3(X) = 0$ and $b_2(X) = 1$.

(2) A smooth $\mathbb{Q}$-homology projective plane must be either the projective plane or a ball quotient with $q = p_g = 0$.

(3) Our definition of $\mathbb{Q}$-homology quadric can also be extended to normal surfaces, requiring however that $b_1(X) = b_3(X) = 0$ and $b_2(X) = 2$. But in this paper a $\mathbb{Q}$-homology quadric means a smooth $\mathbb{Q}$-homology quadric.

For a $\mathbb{Q}$-homology quadric S, condition (1) and the Noether formula imply

$$\chi(\mathcal{O}_S) = 1, c_2(S) = 4, K_S^2 = 8.$$

Also the long exact sequence of cohomology groups associated to the exponential exact sequence shows that

$$c_1 : \mathrm{Pic}(S) \to H^2(S, \mathbb{Z})$$

is an isomorphism. We identify these two groups and denote by $\mathrm{Tors}(S)$ their torsion subgroup. Then

$$\mathrm{Num}(S) \cong \mathrm{Pic}(S)/\mathrm{Tors}(S).$$

In the case of an even homology quadric, condition (2) implies that S is minimal. Hence, by surface classification, for an even homology quadric,

- either S is rational, and then $S \cong \mathbb{F}_{2n}$ for $n \geq 0$;
- or S is a minimal surface of general type.

In the former case S is simply connected, $\mathrm{Tors}(S) = 0$ and K_S is not ample.

In the case of an odd homology quadric

- either S is rational, and then $S \cong \mathbb{F}_{2n+1}$ for $n \geq 0$;
- or S is a (not necessarily minimal) surface of general type.

Definition 2.3 (even fake homology quadric). Let S be a smooth projective surface over $\mathbb{C}$. The surface S will be called[1] an *even fake homology quadric* if it is an even homology quadric and is of general type.

In general, a minimal smooth complex projective surface of general type with $p_g = q = 0$, $K^2 = 8$ has Picard number 2 and K_S is ample by [23, Proposition 2.1.1]. Also $\mathrm{Tors}(S)$ can be nonzero (see [5]).

For $L \in \mathrm{Pic}(S)$, we denote by $[L]$ its class in $\mathrm{Num}(S)$.

Definition 2.4 (odd fake homology Quadric). Let S be a smooth projective surface over $\mathbb{C}$. If S is an odd homology quadric which is of general type, then either S is not minimal and it is a one point blow up of a fake projective plane, or we call the surface S an *odd fake homology quadric*, which means:

(1) S is minimal of general type with $K_S^2 = 8$, $p_g(S) = 0$;

[1] In the literature, sometimes $\mathbb{Q}$-homology quadrics of general type are referred to as fake quadrics, without specification whether the intersection form is even or odd; see [19; 12; 13; 16].

(2) the intersection form on $\mathrm{Num}(S)$ is odd.

Therefore smooth minimal surfaces of general type with $K^2 = 8$ and $p_g = 0$ are divided into two classes: the even and the odd fake homology quadrics.

For the sake of clarity, we give two definitions: the first is meant to be consistent with the current use, the second relates to the original question by Hirzebruch.

Definition 2.5. Let S be a minimal smooth projective surface over $\mathbb{C}$.

(I) Then S is called a fake quadric if and only if it is either an odd fake homology quadric, or an even fake homology quadric.

(II) S is called a fake homotopy quadric if and only if it is a fake quadric and moreover it is simply connected.

By Freedman's theorem [17] a fake homotopy quadric is homeomorphic either to $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ or to $\mathbb{F}_1$.

Lemma 2.6. *Let S be a fake quadric, and let $f : S \to \mathbb{P}^1$ be a fibration, whose general fibre we denote by F. Then:*

(1) *Any fibre $F_t = f^*(t)$ has irreducible support.*

(2) *The multiplicity m_t of any multiple fibre $F_t = m_t F'_t$ divides $g - 1$, where g is the genus of F.*

(3) *The class of the fibre $[F]$ is divisible by d in* $\mathrm{Num}(S)$, *where d is the least common multiple of the exponents m_i of the multiple fibres.*

Proof. (1) follows from Zariski's lemma (the intersection form on the fibre components is seminegative with nullity 1), and from the fact that the Picard number $\rho(S) := \mathrm{rank}(\mathrm{Num}(S))$ equals 2.

(2) follows by adjunction.

(3) If d is the least common multiple of the multiplicities $m_1, \dots, m_s$ of the multiple fibres, then we may write $\frac{1}{m_i} = \frac{r_i}{d}$, where the r_i have GCD equal to 1. Thus, we can write $\frac{1}{d}$ as a sum of the rational numbers $\frac{1}{m_i}$, hence proving that $\frac{1}{d}[F] \in \mathrm{Num}(S)$, since it is an integer linear combination of the classes F'_i.

Therefore, the divisibility of $[F]$ is a multiple of d. □

Lemma 2.7. *Keep the assumptions of Lemma 2.6, and assume that S has another fibration $f' : S \to \mathbb{P}^1$ with general fibre F' of genus g'. Then:*

(1) $FF' = (g-1)(g'-1)$.

(2) $K_S \sim \dfrac{2}{g-1}F + \dfrac{2}{g'-1}F'$, *where $\sim$ denotes numerical equivalence.*

(3) *The intersection form of S is even if* $\dfrac{1}{g-1}F, \dfrac{1}{g'-1}F' \in \mathrm{Num}(S)$.

Proof. The matrix of intersection numbers of K_S, F, F' has determinant

$$8(g-1)(g'-1)x - 8x^2,$$

with $x = FF'$, which must be 0 since $\rho(S) = 2$; hence (1) and (2) follow right away.

For (3), these classes generate a unimodular lattice, hence all of $\mathrm{Num}(S)$. □

Lemma 2.8. *Let S be a fake quadric. Then*[2] *the intersection form is even if and only if K_S is divisible by* 2 *in* $\mathrm{Num}(S)$.

Proof. $K_S D \equiv D^2 (\mathrm{mod}\, 2)$, and the intersection form on

$$\mathrm{Num}(S) = H^2(S, \mathbb{Z})/\mathrm{Tors}(S)$$

is unimodular. □

3. Surfaces isogenous to a product

Definition 3.1. Let S be a smooth projective surface. The surface S is said to be *isogenous to a higher product* [7] if

$$S \cong (C_1 \times C_2)/G,$$

where C_i is a smooth curve with $g(C_i) \geq 2$ and G is a finite group acting faithfully and freely on $C_1 \times C_2$.

If there are respective actions of G on C_1 and C_2 such that G acts by the diagonal action $g(x, y) = (gx, gy)$ on $C_1 \times C_2$, then S is called of *unmixed type*.

If some element of G exchanges the two factors, then S is called of *mixed type*.

Observation. In this paper we shall only consider surfaces isogenous to a higher product with $p_g(S) = 0$.

The next question we ask is to determine which of the $\mathbb{Q}$-homology quadrics which are isogenous to a higher product are even or odd fake quadrics.

Example 3.2. The classical Beauville surface is an even fake quadric.

Proof. Here $C_1 = C_2 = C$ are the Fermat quintic in $\mathbb{P}^2$, and the group $G = \mu_5 \times \mu_5$ acts on C_1 by the linear action

$$(\zeta_1, \zeta_2)(x_0, x_1, x_2) = (x_0, \zeta_1 x_1, \zeta_2 x_2),$$

[2]By Wu's formula, saying that K_S induces the second Stiefel–Whitney class $w_2(S) \in H^2(S, \mathbb{Z}/2)$, and by the universal coefficients formula, S is spin (i.e., $w_2(S) = 0$) if and only if K_S is divisible by 2 in $H^2(S, \mathbb{Z})$, a fact which is often expressed by saying that S is *even*; being an even surface is a stronger notion than requiring that the intersection form is even, as one can see from the example of Enriques surfaces.

while the action on C_2 is twisted by an automorphism of G (in such a way that the action on the product $C_1 \times C_2$ is free; see for instance [7], page 24, for details and a generalization).

Hence $\mathcal{O}_C(1)$ is a G-linearized bundle with square K_C, see also ([10]).

Thus, $\mathcal{O}_{C_1}(1) \otimes \mathcal{O}_{C_2}(1)$ is a G-linearized bundle which descends to a line bundle L with square K_S.

Alternatively, the fibres of both projections are divisible by 5 and yield F'_1, F'_2 such that $F'_1 F'_2 = 1$ and $(F'_j)^2 = 0$. □

We can give a partial answer to the above parity question for surfaces isogenous to a product of unmixed type with $p_g = 0$, which have been classified in [6] (their torsion groups have been classified in [5]).

Theorem 3.3. *Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a product of unmixed type, with $p_g(S) = 0$; then G is one of the groups in the table below and the multiplicities T_1, T_2 of the multiple fibres for the natural fibrations $S \to C_i/G \cong \mathbb{P}^1$ are as listed. For each case in the list we have an irreducible component of the moduli space of surfaces of general type, whose dimension is denoted by D. The property of being an even, respectively an odd homology quadric and the first homology group of S are given in the third last column, respectively in the last column.*[3]

G	Id(G)	T_1	T_2	parity	D	$H_1(S,\mathbb{Z})$
$\mathcal{A}_5$	$\langle 60,5\rangle$	$[2,5,5]$	$[3,3,3,3]$	?	1	$(\mathbb{Z}_3)^2 \times (\mathbb{Z}_{15})$
$\mathcal{A}_5$	$\langle 60,5\rangle$	$[5,5,5]$	$[2,2,2,3]$	?	1	$(\mathbb{Z}_{10})^2$
$\mathcal{A}_5$	$\langle 60,5\rangle$	$[3,3,5]$	$[2,2,2,2,2]$	?	2	$(\mathbb{Z}_2)^3 \times \mathbb{Z}_6$
$\mathcal{S}_4 \times \mathbb{Z}_2$	$\langle 48,48\rangle$	$[2,4,6]$	$[2,2,2,2,2,2]$	?	3	$(\mathbb{Z}_2)^4 \times \mathbb{Z}_4$
G(32)	$\langle 32,27\rangle$	$[2,2,4,4]$	$[2,2,2,4]$	?	2	$(\mathbb{Z}_2)^2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$
$(\mathbb{Z}_5)^2$	$\langle 25,2\rangle$	$[5,5,5]$	$[5,5,5]$	even	0	$(\mathbb{Z}_5)^3$
$\mathcal{S}_4$	$\langle 24,12\rangle$	$[3,4,4]$	$[2,2,2,2,2,2]$	even	3	$(\mathbb{Z}_2)^4 \times \mathbb{Z}_8$
G(16)	$\langle 16,3\rangle$	$[2,2,4,4]$	$[2,2,4,4]$	even	2	$(\mathbb{Z}_2)^2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$
$D_4 \times \mathbb{Z}_2$	$\langle 16,11\rangle$	$[2,2,2,4]$	$[2,2,2,2,2,2]$	?	4	$(\mathbb{Z}_2)^3 \times (\mathbb{Z}_4)^2$
$(\mathbb{Z}_2)^4$	$\langle 16,14\rangle$	$[2,2,2,2,2]$	$[2,2,2,2,2]$	even	4	$(\mathbb{Z}_4)^4$
$(\mathbb{Z}_3)^2$	$\langle 9,2\rangle$	$[3,3,3,3]$	$[3,3,3,3]$	even	2	$(\mathbb{Z}_3)^5$
$(\mathbb{Z}_2)^3$	$\langle 8,5\rangle$	$[2,2,2,2,2]$	$[2,2,2,2,2,2]$	?	5	$(\mathbb{Z}_2)^4 \times (\mathbb{Z}_4)^2$

Remark 3.4. In [11], one of us (Catanese) showed that the intersection form is odd for the first family of surfaces in the above list. The question remains open for the other families in the list.

Proof. In view of the cited results in [6] and [5], it suffices to prove the assertion about the intersection form being even, respectively odd.

[3]The group identity Id(G) consists of the group cardinality $|G|$ followed by the *Atlas* list number.

We use now (3) of Lemma 2.6 showing that $F_j = d_j \Phi_j$, where d_j is the least common multiple of the multiplicities of the fibration with general fibre F_j.

Since $F_1 F_2 = |G|$, we conclude that

$$\Phi_1 \Phi_2 = \frac{|G|}{d_1 d_2}.$$

If $d_1 d_2 = |G|$, then $\Phi_1 \Phi_2 = 1$, and, since $\Phi_j^2 = 0$, we have an even intersection form.

Inspecting the list, we see that $\Phi_1 \Phi_2 \in \{1, 2, 4\}$.

If we have an even intersection form, then we have a basis e_1, e_2 of the lattice $\mathrm{Num}(S)$ with $e_j^2 = 0$, and $e_1 e_2 = 1$.

Hence, without loss, we may assume that $[\Phi_j] = a_j e_j$, and therefore $\Phi_1 \Phi_2 = a_1 a_2$.

If $2 | a_1 a_2$, then $2 | a_j$ for some j, and Φ_j is further divisible by 2.

We could conclude that the intersection form is odd, in case $\Phi_1 \Phi_2 = 2$, if we knew that each Φ_j is not divisible by 2.

In fact, if the intersection form is odd, then we have a basis q_1, q_2 of the lattice $\mathrm{Num}(S)$ with $q_1^2 = 1$, $q_2^2 = -1$, and $q_1 q_2 = 0$.

Then Φ_1, Φ_2 must be multiples of $q_1 + q_2$, respectively $q_1 - q_2$, and indeed $(q_1 + q_2)(q_1 - q_2) = 2$.

In the only case (with group $(\mathbb{Z}/2)^4$) where $\Phi_1 \Phi_2 = 4$, the divisibility index of Φ_1 equals the one of Φ_2 by the symmetry of the roles of the two curves C_1, C_2, including the associated monodromies. Hence either Φ_1 and Φ_2 are both 2-divisible, and the intersection form is even, or the intersection form is odd and Φ_1 and Φ_2 are both indivisible. But then $\{\Phi_1, \Phi_2\} = \{(q_1 + q_2), (q_1 - q_2)\}$ and $(q_1 + q_2)(q_1 - q_2) = 2$ contradicts $\Phi_1 \Phi_2 = 4$. □

Remark 3.5. In the case where $\Phi_1 \Phi_2 = 2$, it is easy to see which of the two divisors may be 2-divisible.

In fact, using the ramification formula for $S \to (C_1/G) \times (C_2/G) = \mathbb{P}^1 \times \mathbb{P}^1$, we see that, setting $m_1, \dots, m_s$ to be the multiplicities of the multiple fibres in the first fibration and $n_1, \dots, n_{s'}$ those in the second, we have

$$K_S = \Big(-2 + \sum_j \Big(1 - \frac{1}{m_j}\Big)\Big) F_1 + \Big(-2 + \sum_i \Big(1 - \frac{1}{n_i}\Big)\Big) F_2$$

and, in $\mathrm{Num}(S)/2\,\mathrm{Num}(S)$, we have

$$K_S \equiv \sum_j (d_1 - r_j) \Phi_1 + \sum_i (d_2 - r_i') \Phi_2 \equiv \delta_1 \Phi_1 + \delta_2 \Phi_2, \quad \delta_1, \delta_2 \in \{0, 1\}.$$

We see by direct inspection that exactly one δ_j equals 1, the other is 0.

Then $[K_S] \in \mathrm{Num}(S)$ is 2-divisible if and only if the $[\Phi_j]$ with $\delta_j = 1$ is 2-divisible.

For instance in the last case we have $\delta_1 = 1$ and $\delta_2 = 0$; indeed, $K_S \sim \Phi_1 + 2\Phi_2$.

Remarks 3.6. (1) In the last case we can prove that K_S is not 2-divisible, since there is no G-linearized theta characteristic on the curve C_1, a hyperelliptic curve of genus $g_1 = 3$. Indeed, the only G-fixed theta characteristics are the hyperelliptic divisor $\mathcal{H}$ (the hyperelliptic divisor class is fixed by any automorphism of the curve), which does not admit a G-linearization, and $P_1 + P_2 + P_3 + P_4 - \mathcal{H}$, where the P_j's are Weierstrass points and their sum is a G-orbit (hence $\mathcal{O}_{C_1}(P_1 + P_2 + P_3 + P_4)$ admits a G-linearization).

(2) On the other hand, showing that $[K_S]$ is not 2-divisible is harder, in view of the existence of torsion divisors of order 4.

(3) Our observation (1) shows that, given a Fuchsian group $\Gamma < \mathbb{P}\,\mathrm{SL}(2, \mathbb{R})$ which is not torsion free, the embedding $\Gamma \hookrightarrow \mathbb{P}\,\mathrm{SL}(2, \mathbb{R})$ does not necessarily lift to $\mathrm{SL}(2, \mathbb{R})$ (unlike the case where Γ is cocompact and torsion free).

4. Even fake quadrics

Assumption. Assumption In this section, we let S be an even fake quadric. Recall that, over the complex number field $\mathbb{C}$, a fake quadric S contains no smooth rational curves [23, Proposition 2.1.1] and in particular K_S is ample.

4.1. *Nef cone of an even fake quadric.*

Lemma 4.1. (1) *There exist $L_1, L_2 \in \mathrm{Pic}(S)$ such that $L_1L_2 = 1$, $K_SL_i = 2$, $L_i^2 = 0$ for $i = 1, 2$.*

(2) *For any L_1, L_2 as in* (1),

$$\mathrm{Num}(S) = \mathbb{Z}[L_1] \oplus \mathbb{Z}[L_2].$$

(3) $K_S \sim 2L_1 + 2L_2$.

The condition that the universal covering of S is the bidisk can be formulated as follows: there exists a 2-torsion divisor η such that

$$H^0(S^2(\Omega^1_S)(-K_S + \eta)) \neq 0;$$

see for instance [8; 9]. This condition is equivalent to the splitting of the cotangent bundle on a suitable unramified double covering of S.

From now on, we fix $L_1, L_2 \in \mathrm{Pic}(S)$ as in 4.1 (1).

Proposition 4.2. (1) *Any effective divisor on S is nef.*

(2) *Any nef and big divisor on S is ample.*

(3) *L_1 and L_2 are nef.*

(4) *For any strictly effective divisor D on S, if $D^2 = 0$, then $D \sim aL_1$ or $D \sim aL_2$ for some $a \in \mathbb{Z}_{>0}$.* (However, we do not claim that such a D exists.)

Proof. For (1), it suffices to show for any irreducible curve C, we have $C^2 \geq 0$. Assume by contradiction that $C^2 < 0$. Assume $C \sim aL_1 + bL_2$ for $a, b \in \mathbb{Z}$. Then

$$C^2 = 2ab, \quad K_S C = 2a + 2b.$$

We may assume that $a > 0$ and $b < 0$. By the adjunction formula,

$$-2 \leq 2p_a(C) - 2 = C^2 + K_S C = 2(a + b + ab) = 2a(1 + b) + 2b.$$

Therefore $b = -1$, $C \cong \mathbb{P}^1$, and this contradicts the fact that S contains no smooth rational curve.

For (2), let D be a nef and big divisor on S. Then $D^2 > 0$. Let C be any irreducible curve. Since C is nef, $DC \geq 0$. Also, since $C^2 \geq 0$, $DC > 0$ by the algebraic index theorem. Therefore D is ample.

For (3) and (4), let D be an effective divisor. Assume that $D \sim aL_1 + bL_2$. Then $D^2 = 2ab$ and $K_S D = 2(a + b)$. Since D is nef and K_S is ample, $a \geq 0$, $b \geq 0$, $a + b > 0$. Then $L_1 D = b \geq 0$, $L_2 D = a \geq 0$. And if $D^2 = 0$, then $a = 0$ or $b = 0$. □

Corollary 4.3. *Let L_1, L_2 as in Lemma 4.1*(1). *In* $\mathrm{Num}(S)_{\mathbb{R}}$,

$$\begin{aligned} \mathrm{Amp}(S) &= \{a[L_1] + b[L_2] \mid a, b \in \mathbb{R}_+\}, \\ \mathrm{Nef}(S) &= \{a[L_1] + b[L_2] \mid a, b \in \mathbb{R}_{\geq 0}\}, \\ \mathrm{Nef}(S) &= \overline{\mathrm{Eff}(S)}. \end{aligned}$$

Lemma 4.4. *Assume that the cotangent sheaf of S splits as the direct sum of two invertible sheaves*:

$$\Omega^1_S = \mathcal{O}_S(A_1) \oplus \mathcal{O}_S(A_2).$$

Then either $[A_1] = 2[L_1]$, $[A_2] = 2[L_2]$ *or* $[A_1] = 2[L_2]$, $[A_2] = 2[L_1]$.

Moreover the universal covering of S is the bidisk $\mathbb{H} \times \mathbb{H}$, and $S = \mathbb{H} \times \mathbb{H}/\Gamma$, where $\Gamma < \mathbb{P}\,\mathrm{SL}(2, \mathbb{R}) \times \mathbb{P}\,\mathrm{SL}(2, \mathbb{R})$.

Proof. Assume that $[A_1] = a[L_1] + b[L_2]$ with $a, b \in \mathbb{Z}$. Since $K_S = A_1 + A_2$ and $[K_S] = 2[L_1] + 2[L_2]$, $[A_2] = (2 - a)[L_1] + (2 - b)[L_2]$.

A Chern class computation shows that $A_1 A_2 = c_2(S) = 4$. That is,

$$a(2 - b) + b(2 - a) = 4, \quad i.e., \ (a - 1)(b - 1) = -1.$$

Therefore either $a = 2, b = 0$ or $a = 0, b = 2$.

The last assertion has been known for a long time; see [26; 3]. □

4.2. ***Even fake quadrics and Mori dream surfaces.*** The following theorem follows from Theorems 3.9 and 3.10 of [21], but we give another proof for the reader's convenience.

Theorem 4.5. *Let S be an even fake quadric. Then S is a Mori dream surface if and only if L_1 and L_2 are semiample, equivalently, if and only if S admits a finite morphism to $\mathbb{P}^1 \times \mathbb{P}^1$. The same is true for any fake quadric that does not contain a negative curve.*

Proof. Note that $\mathrm{Pic}(S) \cong H^2(S, \mathbb{Z})$ is finitely generated. According to [1, Corollary 2.6], S is a Mori dream surface if and only if $\mathrm{Eff}(S)$ is rational polyhedral and $\mathrm{Nef}(S) = \mathrm{SAmp}(S)$.

By 4.3,

$$\mathrm{Nef}(S) = \overline{\mathrm{Eff}}(S) = \{a[L_1] + b[L_2] \mid a, b \in \mathbb{R}_{\geq 0}\}.$$

Since $\mathrm{Eff}(S) \supseteq \mathrm{SAmp}(S)$, it follows that S is Mori dream surface if and only if $\mathrm{Nef}(S) = \mathrm{SAmp}(S)$, if and only if L_1, L_2 are semi-ample.

It follows that S is a Mori dream surface if and only if S has two fibrations $f_1, f_2 : S \to \mathbb{P}^1$. These combine to yield a morphism $f : S \to \mathbb{P}^1 \times \mathbb{P}^1$ which is necessarily finite since the second Betti number of S equals 2.

Conversely, if S has a finite morphism $f : S \to \mathbb{P}^1 \times \mathbb{P}^1$, $q(S) = 0$, and the second Betti number of S equals 2, then S is a Mori dream surface with $q(S) = p_g(S) = 0$, hence in particular it is a $\mathbb{Q}$-homology quadric.

Since the property of being a Mori dream space depends on the structure of $\mathrm{Num}(S) \otimes \mathbb{R}$, by the cited criteria, in the case of an odd fake quadric we take a basis of $\mathrm{Num}(S)$ as in Lemma 6.1 and set $L_1 := Q_1 + Q_2$, $L_2 := Q_1 - Q_2$.

If there are no negative curves, then the cones $\mathrm{Nef}(S)$ and the closure of $\mathrm{Eff}(S)$ are again equal to the first quadrant, and the proof runs exactly as in the even case. □

5. Shavel-type surfaces

Definition 5.1. A smooth projective surface S shall be called a *Shavel surface of special unmixed type* if

$$p_g(S) = q(S) = 0, \; S = \mathbb{H}^2/\Gamma,$$

where Γ is a cocompact discrete, torsion-free (hence acting freely), irreducible subgroup of

$$\mathrm{Aut}(\mathbb{H}^2) \simeq \mathbb{P}\mathrm{SL}(2, \mathbb{R})^2 \rtimes \mathbb{Z}/2,$$

such that

$$\Gamma < \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}).$$

We shall drop the word "special" if

$$\Gamma < \mathbb{P}\mathrm{SL}(2, \mathbb{R})^2.$$

Observe that the group Γ is said to be *reducible* if it contains a normal finite index subgroup which is of the form $\Gamma_1 \times \Gamma_2$: in this case $S = \mathbb{H}^2/\Gamma$ is a finite free quotient $(C_1 \times C_2)/G$, where $C_j := \mathbb{H}/\Gamma_j$, by the action of $G := \Gamma/\Gamma_1 \times \Gamma_2$.

Γ is said to be irreducible if it is not reducible: then the projection of Γ on each of the factors $\mathbb{P}\mathrm{SL}(2, \mathbb{R})$ has dense image.

Note that for a Shavel surface S of unmixed type, we have

$$\gamma z = (\gamma_1 z_1, \gamma_2 z_2) \quad \text{for all } \gamma = (\gamma_1, \gamma_2) \in \Gamma \text{ and all } z = (z_1, z_2) \in \mathbb{H}^2.$$

Hence S admits two smooth foliations and Ω^1_S splits as the direct sum of two invertible sheaves:

$$\Omega^1_S = Ł_1 \oplus Ł_2.$$

Proposition 5.2. *A Shavel surface S of special unmixed type is an even fake quadric and K_S is divisible by* 2 *in $Pic(S)$.*

Proof. It suffices to show K_S is divisible by 2 in $Pic(S)$.

The automorphic factor of the canonical bundle is the inverse of the jacobian determinant

$$\frac{1}{(c_1 z_1 + d_1)^2 (c_2 z_2 + d_2)^2},$$

for $\gamma = (\gamma_1, \gamma_2)$, and where

$$\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \quad i = 1, 2.$$

This shows immediately that, since we assume that $\Gamma < \mathrm{SL}(2, \mathbb{R})$, we have a well defined square root of the jacobian determinant, whence K_S is the square of the automorphic factor $(c_1 z_1 + d_1)(c_2 z_2 + d_2)$; hence $K_S = 2(L_1 + L_2)$, $Ł_j = 2L_j$, and our claim follows. □

Theorem 5.3. *A Shavel surface of unmixed type S is not a Mori-dream surface.*

Proof. We saw in Proposition 5.2 that S is an even fake quadric. We use the notation of Section 2.

It suffices to prove that $|nL_1| = \varnothing$ for any $n \geq 1$.

As remarked above, S admits two smooth foliations and Ω^1_S splits as the sum of two invertible sheaves:

$$\Omega^1_S = Ł_1 \oplus Ł_2,$$

where, by 5.2, we have $Ł_1 = 2L_1, Ł_2 = 2L_2$.

In fact, nL_1 is an automorphic line bundle on $\mathbb{H} \times \mathbb{H}$ corresponding to the following cocycle. We see this as follows: Let

$$p_1 : \mathbb{P}\mathrm{SL}(2, \mathbb{R}) \times \mathbb{P}\mathrm{SL}(2, \mathbb{R}) \to \mathbb{P}\mathrm{SL}(2, \mathbb{R}), \quad \gamma = (\gamma_1, \gamma_2) \mapsto \gamma_1,$$

be the first projection map. Then to γ_1 such that

$$\gamma_1(z_1) = \frac{a(\gamma_1)z_1 + b(\gamma_1)}{c(\gamma_1)z_1 + d(\gamma_1)}$$

we associate the automorphic factor $(c(\gamma_1)z_1 + d(\gamma_1))$.

Since Γ is irreducible, $p_1(\Gamma)$ is dense.

We claim that $H^0(S, nL_1) = 0$ for $n \geq 1$.

In fact, every section of $H^0(S, nL_1)$ is represented by a function f which satisfies the functional equation

$$f(\gamma_1 z_1, \gamma_2 z_2) = (c(\gamma_1)z_1 + d(\gamma_1))^n f(z_1, z_2).$$

By density of $p_1(\Gamma)$, this holds for each $\gamma_1 \in \mathrm{SL}(2, \mathbb{R})$.

Here we have to explain how γ_1, γ_2 are obtained (see [24]): $\mathcal{A}$ is a division quaternion algebra with centre a totally real number field $\mathcal{K}$ of degree 2 over $\mathbb{Q}$.

This means that there are two embeddings $\iota_1, \iota_2 : \mathcal{K} \to \mathbb{R}$, and these determine two homomorphisms

$$\iota_1, \iota_2 : \mathcal{A} \to Mat(2 \times 2, \mathbb{R}).$$

Then $\gamma_j := \iota_j(\gamma)$, for $\gamma \in \Gamma$, where Γ is the group of units lying in a maximal order $\mathfrak{O}$ of $\mathcal{A}$ and having reduced norm 1.

If we take now γ_1 to be in a maximal compact subgroup, the stabilizer of one point, then the same holds for γ_2; hence, using the biholomorphism of $\mathbb{H}$ with the unit disk, and choosing suitable coordinates, we can assume that $\gamma_1(z_1) = \lambda^2 z_1$, $\gamma_2(z_2) = \mu^2 z_2$.

Hence, setting

$$f(z_1, z_2) := \sum_{i,j} a_{i,j} z_1^i z_2^j,$$

we get

$$f(\lambda^2 z_1, \mu^2 z_2) = \lambda^{-n} f(z_1, z_2) \iff a_{i,j}\lambda^{2i}\mu^{2j} - \lambda^{-n} a_{i,j} = 0, \forall i, j \geq 0.$$

Now set $j = 0$: then $a_{i,0}\lambda^{2i} - \lambda^{-2n} a_{i,0} = 0$ for each λ, and we get a Laurent polynomial in λ whose coefficients are all vanishing, hence $a_{i,0} = 0$ for each $i \geq 0$. Therefore $f(z_1, 0)$ vanishes identically, and $f(z_1, z_2)$ vanishes identically for $z_2 = 0$.

Varying now γ_1, we obtain all the maximal compact subgroups to which γ_2 belongs.

Hence we have shown, for each choice of w_2, that $f(z_1, z_2)$ vanishes identically for $z_2 = w_2$.

We conclude that the section determined by the function f vanishes identically on the surface S. □

Remark 5.4. One can formulate the last argument as showing that the Iitaka dimension of L_j is $-\infty$.

Note that a more general result, but with a less elementary proof, is contained in Proposition IV.5.1 of [22], saying that the canonical model of a surface foliation with numerical dimension 1 has Iitaka dimension either $-\infty$ or 1 (and in Example II.2.3 it is stated that the first alternative applies for Hilbert modular surfaces).

6. Odd fake quadrics

In this section, we assume that S is an odd fake quadric. Recall that, over $\mathbb{C}$, S contains no smooth rational curves [23, Proposition 2.1.1] and in particular K_S is ample.

6.1. *The intersection form.*

Lemma 6.1. *There exist $Q_1, Q_2 \in \mathrm{Pic}(S)$ such that*

$$Q_1^2 = 1, \quad Q_2^2 = -1, \quad Q_1 Q_2 = 0, \quad K_S = 3Q_1 - Q_2.$$

The numerical classes $[Q_1]$ and $[Q_2]$ are uniquely determined in $\mathrm{Num}(S)$.

Moreover, for any such Q_1, Q_2,

(1) $h^0(S, 3Q_1) \geq 1$ *and Q_1 is nef and big*;

(2) *Q_1 is ample unless S contains an irreducible curve C such that $C \sim Q_2$*;

(3) *Q_1 is semiample.*

Proof. The intersection form on $\mathrm{Num}(S)$ is $\mathrm{diag}(1, -1)$.

Hence there exist divisors Q_1, Q_2 such that $Q_1^2 = 1$, $Q_2^2 = -1$, $Q_1 Q_2 = 0$.

We may assume $K_S \cdot Q_1 \geq 0$ and $K_S \cdot Q_2 \geq 0$ by possibly replacing Q_i with $-Q_i$. Then $K_S \sim aQ_1 + bQ_2$ with $a, b \in \mathbb{Z}$, $a \geq 0$, $b \leq 0$. Since $K_S^2 = 8$, $a^2 - b^2 = 8$. It follows that $a = 3$, $b = -1$ and $K_S \sim 3Q_1 - Q_2$.

Therefore $K_S = 3Q_1 - Q_2 + \eta$ for some $\eta \in \mathrm{Tors}(S)$, and we can assume $K_S = 3Q_1 - Q_2$ after replacing Q_2.

Note that $h^2(S, 3Q_1) = h^0(S, K_S - 3Q_1) = h^0(S, -Q_2)$ and $K_S(-Q_2) = -1$. Since K_S is ample, $h^2(S, 3Q_1) = 0$. Then the Riemann–Roch theorem shows

$$h^0(S, 3Q_1) \geq \tfrac{1}{2}(3Q_1)(Q_2) + \chi(\mathcal{O}_S) = 1.$$

Let C be an irreducible curve. We write $C \sim aQ_1 + bQ_2$ with $a = CQ_1$ and $b = -CQ_2$. Then $K_S C = 3a + b$ and $C^2 = a^2 - b^2$.

In order to see whether Q_1 is nef, respectively ample, assume that $CQ_1 \leq 0$, i.e., $a \leq 0$.

Then C is a negative curve, and, by Proposition 6.2 below, the class of C equals the class of $(b-1)Q_1 + bQ_2$, and we are done unless $b \leq 1$.

However, since K_S is ample and $3(b-1)+b = K_S C > 0$, $b \geq 1$. If $b = 1$, then $C \sim Q_2$. Then we have shown that Q_1 is nef, and then (1) and (2) are proven.

For (3), we may assume that Q_1 is not ample. Then by (2), there is an irreducible curve $C \sim Q_2$. Note that $p_a(C) = 1$ and thus $\mathcal{O}_C(K_S + C) \cong \omega_C \cong \mathcal{O}_C$. Moreover,

$$3Q_1 = K_S + C + \eta'$$

for some $\eta' \in \mathrm{Tors}(S)$.

There exists $m > 0$ such that

- $m\eta' = 0$, and thus $3mQ_1|_C \cong \mathcal{O}_C$; and
- $h^0(3mQ_1) \gg 0$.

Note that Q_1 is nef and big, and that

$$3mQ_1 - C \sim K_S + 3(m-1)Q_1.$$

By the Kawamata–Viehweg vanishing theorem, we have

$$H^1(S, 3mQ_1 - C) = 0.$$

So the trace (restriction) of $|3mQ_1|$ on C is complete and base-point-free.

Write $|3mQ_1| = |M| + F$, where $|M|$ is the movable part and F is the fixed part. The discussion above shows that $F \not\geq C$ and thus $FC \geq 0$. Since $3mQ_1.C = 0$, we conclude that $M.C = 0$ and $F.C = 0$. It follows that $M \sim \lambda Q_1$ for some positive integer λ. Because $\mathrm{Tors}(S)$ is finite, $|kQ_1|$ has no fixed part for sufficiently large and divisible $k > 0$. By a theorem of Zariski, [27] (see also Theorem 14.19, page 223, of [2]), Q_1 is semiample. □

Unlike the even fake quadric case, we do not know whether S contains a negative curve or not (but in the case it does not contain such a negative curve, we have determined the condition that it is a Mori dream space in Theorem 4.5).

Proposition 6.2. *Let C be an irreducible curve on S. Assume that $C^2 < 0$. Then*:

(1) *$C \sim aQ_1 + (a+1)Q_2$ for some $a \in \mathbb{Z}_{\geq 0}$ and $p_a(C) = a+1$.*

(2) *For any irreducible curve $C_0 \neq C$, $C_0^2 \geq 0$.*

(3) *Set $D := (a+1)Q_1 + aQ_2$. Then $DC = 0$, D is nef and big, moreover D is semiample only if $\mathcal{O}_C(D)$ is a torsion divisor.*

(4) *One of the sides of* $\mathrm{Eff}(S)$ *is* $\mathbb{R}_+[C]$ *and one of the sides of* $\mathrm{Nef}(S)$ *is* $\mathbb{R}_+[D]$.

Proof. We may assume $C \sim aQ_1 + bQ_2$; hence $a = CQ_1$ and $b = -CQ_2$. Then by our assumption $K_S C = 3a + b > 0$, $C^2 = a^2 - b^2 < 0$.

Set $\alpha := |a|$, then $|b| = \alpha + \delta$ with $\delta > 0$.

For some $\epsilon_1, \epsilon_2 \in \{1, -1\}$, we have $a = \epsilon_1\alpha, b = \epsilon_2(\alpha + \delta)$.

The inequalities $K_S C = 3a + b > 0$ and $K_S C + C^2 \geq 0$ (since S contains no rational curve) read out as:

$$3\epsilon_1\alpha + \epsilon_2(\alpha + \delta) > 0, \quad 3\epsilon_1\alpha + \epsilon_2(\alpha + \delta) - 2\delta\alpha - \delta^2 \geq 0.$$

If $\delta \geq 2$, then $-2\delta\alpha + (3\epsilon_1 + \epsilon_2)\alpha \leq 0$, while $\epsilon_2\delta - \delta^2 < 0$; hence this contradiction shows that $\delta = 1$.

The first inequality excludes the possibility $\epsilon_1 = \epsilon_2 = -1$.

If $\epsilon_1 = 1, \epsilon_2 = -1$, then the second inequality tells that $-2 \geq 0$, absurd.

Hence $\epsilon_2 = 1$, and $(3\epsilon_1 + 1 - 2)\alpha \geq 0$ shows that $\epsilon_1 = 1$.

Therefore $C \sim \alpha Q_1 + (\alpha + 1)Q_2$ and (1) is proven.

Next, if C_0 is a different negative curve, we have $C_0 \sim a_0 Q_1 + (a_0 + 1)Q_2$ with $a_0 \geq 0$. Then $CC_0 = aa_0 - (a + 1)(a_0 + 1) < 0$, which is impossible, proving (2).

For (3), clearly we have $D^2 = 2a + 1 > 0$ and $DC = 0$.

From (1) and (2), we see $DC_0 > 0$ for any $C_0 \neq C$. Thus D is nef and big.

By a Theorem of Zariski, saying that a nef and big divisor D is asymptotically base point free if and only if there exists a large multiple $|mD|$ which is without fixed part, D is semiample if and only if for each irreducible curve C', C' is not in the base locus of some $|mD|$ with m positive.

Applying this to $C' = C$ we see that $\mathcal{O}_C(D)$ must be a torsion line bundle.

Let us show (4). We have seen that C is the only irreducible curve which is not inside the closure $\bar{\mathcal{P}}$ of the positive cone $\mathcal{P}$, which is of course contained inside $\mathrm{Eff}(S)$.

Hence $\mathbb{R}_+[C]$ is one of the sides of $\mathrm{Eff}(S)$.

Since the nef cone is the dual of the closure $\overline{\mathrm{Eff}(S)}$, which is the span of $\mathbb{R}_+[C]$ and $\bar{\mathcal{P}}$, and since D is orthogonal to C, and is nef and big, $\mathbb{R}_+[D]$ is one of the sides of $\mathrm{Nef}(S)$. □

Corollary 6.3. *Assume that $f : S \to \mathbb{P}^1$ is a fibration with general fibre F. Then either $F \sim a(Q_1 + Q_2)$ with $a \geq 1$ and $g(F) = 2a + 1$, or $F \sim a(Q_1 - Q_2)$ and $g(F) = a + 1$ with $a \geq 1$.*

Proof. Let $F \sim aQ_1 + bQ_2$: then $F^2 = 0$ amounts to $a^2 = b^2$, that is, $a = \epsilon_1\alpha$, $b = \epsilon_2\alpha$, with $\alpha > 0$, $\epsilon_j \in \{1, -1\}$.

Since $K_S F > 0$, we get $3\epsilon_1 + \epsilon_2 > 0$; hence $\epsilon_1 = 1$, and the two solutions are as stated. □

Corollary 6.4. *Assume that S contains a negative curve. Then S admits at most one fibration, and if there is a fibration $f : S \to \mathbb{P}^1$ with general fibre F, then $F \sim (g(F) - 1)(Q_1 - Q_2)$.*

Proof. To show the last assertion, we observe that $F' := Q_1 + \epsilon Q_2$ is nef; hence $F' \cdot C \geq 0$.

This condition amounts to $a - \epsilon(a + 1) \geq 0$; hence $\epsilon = -1$. □

7. Characteristic p

Let S be an even fake quadric, now over an algebraically closed field of characteristic $p > 0$.

We begin with an easy remark: the quadrant $\{n_1 L_1 + n_2 L_2 \mid n_1, n_2 \geq 0\}$ is contained in the closure of the effective cone, since $\mathcal{P} := \{n_1 L_1 + n_2 L_2 \mid n_1, n_2 > 0\}$ consists of big divisors D (this means, for $n \gg 0$, $nD = A + E$, where A is ample and E effective). Indeed, by Riemann–Roch $D \sim d_1 L_1 + d_2 L_2$ is effective for $d_1, d_2 \geq 2$, $(d_1, d_2) \neq (2, 2)$.

Lemma 7.1. *There are at most two negative irreducible curves C on S.*

The class of C may only be $C \sim -L_1 + bL_2$ or $C \sim aL_1 - L_2$ and $C \cong \mathbb{P}^1$. Moreover, $b \geq 2$, and $a \geq 2$ if K_S is ample.

If K_S is not ample there is a unique irreducible -2-curve C orthogonal to K_S: then either $C \sim -L_1 + L_2$ or $C \sim L_1 - L_2$, but obviously both possibilities cannot occur.

Proof. If C is irreducible with $C \sim c_1 L_1 + c_2 L_2$, if C is negative $c_1 c_2 < 0$, hence we may assume that $c_1 > 0$, $c_2 < 0$.

Since $K_S C \geq 0$, we obtain $c_1 + c_2 \geq 0$.

If C' is another negative irreducible curve, it cannot lie in the same quadrant, since $c_1' > 0$, $c_2' < 0$ implies $CC' = c_1 c_2' + c_1' c_2 < 0$, a contradiction.

Hence there is at most one negative curve, in each of the two quadrants which are neither positive nor negative.

Assume now that we have an irreducible curve C with $C^2 < 0$. Then $C \sim aL_1 + bL_2$ for $a, b \in \mathbb{Z}$ and

$$C^2 = 2ab, \quad K_S C = 2a + 2b.$$

We may assume that $a > 0$ and $b < 0$. By the adjunction formula,

$$-2 \leq 2p_a(C) - 2 = C^2 + K_S C = 2(a + b + ab) = 2a(1 + b) + 2b.$$

Therefore $b = -1$, $C \cong \mathbb{P}^1$. □

Remarks 7.2. (i) In [14], remark after Lemma 6.3, Ekedahl, Shepherd-Barron and Taylor show that, for each prime p which is inert in the quadratic field $\mathcal{K}$ which is the centre of the quaternion algebra $\mathcal{A}$ of a Shavel surface of special unmixed type (see the proof of Proposition 5.2 for more details), the divisors $-2L_1+2pL_2$ and $2pL_1-2L_2$ are effective, since the p-curvature tensor is nonvanishing for both foliations.

(ii) Is it true that the possible numbers a, b in the previous Lemma 7.1 can only be equal to the characteristic p?

If we have two negative curves $C_1 \sim a_1L_1 - L_2$ and $C_2 \sim -L_1 + b_2L_2$, they span the effective cone, which is therefore polyhedral.

The nef cone consists of divisors $D \sim aL_1 + bL_2$ such that

$$a \leq ba_1, b \leq ab_2;$$

hence it is polyhedral and spanned by $D_1 \sim a_1L_1 + L_2$, $D_2 \sim L_1 + b_2L_2$.

Proposition 7.3. *If on S there are two negative curves $C_1 \sim a_1L_1 - L_2$ and $C_2 \sim -L_1 + b_2L_2$, then S is a Mori dream space.*

Proof. By [1] it suffices to show that the divisors $D_1 \sim a_1L_1+L_2$, $D_2 \sim L_1+b_2L_2$ are semiample.

The divisors are both nef and big, and by symmetry, it suffices to show only the first assertion, that D_1 is semiample.

We denote by $\mathbb{E}(D_1)$ the exceptional locus of D_1, i.e., the union of the finite maximal subvarieties Z such that the restriction of D_1 to Z is nonbig. Since D_1 is big and C_1 is the only curve which is orthogonal to D_1, it follows that $\mathbb{E}(D_1) = C_1$. By Lemma 7.1, we have that $C_1 \cong \mathbb{P}^1$ and hence $\mathcal{O}_{C_1}(D_1)$ is semiample.

We apply Theorem 0.2 of [15] (see also [4], Corollary 3.6), stating that if we are in positive characteristic and D_1 is nef and big and the restriction to the exceptional locus $\mathbb{E}(D_1)$ is semiample, then also D_1 is semiample.

Hence we are done. □

Theorem 1.1 follows now immediately from Theorem 5.3, from the fact that S is defined over a number field, from i) of Remarks 7.2, and the previous Proposition 7.3.

8. Problems

Problem 1. Consider all fake $\mathbb{Q}$-homology quadrics S that are isogenous to a product of curves. (a) Determine which ones are even. (b) Determine which ones are odd.

Problem 2. Let S be an odd fake quadric. Does S contain a negative curve?

Problem 3. Let S be an odd fake quadric. Could S have two fibrations?

Remark: this is related to Problem 1 since surfaces isogenous to a product are $\mathbb{Q}$-homology quadrics having two fibrations.

Problem 4′. (Hirzebruch's question) Is every surface homeomorphic to a smooth quadric indeed a deformation of $\mathbb{P}^1 \times \mathbb{P}^1$?

Problem 4″. Is every surface homeomorphic to $\mathbb{F}_1$ indeed a deformation of $\mathbb{F}_1$?

A negative answer to both questions would follow if one could answer positively the next problem 5, or negatively the weaker problem 6: indeed, by a theorem of Michael Freedman [17], a simply connected fake quadric is homeomorphic either to $\mathbb{F}_1$ or to $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

Problem 5. Let S be a fake quadric: is then the universal covering of S biholomorphic to $\mathbb{H} \times \mathbb{H}$?

Problem 6. Is there a simply connected fake quadric?

Problem 7. (raised by Michael Lönne at a seminar talk by the second author): is there a fake quadric with $H_1(S, \mathbb{Z}) = 0$?

Remark 8.1. If a fake quadric S is homeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ then S is spin, that is, K_S is divisible by 2, and one may study its half-canonical ring.

Acknowledgements and addendum

Our collaboration started at the Hefei USTC Conference and continued at the Shanghai Fudan Workshop on birational geometry, held in September 2024; we are very grateful to Meng Chen and Lei Zhang for organizing these events.

The question about the existence of $\mathbb{Q}$-homology fake quadrics of odd type was raised by Jianqiang Yang in email correspondence with the second author (in December 2022).

Cascini is partially supported by a Simons Collaboration grant. Keum is supported by the National Research Foundation of Korea (RS-2022-NR068993).

We thank the referees for pointing out some minor inaccuracies.

A referee pointed out to us the reference [25], as an example of another nonclassical phenomenon that can occur in positive characteristic: the authors construct a minimal surface of general type with $K_S^2 = 8$, and with $e(S) = 4$, equivalently $p_g = q$, in characteristic 3. The interesting fact is that the surface admits a nonarchimedean uniformization yet has $q(S) = h^1(\mathcal{O}_S) = 1$, and trivial Albanese variety.

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Received January 31, 2025. Revised June 23, 2025.

PAOLO CASCINI
DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
LONDON SW7 2AZ
UNITED KINGDOM

p.cascini@imperial.ac.uk

FABRIZIO CATANESE
MATHEMATISCHES INSTITUT
UNIVERSITÄT BAYREUTH
95447 BAYREUTH
GERMANY

fabrizio.catanese@uni-bayreuth.de

and

SCHOOL OF MATHEMATICS
KOREA INSTITUTE FOR ADVANCED STUDY
SEOUL 02455
SOUTH KOREA

YIFAN CHEN
SCHOOL OF MATHEMATICAL SCIENCES
BEIHANG UNIVERSITY
SHAHE HIGHER EDUCATION PARK
BEIJING, 102206
CHINA

chenyifan1984@buaa.edu.cn

JONGHAE KEUM
SCHOOL OF MATHEMATICS
KOREA INSTITUTE FOR ADVANCED STUDY
SEOUL 02455
SOUTH KOREA

jhkeum@kias.re.kr

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 339 No. 2 December 2025