Pacific Journal of Mathematics

FIVE-DIMENSIONAL MINIMAL QUADRATIC AND BILINEAR FORMS OVER FUNCTION FIELDS OF CONICS

ADAM CHAPMAN AND AHMED LAGHRIBI

Volume 339 No. 2 December 2025

PACIFIC JOURNAL OF MATHEMATICS
Vol. 339, No. 2, 2025
https://doi.org/10.2140/pjm.2025.339.243

FIVE-DIMENSIONAL MINIMAL QUADRATIC AND BILINEAR FORMS OVER FUNCTION FIELDS OF CONICS

Adam Chapman and Ahmed Laghribi

Over a field of characteristic 2, we give a complete classification of quadratic and bilinear forms of dimension 5 that are minimal over the function field of an arbitrary conic. This completes the unique known case due to Faivre concerning the classification of minimal quadratic forms of dimension 5 and type (2, 1) over function fields of nondegenerate conics.

1. Introduction

Let F be a field of characteristic 2 and K/F a field extension. An anisotropic F-form (quadratic or bilinear) φ is called K*-minimal* if φ_K is isotropic and any form ψ dominated by φ, such that $\dim\psi < \dim\varphi$, remains anisotropic over F ($\dim\varphi$ denotes the dimension of φ). We refer to Section 2 for the definition of the domination relation which is more refined than the subform relation and is necessary when we take into account singular quadratic forms. Let us mention that the minimality for bilinear forms is equivalent to that of totally singular quadratic forms (Corollary 6). Henceforth, we will restrict ourselves on the minimality for quadratic forms.

In the case of a quadratic extension K/F, any K-minimal form is of dimension 2. Obviously, the same conclusion is true when K is the function field of a 2-dimensional quadratic form. When K is the function field of a conic, then any 3-dimensional anisotropic F-form which becomes isotropic over K is necessarily K-minimal, and there is no K-minimal form of dimension 4. These two facts combine many references that we summarize below (we refer to Sections 2 and 5 for the definition of the type and the norm degree ndeg_F):

(1) For quadratic forms φ of dimension 3, we use [13, théorème 1.4].

(2) For quadratic forms φ of dimension 4, we use

(i) [13, théorème 1.3] for φ of type $(2, 0)$,

MSC2020: 11E04, 11E81.
Keywords: quadratic form, bilinear form, function field of a conic, minimal form, quasi-Pfister neighbor.

(ii) [13, théorème 1.4] for φ of type $(1, 2)$,

(iii) [20, Theorem 1.2] for φ of type $(0, 4)$ and $\mathrm{ndeg}_F(\varphi) = 8$,

(iv) [20, Proposition 1.1 and Theorem 1.2] for φ of type $(0, 4)$ and $\mathrm{ndeg}_F(\varphi) = 4$.

We will distinguish between degenerate and nondegenerate conics. Recall that a conic is called degenerate if it is given by a quadratic form of type $(0, 3)$, otherwise it is given by a quadratic form of type $(1, 1)$ and called nondegenerate. Quadratic forms of dimension 5 which are minimal over function fields of conics were classified first in characteristic not 2 by Hoffmann, Lewis and van Geel [9]. Their result has been extended to characteristic 2 by Faivre in the case of quadratic forms of type $(2, 1)$ and nondegenerate conics. His result states the following:

Theorem 1 [4, Corollary 3.7; 6, Proposition 5.2.12]. *Let φ be an anisotropic F-quadratic form of dimension 5 and type $(2, 1)$, and $\tau = b[1, a] \perp \langle 1\rangle$ an anisotropic F-quadratic form of dimension 3. Then, φ is $F(\tau)$-minimal if and only if these three conditions are satisfied*:

(i) *φ is a Pfister neighbor of a 3-fold Pfister form π.*

(ii) *$\pi \simeq \langle\langle c, b, a]]$ for a suitable $c \in F^*$.*

(iii) $\mathrm{ind}(C_0(\varphi) \otimes C_0(\tau)) = 4$.

For function fields of nondegenerate conics, Faivre proved this general result:

Proposition 1 [6, Propositions 5.2.1, 5.2.8, 5.2.11]. *Let $\tau = b[1, a] \perp \langle 1\rangle$ be an anisotropic F-quadratic form of type $(1, 1)$, and φ an anisotropic F-quadratic form. Suppose that φ is $F(\tau)$-minimal, then we have*:

(1) *φ is singular but not totally singular.*

(2) *If φ is of type $(1, \ell)$, then ℓ is odd.*

(3) *If $\dim \varphi = 5$, then φ is a Pfister neighbor of a 3-fold Pfister form $\langle\langle c, b, a]]$ for some $c \in F^*$.*

The proof of this proposition is mainly based on the fact that the extension given by the function field of a nondegenerate conic is excellent [8, Corollary 5.7] and some arguments similar to those developed by Hoffmann, Lewis and Van Geel in characteristic not 2 [9]. This excellence result is no longer true for degenerate conics as it was proved by Laghribi and Mukhija [19].

To our knowledge, no classification of minimal quadratic forms of type $(2, 1)$ over function fields of degenerate conics, or of type $(1, 3)$ over function fields of arbitrary conics are known. Our aim in this paper is to complete these open cases. The first result in this sense is the following theorem that concerns minimal quadratic forms of type $(2, 1)$ over function fields of degenerate conics.

Theorem 2. *Let φ be an anisotropic F-quadratic form of dimension 5 and type $(2, 1)$, and $\tau = \langle 1, a, b\rangle$ an anisotropic totally singular F-quadratic form of dimension 3. Then, φ is $F(\tau)$-minimal if and only if these three conditions are satisfied*:

(i) *φ is a Pfister neighbor of a 3-fold Pfister form π.*

(ii) *$\pi \simeq \langle\langle a, b, c]]$ for some $c \in F^*$.*

(iii) $\operatorname{ind} C_0(\varphi)_{F(\sqrt{a},\sqrt{b})} = 2$.

Concerning the classification of minimal 5-dimensional quadratic forms of type $(1, 3)$ over function fields of degenerate conics, we prove the following result.

Theorem 3. *Let φ be an anisotropic F-quadratic form of type $(1, 3)$, and $\tau = \langle 1, a, b\rangle$ an anisotropic totally singular quadratic form of dimension 3. Then, φ is $F(\tau)$-minimal if and only if these three conditions are satisfied*:

(i) *φ is a Pfister neighbor of a 3-fold Pfister form π.*

(ii) *$\pi \simeq \langle\langle a, b, c]]$ for some $c \in F^*$.*

(iii) *For any $e \in F^*$, we have either*

  (a) *$i_d(e\tau \perp \mathrm{ql}(\varphi)) \le 1$, or*

  (b) *$i_d(e\tau \perp \mathrm{ql}(\varphi)) = 2$ and $(D_F(\varphi) \cap D_F(e\tau)) \setminus D_F(\mathrm{ql}(\varphi)) = \varnothing$.*

The classifications given in Theorems 1 et 2 are based on the even Clifford algebra $C_0(\varphi)$ of φ. However, for quadratic forms of type $(1, 3)$, another characterization is used in Theorem 3. This is due to the fact that the even Clifford algebra of any quadratic Pfister neighbor of type $(1, 3)$ is split as we state in Corollary 1.

Proposition 2 (compare [21, Lemma 2]). *Let $\varphi = a_1[1, b_1] \perp \cdots \perp a_n[1, b_n] \perp \langle 1, c_1, \cdots, c_s\rangle$ be a singular quadratic form such that $n \ge 1$ and $s \ge 1$, and let $K = F(\sqrt{c_1}, \ldots, \sqrt{c_s})$. Then, $C_0(\varphi)$ is isomorphic to the F-algebra $[b_1, a_1) \otimes_F \cdots \otimes_F [b_n, a_n) \otimes_F K$. In particular, $C_0(\varphi)$ has degree 2^n as a K-algebra.*

Corollary 1. *An anisotropic F-quadratic form φ of type $(1, 3)$ is a Pfister neighbor if and only if φ is similar to $rs[1, u] \perp \langle 1, r, s\rangle$ for suitable scalars $r, s, u \in F^*$. Moreover, $C_0(\varphi)$ is split as a K-algebra, where $K = F(\sqrt{r}, \sqrt{s})$.*

For the classification of minimal 5-dimensional quadratic forms of type $(1, 3)$ over function fields of nondegenerate conics, we prove the following theorem:

Theorem 4. *Let φ be an anisotropic F-quadratic form of type $(1, 3)$, and $\tau = a[1, b] \perp \langle 1\rangle$ an anisotropic quadratic form of dimension 3 and type $(1, 1)$. Then, φ is $F(\tau)$-minimal if and only if these three conditions are satisfied*:

(i) *φ is a Pfister neighbor of a 3-fold Pfister form π.*

(ii) *$\pi \simeq \langle\langle c, a, b]]$ for some $c \in F^*$.*

(iii) *For any $e \in F^*$, if $e[1, b] \subset \varphi$ then $e \notin D_F(a[1, b]) \cdot D_F(\mathrm{ql}(\varphi))$.*

Finally, for the classification of minimal quadratic forms of type (0, 5), we use the language of bilinear forms, which will help us to use a cohomological invariant and a classification parallel to those given in Theorems 1 et 2. Namely, we will prove the following result:

Theorem 5. *Let B be an anisotropic F-bilinear form of dimension 5, and $Q = \langle 1, a, b\rangle$ an anisotropic totally singular quadratic form of dimension 3. Then, the following statements are equivalent*:

(1) *B is $F(Q)$-minimal.*

(2) *There exists an F-bilinear form C of dimension 5 which is a strong Pfister neighbor of a bilinear Pfister form $\langle\langle a, b, c\rangle\rangle_b$ and satisfies the two conditions*:
   (i) $\tilde{B} \simeq \tilde{C}$.
   (ii) *For any $u, v \in F^2(a, b)$ such that $\langle u, v, uv\rangle_b$ is similar to a subform of $\langle\langle a, b, c\rangle\rangle_b$, the invariant $e^2(C \perp \langle \det C\rangle_b \perp \langle\langle u, v\rangle\rangle_b + I^3F)$ has length 2.*

Note that for bilinear forms, nothing happens over the function field of a nondegenerate conic since an anisotropic bilinear form remains anisotropic over such a field. To clarify the notations used in Theorem 5, let us recall that to any bilinear form B of underlying vector space V, we associate a totally singular quadratic form $\tilde{B}$ defined on V by: $\tilde{B}(v) = B(v, v)$ for all $v \in V$. The cohomological invariant e^2 is that due to Kato [10] going from the quotient I^2F/I^3F to $\nu_F(2)$, where $\nu_F(2)$ is the additive group generated by the logarithmic symbols $\frac{da_1}{a_1} \wedge \frac{da_2}{a_2}$ for $a_1, a_2 \in F^*$. This invariant plays the role of the Clifford invariant which is not defined for bilinear forms in characteristic 2, and thus the group $\nu_F(2)$ can be seen as the 2-torsion of the Brauer group. The word "length" that we talk about in the previous theorem concerns the smallest number of logarithmic symbols needed to write the cohomological invariant $e^2(\eta)$ for $\eta \in I^2F/I^3F$. Finally, the notion of a strong Pfister neighbor bilinear form is defined as the classical notion of a quadratic Pfister neighbor. We use the term "strong" since we have another weaker notion of bilinear Pfister neighbor (see Section 5).

2. Background on quadratic and bilinear forms

We refer to [5] for undefined terminologies or facts. Recall that any quadratic form φ decomposes as follows:

$$\varphi \simeq [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp \langle c_1\rangle \perp \cdots \perp \langle c_s\rangle, \qquad (1)$$

where $[a, b]$ (resp. $\langle c\rangle$) denotes the binary quadratic form $ax^2 + xy + by^2$ (resp. cz^2). Here, $\simeq$ and $\perp$ denote the isometry and the orthogonal sum, respectively.

As in (1), the form φ is called of type (r, s). We say that φ is nonsingular (resp. totally singular) if $s = 0$ (resp. $r = 0$). It is called singular if $s > 0$.

The form $\langle c_1\rangle \perp \cdots \perp \langle c_s\rangle$ in (1) is unique up to isometry, we call it the quasilinear part of φ and denote it by $\mathrm{ql}(\varphi)$.

A quadratic form φ of underlying vector space V is called isotropic if there exists $v \in V \setminus \{0\}$ such that $\varphi(v) = 0$. Otherwise, φ is called anisotropic.

Any quadratic form φ uniquely decomposes as follows:

$$\varphi \simeq \varphi_{an} \perp [0, 0] \perp \cdots \perp [0, 0] \perp \langle 0\rangle \perp \cdots \perp \langle 0\rangle \,, \tag{2}$$

where the form φ_{an} is anisotropic that we call the anisotropic part of φ. The number of copies of the hyperbolic plane $[0, 0]$ in (2) is called the Witt index of φ, we denote it by $i_W(\varphi)$. Similarly, the number of $\langle 0\rangle$ in (2) is called the defect index of φ, we denote it by $i_d(\varphi)$. The total index of φ is $i_W(\varphi) + i_d(\varphi)$.

Two quadratic forms φ and ψ are called Witt equivalent if $\varphi \perp m \times [0, 0] \simeq \psi \perp n \times [0, 0]$ for some integers $m, n \geq 0$. In this case, we write $\varphi \sim \psi$.

Let $C(\varphi)$ (resp. $C_0(\varphi)$) denote the Clifford algebra (resp. the even Clifford algebra) of the quadratic form φ. When $\varphi \simeq a_1[1, b_1] \perp \cdots \perp a_r[1, b_r]$ for $a_i, b_i \in F$ such that $a_i \neq 0$ for $1 \leq i \leq r$, its Arf invariant $\Delta(\varphi)$ is the class of $\sum_{i=1}^r b_i$ in $F/\wp(F)$, where $\wp(F) = \{x^2 + x \mid x \in F\}$. In this case, $C(\varphi)$ is isomorphic to $\otimes_{i=1}^r [b_i, a_i)$, where $[b, a)$ denotes the quaternion algebra generated by two elements i and j subject to the relations: $i^2 = a \in F^* := F \setminus \{0\}$, $j^2 + j = b \in F$ and $iji^{-1} = j + 1$.

Let φ and ψ be two quadratic forms over F of underlying vector space V and W, respectively. We say that φ is dominated by ψ if there exists an injective F-linear map $\sigma : V \longrightarrow W$ such that $\varphi(v) = \psi(\sigma(v))$ for any $v \in V$. In this case, we write $\varphi \prec \psi$. We say that φ is weakly dominated by ψ if $\alpha\varphi \prec \psi$ for some $\alpha \in F^*$. The form φ is called a subform of ψ, denoted by $\varphi \subset \psi$, if $\psi \simeq \varphi \perp \varphi'$ for a suitable quadratic form φ'. Clearly, if φ is a subform of ψ, then it is dominated by ψ, but the converse is not true in general. We refer to [7, Lemma 3.1] for more details on the domination relation.

For $a_1, \ldots, a_n \in F^*$, let $\langle a_1, \ldots, a_n\rangle_b$ denote the diagonal bilinear form given by

$$((x_1, \ldots, x_n), (y_1, \cdots, y_n)) \mapsto \sum_{i=1}^n a_i x_i y_i .$$

A metabolic plane is a 2-dimensional bilinear form isometric to $\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ for some $a \in F$; we denote it by $\mathbb{M}(a)$. A bilinear form is called metabolic if it is isometric to a sum of metabolic planes.

Let $W_q(F)$ (resp. $W(F)$) be the Witt group of nonsingular quadratic forms (resp. the Witt ring of nondegenerate symmetric bilinear forms). For any integer $m \geq 1$, let $I^m F$ be the m-th power of the fundamental ideal IF of classes of even-dimensional forms in $W(F)$ (we take $I^0 F = W(F)$). Recall that $W_q(F)$ is endowed with $W(F)$-module structure in a natural way [1]. For any integer $m \geq 2$, let $I_q^m F$ be the submodule $I^{m-1}F \cdot W_q(F)$ of $W_q(F)$. The ideal $I^m F$ is additively generated by the m-fold bilinear Pfister forms $\langle 1, a_1\rangle_b \otimes \cdots \otimes \langle 1, a_m\rangle_b$, that we denote by $\langle\langle a_1, \ldots, a_m\rangle\rangle_b$. The submodule $I_q^m F$ is generated, as a $W(F)$-module, by the quadratic forms $\langle\langle a_1, \ldots, a_{m-1}\rangle\rangle_b \cdot [1, b]$, that we denote by $\langle\langle a_1, \ldots, a_{m-1}, b]]$ and call an m-fold quadratic Pfister form. We have $I_q^1 F = W_q(F)$ and a 1-fold quadratic Pfister form is a form of type $[1, a]$.

Let $P_m F$ be the set of forms isometric to m-fold quadratic Pfister forms, and $GP_m F$ the set of forms similar to forms in $P_m F$. Similarly, let $BP_m F$ be the set of forms isometric to m-fold bilinear Pfister forms, and $GBP_m F$ the set of forms similar to forms in $BP_m F$.

For $m \geq 1$ an integer and $B \in BP_m F$, we have $B \simeq \langle 1\rangle_b \perp B'$ for some bilinear form B'. This form B' is unique, we call it the pure part of B.

The Hauptsatz of Arason–Pfister asserts that any anisotropic form in $I_q^m F$ (or $I^m F$) has dimension $\geq 2^m$. Moreover, if the form has dimension 2^m, then it is similar to a Pfister form (see [16, lemme 4.8] for bilinear forms, and [17, proposition 6.4] for quadratic forms). In this paper, we will only need the Hauptsatz for bilinear forms.

Recall that a quadratic (resp. bilinear) Pfister form Q is isotropic if and only if it is hyperbolic (resp. metabolic). Such a form is also round, meaning that $\alpha \in F^*$ is represented by Q if and only if $Q \simeq \alpha Q$.

For a quadratic form $\varphi \simeq [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp \langle c_1\rangle \perp \cdots \perp \langle c_s\rangle$, we define the polynomial $P_\varphi = \sum_{i=1}^r (a_i x_i^2 + x_i y_i + b_i y_i^2) + \sum_{j=1}^s c_j z_j^2$. This polynomial is reducible if and only if $\varphi \simeq [0, 0] \perp \langle 0\rangle \perp \cdots \perp \langle 0\rangle$ or $\varphi \simeq \langle a\rangle \perp \langle 0\rangle \perp \cdots \perp \langle 0\rangle$ for some $a \in F^*$ [21, Proposition 3]. When P_φ is irreducible, we denote by $F(\varphi)$ the quotient field of the ring $F[x_i, y_i, z_j]/(P_\varphi)$, that we call the function field of φ. When P_φ is reducible or φ is the zero form, then we take $F(\varphi) = F$.

A quadratic form φ is called a Pfister neighbor if there exist $\pi \in P_m F$ such that φ is weakly dominated by π and $2 \dim \varphi > \dim \pi$. Recall that the form π is unique up to isometry, and for any field extension K/F we have that φ_K is isotropic if and only if π_K is isotropic. In particular, $\varphi_{F(\pi)}$ and $\pi_{F(\varphi)}$ are isotropic.

3. Preliminary results

For the proof of Theorems 2, 3 and 4, we give a preparatory result.

Theorem 6. *Let φ and τ be anisotropic quadratic forms of dimension 5 and 3, respectively, with φ not totally singular and $1 \in D_F(\tau)$. Suppose that there exists a 3-fold Pfister form π, some $x \in F^*$ and a 5-dimensional quadratic form φ' of the same type as φ such that τ is weakly dominated by both forms π and φ', and such that $\varphi \sim x\pi \perp \varphi'$. Then, τ is weakly dominated by φ.*

Proof. Let $y \in F^*$ be such that $y\tau \prec \varphi'$. Then, $x\pi \perp y\tau \prec x\pi \perp \varphi'$ is of codimension 2, but the right hand side has Witt index 4, so the left hand side is isotropic. Hence, there exists $z \in D_F(\tau)$ such that $yz \in D_F(x\pi)$. By the roundness of Pfister forms, we get $x\pi \simeq yz\pi$. Let $t \in F^*$ be such that $t\tau \prec \pi$. In particular, $t \in D_F(t\tau) \subset D_F(\pi)$. So $\tau \prec t\pi \simeq \pi$, thus $z \in D_F(\pi)$ and $x\pi \simeq yz\pi \simeq y\pi$. Hence, without loss of generality, we may suppose $x = y = 1$, meaning that τ is dominated by both forms π and φ, and we have

$$\varphi \sim \pi \perp \varphi'. \tag{3}$$

Clearly, from (3) we have $\mathrm{ql}(\varphi) \simeq \mathrm{ql}(\varphi')$. The form φ is of type $(2, 1)$ or $(1, 3)$, and τ is of type $(1, 1)$ or $(0, 3)$. We write

$$\varphi \simeq \begin{cases} R \perp \langle r\rangle & \text{case (a)},\\ R \perp \langle r, s, t\rangle & \text{case (b)},\end{cases}$$

where R is a nonsingular quadratic form and $r, s, t \in F^*$. Obviously, in case (a) the form φ is of type $(2, 1)$ and $\dim R = 4$, while in case (b) the form φ is of type $(1, 3)$ and $\dim R = 2$.

For the proof we will proceed case by case. We have to prove that τ is weakly dominated by φ.

(1) <u>Suppose that τ is of type $(0, 3)$.</u> We write $\tau = \langle 1, a, b\rangle$. The isotropy of $\pi_{F(\tau)}$ implies that $\pi \simeq \langle\langle a, b, c]]$ for some $c \in F^*$. If in case (b), τ is similar to $\langle r, s, t\rangle$, then we are done. So suppose that these two forms are not similar. By the domination relation, there exist $u, v, w \in F^*$ such that $\tau \simeq \langle u, v, w\rangle$ and

$$\varphi' \simeq \begin{cases} u[1, p] \perp v[1, q] \perp \langle w\rangle & \text{in case (a)},\\ u[1, p] \perp \langle v, w, z\rangle & \text{in case (b)},\end{cases}$$

where $p, q, z \in F^*$. Note that in case (a) we may suppose $w = r$. Adding on both sides of (3) the form $\theta := \langle\langle a, b\rangle\rangle$, we get:

$$\begin{cases} R \perp \theta \perp \langle 0\rangle \sim \theta \perp \langle 0\rangle & \text{in case (a)},\\ R \perp \langle z\rangle \perp \theta \perp \langle 0, 0\rangle \sim \langle z\rangle \perp \theta \perp \langle 0, 0\rangle & \text{in case (b)}.\end{cases}$$

The Witt cancellation of the zero form yields

$$\begin{cases} R \perp \theta \sim \theta & \text{in case (a)},\\ R \perp \langle z\rangle \perp \theta \sim \langle z\rangle \perp \theta & \text{in case (b)}.\end{cases}$$

– In case (a) we get $i_W(R \perp \theta) = 2$. Hence, there exists a form $\langle r', s'\rangle$ dominated by R and θ [7, Proposition 3.11]. Then, $\langle r, r', s'\rangle$ is dominated by φ. In particular, $\langle r, r', s'\rangle$ is anisotropic. Since θ represents r', s' and r (because we take $w = r$), it follows that $\langle r, r', s'\rangle$ is dominated by θ. Consequently, $\langle r, r', s'\rangle$ becomes isotropic over $F(\tau)$ because τ and $\langle r, r', s'\rangle$ are quasi-Pfister neighbors (see Section 5 for this notion) of the same quasi-Pfister form $\langle\langle a, b\rangle\rangle$. It follows from [20, Theorem 1.2(1)] that τ is similar to $\langle r, r', s'\rangle$, and thus τ is weakly dominated by φ.

– Similarly, in case (b), we get $i_W(R \perp \langle z\rangle \perp \theta) = 1$, and thus there exists $z' \in D_F(R \perp \langle z\rangle) \cap D_F(\theta)$. Then, $\langle v, w, z'\rangle$ is dominated by φ. In particular, $\langle v, w, z'\rangle$ is anisotropic. Since θ represents v, w and z', it follows that $\langle v, w, z'\rangle$ is dominated by θ. Consequently, $\langle v, w, z'\rangle$ becomes isotropic over $F(\tau)$, and by [20, Theorem 1.2(1)] τ is similar to $\langle v, w, z'\rangle$, thus τ is weakly dominated by φ.

(2) Suppose that τ is of type $(1, 1)$. We write $\tau = b[1, a] \perp \langle 1\rangle$ for some $a, b \in F^*$. The isotropy of $\pi_{F(\tau)}$ implies that $\pi \simeq \langle\langle c, b, a]]$ for a suitable $c \in F^*$. By the domination relation, we get

$$\varphi' \simeq \begin{cases} b[1, a] \perp S \perp \langle r\rangle & \text{in case (a)}, \\ b[1, a] \perp \langle r, s, t\rangle & \text{in case (b)}, \end{cases}$$

where in case (a), S is nonsingular of dimension 2 such that $1 \in D_F(S \perp \langle r\rangle)$; and in case (b) we suppose $r = 1$. The condition $1 \in D_F(S \perp \langle r\rangle)$ implies that $1 = re^2 + f$ for some $e \in F$ and $f \in D_F(S) \cup \{0\}$. If $f = 0$, then $\langle r\rangle \simeq \langle 1\rangle$. If $f \neq 0$, then $S \simeq [f, g]$ for some $g \in F^*$, and thus $S \perp \langle r\rangle \simeq [f + re^2, g] \perp \langle r\rangle = [1, g] \perp \langle r\rangle$. Hence, in case (a), we may suppose $r = 1$ or $1 \in D_F(S)$. When $1 \in D_F(S)$, we get $S \simeq [1, d]$ for some $d \in F$. Inserting the forms φ' and $\pi = \langle\langle c, b, a]]$ in equation (3), we get

$$\varphi \sim \begin{cases} c\,\langle\langle b, a]] \perp T & \text{in case (a)}, \\ c\,\langle\langle b, a]] \perp \langle r, s, t\rangle & \text{in case (b)}, \end{cases} \tag{4}$$

where T is the form $[1, a + d] \perp \langle r\rangle$ or $S \perp \langle 1\rangle$ according as $S \simeq [1, d]$ or $r = 1$. Clearly, the form on the right hand side of (4) is isotropic. Let $b' \in D_F(T)$ (resp. $b' \in D_F(\langle r, s, t\rangle)$) be such that $b' \in D_F(c\,\langle\langle b, a]])$. The existence of b' in case (a) is clear when T is anisotropic. If T is isotropic, then it contains a hyperbolic plane and thus it represents any scalar. The roundness of Pfister forms yields $c\,\langle\langle b, a]] \simeq b'\,\langle\langle b, a]]$.

(1) In case (a), the condition $b' \in D_F(T)$ implies that $b'[1, a] \perp T \sim U$ for some form U of type $(1, 1)$ such that $\langle b'\rangle \prec U$. Since $\varphi \sim bb'[1, a] \perp b'[1, a] \perp T$, it follows that $b'\tau \prec \varphi$.

(2) In case (b), we have $\langle r, s, t\rangle \simeq \langle b', \dots\rangle$; thus $\varphi \sim bb'[1, a] \perp \langle b', \dots\rangle$, and hence $b'\tau \prec \varphi$. □

The rest of this section is devoted to some corollaries that refine some results on isotropy due to the second author. The first one is a refinement of [13, théorème 1.2(3)] and [3, Theorem 1.1(3)].

Corollary 2. *Let φ be an anisotropic quadratic form of type* $(2, 1)$ *or* $(1, 3)$, *and* $\tau = \langle 1, a, b\rangle$ *an anisotropic totally singular quadratic form of dimension* 3. *Suppose that φ is not a Pfister neighbor. Then, $\varphi_{F(\tau)}$ is isotropic if and only if τ is weakly dominated by φ.*

Proof. Suppose that $\varphi_{F(\tau)}$ is isotropic. We have to prove that τ is weakly dominated by φ. If φ is of type $(1, 3)$ and τ is similar to $\mathrm{ql}(\varphi)$, then we are done. So suppose that τ is not similar to $\mathrm{ql}(\varphi)$ when φ is of type $(1, 3)$. Using [13, théorème 1.2(3)] (resp. [3, Theorem 1.1(3)]) when φ is of type $(2, 1)$ (resp. φ is of type $(1, 3)$), we get

$$\varphi \sim x\pi \perp \varphi' \tag{5}$$

where $x \in F^*$, π a 3-fold Pfister form isotropic over $F(\tau)$, and φ' a form of type $(2, 1)$ that weakly dominates τ. The isotropy of $\pi_{F(\tau)}$ is equivalent to saying that τ is weakly dominated by π. Hence, Theorem 6 implies that τ is weakly dominated by φ. Conversely, if τ is weakly dominated by φ, then $\varphi_{F(\tau)}$ is isotropic. □

Corollary 3. *Let φ be an anisotropic quadratic form of type* $(2, 1)$ *or* $(1, 3)$, *and* $\tau = \langle 1, a, b\rangle$ *an anisotropic totally singular quadratic form of dimension* 3. *If φ is $F(\tau)$-minimal, then φ is a Pfister neighbor of a* 3*-fold Pfister form $\langle\langle a, b, c]]$ for some $c \in F^*$.*

Proof. If φ is $F(\tau)$-minimal, then Corollary 2 implies that φ is a Pfister neighbor of some 3-fold Pfister form π. Since $\varphi_{F(\tau)}$ is isotropic, it follows that $\pi_{F(\tau)}$ is isotropic and thus hyperbolic. Hence, $\pi \simeq \langle\langle a, b, c]]$ for some $c \in F^*$. □

The following corollary refines [3, Theorem 1.1(1)].

Corollary 4. *Let φ be an anisotropic quadratic form of type* $(1, 3)$, *and τ an anisotropic quadratic form of type* $(1, 1)$. *Suppose that φ is not a Pfister neighbor. Then, $\varphi_{F(\tau)}$ is isotropic if and only if τ is weakly dominated by φ.*

Proof. Suppose that φ is isotropic over $F(\tau)$. It follows from [3, Theorem 1.1] that there exist $\alpha, \beta, u, v \in F^*$ and R_1, R_2 nonsingular quadratic forms of dimension 2 such that $\alpha\varphi \simeq R_1 \perp \langle 1, u, v\rangle$, $\beta\tau = R_2 \perp \langle 1\rangle$ and

$$R_1 \perp R_2 \perp \rho \sim x\pi, \tag{6}$$

where $x \in F^*$, ρ is a nonsingular complement of $\langle 1, u, v\rangle$ and $\pi \in P_3F$ dominates τ up to a scalar. Adding on both sides of (6) the form $\langle 1, u, v\rangle$ yields

$$\alpha\varphi \sim x\pi \perp \varphi',$$

where $\varphi' = R_2 \perp \langle 1, u, v\rangle$ dominates $\beta\tau$. Theorem 6 implies that τ is weakly dominated by φ. Obviously, if τ is weakly dominated by φ then $\varphi_{F(\tau)}$ is isotropic. □

Corollary 5. *Let φ be an anisotropic F-quadratic form of type* $(1,3)$, *and* $\tau = b[1,a] \perp \langle 1\rangle$ *an anisotropic F-quadratic form of type* $(1,1)$. *If φ is $F(\tau)$-minimal, then it is a Pfister neighbor of a 3-fold quadratic Pfister form $\pi = \langle\langle c,b,a]]$ for some $c \in F^*$.*

Proof. We use Corollary 4 and we proceed as in the proof of Corollary 3. □

Proposition 1 recovers Corollary 5 but the proof given by Faivre uses some arguments different from those developed here.

4. Proof of Theorems 2–4

Proof of Theorem 2. Let φ be an anisotropic F-quadratic form of dimension 5 and type $(2,1)$, and $\tau = \langle 1,a,b\rangle$ an anisotropic totally singular F-quadratic form of dimension 3.

– Suppose that conditions (i)–(iii) from the theorem are satisfied. Since $\pi_{F(\tau)}$ is isotropic and φ is a Pfister neighbor of π, it follows that $\varphi_{F(\tau)}$ is isotropic. Suppose that φ is not $F(\tau)$-minimal. Then, there exists ψ a form dominated by φ of dimension 3 or 4 such that $\psi_{F(\tau)}$ is isotropic. Using [13, théorème 1.3], we can see that the form ψ is neither of type $(2,0)$ nor of type $(1,1)$. Hence, ψ is of type $(0,3)$ or $(1,2)$. In both cases, there exists $x \in F^*$ such that $x\tau \prec \psi$ (use [13, théorème 1.4] for type $(1,2)$, and [20, Theorem 1.2] for type $(0,3)$). Hence, there exist u, v, w such that $x\tau \simeq \langle u,v,w\rangle$ and $\varphi \simeq u[1,p] \perp v[1,q] \perp \langle w\rangle$. We have that $C_0(\varphi)$ is isomorphic to $[p,uw) \otimes_F [q,vw)$ [21, Lemma 2]. Because $x\langle 1,a,b\rangle \simeq \langle u,v,w\rangle$, the scalars uw and vw are squares in $F(\sqrt{a},\sqrt{b})$. Consequently, $C_0(\varphi)_{F(\sqrt{a},\sqrt{b})}$ is split, a contradiction.

– Suppose that φ is $F(\tau)$-minimal. By Corollary 3, we deduce that φ is a Pfister neighbor of a 3-fold Pfister form $\pi = \langle\langle a,b,c]]$ for some $c \in F^*$. Moreover, modulo a scalar, we have $\varphi \simeq \langle\langle s,t]] \perp \langle r\rangle$ (we may use [13, Proposition 3.2]).

Suppose that $C_0(\varphi)_{F(\sqrt{a},\sqrt{b})}$ is split. Then, $\langle\langle s,t]]$ is hyperbolic over $F(\sqrt{a},\sqrt{b})$. By a result of Baeza [1, Corollary 4.16, page 128], we get $\langle\langle s,t]] \sim \langle 1,a\rangle \otimes \varphi_1 \perp \langle 1,b\rangle \otimes \varphi_2$ for suitable $\varphi_1, \varphi_2 \in W_q(F)$. Hence, we get

$$\langle\langle s,t]] \perp \langle\langle a,k_1]] \perp \langle\langle b,k_2]] \in I_q^3 F,$$

where $k_i = \Delta(\varphi_i)$ for $i = 1,2$. Hence, we get

$$r\,\langle\langle s,t]] \perp [1,k_1+k_2] \perp a[1,k_1] \perp b[1,k_2] \in I_q^3 F. \tag{7}$$

It follows from [17, proposition 6.4] that there exists $\rho \in GP_3F$ such that

$$r\,\langle\langle s,t]] \sim [1,k_1+k_2] \perp a[1,k_1] \perp b[1,k_2] \perp \rho.$$

Consequently, by adding $\langle 1\rangle$ on both sides, we get

$$r\varphi \sim a[1, k_1] \perp b[1, k_2] \perp \langle 1\rangle \perp \rho.$$

Since the forms φ and $a[1, k_1] \perp b[1, k_2] \perp \langle 1\rangle$ are isotropic over $F(\tau)$, it follows that $\rho_{F(\tau)}$ is isotropic, and thus τ is weakly dominated by ρ. Theorem 6 implies that τ is weakly dominated by φ, a contradiction to the minimality of φ. □

Let us give an example for which Theorem 2 applies. This example is similar to the one given by Chapman and Quéguiner-Mathieu for the minimality over the function field of a nondegenerate conic [2].

Example 1. Let a, b, c be indeterminates over a field F_0 of characteristic 2. Consider the forms $\varphi = c[1, a+b] \perp b[1, a] \perp \langle 1\rangle$, $\tau = \langle 1, b, ac\rangle$ and $\pi = \langle\langle b, c, a+b]]$ over the rational function field $F := F_0(a, b, c)$. Then:

(1) $\pi \simeq \langle c, bc\rangle_b .[1, a+b] \perp \langle 1, b\rangle_b .[1, a]$ because

$$\langle 1, b\rangle_b .[1, a+b] \simeq \langle 1, b\rangle_b .[1, a].$$

This proves that $\varphi \prec \pi$, and thus φ is a Pfister neighbor of π.

(2) $\varphi_{F(\tau)}$ is isotropic because $c\tau \prec \pi$ as we can see from (1).

(3) Let $L = F(\sqrt{b}, \sqrt{ac})$. We have in the Brauer group of L the following:

$$C_0(\varphi)_L = [a+b, bc)_L = [a+b, a)_L = [b, a)_L.$$

The algebra $[b, a)$ is division over $F_1 := F_0(a, b)(\sqrt{b})$, and it remains division over $L = F_1(\sqrt{ac})$.

Hence, Theorem 2 implies that φ is $F(\tau)$-minimal.

Proof of Theorem 3. Let φ be an anisotropic F-quadratic form of type $(1, 3)$, and $\tau = \langle 1, a, b\rangle$ an anisotropic totally singular quadratic form of dimension 3.

– Suppose that φ is $F(\tau)$-minimal. Then, by Corollary 3, φ is a Pfister neighbor of a 3-fold Pfister form $\pi = \langle\langle a, b, c]]$ for some $c \in F^*$. Suppose that there exists $e \in F^*$ such that: $i_d(e\tau \perp \mathrm{ql}(\varphi)) \geq 2$ and ($i_d(e\tau \perp \mathrm{ql}(\varphi)) \neq 2$ or $(D_F(\varphi) \cap D_F(e\tau)) \setminus D_F(\mathrm{ql}(\varphi)) \neq \varnothing$). This is equivalent to saying: $i_d(e\tau \perp \mathrm{ql}(\varphi)) = 3$ or ($i_d(e\tau \perp \mathrm{ql}(\varphi)) \geq 2$ and $(D_F(\varphi) \cap D_F(e\tau)) \setminus D_F(\mathrm{ql}(\varphi)) \neq \varnothing$). The condition $i_d(e\tau \perp \mathrm{ql}(\varphi)) = 3$ means that $\mathrm{ql}(\varphi)$ is similar to τ, while the second condition means that $e\tau$ is dominated by φ. Hence, φ is not $F(\tau)$-minimal, a contradiction.

– Conversely, suppose that we have the three conditions (i), (ii) and (iii) as described in the theorem. Since φ is a Pfister neighbor of a 3-fold Pfister form $\langle\langle a, b, c]]$ for some $c \in F^*$, it follows that $\varphi_{F(\tau)}$ is isotropic. Suppose that φ is not $F(\tau)$-minimal. Then, there exists ψ a form dominated by φ of dimension 3 or 4 such that $\psi_{F(\tau)}$ is isotropic. Then, $e\tau$ is dominated by ψ for a suitable $e \in F^*$ (we use [20, Theorem

1.2] when ψ is totally singular, and [13, théorème 1.4] when ψ is of type $(1,2)$). This gives two possibilities:

(a) $e\tau$ is isometric to $\mathrm{ql}(\varphi)$, which contradicts the condition (iii), or

(b) there exists $x, y, z, t, u \in F^*$ such that $e\tau \simeq \langle x, y, z\rangle$ and $\varphi \simeq x[1,u] \perp \langle y, z, t\rangle$. This condition also contradicts (iii) because $i_d(e\tau \perp \mathrm{ql}(\varphi)) = 2$ but $x \in (D_F(\varphi) \cap D_F(e\tau)) \setminus D_F(\mathrm{ql}(\varphi))$.

Hence, φ is $F(\tau)$-minimal. □

Proof of Proposition 2. Suppose $\varphi = a_1[1,b_1] \perp \cdots \perp a_n[1,b_n] \perp \langle c_0, c_1, \ldots, c_s\rangle$ for $a_i, b_i, c_j \in F$ such that $c_0 = 1$ and $a_i \neq 0$ for all i. The Clifford algebra of φ is generated by $x_1, y_1, \ldots, x_n, y_n, z_0, \ldots, z_s$ such that z_i commutes with all the generators, x_i commutes with y_j when $i \neq j$, and $x_i y_i + y_i x_i = 1$ and $x_i^2 = a_i$, $y_i^2 = a_i^{-1} b_i$ and $z_i^2 = c_i$. The even Clifford algebra of φ is generated by $u_1, v_1, \ldots, u_n, v_n, w_2, \ldots, w_s$ where $u_i = x_i z_0$, $v_i = y_i z_0$ and $w_i = z_i z_0$. The relations are the following: w_i commutes with all the other generators, u_i commutes with v_j for $i \neq j$, $u_i v_i + v_i u_i = z_0^2 = 1$ and $u_i^2 = a_i$, $v_i^2 = a_i^{-1} b_i$ and $w_j^2 = c_j$. Therefore, the even Clifford algebra of φ is

$$F\langle u_1, v_1\rangle \otimes F\langle u_2, v_2\rangle \otimes \cdots \otimes F\langle u_n, v_n\rangle \otimes F\langle w_1, \ldots, w_s\rangle,$$

which is indeed $[b_1, a_1)_F \otimes \cdots \otimes [b_n, a_n)_F \otimes K$. □

Proof of Corollary 1. Let φ be an anisotropic quadratic form of type $(1,3)$. Suppose that φ is a Pfister neighbor of a 3-fold Pfister form π. Modulo a scalar, we may write $\varphi = R \perp \langle 1, r, s\rangle$ for a suitable nonsingular quadratic form R of dimension 2 and $r, s \in F^*$. On the one hand, since π is isotropic over $F(\langle 1, r, s\rangle)$, it follows that π is also isotropic over $F(\langle\langle r, s\rangle\rangle)$, and thus $\pi \simeq \langle\langle r, s, u]]$ for some $u \in F^*$. On the other hand, the hyperbolicity of $\pi_{F(\varphi)}$ implies that $\pi \simeq R \perp [1,x] \perp r[1,y] \perp s[1,z]$ for some $x, y, z \in F^*$. Hence, we get

$$\langle\langle r, s, u]] \simeq R \perp [1,x] \perp r[1,y] \perp s[1,z].$$

Adding on both sides in the last isometry the form $\langle 1, r, s\rangle$, and canceling the hyperbolic planes, yields that $\varphi \simeq rs[1,u] \perp \langle 1, r, s\rangle$.

The fact that $C_0(\varphi)$ is split as an $F(\sqrt{r}, \sqrt{s})$-algebra is a direct consequence of Proposition 2. □

We finish this section with an example applying the criteria given in Theorem 3.

Example 2. Let r, s, u be indeterminates over a field F_0 of characteristic 2. Let us consider the forms $\varphi = rs[1,u] \perp \langle 1, r, s\rangle$ and $\tau = \big\langle 1, ru, s(r^2 + r + u)\big\rangle$ over the rational function field $F := F_0(r, s, u)$. We have the following statements:

(1) It is clear that φ is a Pfister neighbor of $\pi = \langle\langle r, s, u]]$. Moreover, τ is dominated by π because the scalars 1, ru and $s(r^2+r+u)$ are represented by the forms $[1, u]$, $r[1, u]$ and $s[1, u]$, respectively. Hence, $\varphi_{F(\tau)}$ is isotropic.

(2) For any $e \in F^*$, we have $i_d(e\tau \perp \mathrm{ql}(\varphi)) \leq 1$.

In fact, suppose that $i_d(e\tau \perp \mathrm{ql}(\varphi)) \geq 2$ for some $e \in F^*$. By [7, Proposition 3.2], there exists a totally singular quadratic form of dimension 2 which is dominated by $e\tau$ and $\mathrm{ql}(\varphi)$. Consequently, there exists an inseparable quadratic extension $K = F(\sqrt{d})$ such that $\langle\langle r, s\rangle\rangle_K$ and $\langle\langle ru, s(r^2+r+u)\rangle\rangle_K$ are isotropic, and thus quasi-hyperbolic. This implies that $\langle\langle r, s\rangle\rangle \simeq \langle\langle d, k\rangle\rangle$ and $\langle\langle ru, s(r^2+r+u)\rangle\rangle \simeq \langle\langle d, l\rangle\rangle$ for suitable $k, l \in F^*$. Hence, $\langle\langle r, s\rangle\rangle \perp \langle\langle ru, s(r^2+r+u)\rangle\rangle$ has defect index ≥ 2. In particular, $\theta := \langle\langle r, s\rangle\rangle \perp \langle ru, s(r^2+r+u), rus(r^2+r+u)\rangle$ is isotropic. But, using the classical isometry $\langle a, b\rangle \simeq \langle a, a+b\rangle$ for any $a, b \in F$, we get

$$\begin{aligned}
\theta &= \langle 1, r, s, rs\rangle \perp \langle ru, r^2s + rs + su, rsu(r^2) + su(r^2) + rs(u^2)\rangle \\
&\simeq \langle 1, r, s, rs\rangle \perp \langle ru, su, rsu(r^2) + su(r^2) + rs(u^2)\rangle \\
&\simeq \langle 1, r, s, rs\rangle \perp \langle ru, su, rsu(r^2)\rangle \\
&\simeq \langle 1, r, s, rs\rangle \perp \langle ru, su, rsu\rangle \\
&= \langle 1\rangle \perp \langle 1, u\rangle \cdot \langle r, s, rs\rangle,
\end{aligned}$$

which shows that θ is anisotropic, a contradiction. Hence, $i_d(e\tau \perp \mathrm{ql}(\varphi)) \leq 1$, and thus Theorem 3 implies that φ is $F(\tau)$-minimal. □

It would be interesting to see if there exists an example of an anisotropic quadratic form of type $(1, 3)$ which is minimal over the function field of a degenerate conic and satisfies condition (iii)(b) of Theorem 3.

Proof of Theorem 4. Let φ be an anisotropic F-quadratic form of type $(1, 3)$, and $\tau = a[1, b] \perp \langle 1\rangle$ an anisotropic quadratic form of dimension 3 and type $(1, 1)$.

– Suppose that φ is $F(\tau)$-minimal. It follows from Corollary 5 that φ is a Pfister neighbor of a 3-fold Pfister form $\pi = \langle\langle c, a, b]]$ for some $c \in F^*$. Since π is split by $K = F[\wp^{-1}(b)]$ and φ is its neighbor, the form φ_K is isotropic, which means that $e[1, b] \subset \varphi$ for some $e \in F^*$. Hence, $\varphi \simeq e[1, b] \perp \mathrm{ql}(\varphi)$. Suppose that $e \in D_F(a[1, b]) \cdot D_F(\mathrm{ql}(\varphi))$. Let $x \in D_F([1, b])$ and $y \in D_F(\mathrm{ql}(\varphi))$ be such that $e = axy$. Since $[1, b] \simeq x[1, b]$, it follows that $e[1, b] \simeq axy[1, b] \simeq ay[1, b]$, and thus $y\tau$ is dominated by φ, a contradiction.

– Conversely, suppose that τ satisfies the conditions (i), (ii) and (iii) as described in the theorem. The conditions (i) and (ii) imply that $\varphi_{F(\tau)}$ is isotropic. Suppose that φ is not $F(\tau)$-minimal. Hence, there exists a form ψ of dimension 3 or 4 dominated by φ and isotropic over $F(\tau)$. The form ψ is of type $(1, 1)$ or $(1, 2)$. We use [13, théorème 1.4 (bis) (2)] when $\dim \psi = 3$, and [13, théorème 1.4(2)] when $\dim \psi = 4$ to conclude that τ is weakly dominated by ψ. Hence, there exist $e, f, g \in F^*$ such that $\varphi \simeq ea[1, b] \perp \langle e, f, g\rangle$. This contradicts (iii) because $ea \in D_F(a[1, b]) \cdot D_F(\mathrm{ql}(\varphi))$. □

To give an example of a minimal quadratic form that applies Theorem 4, we need a few notions on the specialization theory due to Knebusch. Let $\lambda : K \longrightarrow L \cup \{\infty\}$ be a place between two fields K and L. Let $\mathbb{O}$ be the valuation ring of λ and $\mathcal{M}$ its maximal ideal. Recall that $\mathbb{O} = \{x \in K \mid \lambda(x) \neq \infty\}$ and $\mathcal{M} = \{x \in \mathbb{O} \mid \lambda(x) = 0\}$. Let μ be the restriction of λ to $\mathbb{O}$ and $k = \mathbb{O}/\mathcal{M}$ the residue field of λ (Note that k can be seen as a subfield of L). We say that a quadratic form φ over K has *nearly good reduction* with respect to λ if there exists a quadratic module ψ over $\mathbb{O}$ such that $\varphi \simeq \psi_K$ and the quasilinear part of the quadratic form ψ_k is anisotropic, where ψ_k is the quadratic form induced by the ring homomorphism $\mathbb{O} \longrightarrow k$. The specialization of φ with respect to λ, denoted by $\lambda_*(\varphi)$, is the L-quadratic form $\mu_*(\psi)$ induced by μ. We refer to [11] for more details.

Example 3. Let $F = \mathbb{F}_2(r, s, u)$ be the rational function field in the indeterminates r, s, u over the field $\mathbb{F}_2$ with two elements. Let $\varphi = rs[1, u] \perp \langle 1, r, s\rangle$, $\pi = \langle\langle r, s, u]]$ and $\tau = su[1, r+u] \perp \langle 1\rangle$. It is clear that φ is a Pfister neighbor of π. Moreover, $\pi \simeq [1, r+u] \perp [r, ur^{-1}+1] \perp s[1, u] \perp rs[1, u]$, and thus $su\tau \prec \pi$. Hence, φ is isotropic over $F(\tau)$. Suppose that φ is not $F(\tau)$-minimal. Then, there exists by Theorem 4 a scalar $e \in F^*$ such that $e[1, r+u] \subset \varphi$ and $e \in D_F(su[1, r+u]) \cdot D_F(\langle 1, r, s\rangle)$. Hence, using the roundness of $[1, r+u]$, we get $\varphi \simeq sut[1, r+u] \perp \langle 1, r, s\rangle$ for a suitable $t \in D_F(\langle 1, r, s\rangle)$. Without loss of generality, we may suppose $t \in \mathbb{F}_2(r, s)[u]$ square free with respect to the indeterminate u. Let $M = \mathbb{F}_2(r, s)$ and consider the M-place λ from F to M with respect to the u-adic valuation of F. We have:

(1) t is a unit for the u-adic valuation because $t \in D_F(\langle 1, r, s\rangle)$.

(2) the form φ has nearly good reduction with respect to λ because it is isometric to $\varphi = rs[1, u] \perp \langle 1, r, s\rangle$ and $\langle 1, r, s\rangle_M$ is anisotropic.

On the one hand, the total index of $\lambda_*(\varphi)$ is equal to 1 because $\lambda_*(rs[1, u]) = [0, 0]$ and $\langle 1, r, s\rangle_M$ is anisotropic. On the other hand, since φ contains $sut[1, r+u]$ as a subform, and $\lambda(\alpha) = 0$ or ∞ for every α represented by $sut[1, r+u]$, we conclude by [11, Proposition 3.4] that the total index of $\lambda_*(\varphi)$ is at least 2, a contradiction.

5. (Quasi-)Pfister neighbor forms

Our aim in this section is to relate the notions of quasi-Pfister neighbor and bilinear (strong)Pfister neighbor. This is useful to classify $F(\tau)$-minimal bilinear forms of dimension 5 when τ is a totally singular quadratic form of dimension 3.

A *quasi-Pfister form* is a totally singular quadratic form $\tilde{B}$ for some bilinear Pfister form B. A totally singular quadratic form Q is called *quasi-Pfister neighbor* if there exists a quasi-Pfister form π such that $2\dim Q > \dim \pi$ and $aQ \subset \pi$ for some $a \in F^*$. In this case, the form π is unique, and for any field extension K/F the form Q_K is isotropic if and only if π_K is isotropic. Thus $Q_{F(\pi)}$ and $\pi_{F(Q)}$ are isotropic.

For any bilinear Pfister form $\langle\langle a_1, \cdots, a_n\rangle\rangle_b$, we write $\widetilde{B} = \langle\langle a_1, \cdots, a_n\rangle\rangle$.

The *norm field* of a nonzero totally singular quadratic form Q is the field $N_F(Q) := F^2(\alpha\beta \mid \alpha, \beta \in D_F(Q))$, where $D_F(Q)$ is the set of scalars in F^* represented by Q. The degree $[N_F(Q) : F^2]$ is called the *norm degree* of Q and it is denoted by $\operatorname{ndeg}_F(Q)$. Clearly, we have $\operatorname{ndeg}_F(Q) = 2^d$ for some integer $d \geq 1$ and $\operatorname{ndeg}_F(Q) \geq \dim Q$. See [7, Section 8] for details on the norm degree and applications.

Here is a characterization of quasi-Pfister neighbors using the norm degree.

Proposition 3 [7, Proposition 8.9(ii)]. *An anisotropic totally singular quadratic form Q is a quasi-Pfister neighbor if and only if* $2 \dim Q > \operatorname{ndeg}_F(Q)$.

The norm degree appears in the description of the Witt kernels for bilinear forms.

Proposition 4 [15, Theorem 1.2]. *Let B be an anisotropic F-bilinear form and Q an anisotropic totally singular form of norm degree 2^d. If B becomes metabolic over $F(Q)$, then* $\dim B$ *is divisible by* 2^d.

A bilinear form B is called a *Pfister neighbor* if $\tilde{B}$ is a quasi-Pfister neighbor. This definition does not imply that B is similar to a subform of a bilinear Pfister form whose dimension is less than twice the dimension of B. For example, over the rational functions field $F(t_1, t_2)$, the bilinear form $B = \langle 1, t_1, t_2, 1 + t_1t_2\rangle_b$ is a Pfister neighbor because $\tilde{B} \simeq \langle\langle t_1, t_2\rangle\rangle$, but B is not similar to a subform of a 2-fold bilinear Pfister form since its determinant is not trivial. See [15] for more on bilinear Pfister neighbors and their splitting properties.

A bilinear form B is called a *strong Pfister neighbor*, or SPN, if there exists a bilinear Pfister form ρ such that $2 \dim B > \dim \rho$ and $\alpha B \subset \rho$ for some $\alpha \in F^*$. In this case, the form ρ is unique. In fact, if B is an SPN of another bilinear Pfister form δ, then there exists $\beta \in F^*$ such that $\beta B \subset \delta$. Hence, $\dim \rho = \dim \delta$ and $i_W(\alpha\rho \perp \beta\delta) \geq \dim B > \frac{1}{2} \dim \rho$, which implies that $\rho \simeq \delta$ since the Witt index $i_W(\alpha\rho \perp \beta\delta)$ is always a power of 2 [17, théorème 3.7]. Obviously, if B is an SPN then it is a Pfister neighbor.

Recall from [10] the Kato isomorphism $e^n : I^nF/I^{n+1}F \longrightarrow \nu_F(n)$ given on generators by

$$e^n(\langle\langle a_1, \ldots, a_n\rangle\rangle_b + I^{n+1}F) = \frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n}.$$

The symbol length (or simply the length) of an element $\theta \in \nu_F(n)$ is the smallest number of n-logarithmic symbols $\frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n}$ needed to write it.

An Albert bilinear form is a 6-dimensional bilinear form whose determinant is trivial.

Lemma 1. *Let γ be an Albert bilinear form and $\tau \in BP_2F$ be such that $\gamma \perp \tau \in I^3F$. Then, γ is isotropic.*

Proof. (1) If τ is isotropic, then it is metabolic, and thus $\gamma \in I^3F$. By the Hauptsatz, the form γ is metabolic, in particular it is isotropic.

(2) If τ is anisotropic, then we get by the previous case that $\gamma_{F(\tau)}$ is metabolic. It follows from Proposition 4 that γ is isotropic because the norm degree of $\widetilde{\tau}$ is 4. □

We give a characterization of SPN of dimension 5. This looks like the characterization of 5-dimensional quadratic Pfister neighbors (due to Knebusch in characteristic not 2 [12, Page 10], and the second author in characteristic 2 [13, Proposition 3.2]).

Proposition 5. *Let B be an anisotropic F-bilinear form of dimension* 5. *The following statements are equivalent*:

(1) *B is an SPN.*

(2) *$B \simeq a\, \langle\langle b, c\rangle\rangle_b \perp \langle d\rangle_b$ for suitable $a, b, c, d \in F^*$.*

(3) *The invariant $e^2(B \perp \langle \det B\rangle_b + I^3F)$ has length* 1.

Proof. Let $d \in F^*$ be such that $\det B = d.F^{*2}$.

(1) $\Longrightarrow$ (2) Suppose that B is an SPN of $\pi \in BP_3F$. Then, we have $x\pi \simeq B \perp \langle y, z, yzd\rangle_b$ for suitable scalars $x, y, z \in F^*$. Hence, $B \perp \langle d\rangle_b \perp d\tau \in I^3F$, where $\tau = \langle\langle dy, dz\rangle\rangle_b$. We conclude by Lemma 1 that $B \perp \langle d\rangle_b$ is isotropic, and thus $B \simeq B' \perp \langle d\rangle_b$ for some bilinear form B' of dimension 4 and trivial determinant, as desired.

(2) $\Longrightarrow$ (3) Suppose that $B \simeq a\, \langle\langle b, c\rangle\rangle_b \perp \langle d\rangle_b$. Clearly, we have $e^2(B \perp \langle d\rangle_b + I^3F) = \frac{\mathrm{d}b}{b} \wedge \frac{\mathrm{d}c}{c}$, which is of length 1 because the anisotropy of $\langle\langle b, c\rangle\rangle_b$ implies that $\frac{\mathrm{d}b}{b} \wedge \frac{\mathrm{d}c}{c} \neq 0$.

(3) $\Longrightarrow$ (1) Suppose that $e^2(B \perp \langle d\rangle_b + I^3F)$ has length 1. Then, there exists an anisotropic 2-fold bilinear Pfister form τ such that $e^2(B \perp \langle d\rangle_b + I^3F) = e^2(\tau + I^3F)$. Hence, $B \perp \langle d\rangle \perp \tau \in I^3F$ using the isomorphism e^2. Consequently, $B \perp d\tau' \in I^3F$, where τ' is the pure part of τ. Then, $B \perp d\tau'$ is similar to a 3-fold bilinear Pfister form because it is of dimension 8, and thus B is an SPN. □

6. *K*-minimal bilinear forms up to dimension 5

Throughout this section we take $Q = \langle 1, a, b\rangle$ an anisotropic totally singular quadratic form over F of dimension 3, and $K = F(Q)$ its function field.

Lemma 2 [18, Lemma 3.7]. *Let B be an anisotropic bilinear form over F. If $\psi = \langle a_1, \ldots, a_n \rangle$ is a subform of $\tilde{B}$, then there exists a bilinear form C over F such that $C \subset B$ and $\tilde{C} \simeq \psi$. Explicitly, we can take $C = \langle b_1, \ldots, b_n \rangle_b$, where $b_i = a_i + \sum_{j=1}^{i-1} a_i x_i^2$ for suitable $x_1, \ldots, x_{i-1} \in F$ (read $b_1 = a_1$).*

Corollary 6. *Let B be an anisotropic F-bilinear form. Then, B is K-minimal if and only if $\tilde{B}$ is K-minimal.*

We recall the isotropy results that we need for the classification of K-minimal bilinear forms of dimension at most 5.

Theorem 7 [20, Proposition 1.1 and Theorem 1.2]. *Let B be an anisotropic F-bilinear form such that* $\dim B \leq 4$ *or* $\dim B = 5$ *and* $\mathrm{ndeg}_F(\tilde{B}) = 16$. *Then, B_K is isotropic if and only if Q is similar to a subform of $\tilde{B}$.*

Corollary 7. *Let B be an anisotropic F-bilinear form of dimension ≤ 5 such that B_K is isotropic. If B is K-minimal, then either* $\dim B = 3$, *or* $\dim B = 5$ *and* $\mathrm{ndeg}_F(\tilde{B}) = 8$.

Proof. Suppose that $\dim B \leq 5$ and B is K-minimal. Since B_K is isotropic, it follows that $\dim B \neq 2$.

– If $\dim B = 3$, then obviously B is K-minimal since any subform of B of dimension 2 is anisotropic over K.

– If $\dim B = 4$, then B_K is isotropic if and only if Q is similar to a subform of $\tilde{B}$ (Theorem 7). Hence, B is not K-minimal (Corollary 6).

– If $\dim B = 5$. In this case, $\mathrm{ndeg}_F(\tilde{B}) \in \{8, 16\}$. If $\mathrm{ndeg}_F(\tilde{B}) = 16$, then Q is similar to a subform of $\tilde{B}$ (Theorem 7). Hence, B is not K-minimal when $\mathrm{ndeg}_F(\tilde{B}) = 16$. □

Corollary 8. *Let B be an anisotropic F-bilinear form of dimension ≤ 5. If B is isotropic over K but not K-minimal, then Q is similar to a subform of $\tilde{B}$.*

Proof. Since B is not K-minimal, there exists C a subform of B such that $\dim C < \dim B$ and C_K is isotropic. It follows from Theorem 7 that Q is similar to a subform of $\tilde{C}$. In particular, Q is similar to a subform of $\tilde{B}$. □

Lemma 3. *Let $\pi_1 \in BP_mF$ and $\pi_2 \in BP_nF$ with $2 \leq m \leq n$. Suppose that π_1' is similar to a subform of π_2, where π_1' denotes the pure part of π_1. Then, $\pi_2 \simeq \pi_1 \otimes \tau$ for some $\tau \in BP_{n-m}F$.*

Proof. We have $i_W(\pi_1 \perp \alpha\pi_2) \geq 2^m - 1$ for some $\alpha \in F^*$. It follows from [17, théorème 3.7] that this Witt index is equal to $\dim \pi_1$, and the forms π_1 and π_2 are m-linked, which means that $\pi_2 \simeq \pi_1 \otimes \tau$ for some $\tau \in BP_{n-m}F$. □

7. Proof of Theorem 5

Let B be an anisotropic bilinear form of dimension 5, and $Q = \langle 1, a, b\rangle$ an anisotropic totally singular quadratic form of dimension 3. Let $K = F(Q)$ be the function field of Q.

(2) $\Longrightarrow$ (1) Suppose that there exists an F-bilinear form C of dimension 5 which is an SPN of a bilinear Pfister form $\rho := \langle\langle a, b, c\rangle\rangle_b$ and satisfies the two conditions:

(a) $\tilde{B} \simeq \tilde{C}$.

(b) For any $u, v \in F^2(a, b)$ such that $\langle u, v, uv\rangle_b$ is similar to a subform of ρ, the invariant $e^2(C \perp \langle \det C\rangle_b \perp \langle\langle u, v\rangle\rangle_b + I^3 F)$ has length 2.

Let $d \in F^*$ be such that $\det C = d.F^{*2}$. The form C_K is isotropic because ρ_K is isotropic and C is an SPN of ρ. In particular, B_K is isotropic.

Suppose that B is not K-minimal. Then, C is not K-minimal as well because $\tilde{B} \simeq \tilde{C}$ (Corollary 6). It follows from Corollary 8 that $Q = \langle 1, a, b\rangle$ is similar to a subform of $\tilde{C}$. By Lemma 2, we conclude that $p\left\langle 1, a + q^2, b + ar^2 + s^2\right\rangle_b$ is a subform of C for suitable $p \neq 0, q, r, s \in F$. In particular,

$$\left\langle a + q^2, b + ar^2 + s^2, (a + q^2)(b + ar^2 + s^2)\right\rangle_b$$

is similar to a subform of ρ, and thus our hypothesis (b) above implies that $e^2(C \perp \langle d\rangle_b \perp \left\langle\!\left\langle a + q^2, b + ar^2 + s^2\right\rangle\!\right\rangle_b + I^3 F)$ has length 2.

Let $u = a + q^2$, $v = b + ar^2 + s^2$. It is easy to see that $C \perp \langle d\rangle_b \sim p\,\langle\langle u, v\rangle\rangle_b \perp \tau$ for some $\tau \in GBP_2F$. Consequently, the invariant $e^2(C \perp \langle d\rangle \perp \langle\langle u, v\rangle\rangle_b + I^3 F)$ has length at most 1, a contradiction. Hence B is K-minimal.

(1) $\Longrightarrow$ (2) Suppose that B is K-minimal. Then, we get by Corollary 7 that $\operatorname{ndeg}_F(\tilde{B}) = 8$. It follows from Proposition 3 that $\tilde{B}$ is quasi-Pfister neighbor of a quasi-Pfister form π. Since $\tilde{B}_K$ is isotropic, it follows that π_K is quasi-hyperbolic. Hence, $\pi \simeq \langle\langle a, b, c\rangle\rangle$ for some $c \in F^*$ [14, Theorem 1.5]. There exists a bilinear form C of dimension 5 similar to a subform of $\rho := \langle\langle a, b, c\rangle\rangle_b$ such that $\tilde{B} \simeq \tilde{C}$ (Lemma 2). In particular, the form C is an SPN of ρ. Modulo a scalar, we may write $C \simeq \langle\langle x, y\rangle\rangle_b \perp \langle z\rangle_b$ for suitable $x, y, z \in F^*$ (Proposition 5).

Let $u, v \in F^2(a, b)$ be such that $\langle u, v, uv\rangle_b$ is similar to a subform of ρ. The form $\langle u, v, uv\rangle$ is anisotropic because ρ is anisotropic. On the one hand, the condition $u, v \in F^2(a, b)$ implies that $\langle 1, u, v\rangle$ becomes isotropic over K, which gives by Theorem 7 that Q is similar to $\langle 1, u, v\rangle$ and thus $K = F(\langle 1, u, v\rangle)$. On the other hand, using Lemma 3, we get $\rho \simeq \langle\langle u, v, w\rangle\rangle_b$ for some $w \in F^*$. Hence, without loss of generality, we may suppose $\langle\langle a, b\rangle\rangle_b \simeq \langle\langle u, v\rangle\rangle_b$ for the rest of the proof.

We have $e^2(C \perp \langle z\rangle_d \perp \langle\langle a, b\rangle\rangle_b + I^3 F) = \frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y} + \frac{\mathrm{d}a}{a} \wedge \frac{\mathrm{d}b}{b}$. Suppose that this invariant has length ≤ 1. Then, there exists a 2-fold bilinear Pfister form τ such that $e^2(C \perp \langle z\rangle_d \perp \langle\langle a, b\rangle\rangle_b + I^3 F) = e^2(\tau + I^3 F)$, which implies that

$\langle\langle x, y\rangle\rangle_b \perp \langle\langle a, b\rangle\rangle_b \perp \tau \in I_q^3 F$. It follows from Lemma 1 that the Albert form $\langle a, b, ab, x, y, xy\rangle_b$ is isotropic. Hence, the forms $\langle\langle x, y\rangle\rangle_b$ and $\langle\langle a, b\rangle\rangle_b$ are 1-linked, meaning that $\langle\langle x, y\rangle\rangle_b \simeq \langle\langle e, r\rangle\rangle_b$ and $\langle\langle a, b\rangle\rangle_b \simeq \langle\langle f, r\rangle\rangle_b$ for suitable $e, f, r \in F^*$. By the uniqueness of the pure part of bilinear Pfister forms, we get $\langle a, b, ab\rangle_b \simeq \langle f, r, fr\rangle_b$, and thus $K = F(\langle 1, f, r\rangle)$. Hence, without loss of generality, we may keep $\langle 1, a, b\rangle$ instead of $\langle 1, r, f\rangle$, and thus suppose that $C \simeq \langle\langle e, b\rangle\rangle_b \perp \langle z\rangle_b$. So the form C is an SPN of $\langle\langle e, b, z\rangle\rangle_b$. But C is also an SPN of $\langle\langle a, b, c\rangle\rangle_b$, it follows that

$$\langle\langle e, b, z\rangle\rangle_b \simeq \langle\langle a, b, c\rangle\rangle_b\,. \tag{8}$$

We continue with some arguments similar to those used by Faivre in his proof. Adding on both sides of (8) the form $\langle 1, b\rangle_b$, we get

$$\mathbb{M}(1) \perp \mathbb{M}(b) \perp \langle z, e, ez\rangle_b \otimes \langle 1, b\rangle_b \simeq \mathbb{M}(1) \perp \mathbb{M}(b) \perp \langle c, a, ac\rangle_b \otimes \langle 1, b\rangle_b\,.$$

By the uniqueness of the anisotropic part, we get

$$\langle z, e, ez\rangle_b \otimes \langle 1, b\rangle_b \simeq \langle c, a, ac\rangle_b \otimes \langle 1, b\rangle_b\,.$$

Adding on both sides $a\,\langle 1, b\rangle_b$, we get

$$a\,\langle\langle ea, b\rangle\rangle_b \perp z\,\langle\langle e, b\rangle\rangle_b \simeq \mathbb{M}(a) \perp \mathbb{M}(ab) \perp c\,\langle\langle a, b\rangle\rangle_b\,.$$

Thus, $a\,\langle\langle ea, b\rangle\rangle_b \perp z\,\langle\langle e, b\rangle\rangle_b$ is isotropic, and thus there exist $r \in D_F(\langle\langle ea, b\rangle\rangle_b)$ and $s \in D_F(\langle\langle e, b\rangle\rangle_b)$ such that $ar = zs$. We have

$$C \simeq \langle\langle e, b\rangle\rangle_b \perp \langle z\rangle_b \simeq s\,\langle\langle e, b\rangle\rangle_b \perp \langle z\rangle_b \simeq s\,\langle 1, b\rangle_b \perp D,$$

where $D = es\,\langle 1, b\rangle_b \perp \langle z\rangle_b$. Let $\beta := as\,\langle 1, b\rangle_b \perp D$. Then, we have

$$\beta \simeq as\,\langle\langle ea, b\rangle\rangle_b \perp \langle z\rangle_b \simeq ars\,\langle\langle ea, b\rangle\rangle_b \perp \langle z\rangle_b \simeq z\,\langle\langle ea, b\rangle\rangle_b \perp \langle z\rangle_b\,.$$

Hence, $\beta \simeq \mathbb{M}(z) \perp \widetilde{\beta} \perp \langle zb\rangle_b$, where $\widetilde{\beta} = zea\,\langle 1, b\rangle_b$. Now, we have

$$\begin{aligned} bC \perp \widetilde{\beta} &\sim s\,\langle\langle e, b\rangle\rangle_b \perp \beta \\ &\sim s\,\langle\langle e, b\rangle\rangle_b \perp as\,\langle 1, b\rangle_b \perp es\,\langle 1, b\rangle_b \perp \langle z\rangle_b \\ &\sim s\,\langle\langle a, b\rangle\rangle_b \perp \langle z\rangle_b\,. \end{aligned}$$

This shows that $bC \perp \widetilde{\beta}$ is isotropic. Then, there exist bilinear forms C_1 and C_2 of dimension 4 and 1, respectively, such that $C_1 \subset bC$, $C_2 \subset \widetilde{\beta}$ and $C_1 \perp C_2 \simeq s\,\langle\langle a, b\rangle\rangle_b \perp \langle z\rangle_b$. Then, $i_W((C_1 \perp C_2)_K) = 2$ and thus $(C_1)_K$ is isotropic, meaning that C is not K-minimal. Since $\tilde{C} \simeq \tilde{B}$, it follows that B is not K-minimal, a contradiction. □

Using Theorem 3, we provide an example of a K-minimal bilinear form of dimension 5, where K is the function field of a degenerate conic. The form we choose in our example is inspired by [2, Proposition 4.1].

Example 4. Let F_0 be a field of characteristic 2, and $k = F_0(a, b, c)$ the rational function field in the indeterminates a, b, c over F_0. Let $B = c\langle 1, a+b\rangle_b \perp b\langle 1, a\rangle_b \perp \langle 1\rangle_b$ and $Q = \langle 1, a, c\rangle$. Then, B is $k(Q)$-minimal.

Proof. Using the isometry $\langle\langle a, b\rangle\rangle_b \simeq \langle\langle ab, a+b\rangle\rangle_b$, we get

$$\begin{aligned} B \perp \langle a\rangle_b \perp abc\langle 1, a+b\rangle_b &= c\langle\langle ab, a+b\rangle\rangle_b \perp \langle\langle a, b\rangle\rangle_b \\ &\simeq c\langle\langle ab, a+b\rangle\rangle_b \perp \langle\langle ab, a+b\rangle\rangle_b \\ &= \langle\langle c, ab, a+b\rangle\rangle_b \\ &\simeq \langle\langle a, b, c\rangle\rangle_b\,. \end{aligned}$$

Hence, B is an SPN of $\langle\langle a, b, c\rangle\rangle_b$, and thus B_K is isotropic. Moreover, B is anisotropic over k since $\langle\langle a, b, c\rangle\rangle_b$ is also anisotropic.

Let $u, v \in k^2(a, c)$ be such that $\langle u, v, uv\rangle_b$ is a subform of $\langle\langle a, b, c\rangle\rangle_b$. We have

$$e^2\left(B \perp \langle \det B\rangle_b \perp \langle\langle u, v\rangle\rangle_b + I^3F\right) = \frac{\mathrm{d}(a+b)}{a+b} \wedge \frac{\mathrm{d}(bc)}{bc} + \frac{\mathrm{d}u}{u} \wedge \frac{\mathrm{d}v}{v}.$$

Suppose that this invariant is of length ≤ 1. Then, the Albert form

$$\langle u, v, uv, a+b, bc, (a+b)bc\rangle_b$$

is isotropic over k. Since the forms $\langle u, v, uv\rangle_b$ and $\langle a+b, bc, (a+b)bc\rangle_b$ are anisotropic over k, there exists $\alpha \in k^*$ represented by both forms. Let us write

$$\begin{aligned} \alpha &= uL^2 + vM^2 + uvN^2 \\ &= (a+b)S^2 + bcT^2 + (a+b)bcU^2 \end{aligned}$$

for suitable $L, M, N, S, T, U \in k$. Hence, we have

$$b(S^2 + cT^2 + acU^2) = uL^2 + vM^2 + uvN^2 + aS^2 + c(bU)^2.$$

The right hand side in this equality and the factor $S^2 + cT^2 + acU^2$ belong to $k^2(a, c)$, but since $b \notin k^2(a, c)$, we necessarily have $S^2 + cT^2 + acU^2 = 0$. Since $\langle 1, c, ac\rangle$ is anisotropic over k, it follows that $S = T = U = 0$ and thus $\alpha = 0$, a contradiction.

Consequently, $e^2(B \perp \langle \det B\rangle_b \perp \langle\langle u, v\rangle\rangle_b + I^3F)$ is of length 2. Since all the conditions of Theorem 5 are satisfied (taking for C the form B itself), we conclude that B is $k(Q)$-minimal. □

Acknowledgement

This work was done in the framework of the project "IEA of CNRS" between the Artois University and the Academic College of Tel-Aviv-Yaffo. The two authors are grateful for the support of both institutions and CNRS.

References

[1] R. Baeza, *Quadratic forms over semilocal rings*, Lecture Notes in Mathematics **655**, Springer, 1978. MR

[2] A. Chapman and A. Quéguiner-Mathieu, "Minimal quadratic forms for the function field of a conic in characteristic 2", pp. 117–127 in *Amitsur Centennial Symposium*, Contemp. Math. **800**, Amer. Math. Soc., [2024] ©2024. MR

[3] A. Dolphin and A. Laghribi, "Isotropy of 5-dimensional quadratic forms over the function field of a quadric in characteristic 2", *Comm. Algebra* **45**:7 (2017), 3034–3044. MR

[4] A. Dolphin and A. Quéguiner-Mathieu, "Symplectic involutions, quadratic pairs and function fields of conics", *J. Pure Appl. Algebra* **221**:8 (2017), 1966–1980. MR

[5] R. Elman, N. Karpenko, and A. Merkurjev, *The algebraic and geometric theory of quadratic forms*, AMS Colloquium Publications **56**, American Mathematical Society, 2008. MR

[6] F. Faivre, *Liaison des formes de Pfister et corps de fonctions de quadriques en caractéristique* 2, PhD thesis, Université de Franche-Comté, 2006.

[7] D. W. Hoffmann and A. Laghribi, "Quadratic forms and Pfister neighbors in characteristic 2", *Trans. Amer. Math. Soc.* **356**:10 (2004), 4019–4053. MR

[8] D. W. Hoffmann and A. Laghribi, "Isotropy of quadratic forms over the function field of a quadric in characteristic 2", *J. Algebra* **295**:2 (2006), 362–386. MR

[9] D. W. Hoffmann, D. W. Lewis, and J. Van Geel, "Minimal forms for function fields of conics", pp. 227–237 in *K-theory and algebraic geometry: connections with quadratic forms and division algebras* (Santa Barbara, CA, 1992), vol. 2, Proc. Sympos. Pure Math. **58**, Amer. Math. Soc., 1995. MR

[10] K. Kato, "Symmetric bilinear forms, quadratic forms and Milnor K-theory in characteristic two", *Invent. Math.* **66**:3 (1982), 493–510. MR

[11] M. Knebusch, "Specialization of quadratic and symmetric bilinear forms, and a norm theorem", *Acta Arith.* **24** (1973), 279–299. MR

[12] M. Knebusch, "Generic splitting of quadratic forms, II", *Proc. London Math. Soc.* (3) **34**:1 (1977), 1–31. MR

[13] A. Laghribi, "Certaines formes quadratiques de dimension au plus 6 et corps des fonctions en caractéristique 2", *Israel J. Math.* **129** (2002), 317–361. MR

[14] A. Laghribi, "Quasi-hyperbolicity of totally singular quadratic forms", pp. 237–248 in *Algebraic and arithmetic theory of quadratic forms*, Contemp. Math. **344**, Amer. Math. Soc., 2004. MR

[15] A. Laghribi, "Witt kernels of function field extensions in characteristic 2", *J. Pure Appl. Algebra* **199**:1-3 (2005), 167–182. MR

[16] A. Laghribi, "Sur le déploiement des formes bilinéaires en caractéristique 2", *Pacific J. Math.* **232**:1 (2007), 207–232. MR

[17] A. Laghribi, "Les formes bilinéaires et quadratiques bonnes de hauteur 2 en caractéristique 2", *Math. Z.* **269**:3-4 (2011), 671–685. MR

[18] A. Laghribi, "Isotropie d'une forme bilinéaire d'Albert sur le corps de fonctions d'une quadrique en caractéristique 2", *J. Algebra* **355** (2012), 1–8. MR

[19] A. Laghribi and D. Mukhija, "The excellence of function fields of conics", *Proc. Amer. Math. Soc.* **152**:5 (2024), 1915–1923. MR

[20] A. Laghribi and U. Rehmann, "Bilinear forms of dimension less than or equal to 5 and function fields of quadrics in characteristic 2", *Math. Nachr.* **286**:11-12 (2013), 1180–1190. MR

[21] P. Mammone, J.-P. Tignol, and A. Wadsworth, "Fields of characteristic 2 with prescribed u-invariants", *Math. Ann.* **290**:1 (1991), 109–128. MR

Received April 9, 2025. Revised August 28, 2025.

ADAM CHAPMAN
SCHOOL OF COMPUTER SCIENCE
ACADEMIC COLLEGE OF TEL-AVIV-YAFFO
6818211 TEL-AVIV-YAFFO
ISRAEL
adam1chapman@yahoo.com

AHMED LAGHRIBI
LABORATOIRE DE MATHÉMATIQUES DE LENS UR2462
UNIVERSITÉ D'ARTOIS
62307 LENS
FRANCE
ahmed.laghribi@univ-artois.fr

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2025 is US $677/year for the electronic version, and $917/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 339 No. 2 December 2025