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We establish precise lower bounds for the eigenvalues and critical values associated with the fractional A-Laplacian operator, where A is a Young function. The obtained bounds are expressed in terms of the domain geometry and the growth properties of the function A. We do not assume that A or its complementary function satisfies the Δ_2 condition.

1. Introduction

One of the central problems in the analysis of p-Laplacian type operators is the study of its eigenvalues, which are closely related to the structure of the underlying domain and the boundary conditions imposed. In particular, the first eigenvalue

$$\lambda_1 = \inf\left\{\int_\Omega |\nabla u|^p\,dx \text{ for } u \in C_c^\infty(\Omega) \text{ such that } \int_\Omega \omega(x)|u|^p\,dx = 1\right\}$$

related to the nonlinear problem defined for $p > 1$ as

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda\omega|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

has been extensively studied, as it provides important information about the behavior of solutions to the geometry of the domain. Here ω is a suitable weight function and $\Omega \subset \mathbb{R}^n$ denotes an open and bounded set. See for instance [29; 30].

While upper bounds for eigenvalues have been established in a variety of settings, obtaining sharp lower bounds remains a challenging and active area of research. Lower bounds are of particular importance because they offer insights into the stability and regularity of solutions, as well as estimates for the oscillatory behavior of eigenfunctions.

In the one-dimensional case with a weight function, lower bounds were obtained in [17; 28; 34; 35; 40]. When $\Omega \subset \mathbb{R}^n$, $n \geq 2$ several results are known. In [6; 16], lower bounds in terms of the measure of the domain were obtained. Indeed, when $\omega \in L^\theta(\Omega)$ then we have

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$$\frac{C}{|\Omega|^{\frac{p}{n}-\frac{1}{\theta}}\|\omega\|_{L^\theta(\Omega)}} \le \lambda_1 \quad \text{where } \theta = \begin{cases} \theta_0 \in \left(\frac{n}{p}, \infty\right] & \text{when } 1 < p < n \\ 1 & \text{when } n < p, \end{cases}$$

where $C = C(n, p) > 0$. In addition, in [33; 35], more accurate bounds involving the inner radius r_Ω of Ω are obtained as a consequence of Lyapunov type inequalities:

$$\text{(1-2)} \qquad \frac{C}{r_\Omega^{p-\frac{n}{\theta}}\|\omega\|_{L^\theta(\Omega)}} \le \lambda_1 \quad \text{where } \theta = \begin{cases} \theta_0 \in \left(\frac{n}{p}, \infty\right] & \text{when } 1 < p < n, \\ 1 & \text{when } n < p, \end{cases}$$

where $C = C(n, p) > 0$ and $r_\Omega := \max\{\operatorname{dist}(x, \partial\Omega) : x \in \Omega\}$ is the inradius of Ω. In [26] these results were extended to the nonlocal case, obtaining that when $\omega \in L^\theta(\Omega)$

$$\text{(1-3)} \qquad \frac{C}{r_\Omega^{sp-\frac{n}{\theta}}\|\omega\|_{L^\theta(\Omega)}} \le \lambda_1^s \quad \text{where } \theta = \begin{cases} \theta_0 \in \left(\frac{n}{sp}, \infty\right] & \text{when } 1 < sp < n, \\ 1 & \text{when } n < sp, \end{cases}$$

where λ_1^s is the first eigenvalue related to the fractional p-Laplacian operator of order $s \in (0, 1)$.

When operators follow a growth more general than a power law, the concept of eigenvalue becomes highly dependent on the normalization of the eigenfunction due to the potential lack of homogeneity. More precisely, equation (1-1) can be generalized by replacing the power p with a so-called Young function: given a Young function A, and a bounded domain $\Omega \subset \mathbb{R}^n$, $n \ge 1$, consider the problem

$$\text{(1-4)} \qquad \begin{cases} -\operatorname{div}\Big(a(|\nabla u|)\dfrac{\nabla u}{|\nabla u|}\Big) = \lambda\omega a(|u|)\dfrac{u}{|u|} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda \in \mathbb{R}$ is the eigenvalue parameter, ω is a suitable weight function and $a(t) = A'(t)$, $t > 0$. Observe that (1-4) boils down to (1-1) when $A(t) = t^p$, $p > 1$.

In this case, in [23; 25] it is proved that given $\alpha > 0$ there exists a critical value $\lambda_{1,\alpha} > 0$ and a function u_α such that $\int_\Omega \omega A(|u_\alpha|)\,dx = \alpha$ and $\frac{1}{\alpha}\int_\Omega A(|\nabla u_\alpha|)\,dx = \lambda_{1,\alpha}$. From this, one can deduce the existence of an eigenvalue $\Lambda_{1,\alpha}$ with corresponding eigenfunction u_α in the sense that pair $(\Lambda_{1,\alpha}, u_\alpha)$ satisfies (1-4) in the weak sense. The quantities Λ_1 and λ_1 are in general different and coincide only when A is homogeneous.

A first result concerning the lower bounds of (1-4) can be found in [32]. In the one-dimensional case, assuming that A satisfies the Δ_2 condition (that is, there exists $c \ge 1$ such that $A(2t) \le cA(t)$ for any $t > 0$), the authors establish that for any $\alpha > 0$,

$$\frac{C_p}{\|\omega\|_{L^1(a,b)}} \le \lambda_{1,\alpha}$$

where $\Omega = (a, b) \subset \mathbb{R}$, and $p > 1$ is defined as $\lim_{r\to\infty} A(rt)/A(r) = t^{p-1}$. Similar bounds in the one-dimensional case were found in [39] when A is a submultiplicative

Young function (that is, there exists $c \geq 1$ such that $A(rt) \leq cA(r)A(t)$ for any $r, t \geq 0$).

When $\Omega \subset \mathbb{R}^n$, $n \geq 1$ and A is a submultiplicative Young function, in Theorems 4.4 and 4.2 of [37] it is proved that given $\alpha > 0$ there exists a computable constant $C > 0$, independent of α and depending only on A and n, such that

$$\frac{1}{\alpha}\left[A\left(\frac{C\sigma(r_\Omega)}{A^{-1}\big(\|\omega\|^{-1}_{L^1(\Omega)}\big)}\right)\right]^{-1} \leq \lambda_{1,\alpha} \quad \text{when } \omega \in L^1(\Omega) \text{ and } \sigma(1) < \infty,$$

$$\frac{1}{\alpha}\left[A\left(\frac{C}{A^{-1}\big(\|\omega\|^{-1}_{L^\infty(\Omega)}/\tau_A(\Omega)\big)}\right)\right]^{-1} \leq \lambda_{1,\alpha} \quad \text{when } \omega \in L^\infty(\Omega) \text{ and } \sigma(1) = \infty,$$

where

$$\sigma(t) = \int_{t^{-n}}^{\infty} A^{-1}(r) r^{-(1+\frac{1}{n})}\, dr \quad \text{and} \quad \tau_A(\Omega) := |\Omega|(\tilde{A})^{-1}(|\Omega|^{-1}),$$

$\tilde{A}$ being the complementary function of A. These inequalities, in the case $A(t) = t^p$, $p > 1$, recover the corresponding inequalities in (1-2).

In the last years nonlocal operators with nonstandard growth have received an increasing amount of attention and an active community is currently working on problems involving operators defined in terms of a Young function $A(t) = \int_0^t a(\tau)\, d\tau$ having the form

$$(-\Delta_a)^s u(x) = \text{p.v.} \int_{\mathbb{R}^n} a(|D^s u|) \frac{D^s u}{|D^s u|} \frac{dy}{|x-y|^n},$$

where $s \in (0, 1)$, $D^s u(x, y) = (u(x) - u(y))/|x - y|^s$ and p.v. stands for "principal value". This nonhomogeneous operator is a generalization of the fractional p-Laplacian of order $s \in (0, 1)$. See also [1; 2; 3; 4; 8; 12; 13; 14; 19; 20; 18; 21; 22; 36; 37; 38]. In particular, the nonlocal version of problem (1-4) takes the form

$$\begin{cases} (-\Delta_a)^s u = \lambda\omega \dfrac{a(|u|)}{|u|} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1-5}$$

where $\lambda \in \mathbb{R}$ is the eigenvalue parameter and ω is a suitable weight function. We refer to [7; 12; 15; 19; 22; 36; 37; 38] for properties and results related with the nonlocal nonstandard growth eigenvalue problem (1-5).

As in the local case, the nonhomogeneity of the problem makes the eigenvalue highly dependent on the normalization of the eigenfunction. More precisely, in [36; 38] it is proved that given $\alpha > 0$ there exists a critical value $\lambda^s_{1,\alpha} > 0$ and u^s_α such that $\int_\Omega \omega A(|u^s_\alpha|)\, dx = \alpha$ and $\frac{1}{\alpha}\int_\Omega A(|D^s u^s_\alpha|)\, dx = \lambda^s_{1,\alpha}$. More precisely, we consider

$$\lambda^s_{1,\alpha} = \inf\left\{\frac{1}{\alpha}\iint_{\mathbb{R}^{2n}} A(|D^s u|)\,d\nu_n : u \in C_c^\infty(\Omega),\ \int_\Omega \omega A(|u|)\,dx = \alpha\right\}, \tag{1-6}$$

where $d\nu_n = |x-y|^{-n}\,dx\,dy$ and ω is a suitable weight function.

From this can be deduced the existence of an eigenvalue $\Lambda^s_{1,\alpha}$ with corresponding eigenfunction u^s_α in the sense that pair $(\Lambda^s_{1,\alpha}, u^s_\alpha)$ satisfies (1-5) in the weak sense, being Λ^s_1 and λ^s_1 different when A is inhomogeneous, but comparable each other when A satisfies the doubling condition.

To the best of our knowledge, estimates for $\Lambda^s_{1,\alpha}$ and $\lambda^s_{1,\alpha}$ have not been previously studied in the literature. The goal of this article is to establish lower bounds for these quantities.

An important aspect of analyzing (1-5) is whether the Young function A satisfies a so-called *doubling condition*. This condition is crucial for controlling constants within the function:

- A satisfies the *doubling condition near infinity* (denoted as $A \in \Delta_2^\infty$) if there exists $C_\infty \geq 2$ such that $A(2t) \leq C_\infty A(t)$ for all $t \geq T_\infty$,
- A satisfies the *doubling condition near zero* (denoted as $A \in \Delta_2^0$) if there exists $C_0 \geq 2$ such that $A(2t) \leq C_0 A(t)$ for all $t \leq T_0$.
- A satisfies the *global doubling condition* (denoted as $A \in \Delta_2$) if the previous condition and fulfilled, and it is denoted $\Delta_2 = \Delta_2^0 \cap \Delta_2^\infty$.

Assuming or relaxing the doubling condition introduces significant technical challenges in the analysis, such as the potential loss of reflexivity in the associated fractional Orlicz–Sobolev spaces. Moreover, imposing this condition on either the function A or its conjugate $\tilde{A}$ is known to yield both upper and lower bounds for the corresponding Young function in terms of power functions. For further details, see Section 2.1.

To characterize the growth of a general Young function A (which may not satisfy the doubling condition), we use the Matuszewska–Orlicz functions associated with A, along with the corresponding indexes, defined as follows:

$$M_A(t) = \sup_{\alpha>0}\frac{A(\alpha t)}{A(\alpha)}, \quad M_0(t,A) = \liminf_{\alpha\to 0^+}\frac{A(\alpha t)}{A(\alpha)}, \quad M_\infty(t,A) = \liminf_{\alpha\to\infty}\frac{A(\alpha t)}{A(\alpha)},$$

$$i(A) = \lim_{t\to\infty}\frac{\log M_A(t)}{\log t}, \quad i_0(A) = \lim_{t\to\infty}\frac{\log M_0(t,A)}{\log t}, \quad i_\infty(A) = \lim_{t\to\infty}\frac{\log M_\infty(t,A)}{\log t}.$$

See Section 2.2 for details and precise definitions.

Main results. We emphasize that, unless explicitly stated otherwise, we do not assume the Δ_2 condition on A or its complementary function $\tilde{A}$.

Theorem 1.1. *Let $s \in (0,1)$ and let A be a Young function satisfying (2-4). Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with inner radius r_Ω. Given $\omega \in L^1(\mathbb{R}^n)$ and $\alpha > 0$, consider the critical value $\lambda^s_{1,\alpha}$ defined in (1-6).*

(i) *There exists a unique $\alpha_0 > 0$ satisfying the equation*

$$\alpha_0 \lambda^s_{1,\alpha_0} = r^n_\Omega.$$

(ii) *Assume that $i_0(A) > \frac{n}{s}$ when $\alpha \le \alpha_0$ or $i_\infty(A) > \frac{n}{s}$ when $\alpha > \alpha_0$. Then, there exists a positive, computable constant $C = C(n, s, A)$ such that*

$$\frac{C}{\|\omega\|_{L^1(\Omega)}} \frac{r^n_\Omega}{M_A(r^s_\Omega)} \le \lambda^s_{1,\alpha}.$$

In particular, this holds when $i_0(A) > \frac{n}{s}$ if $\alpha \ll 1$ or when $i_\infty(A) > \frac{n}{s}$ if $\alpha \gg 1$.

In Section 2.3, we compute the Matuszewska–Orlicz functions and indices for several notable Young functions. With this, we state Theorem 1.1 for some interesting cases:

(i) When $A(t) = t^p$, $p > 1$ the eigenvalue problem becomes homogeneous. Then, for any $\alpha > 0$, when $sp > n$, we have

$$\frac{1}{\|\omega\|_{L^1(\Omega)}} \frac{C}{r^{sp-n}_\Omega} \le \lambda^s_{1,\alpha}$$

which in some extent recovers (1-3).

(ii) Given $1 < p < q < \infty$, consider $A(t) = \frac{t^p}{p} + \frac{t^q}{q}$. Then $A \in \Delta_2$. This gives the eigenvalue problem for the fractional p-q-Laplacian (see for instance [5]). When $\alpha \ll 1$ and $sp > n$, or $\alpha \gg 1$ and $sq > n$, we have

$$\frac{1}{\|\omega\|_{L^1(\Omega)}} \frac{C}{\max\{r^{sp-n}_\Omega, r^{sq-n}_\Omega\}} \le \lambda^s_{1,\alpha}.$$

(iii) Given $p, q, r \ge 1$, consider $A(t) = t^p \ln^r(1 + t^q)$. Then $A \in \Delta_2$. When $\alpha \ll 1$ and $s(p + qr) > n$, or $\alpha \gg 1$ and $sp > n$, then

$$\frac{1}{\|\omega\|_{L^1(\Omega)}} \frac{C}{\max\{r^{s(p-qr)-n}_\Omega, r^{sp-n}_\Omega\}} \le \lambda^s_{1,\alpha}.$$

(iv) For $k \in \mathbb{N}$ define $A(t) = e^t - \sum_{j=0}^{k-1} t^j / j!$. Then $A \in \Delta^0_2$ but $A \notin \Delta^\infty_2$. When $\alpha \ll 1$ and $sk > n$, when $r_\Omega \le 1$ we have

$$\frac{1}{\|\omega\|_{L^1(\Omega)}} \frac{C}{r^{sk-n}_\Omega} \le \lambda^s_{1,\alpha}.$$

(v) Consider the function $A(t)=e^{e^t}-e$. Then $A\in\Delta_2^0$ but $A\notin\Delta_2^\infty$. When $\alpha\gg 1$ we have

$$\frac{1}{\|\omega\|_{L^1(\Omega)}}\frac{C}{r_\Omega^{s-n}}\le\lambda_{1,\alpha}^s.$$

In Theorem 4.1, we also derive a lower bound for the minimizer using the inverse of the Young function instead of the Matuszewska–Orlicz function. Moreover, in Corollary 4.2 we prove that the same lower bounds hold for the eigenvalue $\Lambda_{1,\alpha}^s$ when $A\in\Delta_2$.

Theorem 1.2. *Assume that $\Omega\subset\mathbb{R}^n$ is a bounded domain with diameter d_Ω containing the origin. Let $s\in(0,1)$ and let A be a Young function satisfying conditions (2-6) and $i(A)<\frac{n}{s}$. Given $\omega\in L^\infty(\mathbb{R}^n)$ and $\alpha>0$, consider $\lambda_{1,\alpha}^s$ as in (1-6). Then, there exists a positive constant $C=C(n,s,A)$ such that*

$$\frac{C}{\|\omega\|_{L^\infty(\Omega)}M_A(d_\Omega^s)}\le\lambda_{1,\alpha}^s.$$

To apply Theorem 1.2, there is an implicit growth condition: the condition $i(A)<\frac{n}{s}$ is not satisfied when $A\notin\Delta_2^0$ or $A\notin\Delta_2^\infty$ (see Lemma 2.2 for details). Therefore, this result can be applied only when $A\in\Delta_2$.

Under the assumption of the Δ_2 condition on A, we improve Theorem 1.2 by replacing the diameter with the inner radius.

Theorem 1.3. *Let $s\in(0,1)$ and let $A\in\Delta_2$ be a Young function satisfying (2-7). Assume that $\Omega\subset\mathbb{R}^n$ is a bounded Lipschitz domain with inner radius r_Ω. Given $\omega\in L^\infty(\mathbb{R}^n)$ and $\alpha>0$, consider $\lambda_{1,\alpha}^s$ as in (1-6). Then, there exists a positive constant $C=C(n,s,A)$ such that*

$$\frac{C}{\|\omega\|_{L^\infty(\Omega)}M_A(r_\Omega^s)}\le\lambda_{1,\alpha}^s.$$

Here we state some notable examples derived from Theorem 1.3.

(i) When $A(t)=t^p$, $p>1$, this gives the eigenvalue problem for the fractional p-Laplacian, which is homogeneous. Then for any $\alpha>0$, when $sp<n$:

$$\frac{1}{\|\omega\|_{L^\infty(\Omega)}}\frac{C}{r_\Omega^{sp}}\le\lambda_{1,\alpha}^s$$

which in some extent recovers (1-3).

(ii) Given $1<p<q<\infty$, consider $A(t)=\frac{t^p}{p}+\frac{t^q}{q}$. Then $A\in\Delta_2$. This gives the eigenvalue problem for the fractional $p-q$-Laplacian (see for instance [5]). Then, for $\alpha>0$, when $sq<n$, we have

$$\frac{1}{\|\omega\|_{L^\infty(\Omega)}}\frac{C}{\max\{r_\Omega^{sq},r_\Omega^{sp}\}}\le\lambda_{1,\alpha}^s.$$

(iii) Given $p, q, r \geq 1$, consider $A(t) = t^p \ln^r(1+t^q)$. Then $A \in \Delta_2$. Then, for $\alpha > 0$, when $s(p+qr) < n$,

$$\frac{1}{\|\omega\|_{L^\infty(\Omega)}} \frac{C}{\max\{r_\Omega^{sp}, r_\Omega^{s(p+qr)}\}} \leq \lambda_{1,\alpha}^s.$$

In particular, Theorem 1.2 establishes these same inequalities but with the diameter in place of the inner radius under the same hypothesis on the parameters.

The same lower bounds established in Theorems 1.2 and 1.3 also hold for the eigenvalue $\Lambda_{1,\alpha}^s$, as stated in Corollary 4.3.

2. Preliminaries

2.1. ***Young functions.*** A function $A : [0, \infty) \to [0, \infty]$ is called a *Young function* if it is convex, nonconstant, left continuous and $A(0) = 0$. A function with these properties admits the representation

$$A(t) = \int_0^t a(\tau)\, d\tau \qquad \text{for } t \geq 0,$$

for some nondecreasing, left continuous function $a : [0, \infty) \to [0, \infty]$.

The *complementary function* $\tilde{A}$ of A is the Young function defined as

$$\tilde{A}(t) = \sup\{\tau t - A(\tau) : \tau \geq 0\} \qquad \text{for } t \geq 0.$$

One has

$$t \leq A^{-1}(t)(\tilde{A})^{-1}(t) \leq 2t \qquad \text{for } t \geq 0.$$

From the convexity of the Young function it is immediate that

$$A(rt) \leq rA(t) \quad \text{for } 0 < r < 1, \qquad A(rt) \geq rA(t) \quad \text{for } r > 1. \tag{2-1}$$

From the integral representation of the Young function it follows that

$$G(2t) > tg(t), \qquad G(t) \leq tg(t).$$

2.1.1. *The doubling condition.* A Young function A lies in Δ_2^∞ (or in Δ_2^0) if and only if there exists $p > 1$ and $T_\infty > 0$ (or $T_0 > 0$) such that

$$\frac{ta(t)}{A(t)} \leq p \quad \text{for all } t \geq T_\infty \quad (\text{or } 0 < t \leq T_0). \tag{2-2}$$

It is easy to see that

- $A \in \Delta_2^\infty$ if there exists $C_\infty \geq 2$ such that $A(2t) \leq C_\infty A(t)$ for all $t \geq T_\infty$, and
- $A \in \Delta_2^0$ if there exists $C_0 \geq 2$ such that $A(2t) \leq C_0 A(t)$ for all $t \leq T_0$.

We define $\Delta_2 = \Delta_2^0 \cap \Delta_2^\infty$. The following statements are equivalent:

(i) $A \in \Delta_2$.

(ii) There exists $p > 1$ such that $ta(t)/A(t) \leq p$ for all $t > 0$.

(iii) There exists $C \geq 2$ such that $A(2t) \leq CA(t)$ for all $t > 0$.

Proposition 2.1. *Let A be a Young function such that $A \in \Delta_2^0$ and let $p > 1$ be the number defined in (2-2). Then*

$$\tau^p A(t) \leq A(t\tau) \leq \tau A(t) \quad \text{for } 0 < \tau < 1 \text{ and } t < T_0.$$

Similarly, if $A \in \Delta_2^\infty$ we have

$$\tau A(t) \leq A(t\tau) \leq \tau^p A(t) \quad \text{for } \tau > 1 \text{ and } t > T_\infty.$$

Hence, if $A \in \Delta_2$, then

$$\min\{\tau, \tau^p\} A(t) \leq A(t\tau) \leq \max\{\tau, \tau^p\} A(t) \quad \text{for } \tau > 0.$$

2.1.2. *Ordering of functions.* A Young function A dominates another Young function B near infinity if there exists a positive constant c and t_0 such that

$$B(t) \leq A(ct) \qquad \text{for } t \geq t_0.$$

The functions A and B are called equivalent near infinity if they dominate each other in the respective range of values of their arguments; in this case we write $A \simeq B$.

$A \approx B$ means that A and B are bounded by each other, up to a multiplicative constant.

2.2. ***Matuszewska indexes.*** The Matuszewska–Orlicz functions associated to the Young function A are defined by

$$M_A(t) = \sup_{\alpha > 0} \frac{A(\alpha t)}{A(\alpha)}, \quad M_0(t, A) = \liminf_{\alpha \to 0^+} \frac{A(\alpha t)}{A(\alpha)}, \quad M_\infty(t, A) = \liminf_{\alpha \to \infty} \frac{A(\alpha t)}{A(\alpha)}.$$

They are nondecreasing, submultiplicative in the variable t and equal to 1 at $t = 1$. We also consider the Matuszewska–Orlicz indices at zero and infinity, defined as

$$i(A) = \lim_{t \to \infty} \frac{\log M_A(t)}{\log t}, \quad i_0(A) = \lim_{t \to \infty} \frac{\log M_0(t, A)}{\log t}, \quad i_\infty(A) = \lim_{t \to \infty} \frac{\log M_\infty(t, A)}{\log t}.$$

When there is no confusion, we will remove the dependence on A.

It is easy to see that the Matuszewska function can be bounded in terms of powers if and only if $A, \tilde{A} \in \Delta_2$, that is,

$$\min\{t^{p_A^+}, t^{p_A^-}\} \leq M(t, A) \leq \max\{t^{p_A^-}, t^{p_A^+}\} \tag{2-3}$$

where $p_A^+ = \sup_{t>0} a(t)t/A(t)$ and $p_A^- = \inf_{t>0} a(t)t/A(t)$.

For a comprehensive approach on these functions and indices we refer to the monograph [31].

2.3. ***Examples of Young functions.*** Here we provide for some examples of Young functions and compute their corresponding Matuszewska functions and indexes. For further examples we refer to [31].

Example 1. Let $p > 1$, and assume that

$$A(t) \simeq t^p \qquad \text{when } t \ll 1.$$

Then $A \in \Delta_2^0$. In this case we have $M_0(t) = t^p$ and $i_0(A) = p$. If we assume that

$$A(t) \simeq t^p \qquad \text{when } t \gg 1,$$

then $A \in \Delta_2^\infty$, $M_\infty(t) = t^p$ and $i_\infty(A) = p$.
In particular, when $A(t) = t^p$, $M(t, A) = M_0(t, A) = M_\infty(t, A) = t^p$ and $i(A) = i_0(A) = i_\infty(A) = p$. As a special case, if $1 < p < q < \infty$,

$$A(t) = \frac{t^p}{p} + \frac{t^q}{q};$$

then

$$M_0(t, A) = t^p, \qquad M_\infty(t, A) = t^q, \qquad M(t, A) = \max\{t^p, t^q\},$$

and $i_0(A) = p$, $i_\infty(A) = i(A) = q$.

Example 2. Let $r \geq 0$ and $p \geq 1$. Then if

$$A(t) \simeq t^p \ln^r t \qquad \text{when } t \ll 1$$

then $A \in \Delta_2^0$ and in this case, $M_0(t) = t^p$ and $i_0(A) = p$. If

$$A(t) \simeq t^p \ln^r t \qquad \text{when } t \gg 1,$$

then $M_\infty(t) = t^p$ and $i_\infty(A) = p$.

As a special case, if $p, q, r \geq 0$ and $A(t) = t^p \ln^r(1 + t^q)$ then

$$M_0(t, A) = t^{p+qr}, \qquad M_\infty(t, A) = t^p, \qquad M(t, A) = \max\{t^p, t^{p+qr}\}$$

and $i(A) = i_0(A) = p + qr$, $i_\infty(A) = p$.

Example 3. For $k \in \mathbb{N}$ define $A(t) = e^t - \sum_{j=0}^{k-1} t^j/j!$. Then $A \in \Delta_2^0$ but $A \notin \Delta_2^\infty$, and

$$M_0(t) = t^k, \qquad M_\infty(t) = \begin{cases} 0 & \text{if } 0 < t < 1, \\ \infty & \text{if } t > 1, \end{cases} \qquad M(t) = \begin{cases} t^k & \text{if } 0 < t \leq 1, \\ \infty & \text{if } t > 1, \end{cases}$$

$i_0(A) = k$, $i(A) = i_\infty(A) = \infty$.

Example 4. For $r>0$, assume that $A(t)\simeq e^{-t^{-r}}$ for $t\ll 1$. Then $A\notin\Delta_2^0$ and

$$M_0(t)=\begin{cases}0 & \text{if } 0<t<1,\\ 1 & \text{if } t=1,\\ \infty & \text{if } t>1.\end{cases}$$

In particular, when $A(t)=e^{-t^{-r}}$, we have

$$M_\infty(A)=1,\qquad M_0(t)=\begin{cases}0 & \text{if } 0<t<1,\\ 1 & \text{if } t=1,\\ \infty & \text{if } t>1,\end{cases}\qquad M(t)=\begin{cases}1 & \text{if } 0<t\le 1,\\ \infty & \text{if } t>1.\end{cases}$$

and $i_\infty(A)=i(A)=0$, $i_0(A)=\infty$.

Example 5. Assume that $A(t)\simeq e^{e^t}$ for $t\gg 1$. Then $A\notin\Delta_2^\infty$ and

$$M_\infty(t)=\begin{cases}0 & \text{if } 0<t<1,\\ 1 & \text{if } t=1,\\ \infty & \text{if } t>1.\end{cases}$$

In particular, when $A(t)=e^{e^t}-e$, since $A(t)\simeq et$ when $t\ll 1$,

$$M_0(t)=t,\qquad M_\infty(t)=\begin{cases}0 & \text{if } 0<t<1,\\ 1 & \text{if } t=1,\\ \infty & \text{if } t>1,\end{cases}\qquad M(t)=\begin{cases}t & \text{if } 0<t\le 1,\\ \infty & \text{if } t>1,\end{cases}$$

and $i_0(A)=1$, $i(A)=i_\infty(A)=\infty$.

Lemma 2.2. *Let A be a Young function such that $A\notin\Delta_2^k$ for $k=0$ or $k=\infty$. Then*

$$M(t)=\begin{cases}1 & \text{if } t=1,\\ \infty & \text{if } t>1.\end{cases}$$

When $0<t<1$ we have $M(t)\le t$. Moreover, $i(A)=\infty$.

Proof. First, observe that from (2-1), we have $M(t)\le t$ for $0<t<1$.

In light of [11, Proposition 2.1], if $A\notin\Delta_2^0$ or $A\notin\Delta_2^\infty$ then

$$M_k(t)=\begin{cases}1 & \text{if } t=1,\\ \infty & \text{if } t>1,\end{cases}$$

$k=0,\infty$, respectively. By definition, $M_0(t)\le M(t)$ and $M_\infty(t)\le M(t)$ for any $t>0$. This gives immediately that when $A\notin\Delta_2^k$ for $k=0$ or $k=\infty$, one has $M(1)=1$ and $M(t)=\infty$ when $t>1$.

Since $M(t)=\infty$ for $t>1$, this gives that $i(A)=\infty$. □

Lemma 2.3. *If $A\in\Delta_2$ then there exists $p\ge 1$ such that $M(t)=t^p$.*

Proof. Since $A \in \Delta_2$, there exists $q > 1$ such that $A(rt) \leq \max\{t, t^q\}A(r)$ for any $t, r \geq 0$. Then M is finite for any $t > 0$:

$$M(t) = \sup_{\alpha>0} \frac{A(\alpha t)}{A(\alpha)} \leq \max\{t, t^q\}.$$

Moreover, observe that

$$M(rt) = \sup_{\alpha>0} \frac{A(tr\alpha)}{A(r\alpha)} \frac{A(r\alpha)}{A(\alpha)} \leq \sup_{\alpha>0} M(t) \frac{A(r\alpha)}{A(\alpha)} \leq M(t)M(s)$$

and $M(1) = 1$, that is, $M(t)$ is submultiplicative.

Define $v(t) = \ln(M(e^t))$. This function is additive, that is, $v(r+t) = v(r) + v(t)$ for any $s, t \in \mathbb{R}$. It is well known that measurable additive functions are linear, therefore, there exists $p \in \mathbb{R}$ such that $v(t) = pt$ from there $M(t) = t^p$. Finally, from (2-1),

$$M(t) \leq t \quad \text{for } 0 < t < 1 \quad \text{and} \quad M(t) \geq t \quad \text{for } t > 1.$$

which implies that $p \geq 1$. □

2.4. ***Some useful inequalities.*** Given $s \in (0, 1)$ and a Young function A such that

$$\int^{\infty} \left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} dt < \infty \tag{2-4}$$

consider the Young function E given by

$$E(t) = t^{\frac{n}{n-s}} \int_t^{\infty} \frac{\tilde{A}(\tau)}{\tau^{1+\frac{n}{n-s}}} \, d\tau \qquad \text{for } t \geq 0. \tag{2-5}$$

Consider also $\Psi_s : (0, \infty) \to (0, \infty)$ defined by

$$\Psi_s(r) = \frac{1}{r^{n-s} E^{-1}(r^{-n})}, \qquad \text{for } r > 0.$$

Lemma 2.4 [3, Proposition 2.1]. *Let $s \in (0, 1)$ and let A be a Young function. Assume (2-4). Then*:

(i) *The function Ψ_s is nondecreasing.*

(ii) *Define the Young function $B = \tilde{E}$. Then $\Psi_s(r) \approx r^s B^{-1}(r^{-n})$ for $r > 0$.*

(iii) *If $i_\infty(A) > \frac{n}{s}$ then $B \simeq A$ near infinity, and $\Psi_s(r) \approx r^s A^{-1}(r^{-n})$ for $0 < r \leq 1$.*

(iv) *If $i_0(A) > \frac{n}{s}$ then $B \simeq A$ near 0, and $\Psi_s(r) \approx r^s A^{-1}(r^{-n})$ for $r \geq 1$.*

The following modular Morrey-type inequality is proved in [3, Remark 4.3]:

Proposition 2.5. *Let $s \in (0,1)$ and let A be a Young function satisfying (2-4). Then, $W^sL^A(\mathbb{R}^n) \subset C^{\Psi_s(\cdot)}(\mathbb{R}^n)$. Moreover, for any $u \in W^sL^A(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$,*

$$|u(x)-u(y)| \le C_M |x-y|^s B^{-1}\left(\frac{1}{|x-y|^n}\iint_{\mathbb{R}^{2n}} A(D^s u(z,w))\, d\nu_n(z,w)\right)$$

for some constant C_M depending on n and s, where the Young function B is given by $B(t)=\tilde{E}(t)$, being E the Young function defined in (2-5).

The following Hardy type inequality is proved in Theorem 5.1 and Proposition C in [1].

Proposition 2.6. *Let $s \in (0,1)$ and let A be a Young function satisfying $i(A) < \frac{n}{s}$ and the conditions*

$$\int^{\infty}\left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} dt = \infty, \qquad \int_0\left(\frac{t}{A(t)}\right)^{\frac{s}{n-s}} dt < \infty. \tag{2-6}$$

Then for all $u \in C_c^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} A\left(C_{H_1}\frac{|u(x)|}{|x|^s}\right) dx \le (1-s)\iint_{\mathbb{R}^{2n}} A(C_{H_2}|D^s u|)\, d\nu_n$$

for positive constants C_{H_1} and C_{H_2} depending only on n and s.

Given a bounded domain $\Omega \subset \mathbb{R}^n$, we denote $\delta_\Omega(x) := \inf\{|x-y| : y \in \Omega^c\}$ the distance from x to $\partial\Omega$.

The following Hardy type inequality is proved in [10, Theorem 1.5].

Proposition 2.7. *Let $s \in (0,1)$ and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. If $A \in \Delta_2$ and*

$$\lim_{k\to\infty}\sup_{t\ge 0}\frac{A(kt)}{k^{\frac{n}{s}}A(t)} = 0 = \lim_{k\to 0^+}\sup_{t\ge 0}\frac{A(kt)}{k^{\frac{1}{s}}A(t)}, \tag{2-7}$$

there exists a positive constant C_{H_3} such that for all $u \in C_c^\infty(\Omega)$

$$\int_\Omega A\left(\frac{|u(x)|}{\delta_\Omega(x)}\right) dx \le C_{H_3}\iint_{\mathbb{R}^{2n}} A(|D^s u|)\, d\nu_n.$$

2.5. ***Orlicz and Orlicz–Sobolev spaces.*** The main reference for Orlicz spaces is the book [27]. For Orlicz–Sobolev spaces, the reader can consult [24], for instance. Fractional-order Orlicz–Sobolev spaces, as we will use them here, were introduced in [18] and then further analyze by several authors. The results used in this paper can be found in [1; 2; 19].

2.5.1. *Orlicz spaces.* Given a bounded domain $\Omega \subset \mathbb{R}^n$ and a Young function A, the *Orlicz class* is defined as

$$\mathscr{L}^A(\Omega) := \left\{ u \in L^1_{\mathrm{loc}}(\Omega) : \int_\Omega A(|u|)\, dx < \infty \right\}.$$

The *Orlicz space* $L^A(\Omega)$ is defined as the linear hull of $\mathscr{L}^A(\Omega)$ and is characterized as

$$L^A(\Omega) = \left\{ u \in L^1_{\mathrm{loc}}(\Omega) : \text{ there exists } k > 0 \text{ such that } \int_\Omega A\left(\frac{|u|}{k}\right) dx < \infty \right\}.$$

In general the Orlicz class is strictly smaller than the Orlicz space, and $\mathscr{L}^A(\Omega) = L^A(\Omega)$ if and only if $A \in \Delta_2^\infty$. The space $L^A(\Omega)$ is a Banach space when it is endowed, for instance, with the *Luxemburg norm*, i.e.,

$$\|u\|_{L^A(\Omega)} = \|u\|_A := \inf \left\{ k > 0 : \int_\Omega A\left(\frac{|u|}{k}\right) dx \le 1 \right\}.$$

This space $L^A(\Omega)$ turns out to be separable if and only if $A \in \Delta_2^\infty$.

An important subspace of $L^A(\Omega)$ is $E^A(\Omega)$ that it is defined as the closure of the functions in $L^A(\Omega)$ that are bounded. This space is characterized as

$$E^A(\Omega) = \left\{ u \in L^1_{\mathrm{loc}}(\Omega) : \int_\Omega A\left(\frac{|u|}{k}\right) dx < \infty \text{ for all } k > 0 \right\}.$$

This subspace $E^A(\Omega)$ is separable, and we have the inclusions

$$E^A(\Omega) \subset \mathscr{L}^A(\Omega) \subset L^A(\Omega)$$

with equalities if and only if $A \in \Delta_2^\infty$. Moreover, the following duality relation holds

$$(E^A(\Omega))^* = L^{\tilde{A}}(\Omega),$$

where the equality is understood via the standard duality pairing. This automatically implies that $L^A(\Omega)$ is reflexive if and only if $A, \tilde{A} \in \Delta_2^\infty$.

2.5.2. *Fractional Orlicz–Sobolev spaces.* Given a fractional parameter $s \in (0, 1)$, we define the *Hölder quotient* of a function $u \in L^A(\Omega)$ as

$$D^s u(x, y) = \frac{u(x) - u(y)}{|x - y|^s}.$$

Then, the *fractional Orlicz–Sobolev space* of order s is defined as

$$W^s L^A(\mathbb{R}^n) := \{u \in L^A(\mathbb{R}^n) : D^s u \in L^A(\mathbb{R}^{2n}, d\nu_n)\},$$

where $d\nu_n = |x - y|^{-n}\, dx\, dy$ and

$$W^s E^A(\mathbb{R}^n) := \{u \in E^A(\mathbb{R}^n) : D^s u \in E^A(\mathbb{R}^{2n}, d\nu_n)\}.$$

When $A \in \Delta_2$, these spaces coincide and we write

$$W^{s,A}(\mathbb{R}^n) = W^s L^A(\mathbb{R}^n) = W^s E^A(\mathbb{R}^n).$$

The space $W^s L^A(\mathbb{R}^n)$ is reflexive if and only if $A, \tilde{A} \in \Delta_2$.

In these spaces the norm considered is

$$\|u\|_{W^s L^A(\mathbb{R}^n)} = \|u\|_{s,A} = \|u\|_A + [u]_{s,A,\mathbb{R}^n}$$

with

$$[u]_{s,A,\mathbb{R}^n} = \inf\left\{k > 0 : \iint_{\mathbb{R}^{2n}} A\left(\frac{|D^s u(x,y)|}{k}\right) d\nu_n \le 1\right\}.$$

Again, with this norm, $W^s L^A(\mathbb{R}^n)$ is a Banach space and $W^s E^A(\mathbb{R}^n)$ is a closed subspace. The space $W_0^s L^A(\Omega)$ is then defined as the closure of $C_c^\infty(\Omega)$ with respect to the topology $\sigma(W^s L^A(\mathbb{R}^n), W^s E^{\tilde{A}}(\mathbb{R}^n))$ and $W_0^s E^A(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in norm topology.

3. Eigenvalues and critical points

Let A be a Young function, and let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. For a fixed normalization parameter $\alpha > 0$, we define the *critical point* $\lambda_{1,\alpha}^s$ as

$$\lambda_{1,\alpha}^s = \inf\left\{\frac{1}{\alpha}\iint_{\mathbb{R}^{2n}} A(|D^s u|)\, d\nu_n : u \in C_c^\infty(\Omega), \int_\Omega \omega A(|u|)\, dx = \alpha\right\}. \tag{3-1}$$

Here, ω is a suitable positive weight function. We assume that $\omega \in L^1(\mathbb{R}^n)$ when (2-4) holds, and $\omega \in L^\infty(\mathbb{R}^n)$ when (2-6) holds.

In [38] (see also [36] when $A \in \Delta_2$) it is proved that (3-1) is solvable, that is, there exists a *minimizer* $u_\alpha^s \in W_0^s L^A(\Omega)$ such that $\int_\Omega \omega A(|u_\alpha^s|)\, dx = \alpha$ and

$$\iint_{\mathbb{R}^{2n}} A(|D^s u_\alpha^s|)\, d\nu_n = \lambda_{1,\alpha}^s \int_\Omega \omega A(|u_\alpha^s|)\, dx. \tag{3-2}$$

By applying an appropriate version of the Lagrange multipliers theorem, we can establish the existence of an eigenvalue $\Lambda_{1,\alpha}^s$ with corresponding eigenfunction u_α^s. More precisely, u_α is a weak solution to the following equation, with $\Lambda = \Lambda_{1,\alpha}^s$:

$$\begin{cases} (-\Delta_a)^s u = \Lambda\omega \dfrac{a(|u|)}{|u|} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3-3}$$

with $a(t) = A'(t)$ for $t > 0$, and where the fractional a-Laplacian of order $s \in (0,1)$ is the nonlocal and nonstandard growth operator defined as

$$(-\Delta_a)^s u(x) = \text{p.v.} \int_{\mathbb{R}^n} a(|D^s u|) \frac{D^s u}{|D^s u|}\, d\nu_n,$$

that is, for all $v \in C_c^\infty(\Omega)$

$$\iint_{\mathbb{R}^{2n}} a(|D^s u_\alpha^s|) \frac{D^s u_\alpha^s D^s v}{|D^s u_\alpha^s|} \, d\nu_n = \Lambda_{1,\alpha}^s \int_\Omega \omega a(|u_\alpha^s|) \frac{u_\alpha^s v}{|u_\alpha^s|} \, dx.$$

Refer also to [9; 19] for the existence of higher-order eigenvalues.

Lemma 3.1. *Let A be a Young function such that $A'(t) = a(t)$ for any $t \geq 0$. Then*

$$\frac{1}{p_A} \Lambda_{1,\alpha}^s \leq \lambda_{1,\alpha}^s \leq p_A \Lambda_{1,\alpha}^s.$$

The number $p_A := \sup_{\beta>0} a(\beta)\beta / A(\beta)$ is finite if and only if $A \in \Delta_2$.

Proof. Given $\alpha > 0$, consider the critical point $\lambda_{1,\alpha}^s$ associated to the minimizing function u_α^s such that $\int_\Omega A(|u_\alpha|) \, dx = \alpha$. Observe that

$$\text{(3-4)} \quad \int_\Omega \omega A(|u_\alpha^s|) \, dx \geq \inf_{\beta>0} \frac{A(\beta)}{a(\beta)\beta} \int_\Omega \omega a(|u_\alpha^s|)|u_\alpha^s| \, dx = \frac{1}{p_A} \int_\Omega \omega a(|u_\alpha^s|)|u_\alpha| \, dx.$$

Moreover, since $a(t) = A'(t)$ is increasing, $A(t) = \int_0^t a(\tau) \, d\tau \leq a(t)t$ for any $t > 0$. This fact, together with (3-4) gives that

$$\lambda_{1,\alpha}^s = \frac{\int_\Omega A(|D^s u_\alpha^s|) \, d\nu_n}{\int_\Omega \omega A(|u_\alpha^s|) \, d\nu_n} \leq \frac{\int_\Omega a(|D^s u_\alpha^s|)|D^s u_\alpha^s| \, d\nu_n}{\frac{1}{p_A} \int_\Omega \omega a(|u_\alpha^s|)|u_\alpha^s| \, dx} = p_A \Lambda_{1,\alpha}^s.$$

The other bound is analogous. Finally, note that $p_A < \infty$ if and only if $A \in \Delta_2$. □

For example, the number p_A as defined in Lemma 3.1, takes the following form for the following notable Young functions $A \in \Delta_2$.

(i) Let $p > 1$ and $A(t) = t^p$ then $p_A = p$.

(ii) Let $p, q > 1$, $r \geq 0$ and consider $A(t) = t^p/p + t^q/q$. Then $p_A = q$ when $t \geq 1$ and $p_A = p$ when $t < 1$.

(iii) Let $p, q > 1$, $r \geq 0$ and consider $A(t) = (t^p/p) \ln^r(1 + t^q)$. Then $p_A = p + qr$.

Lemma 3.2. *For $\alpha > 0$ define the function $E(\alpha) := \alpha \lambda_{1,\alpha}^s$. Then E is strictly positive, strictly increasing, $E(0) = 0$, $E(\infty) = \infty$ and E is a Lipschitz function for $\alpha > 0$.*

In particular, that $\lambda_{1,1} \leq \alpha \lambda_{1,\alpha}^s$ when $\alpha > 1$ and $\alpha \lambda_{1,\alpha}^s \leq \lambda_{1,1}$ when $\alpha < 1$.

Proof. Given $\beta > 0$ and a fixed function $u \in C_c^\infty(\Omega)$ such that $\int_\Omega A(|u_\beta|) \, dx = \beta$, since A is continuous and nondecreasing, if we define the function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\phi(r) = \int_\Omega A(r|u_\beta|) \, dx,$$

it follows that ϕ is continuous, nondecreasing, $\phi(0)=0$, $\phi(1)=\beta$ and $\phi(\infty)=\infty$. Hence, for any $\alpha>0$ there exists $r_\alpha>0$ such that $\phi(r_\alpha)=\alpha$ and in particular

$$r_\alpha<1 \text{ when } \alpha<\beta, \qquad r_\alpha>1 \text{ when } \alpha>\beta, \tag{3-5}$$

$$r_\alpha\to 0 \text{ when } \alpha\to 0, \qquad r_\alpha\to\infty \text{ when } \alpha\to\infty. \tag{3-6}$$

Let us see that E is strictly increasing. Let $0<\alpha<\beta$ and in light of (3-2), let $u_\beta\in W^s_0E^A(\Omega)$ be such that

$$\int_\Omega A(|u_\beta|)\,dx=\beta, \qquad \lambda^s_{1,\beta}=\frac{1}{\beta}\iint_{\mathbb{R}^{2n}} A(|D^s u_\beta|)\,d\nu_n.$$

By (3-5) there exists $r_\alpha<1$ such that $\int_\Omega A(r_\alpha|u_\beta|)\,dx=\alpha$. Therefore, using the convexity of A we obtain the desired inequality:

$$\alpha\lambda^s_{1,\alpha}\le\iint_{\mathbb{R}^{2n}} A(r_\alpha|D^s u_\beta|)\,d\nu_n\le r_\alpha\iint_{\mathbb{R}^{2n}} A(|D^s u_\beta|)\,d\nu_n<\beta\lambda^s_{1,\beta}.$$

Moreover, from the previous inequality together with (3-6) we get that

$$0\le\lim_{\alpha\to 0^+}E(\alpha)\le\lim_{\alpha\to 0^+} r_\alpha\iint_{\mathbb{R}^{2n}} A(|D^s u_\beta|)\,d\nu_n=E(\beta)\lim_{\alpha\to 0^+} r_\alpha=0,$$

from where $E(0)=0$. $E(\alpha)$ is lower semicontinuous by [11, Lemma 4.3], and then $\liminf_{\alpha\to\infty}E(\alpha)\ge\infty$. Finally, E is Lipschitz continuous by Theorem 4.5 in [11]. □

Proposition 3.3. *Let A be a Young function and let $\Omega\subset\mathbb{R}^n$ be open and bounded, let $B_1\subset\mathbb{R}^n$ be a ball such that $|\Omega|=|B_1|$ and let $B_2\subset\Omega$ be a ball. Then*

$$\lambda^s_{1,\alpha}(B_1)\le\lambda^s_{1,\alpha}(\Omega)\le\lambda^s_{1,\alpha}(B_2).$$

Proof. Let $u\in W^s_0L^A(\Omega)$ be such that $\int_\Omega A(|u|)\,dx=\alpha$. Denote by u^* the symmetric rearrangement of u. Thus, u^* is radially decreasing about 0 and is equidistributed with u. Using the Pólya–Szegő principle stated in [2, Theorem 3.1] we get

$$\iint_{\mathbb{R}^{2n}} A(|D^s u^*|)\,d\nu_n\le\iint_{\mathbb{R}^{2n}} A(|D^s u|)\,d\nu_n.$$

Hence, if B_1 is a ball of same measure of Ω, since $\int_{B_1} A(|u^*|)\,dx=\alpha$, we get

$$\lambda^s_{1,\alpha}(B_1)\le\lambda^s_{1,\alpha}(\Omega).$$

On the other hand, consider a ball $B_2\subset\Omega$ and the function $u_{B_2}\in W^s_0L^A(B_2)$ such that $\int_B A(|u_{B_2}|)\,dx=\alpha$ to be a minimizer for $\lambda^s_{1,\alpha}(B_2)$. Define $v\in W^{1,A}_0(\Omega)$ defined as the extension of u_{B_2} by zero outside Ω. Then $\int_\Omega A(|v|)\,dx=\int_\Omega A(|u_{B_2}|)\,dx=\alpha$

and therefore v is admissible in the variational characterization of $\lambda^s_{1,\alpha}(\Omega)$. Hence

$$\lambda^s_{1,\alpha}(\Omega) \leq \frac{1}{\alpha} \iint_{\mathbb{R}^{2n}} A(|D^s v|)\, d\nu_n = \lambda^s_{1,\alpha}(B_2),$$

which concludes the proof. □

4. Lower bounds of critical values and eigenvalues

Proof of Theorem 1.1. Fix $\alpha > 0$ and let $u^s_\alpha \in W^s_0 L^A(\Omega)$ be a minimizer of (3-1) such that $\int_\Omega \omega A(|u^s_\alpha|)\, dx = \alpha$, i.e., the pair $(u^s_\alpha, \lambda^s_{1,\alpha})$ satisfies equation (3-2), where $\lambda^s_{1,\alpha}$ is defined in (3-1). Since $s \in (0, 1)$ is fixed, for simplicity we will drop the dependence on s.

In light of Proposition 2.5, u_α is continuous and so there exists $x_0 \in \Omega$ such that

$$|u_\alpha(x_0)| = \max\{|u_\alpha(x)| : x \in \mathbb{R}^n\} > 0.$$

From Proposition 2.5, for any $x, y \in \mathbb{R}^n$ we have

$$|u_\alpha(x) - u_\alpha(y)| \leq C_M |x - y|^s B^{-1}\left(\frac{1}{|x-y|^n} \iint_{\mathbb{R}^{2n}} A(|D^s u_\alpha|)\, d\nu_n\right),$$

where the Young function B complementary to the Young function defined in (2-5). We take $x = x_0$, $y \in \partial\Omega$; the previous expression becomes

$$|u_\alpha(x_0)| \leq C_M |x_0 - y|^s B^{-1}\left(\frac{1}{|x_0-y|^n} \iint_{\mathbb{R}^{2n}} A(|D^s u_\alpha|)\, d\nu_n\right).$$

Using expression (3-2) and item (ii) of Lemma 2.4, since $\Psi_s(r) \approx r^s B^{-1}(r^{-n})$ for all $r > 0$, there exists $c_1 > 0$ depending only on n and s such that

$$\begin{aligned} |u_\alpha(x_0)| &\leq C_M |x_0 - y|^s B^{-1}\left(\frac{\lambda_{1,\alpha}}{|x_0-y|^n} \int_\Omega \omega A(|u_\alpha|)\, dx\right) \\ &= C_M |x_0 - y|^s B^{-1}\left(\frac{\alpha\lambda_{1,\alpha}}{|x_0-y|^n}\right) \\ &= C_M (\alpha\lambda_{1,\alpha})^{\frac{s}{n}} \big((\alpha\lambda_{1,\alpha})^{-\frac{1}{n}} |x_0 - y|\big)^s B^{-1}\left(\left((\alpha\lambda_{1,\alpha}^{-\frac{1}{n}} |x_0 - y|\right)^{-n}\right) \\ &\leq c_1 C_M (\alpha\lambda_{1,\alpha})^{\frac{s}{n}} \Psi_s\big(|x_0 - y| (\alpha\lambda^s_{1,\alpha})^{-\frac{1}{n}}\big). \end{aligned} \tag{4-1}$$

Moreover, by definition of inner radius we get that

$$|x_0 - y| \leq \max\{d(x, \partial\Omega) : x \in \Omega\} = r_\Omega.$$

Hence, since Ψ_s is nondecreasing in light of Lemma 2.4, inequality (4-1) yields

$$|u_\alpha(x_0)| \leq c_1 C_M (\alpha\lambda_{1,\alpha})^{\frac{s}{n}} \Psi_s\big(r_\Omega (\alpha\lambda_{1,\alpha})^{-\frac{1}{n}}\big). \tag{4-2}$$

From Lemma 3.2, the function $E(\alpha) := \alpha\lambda^s_{1,\alpha}$ is strictly increasing, positive and Lipschitz continuous for $\alpha > 0$, and satisfies that $E(0) = 0$, $E(\infty) = \infty$. Hence, defining $f(\alpha) := r_\Omega E(\alpha)^{-\frac{1}{n}}$, $\alpha > 0$, we get that f is a strictly decreasing continuous function such that $f(0) := \lim_{\alpha\to 0^+} f(\alpha) = \infty$ and $f(\infty) := \lim_{\alpha\to\infty} f(\alpha) = 0$. From these properties there exists $\alpha_0 > 0$ such that $f(\alpha_0) = 1$ and

$$f(\alpha) > 1 \quad \text{when } \alpha < \alpha_0, \qquad f(\alpha) < 1 \quad \text{when } \alpha > \alpha_0.$$

In particular, $f(\alpha) > 1$ when $\alpha \ll 1$ and $f(\alpha) < 1$ when $\alpha \gg 1$.

Case $\alpha > \alpha_0$. Since $f(\alpha) < 1$, assuming that $i_\infty(A) > \frac{n}{s}$, by of Lemma 2.4 (iii) we get

$$\Psi_s(f(\alpha)) \approx f(\alpha)^s A^{-1}(f(\alpha)^{-n}).$$

Then, there exists $c_2 > 0$ depending only on n and s, and (4-2) gives

$$|u_\alpha(x_0)| \le c_1 C_M (\alpha\lambda_{1,\alpha})^{\frac{s}{n}} \Psi_s\big(r_\Omega(\alpha\lambda_{1,\alpha})^{-\frac{1}{n}}\big) \le C r^s_\Omega A^{-1}\left(r_\Omega^{-n}\alpha\lambda_{1,\alpha}\right)$$

with $C = c_1 c_2 C_M$. Moreover, since A is nondecreasing, the previous expression yields

$$A(C^{-1} r_\Omega^{-s}|u_\alpha(x_0)|) \le r_\Omega^{-n}\alpha\lambda_{1,\alpha} = r_\Omega^{-n}\lambda_{1,\alpha}\int_\Omega \omega A(|u_\alpha(x)|)\,dx \le r_\Omega^{-n}\lambda_{1,\alpha} A(|u_\alpha(x_0)|)\|\omega\|_{L^1(\Omega)}. \tag{4-3}$$

As a consequence, equation (4-3) yields

$$\frac{r^n_\Omega}{\|\omega\|_{L^1(\Omega)}} \le \lambda_{1,\alpha}\frac{A(|u_\alpha(x_0)|)}{A(C^{-1}r_\Omega^{-s}|u_\alpha(x_0)|)} \le \lambda_{1,\alpha}\sup_{t>0}\frac{A(t)}{A(C^{-1}r_\Omega^{-s}t)} = \lambda_{1,\alpha}\sup_{\tau>0}\frac{A(Cr^s_\Omega\tau)}{A(\tau)} = \lambda_{1,\alpha}M_A(Cr^s_\Omega),$$

where M_A is the Matuszewska–Orlicz function associated to A defined in Section 2.2. Since M is submultiplicative, there is $c > 0$ depending on A such that $M_A(Cr^s_\Omega) \le cM_A(C)M_A(r^s_\Omega)$, and the inequality above leads to the following lower bound for $\lambda_{1,\alpha}$:

$$\frac{r^n_\Omega}{cM_A(C)|\omega\|_{L^1(\Omega)}}\frac{1}{M_A(r^s_\Omega)} \le \lambda_{1,\alpha}.$$

Case $\alpha < \alpha_0$. Here $f(\alpha) > 1$. Then, assuming $i_0(A) > \frac{n}{s}$, by Lemma 2.4(iv) we get

$$\Psi_s(f(\alpha)) \approx f(\alpha)^s A^{-1}(f(\alpha)^{-n}).$$

Proceeding analogously as in the previous case we get the result. □

A similar argument to the proof of Theorem 1.1 yields a lower bound for the critical value, involving the inverse of A instead of the Matuszewska–Orlicz function M_A.

Theorem 4.1. *Let $s \in (0,1)$, $\alpha > 0$ and let A be a Young function satisfying (2-4). Given $\omega \in L^1(\Omega)$ consider the critical value $\lambda^s_{1,\alpha}$ defined in (3-1). Then*

(i) *There exists a unique $\alpha_0 > 0$ satisfying the equation*

$$\alpha_0 \lambda^s_{1,\alpha_0} = r^n_\Omega.$$

(ii) *Assume that $i_0(A) > \frac{n}{s}$ when $\alpha \le \alpha_0$, or $i_\infty(A) > \frac{n}{s}$ when $\alpha > \alpha_0$. Then, there exists a $C > 0$ depending only on s, n and A such that*

$$\frac{r^n_\Omega}{\alpha} A\left(\frac{1}{C r^s_\Omega} A^{-1}\left(\frac{\alpha}{\|\omega\|_{L^1(\Omega)}}\right)\right) \le \lambda^s_{1,\alpha}. \tag{4-4}$$

In particular, this holds when $i_0(A) > \frac{n}{s}$ if $\alpha \ll 1$ and when $i_\infty(A) > \frac{n}{s}$ if $\alpha \gg 1$.

Proof. Fix $\alpha > 0$ and let $u^s_\alpha \in W^s_0 L^A(\Omega)$ be a minimizer of (3-1) such that $\int_\Omega \omega A(|u^s_\alpha|)\,dx = \alpha$, i.e., the pair $(u^s_\alpha, \lambda^s_{1,\alpha})$ satisfies equation (3-2), where $\lambda^s_{1,\alpha}$ is defined in (3-1). Since $s \in (0,1)$ is fixed, for simplicity we will drop the dependence on s.

In light of Proposition 2.5 u_α is continuous and hence there exists $x_0 \in \Omega$ such that $|u_\alpha(x_0)| = \max\{|u_\alpha(x)| : x \in \mathbb{R}^n\} > 0$. Since

$$\alpha = \int_\Omega \omega A(|u_\alpha|)\,dx \le A(|u_\alpha(x_0)|)\|\omega\|_{L^1(\Omega)},$$

using (4-1) and the fact the A is nondecreasing, we get

$$\alpha \le A\big(c_1 C_M (\alpha\lambda_{1,\alpha})^{\frac{s}{n}} \Psi_s\big(|x_0 - y|(\alpha\lambda_{1,\alpha})^{-\frac{1}{n}}\big)\big)\|\omega\|_{L^1(\Omega)}$$

for any $y \in \partial\Omega$, and therefore, denoting by r_Ω the inner radius of Ω, we get

$$\alpha\|\omega\|^{-1}_{L^1(\Omega)} \le A\big(c_1 C_M (\alpha\lambda_{1,\alpha})^{\frac{s}{n}} \Psi_s\big(r_\Omega(\alpha\lambda_{1,\alpha})^{-\frac{1}{n}}\big)\big). \tag{4-5}$$

As in the proof of Theorem 1.1, there exists $\alpha_0 > 0$ such that $r^n_\Omega \le \alpha\lambda_{1,\alpha}$ when $\alpha > \alpha_0$ and $r^n_\Omega \ge \alpha\lambda_{1,\alpha}$ when $\alpha < \alpha_0$.

Case $\alpha > \alpha_0$. Assuming $i_\infty(A) > \frac{n}{s}$, by Lemma 2.4(iii) we get $\Psi_s(t) \approx t^s A^{-1}(t^{-n})$ for any $t > 0$. Then, there exists $c_2 = c_2(n,s) > 0$ for which (4-5) yields

$$\alpha\|\omega\|^{-1}_{L^1(\Omega)} \le A\left(C r^s_\Omega A^{-1}\left(r^{-n}_\Omega \alpha\lambda_{1,\alpha}\right)\right)$$

where $C = c_1 c_2 C_M$. Since A is nondecreasing, the previous expression gives

$$\frac{r^n_\Omega}{\alpha} A\left(\frac{1}{C r^s_\Omega} A^{-1}\left(\frac{\alpha}{\|\omega\|_{L^1(\Omega)}}\right)\right) \le \lambda_{1,\alpha}.$$

Case $\alpha < \alpha_0$. Assuming that $i_0(A) > \frac{n}{s}$, the bound follows analogously. □

As a direct consequence of Theorem 1.1 and Lemma 3.1 we get the following.

Corollary 4.2. *Under the assumptions and notation of Theorem 1.1, if additionally $A \in \Delta_2$, then*

$$\frac{C}{p_A \|\omega\|_{L^1(\Omega)}} \frac{r_\Omega^n}{M(r_\Omega^s)} \leq \Lambda_{1,\alpha}^s$$

where $p_A = \sup_{\beta>0} a(\beta)\beta/A(\beta)$.

Proof. It is direct from Theorem 1.1 by using Lemma 3.1. □

Proof of Theorem 1.2. Fix $\alpha > 0$ and let $u_\alpha^s \in W_0^s L^A(\Omega)$ be a minimizer of (3-1) such that $\int_\Omega \omega A(|u_\alpha^s|)\, dx = \alpha$, that is, the pair $(u_\alpha^s, \lambda_{1,\alpha}^s)$ satisfies (3-2), where $\lambda_{1,\alpha}^s$ is defined in (3-1).

Denote by d_Ω the diameter of Ω. The Hardy inequality given in Proposition 2.6 together with (3-2) and the monotonicity of A gives

$$\int_{\mathbb{R}^n} A\left(\frac{c_1|u_\alpha(x)|}{\mathrm{d}_\Omega^s}\right) dx \leq \int_\Omega A\left(\frac{c_1|u_\alpha(x)|}{|x|^s}\right) dx \leq (1-s) \iint_{\mathbb{R}^{2n}} A(|D^s u_\alpha|)\, d\nu_n = (1-s)\lambda_{1,\alpha}^s \int_\Omega \omega A(|u_\alpha|)\, dx \tag{4-6}$$

where $c_1 = C_{H_1} C_{H_2}^{-1}$, and $C_{H_1}, C_{H_2} > 0$ are the constants given in Proposition 2.6, which depend only on n and s. Now, we compute the following inequality:

$$\begin{aligned} \int_\Omega A(|u_\alpha|)\, dx &= \int_\Omega \frac{A(|u_\alpha|)}{A(c_1 \mathrm{d}_\Omega^{-s}|u_\alpha|)} A(c_1 \mathrm{d}_\Omega^{-s}|u_\alpha|)\, dx \\ &\leq \sup_{t\in(0,\|u\|_\infty)} \frac{A(t)}{A(c_1 \mathrm{d}_\Omega^{-s} t)} \int_\Omega A(c_1 \mathrm{d}_\Omega^{-s}|u_\alpha|)\, dx \\ &\leq \sup_{\tau>0} \frac{A(c^{-1} d_\Omega^s \tau)}{A(\tau)} \int_\Omega A(c_1 \mathrm{d}_\Omega^{-s}|u_\alpha|)\, dx \\ &= M_A(c^{-1} d_\Omega^s) \int_\Omega A\left(\frac{c_1|u_\alpha|}{d_\Omega^s}\right) dx. \end{aligned} \tag{4-7}$$

From (4-6), (4-7) and the fact that M_A is submultiplicative, we get

$$1 \leq (1-s)\lambda_{1,\alpha}^s \|\omega\|_{L^\infty(\Omega)} M_A(c^{-1} d_\Omega^s) \leq (1-s)\lambda_{1,\alpha}^s \|\omega\|_{L^\infty(\Omega)} \tilde{c} M_A(c^{-1}) M_A(d_\Omega^s),$$

with $\tilde{c} = \tilde{c}(c, A)$, which concludes the proof. □

As a direct consequence of Theorem 1.2 and Lemma 3.1 we get the following.

Corollary 4.3. *Under the assumptions and notation of Theorem 1.2, if additionally $A \in \Delta_2$, then*

$$\frac{C}{p_A \|\omega\|_{L^\infty(\Omega)} M_A(C d_\Omega^s)} \leq \Lambda_{1,\alpha}^s$$

where $p_A = \sup_{\beta>0} a(\beta)\beta/A(\beta)$.

When $A \in \Delta_2$ we can improve Theorem 1.2 by replacing d_Ω with r_Ω.

Proof of Theorem 1.3. The proof is analogous to that of Theorem 1.2, noting that in (4-6), the Hardy inequality stated in Proposition 2.7, together with (3-2), leads to

$$\frac{1}{C}\int_{\mathbb{R}^n} A\left(\frac{|u_\alpha(x)|}{r_\Omega^s}\right) dx \le \int_\Omega A\left(\frac{|u_\alpha(x)|}{\delta_\Omega(x)^s}\right) dx \\ \le \iint_{\mathbb{R}^{2n}} A(|D^s u_\alpha|)\, d\nu_n = \lambda^s_{1,\alpha}\int_\Omega \omega A(|u_\alpha|)\, dx,$$

where $\delta_\Omega(x)$ denotes the distance from x to $\partial\Omega$, giving the desired result. □

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