Pacific Journal of Mathematics

Volume 339 No. 2 December 2025

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2025 is US $677/year for the electronic version, and $917/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS
Vol. 339, No. 2, 2025
https://doi.org/10.2140/pjm.2025.339.201

STARTING THE STUDY OF OUTER LENGTH BILLIARDS

LUCA BARACCO, OLGA BERNARDI AND CORENTIN FIEROBE

We focus on outer length billiard dynamics, acting on the exterior of a strictly convex planar domain. We first show that ellipses are totally integrable. We then provide an explicit representation of first order terms for the formal Taylor expansion of the corresponding Mather's β-function. Finally, we provide explicit Lazutkin coordinates up to order 4.

1. Introduction

The aim of the present paper is starting an accurate study of outer length billiards, first described by P. Albers and S. Tabachnikov in 2024, see [2, Section 3.4]. These billiards are the counterpart of Birkhoff ones since the generating function is the outer length instead of the inner length. They are also called "fourth billiards". In fact, two billiards systems — Birkhoff and outer area billiards — have been extensively studied; we refer respectively to [22] and [21] for exhaustive surveys. Another type of billiards, namely symplectic billiards, whose generating function is the inner area, were introduced in 2018 by P. Albers and S. Tabachnikov [1] and their study started to become more intensive only recently. We refer to [4], [6] and [23] for integrability results and to [5] and [12] for area spectral rigidity results for symplectic billiards. Regarding outer length billiards, to the best of our knowledge, they were not studied yet. However, the seminal idea on the base of the definition of this dynamical system (detecting, in particular, circumscribed polygons to a strictly convex domain with minimal perimeter) can already be found in some former papers in convex planar geometry; see [11, Theorem 1] and [10, Section 2], for example.

We first give all the details to introduce this dynamical system, acting on the exterior of a strictly convex planar domain. We then prove, using elementary planar geometry, that ellipses are totally integrable, that is the phase space is fully foliated by continuous invariant curves which are not null-homotopic.

MSC2020: primary 37E40; secondary 37C83.
Keywords: monotone twist maps, mathematical billiards, Mather beta function, geometric approximations.

We successively focus on the main topic of the paper, which is providing an explicit representation of first order terms for the formal Taylor expansion of Mather's β-function (or minimal average function) for outer length billiards. In particular, we write these coefficients (up to order 5) by means of the ordinary curvature and length of the boundary of the billiard table. As already noticed, for such a dynamical system, Mather's β-function is related to the minimal perimeter of polygons circumscribed to a strictly convex domain. These perimeters are special cases (i.e., for periodic trajectories of winding number $= 1$) of the corresponding marked length spectrum for outer length billiards.

Finally, by using the computations we made to obtain minimal average function's coefficients, we provide explicit Lazutkin coordinates up to order 4 and discuss straightforward facts regarding the existence/nonexistence of caustics for outer length billiards.

2. Twist maps and Mather's β-function

Let $\mathbb{T} \times (a, b)$ be the annulus, where $\mathbb{T} = \mathbb{R}/\mathbb{Z} = [0, 1]/\sim$ identifying $0 \sim 1$ and (eventually) $a = -\infty$ and/or $b = +\infty$. Given a diffeomorphism $\Phi : \mathbb{S}^1 \times (a, b) \to \mathbb{S}^1 \times (a, b)$, we denote by

$$\phi : \mathbb{R} \times (a, b) \to \mathbb{R} \times (a, b), \quad (x_0, y_0) \mapsto (x_1, y_1)$$

a lift of Φ to the universal cover. Then ϕ is a diffeomorphism and $\phi(x + 1, y) = \phi(x, y) + (1, 0)$. In the case where a (resp. b) is finite, we assume that ϕ extends continuously to $\mathbb{R} \times \{a\}$ (resp. $\mathbb{R} \times \{b\}$) by a rotation of fixed angle:

$$\phi(x_0, a) = (x_0 + \rho_a, a) \quad (\text{resp. } \phi(x_0, b) = (x_0 + \rho_b, b)). \tag{2-1}$$

Once fixed the lift, the numbers ρ_a, ρ_b are unique. The choice of ρ_a (resp. ρ_b) if $a = -\infty$ (resp. $b = +\infty$) depends on the dynamics at infinity. For example, in the case of outer length billiards, where $b = +\infty$, it is natural to set $\rho_b = \frac{1}{2}$. We refer to point 1 of Section 3 for details.

We recall for convenience the definition of a monotone twist map (see [18, page 2], for instance).

Definition 1. A monotone twist map $\phi : \mathbb{R} \times (a, b) \to \mathbb{R} \times (a, b)$, $(x_0, y_0) \mapsto (x_1, y_1)$ is a diffeomorphism satisfying

1. $\phi(x_0 + 1, y_0) = \phi(x_0, y_0) + (1, 0)$.
2. ϕ preserves orientations and the boundaries of $\mathbb{R} \times (a, b)$.
3. ϕ extends to the boundaries by rotation, as in (2-1).

4. ϕ satisfies a monotone twist condition, that is

$$\frac{\partial x_1}{\partial y_0} > 0. \tag{2-2}$$

5. ϕ is exact symplectic; this means that there exists a generating function $H \in C^2(\mathbb{R} \times \mathbb{R}; \mathbb{R})$ for ϕ such that

$$y_1\, dx_1 - y_0\, dx_0 = dH(x_0, x_1). \tag{2-3}$$

Clearly, $H(x_0+1, x_1+1) = H(x_0, x_1)$ and, due to the twist condition, the domain of H is the strip $\{(x_0, x_1) : \rho_a + x_0 < x_1 < x_0 + \rho_b\}$. Moreover, equality (2-3) reads

$$\begin{cases} y_1 = H_2(x_0, x_1), \\ y_0 = -H_1(x_0, x_1), \end{cases} \tag{2-4}$$

and the twist condition (2-2) becomes $H_{12} < 0$. As a consequence, $\{(x_i, y_i)\}_{i\in\mathbb{Z}}$ is an orbit of ϕ if and only if $H_2(x_{i-1}, x_i) = y_i = -H_1(x_i, x_{i+1})$ for all $i \in \mathbb{Z}$. Formally, this means that the corresponding bi-infinite sequence $x := \{x_i\}_{i\in\mathbb{Z}}$ is a so-called critical configuration of the action functional $\sum_{i\in\mathbb{Z}} H(x_i, x_{i+1})$. In such a setting, minimal orbits play a fundamental role. We recall that a critical configuration x of ϕ is minimal if every finite segment of x minimizes the action functional with fixed end points (we refer to [18, page 7] for details). Clearly, all these facts remain true if we consider a monotone twist map on $\{(x_0, x_1) :\ u_a(x_0) < x_1 < u_b(x_0)\}$, where $u_a, u_b : \mathbb{R} \to \mathbb{R}$ are two continuous 1-periodic functions such that $u_a < u_b$.

For a twist map ϕ generated by H, we finally introduce the rotation number and the Mather β-function (or minimal average action).

Definition 2. The rotation number of an orbit $\{(x_i, y_i)\}_{i\in\mathbb{Z}}$ of ϕ is

$$\rho := \lim_{i\to\pm\infty} \frac{x_i}{i}$$

if such a limit exists.

An important class of monotone twist maps are planar billiard maps. In this setting, the rotation number of a periodic trajectory is the rational number

$$\frac{m}{n} = \frac{\text{winding number}}{\text{number of reflections}} \in \left(0, \tfrac{1}{2}\right];$$

see [18, page 40] for details.

In view of the celebrated Aubry–Mather theory (see [3], for example), a monotone twist map possesses minimal orbits for every rotation number ρ inside the so-called twist interval (ρ_a, ρ_b). As a consequence, we can associate to each ρ the average action of any minimal orbit having that rotation number.

Definition 3. The Mather β-function of ϕ is $\beta : (\rho_a, \rho_b) \to \mathbb{R}$ with

$$\beta(\rho) := \lim_{N\to\infty} \frac{1}{2N} \sum_{i=-N}^{N-1} H(x_i, x_{i+1})$$

where $\{x_i\}_{i\in\mathbb{Z}}$ is any minimal configuration of ϕ with rotation number ρ.

In the framework of Birkhoff billiards, A. Sorrentino in [19] gave an explicit representation of the coefficients of the (formal) Taylor expansion at zero of the corresponding Mather's β-function. More recently, J. Zhang in [24] got (locally) an explicit formula for this function via a Birkhoff normal form. Moreover, M. Bialy in [9] obtained an explicit formula for Mather's β-function for ellipses by using a nonstandard generating function, involving the support function. Regarding symplectic and outer billiards, the first two authors and A. Nardi in [7] computed explicitly the higher order terms of such an expansion, by using tools from affine differential geometry. As anticipated, one of the target of the present paper is writing explicitly these coefficients (up to order 5) in the case of forth billiards.

3. The dynamical system

Let Ω be a strictly convex planar domain with smooth boundary $\partial\Omega$. Assume that the perimeter of $\partial\Omega$ is $\ell = |\partial\Omega|$. Fixing the positive counterclockwise orientation, let $\gamma : \mathbb{T} \to \partial\Omega$ be the smooth arc-length parametrization of $\partial\Omega$. For every $s \in \mathbb{T}$, we denote by $s^* \in \mathbb{T}$ the (unique, by strict convexity) arc-length parameter such that $T_{\gamma(s)}\partial\Omega = T_{\gamma(s^*)}\partial\Omega$. We refer to

$$\mathcal{P} = \{(s,r) \in \mathbb{T}\times\mathbb{T} : s < r < s^*\}$$

as the (open, positive) phase space and we define the outer length billiard map as follows [2, Section 3.4].

Since Ω is strictly convex, to every point $P \in \mathbb{R}^2\setminus \mathrm{cl}(\Omega)$ can be uniquely associated a pair $(s_0, s_1) \in \mathbb{T}\times\mathbb{T}$ with $s_0 < s_1$ and such that the lines $P\gamma(s_0)$ and $P\gamma(s_1)$ are the (negative and positive) tangents to $\partial\Omega$. Consider the circle in $\mathbb{R}^2 \setminus \Omega$ tangent to $\partial\Omega$ at $\gamma(s_1)$ and to the line $P\gamma(s_0)$. Then the image P' of P is defined as the intersection point between the lines $P\gamma(s_1)$ and the other common tangent line of the circle and $\partial\Omega$ (hence passing through P' and $\gamma(s_2)$):

$$T : \mathcal{P} \to \mathcal{P}, \quad (s_0, s_1) \mapsto (s_1, s_2).$$

(We refer to Figure 1.) Setting $\varepsilon_0 = s_1 - s_0$ and

$$\hat{\mathcal{P}} = \{(s,\varepsilon) \in \mathbb{T}\times\mathbb{R} :\ 0 < \varepsilon < s^* - s\},$$

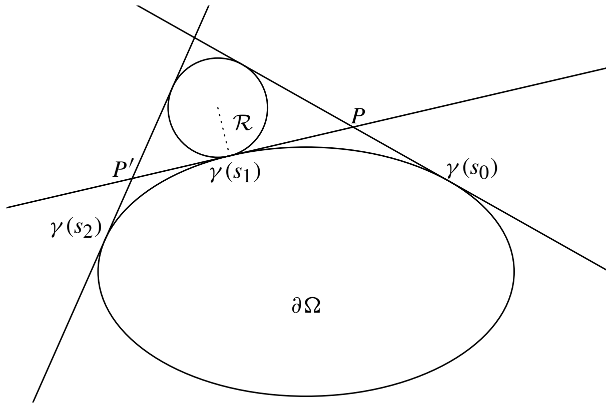


Figure 1. The outer-length billiard map around the domain Ω associates the point P to the point P'.

the outer length billiard map can be equivalently defined as

$$T : \hat{\mathcal{P}} \to \hat{\mathcal{P}}, \quad (s_0, \varepsilon_0) \mapsto (s_1, \varepsilon_1).$$

Here are some properties of the outer length billiard map.

1. T is continuous and can be continuously extended so that $T(s, s) = (s, s)$ and $T(s, s^*) = (s^*, s)$.

2. The function

$$H : \mathcal{P} \to \mathbb{R}, \quad H(s_0, s_1) := |P\gamma(s_0)| + |P\gamma(s_1)|,$$

generates T, that is

$$T(s_0, s_1) = (s_1, s_2) \iff H_2(s_0, s_1) + H_1(s_1, s_2) = 0. \tag{3-1}$$

See [2, Lemma 3.1] for the proof. In view of (3-1), we can equivalently refer to

$$\bar{H} : \mathcal{P} \to \mathbb{R}, \quad \bar{H}(s_0, s_1) := |P\gamma(s_0)| + |P\gamma(s_1)| - s_1 + s_0,$$

as a generating function, which is exactly the Lazutkin parameter of $\partial\Omega$, interpreted as convex caustic for a Birkhoff billiard.

3. T is a twist map preserving the area form $-H_{12}(s_0, s_1)\, ds_0 \wedge ds_1$.

4. By introducing new variables

$$y_0 = -H_1(s_0, s_1), \quad y_1 = H_2(s_0, s_1),$$

(s, y) are coordinates on $\mathcal{P}$ and the outer length billiard map results a (negative) twist map, since

$$\frac{\partial y_1}{\partial s_0} = H_{12}(s_0, s_1) = -\frac{k(s_0)k(s_1)H(s_0, s_1)}{2\sin^2(\varphi/2)} < 0,$$

where φ is the angle between the tangent lines $P\gamma(s_0)$ and $P\gamma(s_1)$ (see also [2, page 11]). In these coordinates, the preserved area form is the standard one: $ds \wedge dy$.

5. The marked length spectrum for the outer length billiard is the map $\mathcal{ML}_o(\Omega) : \mathbb{Q} \cap \left(0, \frac{1}{2}\right) \to \mathbb{R}$ that associates to any m/n in lowest terms the minimal perimeter of the periodic trajectories having rotation number m/n. We refer to [18, Sections 3.1 and 3.2] for a general treatment of the marked spectrum. Clearly, periodic outer length billiard minimal trajectories (with winding number $= 1$) correspond to convex polygons realizing the minimal (circumscribed) perimeter, so that

$$\beta\left(\tfrac{1}{n}\right) = \tfrac{1}{n}\mathcal{ML}_o(\Omega)\left(\tfrac{1}{n}\right). \tag{3-2}$$

3.1. ***Circles and ellipses.*** As expected, the outer length billiard on the circle (of center O) is totally integrable: the phase space is completely foliated by concentric invariant circles. By using as coordinates $(\alpha_0, \alpha_1) \in \mathbb{T} \times \mathbb{T}$, where α_0 and α_1 are respectively the angles of $O\gamma(s_0)$ and $O\gamma(s_1)$ with respect to the positive horizontal direction, the generating function in the case of disk of unit radius is

$$H(\alpha_0, \alpha_1) = 2\tan\frac{\alpha_1 - \alpha_0}{2}.$$

Equivalently, in terms of $(\alpha_0, y_0) = (\alpha_0, -H_1(\alpha_0, \alpha_1)) = \left(\alpha_0, 1 + \tan^2\frac{\alpha_1 - \alpha_0}{2}\right)$, we have

$$H(\alpha_0, y_0) = 2\sqrt{y_0 - 1}$$

and total integrability follows.

An unexpected fact — at least from the authors' point of view, since the billiard dynamics is not invariant by affine transformations — is that also the outer length billiard on the ellipse is totally integrable, as stated in the next proposition.

Proposition 4. *Let $\mathcal{E}$ and Γ be two confocal nested ellipses, $\mathcal{E} \subset \Gamma$. Then Γ is a caustic for the outer-length billiard dynamics outside $\mathcal{E}$.*

The proof of Proposition 4 relies on a lemma from elementary plane geometry:

Lemma 5 [20, Lemma 2.4]. *Let $P_0, P_1 \in \Gamma$ two distinct points such that the line P_0P_1 is tangent to $\mathcal{E}$ at a point Q. Let R be the intersection point of the tangent lines to Γ at P_0 and P_1. Then the lines P_0P_1 and RQ are orthogonal.* (See Figure 2.)

Proof of Proposition 4. Let a point P_0 on Γ. Consider the positive tangent line to $\mathcal{E}$ at a point Q and passing through P_0. Let $P_1 \in \Gamma$ be the intersection point of the latter tangent line P_0Q with Γ, see Figure 3. We need to show that P_1 is the image of P_0 under the outer-length billiard reflection outside $\mathcal{E}$. Consider the point P, such that PP_0 and PP_1 are the two tangent lines to $\mathcal{E}$ passing through P, see Figure 3. Since $\mathcal{E}$ and Γ are confocal, $\mathcal{E}$ is a caustic for the classical billiard in Γ. In particular, the tangent line $T_{P_0}\Gamma$ is a bisector of the angle $\widehat{P_1P_0P}$. With

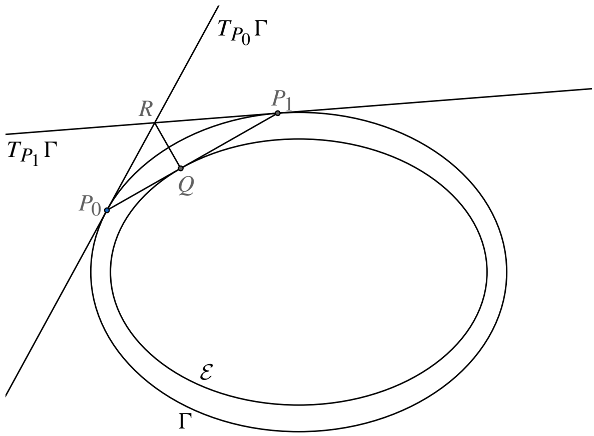


Figure 2. The line RQ is orthogonal to the line P_0P_1.

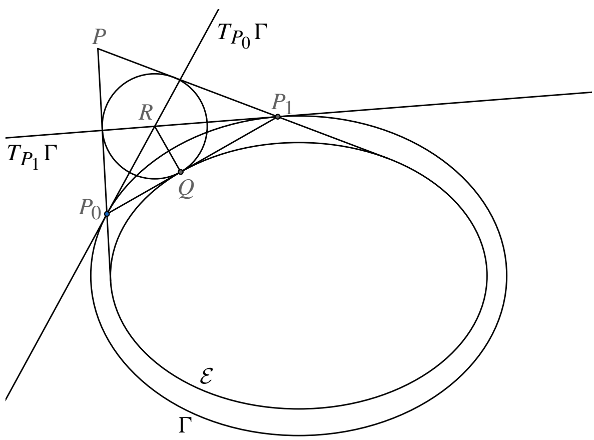


Figure 3. The point $P_0 \in \Gamma$ is reflected to the point $P_1 \in \Gamma$ by the outer-length billiard dynamics around $\mathcal{E}$.

the same argument the tangent line $T_{P_1}\Gamma$ is a bisector of the angle $\widehat{P_0P_1P}$. Hence $T_{P_0}\Gamma$ and $T_{P_1}\Gamma$ intersects at a point R which is the center of the inscribed circle $\mathcal{D}$ to the triangle P_0PP_1. By Lemma 5, the lines RQ and P_0P_1 are orthogonal. In particular $\mathcal{D}$ is tangent to the ellipse $\mathcal{E}$. This implies that P_1 is obtained from P_0 by the outer-length billiard law of reflection. □

It would be interesting to investigate if these are the only cases. This fundamental problem (possibly to be studied by an integral inequality à la Bialy [8]) may present nontrivial difficulties, due to the infinite total area of the phase space.

4. Asymptotic expansions

S. Marvizi and R. Melrose's theory, first stated and proved for Birkhoff billiards [15, Theorem 3.2], can be applied to the general case of (strongly) billiard-like maps, see [13, Section 2.1]. As an outcome, the following expansion at $\rho = 0$ of the corresponding minimal average function holds:

$$\beta(\rho) \sim \beta_1 \rho + \beta_3 \rho^3 + \beta_5 \rho^5 + \cdots$$

in terms of odd powers of ρ. It is well-known (see, e.g., [15, Section 7] again) that for usual billiards the sequence $\{\beta_k\}$ can be interpreted as a spectrum of a differential operator, see also Remark 2.11 in [1]. The question is wide open for other types of billiards, included outer length billiards.

In this section, we gather all the technical results in order to prove the next theorem, providing the coefficient β_5 for the outer length billiard map. This result is a refinement of [17, Theorem 1(iii)]. In fact, in a genuine framework of convex planar geometry, D.E. Vitale and R.A. McClure computed β_3 by using as coordinate the support function and as parameter the angle with respect to a fixed direction.

Theorem 6. *Let Ω be a strictly convex planar domain with smooth boundary $\partial\Omega$. Suppose that $\partial\Omega$ has everywhere positive curvature. Denote by $k(s)$ the (ordinary) curvature of $\partial\Omega$ with arc-length parameter s. Let ℓ be the length of the boundary and*

$$L := \int_0^\ell k^{2/3}(s)\, ds.$$

The formal Taylor expansion at $\rho = 0$ of Mather's β-function for the outer length billiard map has coefficients

$$\beta_{2k} = 0 \textit{ for all } k, \quad \beta_1 = \ell, \quad \beta_3 = \frac{L^3}{12},$$
$$\beta_5 = L^4 \int_0^\ell \left(\frac{k^{4/3}(s)}{120} + \frac{k^{-\frac{8}{3}}(s) k'^2(s)}{2160} \right) ds.$$

As expected, a straightforward consequence of the previous result is that, as for other billiards, also for outer length ones, the two coefficients β_1 and β_3 allow one to recognize a circle among all strictly convex planar domains.

Corollary 7. *The coefficients β_1 and β_3 recognize a circle. In particular,*

$$3\beta_3 + \pi^2 \beta_1 \leq 0$$

with equality if and only if $\partial\Omega$ is a circle.

Proof. We apply Hölder's inequality with $p = 3/2$ and $q = 3$ to obtain

$$L = \int_0^\ell k^{2/3}(s)\,ds \le \Big(\int_0^\ell (k^{2/3}(s))^{3/2} ds\Big)^{2/3} \Big(\int_0^\ell 1^3 ds\Big)^{1/3} = (2\pi)^{2/3}\ell^{1/3}, \tag{4-1}$$

since $\int_0^\ell k(s)\,ds = 2\pi$. Using the expressions of β_1 and β_3 found in Theorem 6, we can write

$$3\beta_3 + \pi^2\beta_1 = \tfrac{1}{4}L^3 - \pi^2\ell \le \tfrac{1}{4}(2\pi)^2\ell - \pi^2\ell = 0.$$

In the case of equality, namely if $3\beta_3 + \pi^2\beta_1 = 0$, then $L = (2\pi)^{2/3}\ell^{1/3}$, and the case of equality is reached in (4-1). In that case, k is constant. Hence Ω is a disk. □

Remark 8. Let $\mathcal{P}_n^c$ be the set of all convex polygons with at most n vertices which are circumscribed to Ω. We define

$$\delta(\Omega; \mathcal{P}_n^c) := \inf\{\ell(P_n) : P_n \in \mathcal{P}_n^c\},$$

where $\ell(P_n)$ is the perimeter length of P_n. Clearly, essentially in view of equality (3-2), Theorem 6 gives also the formal expansion of $\delta(\Omega; \mathcal{P}_n^c)$ at $n \to +\infty$.

Since we use the arc-length parametrization of $\partial\Omega$, it is useful to recall that

$$\begin{cases} \gamma'' = kJ\gamma', \quad \gamma''' = -k^2\gamma' + k'J\gamma', \\ \gamma^{(4)} = -3kk'\gamma' + (-k^3 + k'')J\gamma', \\ \gamma^{(5)} = (k^4 - 4kk'' - 3k'^2)\gamma' + (-6k^2k' + k''')J\gamma', \\ \gamma^{(6)} = (10k^3k' - 10k'k'' - 5kk''')\gamma' + (k^5 - 10k^2k'' - 15kk'^2 + k^{(4)})J\gamma', \end{cases} \tag{4-2}$$

where J is the counterclockwise rotation of angle $\pi/2$.

Proposition 9. *For* $0 \le r \le s \le \ell$, *let* $\delta := s - r$. *Then*

$$H(r,s) = \delta + \frac{k^2(r)}{12}\delta^3 + \frac{k(r)k'(r)}{12}\delta^4 + \frac{2k^4(r) + 4k'^2(r) + 7k(r)k''(r)}{240}\delta^5 + O(\delta^6), \tag{4-3}$$

uniformly as $\delta \to 0$.

Proof. We start by writing separately the Taylor expansions of numerator and denominator of the generating function

$$H(r,s) = \frac{(\gamma(s) - \gamma(r)) \wedge (\gamma'(s) - \gamma'(r))}{\gamma'(r) \wedge \gamma'(s)}. \tag{4-4}$$

From now on, we omit the dependence on r of γ, k and their derivatives. We have

$$\gamma(s) - \gamma(r) = \gamma'\delta + \frac{\gamma''}{2}\delta^2 + \frac{\gamma'''}{6}\delta^3 + \frac{\gamma^{(4)}}{24}\delta^4 + \frac{\gamma^{(5)}}{5!}\delta^5 + O(\delta^6)$$

and likewise for $\gamma'(s) - \gamma'(r)$; thus the Taylor expansion of the numerator of (4-4) is

$$k\delta^2+\frac{k'}{2}\delta^3+\frac{1}{6}\frac{2k''-k^3}{2}\delta^4+\frac{k'''-3k^2k'}{24}\delta^5+\frac{2k^5-48kk'^2-29k^2k''+6k^{(4)}}{720}\delta^6+O(\delta^7),$$

where we have used (4-2). Similarly, the Taylor expansion of the denominator is

$$\begin{aligned}\gamma'(r)\wedge\gamma'(s)&\\ &=\gamma'\wedge\Big(\gamma'+\gamma''\delta+\tfrac{\gamma'''}{2}\delta^2+\tfrac{\gamma^{(4)}}{6}\delta^3+\tfrac{\gamma^{(5)}}{24}\delta^4+\tfrac{\gamma^{(6)}}{5!}\delta^5+O(\delta^6)\Big)\\ &=k\delta+\tfrac{k'}{2}\delta^2+\tfrac{-k^3+k''}{6}\delta^3+\tfrac{-6k^2k'+k'''}{24}\delta^4+\tfrac{k^5-10k^2k''-15kk'^2+k^{(4)}}{5!}\delta^5+O(\delta^6)\\ &=k\delta\Big(1-\tfrac{k'}{2k}\delta+\tfrac{2k^4+3k'^2-2kk''}{12k^2}\delta^2-\tfrac{3k'^3-2k'(k^4+2kk'')+k^2k'''}{24k^3}\delta^3+D\delta^4+O(\delta^5)\Big)^{-1},\end{aligned}$$

where

$$D=\frac{45k'^4-90kk'^2k''+30k^2k'k'''+2k^2(7k^6+10k^3k''+10k''^2-3kk^{(4)})}{720k^4}.$$

Using the above expansions for numerator and denominator, we obtain (4-3). □

Proposition 10. *The outer length billiard map $T:(s_0,\varepsilon_0)\mapsto(s_1,\varepsilon_1)$ has the expansion*

$$\begin{cases}s_1=s_0+\varepsilon_0,\\ \varepsilon_1=\varepsilon_0+A(s_0)\varepsilon_0^2+B(s_0)\varepsilon_0^3+C(s_0)\varepsilon_0^4+O(\varepsilon_0^5),\end{cases}\tag{4-5}$$

where

$$\begin{aligned}&A(s)=-\frac{2k'(s)}{3k(s)},\quad B(s)=\frac{10k'^2(s)}{9k^2(s)}-\frac{2k''(s)}{3k(s)},\\ &C(s)=\frac{-24k^4(s)k'(s)-1160k'^3(s)+1200k(s)k'(s)k''(s)-216k^2(s)k'''(s)}{540k^3(s)}.\end{aligned}\tag{4-6}$$

Proof. We start by writing separately the Taylor expansions of numerator and denominator of the radius $\mathcal{R}$ of the circle in $\mathbb{R}^2\setminus\Omega$ tangent to $\partial\Omega$ at $\gamma(s_1)$ and to the line $P\gamma(s_0)$; see Figure 1.

$$\mathcal{R}=\frac{(\gamma(s_1)-\gamma(s_0))\wedge\gamma'(s_1)}{1+\gamma'(s_1)\cdot\gamma'(s_0)}=\frac{(\gamma(s_2)-\gamma(s_1))\wedge\gamma'(s_2)}{1+\gamma'(s_2)\cdot\gamma'(s_1)}.$$

From now on, we indicate, by subscripting 1, the dependence on s_1 of γ, k and their derivatives. Recall that $\varepsilon_1=s_2-s_1$. The Taylor expansion of the numerator is

$$\begin{aligned}&\Big(\gamma_1'\varepsilon_1+\tfrac{\gamma_1''}{2}\varepsilon_1^2+\tfrac{\gamma_1'''}{6}\varepsilon_1^3+\tfrac{\gamma_1^{(4)}}{24}\varepsilon_1^4+\tfrac{\gamma_1^{(5)}}{5!}\varepsilon_1^5+O(\varepsilon_1^6)\Big)\\ &\qquad\wedge\Big(\gamma_1'+\gamma_1''\varepsilon_1+\tfrac{\gamma_1'''}{2}\varepsilon_1^2+\tfrac{\gamma_1^{(4)}}{6}\varepsilon_1^3+\tfrac{\gamma_1^{(5)}}{24}\varepsilon_1^4+O(\varepsilon_1^5)\Big)\\ &\qquad\qquad=\tfrac{k_1}{2}\varepsilon_1^2+\tfrac{k_1'}{3}\varepsilon_1^3+\Big(\tfrac{-k_1^3+3k_1''}{24}\Big)\varepsilon_1^4+\Big(\tfrac{-9k_1^2k_1'+4k_1'''}{120}\Big)\varepsilon_1^5+O(\varepsilon_1^6),\end{aligned}$$

where we have used (4-2).

Similarly, the Taylor expansion of the denominator is

$$1+\Big(\gamma_1'+\gamma_1''\varepsilon_1+\tfrac{\gamma_1'''}{2}\varepsilon_1^2+\tfrac{\gamma_1^{(4)}}{6}\varepsilon_1^3+\tfrac{\gamma_1^{(5)}}{24}\varepsilon_1^4+O(\varepsilon_1^5)\Big)\cdot\gamma_1'$$
$$=2\Big(1-\tfrac{k_1^2}{4}\varepsilon_1^2-\tfrac{k_1k_1'}{4}\varepsilon_1^3+O(\varepsilon_1^4)\Big)=2\Big(1+\tfrac{k_1^2}{4}\varepsilon_1^2+\tfrac{k_1k_1'}{4}\varepsilon_1^3+O(\varepsilon_1^4)\Big)^{-1}.$$

Using the above expansions for numerator and denominator, we obtain

$$2\mathcal{R}=\frac{k_1}{2}\varepsilon_1^2+\frac{k_1'}{3}\varepsilon_1^3+\frac{2k_1^3+3k_1''}{24}\varepsilon_1^4+\frac{16k_1^2k_1'+4k_1'''}{120}\varepsilon_1^5+O(\varepsilon_1^6) \tag{4-7}$$

or, equivalently,

$$2\mathcal{R}=\frac{k_1}{2}\varepsilon_0^2-\frac{k_1'}{3}\varepsilon_0^3+A_4\varepsilon_0^4+A_5\varepsilon_0^5+O(\varepsilon_0^6), \tag{4-8}$$

with

$$A_4=\frac{2k_1^3+3k_1''}{24},\qquad A_5:=\frac{16k_1^2k_1'+4k_1'''}{120}$$

Substituting the powers of the expansion

$$\varepsilon_1=\varepsilon_0+\alpha(s_1)\varepsilon_0^2+\beta(s_1)\varepsilon_0^3+\gamma(s_1)\varepsilon_0^4+O(\varepsilon_0^5)$$

in (4-7), we obtain (omitting the dependence on s_0 in α, β and γ)

$$2\mathcal{R}=\frac{k_1}{2}\varepsilon_0^2+\left(k_1\alpha+\frac{k_1'}{3}\right)\varepsilon_0^3+\left(\frac{k_1}{2}(\alpha^2+2\beta)+k_1'\alpha+A_4\right)\varepsilon_0^4$$
$$+\big(k_1(\alpha\beta+\gamma)+k_1'(\alpha^2+\beta)+4\alpha A_4+A_5\big)\varepsilon_0^5+O(\varepsilon_0^6).$$

Equaling this to (4-8), we obtain

$$\alpha(s)=-\frac{2k'(s)}{3k(s)},\qquad \beta(s)=\frac{4k'^2(s)}{9k^2(s)},$$
$$\gamma(s)=\frac{-320k'^3(s)+3k'(s)(-8k^4(s)+60k(s)k''(s))-36k^2(s)k'''(s)}{540k^3(s)}.$$

Finally, from

$$\varepsilon_1=\varepsilon_0+\alpha(s_0)\varepsilon_0^2+(\alpha'(s_0)+\beta(s_0))\varepsilon_0^3+\left(\frac{\alpha''(s_0)}{2}+\beta'(s_0)+\gamma(s_0)\right)\varepsilon_0^4+O(\varepsilon_0^5),$$

we obtain the formulas (4-6). □

Proposition 11. *Let $q \geq 3$. The q-periodic orbits of rotation number $1/q$ for the outer length billiard map have the expansion*

$$\begin{cases} s_k = s_0^q + a_0(k/q) + \dfrac{a_1(k/q)}{q} + \dfrac{a_2(k/q)}{q^2} + O\left(\dfrac{1}{q^3}\right) \\ \varepsilon_k = \dfrac{b_1(k/q)}{q} + \dfrac{b_2(k/q)}{q^2} + \dfrac{b_3(k/q)}{q^3} + O\left(\dfrac{1}{q^4}\right) \end{cases} \tag{4-9}$$

where $s_0^q \in \mathbb{R}$ converges to 0 with q, $a_0 : \mathbb{R} \to \mathbb{R}$ is a map such that $a_0(x+1) = a_0(x) + \ell$ for any x and $a_1, a_2, b_1, b_2, b_3 : \mathbb{R} \to \mathbb{R}$ are 1-periodic maps which can be expressed as

$$\begin{cases} a_0^{-1}(s) = \dfrac{1}{L}\displaystyle\int_0^s k^{2/3}(r)\,dr := x(s), \quad L := \int_0^\ell k^{2/3}(r)\,dr, \\ a_1(x) = 0, \\ a_2(x) = k^{-\frac{2}{3}}(a_0(x)) \\ \qquad \times \left(\displaystyle\int_0^x L^3\Big(\frac{1}{810}\big(9k''k^{-\frac{7}{3}} - 12(k')^2k^{-\frac{10}{3}}\big) + \frac{k^{\frac{2}{3}}}{15}\Big)(a_0(t))\,dt + cx\right), \\ b_1(x) = a_0'(x) = Lk^{-2/3}(a_0(x)), \\ b_2(x) = \dfrac{a_0''(x)}{2} = -\dfrac{L^2k'(a_0(x))k^{-7/3}(a_0(x))}{3}, \\ b_3(x) = a_2' + \dfrac{a_0'''}{6}. \end{cases} \tag{4-10}$$

The constant c in the expression of a_2 is such that

$$L^3\Big(\frac{1}{810}\big(9k''k^{-\frac{7}{3}} - 12(k')^2k^{-\frac{10}{3}}\big) + \frac{k^{\frac{2}{3}}}{15}\Big) + c$$

has zero mean.

Proof. Since the points in the orbits are equidistributed as $q \to +\infty$, for any q we can choose the first point of the orbit s_0^q such as $s_0^q \to 0$ for $q \to +\infty$. For simplicity, we omit the dependence of a_i and b_j on k/q.

Combining the expansions in (4-5), we have

$$\begin{aligned} \varepsilon_{k+1} - \varepsilon_k &= A(s_k)\varepsilon_k^2 + B(s_k)\varepsilon_k^3 + C(s_k)\varepsilon_k^4 + O(\varepsilon_k^5) \\ &= \frac{A(s_0^q + a_0)b_1^2}{q^2} + \frac{B(s_0^q + a_0)b_1^3 + A'(s_0^q + a_0)a_1b_1^2 + 2A(s_0^q + a_0)b_1b_2}{q^3} \\ &\qquad + \frac{F(a_i, b_j)}{q^4} + O\left(\frac{1}{q^5}\right), \end{aligned}$$

where

$$F(a_i, b_j) := A(s_0^q+a_0)b_2^2+2A(s_0^q+a_0)b_1b_3+2A'(s_0^q+a_0)a_1b_1b_2+A'(s_0^q+a_0)a_2b_1^2 + A''(s_0^q+a_0)a_1^2b_1^2/2+3B(s_0^q+a_0)b_1^2b_2+B'(s_0^q+a_0)a_1b_1^3+C(s_0^q+a_0)b_1^4.$$

Moreover, directly from the second expansion in (4-9), we have

$$\varepsilon_{k+1}-\varepsilon_k = \frac{b_1'}{q^2}+\frac{b_2'+b_1''/2}{q^4}+\frac{b_3'+b_2''/2+b_1'''/6}{q^5}+O\left(\frac{1}{q^5}\right).$$

Equaling these two expansions, we obtain that a_i and b_j solve

$$\text{(4-11)} \qquad \begin{cases} A(s_0^q+a_0)b_1^2 = b_1', \\ B(s_0^q+a_0)b_1^3 + A'(s_0^q+a_0)a_1b_1^2+2A(s_0^q+a_0)b_1b_2 = b_2'+b_1''/2, \\ F(a_i, b_j) = b_3'+b_2''/2+b_1'''/6. \end{cases}$$

On the other hand, directly from the first expansion in (4-5), we conclude that

$$s_{k+1}-s_k = \frac{a_0'}{q}+\frac{a_1'+a_0''/2}{q^2}+\frac{a_2'+a_1''/2+a_0'''/6}{q^3}+O\left(\frac{1}{q^4}\right),$$

which, compared which the second expansion in (4-5), gives the system

$$\text{(4-12)} \qquad a_0' = b_1, \quad a_1'+a_0''/2 = b_2, \quad a_2'+a_1''/2+a_0'''/6 = b_3.$$

Expressions of a_0 and b_1. To compute a_0 and b_1, we solve the system

$$\text{(4-13)} \qquad b_1 = a_0', \quad b_1' = A(s_0^q+a_0)b_1^2.$$

Replacing b_1 by a_0' in the second equation, we get

$$\text{(4-14)} \qquad a_0'' = (a_0')^2A(s_0^q+a_0).$$

If we let $A_1(s) = -\frac{2}{3}\log k(s)$ be a primitive of A, it follows from (4-14) that

$$\left(a_0'e^{-A_1(s_0^q+a_0)}\right)' = 0.$$

Hence $a_0'e^{-A_1(s_0^q+a_0)}$ is constant. Consider now $A_2(s) = \int_0^s k^{2/3}(r)\,dr$, which is a primitive of $\exp(-A_1)$. We just proved that $A_2(s_0^q+a_0)$ has constant derivative, hence it must be of the form $A_2(s_0^q+a_0(x)) = ux+v$ for any $x \in \mathbb{R}$, where $u, v \in \mathbb{R}$. Since, by definition, $A_2(s_0^q+a_0(0)) = A_2(s_0^q) = v$, we have $v = A_2(s_0^q)$. The expression of u is

$$u = A_2(s_0^q+a_0(1)) - A_2(s_0^q) = A_2(s_0^q+\ell) - A_2(s_0^q) = \int_0^\ell k^{2/3}(r)\,dr.$$

Finally, b_1 follows from $b_1 = a_0'$.

Expressions of a_1 and b_2. To compute a_1 and b_2, we solve the system

$$\begin{cases} b_2 = a_1' + a_0''/2, \\ b_2' + b_1''/2 = B(s_0^q + a_0)b_1^3 + A'(s_0^q + a_0)a_1 b_1^2 + 2A(s_0^q + a_0)b_1 b_2. \end{cases} \tag{4-15}$$

The terms containing a_1 nor b_2 can be computed using the expression of a_0 and b_1 we just obtained. Let us replace in the second equation of (4-15) b_2 by the expression given by the first equation: we obtain an equation for which we split the terms containing a_1 from the others. Namely,

$$\begin{aligned} a_1'' - 2A(s_0^q + a_0)b_1 a_1' - A'(s_0^q + a_0)a_1 b_1^2 \\ = A(s_0^q + a_0)b_1 a_0'' + B(s_0^q + a_0)b_1^3 - \tfrac{1}{2}b_1'' - \tfrac{1}{2}a_0^{(3)}. \end{aligned} \tag{4-16}$$

Replacing a_0 and b_1 by the expressions we just found, the left-hand side of (4-16) can be expressed as

$$a_1'' + \tfrac{4}{3}Lk^{-5/3}k'a_1' + \tfrac{2}{3}L^2\big(k^{-7/3}k'' - k^{-10/3}k'^2\big)a_1 = k^{-2/3}(a_1 k^{2/3})'',$$

where it is understood that k and its derivatives are evaluated at $s_0^q + a_0$. The right-hand side of (4-16) vanishes. Hence (4-16) is equivalent to

$$k^{-2/3}(a_1 k^{2/3})'' = 0.$$

Since a_1 is periodic and vanishes at 0, we necessarily have $a_1 = 0$. The expression of b_2 comes from the first equation of (4-15), namely $b_2 = a_0''/2$.

Expressions of a_2 and b_3. Although not used later, we derive an explicit expression for the coefficient a_2. By making use of (4-11) and (4-12), and taking into account that $a_1 = 0$, we obtain the system

$$\begin{cases} b_3 = a_2' + a_0'''/6, \\ A'(s_0^q + a_0)b_1^2 a_2 + A(s_0^q + a_0)(b_2^2 + 2b_1 b_3) + B(s_0^q + a_0)3b_1^2 b_2 + C(s_0^q + a_0)b_1^4 \\ \qquad = b_1'''/6 + b_2''/2 + b_3' \end{cases} \tag{4-17}$$

From the first equation of (4-17) we have $b_3' = a_2'' + a_0^{(4)}/6$, which in turn gives

$$b_3' = a_2'' + \frac{11k'k''k^{-\frac{10}{3}}}{27} - \frac{8(k')^3k^{\frac{13}{3}}}{27} - \frac{k'''k^{-\frac{7}{3}}}{9}.$$

Replacing into the second of (4-17) and grouping all the terms with a_2, we get

$$(k^{\frac{2}{3}}a_2)'' = L^4\left(\frac{40(k')^3 - 45kk'k'' + 9k^2k''}{810k^5} + \frac{2k'}{45k}\right). \tag{4-18}$$

The right-hand side is the derivative of

$$L^3\left(\frac{1}{810}\left(9k''k^{-\frac{7}{3}} - 12(k')^2k^{-\frac{10}{3}}\right) + \frac{k^{\frac{2}{3}}}{15}\right) + c,$$

where c is a constant such that this function has zero mean. At this point we can integrate again and get for $a_2(x)$ the value

$$k^{-\frac{2}{3}}(a_0(x))\left(\int_0^x L^3\left(\frac{1}{810}\left(9k''k^{-\frac{7}{3}} - 12(k')^2k^{-\frac{10}{3}}\right) + \frac{k^{\frac{2}{3}}}{15}\right)(a_0(t))\,dt + cx\right).$$

The value of b_3 can now be easily derived from the first one of (4-17). □

5. Proof of Theorem 6

This section is entirely devoted to the proof of Theorem 6, providing the coefficient β_5 for the outer length billiard map.

Proof. We start the computation of the beta function by writing its value at rational points of the form $\frac{1}{q}$, which (by the expansion (4-3) of the generating function H) is

$$\begin{aligned}\beta\left(\frac{1}{q}\right) &= \frac{1}{q}\sum_{n=0}^{q-1} H(s_n, s_{n+1}) \\ &= \frac{1}{q}\sum_{n=0}^{q-1} \varepsilon_n + \frac{k^2}{12}\varepsilon_n^3 + \frac{kk'}{12}\varepsilon_n^4 + \frac{2k^4 + 4k'^2 + 7kk''}{240}\varepsilon_n^5 + O(\varepsilon_n^6).\end{aligned} \tag{5-1}$$

Here, the curvature k and its derivatives k' and k'' are to be understood as evaluated in s_n.

Now, we substitute in the above formula s_n and ε_n with their corresponding Taylor expansions obtained in Proposition 11. We then proceed to group the various terms according to their order of magnitude q^k.

First, we observe that the summation of ε_n is simply equal to the perimeter ℓ of D, so that $\beta_1 = \ell$.

By inspecting the formula even before performing the substitution, we see that there are no terms of order q^{-2}, so that $\beta_2 = 0$, as expected by Marvizi–Melrose theory.

The second term of the summation on the right-hand side of (5-1) becomes, after the substitution and after grouping the various powers of q,

$$\begin{aligned}(5\text{-}2)\quad &\frac{1}{12}\sum_{n=0}^{q-1} k^2(s_n)\,\varepsilon_n^3 \\ &= \frac{1}{12}\sum_{n=0}^{q-1} k^2\left(a_0+\frac{a_2}{q^2}+O\left(\frac{1}{q^3}\right)\right)\left(\frac{b_1}{q}+\frac{b_2}{q^2}+\frac{b_3}{q^3}\right)^3 \\ &= \sum_{n=0}^{q-1}\frac{k^2b_1^3}{12}\frac{1}{q^3}+\frac{k^2b_1^2b_2}{4}\frac{1}{q^4}+\frac{2kk'b_1^3a_2+3k^2(b_1^2b_3+b_1b_2^2)}{12}\frac{1}{q^5}+O\Big(\frac{1}{q^6}\Big).\end{aligned}$$

Similarly, we have

$$\begin{aligned}(5\text{-}3)\quad &\frac{1}{12}\sum_{n=0}^{q-1} k(s_n)\,k'(s_n)\,\varepsilon_n^4 = \frac{1}{12}\sum_{n=0}^{q-1} kk'\left(\frac{b_1}{q}+\frac{b_2}{q^2}+O\Big(\frac{1}{q^3}\Big)\right)^4 = \\ &= \sum_{n=0}^{q-1}\frac{kk'b_1^4}{12}\frac{1}{q^4}+\frac{kk'b_1^3b_2}{3}\frac{1}{q^5}+O\Big(\frac{1}{q^6}\Big).\end{aligned}$$

Finally, the last term is

$$(5\text{-}4)\qquad \sum_{n=0}^{q-1}\left(\frac{2k^4+4k'^2+7kk''}{240}\right)b_1^5\frac{1}{q^5}+O\Big(\frac{1}{q^6}\Big).$$

We recall that, in the last three formulas, it is implicitly understood that all functions a_i, b_i are evaluated at n/q, and the curvature k and its derivatives k' and k'', where not explicitly specified, are computed at $s_0^q + a_0(n/q)$. To determine β_3, we compute $\lim_{q\to+\infty} q^3\big(\beta\big(\frac{1}{q}\big)-\frac{\ell}{q}\big)$.

From (5-2), (5-3), and (5-4), we obtain

$$\beta_3 = \lim_{q\to+\infty}\frac{1}{12q}\sum_{n=0}^{q-1}\left(k^2b_1^3+O\Big(\frac{1}{q}\Big)\right)$$

By Proposition 11, we have $b_1 = Lk^{-\frac{2}{3}} \Rightarrow k^2b_1^3 = L^3$, so that

$$\beta_3 = \frac{1}{12}L^3 = \frac{1}{12}\left(\int_0^\ell k^{2/3}(r)dr\right)^3.$$

The leading part of this limit is constant, while the term denoted by $O(1/q)$ contains only higher-order terms. We will take this into account when analyzing $\beta - \ell/q - \beta_3/q^3$, considering only the terms present in $O(1/q)$.

For the terms of order 4, we obtain the expression

$$\sum_{n=0}^{q-1}\left(\frac{k^2b_1^2b_2}{4}+\frac{kk'b_1^4}{12}\right)\frac{1}{q^4}.$$

Since, by Proposition 11, we have $b_1=Lk^{-\frac{2}{3}}$ and $b_2=-\frac{1}{3}L^2k'k^{-\frac{7}{3}}$, we immediately conclude (again, as expected by Marvizi–Melrose theory), that $\beta_4=0$.

The terms of order 5 are

$$\sum_{n=0}^{q-1}\left[\frac{2k^4+4k'^2+7kk''}{240}b_1^5+\frac{1}{12}\left(2kk'b_1^3a_2+3k^2\left(b_3b_1^2+b_2^2b_1\right)+4kk'b_1^3b_2\right)\right]\frac{1}{q^5} = S_1+S_2,$$

where

$$S_1 := \sum_{n=0}^{q-1}\frac{1}{12}\left(2kk'b_1^3a_2+3k^2b_1^2a_2'\right),$$
$$S_2 := \sum_{n=0}^{q-1}\left(\frac{2k^4+4k'^2+7kk''}{240}b_1^5+\frac{1}{12}\left(3k^2b_2^2b_1+4kk'b_1^3b_2+\frac{1}{2}k^2a_0'''b_1^2\right)\right);$$

where we substituted the value $b_3=a_2'+a_0'''/6$ from Proposition 11. We remark that the sum S_1 contains a_2 and the sum S_2 doesn't contain a_2. As established earlier, we have

$$\beta_5=\lim_{q\to+\infty}q^5\left(\beta\left(\frac{1}{q}\right)-\frac{\ell}{q}-\frac{\beta_3}{q^3}\right)=\lim_{q\to+\infty}\frac{1}{q}(S_1+S_2).$$

By studying $\lim_{q\to+\infty}\frac{1}{q}S_1$, we obtain

$$\text{(5-5)}\quad \frac{1}{12}\lim_{q\to+\infty}\frac{1}{q}\sum_{n=0}^{q-1}\left(2kk'k^{-2}a_2L^3+3k^2k^{-\frac{4}{3}}a_2'L^2\right) = \frac{1}{12}\int_0^1\left(\frac{2k'}{k}\left(a_0(x)\right)a_2(x)L^3+3k^{\frac{2}{3}}\left(a_0(x)\right)a_2'(x)L^2\right)dx,$$

where once again in the summations we have used the convention that the functions a_i, b_i are evaluated at n/q, while the functions k, k' are evaluated at $a_0(n/q)$. Similarly, in the integral on the right-hand side, a_i, b_i are evaluated at x, and k, k' at $a_0(x)$. Integrating by parts the second term inside the integral, we have

$$\text{(5-6)}\quad \int_0^1 k^{\frac{2}{3}}\left(a_0(x)\right)a_2'(x)L^2dx = k^{\frac{2}{3}}\left(a_0(x)\right)a_2(x)L^2\Big|_0^1-\int_0^1\frac{2k'}{3k}\left(a_0(x)\right)a_2(x)L^3dx.$$

By periodicity, the first term is 0. By substituting the remaining expression of (5-6) inside (5-5), we conclude that the first limit is 0.

Let us proceed with the calculation of $\lim_{q\to+\infty}\frac{1}{q}S_2$. Recalling from (4-10) that $a_0'(x)=Lk^{-\frac{2}{3}}(a_0(x))$, we have

$$a_0'''=L^3\left(-\frac{2k''}{3k^3}+\frac{14k'^2}{9k^4}\right).$$

Taking into account the expressions of a_i, b_j given in (4-10) and by substituting the previous expression into S_2, we obtain

$$\begin{aligned}(5\text{-}7)\quad &\lim_{q\to+\infty}\frac{1}{q}S_2\\ &=\lim_{q\to+\infty}\frac{1}{q}\sum_{n=0}^{q-1}\frac{2k^4+4k'^2+7kk''}{240}b_1^5+\tfrac{1}{12}\big(3k^2b_2^2b_1+4kk'b_1^3b_2+\tfrac{1}{2}k^2a_0'''b_1^2\big)\\ &=\lim_{q\to+\infty}\frac{L^5}{q}\sum_{n=0}^{q-1}\left(\frac{k^{2/3}}{120}-\frac{k^{-\frac{10}{3}}k'^2}{540}+\frac{k^{-\frac{7}{3}}k''}{720}\right)\\ &=L^5\int_0^1\left(\frac{k^{2/3}}{120}-\frac{k^{-\frac{10}{3}}k'^2}{540}+\frac{k^{-\frac{7}{3}}k''}{720}\right)dx.\end{aligned}$$

We finally integrate by parts the last term of the integral, obtaining

$$\begin{aligned}\int_0^1 L^5k^{-\frac{7}{3}}k''dx=L^4\int_0^1 k^{-5/3}k''(k^{-\frac{2}{3}}L)\,dx=L^4\int_0^1 k^{-\frac{5}{3}}\left(k'\right)'dx\\ =\frac{5L^5}{3}\int_0^1 k^{-\frac{10}{3}}k'^2dx.\end{aligned}$$

Substituting this in (5-7), we conclude that

$$\lim_{q\to+\infty}\frac{1}{q}S_2=L^5\int_0^1\left(\frac{k^{2/3}}{120}+\frac{k^{-\frac{10}{3}}k'^2}{2160}\right)dx.$$

Finally, switching to arc length as the variable of integration, we obtain the desired result:

$$\beta_5=L^4\int_0^\ell\left(\frac{k^{4/3}(s)}{120}+\frac{k^{-\frac{8}{3}}(s)k'^2(s)}{2160}\right)ds. \qquad \square$$

6. Lazutkin coordinates and caustics

A consequence of Proposition 10 is that we can compute explicitly Lazutkin coordinates [14] for order 4 in the case of outer length billiards.

Lemma 12 (Lazutkin for outer length billiards). *The coordinates*

$$x(s) = \frac{1}{L}\int_0^s k^{2/3}(r)\,dr, \quad L := \int_0^\ell k^{2/3}(r)\,dr,$$
$$y(s,\varepsilon) = x(s+\varepsilon) - x(s)$$

are such that the outer length billiard dynamics is given by

$$x \mapsto x + y, \quad y \mapsto y + O(y^4).$$

Proof. Let

$$(s,\varepsilon) \mapsto (x,y) := (f(s), f(s+\varepsilon) - f(s))$$

a change of coordinates so that $(x_k, y_k) \mapsto (x_k + y_k, y_{k+1})$. Then, by using the expansion of ε_1 given in 10, we have

$$\begin{aligned}
y_1 = x_2 - x_1 &= f(s_1+\varepsilon_1) - f(s_1) = f'(s_1)\varepsilon_1 + \frac{f''(s_1)}{2}\varepsilon_1^2 + \frac{f'''(s_1)}{6}\varepsilon_1^3 + O(\varepsilon_1^4)\\
&= \left(f'(s_0) + f''(s_0)\varepsilon_0 + \frac{f'''(s_0)}{2}\varepsilon_0^3\right)\left(\varepsilon_0 + A(s_0)\varepsilon_0^2 + B(s_0)\varepsilon_0^3\right)\\
&\quad + \left(f''(s_0) + f'''(s_0)\varepsilon_0\right)\frac{\left(\varepsilon_0 + A(s_0)\varepsilon_0^2 + B(s_0)\varepsilon_0^3\right)^2}{2} + \frac{f'''(s_0)}{6}\varepsilon_0^3 + O(\varepsilon_0^4)\\
&= \left(f'(s_0)\varepsilon_0 + \frac{f''(s_0)}{2}\varepsilon_0^2 + \frac{f'''(s_0)}{6}\varepsilon_0^3\right) + \left(f''(s_0) + f'(s_0)A(s_0)\right)\varepsilon_0^2\\
&\quad + \left(f'(s_0)B(s_0) + 2f''(s_0)A(s_0) + f'''(s_0)\right)\varepsilon_0^3 + O(\varepsilon_0^4)\\
&= y_0 + \left(f''(s_0) + f'(s_0)A(s_0)\right)\varepsilon_0^2 + \left(f'(s_0)B(s_0) + 2f''(s_0)A(s_0) + f'''(s_0)\right)\varepsilon_0^3\\
&\quad + O(\varepsilon_0^4).
\end{aligned}$$

Thus, to get rid of the ε_0^2 and ε_0^3 terms, we need to choose f solving

$$\begin{cases} f''(s_0) + f'(s_0)A(s_0) = 0,\\ f'(s_0)B(s_0) + 2f''(s_0)A(s_0) + f'''(s_0) = 0.\end{cases}$$

Integrating the first equation, we immediately obtain the desired formula for f, giving, up to normalization,

$$x(s) = \frac{1}{L}\int_0^s k^{2/3}(r)\,dr, \quad L := \int_0^\ell k^{2/3}(r)\,dr.$$

Then, by direct computation, it is easy to check that such a function solves also the second equation. □

As a consequence, the outer length billiard map is a small perturbation of the integrable map

$$(x,y) \mapsto (x+y, y),$$

satisfying the assumptions of Lazutkin's theorem [14, Theorem 1]. Applying this theorem, the next corollary of Proposition 10 immediately follows.

Theorem 13. *Arbitrarily close to the boundary* $\partial\Omega$, *there exist smooth caustics for the outer length billiard map. The union of these caustics has positive measure.*

On the other hand, regarding the nonexistence of caustics, we underline that the following outer length billiard version of Mather s theorem still holds.

Theorem 14. *If the curvature of the boundary* $\partial\Omega$ *vanishes at some point*, *then the outer length billiard in* $\partial\Omega$ *has no caustics.*

Proof. We use Mather's necessary analytic condition for the existence of a caustic [16], that is

$$H_{22}(s_0, s_1) + H_{11}(s_1, s_2) < 0.$$

By using the general expression of the generating function (4-4), it is easily seen that

$$H_1(s_1, s_2) = -1 - \frac{(\gamma(s_2) - \gamma(s_1)) \wedge \gamma''(s_1)}{\gamma'(s_1) \wedge \gamma'(s_2)} - \frac{(\gamma(s_2) - \gamma(s_1)) \wedge (\gamma'(s_2) - \gamma'(s_1)) \cdot (\gamma''(s_1) \wedge \gamma'(s_2))}{(\gamma'(s_1) \wedge \gamma'(s_2))^2}.$$

Hence,

$$\begin{aligned} H_{11}(s_1, s_2) = {} & \frac{\gamma'(s_1)\wedge\gamma''(s_1)}{\gamma'(s_1)\wedge\gamma'(s_2)} - \frac{(\gamma(s_2)-\gamma(s_1))\wedge\gamma'''(s_1)}{\gamma'(s_1)\wedge\gamma'(s_2)} \\ & + 2\frac{(\gamma(s_2)-\gamma(s_1))\wedge\gamma''(s_1)}{(\gamma'(s_1)\wedge\gamma'(s_2))^2}(\gamma''(s_1)\wedge\gamma'(s_2)) + \frac{\gamma''(s_1)\wedge\gamma'(s_2)}{(\gamma'(s_1)\wedge\gamma'(s_2))^2} \\ & - \frac{(\gamma(s_2)-\gamma(s_1))\wedge(\gamma'(s_2)-\gamma'(s_1))}{(\gamma'(s_1)\wedge\gamma'(s_2))^2}(\gamma'''(s_1)\wedge\gamma'(s_2)) \\ & + 2\frac{(\gamma(s_2)-\gamma(s_1))\wedge(\gamma'(s_2)-\gamma'(s_1))}{(\gamma'(s_1)\wedge\gamma'(s_2))^3}(\gamma''(s_1)\wedge\gamma'(s_2))^2. \end{aligned}$$

Now assume that at a point on the boundary corresponding to the arc-length parameter value s_1, the curvature is zero, that is, $k(s_1) = 0$. Since the set is convex, this condition implies that also $k'(s_1) = 0$. From formulas (4-2), it follows that $\gamma''(s_1) = \gamma'''(s_1) = 0$. Substituting into the previous formula, we see that all the terms composing H_{11} vanish, and a similar argument holds for H_{22}. As a consequence, we have $H_{11}(s_1, s_2) + H_{22}(s_0, s_1) = 0$ for every s_0, s_2, and therefore no topologically nontrivial invariant curve can exist. □

References

[1] P. Albers and S. Tabachnikov, "Introducing symplectic billiards", *Adv. Math.* **333** (2018), 822–867. MR

[2] P. Albers and S. Tabachnikov, "Monotone twist maps and Dowker-type theorems", *Pacific J. Math.* **330**:1 (2024), 1–24. MR

[3] V. Bangert, "Mather sets for twist maps and geodesics on tori", pp. 1–56 in *Dynamics reported*, vol. 1, Dynam. Report. Ser. Dynam. Systems Appl., Wiley, Chichester, 1988. MR

[4] L. Baracco and O. Bernardi, "Totally integrable symplectic billiards are ellipses", *Adv. Math.* **454** (2024), art. id. 109873, 17 pp. MR

[5] L. Baracco, O. Bernardi, and A. Nardi, "Area spectral rigidity for axially symmetric and Radon domains", 2024. arXiv 2410.12644

[6] L. Baracco, O. Bernardi, and A. Nardi, "Bialy-Mironov type rigidity for centrally symmetric symplectic billiards", *Nonlinearity* **37**:12 (2024), art. id. 125025, 12 pp. MR

[7] L. Baracco, O. Bernardi, and A. Nardi, "Higher order terms of Mather's β-function for symplectic and outer billiards", *J. Math. Anal. Appl.* **537**:2 (2024), art. id. 128353, 20 pp. MR

[8] M. Bialy, "Convex billiards and a theorem by E. Hopf", *Math. Z.* **214**:1 (1993), 147–154. MR

[9] M. Bialy, "Mather β-function for ellipses and rigidity", *Entropy* **24**:11 (2022), art. id. 1600, 16 pp. MR

[10] W. Cieślak and S. Góźdź, "Extremal polygons circumscribed an oval", *Acta Math. Hungar.* **54**:1-2 (1989), 63–68. MR

[11] D. DeTemple, "The geometry of circumscribing polygons of minimal perimeter", *J. Geom.* **49**:1-2 (1994), 72–89. MR

[12] C. Fierobe, A. Sorrentino, and A. Vig, "Deformational spectral rigidity of axially-symmetric symplectic billiards", 2024. arXiv 2410.13777

[13] A. Glutsyuk, "On infinitely many foliations by caustics in strictly convex open billiards", *Ergodic Theory Dynam. Systems* **44**:5 (2024), 1418–1467. MR

[14] V. F. Lazutkin, "Existence of caustics for the billiard problem in a convex domain", *Izv. Akad. Nauk SSSR Ser. Mat.* **37** (1973), 185–216. In Russian; translated in *Math. USSR Izv.* **7**:1 (1973), 185–214. MR

[15] S. Marvizi and R. Melrose, "Spectral invariants of convex planar regions", *J. Differential Geom.* **17**:3 (1982), 475–502. MR

[16] J. N. Mather, "Glancing billiards", *Ergodic Theory Dynam. Systems* **2**:3-4 (1982), 397–403. MR

[17] D. E. McClure and R. A. Vitale, "Polygonal approximation of plane convex bodies", *J. Math. Anal. Appl.* **51**:2 (1975), 326–358. MR

[18] K. F. Siburg, *The principle of least action in geometry and dynamics*, Lecture Notes in Mathematics **1844**, Springer, 2004. MR

[19] A. Sorrentino, "Computing Mather's β-function for Birkhoff billiards", *Discrete Contin. Dyn. Syst.* **35**:10 (2015), 5055–5082. MR

[20] H. Stachel, "The geometry of billiards in ellipses and their Poncelet grids", *J. Geom.* **112**:3 (2021), art. id. 40, 29 pp. MR

[21] S. Tabachnikov, "On the dual billiard problem", *Adv. Math.* **115**:2 (1995), 221–249. MR

[22] S. Tabachnikov, *Geometry and billiards*, Student Mathematical Library **30**, American Mathematical Society, Providence, RI, 2005. MR

[23] D. Tsodikovich, "Local rigidity for symplectic billiards", *J. Geom. Anal.* **35**:10 (2025), art. id. 306. MR

[24] J. Zhang, "Spectral invariants of convex billiard maps: a viewpoint of Mather's beta function", *Sci. China Math.* (2021). arXiv 2009.12513

Received May 8, 2025. Revised September 8, 2025.

LUCA BARACCO
UNIVERSITY OF PADOVA
ITALY
baracco@math.unipd.it

OLGA BERNARDI
UNIVERSITY OF PADOVA
ITALY
obern@math.unipd.it

CORENTIN FIEROBE
UNIVERSITY OF ROME "TOR VERGATA"
ITALY
cpef@gmx.de

PACIFIC JOURNAL OF MATHEMATICS
Vol. 339, No. 2, 2025
https://doi.org/10.2140/pjm.2025.339.223

MORI DREAM SPACES AND $\mathbb{Q}$-HOMOLOGY QUADRICS

PAOLO CASCINI, FABRIZIO CATANESE, YIFAN CHEN AND JONGHAE KEUM

We show that Shavel-type surfaces are fake $\mathbb{Q}$-homology quadrics of even type which are not Mori dream surfaces, yet there are infinitely many primes p such that the reduction modulo p is a Mori dream surface.

We investigate fake $\mathbb{Q}$-homology quadrics, first concerning the property of being a Mori dream surface, then trying to determine which families are of even type among the surfaces isogenous to a higher product which are fake $\mathbb{Q}$-homology quadrics.

1. Introduction

In [20] the authors considered complex surfaces of general type with $q = p_g = 0$ which are Mori dream surfaces, and asked in section 3.2 whether there are fake quadrics which are not Mori dream surfaces.

We produce here the first examples.

Recall that, for complex surfaces isogenous to a product of curves, it was established in [20] and [16] that they are Mori dream surfaces.

At the Hefei Conference in September 2024, the second author pointed out that such examples should be provided by the surfaces constructed by Shavel in 1978 [24]. Our first aim is therefore to give a complete proof of this assertion.

Our second aim is to discuss several problems related to minimal surfaces of general type which are $\mathbb{Q}$-homology quadrics: that is, smooth surfaces with $q = p_g = 0$, and with second Betti number $b_2(S) = 2$ (equivalently, with $K_S^2 = 8$). All known such examples have universal cover equal to the bidisk $\mathbb{H} \times \mathbb{H}$.

Indeed Hirzebruch ([18] Problem 25; see also pages 779–780 of [19]) was the first to ask the question whether there exists a surface of general type which is homeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and one can respectively ask the same question for a surface homeomorphic to the blow up $\mathbb{F}_1$ of $\mathbb{P}^2$ in one point. The answer is suspected to be negative.

MSC2020: 14J15, 14J25, 14J29, 14J80.

Keywords: surfaces of general type, Mori dream space, homology quadric, fake quadric, group action, surface isogenous to a product, foliation.

The two manifolds above are topologically distinguished by the property that the intersection form on the Severi group Num(S) is even in the first case, odd in the second.

We show here that many of our surfaces (surfaces of general type which are $\mathbb{Q}$-homology quadrics) have even intersection form. The existence of the case of odd intersection form was shown in the affirmative in [11] after the present paper was written.

The third question we consider is: what happens when, instead of complex surfaces, we consider surfaces defined over an algebraically closed field of positive characteristic? When are they Mori dream surfaces?

This is related to a beautiful conjecture by Ekedahl, Shepherd-Barron and Taylor [14] about algebraic integrability of foliations via reduction modulo primes p.

We can summarize our result in this regard as follows.

Theorem 1.1. *A Shavel-type surface S is an even $\mathbb{Q}$-homology fake quadric which is not a Mori dream surface.*

There are infinitely many primes p such that the reduction of S modulo p is a Mori dream surface.

We also give results stating when a fake $\mathbb{Q}$-homology quadric is a Mori dream space.

2. Definitions and basic properties

In these first sections we shall mostly work with projective smooth surfaces defined over the field $\mathbb{C}$, most definitions however make also sense if $\mathbb{C}$ is replaced by an algebraically closed field $\mathcal{K}$ of arbitrary characteristic.

Definition 2.1 ($\mathbb{Q}$-homology quadric). Let S be a smooth projective surface over $\mathbb{C}$.
The surface S is called a $\mathbb{Q}$-*homology quadric* if $q(S) = p_g(S) = 0$, $b_2(S) = 2$.
In turn, it will be called an *even homology quadric* if

(1) $q(S) = p_g(S) = 0$, $b_2(S) = 2$;

(2) the intersection form on Num(S) is even, that is, it is $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

We shall call it an *odd homology quadric* if instead the intersection form is odd, and hence diagonalizable with diagonal entries $(+1, -1)$.

Remarks 2.2. (1) A normal projective surface X is called a $\mathbb{Q}$-homology projective plane if $b_1(X) = b_3(X) = 0$ and $b_2(X) = 1$.

(2) A smooth $\mathbb{Q}$-homology projective plane must be either the projective plane or a ball quotient with $q = p_g = 0$.

(3) Our definition of $\mathbb{Q}$-homology quadric can also be extended to normal surfaces, requiring however that $b_1(X) = b_3(X) = 0$ and $b_2(X) = 2$. But in this paper a $\mathbb{Q}$-homology quadric means a smooth $\mathbb{Q}$-homology quadric.

For a $\mathbb{Q}$-homology quadric S, condition (1) and the Noether formula imply

$$\chi(\mathcal{O}_S) = 1, c_2(S) = 4, K_S^2 = 8.$$

Also the long exact sequence of cohomology groups associated to the exponential exact sequence shows that

$$c_1 : \mathrm{Pic}(S) \to H^2(S, \mathbb{Z})$$

is an isomorphism. We identify these two groups and denote by $\mathrm{Tors}(S)$ their torsion subgroup. Then

$$\mathrm{Num}(S) \cong \mathrm{Pic}(S)/\mathrm{Tors}(S).$$

In the case of an even homology quadric, condition (2) implies that S is minimal. Hence, by surface classification, for an even homology quadric,

- either S is rational, and then $S \cong \mathbb{F}_{2n}$ for $n \geq 0$;
- or S is a minimal surface of general type.

In the former case S is simply connected, $\mathrm{Tors}(S) = 0$ and K_S is not ample.

In the case of an odd homology quadric

- either S is rational, and then $S \cong \mathbb{F}_{2n+1}$ for $n \geq 0$;
- or S is a (not necessarily minimal) surface of general type.

Definition 2.3 (even fake homology quadric). Let S be a smooth projective surface over $\mathbb{C}$. The surface S will be called[1] an *even fake homology quadric* if it is an even homology quadric and is of general type.

In general, a minimal smooth complex projective surface of general type with $p_g = q = 0$, $K^2 = 8$ has Picard number 2 and K_S is ample by [23, Proposition 2.1.1]. Also $\mathrm{Tors}(S)$ can be nonzero (see [5]).

For $L \in \mathrm{Pic}(S)$, we denote by $[L]$ its class in $\mathrm{Num}(S)$.

Definition 2.4 (odd fake homology Quadric). Let S be a smooth projective surface over $\mathbb{C}$. If S is an odd homology quadric which is of general type, then either S is not minimal and it is a one point blow up of a fake projective plane, or we call the surface S an *odd fake homology quadric*, which means:

(1) S is minimal of general type with $K_S^2 = 8$, $p_g(S) = 0$;

[1] In the literature, sometimes $\mathbb{Q}$-homology quadrics of general type are referred to as fake quadrics, without specification whether the intersection form is even or odd; see [19; 12; 13; 16].

(2) the intersection form on $\mathrm{Num}(S)$ is odd.

Therefore smooth minimal surfaces of general type with $K^2 = 8$ and $p_g = 0$ are divided into two classes: the even and the odd fake homology quadrics.

For the sake of clarity, we give two definitions: the first is meant to be consistent with the current use, the second relates to the original question by Hirzebruch.

Definition 2.5. Let S be a minimal smooth projective surface over $\mathbb{C}$.

(I) Then S is called a fake quadric if and only if it is either an odd fake homology quadric, or an even fake homology quadric.

(II) S is called a fake homotopy quadric if and only if it is a fake quadric and moreover it is simply connected.

By Freedman's theorem [17] a fake homotopy quadric is homeomorphic either to $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ or to $\mathbb{F}_1$.

Lemma 2.6. *Let S be a fake quadric, and let $f : S \to \mathbb{P}^1$ be a fibration, whose general fibre we denote by F. Then*:

(1) *Any fibre $F_t = f^*(t)$ has irreducible support.*

(2) *The multiplicity m_t of any multiple fibre $F_t = m_t F'_t$ divides $g - 1$, where g is the genus of F.*

(3) *The class of the fibre $[F]$ is divisible by d in* $\mathrm{Num}(S)$, *where d is the least common multiple of the exponents m_i of the multiple fibres.*

Proof. (1) follows from Zariski's lemma (the intersection form on the fibre components is seminegative with nullity 1), and from the fact that the Picard number $\rho(S) := \mathrm{rank}(\mathrm{Num}(S))$ equals 2.

(2) follows by adjunction.

(3) If d is the least common multiple of the multiplicities $m_1, \ldots, m_s$ of the multiple fibres, then we may write $\frac{1}{m_i} = \frac{r_i}{d}$, where the r_i have GCD equal to 1. Thus, we can write $\frac{1}{d}$ as a sum of the rational numbers $\frac{1}{m_i}$, hence proving that $\frac{1}{d}[F] \in \mathrm{Num}(S)$, since it is an integer linear combination of the classes F'_i.

Therefore, the divisibility of $[F]$ is a multiple of d. □

Lemma 2.7. *Keep the assumptions of Lemma 2.6, and assume that S has another fibration $f' : S \to \mathbb{P}^1$ with general fibre F' of genus g'. Then*:

(1) $FF' = (g-1)(g'-1)$.

(2) $K_S \sim \dfrac{2}{g-1}F + \dfrac{2}{g'-1}F'$, *where $\sim$ denotes numerical equivalence.*

(3) *The intersection form of S is even if* $\dfrac{1}{g-1}F, \dfrac{1}{g'-1}F' \in \mathrm{Num}(S)$.

Proof. The matrix of intersection numbers of K_S, F, F' has determinant

$$8(g-1)(g'-1)x - 8x^2,$$

with $x = FF'$, which must be 0 since $\rho(S) = 2$; hence (1) and (2) follow right away.

For (3), these classes generate a unimodular lattice, hence all of $\text{Num}(S)$. □

Lemma 2.8. *Let S be a fake quadric. Then*[2] *the intersection form is even if and only if K_S is divisible by* 2 *in* $\text{Num}(S)$.

Proof. $K_S D \equiv D^2 (\text{mod}\, 2)$, and the intersection form on

$$\text{Num}(S) = H^2(S, \mathbb{Z})/\,\text{Tors}(S)$$

is unimodular. □

3. Surfaces isogenous to a product

Definition 3.1. Let S be a smooth projective surface. The surface S is said to be *isogenous to a higher product* [7] if

$$S \cong (C_1 \times C_2)/G,$$

where C_i is a smooth curve with $g(C_i) \geq 2$ and G is a finite group acting faithfully and freely on $C_1 \times C_2$.

If there are respective actions of G on C_1 and C_2 such that G acts by the diagonal action $g(x, y) = (gx, gy)$ on $C_1 \times C_2$, then S is called of *unmixed type*.

If some element of G exchanges the two factors, then S is called of *mixed type*.

Observation. In this paper we shall only consider surfaces isogenous to a higher product with $p_g(S) = 0$.

The next question we ask is to determine which of the $\mathbb{Q}$-homology quadrics which are isogenous to a higher product are even or odd fake quadrics.

Example 3.2. The classical Beauville surface is an even fake quadric.

Proof. Here $C_1 = C_2 = C$ are the Fermat quintic in $\mathbb{P}^2$, and the group $G = \mu_5 \times \mu_5$ acts on C_1 by the linear action

$$(\zeta_1, \zeta_2)(x_0, x_1, x_2) = (x_0, \zeta_1 x_1, \zeta_2 x_2),$$

[2] By Wu's formula, saying that K_S induces the second Stiefel–Whitney class $w_2(S) \in H^2(S, \mathbb{Z}/2)$, and by the universal coefficients formula, S is spin (i.e., $w_2(S) = 0$) if and only if K_S is divisible by 2 in $H^2(S, \mathbb{Z})$, a fact which is often expressed by saying that S is *even*; being an even surface is a stronger notion than requiring that the intersection form is even, as one can see from the example of Enriques surfaces.

while the action on C_2 is twisted by an automorphism of G (in such a way that the action on the product $C_1 \times C_2$ is free; see for instance [7], page 24, for details and a generalization).

Hence $\mathcal{O}_C(1)$ is a G-linearized bundle with square K_C, see also ([10]).

Thus, $\mathcal{O}_{C_1}(1) \otimes \mathcal{O}_{C_2}(1)$ is a G-linearized bundle which descends to a line bundle L with square K_S.

Alternatively, the fibres of both projections are divisible by 5 and yield F'_1, F'_2 such that $F'_1 F'_2 = 1$ and $(F'_j)^2 = 0$. □

We can give a partial answer to the above parity question for surfaces isogenous to a product of unmixed type with $p_g = 0$, which have been classified in [6] (their torsion groups have been classified in [5]).

Theorem 3.3. *Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a product of unmixed type, with $p_g(S) = 0$; then G is one of the groups in the table below and the multiplicities T_1, T_2 of the multiple fibres for the natural fibrations $S \to C_i/G \cong \mathbb{P}^1$ are as listed. For each case in the list we have an irreducible component of the moduli space of surfaces of general type, whose dimension is denoted by D. The property of being an even, respectively an odd homology quadric and the first homology group of S are given in the third last column, respectively in the last column.*[3]

G	$\mathrm{Id}(G)$	T_1	T_2	parity	D	$H_1(S,\mathbb{Z})$
$\mathcal{A}_5$	$\langle 60,5\rangle$	$[2,5,5]$	$[3,3,3,3]$	?	1	$(\mathbb{Z}_3)^2 \times (\mathbb{Z}_{15})$
$\mathcal{A}_5$	$\langle 60,5\rangle$	$[5,5,5]$	$[2,2,2,3]$	?	1	$(\mathbb{Z}_{10})^2$
$\mathcal{A}_5$	$\langle 60,5\rangle$	$[3,3,5]$	$[2,2,2,2,2]$	?	2	$(\mathbb{Z}_2)^3 \times \mathbb{Z}_6$
$\mathcal{S}_4 \times \mathbb{Z}_2$	$\langle 48,48\rangle$	$[2,4,6]$	$[2,2,2,2,2,2]$	?	3	$(\mathbb{Z}_2)^4 \times \mathbb{Z}_4$
G(32)	$\langle 32,27\rangle$	$[2,2,4,4]$	$[2,2,2,4]$	?	2	$(\mathbb{Z}_2)^2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$
$(\mathbb{Z}_5)^2$	$\langle 25,2\rangle$	$[5,5,5]$	$[5,5,5]$	even	0	$(\mathbb{Z}_5)^3$
$\mathcal{S}_4$	$\langle 24,12\rangle$	$[3,4,4]$	$[2,2,2,2,2,2]$	even	3	$(\mathbb{Z}_2)^4 \times \mathbb{Z}_8$
G(16)	$\langle 16,3\rangle$	$[2,2,4,4]$	$[2,2,4,4]$	even	2	$(\mathbb{Z}_2)^2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$
$D_4 \times \mathbb{Z}_2$	$\langle 16,11\rangle$	$[2,2,2,4]$	$[2,2,2,2,2,2]$	?	4	$(\mathbb{Z}_2)^3 \times (\mathbb{Z}_4)^2$
$(\mathbb{Z}_2)^4$	$\langle 16,14\rangle$	$[2,2,2,2,2]$	$[2,2,2,2,2]$	even	4	$(\mathbb{Z}_4)^4$
$(\mathbb{Z}_3)^2$	$\langle 9,2\rangle$	$[3,3,3,3]$	$[3,3,3,3]$	even	2	$(\mathbb{Z}_3)^5$
$(\mathbb{Z}_2)^3$	$\langle 8,5\rangle$	$[2,2,2,2,2]$	$[2,2,2,2,2,2]$	?	5	$(\mathbb{Z}_2)^4 \times (\mathbb{Z}_4)^2$

Remark 3.4. In [11], one of us (Catanese) showed that the intersection form is odd for the first family of surfaces in the above list. The question remains open for the other families in the list.

Proof. In view of the cited results in [6] and [5], it suffices to prove the assertion about the intersection form being even, respectively odd.

[3] The group identity $\mathrm{Id}(G)$ consists of the group cardinality $|G|$ followed by the *Atlas* list number.

We use now (3) of Lemma 2.6 showing that $F_j = d_j \Phi_j$, where d_j is the least common multiple of the multiplicities of the fibration with general fibre F_j.

Since $F_1 F_2 = |G|$, we conclude that

$$\Phi_1 \Phi_2 = \frac{|G|}{d_1 d_2}.$$

If $d_1 d_2 = |G|$, then $\Phi_1 \Phi_2 = 1$, and, since $\Phi_j^2 = 0$, we have an even intersection form.

Inspecting the list, we see that $\Phi_1 \Phi_2 \in \{1, 2, 4\}$.

If we have an even intersection form, then we have a basis e_1, e_2 of the lattice $\mathrm{Num}(S)$ with $e_j^2 = 0$, and $e_1 e_2 = 1$.

Hence, without loss, we may assume that $[\Phi_j] = a_j e_j$, and therefore $\Phi_1 \Phi_2 = a_1 a_2$.

If $2 | a_1 a_2$, then $2 | a_j$ for some j, and Φ_j is further divisible by 2.

We could conclude that the intersection form is odd, in case $\Phi_1 \Phi_2 = 2$, if we knew that each Φ_j is not divisible by 2.

In fact, if the intersection form is odd, then we have a basis q_1, q_2 of the lattice $\mathrm{Num}(S)$ with $q_1^2 = 1$, $q_2^2 = -1$, and $q_1 q_2 = 0$.

Then Φ_1, Φ_2 must be multiples of $q_1 + q_2$, respectively $q_1 - q_2$, and indeed $(q_1 + q_2)(q_1 - q_2) = 2$.

In the only case (with group $(\mathbb{Z}/2)^4$) where $\Phi_1 \Phi_2 = 4$, the divisibility index of Φ_1 equals the one of Φ_2 by the symmetry of the roles of the two curves C_1, C_2, including the associated monodromies. Hence either Φ_1 and Φ_2 are both 2-divisible, and the intersection form is even, or the intersection form is odd and Φ_1 and Φ_2 are both indivisible. But then $\{\Phi_1, \Phi_2\} = \{(q_1+q_2), (q_1-q_2)\}$ and $(q_1+q_2)(q_1-q_2) = 2$ contradicts $\Phi_1 \Phi_2 = 4$. □

Remark 3.5. In the case where $\Phi_1 \Phi_2 = 2$, it is easy to see which of the two divisors may be 2-divisible.

In fact, using the ramification formula for $S \to (C_1/G) \times (C_2/G) = \mathbb{P}^1 \times \mathbb{P}^1$, we see that, setting $m_1, \dots, m_s$ to be the multiplicities of the multiple fibres in the first fibration and $n_1, \dots, n_{s'}$ those in the second, we have

$$K_S = \Big(-2 + \sum_j \Big(1 - \frac{1}{m_j}\Big)\Big) F_1 + \Big(-2 + \sum_i \Big(1 - \frac{1}{n_i}\Big)\Big) F_2$$

and, in $\mathrm{Num}(S)/2\,\mathrm{Num}(S)$, we have

$$K_S \equiv \sum_j (d_1 - r_j)\Phi_1 + \sum_i (d_2 - r_i')\Phi_2 \equiv \delta_1 \Phi_1 + \delta_2 \Phi_2, \quad \delta_1, \delta_2 \in \{0, 1\}.$$

We see by direct inspection that exactly one δ_j equals 1, the other is 0.

Then $[K_S] \in \mathrm{Num}(S)$ is 2-divisible if and only if the $[\Phi_j]$ with $\delta_j = 1$ is 2-divisible.

For instance in the last case we have $\delta_1 = 1$ and $\delta_2 = 0$; indeed, $K_S \sim \Phi_1 + 2\Phi_2$.

Remarks 3.6. (1) In the last case we can prove that K_S is not 2-divisible, since there is no G-linearized theta characteristic on the curve C_1, a hyperelliptic curve of genus $g_1 = 3$. Indeed, the only G-fixed theta characteristics are the hyperelliptic divisor $\mathcal{H}$ (the hyperelliptic divisor class is fixed by any automorphism of the curve), which does not admit a G-linearization, and $P_1 + P_2 + P_3 + P_4 - \mathcal{H}$, where the P_j's are Weierstrass points and their sum is a G-orbit (hence $\mathcal{O}_{C_1}(P_1 + P_2 + P_3 + P_4)$ admits a G-linearization).

(2) On the other hand, showing that $[K_S]$ is not 2-divisible is harder, in view of the existence of torsion divisors of order 4.

(3) Our observation (1) shows that, given a Fuchsian group $\Gamma < \mathbb{P}\,\mathrm{SL}(2, \mathbb{R})$ which is not torsion free, the embedding $\Gamma \hookrightarrow \mathbb{P}\,\mathrm{SL}(2, \mathbb{R})$ does not necessarily lift to $\mathrm{SL}(2, \mathbb{R})$ (unlike the case where Γ is cocompact and torsion free).

4. Even fake quadrics

Assumption. Assumption In this section, we let S be an even fake quadric. Recall that, over the complex number field $\mathbb{C}$, a fake quadric S contains no smooth rational curves [23, Proposition 2.1.1] and in particular K_S is ample.

4.1. *Nef cone of an even fake quadric.*

Lemma 4.1. (1) *There exist* $L_1, L_2 \in \mathrm{Pic}(S)$ *such that* $L_1 L_2 = 1$, $K_S L_i = 2$, $L_i^2 = 0$ *for* $i = 1, 2$.

(2) *For any* L_1, L_2 *as in* (1),

$$\mathrm{Num}(S) = \mathbb{Z}[L_1] \oplus \mathbb{Z}[L_2].$$

(3) $K_S \sim 2L_1 + 2L_2$.

The condition that the universal covering of S is the bidisk can be formulated as follows: there exists a 2-torsion divisor η such that

$$H^0(S^2(\Omega^1_S)(-K_S + \eta)) \neq 0;$$

see for instance [8; 9]. This condition is equivalent to the splitting of the cotangent bundle on a suitable unramified double covering of S.

From now on, we fix $L_1, L_2 \in \mathrm{Pic}(S)$ as in 4.1 (1).

Proposition 4.2. (1) *Any effective divisor on S is nef.*

(2) *Any nef and big divisor on S is ample.*

(3) L_1 *and* L_2 *are nef.*

(4) *For any strictly effective divisor D on S, if $D^2 = 0$, then $D \sim aL_1$ or $D \sim aL_2$ for some $a \in \mathbb{Z}_{>0}$.* (However, we do not claim that such a D exists.)

Proof. For (1), it suffices to show for any irreducible curve C, we have $C^2 \geq 0$. Assume by contradiction that $C^2 < 0$. Assume $C \sim aL_1 + bL_2$ for $a, b \in \mathbb{Z}$. Then

$$C^2 = 2ab, \quad K_S C = 2a + 2b.$$

We may assume that $a > 0$ and $b < 0$. By the adjunction formula,

$$-2 \leq 2p_a(C) - 2 = C^2 + K_S C = 2(a + b + ab) = 2a(1 + b) + 2b.$$

Therefore $b = -1$, $C \cong \mathbb{P}^1$, and this contradicts the fact that S contains no smooth rational curve.

For (2), let D be a nef and big divisor on S. Then $D^2 > 0$. Let C be any irreducible curve. Since C is nef, $DC \geq 0$. Also, since $C^2 \geq 0$, $DC > 0$ by the algebraic index theorem. Therefore D is ample.

For (3) and (4), let D be an effective divisor. Assume that $D \sim aL_1 + bL_2$. Then $D^2 = 2ab$ and $K_S D = 2(a + b)$. Since D is nef and K_S is ample, $a \geq 0, b \geq 0, a + b > 0$. Then $L_1 D = b \geq 0$, $L_2 D = a \geq 0$. And if $D^2 = 0$, then $a = 0$ or $b = 0$. □

Corollary 4.3. *Let L_1, L_2 as in Lemma 4.1(1). In* $\mathrm{Num}(S)_{\mathbb{R}}$,

$$\mathrm{Amp}(S) = \{a[L_1] + b[L_2] \mid a, b \in \mathbb{R}_+\},$$
$$\mathrm{Nef}(S) = \{a[L_1] + b[L_2] \mid a, b \in \mathbb{R}_{\geq 0}\},$$
$$\mathrm{Nef}(S) = \overline{\mathrm{Eff}(S)}.$$

Lemma 4.4. *Assume that the cotangent sheaf of S splits as the direct sum of two invertible sheaves*:

$$\Omega_S^1 = \mathcal{O}_S(A_1) \oplus \mathcal{O}_S(A_2).$$

Then either $[A_1] = 2[L_1]$, $[A_2] = 2[L_2]$ or $[A_1] = 2[L_2]$, $[A_2] = 2[L_1]$.

Moreover the universal covering of S is the bidisk $\mathbb{H} \times \mathbb{H}$, and $S = \mathbb{H} \times \mathbb{H}/\Gamma$, where $\Gamma < \mathbb{P}\,\mathrm{SL}(2, \mathbb{R}) \times \mathbb{P}\,\mathrm{SL}(2, \mathbb{R})$.

Proof. Assume that $[A_1] = a[L_1] + b[L_2]$ with $a, b \in \mathbb{Z}$. Since $K_S = A_1 + A_2$ and $[K_S] = 2[L_1] + 2[L_2]$, $[A_2] = (2 - a)[L_1] + (2 - b)[L_2]$.

A Chern class computation shows that $A_1 A_2 = c_2(S) = 4$. That is,

$$a(2 - b) + b(2 - a) = 4, \quad i.e., \ (a - 1)(b - 1) = -1.$$

Therefore either $a = 2, b = 0$ or $a = 0, b = 2$.

The last assertion has been known for a long time; see [26; 3]. □

4.2. ***Even fake quadrics and Mori dream surfaces.*** The following theorem follows from Theorems 3.9 and 3.10 of [21], but we give another proof for the reader's convenience.

Theorem 4.5. *Let S be an even fake quadric. Then S is a Mori dream surface if and only if L_1 and L_2 are semiample, equivalently, if and only if S admits a finite morphism to $\mathbb{P}^1 \times \mathbb{P}^1$. The same is true for any fake quadric that does not contain a negative curve.*

Proof. Note that $\mathrm{Pic}(S) \cong H^2(S, \mathbb{Z})$ is finitely generated. According to [1, Corollary 2.6], S is a Mori dream surface if and only if $\mathrm{Eff}(S)$ is rational polyhedral and $\mathrm{Nef}(S) = \mathrm{SAmp}(S)$.

By 4.3,

$$\mathrm{Nef}(S) = \overline{\mathrm{Eff}}(S) = \{a[L_1] + b[L_2] \mid a, b \in \mathbb{R}_{\geq 0}\}.$$

Since $\mathrm{Eff}(S) \supseteq \mathrm{SAmp}(S)$, it follows that S is Mori dream surface if and only if $\mathrm{Nef}(S) = \mathrm{SAmp}(S)$, if and only if L_1, L_2 are semi-ample.

It follows that S is a Mori dream surface if and only if S has two fibrations $f_1, f_2 : S \to \mathbb{P}^1$. These combine to yield a morphism $f : S \to \mathbb{P}^1 \times \mathbb{P}^1$ which is necessarily finite since the second Betti number of S equals 2.

Conversely, if S has a finite morphism $f : S \to \mathbb{P}^1 \times \mathbb{P}^1$, $q(S) = 0$, and the second Betti number of S equals 2, then S is a Mori dream surface with $q(S) = p_g(S) = 0$, hence in particular it is a $\mathbb{Q}$-homology quadric.

Since the property of being a Mori dream space depends on the structure of $\mathrm{Num}(S) \otimes \mathbb{R}$, by the cited criteria, in the case of an odd fake quadric we take a basis of $\mathrm{Num}(S)$ as in Lemma 6.1 and set $L_1 := Q_1 + Q_2,\ L_2 := Q_1 - Q_2$.

If there are no negative curves, then the cones $\mathrm{Nef}(S)$ and the closure of $\mathrm{Eff}(S)$ are again equal to the first quadrant, and the proof runs exactly as in the even case. □

5. Shavel-type surfaces

Definition 5.1. A smooth projective surface S shall be called a *Shavel surface of special unmixed type* if

$$p_g(S) = q(S) = 0,\ S = \mathbb{H}^2/\Gamma,$$

where Γ is a cocompact discrete, torsion-free (hence acting freely), irreducible subgroup of

$$\mathrm{Aut}(\mathbb{H}^2) \simeq \mathbb{P}\mathrm{SL}(2, \mathbb{R})^2 \rtimes \mathbb{Z}/2,$$

such that

$$\Gamma < \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}).$$

We shall drop the word “special” if

$$\Gamma < \mathbb{P}\mathrm{SL}(2, \mathbb{R})^2.$$

Observe that the group Γ is said to be *reducible* if it contains a normal finite index subgroup which is of the form $\Gamma_1 \times \Gamma_2$: in this case $S = \mathbb{H}^2/\Gamma$ is a finite free quotient $(C_1 \times C_2)/G$, where $C_j := \mathbb{H}/\Gamma_j$, by the action of $G := \Gamma/\Gamma_1 \times \Gamma_2$.

Γ is said to be irreducible if it is not reducible: then the projection of Γ on each of the factors $\mathbb{P}\mathrm{SL}(2, \mathbb{R})$ has dense image.

Note that for a Shavel surface S of unmixed type, we have

$$\gamma z = (\gamma_1 z_1, \gamma_2 z_2) \quad \text{for all } \gamma = (\gamma_1, \gamma_2) \in \Gamma \text{ and all } z = (z_1, z_2) \in \mathbb{H}^2.$$

Hence S admits two smooth foliations and Ω^1_S splits as the direct sum of two invertible sheaves:

$$\Omega^1_S = Ł_1 \oplus Ł_2.$$

Proposition 5.2. *A Shavel surface S of special unmixed type is an even fake quadric and K_S is divisible by 2 in $Pic(S)$.*

Proof. It suffices to show K_S is divisible by 2 in $Pic(S)$.

The automorphic factor of the canonical bundle is the inverse of the jacobian determinant

$$\frac{1}{(c_1 z_1 + d_1)^2 (c_2 z_2 + d_2)^2},$$

for $\gamma = (\gamma_1, \gamma_2)$, and where

$$\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, \quad i = 1, 2.$$

This shows immediately that, since we assume that $\Gamma < \mathrm{SL}(2, \mathbb{R})$, we have a well defined square root of the jacobian determinant, whence K_S is the square of the automorphic factor $(c_1 z_1 + d_1)(c_2 z_2 + d_2)$; hence $K_S = 2(L_1 + L_2)$, $Ł_j = 2L_j$, and our claim follows. □

Theorem 5.3. *A Shavel surface of unmixed type S is not a Mori-dream surface.*

Proof. We saw in Proposition 5.2 that S is an even fake quadric. We use the notation of Section 2.

It suffices to prove that $|nL_1| = \varnothing$ for any $n \geq 1$.

As remarked above, S admits two smooth foliations and Ω^1_S splits as the sum of two invertible sheaves:

$$\Omega^1_S = Ł_1 \oplus Ł_2,$$

where, by 5.2, we have $Ł_1 = 2L_1, Ł_2 = 2L_2$.

In fact, nL_1 is an automorphic line bundle on $\mathbb{H} \times \mathbb{H}$ corresponding to the following cocycle. We see this as follows: Let

$$p_1 : \mathbb{P}\mathrm{SL}(2, \mathbb{R}) \times \mathbb{P}\mathrm{SL}(2, \mathbb{R}) \to \mathbb{P}\mathrm{SL}(2, \mathbb{R}), \quad \gamma = (\gamma_1, \gamma_2) \mapsto \gamma_1,$$

be the first projection map. Then to γ_1 such that

$$\gamma_1(z_1) = \frac{a(\gamma_1)z_1 + b(\gamma_1)}{c(\gamma_1)z_1 + d(\gamma_1)}$$

we associate the automorphic factor $(c(\gamma_1)z_1 + d(\gamma_1))$.

Since Γ is irreducible, $p_1(\Gamma)$ is dense.

We claim that $H^0(S, nL_1) = 0$ for $n \geq 1$.

In fact, every section of $H^0(S, nL_1)$ is represented by a function f which satisfies the functional equation

$$f(\gamma_1 z_1, \gamma_2 z_2) = (c(\gamma_1)z_1 + d(\gamma_1))^n f(z_1, z_2).$$

By density of $p_1(\Gamma)$, this holds for each $\gamma_1 \in \mathrm{SL}(2, \mathbb{R})$.

Here we have to explain how γ_1, γ_2 are obtained (see [24]): $\mathcal{A}$ is a division quaternion algebra with centre a totally real number field $\mathcal{K}$ of degree 2 over $\mathbb{Q}$.

This means that there are two embeddings $\iota_1, \iota_2 : \mathcal{K} \to \mathbb{R}$, and these determine two homomorphisms

$$\iota_1, \iota_2 : \mathcal{A} \to Mat(2 \times 2, \mathbb{R}).$$

Then $\gamma_j := \iota_j(\gamma)$, for $\gamma \in \Gamma$, where Γ is the group of units lying in a maximal order $\mathfrak{O}$ of $\mathcal{A}$ and having reduced norm 1.

If we take now γ_1 to be in a maximal compact subgroup, the stabilizer of one point, then the same holds for γ_2; hence, using the biholomorphism of $\mathbb{H}$ with the unit disk, and choosing suitable coordinates, we can assume that $\gamma_1(z_1) = \lambda^2 z_1$, $\gamma_2(z_2) = \mu^2 z_2$.

Hence, setting

$$f(z_1, z_2) := \sum_{i,j} a_{i,j} z_1^i z_2^j,$$

we get

$$f(\lambda^2 z_1, \mu^2 z_2) = \lambda^{-n} f(z_1, z_2) \iff a_{i,j}\lambda^{2i}\mu^{2j} - \lambda^{-n} a_{i,j} = 0, \forall i, j \geq 0.$$

Now set $j = 0$: then $a_{i,0}\lambda^{2i} - \lambda^{-2n} a_{i,0} = 0$ for each λ, and we get a Laurent polynomial in λ whose coefficients are all vanishing, hence $a_{i,0} = 0$ for each $i \geq 0$. Therefore $f(z_1, 0)$ vanishes identically, and $f(z_1, z_2)$ vanishes identically for $z_2 = 0$.

Varying now γ_1, we obtain all the maximal compact subgroups to which γ_2 belongs.

Hence we have shown, for each choice of w_2, that $f(z_1, z_2)$ vanishes identically for $z_2 = w_2$.

We conclude that the section determined by the function f vanishes identically on the surface S. □

Remark 5.4. One can formulate the last argument as showing that the Iitaka dimension of L_j is $-\infty$.

Note that a more general result, but with a less elementary proof, is contained in Proposition IV.5.1 of [22], saying that the canonical model of a surface foliation with numerical dimension 1 has Iitaka dimension either $-\infty$ or 1 (and in Example II.2.3 it is stated that the first alternative applies for Hilbert modular surfaces).

6. Odd fake quadrics

In this section, we assume that S is an odd fake quadric. Recall that, over $\mathbb{C}$, S contains no smooth rational curves [23, Proposition 2.1.1] and in particular K_S is ample.

6.1. *The intersection form.*

Lemma 6.1. *There exist* $Q_1, Q_2 \in \mathrm{Pic}(S)$ *such that*

$$Q_1^2 = 1, \quad Q_2^2 = -1, \quad Q_1 Q_2 = 0, \quad K_S = 3Q_1 - Q_2.$$

The numerical classes $[Q_1]$ *and* $[Q_2]$ *are uniquely determined in* $\mathrm{Num}(S)$.

Moreover, for any such Q_1, Q_2,

(1) $h^0(S, 3Q_1) \geq 1$ *and* Q_1 *is nef and big*;

(2) Q_1 *is ample unless* S *contains an irreducible curve* C *such that* $C \sim Q_2$;

(3) Q_1 *is semiample.*

Proof. The intersection form on $\mathrm{Num}(S)$ is $\mathrm{diag}(1, -1)$.

Hence there exist divisors Q_1, Q_2 such that $Q_1^2 = 1$, $Q_2^2 = -1$, $Q_1 Q_2 = 0$.

We may assume $K_S \cdot Q_1 \geq 0$ and $K_S \cdot Q_2 \geq 0$ by possibly replacing Q_i with $-Q_i$. Then $K_S \sim aQ_1 + bQ_2$ with $a, b \in \mathbb{Z}$, $a \geq 0$, $b \leq 0$. Since $K_S^2 = 8$, $a^2 - b^2 = 8$. It follows that $a = 3$, $b = -1$ and $K_S \sim 3Q_1 - Q_2$.

Therefore $K_S = 3Q_1 - Q_2 + \eta$ for some $\eta \in \mathrm{Tors}(S)$, and we can assume $K_S = 3Q_1 - Q_2$ after replacing Q_2.

Note that $h^2(S, 3Q_1) = h^0(S, K_S - 3Q_1) = h^0(S, -Q_2)$ and $K_S(-Q_2) = -1$. Since K_S is ample, $h^2(S, 3Q_1) = 0$. Then the Riemann–Roch theorem shows

$$h^0(S, 3Q_1) \geq \tfrac{1}{2}(3Q_1)(Q_2) + \chi(\mathcal{O}_S) = 1.$$

Let C be an irreducible curve. We write $C \sim aQ_1 + bQ_2$ with $a = CQ_1$ and $b = -CQ_2$. Then $K_S C = 3a + b$ and $C^2 = a^2 - b^2$.

In order to see whether Q_1 is nef, respectively ample, assume that $CQ_1 \leq 0$, i.e., $a \leq 0$.

Then C is a negative curve, and, by Proposition 6.2 below, the class of C equals the class of $(b-1)Q_1 + bQ_2$, and we are done unless $b \leq 1$.

However, since K_S is ample and $3(b-1)+b = K_S C > 0$, $b \geq 1$. If $b=1$, then $C \sim Q_2$. Then we have shown that Q_1 is nef, and then (1) and (2) are proven.

For (3), we may assume that Q_1 is not ample. Then by (2), there is an irreducible curve $C \sim Q_2$. Note that $p_a(C) = 1$ and thus $\mathcal{O}_C(K_S + C) \cong \omega_C \cong \mathcal{O}_C$. Moreover,

$$3Q_1 = K_S + C + \eta'$$

for some $\eta' \in \mathrm{Tors}(S)$.

There exists $m > 0$ such that

- $m\eta' = 0$, and thus $3mQ_1|_C \cong \mathcal{O}_C$; and
- $h^0(3mQ_1) \gg 0$.

Note that Q_1 is nef and big, and that

$$3mQ_1 - C \sim K_S + 3(m-1)Q_1.$$

By the Kawamata–Viehweg vanishing theorem, we have

$$H^1(S, 3mQ_1 - C) = 0.$$

So the trace (restriction) of $|3mQ_1|$ on C is complete and base-point-free.

Write $|3mQ_1| = |M| + F$, where $|M|$ is the movable part and F is the fixed part. The discussion above shows that $F \not\geq C$ and thus $FC \geq 0$. Since $3mQ_1.C = 0$, we conclude that $M.C = 0$ and $F.C = 0$. It follows that $M \sim \lambda Q_1$ for some positive integer λ. Because $\mathrm{Tors}(S)$ is finite, $|kQ_1|$ has no fixed part for sufficiently large and divisible $k > 0$. By a theorem of Zariski, [27] (see also Theorem 14.19, page 223, of [2]), Q_1 is semiample. □

Unlike the even fake quadric case, we do not know whether S contains a negative curve or not (but in the case it does not contain such a negative curve, we have determined the condition that it is a Mori dream space in Theorem 4.5).

Proposition 6.2. *Let C be an irreducible curve on S. Assume that $C^2 < 0$. Then*:

(1) *$C \sim aQ_1 + (a+1)Q_2$ for some $a \in \mathbb{Z}_{\geq 0}$ and $p_a(C) = a+1$.*

(2) *For any irreducible curve $C_0 \neq C$, $C_0^2 \geq 0$.*

(3) *Set $D := (a+1)Q_1 + aQ_2$. Then $DC = 0$, D is nef and big, moreover D is semiample only if $\mathcal{O}_C(D)$ is a torsion divisor.*

(4) *One of the sides of $\mathrm{Eff}(S)$ is $\mathbb{R}_+[C]$ and one of the sides of $\mathrm{Nef}(S)$ is $\mathbb{R}_+[D]$.*

Proof. We may assume $C \sim aQ_1 + bQ_2$; hence $a = CQ_1$ and $b = -CQ_2$. Then by our assumption $K_S C = 3a + b > 0$, $C^2 = a^2 - b^2 < 0$.

Set $\alpha := |a|$, then $|b| = \alpha + \delta$ with $\delta > 0$.

For some $\epsilon_1, \epsilon_2 \in \{1, -1\}$, we have $a = \epsilon_1 \alpha$, $b = \epsilon_2(\alpha + \delta)$.

The inequalities $K_S C = 3a + b > 0$ and $K_S C + C^2 \geq 0$ (since S contains no rational curve) read out as:

$$3\epsilon_1\alpha + \epsilon_2(\alpha + \delta) > 0, \quad 3\epsilon_1\alpha + \epsilon_2(\alpha + \delta) - 2\delta\alpha - \delta^2 \geq 0.$$

If $\delta \geq 2$, then $-2\delta\alpha + (3\epsilon_1 + \epsilon_2)\alpha \leq 0$, while $\epsilon_2\delta - \delta^2 < 0$; hence this contradiction shows that $\delta = 1$.

The first inequality excludes the possibility $\epsilon_1 = \epsilon_2 = -1$.

If $\epsilon_1 = 1$, $\epsilon_2 = -1$, then the second inequality tells that $-2 \geq 0$, absurd.

Hence $\epsilon_2 = 1$, and $(3\epsilon_1 + 1 - 2)\alpha \geq 0$ shows that $\epsilon_1 = 1$.

Therefore $C \sim \alpha Q_1 + (\alpha + 1)Q_2$ and (1) is proven.

Next, if C_0 is a different negative curve, we have $C_0 \sim a_0 Q_1 + (a_0 + 1)Q_2$ with $a_0 \geq 0$. Then $CC_0 = aa_0 - (a + 1)(a_0 + 1) < 0$, which is impossible, proving (2).

For (3), clearly we have $D^2 = 2a + 1 > 0$ and $DC = 0$.

From (1) and (2), we see $DC_0 > 0$ for any $C_0 \neq C$. Thus D is nef and big.

By a Theorem of Zariski, saying that a nef and big divisor D is asymptotically base point free if and only if there exists a large multiple $|mD|$ which is without fixed part, D is semiample if and only if for each irreducible curve C', C' is not in the base locus of some $|mD|$ with m positive.

Applying this to $C' = C$ we see that $\mathcal{O}_C(D)$ must be a torsion line bundle.

Let us show (4). We have seen that C is the only irreducible curve which is not inside the closure $\bar{\mathcal{P}}$ of the positive cone $\mathcal{P}$, which is of course contained inside $\mathrm{Eff}(S)$.

Hence $\mathbb{R}_+[C]$ is one of the sides of $\mathrm{Eff}(S)$.

Since the nef cone is the dual of the closure $\overline{\mathrm{Eff}(S)}$, which is the span of $\mathbb{R}_+[C]$ and $\bar{\mathcal{P}}$, and since D is orthogonal to C, and is nef and big, $\mathbb{R}_+[D]$ is one of the sides of $\mathrm{Nef}(S)$. □

Corollary 6.3. *Assume that $f : S \to \mathbb{P}^1$ is a fibration with general fibre F. Then either $F \sim a(Q_1 + Q_2)$ with $a \geq 1$ and $g(F) = 2a + 1$, or $F \sim a(Q_1 - Q_2)$ and $g(F) = a + 1$ with $a \geq 1$.*

Proof. Let $F \sim aQ_1 + bQ_2$: then $F^2 = 0$ amounts to $a^2 = b^2$, that is, $a = \epsilon_1\alpha$, $b = \epsilon_2\alpha$, with $\alpha > 0$, $\epsilon_j \in \{1, -1\}$.

Since $K_S F > 0$, we get $3\epsilon_1 + \epsilon_2 > 0$; hence $\epsilon_1 = 1$, and the two solutions are as stated. □

Corollary 6.4. *Assume that S contains a negative curve. Then S admits at most one fibration, and if there is a fibration $f : S \to \mathbb{P}^1$ with general fibre F, then $F \sim (g(F)-1)(Q_1 - Q_2)$.*

Proof. To show the last assertion, we observe that $F' := Q_1 + \epsilon Q_2$ is nef; hence $F' \cdot C \geq 0$.

This condition amounts to $a - \epsilon(a+1) \geq 0$; hence $\epsilon = -1$. □

7. Characteristic p

Let S be an even fake quadric, now over an algebraically closed field of characteristic $p > 0$.

We begin with an easy remark: the quadrant $\{n_1 L_1 + n_2 L_2 \mid n_1, n_2 \geq 0\}$ is contained in the closure of the effective cone, since $\mathcal{P} := \{n_1 L_1 + n_2 L_2 \mid n_1, n_2 > 0\}$ consists of big divisors D (this means, for $n \gg 0$, $nD = A + E$, where A is ample and E effective). Indeed, by Riemann–Roch $D \sim d_1 L_1 + d_2 L_2$ is effective for $d_1, d_2 \geq 2$, $(d_1, d_2) \neq (2, 2)$.

Lemma 7.1. *There are at most two negative irreducible curves C on S.*

The class of C may only be $C \sim -L_1 + bL_2$ or $C \sim aL_1 - L_2$ and $C \cong \mathbb{P}^1$. Moreover, $b \geq 2$, and $a \geq 2$ if K_S is ample.

If K_S is not ample there is a unique irreducible -2-curve C orthogonal to K_S: then either $C \sim -L_1 + L_2$ or $C \sim L_1 - L_2$, but obviously both possibilities cannot occur.

Proof. If C is irreducible with $C \sim c_1 L_1 + c_2 L_2$, if C is negative $c_1 c_2 < 0$, hence we may assume that $c_1 > 0$, $c_2 < 0$.

Since $K_S C \geq 0$, we obtain $c_1 + c_2 \geq 0$.

If C' is another negative irreducible curve, it cannot lie in the same quadrant, since $c'_1 > 0$, $c'_2 < 0$ implies $CC' = c_1 c'_2 + c'_1 c_2 < 0$, a contradiction.

Hence there is at most one negative curve, in each of the two quadrants which are neither positive nor negative.

Assume now that we have an irreducible curve C with $C^2 < 0$. Then $C \sim aL_1 + bL_2$ for $a, b \in \mathbb{Z}$ and

$$C^2 = 2ab, \qquad K_S C = 2a + 2b.$$

We may assume that $a > 0$ and $b < 0$. By the adjunction formula,

$$-2 \leq 2p_a(C) - 2 = C^2 + K_S C = 2(a + b + ab) = 2a(1+b) + 2b.$$

Therefore $b = -1$, $C \cong \mathbb{P}^1$. □

Remarks 7.2. (i) In [14], remark after Lemma 6.3, Ekedahl, Shepherd-Barron and Taylor show that, for each prime p which is inert in the quadratic field $\mathcal{K}$ which is the centre of the quaternion algebra $\mathcal{A}$ of a Shavel surface of special unmixed type (see the proof of Proposition 5.2 for more details), the divisors $-2L_1+2pL_2$ and $2pL_1-2L_2$ are effective, since the p-curvature tensor is nonvanishing for both foliations.

(ii) Is it true that the possible numbers a, b in the previous Lemma 7.1 can only be equal to the characteristic p?

If we have two negative curves $C_1 \sim a_1L_1 - L_2$ and $C_2 \sim -L_1+b_2L_2$, they span the effective cone, which is therefore polyhedral.

The nef cone consists of divisors $D \sim aL_1+bL_2$ such that

$$a \leq ba_1, b \leq ab_2;$$

hence it is polyhedral and spanned by $D_1 \sim a_1L_1+L_2$, $D_2 \sim L_1+b_2L_2$.

Proposition 7.3. *If on S there are two negative curves $C_1 \sim a_1L_1 - L_2$ and $C_2 \sim -L_1+b_2L_2$, then S is a Mori dream space.*

Proof. By [1] it suffices to show that the divisors $D_1 \sim a_1L_1+L_2$, $D_2 \sim L_1+b_2L_2$ are semiample.

The divisors are both nef and big, and by symmetry, it suffices to show only the first assertion, that D_1 is semiample.

We denote by $\mathbb{E}(D_1)$ the exceptional locus of D_1, i.e., the union of the finite maximal subvarieties Z such that the restriction of D_1 to Z is nonbig. Since D_1 is big and C_1 is the only curve which is orthogonal to D_1, it follows that $\mathbb{E}(D_1) = C_1$. By Lemma 7.1, we have that $C_1 \cong \mathbb{P}^1$ and hence $\mathcal{O}_{C_1}(D_1)$ is semiample.

We apply Theorem 0.2 of [15] (see also [4], Corollary 3.6), stating that if we are in positive characteristic and D_1 is nef and big and the restriction to the exceptional locus $\mathbb{E}(D_1)$ is semiample, then also D_1 is semiample.

Hence we are done. □

Theorem 1.1 follows now immediately from Theorem 5.3, from the fact that S is defined over a number field, from i) of Remarks 7.2, and the previous Proposition 7.3.

8. Problems

Problem 1. Consider all fake $\mathbb{Q}$-homology quadrics S that are isogenous to a product of curves. (a) Determine which ones are even. (b) Determine which ones are odd.

Problem 2. Let S be an odd fake quadric. Does S contain a negative curve?

Problem 3. Let S be an odd fake quadric. Could S have two fibrations?

Remark: this is related to Problem 1 since surfaces isogenous to a product are $\mathbb{Q}$-homology quadrics having two fibrations.

Problem 4′. (Hirzebruch's question) Is every surface homeomorphic to a smooth quadric indeed a deformation of $\mathbb{P}^1 \times \mathbb{P}^1$?

Problem 4″. Is every surface homeomorphic to $\mathbb{F}_1$ indeed a deformation of $\mathbb{F}_1$?

A negative answer to both questions would follow if one could answer positively the next problem 5, or negatively the weaker problem 6: indeed, by a theorem of Michael Freedman [17], a simply connected fake quadric is homeomorphic either to $\mathbb{F}_1$ or to $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

Problem 5. Let S be a fake quadric: is then the universal covering of S biholomorphic to $\mathbb{H} \times \mathbb{H}$?

Problem 6. Is there a simply connected fake quadric?

Problem 7. (raised by Michael Lönne at a seminar talk by the second author): is there a fake quadric with $H_1(S, \mathbb{Z}) = 0$?

Remark 8.1. If a fake quadric S is homeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ then S is spin, that is, K_S is divisible by 2, and one may study its half-canonical ring.

Acknowledgements and addendum

Our collaboration started at the Hefei USTC Conference and continued at the Shanghai Fudan Workshop on birational geometry, held in September 2024; we are very grateful to Meng Chen and Lei Zhang for organizing these events.

The question about the existence of $\mathbb{Q}$-homology fake quadrics of odd type was raised by Jianqiang Yang in email correspondence with the second author (in December 2022).

Cascini is partially supported by a Simons Collaboration grant. Keum is supported by the National Research Foundation of Korea (RS-2022-NR068993).

We thank the referees for pointing out some minor inaccuracies.

A referee pointed out to us the reference [25], as an example of another nonclassical phenomenon that can occur in positive characteristic: the authors construct a minimal surface of general type with $K_S^2 = 8$, and with $e(S) = 4$, equivalently $p_g = q$, in characteristic 3. The interesting fact is that the surface admits a nonarchimedean uniformization yet has $q(S) = h^1(\mathcal{O}_S) = 1$, and trivial Albanese variety.

References

[1] Michela Artebani, Jürgen Hausen and Antonio Laface, *On Cox rings of K3 surfaces*, Compos. Math. **146 (4)** (2010), 964-998.

[2] Lucian Badescu, Algebraic surfaces. Transl. from the Romanian by V. Macek. Universitext. New York, NY: Springer. x, 258 p. (2001).

[3] Arnaud Beauville, *Complex manifolds with split tangent bundle.* Peternell, Thomas (ed.) et al., Complex analysis and algebraic geometry. A volume in memory of Michael Schneider. Berlin: Walter de Gruyter. 61–70 (2000).

[4] Paolo Cascini, James M^cKernan, Mircea Mustata, *The augmented base locus in positive characteristic.* Proc. Edinb. Math. Soc., II. Ser. **57**, No. 1, 79–87 (2014).

[5] Ingrid Bauer, Fabrizio Catanese, and Davide Frapporti, *The fundamental group and torsion group of Beauville surfaces.* Bauer, Ingrid (ed.) et al., Beauville surfaces and groups. Proceedings of the conference, Newcastle, UK, June 7–9, 2012. Springer Proceedings in Mathematics and Statistics **123** (2015), 1-14.

[6] Ingrid Bauer, Fabrizio Catanese, and Fritz Grunewald, *The classification of surfaces with* $p_g = q = 0$ *isogenous to a product of curves*, Pure Appl. Math. Q. **4** (2) (2008), 547-586.

[7] Fabrizio Catanese, *Fibred surfaces, varieties isogenous to a product and related moduli spaces.* Am. J. Math. **122**, No. 1, 1–44 (2000).

[8] Fabrizio Catanese, Marco Franciosi, [Antonio J. Di Scala] *On varieties whose universal cover is a product of curves. (With an appendix by Antonio J. Di Scala).* Bates, Daniel J. (ed.) et al., Interactions of classical and numerical algebraic geometry. A conference in honor of Andrew Sommese, Notre Dame, IN, USA, May 22-24, 2008. Providence, RI: American Mathematical Society (AMS) Contemporary Mathematics **496**, 157–179 (2009).

[9] Fabrizio Catanese, Antonio J. Di Scala, *A characterization of varieties whose universal cover is the polydisk or a tube domain.* Math. Ann. **356**, No. 2, 419–438 (2013).

[10] Fabrizio Catanese, *Subcanonical graded rings which are not Cohen-Macaulay.* Hacon, Christopher D. (ed.) et al., Recent advances in algebraic geometry. A volume in honor of Rob Lazarsfeld' s 60th birthday. Based on the conference, Ann Arbor, MI, USA, May 16–19, 2013. Cambridge: Cambridge University Press London Mathematical Society Lecture Note Series **417**, (2014), 92–101 .

[11] Fabrizio Catanese, *Odd fake* $\mathbb{Q}$*-homology quadrics exist*, arXiv: 2504.17475.

[12] Amir Dzambić, *Fake quadrics from irreducible lattices acting on the product of upper half planes.* Math. Ann. **360**, No. 1-2, 23–51 (2014).

[13] Amir Dzambić, Xavier Roulleau, *Minimal surfaces of general type with* $p_g = q = 0$ *arising from Shimura surfaces.* Asian J. Math. **21**, No. 4, 775–790 (2017).

[14] Torsten Ekedahl, Nick Shepherd Barron, Richard L. Taylor *A conjecture on the existence of compact leaves of algebraic foliations*, preprint (1999).

[15] Sean Keel, *Basepoint freeness for nef and big line bundles in positive characteristic.* Ann. Math. (2) **149**, No. 1, 253–286 (1999).

[16] Davide Frapporti, Kyoung-Seog Lee, *Divisors on surfaces isogenous to a product of mixed type with* $p_g = 0$. Pac. J. Math. **318**, No. 2, 233–247 (2022).

[17] Michael Hartley Freedman, *The topology of four-dimensional manifolds.* J. Differ. Geom. 17, 357–453 (1982).

[18] Friedrich Hirzebruch, *Some problems on differentiable and complex manifolds.* Ann. Math. (2) 60, 213–236 (1954).

[19] Friedrich Hirzebruch, *'Kommentare'*, Gesammelte Abhandlungen, vol. I. Springer, Berlin 755-804 (1987).

[20] JongHae Keum and Kyoung-Seog Lee, *Examples of Mori dream surfaces of general type with* $p_g = 0$, Adv. Math. **347** (2019), 708-738.

[21] JongHae Keum and Kyoung-Seog Lee, *Combinatorially minimal Mori dream surfaces of general type.* Preprint, arXiv:2206.02913.

[22] Michael McQuillan, *Canonical Models of Foliations*, Pure and Applied Math. Quart. **4**, Number 3, (2008) 877–1012.

[23] Yoichi Miyaoka, *The maximal number of quotient singularities on surfaces with given numerical invariants*, Math. Ann. **268** (1984), 159-171.

[24] Ira H. Shavel, *A class of algebraic surfaces of general type constructed from quaternion algebras.* Pac. J. Math. **76**, 221–245 (1978).

[25] Jakob Stix, Alina Vdovina, *Simply transitive quaternionic lattices of rank* 2 *over* $\mathbb{F}_q(t)$ *and a non-classical fake quadric.* Math. Proc. Camb. Philos. Soc. **163**, No. 3, 453–498 (2017).

[26] Shing Tung Yau, *A splitting theorem and an algebraic geometric characterization of locally Hermitian symmetric spaces.* Comm. Anal. Geom. **1**(3–4), 473–486 (1993).

[27] Oscar Zariski, The theorem of Riemann–Roch for high multiples of an effective divisor on an algebraic surface. Ann. Math. (2) 76, 560–615 (1962).

Received January 31, 2025. Revised June 23, 2025.

PAOLO CASCINI
DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE LONDON
LONDON SW7 2AZ
UNITED KINGDOM

p.cascini@imperial.ac.uk

FABRIZIO CATANESE
MATHEMATISCHES INSTITUT
UNIVERSITÄT BAYREUTH
95447 BAYREUTH
GERMANY

fabrizio.catanese@uni-bayreuth.de

and

SCHOOL OF MATHEMATICS
KOREA INSTITUTE FOR ADVANCED STUDY
SEOUL 02455
SOUTH KOREA

YIFAN CHEN
SCHOOL OF MATHEMATICAL SCIENCES
BEIHANG UNIVERSITY
SHAHE HIGHER EDUCATION PARK
BEIJING, 102206
CHINA

chenyifan1984@buaa.edu.cn

JONGHAE KEUM
SCHOOL OF MATHEMATICS
KOREA INSTITUTE FOR ADVANCED STUDY
SEOUL 02455
SOUTH KOREA

jhkeum@kias.re.kr

PACIFIC JOURNAL OF MATHEMATICS
Vol. 339, No. 2, 2025
https://doi.org/10.2140/pjm.2025.339.243

FIVE-DIMENSIONAL MINIMAL QUADRATIC AND BILINEAR FORMS OVER FUNCTION FIELDS OF CONICS

ADAM CHAPMAN AND AHMED LAGHRIBI

Over a field of characteristic 2, we give a complete classification of quadratic and bilinear forms of dimension 5 that are minimal over the function field of an arbitrary conic. This completes the unique known case due to Faivre concerning the classification of minimal quadratic forms of dimension 5 and type (2, 1) over function fields of nondegenerate conics.

1. Introduction

Let F be a field of characteristic 2 and K/F a field extension. An anisotropic F-form (quadratic or bilinear) φ is called *K-minimal* if φ_K is isotropic and any form ψ dominated by φ, such that $\dim\psi < \dim\varphi$, remains anisotropic over F ($\dim\varphi$ denotes the dimension of φ). We refer to Section 2 for the definition of the domination relation which is more refined than the subform relation and is necessary when we take into account singular quadratic forms. Let us mention that the minimality for bilinear forms is equivalent to that of totally singular quadratic forms (Corollary 6). Henceforth, we will restrict ourselves on the minimality for quadratic forms.

In the case of a quadratic extension K/F, any K-minimal form is of dimension 2. Obviously, the same conclusion is true when K is the function field of a 2-dimensional quadratic form. When K is the function field of a conic, then any 3-dimensional anisotropic F-form which becomes isotropic over K is necessarily K-minimal, and there is no K-minimal form of dimension 4. These two facts combine many references that we summarize below (we refer to Sections 2 and 5 for the definition of the type and the norm degree ndeg_F):

(1) For quadratic forms φ of dimension 3, we use [13, théorème 1.4].

(2) For quadratic forms φ of dimension 4, we use

(i) [13, théorème 1.3] for φ of type $(2, 0)$,

MSC2020: 11E04, 11E81.

Keywords: quadratic form, bilinear form, function field of a conic, minimal form, quasi-Pfister neighbor.

(ii) [13, théorème 1.4] for φ of type $(1, 2)$,

(iii) [20, Theorem 1.2] for φ of type $(0, 4)$ and $\mathrm{ndeg}_F(\varphi) = 8$,

(iv) [20, Proposition 1.1 and Theorem 1.2] for φ of type $(0, 4)$ and $\mathrm{ndeg}_F(\varphi) = 4$.

We will distinguish between degenerate and nondegenerate conics. Recall that a conic is called degenerate if it is given by a quadratic form of type (0, 3), otherwise it is given by a quadratic form of type (1, 1) and called nondegenerate. Quadratic forms of dimension 5 which are minimal over function fields of conics were classified first in characteristic not 2 by Hoffmann, Lewis and van Geel [9]. Their result has been extended to characteristic 2 by Faivre in the case of quadratic forms of type (2, 1) and nondegenerate conics. His result states the following:

Theorem 1 [4, Corollary 3.7; 6, Proposition 5.2.12]. *Let φ be an anisotropic F-quadratic form of dimension 5 and type $(2, 1)$, and $\tau = b[1, a] \perp \langle 1\rangle$ an anisotropic F-quadratic form of dimension 3. Then, φ is $F(\tau)$-minimal if and only if these three conditions are satisfied*:

(i) *φ is a Pfister neighbor of a 3-fold Pfister form π.*

(ii) *$\pi \simeq \langle\langle c, b, a]]$ for a suitable $c \in F^*$.*

(iii) $\mathrm{ind}(C_0(\varphi) \otimes C_0(\tau)) = 4$.

For function fields of nondegenerate conics, Faivre proved this general result:

Proposition 1 [6, Propositions 5.2.1, 5.2.8, 5.2.11]. *Let $\tau = b[1, a] \perp \langle 1\rangle$ be an anisotropic F-quadratic form of type $(1, 1)$, and φ an anisotropic F-quadratic form. Suppose that φ is $F(\tau)$-minimal, then we have*:

(1) *φ is singular but not totally singular.*

(2) *If φ is of type $(1, \ell)$, then ℓ is odd.*

(3) *If $\dim\varphi = 5$, then φ is a Pfister neighbor of a 3-fold Pfister form $\langle\langle c, b, a]]$ for some $c \in F^*$.*

The proof of this proposition is mainly based on the fact that the extension given by the function field of a nondegenerate conic is excellent [8, Corollary 5.7] and some arguments similar to those developed by Hoffmann, Lewis and Van Geel in characteristic not 2 [9]. This excellence result is no longer true for degenerate conics as it was proved by Laghribi and Mukhija [19].

To our knowledge, no classification of minimal quadratic forms of type (2, 1) over function fields of degenerate conics, or of type $(1, 3)$ over function fields of arbitrary conics are known. Our aim in this paper is to complete these open cases. The first result in this sense is the following theorem that concerns minimal quadratic forms of type (2, 1) over function fields of degenerate conics.

Theorem 2. *Let φ be an anisotropic F-quadratic form of dimension 5 and type $(2, 1)$, and $\tau = \langle 1, a, b\rangle$ an anisotropic totally singular F-quadratic form of dimension 3. Then, φ is $F(\tau)$-minimal if and only if these three conditions are satisfied*:

(i) *φ is a Pfister neighbor of a 3-fold Pfister form π.*

(ii) *$\pi \simeq \langle\langle a, b, c]]$ for some $c \in F^*$.*

(iii) $\operatorname{ind} C_0(\varphi)_{F(\sqrt{a},\sqrt{b})} = 2$.

Concerning the classification of minimal 5-dimensional quadratic forms of type $(1, 3)$ over function fields of degenerate conics, we prove the following result.

Theorem 3. *Let φ be an anisotropic F-quadratic form of type $(1, 3)$, and $\tau = \langle 1, a, b\rangle$ an anisotropic totally singular quadratic form of dimension 3. Then, φ is $F(\tau)$-minimal if and only if these three conditions are satisfied*:

(i) *φ is a Pfister neighbor of a 3-fold Pfister form π.*

(ii) *$\pi \simeq \langle\langle a, b, c]]$ for some $c \in F^*$.*

(iii) *For any $e \in F^*$, we have either*
   (a) *$i_d(e\tau \perp \mathrm{ql}(\varphi)) \leq 1$, or*
   (b) *$i_d(e\tau \perp \mathrm{ql}(\varphi)) = 2$ and $(D_F(\varphi) \cap D_F(e\tau)) \setminus D_F(\mathrm{ql}(\varphi)) = \varnothing$.*

The classifications given in Theorems 1 et 2 are based on the even Clifford algebra $C_0(\varphi)$ of φ. However, for quadratic forms of type $(1, 3)$, another characterization is used in Theorem 3. This is due to the fact that the even Clifford algebra of any quadratic Pfister neighbor of type $(1, 3)$ is split as we state in Corollary 1.

Proposition 2 (compare [21, Lemma 2]). *Let $\varphi = a_1[1, b_1] \perp \cdots \perp a_n[1, b_n] \perp \langle 1, c_1, \cdots, c_s\rangle$ be a singular quadratic form such that $n \geq 1$ and $s \geq 1$, and let $K = F(\sqrt{c_1}, \ldots, \sqrt{c_s})$. Then, $C_0(\varphi)$ is isomorphic to the F-algebra $[b_1, a_1) \otimes_F \cdots \otimes_F [b_n, a_n) \otimes_F K$. In particular, $C_0(\varphi)$ has degree 2^n as a K-algebra.*

Corollary 1. *An anisotropic F-quadratic form φ of type $(1, 3)$ is a Pfister neighbor if and only if φ is similar to $rs[1, u] \perp \langle 1, r, s\rangle$ for suitable scalars $r, s, u \in F^*$. Moreover, $C_0(\varphi)$ is split as a K-algebra, where $K = F(\sqrt{r}, \sqrt{s})$.*

For the classification of minimal 5-dimensional quadratic forms of type $(1, 3)$ over function fields of nondegenerate conics, we prove the following theorem:

Theorem 4. *Let φ be an anisotropic F-quadratic form of type $(1, 3)$, and $\tau = a[1, b] \perp \langle 1\rangle$ an anisotropic quadratic form of dimension 3 and type $(1, 1)$. Then, φ is $F(\tau)$-minimal if and only if these three conditions are satisfied*:

(i) *φ is a Pfister neighbor of a 3-fold Pfister form π.*

(ii) *$\pi \simeq \langle\langle c, a, b]]$ for some $c \in F^*$.*

(iii) *For any $e \in F^*$, if $e[1, b] \subset \varphi$ then $e \notin D_F(a[1, b]) \cdot D_F(\mathrm{ql}(\varphi))$.*

Finally, for the classification of minimal quadratic forms of type (0, 5), we use the language of bilinear forms, which will help us to use a cohomological invariant and a classification parallel to those given in Theorems 1 et 2. Namely, we will prove the following result:

Theorem 5. *Let B be an anisotropic F-bilinear form of dimension 5, and $Q = \langle 1, a, b\rangle$ an anisotropic totally singular quadratic form of dimension 3. Then, the following statements are equivalent*:

(1) *B is $F(Q)$-minimal.*

(2) *There exists an F-bilinear form C of dimension 5 which is a strong Pfister neighbor of a bilinear Pfister form $\langle\langle a, b, c\rangle\rangle_b$ and satisfies the two conditions*:

  (i) *$\tilde{B} \simeq \tilde{C}$.*

  (ii) *For any $u, v \in F^2(a, b)$ such that $\langle u, v, uv\rangle_b$ is similar to a subform of $\langle\langle a, b, c\rangle\rangle_b$, the invariant $e^2(C \perp \langle \det C\rangle_b \perp \langle\langle u, v\rangle\rangle_b + I^3 F)$ has length 2.*

Note that for bilinear forms, nothing happens over the function field of a nondegenerate conic since an anisotropic bilinear form remains anisotropic over such a field. To clarify the notations used in Theorem 5, let us recall that to any bilinear form B of underlying vector space V, we associate a totally singular quadratic form $\tilde{B}$ defined on V by: $\tilde{B}(v) = B(v, v)$ for all $v \in V$. The cohomological invariant e^2 is that due to Kato [10] going from the quotient $I^2 F/I^3 F$ to $\nu_F(2)$, where $\nu_F(2)$ is the additive group generated by the logarithmic symbols $\frac{da_1}{a_1} \wedge \frac{da_2}{a_2}$ for $a_1, a_2 \in F^*$. This invariant plays the role of the Clifford invariant which is not defined for bilinear forms in characteristic 2, and thus the group $\nu_F(2)$ can be seen as the 2-torsion of the Brauer group. The word "length" that we talk about in the previous theorem concerns the smallest number of logarithmic symbols needed to write the cohomological invariant $e^2(\eta)$ for $\eta \in I^2 F/I^3 F$. Finally, the notion of a strong Pfister neighbor bilinear form is defined as the classical notion of a quadratic Pfister neighbor. We use the term "strong" since we have another weaker notion of bilinear Pfister neighbor (see Section 5).

2. Background on quadratic and bilinear forms

We refer to [5] for undefined terminologies or facts. Recall that any quadratic form φ decomposes as follows:

$$\varphi \simeq [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp \langle c_1\rangle \perp \cdots \perp \langle c_s\rangle \,, \tag{1}$$

where $[a, b]$ (resp. $\langle c\rangle$) denotes the binary quadratic form $ax^2 + xy + by^2$ (resp. cz^2). Here, $\simeq$ and $\perp$ denote the isometry and the orthogonal sum, respectively.

As in (1), the form φ is called of type (r, s). We say that φ is nonsingular (resp. totally singular) if $s = 0$ (resp. $r = 0$). It is called singular if $s > 0$.

The form $\langle c_1\rangle \perp \cdots \perp \langle c_s\rangle$ in (1) is unique up to isometry, we call it the quasilinear part of φ and denote it by $\mathrm{ql}(\varphi)$.

A quadratic form φ of underlying vector space V is called isotropic if there exists $v \in V \setminus \{0\}$ such that $\varphi(v) = 0$. Otherwise, φ is called anisotropic.

Any quadratic form φ uniquely decomposes as follows:

$$\varphi \simeq \varphi_{an} \perp [0,0] \perp \cdots \perp [0,0] \perp \langle 0\rangle \perp \cdots \perp \langle 0\rangle\,, \tag{2}$$

where the form φ_{an} is anisotropic that we call the anisotropic part of φ. The number of copies of the hyperbolic plane $[0,0]$ in (2) is called the Witt index of φ, we denote it by $i_W(\varphi)$. Similarly, the number of $\langle 0\rangle$ in (2) is called the defect index of φ, we denote it by $i_d(\varphi)$. The total index of φ is $i_W(\varphi) + i_d(\varphi)$.

Two quadratic forms φ and ψ are called Witt equivalent if $\varphi \perp m \times [0,0] \simeq \psi \perp n \times [0,0]$ for some integers $m, n \geq 0$. In this case, we write $\varphi \sim \psi$.

Let $C(\varphi)$ (resp. $C_0(\varphi)$) denote the Clifford algebra (resp. the even Clifford algebra) of the quadratic form φ. When $\varphi \simeq a_1[1,b_1] \perp \cdots \perp a_r[1,b_r]$ for $a_i, b_i \in F$ such that $a_i \neq 0$ for $1 \leq i \leq r$, its Arf invariant $\Delta(\varphi)$ is the class of $\sum_{i=1}^r b_i$ in $F/\wp(F)$, where $\wp(F) = \{x^2 + x \mid x \in F\}$. In this case, $C(\varphi)$ is isomorphic to $\otimes_{i=1}^r [b_i, a_i)$, where $[b,a)$ denotes the quaternion algebra generated by two elements i and j subject to the relations: $i^2 = a \in F^* := F \setminus \{0\}$, $j^2 + j = b \in F$ and $iji^{-1} = j + 1$.

Let φ and ψ be two quadratic forms over F of underlying vector space V and W, respectively. We say that φ is dominated by ψ if there exists an injective F-linear map $\sigma : V \longrightarrow W$ such that $\varphi(v) = \psi(\sigma(v))$ for any $v \in V$. In this case, we write $\varphi \prec \psi$. We say that φ is weakly dominated by ψ if $\alpha\varphi \prec \psi$ for some $\alpha \in F^*$. The form φ is called a subform of ψ, denoted by $\varphi \subset \psi$, if $\psi \simeq \varphi \perp \varphi'$ for a suitable quadratic form φ'. Clearly, if φ is a subform of ψ, then it is dominated by ψ, but the converse is not true in general. We refer to [7, Lemma 3.1] for more details on the domination relation.

For $a_1, \ldots, a_n \in F^*$, let $\langle a_1, \ldots, a_n\rangle_b$ denote the diagonal bilinear form given by

$$((x_1, \ldots, x_n), (y_1, \cdots, y_n)) \mapsto \sum_{i=1}^n a_i x_i y_i.$$

A metabolic plane is a 2-dimensional bilinear form isometric to $\left(\begin{smallmatrix} a & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for some $a \in F$; we denote it by $\mathbb{M}(a)$. A bilinear form is called metabolic if it is isometric to a sum of metabolic planes.

Let $W_q(F)$ (resp. $W(F)$) be the Witt group of nonsingular quadratic forms (resp. the Witt ring of nondegenerate symmetric bilinear forms). For any integer $m \geq 1$, let $I^m F$ be the m-th power of the fundamental ideal IF of classes of even-dimensional forms in $W(F)$ (we take $I^0 F = W(F)$). Recall that $W_q(F)$ is endowed with $W(F)$-module structure in a natural way [1]. For any integer $m \geq 2$, let $I_q^m F$ be the submodule $I^{m-1}F \cdot W_q(F)$ of $W_q(F)$. The ideal $I^m F$ is additively generated by the m-fold bilinear Pfister forms $\langle 1, a_1\rangle_b \otimes \cdots \otimes \langle 1, a_m\rangle_b$, that we denote by $\langle\langle a_1, \ldots, a_m\rangle\rangle_b$. The submodule $I_q^m F$ is generated, as a $W(F)$-module, by the quadratic forms $\langle\langle a_1, \ldots, a_{m-1}\rangle\rangle_b \cdot [1, b]$, that we denote by $\langle\langle a_1, \ldots, a_{m-1}, b]]$ and call an m-fold quadratic Pfister form. We have $I_q^1 F = W_q(F)$ and a 1-fold quadratic Pfister form is a form of type $[1, a]$.

Let $P_m F$ be the set of forms isometric to m-fold quadratic Pfister forms, and $GP_m F$ the set of forms similar to forms in $P_m F$. Similarly, let $BP_m F$ be the set of forms isometric to m-fold bilinear Pfister forms, and $GBP_m F$ the set of forms similar to forms in $BP_m F$.

For $m \geq 1$ an integer and $B \in BP_m F$, we have $B \simeq \langle 1\rangle_b \perp B'$ for some bilinear form B'. This form B' is unique, we call it the pure part of B.

The Hauptsatz of Arason–Pfister asserts that any anisotropic form in $I_q^m F$ (or $I^m F$) has dimension $\geq 2^m$. Moreover, if the form has dimension 2^m, then it is similar to a Pfister form (see [16, lemme 4.8] for bilinear forms, and [17, proposition 6.4] for quadratic forms). In this paper, we will only need the Hauptsatz for bilinear forms.

Recall that a quadratic (resp. bilinear) Pfister form Q is isotropic if and only if it is hyperbolic (resp. metabolic). Such a form is also round, meaning that $\alpha \in F^*$ is represented by Q if and only if $Q \simeq \alpha Q$.

For a quadratic form $\varphi \simeq [a_1, b_1] \perp \cdots \perp [a_r, b_r] \perp \langle c_1\rangle \perp \cdots \perp \langle c_s\rangle$, we define the polynomial $P_\varphi = \sum_{i=1}^{r}(a_i x_i^2 + x_i y_i + b_i y_i^2) + \sum_{j=1}^{s} c_j z_j^2$. This polynomial is reducible if and only if $\varphi \simeq [0, 0] \perp \langle 0\rangle \perp \cdots \perp \langle 0\rangle$ or $\varphi \simeq \langle a\rangle \perp \langle 0\rangle \perp \cdots \perp \langle 0\rangle$ for some $a \in F^*$ [21, Proposition 3]. When P_φ is irreducible, we denote by $F(\varphi)$ the quotient field of the ring $F[x_i, y_i, z_j]/(P_\varphi)$, that we call the function field of φ. When P_φ is reducible or φ is the zero form, then we take $F(\varphi) = F$.

A quadratic form φ is called a Pfister neighbor if there exist $\pi \in P_m F$ such that φ is weakly dominated by π and $2\dim\varphi > \dim\pi$. Recall that the form π is unique up to isometry, and for any field extension K/F we have that φ_K is isotropic if and only if π_K is isotropic. In particular, $\varphi_{F(\pi)}$ and $\pi_{F(\varphi)}$ are isotropic.

3. Preliminary results

For the proof of Theorems 2, 3 and 4, we give a preparatory result.

Theorem 6. *Let φ and τ be anisotropic quadratic forms of dimension* 5 *and* 3*, respectively, with φ not totally singular and* $1 \in D_F(\tau)$*. Suppose that there exists a* 3*-fold Pfister form π, some $x \in F^*$ and a* 5*-dimensional quadratic form φ' of the same type as φ such that τ is weakly dominated by both forms π and φ', and such that $\varphi \sim x\pi \perp \varphi'$. Then, τ is weakly dominated by φ.*

Proof. Let $y \in F^*$ be such that $y\tau \prec \varphi'$. Then, $x\pi \perp y\tau \prec x\pi \perp \varphi'$ is of codimension 2, but the right hand side has Witt index 4, so the left hand side is isotropic. Hence, there exists $z \in D_F(\tau)$ such that $yz \in D_F(x\pi)$. By the roundness of Pfister forms, we get $x\pi \simeq yz\pi$. Let $t \in F^*$ be such that $t\tau \prec \pi$. In particular, $t \in D_F(t\tau) \subset D_F(\pi)$. So $\tau \prec t\pi \simeq \pi$, thus $z \in D_F(\pi)$ and $x\pi \simeq yz\pi \simeq y\pi$. Hence, without loss of generality, we may suppose $x = y = 1$, meaning that τ is dominated by both forms π and φ, and we have

$$\varphi \sim \pi \perp \varphi'. \tag{3}$$

Clearly, from (3) we have $\mathrm{ql}(\varphi) \simeq \mathrm{ql}(\varphi')$. The form φ is of type $(2, 1)$ or $(1, 3)$, and τ is of type $(1, 1)$ or $(0, 3)$. We write

$$\varphi \simeq \begin{cases} R \perp \langle r\rangle & \text{case (a)}, \\ R \perp \langle r, s, t\rangle & \text{case (b)}, \end{cases}$$

where R is a nonsingular quadratic form and $r, s, t \in F^*$. Obviously, in case (a) the form φ is of type $(2, 1)$ and $\dim R = 4$, while in case (b) the form φ is of type $(1, 3)$ and $\dim R = 2$.

For the proof we will proceed case by case. We have to prove that τ is weakly dominated by φ.

(1) Suppose that τ is of type $(0, 3)$. We write $\tau = \langle 1, a, b\rangle$. The isotropy of $\pi_{F(\tau)}$ implies that $\pi \simeq \langle\langle a, b, c]]$ for some $c \in F^*$. If in case (b), τ is similar to $\langle r, s, t\rangle$, then we are done. So suppose that these two forms are not similar. By the domination relation, there exist $u, v, w \in F^*$ such that $\tau \simeq \langle u, v, w\rangle$ and

$$\varphi' \simeq \begin{cases} u[1, p] \perp v[1, q] \perp \langle w\rangle & \text{in case (a)}, \\ u[1, p] \perp \langle v, w, z\rangle & \text{in case (b)}, \end{cases}$$

where $p, q, z \in F^*$. Note that in case (a) we may suppose $w = r$. Adding on both sides of (3) the form $\theta := \langle\langle a, b\rangle\rangle$, we get:

$$\begin{cases} R \perp \theta \perp \langle 0\rangle \sim \theta \perp \langle 0\rangle & \text{in case (a)}, \\ R \perp \langle z\rangle \perp \theta \perp \langle 0, 0\rangle \sim \langle z\rangle \perp \theta \perp \langle 0, 0\rangle & \text{in case (b)}. \end{cases}$$

The Witt cancellation of the zero form yields

$$\begin{cases} R \perp \theta \sim \theta & \text{in case (a)}, \\ R \perp \langle z\rangle \perp \theta \sim \langle z\rangle \perp \theta & \text{in case (b)}. \end{cases}$$

– In case (a) we get $i_W(R \perp \theta) = 2$. Hence, there exists a form $\langle r', s'\rangle$ dominated by R and θ [7, Proposition 3.11]. Then, $\langle r, r', s'\rangle$ is dominated by φ. In particular, $\langle r, r', s'\rangle$ is anisotropic. Since θ represents r', s' and r (because we take $w = r$), it follows that $\langle r, r', s'\rangle$ is dominated by θ. Consequently, $\langle r, r', s'\rangle$ becomes isotropic over $F(\tau)$ because τ and $\langle r, r', s'\rangle$ are quasi-Pfister neighbors (see Section 5 for this notion) of the same quasi-Pfister form $\langle\langle a, b\rangle\rangle$. It follows from [20, Theorem 1.2(1)] that τ is similar to $\langle r, r', s'\rangle$, and thus τ is weakly dominated by φ.

– Similarly, in case (b), we get $i_W(R \perp \langle z\rangle \perp \theta) = 1$, and thus there exists $z' \in D_F(R \perp \langle z\rangle) \cap D_F(\theta)$. Then, $\langle v, w, z'\rangle$ is dominated by φ. In particular, $\langle v, w, z'\rangle$ is anisotropic. Since θ represents v, w and z', it follows that $\langle v, w, z'\rangle$ is dominated by θ. Consequently, $\langle v, w, z'\rangle$ becomes isotropic over $F(\tau)$, and by [20, Theorem 1.2(1)] τ is similar to $\langle v, w, z'\rangle$, thus τ is weakly dominated by φ.

(2) Suppose that τ is of type (1, 1). We write $\tau = b[1, a] \perp \langle 1\rangle$ for some $a, b \in F^*$. The isotropy of $\pi_{F(\tau)}$ implies that $\pi \simeq \langle\langle c, b, a]]$ for a suitable $c \in F^*$. By the domination relation, we get

$$\varphi' \simeq \begin{cases} b[1, a] \perp S \perp \langle r\rangle & \text{in case (a)}, \\ b[1, a] \perp \langle r, s, t\rangle & \text{in case (b)}, \end{cases}$$

where in case (a), S is nonsingular of dimension 2 such that $1 \in D_F(S \perp \langle r\rangle)$; and in case (b) we suppose $r = 1$. The condition $1 \in D_F(S \perp \langle r\rangle)$ implies that $1 = re^2 + f$ for some $e \in F$ and $f \in D_F(S) \cup \{0\}$. If $f = 0$, then $\langle r\rangle \simeq \langle 1\rangle$. If $f \neq 0$, then $S \simeq [f, g]$ for some $g \in F^*$, and thus $S \perp \langle r\rangle \simeq [f + re^2, g] \perp \langle r\rangle = [1, g] \perp \langle r\rangle$. Hence, in case (a), we may suppose $r = 1$ or $1 \in D_F(S)$. When $1 \in D_F(S)$, we get $S \simeq [1, d]$ for some $d \in F$. Inserting the forms φ' and $\pi = \langle\langle c, b, a]]$ in equation (3), we get

$$\varphi \sim \begin{cases} c\,\langle\langle b, a]] \perp T & \text{in case (a)}, \\ c\,\langle\langle b, a]] \perp \langle r, s, t\rangle & \text{in case (b)}, \end{cases} \tag{4}$$

where T is the form $[1, a + d] \perp \langle r\rangle$ or $S \perp \langle 1\rangle$ according as $S \simeq [1, d]$ or $r = 1$. Clearly, the form on the right hand side of (4) is isotropic. Let $b' \in D_F(T)$ (resp. $b' \in D_F(\langle r, s, t\rangle)$) be such that $b' \in D_F(c\,\langle\langle b, a]])$. The existence of b' in case (a) is clear when T is anisotropic. If T is isotropic, then it contains a hyperbolic plane and thus it represents any scalar. The roundness of Pfister forms yields $c\,\langle\langle b, a]] \simeq b'\,\langle\langle b, a]]$.

(1) In case (a), the condition $b' \in D_F(T)$ implies that $b'[1, a] \perp T \sim U$ for some form U of type (1, 1) such that $\langle b'\rangle \prec U$. Since $\varphi \sim bb'[1, a] \perp b'[1, a] \perp T$, it follows that $b'\tau \prec \varphi$.

(2) In case (b), we have $\langle r, s, t\rangle \simeq \langle b', \dots\rangle$; thus $\varphi \sim bb'[1, a] \perp \langle b', \dots\rangle$, and hence $b'\tau \prec \varphi$. □

The rest of this section is devoted to some corollaries that refine some results on isotropy due to the second author. The first one is a refinement of [13, théorème 1.2(3)] and [3, Theorem 1.1(3)].

Corollary 2. *Let φ be an anisotropic quadratic form of type* $(2, 1)$ *or* $(1, 3)$, *and* $\tau = \langle 1, a, b\rangle$ *an anisotropic totally singular quadratic form of dimension* 3. *Suppose that φ is not a Pfister neighbor. Then, $\varphi_{F(\tau)}$ is isotropic if and only if τ is weakly dominated by φ.*

Proof. Suppose that $\varphi_{F(\tau)}$ is isotropic. We have to prove that τ is weakly dominated by φ. If φ is of type $(1, 3)$ and τ is similar to $\mathrm{ql}(\varphi)$, then we are done. So suppose that τ is not similar to $\mathrm{ql}(\varphi)$ when φ is of type $(1, 3)$. Using [13, théorème 1.2(3)] (resp. [3, Theorem 1.1(3)]) when φ is of type $(2, 1)$ (resp. φ is of type $(1, 3)$), we get

$$\varphi \sim x\pi \perp \varphi' \tag{5}$$

where $x \in F^*$, π a 3-fold Pfister form isotropic over $F(\tau)$, and φ' a form of type $(2, 1)$ that weakly dominates τ. The isotropy of $\pi_{F(\tau)}$ is equivalent to saying that τ is weakly dominated by π. Hence, Theorem 6 implies that τ is weakly dominated by φ. Conversely, if τ is weakly dominated by φ, then $\varphi_{F(\tau)}$ is isotropic. □

Corollary 3. *Let φ be an anisotropic quadratic form of type* $(2, 1)$ *or* $(1, 3)$, *and* $\tau = \langle 1, a, b\rangle$ *an anisotropic totally singular quadratic form of dimension* 3. *If φ is $F(\tau)$-minimal, then φ is a Pfister neighbor of a 3-fold Pfister form $\langle\langle a, b, c]]$ for some $c \in F^*$.*

Proof. If φ is $F(\tau)$-minimal, then Corollary 2 implies that φ is a Pfister neighbor of some 3-fold Pfister form π. Since $\varphi_{F(\tau)}$ is isotropic, it follows that $\pi_{F(\tau)}$ is isotropic and thus hyperbolic. Hence, $\pi \simeq \langle\langle a, b, c]]$ for some $c \in F^*$. □

The following corollary refines [3, Theorem 1.1(1)].

Corollary 4. *Let φ be an anisotropic quadratic form of type* $(1, 3)$, *and τ an anisotropic quadratic form of type* $(1, 1)$. *Suppose that φ is not a Pfister neighbor. Then, $\varphi_{F(\tau)}$ is isotropic if and only if τ is weakly dominated by φ.*

Proof. Suppose that φ is isotropic over $F(\tau)$. It follows from [3, Theorem 1.1] that there exist $\alpha, \beta, u, v \in F^*$ and R_1, R_2 nonsingular quadratic forms of dimension 2 such that $\alpha\varphi \simeq R_1 \perp \langle 1, u, v\rangle$, $\beta\tau = R_2 \perp \langle 1\rangle$ and

$$R_1 \perp R_2 \perp \rho \sim x\pi, \tag{6}$$

where $x \in F^*$, ρ is a nonsingular complement of $\langle 1, u, v\rangle$ and $\pi \in P_3F$ dominates τ up to a scalar. Adding on both sides of (6) the form $\langle 1, u, v\rangle$ yields

$$\alpha\varphi \sim x\pi \perp \varphi',$$

where $\varphi' = R_2 \perp \langle 1, u, v\rangle$ dominates $\beta\tau$. Theorem 6 implies that τ is weakly dominated by φ. Obviously, if τ is weakly dominated by φ then $\varphi_{F(\tau)}$ is isotropic. □

Corollary 5. *Let φ be an anisotropic F-quadratic form of type $(1,3)$, and $\tau = b[1,a] \perp \langle 1\rangle$ an anisotropic F-quadratic form of type $(1,1)$. If φ is $F(\tau)$-minimal, then it is a Pfister neighbor of a 3-fold quadratic Pfister form $\pi = \langle\langle c,b,a]]$ for some $c \in F^*$.*

Proof. We use Corollary 4 and we proceed as in the proof of Corollary 3. □

Proposition 1 recovers Corollary 5 but the proof given by Faivre uses some arguments different from those developed here.

4. Proof of Theorems 2–4

Proof of Theorem 2. Let φ be an anisotropic F-quadratic form of dimension 5 and type $(2,1)$, and $\tau = \langle 1,a,b\rangle$ an anisotropic totally singular F-quadratic form of dimension 3.

– Suppose that conditions (i)–(iii) from the theorem are satisfied. Since $\pi_{F(\tau)}$ is isotropic and φ is a Pfister neighbor of π, it follows that $\varphi_{F(\tau)}$ is isotropic. Suppose that φ is not $F(\tau)$-minimal. Then, there exists ψ a form dominated by φ of dimension 3 or 4 such that $\psi_{F(\tau)}$ is isotropic. Using [13, théorème 1.3], we can see that the form ψ is neither of type $(2,0)$ nor of type $(1,1)$. Hence, ψ is of type $(0,3)$ or $(1,2)$. In both cases, there exists $x \in F^*$ such that $x\tau \prec \psi$ (use [13, théorème 1.4] for type $(1,2)$, and [20, Theorem 1.2] for type $(0,3)$). Hence, there exist u, v, w such that $x\tau \simeq \langle u,v,w\rangle$ and $\varphi \simeq u[1,p] \perp v[1,q] \perp \langle w\rangle$. We have that $C_0(\varphi)$ is isomorphic to $[p,uw) \otimes_F [q,vw)$ [21, Lemma 2]. Because $x\langle 1,a,b\rangle \simeq \langle u,v,w\rangle$, the scalars uw and vw are squares in $F(\sqrt{a},\sqrt{b})$. Consequently, $C_0(\varphi)_{F(\sqrt{a},\sqrt{b})}$ is split, a contradiction.

– Suppose that φ is $F(\tau)$-minimal. By Corollary 3, we deduce that φ is a Pfister neighbor of a 3-fold Pfister form $\pi = \langle\langle a,b,c]]$ for some $c \in F^*$. Moreover, modulo a scalar, we have $\varphi \simeq \langle\langle s,t]] \perp \langle r\rangle$ (we may use [13, Proposition 3.2]).

Suppose that $C_0(\varphi)_{F(\sqrt{a},\sqrt{b})}$ is split. Then, $\langle\langle s,t]]$ is hyperbolic over $F(\sqrt{a},\sqrt{b})$. By a result of Baeza [1, Corollary 4.16, page 128], we get $\langle\langle s,t]] \sim \langle 1,a\rangle \otimes \varphi_1 \perp \langle 1,b\rangle \otimes \varphi_2$ for suitable $\varphi_1, \varphi_2 \in W_q(F)$. Hence, we get

$$\langle\langle s,t]] \perp \langle\langle a,k_1]] \perp \langle\langle b,k_2]] \in I_q^3 F,$$

where $k_i = \Delta(\varphi_i)$ for $i = 1,2$. Hence, we get

$$r\,\langle\langle s,t]] \perp [1,k_1+k_2] \perp a[1,k_1] \perp b[1,k_2] \in I_q^3 F. \tag{7}$$

It follows from [17, proposition 6.4] that there exists $\rho \in GP_3F$ such that

$$r\,\langle\langle s,t]] \sim [1,k_1+k_2] \perp a[1,k_1] \perp b[1,k_2] \perp \rho.$$

Consequently, by adding $\langle 1\rangle$ on both sides, we get

$$r\varphi \sim a[1, k_1] \perp b[1, k_2] \perp \langle 1\rangle \perp \rho.$$

Since the forms φ and $a[1, k_1] \perp b[1, k_2] \perp \langle 1\rangle$ are isotropic over $F(\tau)$, it follows that $\rho_{F(\tau)}$ is isotropic, and thus τ is weakly dominated by ρ. Theorem 6 implies that τ is weakly dominated by φ, a contradiction to the minimality of φ. □

Let us give an example for which Theorem 2 applies. This example is similar to the one given by Chapman and Quéguiner-Mathieu for the minimality over the function field of a nondegenerate conic [2].

Example 1. Let a, b, c be indeterminates over a field F_0 of characteristic 2. Consider the forms $\varphi = c[1, a+b] \perp b[1, a] \perp \langle 1\rangle$, $\tau = \langle 1, b, ac\rangle$ and $\pi = \langle\langle b, c, a+b]]$ over the rational function field $F := F_0(a, b, c)$. Then:

(1) $\pi \simeq \langle c, bc\rangle_b .[1, a+b] \perp \langle 1, b\rangle_b .[1, a]$ because

$$\langle 1, b\rangle_b .[1, a+b] \simeq \langle 1, b\rangle_b .[1, a].$$

This proves that $\varphi \prec \pi$, and thus φ is a Pfister neighbor of π.

(2) $\varphi_{F(\tau)}$ is isotropic because $c\tau \prec \pi$ as we can see from (1).

(3) Let $L = F(\sqrt{b}, \sqrt{ac})$. We have in the Brauer group of L the following:

$$C_0(\varphi)_L = [a+b, bc)_L = [a+b, a)_L = [b, a)_L.$$

The algebra $[b, a)$ is division over $F_1 := F_0(a, b)(\sqrt{b})$, and it remains division over $L = F_1(\sqrt{ac})$.

Hence, Theorem 2 implies that φ is $F(\tau)$-minimal.

Proof of Theorem 3. Let φ be an anisotropic F-quadratic form of type $(1, 3)$, and $\tau = \langle 1, a, b\rangle$ an anisotropic totally singular quadratic form of dimension 3.

– Suppose that φ is $F(\tau)$-minimal. Then, by Corollary 3, φ is a Pfister neighbor of a 3-fold Pfister form $\pi = \langle\langle a, b, c]]$ for some $c \in F^*$. Suppose that there exists $e \in F^*$ such that: $i_d(e\tau \perp \mathrm{ql}(\varphi)) \geq 2$ and $(i_d(e\tau \perp \mathrm{ql}(\varphi)) \neq 2$ or $(D_F(\varphi) \cap D_F(e\tau)) \setminus D_F(\mathrm{ql}(\varphi)) \neq \varnothing)$. This is equivalent to saying: $i_d(e\tau \perp \mathrm{ql}(\varphi)) = 3$ or $(i_d(e\tau \perp \mathrm{ql}(\varphi)) \geq 2$ and $(D_F(\varphi) \cap D_F(e\tau)) \setminus D_F(\mathrm{ql}(\varphi)) \neq \varnothing)$. The condition $i_d(e\tau \perp \mathrm{ql}(\varphi)) = 3$ means that $\mathrm{ql}(\varphi)$ is similar to τ, while the second condition means that $e\tau$ is dominated by φ. Hence, φ is not $F(\tau)$-minimal, a contradiction.

– Conversely, suppose that we have the three conditions (i), (ii) and (iii) as described in the theorem. Since φ is a Pfister neighbor of a 3-fold Pfister form $\langle\langle a, b, c]]$ for some $c \in F^*$, it follows that $\varphi_{F(\tau)}$ is isotropic. Suppose that φ is not $F(\tau)$-minimal. Then, there exists ψ a form dominated by φ of dimension 3 or 4 such that $\psi_{F(\tau)}$ is isotropic. Then, $e\tau$ is dominated by ψ for a suitable $e \in F^*$ (we use [20, Theorem

1.2] when ψ is totally singular, and [13, théorème 1.4] when ψ is of type $(1,2)$). This gives two possibilities:

(a) $e\tau$ is isometric to $\mathrm{ql}(\varphi)$, which contradicts the condition (iii), or

(b) there exists $x, y, z, t, u \in F^*$ such that $e\tau \simeq \langle x, y, z\rangle$ and $\varphi \simeq x[1,u] \perp \langle y, z, t\rangle$. This condition also contradicts (iii) because $i_d(e\tau \perp \mathrm{ql}(\varphi)) = 2$ but $x \in (D_F(\varphi) \cap D_F(e\tau)) \setminus D_F(\mathrm{ql}(\varphi))$.

Hence, φ is $F(\tau)$-minimal. □

Proof of Proposition 2. Suppose $\varphi = a_1[1,b_1] \perp \cdots \perp a_n[1,b_n] \perp \langle c_0, c_1, \dots, c_s\rangle$ for $a_i, b_i, c_j \in F$ such that $c_0 = 1$ and $a_i \neq 0$ for all i. The Clifford algebra of φ is generated by $x_1, y_1, \dots, x_n, y_n, z_0, \dots, z_s$ such that z_i commutes with all the generators, x_i commutes with y_j when $i \neq j$, and $x_i y_i + y_i x_i = 1$ and $x_i^2 = a_i$, $y_i^2 = a_i^{-1} b_i$ and $z_i^2 = c_i$. The even Clifford algebra of φ is generated by $u_1, v_1, \dots, u_n, v_n, w_2, \dots, w_s$ where $u_i = x_i z_0$, $v_i = y_i z_0$ and $w_i = z_i z_0$. The relations are the following: w_i commutes with all the other generators, u_i commutes with v_j for $i \neq j$, $u_i v_i + v_i u_i = z_0^2 = 1$ and $u_i^2 = a_i$, $v_i^2 = a_i^{-1} b_i$ and $w_j^2 = c_j$. Therefore, the even Clifford algebra of φ is

$$F\langle u_1, v_1\rangle \otimes F\langle u_2, v_2\rangle \otimes \cdots \otimes F\langle u_n, v_n\rangle \otimes F\langle w_1, \dots, w_s\rangle,$$

which is indeed $[b_1, a_1)_F \otimes \cdots \otimes [b_n, a_n)_F \otimes K$. □

Proof of Corollary 1. Let φ be an anisotropic quadratic form of type $(1,3)$. Suppose that φ is a Pfister neighbor of a 3-fold Pfister form π. Modulo a scalar, we may write $\varphi = R \perp \langle 1, r, s\rangle$ for a suitable nonsingular quadratic form R of dimension 2 and $r, s \in F^*$. On the one hand, since π is isotropic over $F(\langle 1, r, s\rangle)$, it follows that π is also isotropic over $F(\langle\langle r, s\rangle\rangle)$, and thus $\pi \simeq \langle\langle r, s, u]]$ for some $u \in F^*$. On the other hand, the hyperbolicity of $\pi_{F(\varphi)}$ implies that $\pi \simeq R \perp [1,x] \perp r[1,y] \perp s[1,z]$ for some $x, y, z \in F^*$. Hence, we get

$$\langle\langle r, s, u]] \simeq R \perp [1,x] \perp r[1,y] \perp s[1,z].$$

Adding on both sides in the last isometry the form $\langle 1, r, s\rangle$, and canceling the hyperbolic planes, yields that $\varphi \simeq rs[1,u] \perp \langle 1, r, s\rangle$.

The fact that $C_0(\varphi)$ is split as an $F(\sqrt{r}, \sqrt{s})$-algebra is a direct consequence of Proposition 2. □

We finish this section with an example applying the criteria given in Theorem 3.

Example 2. Let r, s, u be indeterminates over a field F_0 of characteristic 2. Let us consider the forms $\varphi = rs[1,u] \perp \langle 1, r, s\rangle$ and $\tau = \big\langle 1, ru, s(r^2 + r + u)\big\rangle$ over the rational function field $F := F_0(r, s, u)$. We have the following statements:

(1) It is clear that φ is a Pfister neighbor of $\pi = \langle\langle r, s, u]]$. Moreover, τ is dominated by π because the scalars 1, ru and $s(r^2+r+u)$ are represented by the forms $[1, u]$, $r[1, u]$ and $s[1, u]$, respectively. Hence, $\varphi_{F(\tau)}$ is isotropic.

(2) For any $e \in F^*$, we have $i_d(e\tau \perp \mathrm{ql}(\varphi)) \leq 1$.

In fact, suppose that $i_d(e\tau \perp \mathrm{ql}(\varphi)) \geq 2$ for some $e \in F^*$. By [7, Proposition 3.2], there exists a totally singular quadratic form of dimension 2 which is dominated by $e\tau$ and $\mathrm{ql}(\varphi)$. Consequently, there exists an inseparable quadratic extension $K = F(\sqrt{d})$ such that $\langle\langle r, s\rangle\rangle_K$ and $\langle\langle ru, s(r^2+r+u)\rangle\rangle_K$ are isotropic, and thus quasi-hyperbolic. This implies that $\langle\langle r, s\rangle\rangle \simeq \langle\langle d, k\rangle\rangle$ and $\langle\langle ru, s(r^2+r+u)\rangle\rangle \simeq \langle\langle d, l\rangle\rangle$ for suitable $k, l \in F^*$. Hence, $\langle\langle r, s\rangle\rangle \perp \langle\langle ru, s(r^2+r+u)\rangle\rangle$ has defect index ≥ 2. In particular, $\theta := \langle\langle r, s\rangle\rangle \perp \langle ru, s(r^2+r+u), rus(r^2+r+u)\rangle$ is isotropic. But, using the classical isometry $\langle a, b\rangle \simeq \langle a, a+b\rangle$ for any $a, b \in F$, we get

$$\begin{aligned}
\theta &= \langle 1, r, s, rs\rangle \perp \langle ru, r^2s+rs+su, rsu(r^2)+su(r^2)+rs(u^2)\rangle \\
&\simeq \langle 1, r, s, rs\rangle \perp \langle ru, su, rsu(r^2)+su(r^2)+rs(u^2)\rangle \\
&\simeq \langle 1, r, s, rs\rangle \perp \langle ru, su, rsu(r^2)\rangle \\
&\simeq \langle 1, r, s, rs\rangle \perp \langle ru, su, rsu\rangle \\
&= \langle 1\rangle \perp \langle 1, u\rangle \cdot \langle r, s, rs\rangle \,,
\end{aligned}$$

which shows that θ is anisotropic, a contradiction. Hence, $i_d(e\tau \perp \mathrm{ql}(\varphi)) \leq 1$, and thus Theorem 3 implies that φ is $F(\tau)$-minimal. □

It would be interesting to see if there exists an example of an anisotropic quadratic form of type $(1, 3)$ which is minimal over the function field of a degenerate conic and satisfies condition (iii)(b) of Theorem 3.

Proof of Theorem 4. Let φ be an anisotropic F-quadratic form of type $(1, 3)$, and $\tau = a[1, b] \perp \langle 1\rangle$ an anisotropic quadratic form of dimension 3 and type $(1, 1)$.

– Suppose that φ is $F(\tau)$-minimal. It follows from Corollary 5 that φ is a Pfister neighbor of a 3-fold Pfister form $\pi = \langle\langle c, a, b]]$ for some $c \in F^*$. Since π is split by $K = F[\wp^{-1}(b)]$ and φ is its neighbor, the form φ_K is isotropic, which means that $e[1, b] \subset \varphi$ for some $e \in F^*$. Hence, $\varphi \simeq e[1, b] \perp \mathrm{ql}(\varphi)$. Suppose that $e \in D_F(a[1, b]) \cdot D_F(\mathrm{ql}(\varphi))$. Let $x \in D_F([1, b])$ and $y \in D_F(\mathrm{ql}(\varphi))$ be such that $e = axy$. Since $[1, b] \simeq x[1, b]$, it follows that $e[1, b] \simeq axy[1, b] \simeq ay[1, b]$, and thus $y\tau$ is dominated by φ, a contradiction.

– Conversely, suppose that τ satisfies the conditions (i), (ii) and (iii) as described in the theorem. The conditions (i) and (ii) imply that $\varphi_{F(\tau)}$ is isotropic. Suppose that φ is not $F(\tau)$-minimal. Hence, there exists a form ψ of dimension 3 or 4 dominated by φ and isotropic over $F(\tau)$. The form ψ is of type $(1, 1)$ or $(1, 2)$. We use [13, théorème 1.4 (bis) (2)] when $\dim\psi = 3$, and [13, théorème 1.4(2)] when $\dim\psi = 4$ to conclude that τ is weakly dominated by ψ. Hence, there exist $e, f, g \in F^*$ such that $\varphi \simeq ea[1, b] \perp \langle e, f, g\rangle$. This contradicts (iii) because $ea \in D_F(a[1, b]) \cdot D_F(\mathrm{ql}(\varphi))$. □

To give an example of a minimal quadratic form that applies Theorem 4, we need a few notions on the specialization theory due to Knebusch. Let $\lambda : K \longrightarrow L \cup \{\infty\}$ be a place between two fields K and L. Let $\mathbb{O}$ be the valuation ring of λ and $\mathcal{M}$ its maximal ideal. Recall that $\mathbb{O} = \{x \in K \mid \lambda(x) \neq \infty\}$ and $\mathcal{M} = \{x \in \mathbb{O} \mid \lambda(x) = 0\}$. Let μ be the restriction of λ to $\mathbb{O}$ and $k = \mathbb{O}/\mathcal{M}$ the residue field of λ (Note that k can be seen as a subfield of L). We say that a quadratic form φ over K has *nearly good reduction* with respect to λ if there exists a quadratic module ψ over $\mathbb{O}$ such that $\varphi \simeq \psi_K$ and the quasilinear part of the quadratic form ψ_k is anisotropic, where ψ_k is the quadratic form induced by the ring homomorphism $\mathbb{O} \longrightarrow k$. The specialization of φ with respect to λ, denoted by $\lambda_*(\varphi)$, is the L-quadratic form $\mu_*(\psi)$ induced by μ. We refer to [11] for more details.

Example 3. Let $F = \mathbb{F}_2(r, s, u)$ be the rational function field in the indeterminates r, s, u over the field $\mathbb{F}_2$ with two elements. Let $\varphi = rs[1, u] \perp \langle 1, r, s\rangle$, $\pi = \langle\langle r, s, u]]$ and $\tau = su[1, r+u] \perp \langle 1\rangle$. It is clear that φ is a Pfister neighbor of π. Moreover, $\pi \simeq [1, r+u] \perp [r, ur^{-1}+1] \perp s[1, u] \perp rs[1, u]$, and thus $su\tau \prec \pi$. Hence, φ is isotropic over $F(\tau)$. Suppose that φ is not $F(\tau)$-minimal. Then, there exists by Theorem 4 a scalar $e \in F^*$ such that $e[1, r+u] \subset \varphi$ and $e \in D_F(su[1, r+u]) \cdot D_F(\langle 1, r, s\rangle)$. Hence, using the roundness of $[1, r+u]$, we get $\varphi \simeq sut[1, r+u] \perp \langle 1, r, s\rangle$ for a suitable $t \in D_F(\langle 1, r, s\rangle)$. Without loss of generality, we may suppose $t \in \mathbb{F}_2(r, s)[u]$ square free with respect to the indeterminate u. Let $M = \mathbb{F}_2(r, s)$ and consider the M-place λ from F to M with respect to the u-adic valuation of F. We have:

(1) t is a unit for the u-adic valuation because $t \in D_F(\langle 1, r, s\rangle)$.

(2) the form φ has nearly good reduction with respect to λ because it is isometric to $\varphi = rs[1, u] \perp \langle 1, r, s\rangle$ and $\langle 1, r, s\rangle_M$ is anisotropic.

On the one hand, the total index of $\lambda_*(\varphi)$ is equal to 1 because $\lambda_*(rs[1, u]) = [0, 0]$ and $\langle 1, r, s\rangle_M$ is anisotropic. On the other hand, since φ contains $sut[1, r+u]$ as a subform, and $\lambda(\alpha) = 0$ or ∞ for every α represented by $sut[1, r+u]$, we conclude by [11, Proposition 3.4] that the total index of $\lambda_*(\varphi)$ is at least 2, a contradiction.

5. (Quasi-)Pfister neighbor forms

Our aim in this section is to relate the notions of quasi-Pfister neighbor and bilinear (strong)Pfister neighbor. This is useful to classify $F(\tau)$-minimal bilinear forms of dimension 5 when τ is a totally singular quadratic form of dimension 3.

A *quasi-Pfister form* is a totally singular quadratic form $\tilde{B}$ for some bilinear Pfister form B. A totally singular quadratic form Q is called *quasi-Pfister neighbor* if there exists a quasi-Pfister form π such that $2 \dim Q > \dim \pi$ and $aQ \subset \pi$ for some $a \in F^*$. In this case, the form π is unique, and for any field extension K/F the form Q_K is isotropic if and only if π_K is isotropic. Thus $Q_{F(\pi)}$ and $\pi_{F(Q)}$ are isotropic.

For any bilinear Pfister form $\langle\langle a_1, \cdots, a_n\rangle\rangle_b$, we write $\widetilde{B} = \langle\langle a_1, \cdots, a_n\rangle\rangle$.

The *norm field* of a nonzero totally singular quadratic form Q is the field $N_F(Q) := F^2(\alpha\beta \mid \alpha, \beta \in D_F(Q))$, where $D_F(Q)$ is the set of scalars in F^* represented by Q. The degree $[N_F(Q) : F^2]$ is called the *norm degree* of Q and it is denoted by $\mathrm{ndeg}_F(Q)$. Clearly, we have $\mathrm{ndeg}_F(Q) = 2^d$ for some integer $d \geq 1$ and $\mathrm{ndeg}_F(Q) \geq \dim Q$. See [7, Section 8] for details on the norm degree and applications.

Here is a characterization of quasi-Pfister neighbors using the norm degree.

Proposition 3 [7, Proposition 8.9(ii)]. *An anisotropic totally singular quadratic form Q is a quasi-Pfister neighbor if and only if* $2\dim Q > \mathrm{ndeg}_F(Q)$.

The norm degree appears in the description of the Witt kernels for bilinear forms.

Proposition 4 [15, Theorem 1.2]. *Let B be an anisotropic F-bilinear form and Q an anisotropic totally singular form of norm degree 2^d. If B becomes metabolic over $F(Q)$, then* $\dim B$ *is divisible by* 2^d.

A bilinear form B is called a *Pfister neighbor* if $\tilde{B}$ is a quasi-Pfister neighbor. This definition does not imply that B is similar to a subform of a bilinear Pfister form whose dimension is less than twice the dimension of B. For example, over the rational functions field $F(t_1, t_2)$, the bilinear form $B = \langle 1, t_1, t_2, 1 + t_1t_2\rangle_b$ is a Pfister neighbor because $\tilde{B} \simeq \langle\langle t_1, t_2\rangle\rangle$, but B is not similar to a subform of a 2-fold bilinear Pfister form since its determinant is not trivial. See [15] for more on bilinear Pfister neighbors and their splitting properties.

A bilinear form B is called a *strong Pfister neighbor*, or SPN, if there exists a bilinear Pfister form ρ such that $2\dim B > \dim\rho$ and $\alpha B \subset \rho$ for some $\alpha \in F^*$. In this case, the form ρ is unique. In fact, if B is an SPN of another bilinear Pfister form δ, then there exists $\beta \in F^*$ such that $\beta B \subset \delta$. Hence, $\dim\rho = \dim\delta$ and $i_W(\alpha\rho \perp \beta\delta) \geq \dim B > \frac{1}{2}\dim\rho$, which implies that $\rho \simeq \delta$ since the Witt index $i_W(\alpha\rho \perp \beta\delta)$ is always a power of 2 [17, théorème 3.7]. Obviously, if B is an SPN then it is a Pfister neighbor.

Recall from [10] the Kato isomorphism $e^n : I^nF/I^{n+1}F \longrightarrow \nu_F(n)$ given on generators by

$$e^n(\langle\langle a_1, \ldots, a_n\rangle\rangle_b + I^{n+1}F) = \frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n}.$$

The symbol length (or simply the length) of an element $\theta \in \nu_F(n)$ is the smallest number of n-logarithmic symbols $\frac{\mathrm{d}a_1}{a_1} \wedge \cdots \wedge \frac{\mathrm{d}a_n}{a_n}$ needed to write it.

An Albert bilinear form is a 6-dimensional bilinear form whose determinant is trivial.

Lemma 1. *Let γ be an Albert bilinear form and $\tau \in BP_2F$ be such that $\gamma \perp \tau \in I^3F$. Then, γ is isotropic.*

Proof. (1) If τ is isotropic, then it is metabolic, and thus $\gamma \in I^3F$. By the Hauptsatz, the form γ is metabolic, in particular it is isotropic.

(2) If τ is anisotropic, then we get by the previous case that $\gamma_{F(\tau)}$ is metabolic. It follows from Proposition 4 that γ is isotropic because the norm degree of $\widetilde{\tau}$ is 4. □

We give a characterization of SPN of dimension 5. This looks like the characterization of 5-dimensional quadratic Pfister neighbors (due to Knebusch in characteristic not 2 [12, Page 10], and the second author in characteristic 2 [13, Proposition 3.2]).

Proposition 5. *Let B be an anisotropic F-bilinear form of dimension 5. The following statements are equivalent*:

(1) *B is an SPN.*

(2) *$B \simeq a\,\langle\langle b, c\rangle\rangle_b \perp \langle d\rangle_b$ for suitable $a, b, c, d \in F^*$.*

(3) *The invariant $e^2(B \perp \langle \det B\rangle_b + I^3F)$ has length 1.*

Proof. Let $d \in F^*$ be such that $\det B = d.F^{*2}$.

(1) $\Longrightarrow$ (2) Suppose that B is an SPN of $\pi \in BP_3F$. Then, we have $x\pi \simeq B \perp \langle y, z, yzd\rangle_b$ for suitable scalars $x, y, z \in F^*$. Hence, $B \perp \langle d\rangle_b \perp d\tau \in I^3F$, where $\tau = \langle\langle dy, dz\rangle\rangle_b$. We conclude by Lemma 1 that $B \perp \langle d\rangle_b$ is isotropic, and thus $B \simeq B' \perp \langle d\rangle_b$ for some bilinear form B' of dimension 4 and trivial determinant, as desired.

(2) $\Longrightarrow$ (3) Suppose that $B \simeq a\,\langle\langle b, c\rangle\rangle_b \perp \langle d\rangle_b$. Clearly, we have $e^2(B \perp \langle d\rangle_b + I^3F) = \frac{\mathrm{d}b}{b} \wedge \frac{\mathrm{d}c}{c}$, which is of length 1 because the anisotropy of $\langle\langle b, c\rangle\rangle_b$ implies that $\frac{\mathrm{d}b}{b} \wedge \frac{\mathrm{d}c}{c} \neq 0$.

(3) $\Longrightarrow$ (1) Suppose that $e^2(B \perp \langle d\rangle_b + I^3F)$ has length 1. Then, there exists an anisotropic 2-fold bilinear Pfister form τ such that $e^2(B \perp \langle d\rangle_b + I^3F) = e^2(\tau + I^3F)$. Hence, $B \perp \langle d\rangle \perp \tau \in I^3F$ using the isomorphism e^2. Consequently, $B \perp d\tau' \in I^3F$, where τ' is the pure part of τ. Then, $B \perp d\tau'$ is similar to a 3-fold bilinear Pfister form because it is of dimension 8, and thus B is an SPN. □

6. K-minimal bilinear forms up to dimension 5

Throughout this section we take $Q = \langle 1, a, b\rangle$ an anisotropic totally singular quadratic form over F of dimension 3, and $K = F(Q)$ its function field.

Lemma 2 [18, Lemma 3.7]. *Let B be an anisotropic bilinear form over F. If $\psi = \langle a_1, \dots, a_n\rangle$ is a subform of $\tilde{B}$, then there exists a bilinear form C over F such that $C \subset B$ and $\tilde{C} \simeq \psi$. Explicitly, we can take $C = \langle b_1, \dots, b_n\rangle_b$, where $b_i = a_i + \sum_{j=1}^{i-1} a_i x_i^2$ for suitable $x_1, \dots, x_{i-1} \in F$ (read $b_1 = a_1$).*

Corollary 6. *Let B be an anisotropic F-bilinear form. Then, B is K-minimal if and only if $\tilde{B}$ is K-minimal.*

We recall the isotropy results that we need for the classification of K-minimal bilinear forms of dimension at most 5.

Theorem 7 [20, Proposition 1.1 and Theorem 1.2]. *Let B be an anisotropic F-bilinear form such that* $\dim B \leq 4$ *or* $\dim B = 5$ *and* $\operatorname{ndeg}_F(\tilde{B}) = 16$. *Then, B_K is isotropic if and only if Q is similar to a subform of $\tilde{B}$.*

Corollary 7. *Let B be an anisotropic F-bilinear form of dimension ≤ 5 such that B_K is isotropic. If B is K-minimal, then either* $\dim B = 3$, *or* $\dim B = 5$ *and* $\operatorname{ndeg}_F(\tilde{B}) = 8$.

Proof. Suppose that $\dim B \leq 5$ and B is K-minimal. Since B_K is isotropic, it follows that $\dim B \neq 2$.

– If $\dim B = 3$, then obviously B is K-minimal since any subform of B of dimension 2 is anisotropic over K.

– If $\dim B = 4$, then B_K is isotropic if and only if Q is similar to a subform of $\tilde{B}$ (Theorem 7). Hence, B is not K-minimal (Corollary 6).

– If $\dim B = 5$. In this case, $\operatorname{ndeg}_F(\tilde{B}) \in \{8, 16\}$. If $\operatorname{ndeg}_F(\tilde{B}) = 16$, then Q is similar to a subform of $\tilde{B}$ (Theorem 7). Hence, B is not K-minimal when $\operatorname{ndeg}_F(\tilde{B}) = 16$. □

Corollary 8. *Let B be an anisotropic F-bilinear form of dimension ≤ 5. If B is isotropic over K but not K-minimal, then Q is similar to a subform of $\tilde{B}$.*

Proof. Since B is not K-minimal, there exists C a subform of B such that $\dim C < \dim B$ and C_K is isotropic. It follows from Theorem 7 that Q is similar to a subform of $\tilde{C}$. In particular, Q is similar to a subform of $\tilde{B}$. □

Lemma 3. *Let $\pi_1 \in BP_mF$ and $\pi_2 \in BP_nF$ with $2 \leq m \leq n$. Suppose that π_1' is similar to a subform of π_2, where π_1' denotes the pure part of π_1. Then, $\pi_2 \simeq \pi_1 \otimes \tau$ for some $\tau \in BP_{n-m}F$.*

Proof. We have $i_W(\pi_1 \perp \alpha\pi_2) \geq 2^m - 1$ for some $\alpha \in F^*$. It follows from [17, théorème 3.7] that this Witt index is equal to $\dim \pi_1$, and the forms π_1 and π_2 are m-linked, which means that $\pi_2 \simeq \pi_1 \otimes \tau$ for some $\tau \in BP_{n-m}F$. □

7. Proof of Theorem 5

Let B be an anisotropic bilinear form of dimension 5, and $Q = \langle 1, a, b\rangle$ an anisotropic totally singular quadratic form of dimension 3. Let $K = F(Q)$ be the function field of Q.

(2) $\Longrightarrow$ (1) Suppose that there exists an F-bilinear form C of dimension 5 which is an SPN of a bilinear Pfister form $\rho := \langle\langle a, b, c\rangle\rangle_b$ and satisfies the two conditions:

(a) $\tilde{B} \simeq \tilde{C}$.

(b) For any $u, v \in F^2(a, b)$ such that $\langle u, v, uv\rangle_b$ is similar to a subform of ρ, the invariant $e^2(C \perp \langle \det C\rangle_b \perp \langle\langle u, v\rangle\rangle_b + I^3F)$ has length 2.

Let $d \in F^*$ be such that $\det C = d.F^{*2}$. The form C_K is isotropic because ρ_K is isotropic and C is an SPN of ρ. In particular, B_K is isotropic.

Suppose that B is not K-minimal. Then, C is not K-minimal as well because $\tilde{B} \simeq \tilde{C}$ (Corollary 6). It follows from Corollary 8 that $Q = \langle 1, a, b\rangle$ is similar to a subform of $\tilde{C}$. By Lemma 2, we conclude that $p\left\langle 1, a + q^2, b + ar^2 + s^2\right\rangle_b$ is a subform of C for suitable $p \neq 0, q, r, s \in F$. In particular,

$$\left\langle a + q^2, b + ar^2 + s^2, (a + q^2)(b + ar^2 + s^2)\right\rangle_b$$

is similar to a subform of ρ, and thus our hypothesis (b) above implies that $e^2(C \perp \langle d\rangle_b \perp \left\langle\!\left\langle a + q^2, b + ar^2 + s^2\right\rangle\!\right\rangle_b + I^3F)$ has length 2.

Let $u = a + q^2$, $v = b + ar^2 + s^2$. It is easy to see that $C \perp \langle d\rangle_b \sim p\,\langle\langle u, v\rangle\rangle_b \perp \tau$ for some $\tau \in GBP_2F$. Consequently, the invariant $e^2(C \perp \langle d\rangle \perp \langle\langle u, v\rangle\rangle_b + I^3F)$ has length at most 1, a contradiction. Hence B is K-minimal.

(1) $\Longrightarrow$ (2) Suppose that B is K-minimal. Then, we get by Corollary 7 that $\operatorname{ndeg}_F(\tilde{B}) = 8$. It follows from Proposition 3 that $\tilde{B}$ is quasi-Pfister neighbor of a quasi-Pfister form π. Since $\tilde{B}_K$ is isotropic, it follows that π_K is quasi-hyperbolic. Hence, $\pi \simeq \langle\langle a, b, c\rangle\rangle$ for some $c \in F^*$ [14, Theorem 1.5]. There exists a bilinear form C of dimension 5 similar to a subform of $\rho := \langle\langle a, b, c\rangle\rangle_b$ such that $\tilde{B} \simeq \tilde{C}$ (Lemma 2). In particular, the form C is an SPN of ρ. Modulo a scalar, we may write $C \simeq \langle\langle x, y\rangle\rangle_b \perp \langle z\rangle_b$ for suitable $x, y, z \in F^*$ (Proposition 5).

Let $u, v \in F^2(a, b)$ be such that $\langle u, v, uv\rangle_b$ is similar to a subform of ρ. The form $\langle u, v, uv\rangle$ is anisotropic because ρ is anisotropic. On the one hand, the condition $u, v \in F^2(a, b)$ implies that $\langle 1, u, v\rangle$ becomes isotropic over K, which gives by Theorem 7 that Q is similar to $\langle 1, u, v\rangle$ and thus $K = F(\langle 1, u, v\rangle)$. On the other hand, using Lemma 3, we get $\rho \simeq \langle\langle u, v, w\rangle\rangle_b$ for some $w \in F^*$. Hence, without loss of generality, we may suppose $\langle\langle a, b\rangle\rangle_b \simeq \langle\langle u, v\rangle\rangle_b$ for the rest of the proof.

We have $e^2(C \perp \langle z\rangle_d \perp \langle\langle a, b\rangle\rangle_b + I^3F) = \frac{dx}{x} \wedge \frac{dy}{y} + \frac{da}{a} \wedge \frac{db}{b}$. Suppose that this invariant has length ≤ 1. Then, there exists a 2-fold bilinear Pfister form τ such that $e^2(C \perp \langle z\rangle_d \perp \langle\langle a, b\rangle\rangle_b + I^3F) = e^2(\tau + I^3F)$, which implies that

$\langle\langle x, y\rangle\rangle_b \perp \langle\langle a, b\rangle\rangle_b \perp \tau \in I_q^3 F$. It follows from Lemma 1 that the Albert form $\langle a, b, ab, x, y, xy\rangle_b$ is isotropic. Hence, the forms $\langle\langle x, y\rangle\rangle_b$ and $\langle\langle a, b\rangle\rangle_b$ are 1-linked, meaning that $\langle\langle x, y\rangle\rangle_b \simeq \langle\langle e, r\rangle\rangle_b$ and $\langle\langle a, b\rangle\rangle_b \simeq \langle\langle f, r\rangle\rangle_b$ for suitable $e, f, r \in F^*$. By the uniqueness of the pure part of bilinear Pfister forms, we get $\langle a, b, ab\rangle_b \simeq \langle f, r, fr\rangle_b$, and thus $K = F(\langle 1, f, r\rangle)$. Hence, without loss of generality, we may keep $\langle 1, a, b\rangle$ instead of $\langle 1, r, f\rangle$, and thus suppose that $C \simeq \langle\langle e, b\rangle\rangle_b \perp \langle z\rangle_b$. So the form C is an SPN of $\langle\langle e, b, z\rangle\rangle_b$. But C is also an SPN of $\langle\langle a, b, c\rangle\rangle_b$, it follows that

$$\langle\langle e, b, z\rangle\rangle_b \simeq \langle\langle a, b, c\rangle\rangle_b . \tag{8}$$

We continue with some arguments similar to those used by Faivre in his proof. Adding on both sides of (8) the form $\langle 1, b\rangle_b$, we get

$$\mathbb{M}(1) \perp \mathbb{M}(b) \perp \langle z, e, ez\rangle_b \otimes \langle 1, b\rangle_b \simeq \mathbb{M}(1) \perp \mathbb{M}(b) \perp \langle c, a, ac\rangle_b \otimes \langle 1, b\rangle_b .$$

By the uniqueness of the anisotropic part, we get

$$\langle z, e, ez\rangle_b \otimes \langle 1, b\rangle_b \simeq \langle c, a, ac\rangle_b \otimes \langle 1, b\rangle_b .$$

Adding on both sides $a\,\langle 1, b\rangle_b$, we get

$$a\,\langle\langle ea, b\rangle\rangle_b \perp z\,\langle\langle e, b\rangle\rangle_b \simeq \mathbb{M}(a) \perp \mathbb{M}(ab) \perp c\,\langle\langle a, b\rangle\rangle_b .$$

Thus, $a\,\langle\langle ea, b\rangle\rangle_b \perp z\,\langle\langle e, b\rangle\rangle_b$ is isotropic, and thus there exist $r \in D_F(\langle\langle ea, b\rangle\rangle_b)$ and $s \in D_F(\langle\langle e, b\rangle\rangle_b)$ such that $ar = zs$. We have

$$C \simeq \langle\langle e, b\rangle\rangle_b \perp \langle z\rangle_b \simeq s\,\langle\langle e, b\rangle\rangle_b \perp \langle z\rangle_b \simeq s\,\langle 1, b\rangle_b \perp D,$$

where $D = es\,\langle 1, b\rangle_b \perp \langle z\rangle_b$. Let $\beta := as\,\langle 1, b\rangle_b \perp D$. Then, we have

$$\beta \simeq as\,\langle\langle ea, b\rangle\rangle_b \perp \langle z\rangle_b \simeq ars\,\langle\langle ea, b\rangle\rangle_b \perp \langle z\rangle_b \simeq z\,\langle\langle ea, b\rangle\rangle_b \perp \langle z\rangle_b .$$

Hence, $\beta \simeq \mathbb{M}(z) \perp \widetilde{\beta} \perp \langle zb\rangle_b$, where $\widetilde{\beta} = zea\,\langle 1, b\rangle_b$. Now, we have

$$\begin{aligned} bC \perp \widetilde{\beta} &\sim s\,\langle\langle e, b\rangle\rangle_b \perp \beta \\ &\sim s\,\langle\langle e, b\rangle\rangle_b \perp as\,\langle 1, b\rangle_b \perp es\,\langle 1, b\rangle_b \perp \langle z\rangle_b \\ &\sim s\,\langle\langle a, b\rangle\rangle_b \perp \langle z\rangle_b . \end{aligned}$$

This shows that $bC \perp \widetilde{\beta}$ is isotropic. Then, there exist bilinear forms C_1 and C_2 of dimension 4 and 1, respectively, such that $C_1 \subset bC$, $C_2 \subset \widetilde{\beta}$ and $C_1 \perp C_2 \simeq s\,\langle\langle a, b\rangle\rangle_b \perp \langle z\rangle_b$. Then, $i_W((C_1 \perp C_2)_K) = 2$ and thus $(C_1)_K$ is isotropic, meaning that C is not K-minimal. Since $\tilde{C} \simeq \tilde{B}$, it follows that B is not K-minimal, a contradiction. □

Using Theorem 3, we provide an example of a K-minimal bilinear form of dimension 5, where K is the function field of a degenerate conic. The form we choose in our example is inspired by [2, Proposition 4.1].

Example 4. Let F_0 be a field of characteristic 2, and $k = F_0(a, b, c)$ the rational function field in the indeterminates a, b, c over F_0. Let $B = c\,\langle 1, a+b\rangle_b \perp b\,\langle 1, a\rangle_b \perp \langle 1\rangle_b$ and $Q = \langle 1, a, c\rangle$. Then, B is $k(Q)$-minimal.

Proof. Using the isometry $\langle\langle a, b\rangle\rangle_b \simeq \langle\langle ab, a+b\rangle\rangle_b$, we get

$$\begin{aligned} B \perp \langle a\rangle_b \perp abc\,\langle 1, a+b\rangle_b &= c\,\langle\langle ab, a+b\rangle\rangle_b \perp \langle\langle a, b\rangle\rangle_b \\ &\simeq c\,\langle\langle ab, a+b\rangle\rangle_b \perp \langle\langle ab, a+b\rangle\rangle_b \\ &= \langle\langle c, ab, a+b\rangle\rangle_b \\ &\simeq \langle\langle a, b, c\rangle\rangle_b\,. \end{aligned}$$

Hence, B is an SPN of $\langle\langle a, b, c\rangle\rangle_b$, and thus B_K is isotropic. Moreover, B is anisotropic over k since $\langle\langle a, b, c\rangle\rangle_b$ is also anisotropic.

Let $u, v \in k^2(a, c)$ be such that $\langle u, v, uv\rangle_b$ is a subform of $\langle\langle a, b, c\rangle\rangle_b$. We have

$$e^2\left(B \perp \langle \det B\rangle_b \perp \langle\langle u, v\rangle\rangle_b + I^3F\right) = \frac{\mathrm{d}(a+b)}{a+b} \wedge \frac{\mathrm{d}(bc)}{bc} + \frac{\mathrm{d}u}{u} \wedge \frac{\mathrm{d}v}{v}.$$

Suppose that this invariant is of length ≤ 1. Then, the Albert form

$$\langle u, v, uv, a+b, bc, (a+b)bc\rangle_b$$

is isotropic over k. Since the forms $\langle u, v, uv\rangle_b$ and $\langle a+b, bc, (a+b)bc\rangle_b$ are anisotropic over k, there exists $\alpha \in k^*$ represented by both forms. Let us write

$$\begin{aligned} \alpha &= uL^2 + vM^2 + uvN^2 \\ &= (a+b)S^2 + bcT^2 + (a+b)bcU^2 \end{aligned}$$

for suitable $L, M, N, S, T, U \in k$. Hence, we have

$$b(S^2 + cT^2 + acU^2) = uL^2 + vM^2 + uvN^2 + aS^2 + c(bU)^2.$$

The right hand side in this equality and the factor $S^2 + cT^2 + acU^2$ belong to $k^2(a, c)$, but since $b \notin k^2(a, c)$, we necessarily have $S^2 + cT^2 + acU^2 = 0$. Since $\langle 1, c, ac\rangle$ is anisotropic over k, it follows that $S = T = U = 0$ and thus $\alpha = 0$, a contradiction.

Consequently, $e^2(B \perp \langle \det B\rangle_b \perp \langle\langle u, v\rangle\rangle_b + I^3F)$ is of length 2. Since all the conditions of Theorem 5 are satisfied (taking for C the form B itself), we conclude that B is $k(Q)$-minimal. □

Acknowledgement

This work was done in the framework of the project "IEA of CNRS" between the Artois University and the Academic College of Tel-Aviv-Yaffo. The two authors are grateful for the support of both institutions and CNRS.

References

[1] R. Baeza, *Quadratic forms over semilocal rings*, Lecture Notes in Mathematics **655**, Springer, 1978. MR

[2] A. Chapman and A. Quéguiner-Mathieu, "Minimal quadratic forms for the function field of a conic in characteristic 2", pp. 117–127 in *Amitsur Centennial Symposium*, Contemp. Math. **800**, Amer. Math. Soc., [2024] ©2024. MR

[3] A. Dolphin and A. Laghribi, "Isotropy of 5-dimensional quadratic forms over the function field of a quadric in characteristic 2", *Comm. Algebra* **45**:7 (2017), 3034–3044. MR

[4] A. Dolphin and A. Quéguiner-Mathieu, "Symplectic involutions, quadratic pairs and function fields of conics", *J. Pure Appl. Algebra* **221**:8 (2017), 1966–1980. MR

[5] R. Elman, N. Karpenko, and A. Merkurjev, *The algebraic and geometric theory of quadratic forms*, AMS Colloquium Publications **56**, American Mathematical Society, 2008. MR

[6] F. Faivre, *Liaison des formes de Pfister et corps de fonctions de quadriques en caractéristique* 2, PhD thesis, Université de Franche-Comté, 2006.

[7] D. W. Hoffmann and A. Laghribi, "Quadratic forms and Pfister neighbors in characteristic 2", *Trans. Amer. Math. Soc.* **356**:10 (2004), 4019–4053. MR

[8] D. W. Hoffmann and A. Laghribi, "Isotropy of quadratic forms over the function field of a quadric in characteristic 2", *J. Algebra* **295**:2 (2006), 362–386. MR

[9] D. W. Hoffmann, D. W. Lewis, and J. Van Geel, "Minimal forms for function fields of conics", pp. 227–237 in *K-theory and algebraic geometry*: *connections with quadratic forms and division algebras* (Santa Barbara, CA, 1992), vol. 2, Proc. Sympos. Pure Math. **58**, Amer. Math. Soc., 1995. MR

[10] K. Kato, "Symmetric bilinear forms, quadratic forms and Milnor K-theory in characteristic two", *Invent. Math.* **66**:3 (1982), 493–510. MR

[11] M. Knebusch, "Specialization of quadratic and symmetric bilinear forms, and a norm theorem", *Acta Arith.* **24** (1973), 279–299. MR

[12] M. Knebusch, "Generic splitting of quadratic forms, II", *Proc. London Math. Soc.* (3) **34**:1 (1977), 1–31. MR

[13] A. Laghribi, "Certaines formes quadratiques de dimension au plus 6 et corps des fonctions en caractéristique 2", *Israel J. Math.* **129** (2002), 317–361. MR

[14] A. Laghribi, "Quasi-hyperbolicity of totally singular quadratic forms", pp. 237–248 in *Algebraic and arithmetic theory of quadratic forms*, Contemp. Math. **344**, Amer. Math. Soc., 2004. MR

[15] A. Laghribi, "Witt kernels of function field extensions in characteristic 2", *J. Pure Appl. Algebra* **199**:1-3 (2005), 167–182. MR

[16] A. Laghribi, "Sur le déploiement des formes bilinéaires en caractéristique 2", *Pacific J. Math.* **232**:1 (2007), 207–232. MR

[17] A. Laghribi, "Les formes bilinéaires et quadratiques bonnes de hauteur 2 en caractéristique 2", *Math. Z.* **269**:3-4 (2011), 671–685. MR

[18] A. Laghribi, "Isotropie d'une forme bilinéaire d'Albert sur le corps de fonctions d'une quadrique en caractéristique 2", *J. Algebra* **355** (2012), 1–8. MR

[19] A. Laghribi and D. Mukhija, "The excellence of function fields of conics", *Proc. Amer. Math. Soc.* **152**:5 (2024), 1915–1923. MR

[20] A. Laghribi and U. Rehmann, "Bilinear forms of dimension less than or equal to 5 and function fields of quadrics in characteristic 2", *Math. Nachr.* **286**:11-12 (2013), 1180–1190. MR

[21] P. Mammone, J.-P. Tignol, and A. Wadsworth, "Fields of characteristic 2 with prescribed u-invariants", *Math. Ann.* **290**:1 (1991), 109–128. MR

Received April 9, 2025. Revised August 28, 2025.

ADAM CHAPMAN
SCHOOL OF COMPUTER SCIENCE
ACADEMIC COLLEGE OF TEL-AVIV-YAFFO
6818211 TEL-AVIV-YAFFO
ISRAEL
adam1chapman@yahoo.com

AHMED LAGHRIBI
LABORATOIRE DE MATHÉMATIQUES DE LENS UR2462
UNIVERSITÉ D'ARTOIS
62307 LENS
FRANCE
ahmed.laghribi@univ-artois.fr

PACIFIC JOURNAL OF MATHEMATICS
Vol. 339, No. 2, 2025
https://doi.org/10.2140/pjm.2025.339.265

THE MANAKOV EQUATION OF MIXED TYPE AND ITS MATRIX GENERALIZATION

QING DING, CHAOHAO YE AND SHIPING ZHONG

We introduce the matrix Manakov equation of mixed type by using algebraic properties of the Lie algebra $u(k, n-k)$, which is a nice supplementary to the matrix nonlinear Schrödinger equation. As a consequence, the general Manakov equation is generalized to the matrix case. By making use of some peculiar properties of $u(k, n-k)$, we derive both the geometric realization and Darboux transformation of the matrix Manakov equation of mixed type.

1. Introduction

The general Manakov equation reads

$$\begin{cases} i\varphi_{1t} + \varphi_{1xx} + (b_1|\varphi_1|^2 + b_2|\varphi_2|^2)\varphi_1 = 0, \\ i\varphi_{2t} + \varphi_{2xx} + (c_1|\varphi_1|^2 + c_2|\varphi_2|^2)\varphi_2 = 0, \end{cases} \tag{1}$$

where $\varphi_1 = \varphi_1(t, x)$, $\varphi_2 = \varphi_2(t, x)$ are unknown complex functions, subscript t and x denote differentiation with respect to time and position, respectively, and b_1, b_2, c_1, c_2 are nonzero real parameters. This differential system models the evolution of multicomponent weakly nonlinear dispersive wave trains in nonlinear optics, superfluid, plasma, Bose–Einstein condensed matter physics etc (refer to [2; 3; 4; 5; 12; 15; 17; 20; 22]). Equation (1) was first introduced by Manakov in 1974 (refer to [14]) and thus is named after him. Although involving 4 free real parameters and looking complicated, the analytic properties of equation (1) have been explored deeply and summarized in [11]. One notes that, with parameters being suitably chosen, equation (1) contains the three integrable equations

$$\begin{cases} i\varphi_{1t} + \varphi_{1xx} + 2(|\varphi_1|^2 + |\varphi_2|^2)\varphi_1 = 0, \\ i\varphi_{2t} + \varphi_{2xx} + 2(|\varphi_1|^2 + |\varphi_2|^2)\varphi_2 = 0, \end{cases} \tag{2}$$

$$\begin{cases} i\varphi_{1t} + \varphi_{1xx} - 2(|\varphi_1|^2 + |\varphi_2|^2)\varphi_1 = 0, \\ i\varphi_{2t} + \varphi_{2xx} - 2(|\varphi_1|^2 + |\varphi_2|^2)\varphi_2 = 0, \end{cases} \tag{3}$$

Qing Ding is the corresponding author.

MSC2020: primary 53C30, 53E30, 37K25; secondary 35Q55, 35Q60.

Keywords: Manakov system, matrix generalization, geometric realization, Darboux transformation.

and

$$\begin{cases} i\varphi_{1t} + \varphi_{1xx} \pm 2(|\varphi_1|^2 - |\varphi_2|^2)\varphi_1 = 0, \\ i\varphi_{2t} + \varphi_{2xx} \pm 2(|\varphi_1|^2 - |\varphi_2|^2)\varphi_2 = 0, \end{cases} \tag{4}$$

which are called the integrable Manakov equations, or 2-component nonlinear Schrödinger equations, of focusing, defocusing and mixed types, respectively. The two systems in equation (4) are actually equivalent to each other by the change of variables $\varphi_1 \to \varphi_2$ and $\varphi_2 \to \varphi_1$. Therefore, one may choose the equation

$$\begin{cases} i\varphi_{1t} + \varphi_{1xx} + 2(|\varphi_1|^2 - |\varphi_2|^2)\varphi_1 = 0, \\ i\varphi_{2t} + \varphi_{2xx} + 2(|\varphi_1|^2 - |\varphi_2|^2)\varphi_2 = 0, \end{cases} \tag{5}$$

as a representation of equation (4).

On the other hand, in the past decades, many efforts have been devoted to extending the theory of nonlinear Schrödinger hierarchy to higher dimensional cases. A successful generalization in literature seems to be the "matrix nonlinear Schrödinger hierarchy" associated to a Hermitian symmetric Lie algebra (see, for example, [6; 9; 13; 19]), which depends seriously on the integrability. Fordy and Kulish constructed in [9] the matrix nonlinear Schrödinger equation

$$i\Omega_t + \Omega_{xx} \pm 2\Omega\Omega^*\Omega = 0, \tag{6}$$

associated to the Hermitian symmetric Lie algebra $\boldsymbol{g} = u(n)$ (resp. $u(k, n-k)$) of index k ($1 \le k < n$), where Ω is a $k \times (n-k)$ complex matrix-valued function and Ω^* denotes the transposed conjugate of Ω. Equation (6) is also called the Fordy–Kulish equation in literature (refer to [13]). When $\boldsymbol{g} = u(2)$ or $u(1,1)$, equation (6) returns respectively to the nonlinear Schrödinger equation of focusing or defocusing types: $iq_t + q_{xx} \pm 2|q|^2 q = 0$. It is clear that equations (2) and (3) have their matrix generalizations which are included as special cases of the matrix nonlinear Schrödinger equation (6). In fact, if we take $\Omega = (\Omega_1\ \Omega_2)$ as a block matrix with Ω_1 and Ω_2 being respectively $k \times \mu$ and $k \times \nu$ complex matrices, in which $1 \le \mu, \nu \le n-k$ and $\mu + \nu = n-k$, then equation (6) becomes

$$\begin{cases} i\Omega_{1t} + \Omega_{1xx} \pm 2(\Omega_1\Omega_1^* + \Omega_2\Omega_2^*)\Omega_1 = 0, \\ i\Omega_{2t} + \Omega_{2xx} \pm 2(\Omega_1\Omega_1^* + \Omega_2\Omega_2^*)\Omega_2 = 0. \end{cases} \tag{7}$$

Equation (7) reverts to equations (2) and (3) when $n = 3$ and $k = \mu = \nu = 1$, respectively. But equation (5) of mixed type cannot be treated as a special case of matrix nonlinear Schrödinger equations (6). The problem is that the third term in the left-hand side of equation (5) involves a difference of two square norms of the unknown functions. This arises an interesting question: does the Manakov equation (5) of mixed type have a matrix generalization? To our surprise and to our best knowledge, such a question is not answered up to now in literature.

The aim of this paper is to give an affirmative answer to the above question based on exploiting some peculiar algebraic properties (see Lemmas 1–3 below) of the Lie algebra $u(k, n-k)$ of the semi-unitary group $U(k, n-k)$ ($n \geq 3$, $1 < k < n$), where $U(k, n-k)$ is the set of all $\mathbb{C}$-linear isometries of $\mathbb{C}^n_k$, here $\mathbb{C}^n_k$ denotes the space $\mathbb{C}^n$ with the semi-Hermitian product $\langle u, v\rangle = \sum_{j=1}^{k} u_j \bar{v}_j - \sum_{j=k+1}^{n} u_j \bar{v}_j$. It follows that $U(k, n-k) = \{g \in GL(n, \mathbb{C}) | g^* = \varepsilon g^{-1} \varepsilon\}$, in which the signature matrix $\varepsilon = (\delta_{ij}\varepsilon_j)$ whose diagonal entries are $\varepsilon_1 = \cdots = \varepsilon_k = 1$ and $\varepsilon_{k+1} = \cdots = \varepsilon_n = -1$, and $u(k, n-k) = \{S \in gl(n, \mathbb{C}) | S^* = -\varepsilon S \varepsilon\}$ (refer to [16]).

After deriving the matrix Manakov equation of mixed type (see below), we shall investigate some of its geometric and analytic properties. We first demonstrate the geometric realization of the matrix Manakov equation of mixed type in $u(k, n-k)$ (for the concept of geometric realization, see [13]). Then we deduce a Darboux transformation for the matrix Manakov equation of mixed type under some additional conditions. Based on the above exploitation, the general matrix Manakov equation is proposed.

Outline. Section 2 is devoted to introducing the matrix Manakov equation of mixed type based on exploiting some algebraic properties of the Lie algebra $u(k, n-k)$. In Section 3, we derive a model of moving curves in the Lie algebra $u(k, n-k)$ by Schrödinger flows such that it is a geometric realization of the matrix Manakov equation of mixed type. In Section 4, with some restrictions on the orders of matrices, we deduce a Darboux transformation for the matrix Manakov equation of mixed type. 1-soliton solutions are constructed explicitly in that section.

2. Decomposition of the Lie algebra $u(k, n-k)$

In order to generalize the Manakov equation (5) of mixed type to the matrix case, let's first explore some properties of the Lie algebra $u(k, n-k)$ of the semi-unitary group $U(k, n-k)$. As mentioned in Introduction, $u(k, n-k)$ is the subalgebra of $gl(n, \mathbb{C})$ consisting of all S which $S^* = -\varepsilon S \varepsilon$, in other words, $S \in u(k, n-k)$ has the form

(8) $$S = \begin{pmatrix} A & \Gamma \\ \Gamma^* & B_0 \end{pmatrix},$$

for some $A \in u(k)$, $B_0 \in u(n-k)$, and a $k \times (n-k)$ complex matrix Γ. Recall that an element $A \in u(k)$ satisfies $A^* = -A$.

Now for an integer ν satisfying $1 \leq \nu \leq k$, we come to introduce a direct decomposition of $u(k, n-k)$ with respect to ν by the following key observation. Since $A \in u(k)$ is presented by

$$A = \begin{pmatrix} A_1 & \Omega_1 \\ -\Omega_1^* & B_1 \end{pmatrix}$$

for some $A_1 \in u(\nu)$, $B_1 \in u(k-\nu)$, and a $\nu \times (k-\nu)$ complex matrix Ω_1, it follows that an element $S \in u(k, n-k)$ of the form (8) can be expressed by

$$S = \begin{pmatrix} \begin{pmatrix} A_1 & \Omega_1 \\ -\Omega_1^* & B_1 \end{pmatrix} & \begin{pmatrix} \Omega_2 \\ Q_0 \end{pmatrix} \\ \begin{pmatrix} \Omega_2^* & Q_0^* \end{pmatrix} & B_0 \end{pmatrix}, \tag{9}$$

for some $\nu \times (n-k)$ complex matrix Ω_2 and $(k-\nu) \times (n-k)$ complex matrix Q_0. Hence, S has an alternative expression as a block matrix:

$$\begin{pmatrix} A_1 & \begin{pmatrix} \Omega_1 & \Omega_2 \end{pmatrix} \\ \begin{pmatrix} -\Omega_1^* \\ \Omega_2^* \end{pmatrix} & \begin{pmatrix} B_1 & Q_0 \\ Q_0^* & B_0 \end{pmatrix} \end{pmatrix} =: \begin{pmatrix} A_1 & \Omega \\ \Omega^\dagger & B \end{pmatrix}, \tag{10}$$

where $\Omega = \begin{pmatrix} \Omega_1 & \Omega_2 \end{pmatrix}$, $\Omega^\dagger = \begin{pmatrix} -\Omega_1^* & \Omega_2^* \end{pmatrix}^\top$ is called the semicomposed conjugate of the block matrix Ω, and

$$B = \begin{pmatrix} B_1 & Q_0 \\ Q_0^* & B_0 \end{pmatrix}.$$

It is obvious that $B \in u(k-\nu, n-k)$. This leads to the following decomposition of the Lie algebra $u(k, n-k)$.

Lemma 1. *For a given integer ν with $1 \le \nu \le k$, the Lie algebra $u(k, n-k)$ ($n \ge 3$ and $1 \le k < n$) has the decomposition $u(k, n-k) = \boldsymbol{k} \oplus \boldsymbol{m}$, where*

$$\boldsymbol{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \;\middle|\; A \in u(\nu),\ B \in u(k-\nu, n-k) \right\} \tag{11}$$

and

$$\boldsymbol{m} = \left\{ \begin{pmatrix} 0 & \Omega \\ \Omega^\dagger & 0 \end{pmatrix} \;\middle|\; \Omega = \begin{pmatrix} \Omega_1 & \Omega_2 \end{pmatrix} \text{ is indicated in (10)} \right\}. \tag{12}$$

The decomposition satisfies the symmetric conditions

$$[\boldsymbol{k}, \boldsymbol{k}] \subset \boldsymbol{k}, \quad [\boldsymbol{k}, \boldsymbol{m}] \subset \boldsymbol{m}, \quad [\boldsymbol{m}, \boldsymbol{m}] \subset \boldsymbol{k}.$$

Proof. With the description mentioned above, what remains for us to do is to verify that the decomposition $u(k, n-k) = \boldsymbol{k} \oplus \boldsymbol{m}$ satisfies the symmetric conditions. This is direct and we omit it here. □

We emphasize that the semicomposed conjugate $\Omega^\dagger$ of a block matrix $\Omega = \begin{pmatrix} \Omega_1 & \Omega_2 \end{pmatrix}$ is very different from the usual one $*$. One will see in the sequel that, because of the decomposition of $u(k, n-k)$ in Lemma 1, we find the matrix generalization of equation (5) of mixed type. Before going further, the Lie algebra $u(k, n-k)$ with the decomposition displayed in Lemma 1 is now denoted by $u(\nu, k-\nu, n-k)$, here ν denotes the rank of the decomposition, $k-\nu$ and $n-k$

indicate that the first and second matrix in Ω are of $\nu\times(k-\nu)$ and $\nu\times(n-k)$ respectively. The following lemmas present interesting and important properties of the semicomposed conjugate operation $\dagger: Q\to Q^\dagger$.

Lemma 2. (i) *For the block matrices* $\Omega=\begin{pmatrix}\Omega_1 & \Omega_2\end{pmatrix}$ *and* $\Gamma=\begin{pmatrix}\Gamma_1 & \Gamma_2\end{pmatrix}$, *the identity* $\left(\Omega\Gamma^\dagger\right)^*=\Gamma\Omega^\dagger$ *is valid.*

(ii) *For the block matrix* $\Omega=\begin{pmatrix}\Omega_1 & \Omega_2\end{pmatrix}$, *the identity* $(\Omega\Omega^\dagger\Omega)^\dagger=\Omega^\dagger\Omega\Omega^\dagger$ *is valid as well as the usual composed conjugate operation* $*$.

Proof. Part (i) is obvious. For (ii), a direct computation shows that

$$\Omega\Omega^\dagger\Omega=\begin{pmatrix}(-\Omega_1\Omega_1^*+\Omega_2\Omega_2^*)\Omega_1 & (-\Omega_1\Omega_1^*+\Omega_2\Omega_2^*)\Omega_2\end{pmatrix}$$

and

$$\Omega^\dagger\Omega\Omega^\dagger=\begin{pmatrix}-\Omega_1^*(-\Omega_1\Omega_1^*+\Omega_2\Omega_2^*)\\ \Omega_2^*(-\Omega_1\Omega_1^*+\Omega_2\Omega_2^*)\end{pmatrix}.$$

Hence, by definition, we have $(\Omega\Omega^\dagger\Omega)^\dagger=\Omega^\dagger\Omega\Omega^\dagger$. □

Lemma 3. *For the Lie algebra* $u(\nu,k-\nu,n-k):=u(k,n-k)$ *with the decomposition given in Lemma 1, we set*

$$u^*(\nu,k-\nu,n-k)=\left\{\begin{pmatrix}A & \Omega\\ -\Omega^\dagger & B\end{pmatrix}\,\middle|\, A\in u(\nu),\,B\in u(n-k,k-\nu),\,\Omega=(\Omega_2\;\Omega_1)\right\}. \tag{13}$$

Then $u^*(\nu,k-\nu,n-k)$ *is isomorphic to* $u(\nu,k-\nu,n-k)$ *and* $u^*(\nu,k-\nu,n-k)$ *has similarly a symmetric decomposition* $u^*(\nu,k-\nu,n-k)=\boldsymbol{k}^*\oplus\boldsymbol{m}^*$, *where*

$$\boldsymbol{k}^*=\left\{\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix}\,\middle|\, A\in u(\nu),\,B\in u(n-k,k-\nu)\right\},$$

$$\boldsymbol{m}^*=\left\{\begin{pmatrix}0 & \Omega\\ -\Omega^\dagger & 0\end{pmatrix}\,\middle|\, \Omega=(\Omega_2\;\Omega_1)\right\}.$$

It is obvious that Lemma 3 is not true if we replace the semicomposed conjugate operation $\dagger$ by the usual composed conjugate operation $*$. This is a peculiar property of the operation $\dagger$. Lemma 3 indicates that the Lie algebra $u(k,n-k)$ of $U(k,n-k)$ can be presented not only by $u(\nu,k-\nu,n-k)$, but also by $u^*(\nu,k-\nu,n-k)$. This fact will be used very technically in Section 4 to establish the Darboux transformation.

Proof of Lemma 3. It is easy to verify that $u^*(\nu,k-\nu,n-k)$ is a Lie algebra, in this process, Lemma 2 is applied. Now we define a map φ from $u(\nu,k-\nu,n-k)$ to $u^*(\nu,k-\nu,n-k)$ by

$$\varphi\left(\begin{pmatrix}A & \Omega\\ \Omega^\dagger & B\end{pmatrix}\right)=\begin{pmatrix}A & \sigma(\Omega)\\ -(\sigma(\Omega))^\dagger & \sigma(B)\end{pmatrix},$$

where

$$\sigma(\Omega) = \begin{pmatrix} \Omega_2 & \Omega_1 \end{pmatrix} \qquad \text{when } \Omega = \begin{pmatrix} \Omega_1 & \Omega_2 \end{pmatrix},$$
$$\sigma(B) = \begin{pmatrix} B_{00} & Q_0^* \\ Q_0 & B_{01} \end{pmatrix} \quad \text{when } B = \begin{pmatrix} B_{01} & Q_0 \\ Q_0^* & B_{00} \end{pmatrix} \in u(k-\nu, n-k).$$

What remains for us to do is to verify φ is an isomorphism. First, one notes that the inverse map of φ is

$$\varphi^{-1}\left(\begin{pmatrix} A & \Omega \\ -\Omega^\dagger & B \end{pmatrix}\right) = \begin{pmatrix} A & \sigma(\Omega) \\ (\sigma(\Omega))^\dagger & \sigma(B) \end{pmatrix}.$$

Hence φ is a linear bijective map. Next, we verify that φ preserves the Lie bracket operation. For two matrices

$$\begin{pmatrix} A_1 & \Omega \\ \Omega^\dagger & B_1 \end{pmatrix}, \begin{pmatrix} A_2 & \Gamma \\ \Gamma^\dagger & B_2 \end{pmatrix} \in u(\nu, k-\nu, n-k),$$

we have

$$\left[\begin{pmatrix} A_1 & \Omega \\ \Omega^\dagger & B_1 \end{pmatrix}, \begin{pmatrix} A_2 & \Gamma \\ \Gamma^\dagger & B_2 \end{pmatrix}\right]$$
$$= \begin{pmatrix} A_1A_2 - A_2A_1 + \Omega\Gamma^\dagger - \Gamma\Omega^\dagger & A_1\Gamma - A_2\Omega + \Omega B_2 - \Gamma B_1 \\ \Omega^\dagger A_2 - \Gamma^\dagger A_1 + B_1\Gamma^\dagger - B_2\Omega^\dagger & \Omega^\dagger\Gamma - \Gamma^\dagger\Omega + B_1B_2 - B_2B_1 \end{pmatrix}$$

and

$$\left[\begin{pmatrix} A_1 & \sigma(\Omega) \\ -(\sigma(\Omega))^\dagger & \sigma(B_1) \end{pmatrix}, \begin{pmatrix} A_2 & \sigma(\Gamma) \\ -(\sigma(\Gamma))^\dagger & \sigma(B_2) \end{pmatrix}\right] = \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_3 & \Sigma_4 \end{pmatrix},$$

where

$$\Sigma_1 = A_1A_2 - A_2A_1 - \sigma(\Omega)(\sigma(\Gamma))^\dagger + \sigma(\Gamma)(\sigma(\Omega))^\dagger,$$
$$\Sigma_2 = A_1\sigma(\Gamma) - A_2\sigma(\Omega) + \sigma(\Omega)\sigma(B_2) - \sigma(\Gamma)\sigma(B_1),$$
$$\Sigma_3 = -(\sigma(\Omega))^\dagger A_2 + (\sigma(\Gamma))^\dagger A_1 - \sigma(B_1)(\sigma(\Gamma))^\dagger + \sigma(B_2)(\sigma(\Omega))^\dagger,$$
$$\Sigma_4 = -(\sigma(\Omega))^\dagger\sigma(\Gamma) + (\sigma(\Gamma))^\dagger\sigma(\Omega) + \sigma(B_1)\sigma(B_2) - \sigma(B_2)\sigma(B_1).$$

It is direct to verify that

$$-\sigma(\Omega)(\sigma(\Gamma))^\dagger + \sigma(\Gamma)(\sigma(\Omega))^\dagger = \Omega\Gamma^\dagger - \Gamma\Omega^\dagger,$$
$$\Sigma_4 = \sigma(\Omega^\dagger\Gamma - \Gamma^\dagger\Omega + B_1B_2 - B_2B_1),$$
$$\Sigma_2 = \sigma(A_1\Gamma - A_2\Omega + \Omega B_2 - \Gamma B_1),$$
$$\Sigma_2^\dagger = -\Sigma_3.$$

Hence, we obtain that

$$\varphi\left(\left[\begin{pmatrix} A_1 & \Omega \\ \Omega^\dagger & B_1 \end{pmatrix}, \begin{pmatrix} A_2 & \Gamma \\ \Gamma^\dagger & B_2 \end{pmatrix}\right]\right) = \left[\varphi\left(\begin{pmatrix} A_1 & \Omega \\ \Omega^\dagger & B_1 \end{pmatrix}\right), \varphi\left(\begin{pmatrix} A_2 & \Gamma \\ \Gamma^\dagger & B_2 \end{pmatrix}\right)\right]. \tag{14}$$

In the verifications, we have used that $A_1, A_2 \in u(\nu)$ and that

$$B_{01} \in u(k-\nu),\ \ B_{00} \in u(n-k) \quad \text{for } B = \begin{pmatrix} B_{01} & Q_0 \\ Q_0^* & B_{00} \end{pmatrix} \in u(k-\nu, n-k).$$

This proves that $u(\nu, k-\nu, n-k)$ is isomorphic to $u^*(\nu, k-\nu, n-k)$.

The verification of the symmetric decomposition $u^*(\nu, k-\nu, n-k) = \boldsymbol{k}^* \oplus \boldsymbol{m}^*$ is direct and we omit it here. The proof of Lemma 3 is completed. □

Corollary 1. *When $\mu := k-\nu = n-k \geq 1$—in other words, when Ω_1, Ω_2 in Ω are the same $\nu \times \mu$ matrices $(1 \leq \nu < k < n)$—the semi-unitary group $U(\mu+\nu, \mu)$ has two isomorphic Lie algebras $u(\nu, \mu, \mu)$ and $u^*(\nu, \mu, \mu)$.*

Theorem 1. *The matrix Manakov equation of mixed type, with model*

$$\begin{cases} i\Omega_{1_t} + \Omega_{1xx} + 2(\Omega_1\Omega_1^* - \Omega_2\Omega_2^*)\Omega_1 = 0, \\ i\Omega_{2_t} + \Omega_{2xx} + 2(\Omega_1\Omega_1^* - \Omega_2\Omega_2^*)\Omega_2 = 0, \end{cases} \tag{15}$$

is an integrable generalization of the Manakov equation (5) of mixed type, where Ω_1 and Ω_2 are unknown complex $\nu \times (k-\nu)$ and $\nu \times (n-k)$ matrices, respectively $(1 \leq \nu < k < n)$.

When $\nu = 1$, $k = 2$ and $n = 3$, equation (15) reverts to (5). Equation (15) includes the k-component $(k \geq 3)$ nonlinear Schrödinger equations of mixed type as a special case. For the physical significance of general k-component $(k \geq 2)$ nonlinear Schrödinger equations, one may refer to [1; 21].

When $\nu = k$, equation (15) reverts to the matrix nonlinear Schrödinger equation of defocusing type and when $k = n$, to the matrix nonlinear Schrödinger equation of focusing type. Therefore, equation (15) is a nice supplementary equation to the general integrable matrix nonlinear Schrödinger equations.

Proof of Theorem 1. For the Lie symmetric algebra $u(\nu, k-\nu, n-k)$, we set

$$\sigma_3 = \frac{i}{2}\begin{pmatrix} I_\nu & 0 \\ 0 & -I_{n-\nu} \end{pmatrix}, \tag{16}$$

where I_n stands for the $n \times n$ unit matrix and consider the linear equations

$$\begin{cases} \phi_x = -(\lambda\sigma_3 + Q)\phi, \\ \phi_t = P\phi, \end{cases} \tag{17}$$

where λ is the spectral parameter, $Q = Q(t, x) = \begin{pmatrix} 0 & \Omega \\ \Omega^\dagger & 0 \end{pmatrix} \in \boldsymbol{m}$, with $\Omega = \begin{pmatrix} \Omega_1^* & \Omega_2^* \end{pmatrix}$, and

$$P = \sum_{j=0}^{2} P_j(t, x)\lambda^j$$

is a *polynomial ansatz*, with the $P_j(t,x) \in u(\nu, k-\nu, n-k)$ $(j=0,1,2)$, to be determined later, being functions of Q and its derivatives, but independent of λ. From the integrability condition of (17), namely

$$-(\lambda\sigma_3 + Q)_t - P_x - [(\lambda\sigma_3 + Q), P] = 0$$

and identifying the coefficients of λ^j $(j = 0, 1, 2, 3)$ in the left-hand-side of the above equation to be zero, we find that $P_2 = \sigma_3$, $P_1 = Q$ and $P_0 = -2(Q_x - Q^2)\sigma_3$, i.e.,

$$P = \lambda^2\sigma_3 + \lambda Q - 2(Q_x - Q^2)\sigma_3$$

and Q satisfies

$$-Q_t + 2Q_{xx}\sigma_3 - 4Q^3\sigma_3 = 0, \tag{18}$$

which is exactly equation (15). Here we have used of Lemma 2(ii) in the computation. The proof of Theorem 1 is completed. □

Combining the matrix Manakov equations (7) of focusing and defocusing types with the matrix Manakov equation (15) of mixed type, the general matrix Manakov equation is now proposed as

$$\begin{cases} i\Omega_{1_t} + \Omega_{1xx} + (B_1\Omega_1\Omega_1^* + B_2\Omega_2\Omega_2^*)\Omega_1 = 0, \\ i\Omega_{2t} + \Omega_{2xx} + (C_1\Omega_1\Omega_1^* + C_2\Omega_2\Omega_2^*)\Omega_2 = 0, \end{cases} \tag{19}$$

where Ω_1 and Ω_2 are respectively unknown complex $(\nu \times (k-\nu))$ and $(\nu \times (n-k))$ matrices $(1 \le \nu < k < n)$, B_1, B_2, C_1 and C_2 are given real nonsingular $\nu \times \nu$ matrices. When $n = 3$, $k = 2$ and $\nu = 1$, equation (19) returns to the general Manakov equation (1). It is very interesting to explore analytic and geometric properties of equation (19). One may refer to [7; 8; 11] for such investigations in the case of the general Manakov equation (1).

3. Geometric realization

We will now derive a model of moving curves in the Lie algebra $u(k, n-k)$ such that it is a geometric realization of the matrix Manakov equation (15) of mixed type.

Recall that a (compact) Grassmannian manifold can be represented as an adjoint orbit space in the unitary algebra $u(n)$. From this representation for Grassmannians, Terng and Uhlenbeck showed in [19] that the matrix nonlinear Schrödinger equation of focusing type is gauge equivalent to the equation of 1-d Schrödinger flow to a Grassmannian manifold. In the semi-unitary Lie algebra $u(k, n-k) = u(\nu, k-\nu, n-k) = \boldsymbol{k} \oplus \boldsymbol{m}$ indicated in Lemma 1, the adjoint orbit space

$$U(k, n-k)/U(\nu) \times U(k-\nu, n-k) = \{E^{-1}\sigma_3 E \mid E \in U(k, n-k)\}$$

represents a pseudosymmetric space, where σ_3 is given by (16). This pseudo-symmetric space is denoted by $G_{\nu,k,n}$ and called a semi-Grassmannian manifold. Recall that the standard Killing metric (i.e., a bi-invariant inner product) $\langle \cdot, \cdot \rangle$ on $u(k, n-k)$ is given by $\langle A, B\rangle = -\mathrm{tr}(AB)$, $A, B \in u(k, n-k)$.

Theorem 2. *For the semi-unitary Lie algebra $u(k, n-k) = u(\nu, k-\nu, n-k) = \boldsymbol{k} \oplus \boldsymbol{m}$ indicated in Lemma 1 and the standard Killing matric $\langle \cdot, \cdot \rangle$ indicated above, the following model of moving curves $\tilde{\gamma}(t, x)$ in $u(k, n-k)$,*

$$\tilde{\gamma}_t = [\tilde{\gamma}_x, \tilde{\gamma}_{xx}] \tag{20}$$

with $\tilde{\gamma}_x(t, x) \in G_{\nu,k,n}$, is gauge equivalent to the matrix Manakov equation (15) of mixed type.

Remarks. (i) The model (20) in Theorem 2 preserves the arc-length x of $\tilde{\gamma}(t, x)$ invariant when t involves. In fact,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(|\tilde{\gamma}_x|^2) &= \langle \tilde{\gamma}_x, \tilde{\gamma}_{xt}\rangle = \langle \tilde{\gamma}_x, [\tilde{\gamma}_x \tilde{\gamma}_{xx}]_x\rangle = \langle \tilde{\gamma}_x, [\tilde{\gamma}_x, \tilde{\gamma}_{xxx}]\rangle \\
&= \langle \tilde{\gamma}_x, \tilde{\gamma}_x \gamma_{xxx} - \tilde{\gamma}_{xxx}\tilde{\gamma}_x\rangle = -\mathrm{tr}(\tilde{\gamma}_x\tilde{\gamma}_x\tilde{\gamma}_{xxx}) + \mathrm{tr}(\tilde{\gamma}_x\tilde{\gamma}_{xxx}\tilde{\gamma}_x) \\
&= -\mathrm{tr}(\tilde{\gamma}_x\tilde{\gamma}_x\tilde{\gamma}_{xxx}) + \mathrm{tr}(\tilde{\gamma}_x\tilde{\gamma}_x\tilde{\gamma}_{xxx}) \\
&= 0.
\end{aligned}$$

Here we have used the metric formula $\langle A, B\rangle = -\mathrm{tr}(AB)$ and the fact that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. This implies that $|\tilde{\gamma}_x(t, x)|^2 = |\tilde{\gamma}_x(0, x)|^2$; in other words, the arc-length x is invariant when t evolves.

(ii) Taking the derivative with respect to x on both sides of equation (20) and presenting $\tilde{\gamma}_x$ by γ, we obtain

$$\gamma_t = [\gamma, \gamma_{xx}], \tag{21}$$

where $\gamma(t, x) \in G_{\nu,k,n}$. It is a straightforward verification that the tangent space at any point of $G_{\nu,k,n}$ can be identified with $\boldsymbol{m}$ and equation (21) is just the equation of 1-d Schrödinger flow to the semi-Grassmannian manifold $G_{\nu,k,n}$. The details are omitted here.

(iii) It is obvious that equation (20) is equivalent to equation (21). Indeed, if $\tilde{\gamma} \in u(k, n-k)$ satisfies equation (20) with $\tilde{\gamma}_x(t, x) \in G_{\nu,k,n}$, by taking the derivative with respect to x in the both hand-sides of equation (20) , we see that $\gamma = \tilde{\gamma}_x$ satisfies equation (21). Conversely, if $\gamma(t, x) = E^{-1}(t, x)\sigma_3 E(t, x)$ for some $E(t, x) \in U(k, n-k)$ satisfies equation (21), then it is easy to see that $\tilde{\gamma}(t, x) = \int^x \gamma(t, s)\, ds$ solves equation (20) and satisfies

$$\tilde{\gamma}_x(t, x) \in G_{\nu,k,n} = \left\{E^{-1}(t, x)\sigma_3 E(t, x) \mid E \in U(k, n-k)\right\}.$$

As a consequence, equation (20) keeps $\tilde{\gamma}_x(t,x)\in G_{\nu,k,n}$ invariant when t involves once $\tilde{\gamma}_x(t_0,x)\in G_{\nu,k,n}$ for some t_0.

Proof of Theorem 2. We only need to show that equation (21) is gauge equivalent to the matrix Manakov equation (15) of mixed type.

We first show that equation (15) is gauge transformed to equation (21). For this purpose, let $(\Omega_1\ \Omega_2)$ be a solution to equation (15) and set

$$Q=Q(t,x)=\begin{pmatrix} 0 & \Omega \\ \Omega^\dagger & 0\end{pmatrix}\in \boldsymbol{m} \quad \text{with } \Omega=\begin{pmatrix}\Omega_1^* & \Omega_2^*\end{pmatrix}.$$

Notice that equation (15) has a zero curvature representation; namely, if we set

$$A=(\lambda\sigma_3+Q)\,dx-(\lambda^2\sigma_3+\lambda Q-V)\,dt,\quad \text{with } V=2(Q_x-Q^2)\sigma_3, \tag{22}$$

to be a connection of the trivial bundle $\mathbb{R}^2\times u(k,n-k)$, where λ is a spectral parameter, then equation (15) is equivalent to

$$F_A=dA+A\wedge A=0.$$

Now we choose a fundamental solution $E=E(t,x)$ to

$$\begin{cases} E_x=-(\lambda\sigma_3+Q)E,\\ E_t=(\lambda^2\sigma_3+\lambda Q-V)E,\end{cases} \tag{23}$$

at $\lambda=0$ and make a gauge transformation for the connection A given by (22) by using $E\in U(k,n-k)$:

$$A\mapsto \tilde{A}=E^{-1}dE+E^{-1}AE. \tag{24}$$

From the theory of Yang–Mills we know that $F_{\tilde{A}}=E^{-1}F_AE=0$. By a direct computation, we obtain from (24) that

$$\tilde{A}=E^{-1}dE+E^{-1}AE=\lambda\gamma\,dx-(\lambda^2\gamma+\lambda[\gamma,\gamma_x])\,dt, \tag{25}$$

where $\gamma=E^{-1}\sigma_3E$. Here we have used the relations

$$\gamma_x=-E^{-1}[\sigma_3,Q]E,\quad [\gamma,\gamma_x]=E^{-1}QE.$$

By using (25), the zero curvature condition of $\tilde{A}$,

$$F_{\tilde{A}}=d\tilde{A}+\tilde{A}\wedge\tilde{A}=0$$

is exactly equation (21). This proves that equation (15) is gauge transformed to equation (21) of 1-d Schrödinger flow to the semi-Grassmannian $G_{\nu,k,n}$.

Next, we shall show that the above conclusion is reversible, i.e., equation (21) is also gauge transformed to equation (15). In fact, for a given solution $\gamma=E^{-1}\sigma_3E$ to equation (21), where $E=E(t,x)\in U(k,n-k)$. Without loss of generality,

E may be assumed to satisfy $E_x = -QE$ for some

$$Q = \begin{pmatrix} 0 & \Omega \\ \Omega^\dagger & 0 \end{pmatrix} \in \boldsymbol{m}$$

with $\Omega = \begin{pmatrix} \Omega_1^* & \Omega_2^* \end{pmatrix}$: indeed, if $E_x = PE$ holds for some $P \in u(k, n-k)$ with $P = P_{\boldsymbol{k}} + P_{\boldsymbol{m}}$ and $P_{\boldsymbol{k}} \neq 0$, one may choose a matrix $B \in U(\nu) \times U(k-\nu, n-k)$ by solving the ordinary differential equation $B_x = -BP_{\boldsymbol{k}}$. Then, from the transform $E \to \tilde{E} = BE$, we still have $\gamma = E^{-1}\sigma_3 E = \tilde{E}^{-1}\sigma_3\tilde{E}$, since $B \in U(\nu) \times U(k-\nu, n-k)$ commutes with σ_3. Moreover, $\tilde{E}_x = B_x E + BE_x = BP_{\boldsymbol{m}}B^{-1}\tilde{E}$ with $BP_{\boldsymbol{m}}B^{-1} \in \boldsymbol{m}$, and the claim is justified.

Now, noting that equation (21) possesses the zero curvature representation by the connection $\tilde{A}$ given in (25) with $\gamma = E^{-1}\sigma_3 E$, we make the following gauge transformation for the connection $\tilde{A}$ via $E_1 = E^{-1}$:

$$\tilde{A} \mapsto A = E_1^{-1} dE_1 + E_1^{-1}\tilde{A}E_1 = -(dE)E^{-1} + E\tilde{A}E^{-1}. \tag{26}$$

By a direct computation, we have from (26) that

$$A = -(dE)E^{-1} + E\tilde{A}E^{-1} = (\lambda\sigma_3 + Q)\,dx - (\lambda^2\sigma_3 + \lambda Q + E_t E^{-1})\,dt, \tag{27}$$

where $E^{-1}E_t$ independent of λ will be determined later. By using (27), a direct calculation show that

$$\begin{aligned} F_A &= dA + A \wedge A \\ &= \big\{(-Q_t - (E_t E^{-1})_x - [Q,\, E_t E^{-1}]) - \lambda(Q_x + [\sigma_3,\, E_t E^{-1}])\big\} dx \wedge dt. \end{aligned} \tag{28}$$

The coefficients of λ^1 and λ^0 in the right-hand-side of (28) are zero implies that

$$Q_x + [\sigma_3,\, E_t E^{-1}] = 0, \tag{29}$$

$$Q_t + (E_t E^{-1})_x + [Q,\, E_t E^{-1}] = 0. \tag{30}$$

From equation (29), we have

$$(E_t E^{-1})_{\boldsymbol{m}} = -V_{\boldsymbol{m}}, \tag{31}$$

where $V = 2(Q_x - Q^2)\sigma_3$ is given in (22). Taking the $\boldsymbol{k}$ part in equation (30), we have

$$(E_t E^{-1})_{\boldsymbol{k}x} + [Q,\, (E_t E^{-1})_{\boldsymbol{m}}] = 0, \tag{32}$$

which implies that $(E_t E^{-1})_{\boldsymbol{k}} = -V_{\boldsymbol{k}} + \mathrm{diag}(\alpha_1(t), \alpha_2(t))$ for some $\alpha_1(t) \in u(\nu)$ and $\alpha_2(t) \in u(k-\nu, n-k)$ that independent of x. Hence we obtain

$$E_t E^{-1} = (E_t E^{-1})_{\boldsymbol{k}} + (E_t E^{-1})_{\boldsymbol{m}} = -V_{\boldsymbol{k}} - V_{\boldsymbol{m}} = -V + \mathrm{diag}(\alpha_1(t), \alpha_2(t)). \tag{33}$$

In order to cancel $\mathrm{diag}(\alpha_1(t), \alpha_2(t))$ in (33), we modify E by

$$E \to \tilde{E} = \beta(t)E,$$

where $\beta(t) = \mathrm{diag}(\beta_1(t), \beta_2(t)) \in U(\nu) \times U(k-\nu, n-k)$ depends only on t and satisfy $\frac{d\beta_j}{dt}(t) + \alpha_j(t)\beta_j(t) = 0$ $(j = 1, 2)$. The existence of β_j $(j = 1, 2)$ are obvious. One may verify straightforwardly that with E being modified by $\tilde{E}$, the second term on the right-hand-side of (33) vanishes identically and A given by equation (27) is exactly that given in (22) with Q be replaced by $\tilde{Q} = \beta(t)Q\beta(t)^{-1}$. The corresponding (Ω_1, Ω_2) obtained from $\gamma = E^{-1}\sigma_3 E$ is a solution to the matrix Manakov systems (15) of mixed type. This proves that equation (21) is also gauge transformed to equation (15). The proof of Theorem 2 is completed. □

From Theorem 2 we see that the matrix Manakov equation (15) of mixed type admits a geometric interpretation, that is to say, it is gauge equivalent to the 1-d Schrödinger flow to the semi-Grassmannian $G_{\nu,k,n}$. Furthermore, the model (20) of moving curves in the Lie algebra $u(k, n-k)$ is a geometric realization of the matrix Manakov equation (15) of mixed type.

4. Darboux transformation

We now exploit some analytic aspects of the matrix Manakov equation (15) of mixed type. More precisely, we establish a Darboux transformation and construct soliton solutions.

Darboux transformation is a useful tool in constructing new solutions from given ones to an integrable system/equation. For a given integrable system/equation, usually there are two effective ways in deducing a Darboux transformation. One is the usual way (see [10]), that is, if $\Phi(t, x, \lambda)$ is a potential (matrix-valued) function to a Lax pair of the integrable system/equation with a given solution $Q = Q(t, x)$, where λ is the spectral parameter, the idea is to find a (matrix-valued) function S in an algebraic way such that $\tilde{\Phi}(t, x, \lambda) = (\lambda - S)\Phi(t, x, \lambda)$ is a solution to the Lax pair with some $\tilde{Q}$ depending on Q and S so that it is a new solution to the integrable system/equation. Then $Q \to \tilde{Q}$ is the desired Darboux transformation. Another way is given by Terng and Uhlenbeck in [18] who used the loop-group technique in constructing new potential functions and hence the corresponding Darboux transformation. In this section, we deduce a Darboux transformation for the matrix Manakov equation of mixed type equation (15) (or equivalently equation (18)) via the usual way. The difficulty here is whether the obtained $\tilde{Q}$ still possesses the form

$$\tilde{Q} = \tilde{Q}(t, x) = \begin{pmatrix} 0 & \tilde{\Omega} \\ \tilde{\Omega}^{\dagger} & 0 \end{pmatrix}$$

for some new $\tilde{\Omega}$. To overcome this crucial difficulty, we need to apply Lemma 3 and its corollary.

We first rewrite the Lax pair (17) of equation (15) (or equivalently (18)) as

$$\begin{cases}\Phi_x = -(i\lambda \boldsymbol{l}_3 + Q)\Phi, \\ \Phi_t = \left(\sum_{j=0}^{2} P_j(t,x)\lambda^j\right)\Phi,\end{cases} \tag{34}$$

in which Φ is regarded as a $U(k, n-k)$-matrix-valued potential function, $\boldsymbol{l}_3 = \begin{pmatrix} I_\nu & 0 \\ 0 & -I_{n-\nu}\end{pmatrix}$, $Q = \begin{pmatrix} 0 & \Omega \\ \Omega^\dagger & 0\end{pmatrix}$ is a solution to equation (18) with $\Omega = (\Omega_1^* \ \Omega_2^*)$, $P_2 = 2i\boldsymbol{l}_3$, $P_1 = 2Q$ and $P_0 = -i(Q_x - Q^2)\boldsymbol{l}_3$.

Next, according to the usual idea for constructing a Darboux transformation $Q \to \tilde{Q}$, we must find a Darboux matrix $(i\lambda I_n - S)$ such that $\tilde{\Phi} = (i\lambda I_n - S)\Phi$ still solves equation (34) with some new $\tilde{Q}$ which will be determined later. We would point out that the Darboux matrix displayed in [10] is of the form $\lambda I_n - S$. Here we modify it so that $i\lambda I_n - S \in u(\nu, k-\nu, n-k)$ when S is suitably chosen. From the relation

$$\begin{aligned}(-i\lambda \boldsymbol{l}_3 - \tilde{Q})(i\lambda I_n - S)\Phi = (\tilde{\Phi})_x &= ((i\lambda I - S)\Phi)_x \\ &= (i\lambda I_n - S)(-i\lambda \boldsymbol{l}_3 - Q)\Phi - S_x\Phi,\end{aligned} \tag{35}$$

we obtain, from identifying the coefficients of λ^1 and λ^0,

$$\tilde{Q} = Q + [\boldsymbol{l}_3, S], \tag{36}$$

$$S_x = [S, Q + \boldsymbol{l}_3 S]. \tag{37}$$

Similarly, we have

$$\Big(\sum_{j=1}^{2} \tilde{P}_j\lambda^j\Big)(i\lambda I_n - S)\Phi = (\tilde{\Phi})_t = (i\lambda I_n - S)\Big(\sum_{j=1}^{2} P_j\lambda^j\Big)\Phi - S_t\Phi. \tag{38}$$

By identifying the coefficients of λ^j $(j = 0, 1, 2, 3)$, we see that

$$\tilde{P}_2 = P_2 = 2i\boldsymbol{l}_3, \quad \tilde{P}_1 = P_1 + 2[\boldsymbol{l}_3, S] = 2\tilde{Q}, \quad \tilde{P}_0 = P_0 - i[P_1, S] - [P_2, S]S \tag{39}$$

and

$$S_t + [S, P_0 - iP_1S - P_2S^2] = 0. \tag{40}$$

Lemma 4. *$i\lambda I_n - S$ is a Darboux matrix of the Lax pair (34) if and only if S satisfies equations (37) and (40).*

Proof. If $i\lambda I_n - S$ is a Darboux matrix of the Lax pair (34), then the above exploitation indicates that S satisfies equation (37) and equation (40). Inversely, if a matrix S satisfies equation (37) and equation (40). Then for a solution Φ to equation (34) with a given Q fulfilling equation (18), we have the validity of equation (35) and equation (38). Hence, with $\tilde{Q}$ determined by (36) and $\tilde{P}_j$ $(j=0, 1, 2)$ by (39), we see that $\tilde{\Phi}=(i\lambda I_n - S)\Phi$ is a solution to

$$\begin{cases} \tilde{\Phi}_x = -(i\lambda \boldsymbol{I}_3 + \tilde{Q})\tilde{\Phi}, \\ \tilde{\Phi}_t = \left(\sum_{j=0}^{2} \tilde{P}_j(t,x)\lambda^j\right)\tilde{\Phi}. \end{cases} \tag{41}$$

This implies that $i\lambda I_n - S$ is a Darboux matrix of the Lax pair (34). Lemma 4 is proved. □

Now we will construct the Darboux matrix by making use of (37) and (40). Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be real numbers, at least two of which are different. Set $\Lambda = \operatorname{diag}(i\lambda_1, i\lambda_2, \dots, i\lambda_n)$. For a given solution Q to equation (18), we first choose h_1 to be a unit column $\mathbb{C}^n_k$-vector-valued solution to equation (34) at $\lambda=\lambda_1$. Then, in the complementary space of $\mathbb{C}h_1$ in $\mathbb{C}^n_k$, we choose h_2 to be a unit column $\mathbb{C}^n_k$-vector-valued solution to equation (34) at $\lambda=\lambda_2$. Going on this way, we choose h_j to be a unit column $\mathbb{C}^n_k$-vector-valued solution to equation (34) at $\lambda=\lambda_j$ in the complementary space of $\operatorname{span}_{\mathbb{C}}(h_1, h_2, \dots, h_{j-1})$ $(j=3, , \dots, n)$ in $\mathbb{C}^n_k$. The above process can be continuously displayed to $j=n$ since for any real spectral parameter λ, two coefficients on the right-hand side of equation (34) belong to $u(k, n-k) = u(\nu, k-\nu, n-k)$. The obtained matrix $H=(h_1, h_2, \dots, h_n)$ is thus of $U(k, n-k)$-valued. We claim that

$$S = H\Lambda H^{-1} \tag{42}$$

is a solution to (37) and (40). In fact, since h_j is a solution to equation (34) at $\lambda=\lambda_j$, we have $(h_j)_x = -i\lambda_j \boldsymbol{I}_3 h_j - Qh_j$ and

$$(h_j)_t = \sum_{k=0}^{2} P_k \lambda_j^k h_j,$$

which means that

$$H_x = -\boldsymbol{I}_3 H\Lambda - QH, \quad H_t = P_0 H - iP_1 H\Lambda - P_2 H\Lambda^2.$$

Hence

$$S_x = H_x\Lambda H^{-1} - H\Lambda H^{-1} H_x H^{-1} = [H_x H^{-1}, S] = -[\boldsymbol{I}_3 S + Q, S],$$
$$S_t = [H_t H^{-1}, S] = \left[P_0 - iP_1 S - P_2 S^2, S\right],$$

which indicates that S satisfies equations (37) and (40). This verifies the claim. Hence $(i\lambda I_n - S)$ with S given in (42) is a Darboux matrix.

Finally, we must to show that $\tilde{Q}$ given by (36) possesses the form

$$\tilde{Q} = \tilde{Q}(t,x) = \begin{pmatrix} 0 & \tilde{\Omega} \\ \tilde{\Omega}^\dagger & 0 \end{pmatrix}$$

for some block matrix $\tilde{\Omega}$, under an additional condition that $\mu := k-\nu = n-k \geq 1$.

In fact, since $H \in U(k, n-k)$, we have $H^{-1} = \varepsilon H^* \varepsilon$. Therefore, $S = H \Lambda H^{-1} = H \Lambda \varepsilon H^* \varepsilon$ admits

$$S^* = -\varepsilon H \varepsilon \Lambda H^* = -\varepsilon H \Lambda \varepsilon H^* = -\varepsilon H \Lambda \varepsilon H^* \varepsilon^2 = -\varepsilon S \varepsilon.$$

This implies that $S \in u(\mu+\nu, \mu) = u(\nu, \mu, \mu) \cong u^*(\nu, \mu, \mu)$. By applying Lemma 3, we take S to be of the form

$$\begin{pmatrix} A & \Gamma \\ -\Gamma^\dagger & B \end{pmatrix}$$

for some $A \in u(\nu)$, $B \in u(\mu)$ and a $\nu \times (2\mu)$ matrix $\Gamma = \begin{pmatrix} \Gamma_1^* & \Gamma_2^* \end{pmatrix}$ with Γ_1 and Γ_2 being of $\nu \times \mu$ complex matrices. From this choice we obtain that

$$[\boldsymbol{l}_3, S] = 2 \begin{pmatrix} 0 & \Gamma \\ \Gamma^\dagger & 0 \end{pmatrix}$$

and hence by (36) we see that

$$\tilde{Q} = Q + [\boldsymbol{l}_3, S] = \begin{pmatrix} 0 & \tilde{\Omega} \\ \tilde{\Omega}^\dagger & 0 \end{pmatrix}$$

for $\tilde{\Omega} = \Omega + 2\Gamma$. One notes that the linear operation $\Omega + 2\Gamma$ between Ω and Γ works well under the additional condition: $k-\nu = n-k \geq 1$. Without this condition, the linear operation $\Omega + 2\Gamma$ does not work in general and hence such a Darboux transformation $Q \to \tilde{Q}$ is not obtained.

To summarize, we obtain the following Darboux transformation.

Theorem 3. *The Darboux transformation for equation* (15) (*or equivalently equation* (18)) *with* $\mu := k-\nu = n-k \geq 1$ *is*

$$Q \to \tilde{Q} = Q + [\boldsymbol{l}_3, S], \tag{43}$$

where $Q \in \boldsymbol{m} \subset u(\nu, \mu, \mu)$ *is a given solution to equation* (18), $S = H \Lambda H^{-1} \in u(\mu + \nu, \mu)$ *is constructed by* (42) *and represented by* $u^*(\nu, \mu, \mu)$.

We end this section with the construction of 1-soliton solutions by the Darboux transformation (43). One knows that 1-soliton solutions come from the trivial solution $Q = 0$ to equation (18) by Darboux transformation. For the Lax pair (34) of equation (18) with $Q = 0$,

$$\begin{cases} \Phi_x = -i\lambda \boldsymbol{l}_3 \Phi, \\ \Phi_t = 2i\lambda^2 \boldsymbol{l}_3 \Phi, \end{cases}$$

one obtains $U(k, n-k)$-solutions at $\lambda = \lambda_j$ $(j = 1, 2, \dots, n)$:

$$H = \operatorname{diag}\left(e^{-i\Lambda_1 x + 2i\Lambda_1^2 t}, e^{i\Lambda_2 x - 2i\Lambda_2^2 t}, e^{i\Lambda_3 x - 2i\Lambda_3^2 t}\right) U_0,$$

where $\Lambda_1 = \operatorname{diag}(\lambda_1, \dots, \lambda_\nu)$, $\Lambda_2 = \operatorname{diag}(\lambda_{\nu+1}, \dots, \lambda_k)$, $\Lambda_3 = \operatorname{diag}(\lambda_{k+1}, \dots, \lambda_n)$ and U_0 is a $U(k, n-k)$-matrix independent of x and t.

Writing $U_0 = \begin{pmatrix} U_1 & U_2 & U_3 \\ U_4 & U_5 & U_6 \\ U_7 & U_8 & U_9 \end{pmatrix}$ and noting that $\Lambda = \mathrm{diag}(i\Lambda_1, i\Lambda_2, i\Lambda_3)$, we have

$$U_0 \Lambda U_0^{-1} = \begin{pmatrix} \alpha_0 & \Gamma_0 \\ \Gamma_0^\dagger & \beta_0 \end{pmatrix},$$

where $\Gamma_0 = \left(\Gamma_1^0,\ \Gamma_2^0\right)$ and

$$\alpha_0 = U_1 i\Lambda_1 U_1^* + U_2 i\Lambda_2 U_2^* - U_3 i\Lambda_3 U_3^*,$$
$$\Gamma_1^0 = U_1 i\Lambda_1 U_4^* + U_2 i\Lambda_2 U_5^* - U_3 i\Lambda_3 U_6^*,$$
$$\Gamma_2^0 = -U_1 i\Lambda_1 U_7^* - U_2 i\Lambda_2 U_8^* + U_3 i\Lambda_3 U_9^*,$$
$$\beta_0 = \begin{pmatrix} U_4 i\Lambda_1 U_4^* + U_5 i\Lambda_2 U_5^* - U_6 i\Lambda_3 U_6^* & -U_4 i\Lambda_1 U_7^* - U_5 i\Lambda_2 U_8^* + U_6 i\Lambda_3 U_9^* \\ U_7 i\Lambda_1 U_4^* + U_8 i\Lambda_2 U_5^* - U_9 i\Lambda_3 U_6^* & -U_7 i\Lambda_1 U_7^* - U_8 i\Lambda_2 U_8^* + U_9 i\Lambda_3 U_9^* \end{pmatrix}.$$

Hence, by (42),

$$S = H\Lambda H^{-1} = \begin{pmatrix} e^{-i\Lambda_1 x + 2i\Lambda_1^2 t}\ \alpha_0\ e^{i\Lambda_1 x - 2i\Lambda_1^2 t} & e^{-i\Lambda_1 x + 2i\Lambda_1^2 t}\ \Gamma_0\ e^{-i\Lambda_{23} x + 2i\Lambda_{23}^2 t} \\ e^{i\Lambda_{23} x - 2i\Lambda_{23}^2 t}\ \Gamma_0^\dagger\ e^{i\Lambda_1 x - 2i\Lambda_1^2 t} & e^{i\Lambda_{23} x - 2i\Lambda_{23}^2 t}\ \beta_0\ e^{-i\Lambda_{23} x + 2i\Lambda_{23}^2 t} \end{pmatrix},$$

in which $\Lambda_{23} = \mathrm{diag}\,(\Lambda_2, \Lambda_3)$. Therefore, under the condition that $k - \nu = n - k$ and applying Lemma 3 (notice that Ω is represented as $\Omega = (\Omega_1^*\ \Omega_2^*)$), (43) gives general 1-soliton solutions to equation (15) as follows:

$$\Omega_1 = 2e^{i\Lambda_3 x - 2i\Lambda_3^2 t}\ \Gamma_2^{0*}\ e^{i\Lambda_1 x - 2i\Lambda_1^2 t}, \quad \Omega_2 = 2e^{i\Lambda_2 x - 2i\Lambda_2^2 t}\ \Gamma_1^{0*}\ e^{i\Lambda_1 x - 2i\Lambda_1^2 t}.$$

Particularly, when $n = 3$, $k = 2$, $\nu = 1$ and taking U_0 to be

$$U_0 = \begin{pmatrix} \cos\theta_0 & \sin\theta_0 \cosh\varphi_0 & \sin\theta_0 \sinh\varphi_0 \\ -\sin\theta_0 & \cos\theta_0 \cosh\varphi_0 & \cos\theta_0 \sinh\varphi_0 \\ 0 & \sinh\varphi_0 & \cosh\varphi_0 \end{pmatrix}$$

for some $\theta_0 \in [0, 2\pi)$ and $\varphi_0 \in (-\infty, +\infty)$, we have 1-soliton solutions to (5) as follows:

$$\varphi_1 = 2ie^{i(\lambda_1+\lambda_3)x - 2i(\lambda_1^2+\lambda_3^2)t}(\lambda_2 - \lambda_3)\sin\theta_0 \sinh\varphi_0 \cosh\varphi_0,$$
$$\varphi_2 = 2ie^{i(\lambda_1+\lambda_2)x - 2i(\lambda_1^2+\lambda_2^2)t}\cos\theta_0 \sin\theta_0(\lambda_1 - \lambda_2\cosh^2\varphi_0 + \lambda_3\sinh^2\varphi_0).$$

One may continue to construct 2-soliton solutions and so on based on the Darboux transformation (43) and 1-solitons. But the complexity of computations will be exponentially increased with the order of solitons and we leave it for future study. The analytic and geometric properties of the general matrix Manakov equation (19) also deserve to be future investigated.

Acknowledgement

Qing Ding is supported by the National Natural Science Foundation of China (no. 12141104). Shiping Zhong is supported by the Natural Science Foundation of Jiangxi Province (no. 20232BAB201006), the National Natural Science Foundation of China (no. 12561010) and the Degree and Graduate Education Teaching Reform Research Project of Jiangxi Province (no. JXYJG-2023-169).

References

[1] A. Agalarov, V. Zhulego, and T. Gadzhimuradov, "Bright, dark, and mixed vector soliton solutions of the general coupled nonlinear Schrödinger equations", *Phys. Rev. E* **91**:4 (2015), art. id. 042909, 7 pp. MR

[2] N. Akhmediev, W. Krolikowski, and A. W. Snyder, "Partially coherent solitons of variable shape", *Phys. Rev. Lett.* **81**:21 (1998), 4632–4635.

[3] F. Baronio, A. Degasperis, M. Conforti, and S. Wabnitz, "Solutions of the vector nonlinear Schrödinger equations: evidence for deterministic rogue waves", *Phys. Rev. Lett.* **109**:4 (2012), art. id. 044102.

[4] F. Baronio, M. Conforti, A. Degasperis, S. Lombardo, M. Onorato, and S. Wabnitz, "Vector rogue waves and baseband modulation instability in the defocusing regime", *Phys. Rev. Lett.* **113**:3 (2014), art. id. 034101.

[5] W.-J. Che, S.-C. Chen, C. Liu, L.-C. Zhao, and N. Akhmediev, "Nondegenerate Kuznetsov–Ma solitons of Manakov equations and their physical spectra", *Phys. Rev. A* **105**:4 (2022), art. id. 043526, 14 pp. MR

[6] Q. Ding and Y. Wang, "Vortex filament on symmetric Lie algebras and generalized bi-Schrödinger flows", *Math. Z.* **290**:1-2 (2018), 167–193. MR

[7] Q. Ding and S. Zhong, "On geometric realization of the general Manakov system", *Chinese Ann. Math. Ser. B* **44**:5 (2023), 753–764. MR

[8] Q. Ding, S. Zhong, and D. Ma, "A geometric characterization of a kind of Manakov systems", *Scientia Sinica Math.* **54**:10 (2024), 1509–1520. In Chinese.

[9] A. P. Fordy and P. P. Kulish, "Nonlinear Schrödinger equations and simple Lie algebras", *Comm. Math. Phys.* **89**:3 (1983), 427–443. MR

[10] C. Gu, H. Hu, and Z. Zhou, *Darboux transformations in integrable systems: theory and their applications to geometry*, Mathematical Physics Studies **26**, Springer, 2005. MR

[11] U. A. Kahwaja and L. A. Sakkaf, *Handbook of exact solutions to the nonlinear Schrödinger equations*, 2nd ed., IOP Publising, Bristol, 2020.

[12] T. Kanna, M. Lakshmanan, P. Tchofo Dinda, and N. Akhmediev, "Soliton collisions with shape change by intensity redistribution in mixed coupled nonlinear Schrödinger equations", *Phys. Rev. E* **73**:2 (2006), art. id. 026604, 15 pp. MR

[13] J. Langer and R. Perline, "Geometric realizations of Fordy–Kulish nonlinear Schrödinger systems", *Pacific J. Math.* **195**:1 (2000), 157–178. MR

[14] S. V. Manakov, "On the theory of two-dimensional stationary self-focusing electro-magnetic waves", *Sov. Phys. JETP* **38**:2 (1974), 248–253.

[15] Y. Nogami and C. S. Warke, "Soliton solutions of multicomponent nonlinear Schrödinger equation", *Phys. Lett. A* **59**:4 (1976), 251–253. MR

[16] B. O'Neill, *Semi-Riemannian geometry*: *with applications to relativity*, Pure and Applied Mathematics **103**, Academic Press, New York, 1983. MR

[17] R. Radha, P. S. Vinayagam, and K. Porsezian, "Rotation of the trajectories of bright solitons and realignment of intensity distribution in the coupled nonlinear Schrödinger equation", *Phys. Rev. E* **88**:3 (2013), art. id. 032903.

[18] C.-L. Terng and K. Uhlenbeck, "Bäcklund transformations and loop group actions", *Comm. Pure Appl. Math.* **53**:1 (2000), 1–75. MR

[19] C.-L. Terng and K. Uhlenbeck, "Schrödinger flows on Grassmannians", pp. 235–256 in *Integrable systems*, *geometry*, *and topology*, AMS/IP Stud. Adv. Math. **36**, Amer. Math. Soc., Providence, RI, 2006. MR

[20] M. Vijayajayanthi, T. Kanna, and M. Lakshmanan, "Bright-dark solitons and their collisions in mixed N-coupled nonlinear Schrödinger equations", *Phys. Rev. A* **77**:1 (2008), art. id. 013820.

[21] N. Vishnu Priya, M. Senthilvelan, and M. Lakshmanan, "Dark solitons, breathers, and rogue wave solutions of the coupled generalized nonlinear Schrödinger equations", *Phys. Rev. E* **89**:6 (2014), art. id. 062901.

[22] C. Yeh and L. Bergman, "Enhanced pulse compression in a nonlinear fiber by a wavelength division multiplexed optical pulse", *Phys. Rev. E* **57**:2 (1998), 2398–2404.

Received May 19, 2025. Revised August 21, 2025.

QING DING
DEPARTMENT OF MATHEMATICS
WENZHOU UNIVERSITY
WENZHOU, 325035
CHINA

and

SCHOOL OF MATHEMATICAL SCIENCES
FUDAN UNIVERSITY
SHANGHAI, 200433
CHINA

qding@fudan.edu.cn

CHAOHAO YE
DEPARTMENT OF MATHEMATICS
WENZHOU UNIVERSITY
WENZHOU, 325035
CHINA

23450829032@stu.wzu.edu.cn

SHIPING ZHONG
SCHOOL OF MATHEMATICS AND COMPUTER SCIENCES
GANNAN NORMAL UNIVERSITY
GANZHOU, 341000
CHINA

zhongshiping@gnnu.edu.cn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 339, No. 2, 2025
https://doi.org/10.2140/pjm.2025.339.283

MAPPING CLASSES FIXING AN ISOTROPIC HOMOLOGY CLASS OF MINIMAL GENUS 0 IN RATIONAL 4-MANIFOLDS

SERAPHINA EUN BI LEE

For any $N \geq 1$, let M_N denote the rational 4-manifold $\mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$. We study the stabilizer $\mathrm{Stab}(w)$ of a primitive, isotropic class $w \in H_2(M_N; \mathbb{Z})$ of minimal genus 0 under the natural action of the topological mapping class group $\mathrm{Mod}(M_N)$ on $H_2(M_N; \mathbb{Z})$. Although most elements of $\mathrm{Stab}(w)$ cannot be represented by homeomorphisms that preserve any Lefschetz fibration $M_N \to \Sigma$, we show that every element of $\mathrm{Stab}(w)$ can be represented by a diffeomorphism that *almost preserves* a holomorphic, genus-0 Lefschetz fibration $\mathrm{proj}: M_N \to \mathbb{CP}^1$ whose generic fibers represent the homology class w. We also answer the Nielsen realization problem for a certain maximal torsion-free, abelian subgroup Λ_w of $\mathrm{Mod}(M_N)$ by finding a lift of Λ_w to $\mathrm{Diff}^+(M_N) \leq \mathrm{Homeo}^+(M_N)$ under the quotient map $q : \mathrm{Homeo}^+(M_N) \to \mathrm{Mod}(M_N)$. This lift of Λ_w can be made to almost preserve $\mathrm{proj}: M_N \to \mathbb{CP}^1$. All results of this paper also hold for every primitive, isotropic class $w \in H_2(M_N; \mathbb{Z})$ if $N \leq 8$ because any such class has minimal genus 0.

1. Introduction

The (topological) *mapping class group* $\mathrm{Mod}(M)$ of a closed, oriented manifold M is the group

$$\mathrm{Mod}(M) := \pi_0(\mathrm{Homeo}^+(M))$$

of isotopy classes of orientation-preserving homeomorphisms of M. There is a natural action of $\mathrm{Mod}(M)$ on $H_2(M; \mathbb{Z})$ preserving the intersection form Q_M and we consider the stabilizer $\mathrm{Stab}(w) \leq \mathrm{Mod}(M)$ of any class $w \in H_2(M; \mathbb{Z})$.

Suppose M is a smooth, simply connected 4-manifold. If $w \in H_2(M; \mathbb{Z})$ is a nonzero homology class with self-intersection 0 then w is called *isotropic*. One way in which isotropic classes arise are as the homology class of the generic fibers of a Lefschetz fibration $p : M \to \Sigma$ where Σ is a closed, oriented surface. We say that a diffeomorphism φ of M *preserves* p if there exists some diffeomorphism ψ of Σ such that $p \circ \varphi = \psi \circ p$.

MSC2020: 57K40, 57S25.
Keywords: rational surface, mapping class group, Nielsen realization problem, Lefschetz fibration.

In some settings, elements $g \in \mathrm{Stab}(w) \leq \mathrm{Mod}(M)$ are known to admit representative maps φ that preserve some Lefschetz fibration $p: M \to \Sigma$ whose generic fibers represent the homology class w. For example, Gizatullin ([12]) showed that any parabolic automorphism of a compact Kähler surface M must preserve some elliptic fibration $M \to \Sigma$ (also see [4, Proposition 1.4] or [7, Theorem 4.3, Appendix]). In the smooth setting, Farb–Looijenga ([9, Theorem 1.11]) showed that on a K3 manifold, any $g \in \mathrm{Stab}(w)$ can be represented by a diffeomorphism preserving the fibers of some holomorphic elliptic fibration $M \to \mathbb{CP}^1$; this result is an application of their study of the moduli space of genus one fibered K3 surfaces with only nodal singular fibers. As another application, Farb–Looijenga ([9, Corollary 1.12]) study the Nielsen realization problem for a certain rank-20 free abelian subgroup of $\mathrm{Stab}(w)$ by diffeomorphisms preserving the fibers of a given genus one fibration of a K3 manifold.

In this paper we study representative maps of the stabilizers of isotropic classes of *rational manifolds* M and their relationships to genus-0 Lefschetz fibrations $M \to \Sigma$. More specifically, we study manifolds of the form

$$M_N := \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2} \qquad \text{for } N \geq 1,$$

which are the underlying smooth 4-manifolds of the blowup of $\mathbb{CP}^2$ at N points. The total space M of a nontrivial genus-0 Lefschetz fibration $M \to \Sigma$ is a rational manifold M_N for some N if M is simply connected. If $N \leq 8$, all primitive, isotropic classes $w \in H_2(M_N; \mathbb{Z})$ are represented by generic fibers of a genus-0, holomorphic Lefschetz fibration $p: M_N \to \mathbb{CP}^1$. We sometimes refer to such a Lefschetz fibration as a *conic bundle structure on M_N*. Note that these Lefschetz fibrations are not relatively minimal unless $N = 1$. See Section 2.3.

Representing* $\mathbf{Stab}(w)$ *by diffeomorphisms. Let $N \geq 1$ and let $w \in H_2(M_N; \mathbb{Z})$ be any primitive, isotropic class of minimal genus 0. Although any such class w is represented by a generic fiber of a genus-0 Lefschetz fibration $p: M_N \to \mathbb{CP}^1$, the following proposition shows that there does not exist any $\varphi \in \mathrm{Homeo}^+(M_N)$ with $[\varphi] \in \mathrm{Stab}(w)$ that preserves such a fibration p if $[\varphi]$ has infinite order in $\mathrm{Mod}(M_N)$.

Proposition 1.1. *Let $N \geq 1$ and let $w \in H_2(M_N; \mathbb{Z})$ be a primitive, isotropic class of minimal genus 0. Let $\varphi \in \mathrm{Homeo}^+(M_N)$ represent an infinite-order mapping class $[\varphi] \in \mathrm{Stab}(w) \leq \mathrm{Mod}(M_N)$. There does not exist any Lefschetz fibration $p: M_N \to \Sigma$ where Σ is a closed, oriented surface and where the generic fiber represents w such that φ preserves p.*

For a proof, see Section 2.3. In this paper we ask instead that any diffeomorphism representing any infinite-order mapping class $f \in \mathrm{Stab}(w) \leq \mathrm{Mod}(M_N)$ *almost preserves* some Lefschetz fibration $p: M_N \to \Sigma$.

Definition 1.2 (almost preserving a Lefschetz fibration). A group of diffeomorphisms $G \leq \mathrm{Diff}^+(M)$ *almost preserves* a Lefschetz fibration $p : M \to \Sigma$ if the elements of G act on the fibers of p outside of disjoint neighborhoods of the singular fibers of p. More precisely, there exist

(a) disjoint, open neighborhoods $V_1, \dots, V_m \subseteq \Sigma$ of the images of the singular points $z_1, \dots, z_m \in \Sigma$, and
(b) a homomorphism $i : G \to \mathrm{Diff}^+\left(\Sigma - \bigcup_{k=1}^m V_k\right)$

such that for all $\varphi \in G$, the following commutes:

$$\begin{array}{ccc} M - \bigcup_{k=1}^m p^{-1}(V_k) & \xrightarrow{\varphi} & M - \bigcup_{k=1}^m p^{-1}(V_k) \\ \downarrow p & & \downarrow p \\ \Sigma - \bigcup_{k=1}^m V_k & \xrightarrow{i(\varphi)} & \Sigma - \bigcup_{k=1}^m V_k \end{array}$$

A diffeomorphism $\varphi \in \mathrm{Diff}^+(M)$ *almost preserves* a Lefschetz fibration $p : M \to \Sigma$ if the group $\langle \varphi \rangle \leq \mathrm{Diff}^+(M)$ almost preserves $p : M \to \Sigma$.

On the other hand, any element of $\mathrm{Stab}(w) \leq \mathrm{Mod}(M_N)$ with $N \geq 2$ must preserve the following subgroup of $H_2(M_N; \mathbb{Z})$:

$$w^\perp := \{w_0 \in H_2(M_N; \mathbb{Z}) : Q_{M_N}(w, w_0) = 0\} \cong \mathbb{Z}^N.$$

Thus $\mathrm{Stab}(w)$ acts on the lattice $(w^\perp/\mathbb{Z}\{w\}, \overline{Q}_{M_N})$ where $\overline{Q}_{M_N}$ is the unimodular, symmetric, bilinear form on $w^\perp/\mathbb{Z}\{w\}$ induced by Q_{M_N}. Since $(H_2(M_N; \mathbb{Z}), Q_{M_N})$ has signature $(1, N)$, $(w^\perp/\mathbb{Z}\{w\}, \overline{Q}_{M_N})$ must be negative definite of rank $N - 1$.

Definition 1.3. Let Λ_w be the kernel of the map $\mathrm{Stab}(w) \to \mathrm{Aut}(w^\perp/\mathbb{Z}\{w\}, \overline{Q}_{M_N})$.

There is an identification of Λ_w with the subgroup of even elements of the lattice $(w^\perp/\mathbb{Z}\{w\}, \overline{Q}_{M_N})$, and $\mathrm{Stab}(w)$ fits into a split short exact sequence

$$0 \to \underbrace{\Lambda_w}_{\cong \mathbb{Z}^{N-1} \leq w^\perp/\mathbb{Z}\{w\}} \to \mathrm{Stab}(w) \to \mathrm{Aut}(w^\perp/\mathbb{Z}\{w\}, \overline{Q}_{M_N}) \to 0. \tag{1}$$

Two properties of Λ_w are that it is a maximal torsion-free, abelian subgroup of $\mathrm{Mod}(M_N)$ and that it has finite index in $\mathrm{Stab}(w)$. See Lemmas 2.5 and 2.6.

With the preliminaries above in hand, we state our main result concerning the Nielsen realization problem for Λ_w.

Theorem 1.4 (realizing Λ_w by diffeomorphisms). *Let $N \geq 2$ and let $w \in H_2(M_N; \mathbb{Z})$ be a primitive, isotropic class of minimal genus* 0. *There exists a homomorphism*

$\rho_w : \Lambda_w \to \mathrm{Diff}^+(M_N)$ *such that the following diagram commutes*:

$$\begin{array}{ccc} & & \mathrm{Diff}^+(M_N) \\ & \overset{\rho_w}{\nearrow} & \downarrow q \\ \Lambda_w & \hookrightarrow & \mathrm{Mod}(M_N) \end{array}$$

The image $\rho_w(\Lambda_w)$ *almost preserves a holomorphic genus-*0 *Lefschetz fibration* $p : M_N \to \mathbb{CP}^1$ *whose generic fiber represents the homology class* w.

If $N \le 8$, Theorem 1.4 holds for any primitive, isotropic class $w \in H_2(M_N; \mathbb{Z})$ because any such class has minimal genus 0. See Corollary 3.10.

Compare Theorem 1.4 to the case of the K3 manifold M for which the subgroup $\Lambda_w \le \mathrm{Mod}(M)$ is isomorphic to $\mathbb{Z}^{20}$, where $w \in H_2(M; \mathbb{Z})$ is a fiber class of a genus-1 fibration of M with only nodal fibers. As mentioned above, Farb–Looijenga ([9, Corollary 1.12]) showed that Λ_w lifts to the group of diffeomorphisms preserving the fibers of the given genus-1 fibration. In contrast to the case of the K3 manifold, Theorem 1.4 shows that Nielsen realization for Λ_w holds in our setting despite the fact that no element of Λ_w can preserve any genus-0 Lefschetz fibration of M_N (Proposition 1.1).

The next theorem uses the short exact sequence (1) and the diffeomorphisms constructed in the proof of Theorem 1.4 to find a diffeomorphism representative of any element of $\mathrm{Stab}(w)$ that almost preserves a genus-0 Lefschetz fibration.

Theorem 1.5 (mapping classes fixing an isotropic class). *Let* $N \ge 1$ *and let* $w \in H_2(M_N; \mathbb{Z})$ *be a primitive, isotropic class of minimal genus* 0. *For any* $h \in \mathrm{Stab}(w)$, *there exists* $\varphi \in \mathrm{Diff}^+(M_N)$ *almost preserving a holomorphic genus-*0 *Lefschetz fibration* $p : M_N \to \mathbb{CP}^1$ *whose generic fiber represents the homology class* w *such that* $[\varphi] = h \in \mathrm{Mod}(M_N)$.

Similarly as with Theorem 1.4, Theorem 1.5 holds for any primitive, isotropic class $w \in H_2(M_N; \mathbb{Z})$ if $2 \le N \le 8$. See Corollary 4.2.

A large part of the work of this paper is to ensure that the diffeomorphisms constructed in Section 3 commute as diffeomorphisms of M_N. We point out that the calculations of Section 3 are essential to the proof of Theorem 1.4 regarding the Nielsen realization problem for Λ_w although Theorem 1.5 alone may be proven more succinctly. For the sake of concreteness, we give explicit constructions of all diffeomorphisms used in this paper.

One way to interpret the results of this paper is via the natural action of (an index-2 subgroup of) $\mathrm{Mod}(M_N)$ on $\mathbb{H}^N$ and the classification of hyperbolic isometries into three types: elliptic, parabolic, and hyperbolic. Infinite-order elements of the stabilizer $\mathrm{Stab}(w)$ for an isotropic class $w \in H_2(M_N; \mathbb{Z})$ are precisely the elements of $\mathrm{Mod}(M_N)$ acting by parabolic isometries on $\mathbb{H}^N$ (Lemma 2.2). Therefore the following is an immediate corollary of Theorem 1.5.

Corollary 1.6. *Let* $2 \le N \le 8$. *If* $g \in \mathrm{Mod}(M_N)$ *acts by a parabolic isometry on* $\mathbb{H}^N$ *then there exists* $\varphi \in \mathrm{Diff}^+(M_N)$ *that almost preserves a holomorphic genus-*0 *Lefschetz fibration* $p : M_N \to \mathbb{CP}^1$ *such that* $[\varphi] = g$.

Related work. The relationship between mapping classes of 4-manifolds fixing an isotropic class and Lefschetz fibrations with the prescribed generic fiber has been studied in some settings. As mentioned above, see Gizatullin [12] and Cantat [4] for the case of compact, Kähler surfaces and elliptic fibrations and Farb–Looijenga [9] for the case of K3 manifolds; [9] was an inspiration for this current paper.

Automorphisms preserving a genus-0 Lefschetz fibration (or a conic bundle structure) also play an important role in the study of finite groups of automorphisms of M_N. An example of such a complex automorphism is the *de Jonquières* involution, which is a main tool for this paper. Some examples of work in this direction include the classification of order-2 birational automorphisms of $\mathbb{CP}^2$ up to conjugacy (Bertini [2], Bayle–Beauville [1]) and finite subgroups of birational automorphisms of $\mathbb{CP}^2$ in general (Dolgachev–Iskovskikh [8], Blanc [3]) in the complex category and a study of finite groups of symplectomorphisms of rational surfaces (Chen–Li–Wu [5]) in the symplectic category.

Outline. In Section 2, we recall relevant facts about the mapping class group $\mathrm{Mod}(M_N)$ of rational manifolds and deduce basic facts about isotropic classes $w \in H_2(M_N; \mathbb{Z})$, including the proof of Proposition 1.1. In Section 3, we prove Theorem 1.4 by explicitly constructing the necessary diffeomorphisms. Using these diffeomorphisms from Section 3, we prove Theorem 1.5 in Section 4.

2. Isotropic homology classes and their stabilizers in $\mathrm{Mod}(M_N)$

We collect useful properties of the mapping class groups of 4-manifolds, isotropic classes in $H_2(M_N; \mathbb{Z})$, and certain Lefschetz fibrations.

2.1. ***The mapping class group of M_N.*** For any 4-manifold M, let Q_M denote the intersection form on $H_2(M; \mathbb{Z})$. The form Q_M is an integral, unimodular, nondegenerate, symmetric bilinear form on $H_2(M; \mathbb{Z})/\mathrm{Torsion}$; we denote the lattice $(H_2(M; \mathbb{Z})/\mathrm{Torsion}, Q_M)$ by H_M. The automorphism group of the lattice H_M is denoted $\mathrm{O}(\mathrm{H}_M)$.

The mapping class group $\mathrm{Mod}(M) := \pi_0(\mathrm{Homeo}^+(M))$ of a closed, oriented, simply connected 4-manifold M is computable due to the following theorems of Freedman [10], Perron [18], Quinn [19], Cochran–Habegger [6], and Gabai–Gay–Hartman–Krushkal–Powell [11]. (For a more detailed history of this theorem, see [11, Section 1.3].)

Theorem 2.1. *Let M^4 be a closed, oriented, and simply connected manifold. The map*

$$\Phi : \mathrm{Mod}(M) \to \mathrm{O}(\mathrm{H}_M)$$

given by $\Phi : [\varphi] \mapsto \varphi_*$ is an isomorphism of groups.

By the Mayer–Vietoris sequence, $H_2(M_N; \mathbb{Z}) = H_2(\mathbb{CP}^2; \mathbb{Z}) \oplus H_2(\overline{\mathbb{CP}^2}; \mathbb{Z})^{\oplus N}$, and we have the usual $\mathbb{Z}$-basis $\{H, E_1, \dots, E_N\}$. The intersection form Q_{M_N} is given by the diagonal, $(N+1) \times (N+1)$ matrix

$$\mathrm{diag}(1, -1, \dots, -1)$$

with respect to the $\mathbb{Z}$-basis $\{H, E_1, \dots, E_N\}$. On the other hand, there is a natural $\mathbb{Z}$-basis

$$\{s, v, e_1, \dots, e_{N-1}\} \tag{2}$$

of $H_2((\mathbb{CP}^1 \times \mathbb{CP}^1)\#(N-1)\overline{\mathbb{CP}^2}; \mathbb{Z})$ via the Mayer–Vietoris sequence; here, s and v correspond to the first and second factors of $\mathbb{CP}^1 \times \mathbb{CP}^1$ respectively. There is a diffeomorphism $(\mathbb{CP}^1 \times \mathbb{CP}^1)\#(N-1)\overline{\mathbb{CP}^2} \cong M_N$ for all $N \geq 2$ giving an identification

$$v = H - E_1, \quad s = H - E_2, \quad e_1 = H - E_1 - E_2, \quad e_k = E_{k+1} \text{ for all } 2 \leq k \leq N-1.$$

We will mostly work with the $\mathbb{Z}$-basis $\{s, v, e_1, \dots, e_{N-1}\}$ of $H_2(M_N; \mathbb{Z})$.

Therefore by Theorem 2.1,

$$\mathrm{Mod}(M_N) \cong \mathrm{O}(1, N)(\mathbb{Z}) := \mathrm{O}(\mathrm{H}_{M_N})$$

We will identify $\mathrm{O}(\mathrm{H}_{M_N})$ and $\mathrm{Mod}(M_N)$ throughout this paper.

On the other hand, consider $\mathbb{E}^{1,N} := (\mathbb{R}^{N+1}, Q_N)$, where Q_N is the diagonal bilinear symmetric form of signature $(1, N)$:

$$Q_N((x_0, x_1, \dots, x_N), (y_0, y_1, \dots, y_N)) = x_0 y_0 - x_1 y_1 - \cdots - x_N y_N.$$

There is a natural identification of the $\mathbb{R}$-span of the $\mathbb{Z}$-basis $\{H, E_1, \dots, E_N\}$ of $H_2(M_N; \mathbb{Z})$ with $\mathbb{R}^{N+1}$, under which the $\mathbb{R}$-bilinear extension of Q_{M_N} coincides with Q_N. The hyperboloid model for $\mathbb{H}^N$ sits in $\mathbb{E}^{1,N}$ by

$$\mathbb{H}^N = \{w = (w_0, w_1, \dots, w_N) \in \mathbb{R}^{N+1} : Q_N(w, w) = 1,\ w_0 > 0\}.$$

where the Riemannian metric is defined by the restriction of $-Q_N$ to $\mathbb{H}^N$ (see [20, Chapter 2]). Because $\mathrm{O}(1, N)(\mathbb{Z})$ acts on $\mathbb{R}^{N+1}$ and preserves Q_N, it contains an index-2 subgroup $\mathrm{O}^+(1, N)(\mathbb{Z})$ acting by isometries on $\mathbb{H}^N$.

The boundary sphere of $\mathbb{H}^N$ corresponds to

$$\partial\mathbb{H}^N = \{w = (w_0, w_1, \dots, w_N) \in \mathbb{R}^{N+1} : Q_N(w, w) = 0,\ w_0 > 0\}/\sim$$

where $aw \sim w$ for all $a \in \mathbb{R}_{>0}$. Parabolic isometries of $\mathbb{H}^N$ are those that fix a unique point of $\partial\mathbb{H}^n$ and no point in $\mathbb{H}^n$. By [20, Problem 2.5.24(g)], parabolic isometries not only preserve some line in $\mathbb{R}^{N+1}$ but fix it pointwise. Moreover, parabolic isometries in $\mathrm{O}^+(1, N)(\mathbb{Z})$ must fix a nonzero, isotropic vector with integral entries, i.e., some nonzero $w \in H_2(M_N; \mathbb{Z})$ with $Q_{M_N}(w, w) = 0$, and have infinite order ([20, Exercise 2.5.20]). The converse is true as well:

Lemma 2.2. *Let $N \geq 2$. An element $f \in \mathrm{O}^+(1, N)(\mathbb{Z})$ acts by a parabolic isometry if and only if f has infinite order and there exists some primitive, isotropic class $w \in H_2(M_N; \mathbb{Z})$ such that $f \in \mathrm{Stab}(w) \leq \mathrm{O}(\mathrm{H}_{M_N})$.*

Proof. One direction holds by the discussion preceding the statement of the lemma, so it suffices to prove that if $f \in \mathrm{Stab}(w)$ has infinite order then f acts on $\mathbb{H}^N$ by a parabolic isometry.

Note that f fixes the point of $\partial\mathbb{H}^N$ corresponding to $w \in H_2(M_N; \mathbb{Z})$. Because stabilizers of points in $\mathbb{H}^N$ in $\mathrm{O}(\mathrm{H}_{M_N})$ have finite order, we only need to show that this is the unique point of $\partial\mathbb{H}^N$ fixed by f. To do this, let $w_0 \in \mathbb{E}^{1,N}$ be an isotropic vector such that $f(w_0) = \lambda w_0$ for some $\lambda \in \mathbb{R}$. If $w_0 \in \mathbb{R}\{w\}^\perp$ then w_0 must be a scalar multiple of w because the restriction of Q_{M_N} to $\mathbb{R}\{w\}^\perp/\mathbb{R}\{w\}$ is negative definite. If $Q_{M_N}(w, w_0) =: a \neq 0$ then $\lambda = 1$ because

$$a = Q_{M_N}(w, w_0) = Q_{M_N}(f(w), f(w_0)) = Q_{M_N}(w, \lambda w_0) = \lambda a.$$

Then $f(aw + w_0) = aw + w_0$ and

$$Q_{M_N}(aw + w_0, aw + w_0) = 2a Q_{M_N}(w, w_0) = 2a^2 > 0.$$

A scalar multiple of $aw + w_0$ lies in $\mathbb{H}^N$, meaning f acts on $\mathbb{H}^N$ by an elliptic isometry, and all such isometries of $\mathbb{H}^N$ in $\mathrm{O}(\mathrm{H}_{M_N})$ have finite order. Therefore, w_0 must be a scalar multiple of w and hence f fixes a unique point in $\partial\mathbb{H}^N$. □

2.2. *Primitive, isotropic classes $w \in H_2(M_N; \mathbb{Z})$ and* $\mathrm{Stab}(w) \leq \mathrm{Mod}(M_N)$. Consider lattices (L, Q), where $L \cong \mathbb{Z}^r$ as an abelian group for some $r \in \mathbb{N}$ and Q is an integral, unimodular, nondegenerate, symmetric, bilinear form on L. For each primitive, isotropic vector $w \in L$, there exists $u \in L$ such that $Q(w, u) = 1$ by unimodularity of Q. There is an orthogonal decomposition

$$L = \mathbb{Z}\{u, w\} \oplus \mathbb{Z}\{u, w\}^\perp$$

to which Q restricts to a unimodular form on each factor. The restriction of Q to $\mathbb{Z}\{u, w\}$ has signature $(1, 1)$. Note that $\mathbb{Z}\{u, w\}^\perp$ is a lift of $w^\perp/\mathbb{Z}\{w\}$ under the natural quotient $w^\perp \to w^\perp/\mathbb{Z}\{w\}$. This means that $(\mathbb{Z}\{u, w\}^\perp, Q|_{\mathbb{Z}\{u,w\}^\perp})$ is isometric (i.e., isomorphic as a lattice) to $(w^\perp/\mathbb{Z}\{w\}, \overline{Q})$ via this quotient, where $\overline{Q}$ is the induced bilinear form on $w^\perp/\mathbb{Z}\{w\}$. We fix the above notation throughout this subsection.

Lemma 2.3. *Let $w \in L$ be a primitive, isotropic vector. If $h_1, h_2 \in \mathrm{Stab}(w) \leq \mathrm{O}(L, Q)$ and $h_1|_{w^\perp} = h_2|_{w^\perp}$ then $h_1 = h_2$. In particular, for any $h_1, h_2 \in \mathrm{Stab}(v) \leq \mathrm{O}(\mathrm{H}_{M_N})$, where v is the homology class as given in (2) and $N \geq 2$, if $h_1(e_k) = h_2(e_k)$ for all $1 \leq k \leq N-1$ then $h_1 = h_2$.*

Proof. Observe that $h_1^{-1} \circ h_2$ acts as the identity on $\mathbb{Z}\{u, w\}^\perp \leq w^\perp$. We claim that $h_1^{-1} \circ h_2(u) = u$. To see this, write $h_1^{-1} \circ h_2(u) = au + bw + v_0$ for some $v_0 \in \mathbb{Z}\{u, w\}^\perp$ and some $a, b \in \mathbb{Z}$. For any $v_1 \in \mathbb{Z}\{u, w\}^\perp$,

$$0 = Q(u, v_1) = Q((h_1^{-1} \circ h_2)(u), (h_1^{-1} \circ h_2)(v_1)) = Q(au + bw + v_0, v_1) = Q(v_0, v_1).$$

Therefore, $v_0 = 0$ by unimodularity of $Q|_{\mathbb{Z}\{u,w\}^\perp}$. Moreover, $(h_1^{-1} \circ h_2)(w) = w$, so

$$1 = Q((h_1^{-1} \circ h_2)(u), (h_1^{-1} \circ h_2)(w)) = Q(au + bw, w) = a,$$
$$0 = Q((h_1^{-1} \circ h_2)(u), (h_1^{-1} \circ h_2)(u)) = Q(u + bw, u + bw) = 2b.$$

Therefore, $(h_1^{-1} \circ h_2)(u) = u$ and $h_1^{-1} \circ h_2$ restricts to the identity on $\mathbb{Z}\{u, w\}$. In the case of $v \in H_2(M_N; \mathbb{Z})$ for any $N \geq 2$, apply the above argument with $w = v$, $u = s$ and $\mathbb{Z}\{u, w\}^\perp = \mathbb{Z}\{e_1, \ldots, e_{N-1}\}$. □

Let Λ_w denote the kernel of the natural map $h_w : \mathrm{Stab}(w) \to \mathrm{O}(w^\perp/\mathbb{Z}\{w\}, \overline{Q})$ (cf. Definition 1.3). In order to describe Λ_w, we introduce an important type of element of $\mathrm{O}(\mathrm{H}_{M_N})$ used throughout this paper.

Definition 2.4. Let $N \geq 2$ and $u \in H_2(M_N; \mathbb{Z})$ satisfy $Q_{M_N}(u, u) = \pm 1$ or ± 2. The *reflection* Ref_u about u is an element of $\mathrm{O}(\mathrm{H}_{M_N})$ defined by

$$\mathrm{Ref}_u(x) = x - \frac{2Q_{M_N}(x, u)}{Q_{M_N}(u, u)} u.$$

We now use reflections and Eichler transformations to give generators for Λ_w.

Lemma 2.5. *Let (L, Q) be any lattice and $w \in L$ be a primitive, isotropic vector. Let $A \leq w^\perp/\mathbb{Z}\{w\}$ denote the subgroup of even elements with respect to $\overline{Q}$. Then there is an isomorphism of groups*

$$E(w, \cdot) : A \to \Lambda_w.$$

In the case that $(L, Q) = \mathrm{H}_{M_N}$ for any $N \geq 2$ and $w = v$, the group Λ_w is generated by

$$f_k := \mathrm{Ref}_{e_k} \circ \mathrm{Ref}_{e_{k+1}} \circ \mathrm{Ref}_{v - e_k - e_{k+1}} \circ \mathrm{Ref}_{e_k - e_{k+1}}$$

for $1 \leq k \leq N-2$ and $g := \mathrm{Ref}_{e_1} \circ \mathrm{Ref}_{v - e_1}$.

Proof. For any $f \in \Lambda_w$, there exists $c(f) \in w^\perp/\mathbb{Z}\{w\}$ such that for any $e \in w^\perp$,

$$f(e) = e - \overline{Q}(c(f), e)w$$

by the definition of Λ_w and the unimodularity of $\overline{Q}$. This defines a homomorphism $c : \Lambda_w \to w^{\perp}/\mathbb{Z}\{w\}$ which is injective by Lemma 2.3.

For any $f \in \Lambda_w$, there exists $a \in \mathbb{Z}$ and $e \in \mathbb{Z}\{u, w\}^{\perp}$ such that

$$f(u) = u + aw + e$$

because $Q(f(u), w) = 1$. (Here, $u \in L$ is as chosen at the beginning of this subsection.) Moreover,

$$Q(u, u) = Q(f(u), f(u)) = Q(u, u) + 2a + Q(e, e)$$

and so $Q(e, e)$ is even. Because $c(f), e$ are contained in $\mathbb{Z}\{u, w\}^{\perp}$ where $c(f) \in w^{\perp}/\mathbb{Z}\{w\}$ is identified with its lift in $\mathbb{Z}\{u, w\}^{\perp}$,

$$\begin{aligned} 0 &= Q(f(u), f(c(f))) = Q(f(u), c(f) - Q(c(f), c(f))w) \\ &= Q(e, c(f)) - Q(c(f), c(f)), \\ 0 &= Q(f(u), f(e)) = Q(f(u), e - Q(e, c(f))w) = Q(e, e) - Q(e, c(f)). \end{aligned}$$

By the second string of equalities, $Q(e, c(f))$ is even, and by the first, $Q(c(f), c(f)) = \overline{Q}(c(f), c(f))$ is even. Hence $c(\Lambda_w) \leq A$.

Consider the homomorphism $E(w, \cdot\,) : A \to \Lambda_w$ defined by

$$E(w, e) : x \mapsto x + Q(w, x)e - Q(e, x)w - \frac{1}{2}Q(e, e)Q(w, x)w$$

for each $[e] \in A \leq w^{\perp}/\mathbb{Z}\{w\}$ with $e \in w^{\perp}$, where $E(w, e)$ is an *Eichler transformation*. A computation shows that $E(w, \cdot\,)$ does not depend on the choice of lift $e \in w^{\perp}$, and hence descends to a well-defined homomorphism on $A \leq w^{\perp}/\mathbb{Z}\{w\}$. Another computation shows that $c \circ E(w, \cdot\,) = \mathrm{Id}\,|_A$. Finally, if $(L, Q) = (H_2(M_N; \mathbb{Z}), Q_{M_N})$ and $w = v$, compute that $f_k = E(w, e_k + e_{k+1})$ for each $1 \leq k \leq N - 2$ and $g = E(w, 2e_1)$, which together generate Λ_w as $e_k + e_{k+1}$ with $1 \leq k \leq N - 2$ and $2e_1$ generate A. □

We combine the results of this subsection and record an important algebraic property of $\mathrm{Stab}(w)$. Below, $\mathrm{O}(r)(\mathbb{Z})$ denotes the automorphism group of the diagonal lattice $(\mathbb{Z}^r, \mathrm{diag}(1, \ldots, 1))$, or equivalently, the automorphism group of the diagonal lattice $(\mathbb{Z}^r, \mathrm{diag}(-1, \ldots, -1))$.

Lemma 2.6. *For any primitive, isotropic vector $w \in L$, there is a split short exact sequence*

$$0 \to \Lambda_w \to \mathrm{Stab}(w) \xrightarrow{h_w} \mathrm{O}(w^{\perp}/\mathbb{Z}\{w\}, \overline{Q}) \to 0.$$

In the case that $(L, Q) = \mathrm{H}_{M_N}$ for any $N \geq 2$ and $w = f(v)$ for any $f \in \mathrm{O}(\mathrm{H}_{M_N})$, the split short exact sequence above is isomorphic to

$$0 \to \mathbb{Z}^{N-1} \to \mathrm{Stab}(w) \xrightarrow{h_w} \mathrm{O}(N-1)(\mathbb{Z}) \to 0.$$

There is an equality of subgroups $\Lambda_w = f \circ \Lambda_v \circ f^{-1}$. *Moreover*, $\Lambda_w \cong \mathbb{Z}^{N-1}$ *is a finite-index maximal torsion-free subgroup of* $\mathrm{Stab}(w)$ *and a maximal torsion-free, abelian subgroup of* $\mathrm{O}(\mathrm{H}_{M_N})$.

Proof. There is a section ℓ of h_w defined by

$$\ell : f \mapsto \mathrm{Id} \oplus f \in \mathrm{O}(\mathbb{Z}\{u, w\} \oplus \mathbb{Z}\{u, w\}^\perp, Q) = \mathrm{O}(L, Q)$$

which shows that h_w is surjective and the sequence is split.

In the case of $(L, Q) = \mathrm{H}_{M_N}$ with $N \geq 2$ and $w = f(v)$ for any $f \in \mathrm{O}(\mathrm{H}_{M_N})$, we can let $u = f(s)$, in which case

$$(w^\perp/\mathbb{Z}\{w\}, \bar{Q}) \cong (\mathbb{Z}\{f(e_1), \ldots, f(e_{N-1})\}, Q_{M_N})$$

and so $\mathrm{O}(w^\perp/\mathbb{Z}\{w\}, \bar{Q}) \cong \mathrm{O}(N-1)(\mathbb{Z})$ is finite. The subgroup $A \leq w^\perp/\mathbb{Z}\{w\}$ of even elements with respect to $\bar{Q}$ has index 2 in $w^\perp/\mathbb{Z}\{w\}$ which has rank $N-1$, and so $\Lambda_w \cong A \cong \mathbb{Z}^{N-1}$. Because the sequence is split, the subgroup $\langle \Lambda_w, h\rangle$ of $\mathrm{Stab}(w)$ generated by Λ_w and h must have torsion for any $h \in \mathrm{Stab}(w)$ with $h \notin \Lambda_w$ and so Λ_w is a maximal torsion-free subgroup of $\mathrm{Stab}(w)$.

To see that $\Lambda_w = f \circ \Lambda_v \circ f^{-1}$, compute for any $h \in \Lambda_v$ and $e \in v^\perp$ that

$$(f \circ h \circ f^{-1})(f(e)) = f(h(e)) = f(e - \bar{Q}(c(h), e)v) = f(e) - \bar{Q}(c(h), e)w$$

for some $c(h) \in v^\perp/\mathbb{Z}\{v\}$ as in the proof of Lemma 2.5 and where $\bar{Q}$ is the bilinear form on $v^\perp/\mathbb{Z}\{v\}$ induced by Q. Because $w^\perp = f(v^\perp)$, we see that $f \circ h \circ f^{-1}$ induces the identity map on $w^\perp/\mathbb{Z}\{w\}$, showing that $f \circ \Lambda_v \circ f^{-1} \subseteq \Lambda_w$. By symmetry, it follows that $f \circ \Lambda_v \circ f^{-1} = \Lambda_w$. Each of the generators of Λ_v given in Lemma 2.5 is contained in $\mathrm{O}^+(1, N)(\mathbb{Z})$. Because $\Lambda_w = f \circ \Lambda_v \circ f^{-1}$ and $\mathrm{O}^+(1, N)(\mathbb{Z})$ is a normal subgroup of $\mathrm{O}(\mathrm{H}_{M_N})$, we conclude that Λ_w is contained in $\mathrm{O}^+(1, N)(\mathbb{Z})$.

It remains to show that Λ_w is a maximal torsion-free, abelian subgroup of $\mathrm{O}(\mathrm{H}_{M_N})$. To this end, consider any $h \in \Lambda_w$ with $h \neq \mathrm{Id}$. Because $\Lambda_w \leq \mathrm{O}^+(1, N)(\mathbb{Z})$ and Λ_w is torsion-free, Lemma 2.2 shows that h is parabolic and $w \in H_2(M_N; \mathbb{Z})$ is the unique isotropic element of $H_2(M_N; \mathbb{Z})$ fixed by h, up to scaling. Suppose $k \in \mathrm{O}(\mathrm{H}_{M_N})$ commutes with some $h \in \Lambda_w$ and that $\langle k, \Lambda_w\rangle$ is torsion-free. Note that then $\langle -k, \Lambda_w\rangle$ is also torsion-free because $-\mathrm{Id} \in \mathrm{O}(\mathrm{H}_{M_N})$ is in the center of $\mathrm{O}(\mathrm{H}_{M_N})$ and has order 2. Moreover, $k(w) = \pm w$ because h fixes $k(w)$, so $k \in \mathrm{Stab}(w)$ or $-k \in \mathrm{Stab}(w)$. If $-k \in \mathrm{Stab}(w)$ then $\langle -k, \Lambda_w\rangle = \Lambda_w$ because $\langle -k, \Lambda_w\rangle$ is torsion-free and Λ_w is a maximal torsion-free subgroup of $\mathrm{Stab}(w)$. However, $-k \notin \Lambda_w$ because $-k \circ k^{-1} = -\mathrm{Id}$ is torsion and $\langle k, \Lambda_w\rangle$ is torsion-free. Therefore, $k \in \mathrm{Stab}(w)$ and $k \in \Lambda_w$ since $\langle k, \Lambda_w\rangle$ is a torsion-free subgroup of $\mathrm{Stab}(w)$. ∎

To use Lemma 2.6, we apply a theorem of Li–Li [15, Theorem 4.2] which says that for any $N \geq 2$ and any primitive, isotropic class $w \in H_2(M_N; \mathbb{Z})$ of minimal genus 0, there exists $\varphi \in \mathrm{Diff}^+(M_N)$ such that $[\varphi](v) = w$. Moreover, following elementary lemma strengthens this theorem in the case $2 \leq N \leq 8$. Recall the fixed $\mathbb{Z}$-basis $\{s, v, e_1, \ldots, e_{N-1}\}$ of $H_2(M_N; \mathbb{Z})$ given in (2).

Lemma 2.7. *If* $2 \leq N \leq 8$ *and* $w \in H_2(M_N; \mathbb{Z})$ *is an isotropic class, then*

(a) *there exists* $f \in \mathrm{O}(\mathrm{H}_{M_N})$ *such that* $f(v) = w$ *if* w *is primitive, and*

(b) w *has minimal genus* 0.

Proof. To prove (a), suppose w is primitive and $u \in \mathrm{H}_{M_N}$ is chosen as in the beginning of this subsection. The restriction of Q_{M_N} to $\mathbb{Z}\{w, u\}$ is unimodular and indefinite so $\mathbb{Z}\{w, u\}^\perp$ is negative definite of rank $N-1 < 8$. There exists a unique unimodular and negative definite lattice of rank r if $r \leq 7$; see [16, p. 1], for example. Therefore, $(\mathbb{Z}\{w, u\}^\perp, Q_{M_N}|_{\mathbb{Z}\{w,u\}^\perp})$ is isometric to $(\mathbb{Z}\{e_1, \ldots, e_{N-1}\}, Q_{M_N}|_{\mathbb{Z}\{e_1,\ldots,e_{N-1}\}})$; let $w_0 \in \mathbb{Z}\{w, u\}^\perp$ satisfy $Q_{M_N}(w_0, w_0) = -1$.

With $a := Q_{M_N}(u, u)$, we have $Q_{M_N}(w, u - aw_0) = 1$ and

$$Q_{M_N}(u - aw_0, u - aw_0) = a - a^2 \equiv 0 \pmod 2.$$

So $\mathbb{Z}\{w, u - aw_0\}$ is unimodular, even, and indefinite. Again, $\mathbb{Z}\{w, u - aw_0\}^\perp$ is negative definite of rank $N - 1 < 8$, and so $(\mathbb{Z}\{w, u - aw_0\}^\perp, Q_{M_N}|_{\mathbb{Z}\{w,u-aw_0\}^\perp})$ is isometric to $(\mathbb{Z}\{e_1, \ldots, e_{N-1}\}, Q_{M_N}|_{\mathbb{Z}\{e_1,\ldots,e_{N-1}\}})$. There exists $f \in \mathrm{O}(\mathrm{H}_{M_N})$ that preserves the orthogonal direct sums below

$$f : \mathbb{Z}\{v, s\} \oplus \mathbb{Z}\{e_1, \ldots, e_{N-1}\} \to \mathbb{Z}\{w, u - aw_0\} \oplus \mathbb{Z}\{w, u - aw_0\}^\perp$$

such that $f(v) = w$. This proves (a).

To prove (b), we may assume that $w \neq 0$. Suppose $w_1 \in H_2(M_N; \mathbb{Z})$ is a primitive isotropic class such that $aw_1 = w$ for some $a \in \mathbb{Z}$. By (a), there exists some $f \in \mathrm{Mod}(M_N)$ such that $f(v) = w_1$. Because $N \leq 9$, there exists a diffeomorphism $\varphi \in \mathrm{Diff}^+(M_N)$ such that $[\varphi] = f$ by [21, Theorem 2], and so the minimal genus of w and the minimal genus of av are equal, and the minimal genus of $av = a(H - E_1)$ is 0 (cf. [15, Theorem 4.2]). □

2.3. ***Lefschetz fibrations, conic bundles, and de Jonquières involutions.*** Let $N = 2m+1 \geq 3$ be odd and fix some distinct complex numbers $a_1, \ldots, a_{2m} \in \mathbb{C}$. Consider the birational map $\mathcal{J}_0 : \mathbb{CP}^1 \times \mathbb{CP}^1 \dashrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ given by

$$([X_1 : X_2], [Y_1 : Y_2]) \mapsto \left([X_1 : X_2], \left[Y_2 \prod_{i=m+1}^{2m} (X_1 - a_i X_2) : Y_1 \prod_{i=1}^{m} (X_1 - a_i X_2)\right]\right).$$

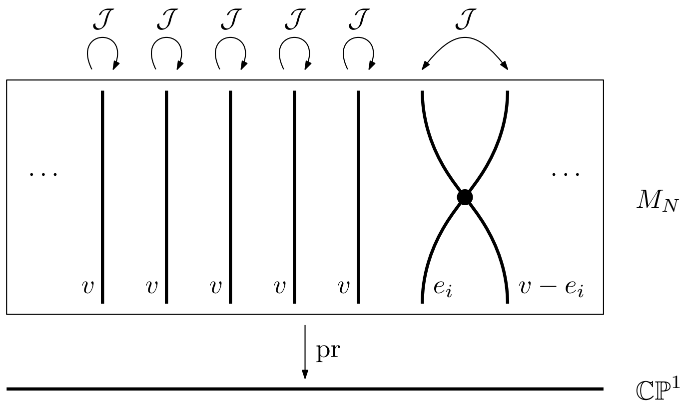


Figure 1. Each line represents a copy of $\mathbb{CP}^1$ and is labeled with its homology class in M_N. The rightmost fiber, for $1 \leq i \leq N-1$, is a singular fiber. Each singular fiber is a union of two (-1)-spheres intersecting transversely once.

Then $\mathcal{J}_0$ lifts to an automorphism $\mathcal{J}$ of order 2 called a *de Jonquières involution* of $X := \mathrm{Bl}_P(\mathbb{CP}^1 \times \mathbb{CP}^1)$ where

$$P := \Big\{([a_i:1],\ [1:0]) : 1 \leq i \leq m\Big\} \cup \Big\{([a_i:1],\ [0:1]) : m+1 \leq i \leq 2m\Big\}$$

is a set in $\mathbb{CP}^1 \times \mathbb{CP}^1$ with $2m$ points. Note that X is diffeomorphic to M_N. Under this identification, $e_k \in H_2(M_N;\mathbb{Z})$ is the class of the exceptional fiber above $([a_k:1],[1:0])$ for each $1 \leq k \leq m$ and the class of the exceptional fiber above $([a_k:1],[0:1])$ for each $m+1 \leq k \leq m$.

The projection map $\mathrm{pr}_0 : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ onto the first coordinate extends to a map $\mathrm{pr} : X \to \mathbb{CP}^1$ defining a holomorphic genus-0 Lefschetz fibration (in other words, a conic bundle). By construction, $\mathrm{pr} \circ \mathcal{J} = \mathrm{pr}$.

If $z \neq z_k := [a_k : 1] \in \mathbb{CP}^1$ for any k, the fiber of pr over a point $z \in \mathbb{CP}^1$ is $\{z\} \times \mathbb{CP}^1$ which is in the homology class $v \in H_2(M_N;\mathbb{Z})$. Because $\mathcal{J}$ acts on each such $\mathrm{pr}^{-1}(z)$ in an orientation-preserving way, $[\mathcal{J}] \in \mathrm{Stab}(v) \leq \mathrm{Mod}(M_N)$. Moreover for all $1 \leq k \leq 2m$ and all $([a_k:1],[Y_1:Y_2]) \notin P$,

$$\mathcal{J}_0 : ([a_k:1],[Y_1:Y_2]) \mapsto \begin{cases} ([a_k:1],[1:0]) & \text{if } 1 \leq k \leq m, \\ ([a_k:1],[0:1]) & \text{if } m+1 \leq k \leq 2m. \end{cases}$$

Therefore, $[\mathcal{J}]$ must send the homology class $v - e_k$ of the strict transform of $\mathrm{pr}^{-1}([a_k:1])$ in X to the exceptional divisor e_k. See Figure 1 for an illustration of the action of $\mathcal{J}$ on the fibers of pr.

The maps pr and $\mathcal{J}$ will be used in the explicit constructions in Sections 3 and 4. The goal of the rest of this section is to show that it suffices to only consider the Lefschetz fibration $\mathrm{pr} : M_N \to \mathbb{CP}^1$ for our setting and to prove Proposition 1.1.

Proposition 2.8. *Let $p : M_N \to \Sigma$ be a Lefschetz fibration where Σ is a closed, oriented surface and the generic fiber F satisfies $[F] \neq 0 \in H_2(M_N; \mathbb{Z})$. If $[F]$ has minimal genus* 0 *then $\Sigma = \mathbb{CP}^1$ and $F = \mathbb{CP}^1$.*

Proof. Because M_N is closed, a generic fiber F is a compact submanifold of M and has finitely many connected components, i.e., $\pi_0(F)$ is finite. By the exact sequence of the fibration [14, Proposition 8.1.9], there is a bijection $\pi_1(\Sigma) \to \pi_0(F)$ because $\pi_1(M_N) = 0$. Therefore, $\pi_1(\Sigma)$ is finite because $\pi_0(F)$ is finite. Because Σ is a closed, oriented surface, this implies that $\Sigma = \mathbb{CP}^1$. Furthermore, F is connected since $\pi_0(F) = \pi_1(\mathbb{CP}^1) = 0$.

Because $[F]$ is nontrivial in $H_2(M_N; \mathbb{R})$ since $H_2(M_N; \mathbb{Z})$ has no torsion, M_N can be given a symplectic structure such that F is a symplectic submanifold (Gompf [14, Theorem 10.2.18], [13, Theorem 1.2]) and so F must achieve the minimal genus in its homology class by the solution to the symplectic Thom conjecture (Oszváth–Szabó [17, Theorem 1.1]). □

Proof of Proposition 1.1. Suppose there exists such a Lefschetz fibration $p : M_N \to \Sigma$ whose generic fiber represents w and a homeomorphism $h : \Sigma \to \Sigma$ with $p \circ \varphi = h \circ p$. Because w is nonzero and has minimal genus 0, Proposition 2.8 says that $\Sigma = \mathbb{CP}^1$ and the generic fiber of p has genus 0. After blowing down the (-1)-spheres contained in the fibers of p, we see that p must be a $\mathbb{CP}^1$-bundle over Σ by [14, Proposition 8.1.7]. Because all $\mathbb{CP}^1$-bundles over $\mathbb{CP}^1$ are holomorphic, M_N gets a complex structure as a rational surface and p is holomorphic.

We prove by induction on N that if some homeomorphism $\varphi \in \mathrm{Homeo}^+(M_N)$ preserves a genus-0 Lefschetz fibration $p : M_N \to \mathbb{CP}^1$ then $[\varphi] \in \mathrm{Mod}(M_N)$ has finite order. If $N = 1$ then it is easy to see $\mathrm{Mod}(M_N) = \mathrm{O}(H_{M_N})$ is finite. Now assume for some $N_0 > 1$ that the claim holds for any $1 \leq N < N_0$.

Let $N = N_0$ and suppose $\varphi \in \mathrm{Homeo}^+(M_N)$ preserves a genus-0 Lefschetz fibration $p : M_N \to \mathbb{CP}^1$. Then φ must permute the singular fibers because none of the singular fibers are homeomorphic to a generic fiber $\mathbb{CP}^1$. There are finitely many singular fibers, so some power φ^k must preserve each singular fiber. Each singular fiber F of p is a union of finitely many spheres of negative self-intersection intersecting transversely at finitely many points $q_1, \dots, q_m$. Because φ^k restricts to a homeomorphism of each singular fiber, φ^k must permute the points $q_1, \dots, q_m$. Moreover, φ^k also restricts to a homeomorphism on $F - \{q_1, \dots, q_m\}$, a disjoint union of finitely many spheres with punctures. Therefore, a further power $\varphi^{k\ell}$ must preserve each component of $F - \{q_1, \dots, q_m\}$ and its orientation.

Let $S \subseteq F$ be an embedded (-1)-sphere in M_N. Because the homeomorphism $\varphi^{k\ell}$ fixes each point $q_1, \dots, q_m$ and preserves $S - (S \cap \{q_1, \dots, q_m\}) \subseteq F - \{q_1, \dots, q_m\}$, it must preserve $S \subseteq M_N$. Let $b : M_N \to M$ be the map that blows down S to a point $q \in M$. Being a rational surface, M is diffeomorphic to M_{N-1} or $\mathbb{CP}^1 \times \mathbb{CP}^1$.

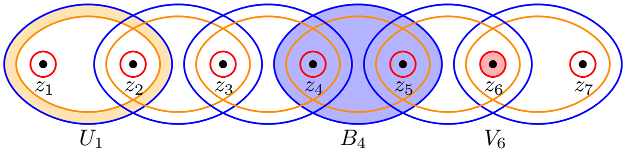


Figure 2. The sets U_k, B_k, and V_k in the case $n = 7$. The set B_4, shaded in blue, surrounds z_4 and z_5. The annulus U_1, shaded in orange, is the collar neighborhood of B_1 surrounding z_1 and z_2. The disk V_6, shaded in red, contains z_6 and contained in $(B_5 - U_5) \cap (B_6 - U_6)$.

Because $\varphi^{k\ell}$ defines a homeomorphism on $M_N - S$, it induces a homeomorphism of $M - q$ that extends to a homeomorphism ψ of M and preserves the Lefschetz fibration $p' : M \to \mathbb{CP}^1$ such that $p = p' \circ b$. If M is diffeomorphic to M_{N-1} then $[\psi]$ has finite order in $\mathrm{Mod}(M)$ by the inductive hypothesis. Otherwise, M is diffeomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$ and so $\mathrm{Mod}(M)$ is finite. Therefore, $[\psi]$ also has finite order in $\mathrm{Mod}(M)$.

Finally, note that $b_* : H_2(M_N; \mathbb{Z}) \to H_2(M; \mathbb{Z})$ induces the quotient map

$$H_2(M_N; \mathbb{Z}) \cong \mathbb{Z}\{[S]\}^\perp \oplus \mathbb{Z}\{[S]\} \to \mathbb{Z}\{[S]\}^\perp \cong H_2(M; \mathbb{Z}).$$

Because $\psi \circ b = b \circ \varphi^{k\ell}$ and $\varphi_*^{k\ell}([S]) = [S]$, the restriction of $\varphi_*^{k\ell}$ to $\mathbb{Z}\{[S]\}^\perp$ must have the same order as ψ_*. Finally, this shows that $[\varphi^{k\ell}]$, and therefore $[\varphi]$, has finite order in $\mathrm{Mod}(M_N)$. □

3. Theorem 1.4: lifting Λ_w to $\mathrm{Diff}^+(M_N)$

We turn to the proof of Theorem 1.4, after fixing some notation regarding certain subsets of $\mathbb{CP}^1$ illustrated in Figure 2. Let $N = n+1$ and $m = \lceil \frac{n}{2} \rceil$; thus $N = 2m+1$ if n is even and $N = 2m$ if n is odd. Fix distinct complex numbers $a_1, \ldots, a_{2m} \in \mathbb{C}$ and let $z_k := [a_k : 1]$ for all $k = 1, \ldots, n$. Then:

(a) For each $1 \le k \le n-1$, let $B_k \cong \mathbb{D}^2$ denote a closed disk in $\mathbb{CP}^1 - \{[1:0]\}$ containing z_k and z_{k+1} and no other points z_j for $j \ne k, k+1$ so that $B_k \cap B_{k'} = \varnothing$ if $|k - k'| > 1$.

(b) For each $1 \le k \le n-1$, let $U_k \cong [0,1] \times \mathbb{S}^1 \subseteq B_k$ denote a collar neighborhood of B_k that does not contain z_k and z_{k+1} where $\{0\} \times \mathbb{S}^1$ corresponds to ∂B_k.

(c) For each $1 \le k \le n$, let $V_k \cong \mathbb{D}^2$ denote a closed disk containing z_k in $B_1 - U_1 - B_2$ if $k = 1$, in $(B_k - U_k) \cap (B_{k-1} - U_{k-1})$ if $2 \le k \le n-1$, or in $B_{n-1} - U_{n-1} - B_{n-2}$ if $k = n$.

As in Section 2.3, let

$$P = \Big\{([a_i : 1], [1:0]) : 1 \le i \le m\Big\} \cup \Big\{([a_i : 1], [0:1]) : m+1 \le i \le 2m\Big\}$$

and consider the de Jonquières involution $\mathcal{J}$ on $\mathrm{Bl}_P(\mathbb{CP}^1 \times \mathbb{CP}^1)$. Identify M_{n+1} with

(a) $\mathrm{Bl}_P(\mathbb{CP}^1 \times \mathbb{CP}^1)$ if n is even, and

(b) $\mathrm{Bl}_{P-\{([a_{2m}:1],\,[0:1])\}}(\mathbb{CP}^1 \times \mathbb{CP}^1)$ if n is odd.

In both cases, consider $\mathrm{pr} : M_{n+1} \to \mathbb{CP}^1$ defined in Section 2.3. There is a natural inclusion

$$\mathrm{pr}^{-1}(B_k) \hookrightarrow \mathrm{Bl}_P(\mathbb{CP}^1 \times \mathbb{CP}^1)$$

that is preserved by $\mathcal{J}$ on $\mathrm{Bl}_P(\mathbb{CP}^1 \times \mathbb{CP}^1)$ for all $1 \le k \le n-1$. We use this inclusion to define $\mathcal{J}|_{\mathrm{pr}^{-1}(B_k)}$ on each $\mathrm{pr}^{-1}(B_k) \subseteq M_{n+1}$ regardless of the parity of n. Note that $\mathcal{J}|_{\mathrm{pr}^{-1}(B_k)} = \mathcal{J}|_{\mathrm{pr}^{-1}(B_{k+1})}$ when restricted to $\mathrm{pr}^{-1}(B_k) \cap \mathrm{pr}^{-1}(B_{k+1})$ for all $1 \le k \le n-2$.

There are four main steps to the proof of Theorem 1.4.

(1) Constructing commuting diffeomorphisms $\gamma_1, \ldots, \gamma_{n-1} \in \mathrm{Diff}^+(M_N)$ that preserve the genus-0 holomorphic Lefschetz fibration $\mathrm{pr} : M_N \to \mathbb{CP}^1$ such that $\mathrm{supp}(\gamma_k) \subseteq \mathrm{pr}^{-1}(B_k)$ and γ_k agrees with $\mathcal{J}$ on $\mathrm{pr}^{-1}(B_k - U_k)$ for each $1 \le k \le n-1$. These maps should be thought of as *local de Jonquières maps*.

(2) Constructing commuting diffeomorphisms $r_1, \ldots, r_n \in \mathrm{Diff}^+(M_N)$ satisfying $\mathrm{supp}(r_k) \subseteq \mathrm{pr}^{-1}(V_k)$ and $[r_k] = \mathrm{Ref}_{e_k}$ for each $1 \le k \le n$.

(3) Defining a homomorphism $\rho_v : \Lambda_v \to \mathrm{Diff}^+(M_N)$ using the diffeomorphisms above so that ρ_v is a section of $q : \mathrm{Diff}^+(M_N) \to \mathrm{Mod}(M_N)$ restricted to Λ_v and $\rho_v(\Lambda_v)$ almost preserves $\mathrm{pr} : M_N \to \mathbb{CP}^1$.

(4) Defining a homomorphism $\rho_w : \Lambda_w \to \mathrm{Diff}^+(M_N)$ for any other primitive, isotropic class w of minimal genus 0 by pre- and post-composing ρ_v by conjugation in $\mathrm{Mod}(M_N)$ and $\mathrm{Diff}^+(M_N)$.

Step 1: Constructing local de Jonquières maps $\gamma_1, \ldots, \gamma_{n-1} \in \mathrm{Diff}^+(M_N)$. First, recall by construction that for each $1 \le k \le n-1$, the disk B_k is a closed subset of $\mathbb{CP}^1 - \{[1:0]\}$. Throughout this section, we identify $\mathbb{CP}^1 - \{[1:0]\}$ with $\mathbb{C}$ by the diffeomorphism $[a:1] \mapsto a$. Then the disk B_k and the annulus U_k are subsets of $\mathbb{C}$ and the point $z_k = [a_k : 1] \in \mathbb{CP}^1 - \{[0:1]\}$ corresponds to $a_k \in \mathbb{C}$ under this identification.

For each $1 \le k \le n-1$, define $\lambda_k : U_k \to \mathbb{C}^\times$ by

$$\lambda_k(x) := \sqrt{\frac{\prod_{i=1}^{m}(x-a_i)}{\prod_{i=m+1}^{2m}(x-a_i)}}$$

with any smooth choice of square root. A computation of fundamental groups shows that such a choice exists. For completeness, we include a proof below.

Lemma 3.1. *For each* $1 \leq k \leq n-1$, *there exists a smooth map* $\lambda_k : U_k \to \mathbb{C}^\times$ *so that*

$$\lambda_k(x)^2 = \frac{\prod_{i=1}^{m}(x-a_i)}{\prod_{i=m+1}^{2m}(x-a_i)}.$$

Proof. Consider a function $\mu_k : B_k - \{a_k, a_{k+1}\} \to \mathbb{C}^\times$ defined by

$$\mu_k(x) := \Bigl(\prod_{i=1}^{m}(x-a_i)\Bigr)\Bigl(\prod_{i=m+1}^{2m}(x-a_i)^{-1}\Bigr).$$

Then $\mu_k(x)$ is well-defined and nonzero for any $x \in B_k - \{a_k, a_{k+1}\}$ because $x - a_i \neq 0$ for any $1 \leq i \leq n$. It suffices to show that there exists a lift $\lambda_k : U_k \to \mathbb{C}^\times$ of the restriction $\mu_k|_{U_k} : U_k \to \mathbb{C}^\times$ under the double cover $\mathbb{C}^\times \to \mathbb{C}^\times$ given by $a \mapsto a^2$. In other words, we will show that $(\mu_k)_*(\pi_1(U_k))$ is contained in $2\mathbb{Z} \leq \mathbb{Z} \cong \pi_1(\mathbb{C}^\times)$.

Let $\delta_k, \delta_{k+1} \in \pi_1(B_k - \{a_k, a_{k+1}\})$ be generators so that $\delta_k\delta_{k+1}$ is a generator of $\pi_1(U_k) \leq \pi_1(B_k - \{a_k, a_{k+1}\})$ and so that δ_k (resp. δ_{k+1}) is freely homotopic in $B_k - \{a_k, a_{k+1}\}$ to a loop $S^1 \to B_k - \{a_k, a_{k+1}\}$ given by $\theta \mapsto a_k + \varepsilon e^{2\pi\theta\sqrt{-1}} \in V_k$ (resp. $\theta \mapsto a_{k+1} + \varepsilon e^{2\pi\theta\sqrt{-1}} \in V_{k+1}$) for some $0 < \varepsilon \ll 1$.

The restrictions of μ_k to $V_k - \{a_k\}$ and $V_{k+1} - \{a_{k+1}\}$ take the forms

$$\mu_k|_{V_k - \{a_k\}}(x) = \eta_k(x)(x-a_k)^{\pm 1}, \quad \mu_k|_{V_{k+1} - \{a_{k+1}\}} = \eta_{k+1}(x)(x-a_{k+1})^{\pm 1},$$

where $\eta_i : V_i \to \mathbb{C}^\times$ are nonvanishing functions for each $1 \leq i \leq n$. Each of $\mu_k(\delta_k)$ and $\mu_k(\delta_{k+1})$ is freely homotopic to the loop $S^1 \to \mathbb{C}^\times$ given by $\theta \mapsto e^{2\pi\theta\sqrt{-1}}$ or by $\theta \mapsto e^{-2\pi\theta\sqrt{-1}}$, depending on the exponent of $(x-a_k)$ and $(x-a_{k+1})$ in $\mu_k(x)$. Therefore, $(\mu_k)_*(\delta_k\delta_{k+1})$ is an element of $2\mathbb{Z} \leq \pi_1(\mathbb{C}^\times)$. □

For such a choice of λ_k, consider the map $M_{\lambda_k} : U_k \to \mathrm{PGL}_2(\mathbb{C})$ given by

$$M_{\lambda_k}(x) = \begin{pmatrix} 1 & 1 \\ -\lambda_k(x) & \lambda_k(x) \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C}).$$

We also record the inverse of $M_{\lambda_k}(x)$ for later use:

$$M_{\lambda_k}(x)^{-1} = \begin{pmatrix} 1 & -1/\lambda_k(x) \\ 1 & 1/\lambda_k(x) \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C}).$$

Viewing $M_{\lambda_k}(x)$ and $M_{\lambda_k}(x)^{-1}$ as automorphisms of $\mathbb{CP}^1$, define a diffeomorphism u_{λ_k} of $\mathrm{pr}^{-1}(U_k) = U_k \times \mathbb{CP}^1$ by

$$u_{\lambda_k}([x:1], [Y_1:Y_2]) = ([x:1],\ M_{\lambda_k}(x) \cdot [Y_1:Y_2]).$$

Let $T : [0,1] \to [0,1]$ be a smooth, nondecreasing function such that $T|_{[0,\varepsilon]} \equiv 0$ and $T|_{[1-\varepsilon,1]} \equiv 1$ for some $0 < \varepsilon \ll 1$. Identifying $\mathrm{pr}^{-1}(U_k) = U_k \times \mathbb{CP}^1$ with $[0,1] \times \partial B_k \times \mathbb{CP}^1$ (cf. Figure 2), we define a diffeomorphism j_k of $\mathrm{pr}^{-1}(U_k)$ by

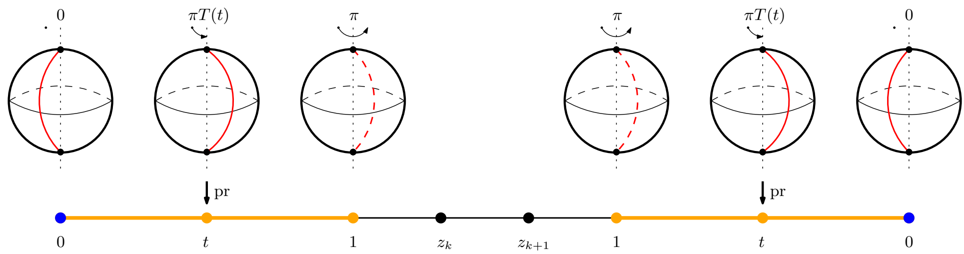


Figure 3. Illustrating the action of j_k on $\mathrm{pr}^{-1}(U_k) \cong [0, 1] \times \partial B_k \times \mathbb{CP}^1$. The horizontal line represents $B_k \subseteq \mathbb{CP}^1$ and the orange portion represents the annulus $U_k \subseteq B_k \subseteq \mathbb{CP}^1$ whose width is parametrized by $t \in [0, 1]$. The two blue points represent $\partial B_k \cong \mathbb{S}^1$. The diffeomorphism j_k acts by rotation-by-$\pi T(t)$ on the sphere lying above a point $(t, x) \in [0, 1] \times \partial B_k \cong U_k$.

$$j_k(t,\ \theta,\ [Y_1 : Y_2]) = (t,\ \theta,\ [e^{\sqrt{-1}\pi T(t)} Y_1 : Y_2]).$$

Roughly, j_k is a map on $[0, 1] \times \mathbb{S}^1 \times \mathbb{CP}^1$ induced by an isotopy of $\mathbb{S}^1 \times \mathbb{CP}^1$ from the $\mathrm{Id} \times \mathrm{Id}$ to $\mathrm{Id} \times R(\pi)$, where $R(\pi)$ is a rotation-by-π map on $\mathbb{CP}^1$. See Figure 3.

In the next lemma, we show that the de Jonquières map is conjugate to $\mathrm{Id} \times R(\pi)$ on each $\{t\} \times \partial B_k \times \mathbb{CP}^1$, which will be used to modify $\mathcal{J}|_{\mathrm{pr}^{-1}(U_k)}$ to be the identity near the boundary $\mathrm{pr}^{-1}(\partial B_k)$.

Lemma 3.2. *Let* $1 \le k \le n-1$. *On* $\mathrm{pr}^{-1}(U_k) = [0, 1] \times \partial B_k \times \mathbb{CP}^1$,

$$u_{\lambda_k} \circ j_k \circ u_{\lambda_k}^{-1} = \begin{cases} \mathcal{J} & \text{on } \mathrm{pr}^{-1}([1-\varepsilon, 1] \times \partial B_k), \\ \mathrm{Id} & \text{on } \mathrm{pr}^{-1}([0, \varepsilon] \times \partial B_k). \end{cases}$$

Proof. On $\mathrm{pr}^{-1}([0, \varepsilon] \times \partial B_k)$, note that $j_k \equiv \mathrm{Id}$. On $\mathrm{pr}^{-1}([1-\varepsilon, 1] \times \partial B_k)$,

$$j_k([x : 1], [Y_1 : Y_2]) = ([x : 1], [-Y_1 : Y_2]).$$

For all $[x : 1] \in [1-\varepsilon, 1] \times \partial B_k \subseteq U_k$

$$M_{\lambda_k}(x) \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} M_{\lambda_k}(x)^{-1} = \begin{pmatrix} 0 & 1 \\ \lambda_k(x)^2 & 0 \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C}),$$

and so

$$u_{\lambda_k} \circ j_k \circ u_{\lambda_k}^{-1}([x:1], [Y_1 : Y_2]) = \big([x : 1], [Y_2 : \lambda_k(x)^2 Y_1]\big) = \mathcal{J}([x:1], [Y_1 : Y_2]). \quad \square$$

The diffeomorphisms γ_k below should be thought of as *local* de Jonquières maps, acting only on a single pair of singular fibers of pr.

Definition 3.3. For $1 \le k \le n-1$, let γ_k be the diffeomorphism of M_N given by

$$\gamma_k = \begin{cases} \mathcal{J} & \text{on } \mathrm{pr}^{-1}(B_k - U_k), \\ u_{\lambda_k} \circ j_k \circ u_{\lambda_k}^{-1} & \text{on } \mathrm{pr}^{-1}(U_k), \\ \mathrm{Id} & \text{on } \mathrm{pr}^{-1}(\mathbb{CP}^1 - B_k). \end{cases}$$

Proposition 3.4. *The diffeomorphisms γ_k satisfy the following properties*:

(a) *The diffeomorphism γ_k preserves* pr *for all* $1 \le k \le n-1$. *In fact*, $\mathrm{pr} \circ \gamma_k = \mathrm{pr}$.

(b) *The diffeomorphisms γ_i and γ_j commute for all* $1 \le i, j \le n-1$.

(c) *As mapping classes*, $[\gamma_k] = \mathrm{Ref}_{v-e_k-e_{k+1}} \circ \mathrm{Ref}_{e_k-e_{k+1}}$ *for all* $1 \le k \le n-1$.

Proof. For each k, $\mathrm{pr} \circ u_{\lambda_k} = \mathrm{pr}$ and $\mathrm{pr} \circ j_k = \mathrm{pr}$ by construction of u_{λ_k} and j_k when restricted to $\mathrm{pr}^{-1}(U_k)$. Therefore,

$$(\mathrm{pr} \circ \gamma_k)|_{\mathrm{pr}^{-1}(U_k)} = \big(\mathrm{pr} \circ (u_{\lambda_k} \circ j_k \circ u_{\lambda_k}^{-1})\big)\big|_{\mathrm{pr}^{-1}(U_k)} = \mathrm{pr}\,|_{\mathrm{pr}^{-1}(U_k)}$$

and γ_k preserves the fibers of pr on $\mathrm{pr}^{-1}(U_k)$ for all k. The same is clearly true on $\mathrm{pr}^{-1}(\mathbb{CP}^1 - B_k)$ and true on $\mathrm{pr}^{-1}(B_k - U_k)$ by construction of $\mathcal{J}$. This proves (a).

If $|i-j| > 1$ then $\mathrm{supp}(\gamma_i) \cap \mathrm{supp}(\gamma_j) = \varnothing$ so γ_i and γ_j commute. To show that γ_i and γ_{i+1} commute for $1 \le i \le n-2$, we will consider the action of these diffeomorphisms on $\mathrm{pr}^{-1}(B_i \cap B_{i+1})$, which contains $\mathrm{supp}(\gamma_i) \cap \mathrm{supp}(\gamma_{i+1})$. We split $B_i \cap B_{i+1}$ as a union of $\mathcal{C}_i := (B_i \cap B_{i+1}) \cap (U_i \cup U_{i+1})$ and $\mathcal{V}_i := (B_i \cap B_{i+1}) - (U_i \cup U_{i+1})$ (see Figure 4), so

$$\mathrm{pr}^{-1}(B_i \cap B_{i+1}) = \mathrm{pr}^{-1}(\mathcal{C}_i) \cup \mathrm{pr}^{-1}(\mathcal{V}_i).$$

By construction, $\gamma_i|_{\mathrm{pr}^{-1}(\mathcal{V}_i)} = \mathcal{J}\,|_{\mathrm{pr}^{-1}(\mathcal{V}_i)} = \gamma_{i+1}|_{\mathrm{pr}^{-1}(\mathcal{V}_i)}$, and so γ_i and γ_{i+1} commute on $\mathrm{pr}^{-1}(\mathcal{V}_i)$.

For any $[x:1] \in \mathcal{C}_i$, both γ_i and γ_{i+1} act on $\mathrm{pr}^{-1}([x:1])$ by (a). If $[x:1] \in U_i \cap U_{i+1}$ then for some $t, T \in [0,1]$ depending on x, we have

$$\gamma_i([x:1],[Y_1:Y_2]) = \left([x:1],\ \left(M_{\lambda_i}(x)\begin{pmatrix} e^{\sqrt{-1}\pi t} & 0 \\ 0 & 1\end{pmatrix} M_{\lambda_i}(x)^{-1}\right)\cdot [Y_1:Y_2]\right),$$

$$\gamma_{i+1}([x:1],[Y_1:Y_2]) = \left([x:1],\ \left(M_{\lambda_{i+1}}(x)\begin{pmatrix} e^{\sqrt{-1}\pi T} & 0 \\ 0 & 1\end{pmatrix} M_{\lambda_{i+1}}(x)^{-1}\right)\cdot [Y_1:Y_2]\right).$$

Moreover, $\lambda_i(x) = \lambda_{i+1}(x)$ or $-\lambda_{i+1}(x)$. In the first case, $M_{\lambda_i}(x) = M_{\lambda_{i+1}}(x)$, so γ_i and γ_{i+1} commute on $\mathrm{pr}^{-1}([x:1])$. In the second case, we compute for each $[x:1] \in U_i$ that

$$M_{\lambda_i}(x) = M_{-\lambda_i}(x)\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} \qquad \text{and} \qquad M_{\lambda_i}(x)^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix} M_{-\lambda_i}(x)^{-1}$$

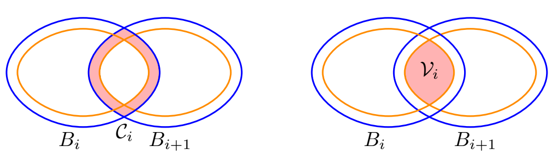


Figure 4. For each $1 \le i \le n-2$, the sets $\mathcal{C}_i$ (left) and $\mathcal{V}_i$ (right) are contained in $B_i \cap B_{i+1}$.

and so

$$M_{-\lambda_i}(x)\begin{pmatrix} e^{\sqrt{-1}\pi T} & 0 \\ 0 & 1 \end{pmatrix} M_{-\lambda_i}(x)^{-1} = M_{\lambda_i}(x)\begin{pmatrix} 1 & 0 \\ 0 & e^{\sqrt{-1}\pi T} \end{pmatrix} M_{\lambda_i}(x)^{-1}.$$

It is clear that

$$M_{\lambda_i}(x)\begin{pmatrix} e^{\sqrt{-1}\pi t} & 0 \\ 0 & 1 \end{pmatrix} M_{\lambda_i}(x)^{-1} \qquad \text{and} \qquad M_{\lambda_i}(x)\begin{pmatrix} 1 & 0 \\ 0 & e^{\sqrt{-1}\pi T} \end{pmatrix} M_{\lambda_i}(x)^{-1}$$

commute in $\mathrm{PGL}_2(\mathbb{C})$, which shows that γ_i and γ_{i+1} commute on $\mathrm{pr}^{-1}([x:1])$ in this case.

If $[x:1] \in U_i$ and $[x:1] \notin U_{i+1}$ then for some $t \in [0,1]$ depending on x and for all $[Y_1:Y_2] \in \mathbb{CP}^1$,

$$\gamma_i([x:1],[Y_1:Y_2]) = \left([x:1],\ \left(M_{\lambda_i}(x)\begin{pmatrix} e^{\sqrt{-1}\pi t} & 0 \\ 0 & 1 \end{pmatrix} M_{\lambda_i}(x)^{-1}\right)\cdot[Y_1:Y_2]\right)$$

$$\begin{aligned}\gamma_{i+1}([x:1],[Y_1:Y_2]) &= \mathcal{J}([x:1],[Y_1:Y_2]) \\ &= \left([x:1],\ \left(M_{\lambda_{i+1}}(x)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} M_{\lambda_{i+1}}(x)^{-1}\right)\cdot[Y_1:Y_2]\right)\end{aligned}$$

where the second equality follows from (the proof of) Lemma 3.2. Therefore, we can show that γ_i and γ_{i+1} commute on $\mathrm{pr}^{-1}([x:1])$ similarly as in the previous case. By analogous computations, γ_i and γ_{i+1} commute on $\mathrm{pr}^{-1}([x:1])$ if $[x:1] \in U_{i+1}$ and $[x:1] \notin U_i$. This proves (b).

Finally, note that for all $j \neq k, k+1$, the map γ_k restricts to the identity on e_j and on $\mathrm{pr}^{-1}(z)$ for any $z \notin B_k$ so $(\gamma_k)_*(e_j) = e_j$ and $(\gamma_k)_*(v) = v$. Moreover, γ_k agrees with $\mathcal{J}$ on $\mathrm{pr}^{-1}(B_k)$, meaning that $(\gamma_k)_*(e_j) = v - e_j$ for $j = k$ and $j = k+1$. This then determines $[\gamma_k] \in \mathrm{Mod}(M_{n+1})$ by Lemma 2.3. A computation shows that the same holds for $\mathrm{Ref}_{v-e_k-e_{k+1}} \circ \mathrm{Ref}_{e_k-e_{k+1}}$. □

Step 2: Constructing $r_1, \ldots, r_n \in \mathrm{Diff}^+(M_N)$. For each $1 \le k \le n$, the exceptional divisor e_k has a tubular neighborhood ν_k in $\mathrm{pr}^{-1}(V_k)$ that is diffeomorphic to $\overline{\mathbb{CP}^2} - \{[0:0:1]\}$. Let $i_k : \overline{\mathbb{CP}^2} - \{[0:0:1]\} \to \nu_k$ be this diffeomorphism and let τ_0 be a diffeomorphism of $\overline{\mathbb{CP}^2} - \{[0:0:1]\}$ given by complex conjugation, $\tau_0 : [X:Y:Z] \mapsto [\bar{X}:\bar{Y}:\bar{Z}]$.

Consider a smooth path $\eta : (0,1) \to \mathrm{SO}(4)$ such that

$$\eta(t) = \begin{cases} \mathrm{diag}(1,-1,1,-1) & \text{if } t \in (1-\varepsilon, 1) \\ \mathrm{Id} & \text{if } t \in (0,\varepsilon) \end{cases}$$

for some $0 < \varepsilon \ll 1$. Let B denote the punctured ball in $\overline{\mathbb{CP}^2} - \{[0:0:1]\}$ given by

$$B := \{[a + b\sqrt{-1} : c + d\sqrt{-1} : 1] \in \overline{\mathbb{CP}^2} : 0 < \|(a + b\sqrt{-1}, c + d\sqrt{-1})\| < 1\},$$

identifying it with $(0,1)\times\mathbb{S}^3\subseteq\mathbb{R}^4$, and define $\tau\in\mathrm{Diff}^+(\overline{\mathbb{CP}^2}-\{[0:0:1]\})$ by

$$\tau=\begin{cases}\tau_0 & \text{on } \overline{\mathbb{CP}^2}-B,\\ (t,x)\mapsto(t,\eta(t)x) & \text{on } B\cong(0,1)\times\mathbb{S}^3.\end{cases}$$

Then τ is compactly supported in $\overline{\mathbb{CP}^2}-\{[0:0:1]\}$.

Definition 3.5. For all $1\le k\le n$, let $r_k\in\mathrm{Diff}^+(M_N)$ be

$$r_k:=\begin{cases}i_k\circ\tau\circ i_k^{-1} & \text{on } \nu_k,\\ \mathrm{Id} & \text{on } M_N-\nu_k.\end{cases}$$

Remark 3.6. By construction, the diffeomorphism r_k restricts to an orientation-reversing diffeomorphism of e_k and preserves the homology classes e_i for all $i\ne k$ and v. This forces $[r_k]=\mathrm{Ref}_{e_k}\in\mathrm{Mod}(M_N)$ by Lemma 2.3. Moreover, $\mathrm{supp}(r_k)\subseteq\nu_k\subseteq\mathrm{pr}^{-1}(V_k)$ and r_k preserves $\nu_k\subseteq\mathrm{pr}^{-1}(V_k)$.

Step 3: Constructing $\rho_v:\Lambda_v\to\mathrm{Diff}^+(M_N)$. The generators $f_1,\dots,f_{n-1}$ of Λ_v (cf. Lemma 2.5) will be mapped under ρ_v to the following diffeomorphisms.

Lemma 3.7. *For each $1\le k\le n-1$, let*

$$\varphi_k:=r_k\circ r_{k+1}\circ\gamma_k.$$

Then $\varphi_i\circ\varphi_j=\varphi_j\circ\varphi_i$ for any $1\le i,j\le n-1$.

Proof. For any $1\le i,j\le n-1$, the diffeomorphisms φ_i and φ_j commute if $|i-j|>1$ because they have disjoint support. For any $1\le i\le n-2$,

$$\varphi_i|_{\mathrm{pr}^{-1}(V_{i+1})}=(r_{i+1}\circ\mathcal{J})|_{\mathrm{pr}^{-1}(V_{i+1})}=\varphi_{i+1}|_{\mathrm{pr}^{-1}(V_{i+1})}$$

so φ_i and φ_{i+1} commute on $\mathrm{pr}^{-1}(V_{i+1})$. Moreover on $S_i:=\mathrm{pr}^{-1}(B_i\cap B_{i+1})-\mathrm{pr}^{-1}(V_{i+1})$

$$\varphi_i|_{S_i}=\gamma_i,\qquad \varphi_{i+1}|_{S_i}=\gamma_{i+1}$$

and so φ_i and φ_{i+1} commute on $\mathrm{pr}^{-1}(B_i\cap B_{i+1})$ by Proposition 3.4(b). Finally, φ_i and φ_{i+1} commute on $M_{n+1}-\mathrm{pr}^{-1}(B_i\cap B_{i+1})$ because $\mathrm{supp}(\varphi_i)\cap\mathrm{supp}(\varphi_{i+1})$ is contained in $\mathrm{pr}^{-1}(B_i\cap B_{i+1})$. □

It remains to construct the image of the last generator g of Λ_v under ρ_v.

Lemma 3.8. *The map*

$$\psi=\begin{cases}\varphi_1\circ\varphi_1 & \textit{on } \mathrm{pr}^{-1}(V_1)\\ \mathrm{Id} & \textit{on } M_{n+1}-\mathrm{pr}^{-1}(V_1).\end{cases}$$

is a well-defined diffeomorphism, which commutes with φ_k for all $1\le k\le n-1$ and satisfies $[\psi]=g=\mathrm{Ref}_{e_1}\circ\mathrm{Ref}_{v-e_1}$ in $\mathrm{Mod}(M_{n+1})$, *Proof.* By definition, r_k has support contained in the interior of $\mathrm{pr}^{-1}(V_k)$ for all $1 \le k \le n$. So $r_1|_C = \mathrm{Id}|_C$ on some collar neighborhood C of $\mathrm{pr}^{-1}(V_1)$, and

$$\psi|_C = (\varphi_1 \circ \varphi_1)|_C = (r_1 \circ \mathcal{J} \circ r_1 \circ \mathcal{J})|_C = (\mathcal{J} \circ \mathcal{J})|_C = \mathrm{Id}|_C.$$

Moreover, $\mathcal{J}$, r_1, and r_2 all preserve $\mathrm{pr}^{-1}(V_1)$, so the map ψ is indeed a diffeomorphism.

The diffeomorphisms ψ and φ_k have disjoint supports for all $k > 1$. Considering the subsets $\mathrm{pr}^{-1}(V_1)$ and $M_{n+1} - \mathrm{pr}^{-1}(V_1)$ separately shows that φ_1 and ψ commute as well.

Compute for all $2 \le k \le n$ that $[\psi](e_k) = e_k$ because $\mathrm{supp}(\psi) \subseteq \mathrm{pr}^{-1}(V_1)$. Moreover, ψ agrees with φ_1^2 on $\mathrm{pr}^{-1}(V_1)$, meaning that

$$[\psi](e_1) = [\varphi_1^2](e_1) = e_1 + 2v.$$

Computing that

$$\mathrm{Ref}_{e_1} \circ \mathrm{Ref}_{v-e_1}(e_k) = [\psi](e_k)$$

for all $1 \le k \le n$ and applying Lemma 2.3 shows that $[\psi] = g$. □

Proposition 3.9. *There is a homomorphism $\rho_v : \Lambda_v \to \mathrm{Diff}^+(M_N)$ defined by*

$$\rho_v(f_k) := \varphi_k \quad \textit{for all } 1 \le k \le n-1, \qquad \rho_v(g) := \psi,$$

where g and f_k for $1 \le k \le n-1$ are the generators of Λ_v as given in Lemma 2.5. Moreover,

(a) *ρ_v is a section of the map $q : \mathrm{Diff}^+(M_N) \to \mathrm{Mod}(M_N)$ restricted to $\Lambda_v \le \mathrm{Mod}(M_N)$, and*

(b) *for all $\varphi \in \rho_v(\Lambda_v)$,*

$$(\mathrm{pr} \circ \varphi)|_{M_N - \bigcup_{i=1}^{n} \mathrm{pr}^{-1}(V_i)} = \mathrm{pr}|_{M_N - \bigcup_{i=1}^{n} \mathrm{pr}^{-1}(V_i)}.$$

Hence $\rho_v(\Lambda_v)$ almost preserves the Lefschetz fibration $\mathrm{pr} : M_N \to \mathbb{CP}^1$.

Proof. By Lemma 2.5, $\Lambda_v \cong \mathbb{Z}^n$ is generated by $f_1, \dots, f_{n-1}, g$. By Lemmas 3.7 and 3.8, the image of ρ_v is abelian and therefore ρ_v is a well-defined homomorphism.

Compute using Proposition 3.4(c) and Remark 3.6 that

$$[\rho_v(f_k)] = [r_k] \circ [r_{k+1}] \circ [\gamma_k] = \mathrm{Ref}_{e_k} \circ \mathrm{Ref}_{e_{k+1}} \circ \mathrm{Ref}_{v-e_k-e_{k+1}} \circ \mathrm{Ref}_{e_k-e_{k+1}} = f_k \in \Lambda_v.$$

Lemma 3.8 shows that $[\rho_v(g)] = g$. Therefore, ρ_v is a section of the quotient map $q : \mathrm{Diff}^+(M_N) \to \mathrm{Mod}(M_N)$ restricted to $\Lambda_v \le \mathrm{Mod}(M_N)$.

Finally, $\mathrm{supp}(r_k) \subseteq \mathrm{pr}^{-1}(V_k)$ for all $1 \le k \le n$ (cf. Remark 3.6). By Proposition 3.4(a), $\mathrm{pr} \circ \gamma_k = \mathrm{pr}$ for all $1 \le k \le n-1$, so

$$\varphi_k|_{M_N - \bigcup_{i=1}^{n} \mathrm{pr}^{-1}(V_i)} = \gamma_k|_{M_N - \bigcup_{i=1}^{n} \mathrm{pr}^{-1}(V_i)}.$$

By Lemma 3.8,

$$\psi|_{M_N-\bigcup_{i=1}^n \mathrm{pr}^{-1}(V_i)} = \mathrm{Id}|_{M_N-\bigcup_{i=1}^n \mathrm{pr}^{-1}(V_i)}.$$

Hence $\mathrm{pr}\circ\varphi = \mathrm{pr}$ restricted to $M_N - \bigcup_{i=1}^n \mathrm{pr}^{-1}(V_i)$ for all $\varphi \in \rho_v(\Lambda_v)$ and so $\rho_v(\Lambda_v)$ almost preserves pr. □

***Step 4: Extension to any primitive, isotropic class w of minimal genus* 0.** With the constructions above in hand, we conclude the proof of Theorem 1.4.

Proof of Theorem 1.4. Because w is a primitive, isotropic class of minimal genus 0, there exists some $\alpha \in \mathrm{Diff}^+(M_N)$ such that $[\alpha](w) = v$ by a theorem of Li–Li ([15, Theorem 4.2]). Using the definitions of h_v and h_w (cf. Lemma 2.6), compute that

$$\Lambda_v = [\alpha]\circ\Lambda_w\circ[\alpha^{-1}]$$

and define $\rho_w : \Lambda_w \to \mathrm{Diff}^+(M_N)$ by

$$\rho_w(f) = \alpha^{-1}\circ\rho_v([\alpha]\circ f\circ[\alpha^{-1}])\circ\alpha$$

where $\rho_v : \Lambda_v \to \mathrm{Diff}^+(M_N)$ is the homomorphism constructed in Proposition 3.9. Compute that for all $x \in M_N - \bigcup_{k=1}^n (\mathrm{pr}\circ\alpha)^{-1}(V_k)$,

$$(\mathrm{pr}\circ\alpha)\circ\rho_w(f)(x) = \mathrm{pr}\circ\rho_v([\alpha]\circ f\circ[\alpha^{-1}])\circ\alpha(x) = (\mathrm{pr}\circ\alpha)(x)$$

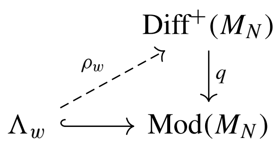
because $\mathrm{pr}\circ\rho_v([\alpha]\circ f\circ[\alpha^{-1}]) = \mathrm{pr}$ on $M_N - \bigcup_{k=1}^n \mathrm{pr}^{-1}(V_k)$ by Proposition 3.9(b). Hence $\rho_w(\Lambda_w)$ almost preserves $\mathrm{pr}\circ\alpha$, which is holomorphic for some complex structure on M_N. Finally, compute by Proposition 3.9(a) that for any $f \in \Lambda_w$,

$$\begin{aligned} q\circ\rho_w(f) &= q(\alpha^{-1}\circ\rho_v([\alpha]\circ f\circ[\alpha^{-1}])\circ\alpha) \\ &= [\alpha^{-1}]\circ(q\circ\rho_v)([\alpha]\circ f\circ[\alpha^{-1}])\circ[\alpha] = f. \end{aligned}$$

□

If $2 \le N \le 8$, Theorem 1.4 holds for any primitive, isotropic class in $H_2(M_N;\mathbb{Z})$.

Corollary 3.10. *Let $2 \le N \le 8$ and let $w \in H_2(M_N;\mathbb{Z})$ be any primitive, isotropic class. There exists a homomorphism $\rho_w : \Lambda_w \to \mathrm{Diff}^+(M_N)$ such that the following diagram commutes*:



*The image $\rho_w(\Lambda_w)$ almost preserves a holomorphic genus-*0 *Lefschetz fibration $p : M_N \to \mathbb{CP}^1$ whose generic fiber represents the homology class w.*

Proof. If $2 \le N \le 8$ and $w \in H_2(M_N;\mathbb{Z})$ is a primitive, isotropic class then Lemma 2.7(b) says that the minimal genus of w is 0. Now apply Theorem 1.4. □

4. Theorem 1.5: individual elements of the stabilizer $\mathrm{Stab}(w)$ of w

We next prove Theorem 1.5 using the diffeomorphisms constructed in Section 3. The following lemma considers the subgroup $S_n \leq \mathrm{Stab}(v) \cap \mathrm{Stab}(s) \leq \mathrm{Mod}(M_N)$ given by permuting the classes $e_1, \ldots, e_n$. In other words, we consider the subgroup $\langle \mathrm{Ref}_{e_k - e_{k+1}} : 1 \leq k \leq n-1 \rangle$ of $\mathrm{Mod}(M_N)$.

Lemma 4.1. *For each* $1 \leq k \leq n-1$, *there exist* $s_k \in \mathrm{Diff}^+(M_N)$ *and* $\tau_k \in \mathrm{Diff}^+(\mathbb{CP}^1)$ *such that*

(a) $\mathrm{pr} \circ s_k = \tau_k \circ \mathrm{pr}$, *and*

(b) $[s_k] = \mathrm{Ref}_{e_k - e_{k+1}} \in \mathrm{Mod}(M_N)$.

Proof. Let $A := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{PGL}_2(\mathbb{C})$, so A has order 2 and $A([1:1]) = [-1:1] \in \mathbb{CP}^1$. There exists a neighborhood D diffeomorphic to a disk $\mathbb{D}^2$ of the path $\{[t:1] : t \in [-1,1]\} \subseteq \mathbb{CP}^1$ that is preserved by A. Let $\iota_k : D \hookrightarrow \mathbb{CP}^1$ be a smooth embedding with image contained in $B_k - U_k$ and

$$\iota_k([-1:1]) = [a_k : 1], \quad \iota_k([1:1]) = [a_{k+1} : 1],$$

so that ι_k is holomorphic if restricted to small neighborhoods of $[1:1]$ and $[-1:1]$ in D. Now let $\tau_k \in \mathrm{Diff}^+(\mathbb{CP}^1)$ be a diffeomorphism such that

$$\tau_k = \begin{cases} \iota_k \circ A \circ \iota_k^{-1} & \text{on } \iota_k(D) \subseteq B_k - U_k, \\ \mathrm{Id} & \text{on } \mathbb{CP}^1 - B_k. \end{cases}$$

Consider the diffeomorphism $s : (X, Y) \mapsto (\tau_k(X), Y)$ of $\mathbb{CP}^1 \times \mathbb{CP}^1$ which extends to a diffeomorphism s_k of M_N because s is holomorphic on a neighborhood of $\mathrm{pr}^{-1}([a_i : 1])$ for all $1 \leq i \leq n$. By construction, $\mathrm{pr} \circ s_k = \tau_k \circ \mathrm{pr}$. Moreover, if $i \neq k$ or $k+1$ then s_k acts as the identity on e_i but $s_k(e_k) = e_{k+1}$ and $s_k(e_{k+1}) = e_k$. Hence $[s_k] = \mathrm{Ref}_{e_k - e_{k+1}}$ by Lemma 2.3 because $[s_k] \in \mathrm{Stab}(v)$. ☐

We may assume that for all $1 \leq k \leq n-1$, the choice of $V_k, V_{k+1} \subseteq \mathbb{CP}^1$ satisfies $V_k, V_{k+1} \subseteq \iota_k(D)$ and $\tau_k(V_k) = V_{k+1}$, where $\iota_k : D \hookrightarrow \mathbb{CP}^1$ is the embedding defined in the proof of Lemma 4.1. This also implies that $\tau_k(V_{k+1}) = V_k$ because $\tau_k|_{\iota_k(D)}$ has order 2.

Proof of Theorem 1.5. The theorem holds for $N = 1$ because then $\mathrm{Stab}(w) = 1$. Now assume that $N \geq 2$ and that $w = v$. Since $h \in \mathrm{Stab}(v)$, we may write $h = f \circ \sigma$ where $f \in \Lambda_v$ and $\sigma \in \mathrm{Aut}(\mathbb{Z}\{e_1, \ldots, e_n\}, Q_{M_N}) \cong \mathrm{O}(n)(\mathbb{Z})$ by Lemma 2.6. Furthermore, the action of $\mathrm{O}(n)(\mathbb{Z})$ on $\mathbb{Z}\{e_1, \ldots, e_n\}$ preserves the set $\{e_1, \ldots, e_n, -e_1, \ldots, -e_n\}$ of classes of norm -1. The action of $\mathrm{O}(n)(\mathbb{Z})$ on the set of n unordered pairs $\{e_k, -e_k\}$ with $1 \leq k \leq n$ defines a homomorphism $\mathrm{O}(n)(\mathbb{Z}) \to S_n$ with kernel $\langle \mathrm{Ref}_{e_k} : 1 \leq k \leq n \rangle \cong (\mathbb{Z}/2\mathbb{Z})^n$. Moreover, this homomorphism admits a section with image $\langle \mathrm{Ref}_{e_k - e_{k+1}} : 1 \leq k \leq n-1 \rangle \leq \mathrm{O}(n)(\mathbb{Z})$. In other words, any element $\sigma \in \mathrm{Aut}(\mathbb{Z}\{e_1, \ldots, e_n\}, Q_{M_N})$ can be written as a product $[r] \circ [s] \in \mathrm{Aut}(\mathbb{Z}\{e_1, \ldots, e_n\}, Q_{M_N})$ where

$$r \in \langle r_k : 1 \leq k \leq n \rangle \leq \mathrm{Diff}^+(M_N) \quad \text{and} \quad s \in \langle s_k : 1 \leq k \leq n-1 \rangle \leq \mathrm{Diff}^+(M_N),$$

by Remark 3.6 and Lemma 4.1(b). Let

$$\varphi := \rho_v(f) \circ r \circ s \in \mathrm{Diff}^+(M_N)$$

where $\rho_v : \Lambda_v \to \mathrm{Diff}^+(M_N)$ is the homomorphism from Proposition 3.9. By construction, $[\varphi] = h$.

Note that

$$\mathrm{pr} \circ \rho_v(f) \circ r|_{M_N - \bigcup_{i=1}^n \mathrm{pr}^{-1}(V_i)} = \mathrm{pr}\,|_{M_N - \bigcup_{i=1}^n \mathrm{pr}^{-1}(V_i)}$$

by Proposition 3.9(b) and by Remark 3.6. By Lemma 4.1(a), there exists $\tau \in \mathrm{Diff}^+(\mathbb{CP}^1)$ such that $\mathrm{pr} \circ s = \tau \circ \mathrm{pr}$ and τ preserves $\bigcup_{i=1}^n \mathrm{pr}^{-1}(V_i)$. Hence

$$\mathrm{pr} \circ \varphi|_{M_N - \bigcup\limits_{i=1}^{n} \mathrm{pr}^{-1}(V_i)} = \mathrm{pr} \circ (\rho_v(f) \circ r \circ s)|_{M_N - \bigcup\limits_{i=1}^{n} \mathrm{pr}^{-1}(V_i)} = \tau \circ \mathrm{pr}\,|_{M_N - \bigcup\limits_{i=1}^{n} \mathrm{pr}^{-1}(V_i)},$$

which shows that φ almost preserves pr.

Take any other primitive, isotropic class $w \in H_2(M_N; \mathbb{Z})$ with minimal genus 0; we proceed similarly as in the proof of Theorem 1.4. Apply Li–Li [15, Theorem 4.2] to obtain $\alpha \in \mathrm{Diff}^+(M_N)$ such that $[\alpha](w) = v$ and $[\alpha] \circ h \circ [\alpha^{-1}] \in \mathrm{Stab}(v)$. There exists a diffeomorphism $\varphi \in \mathrm{Diff}^+(M_N)$ almost preserving $\mathrm{pr} : M_N \to \mathbb{CP}^1$ with $[\varphi] = [\alpha] \circ h \circ [\alpha^{-1}]$. Then $\alpha^{-1} \circ \varphi \circ \alpha$ almost preserves $\mathrm{pr} \circ \alpha$, and $[\alpha^{-1} \circ \varphi \circ \alpha] = h$. □

Corollary 4.2. *Let $2 \leq N \leq 8$ and let $w \in H_2(M_N)$ be any primitive, isotropic class. For any $h \in \mathrm{Stab}(w)$, there exists $\varphi \in \mathrm{Diff}^+(M_N)$ almost preserving some holomorphic genus-0 Lefschetz fibration $p : M_N \to \mathbb{CP}^1$ whose generic fiber represents the homology class w such that $[\varphi] = h \in \mathrm{Mod}(M_N)$.*

Proof. If $2 \leq N \leq 8$ and $w \in H_2(M_N; \mathbb{Z})$ is a primitive, isotropic class then Lemma 2.7(b) says that the minimal genus of w is 0. Now apply Theorem 1.5. □

Acknowledgments

I thank Benson Farb for suggesting this problem, for his continuous encouragement, advice, and guidance throughout this project, and for many helpful comments on an earlier draft of this paper. I would also like to thank Farb and Eduard Looijenga for sharing their preliminary results on K3 manifolds (which appeared in [9]) with me. I would also like to thank Carlos A. Serván for many useful conversations about rational 4-manifolds and Lefschetz fibrations. I thank Dan Margalit for comments on an earlier draft that improved the exposition of this paper. Finally, I thank anonymous referees for their careful reading, corrections, and extensive suggestions.

References

[1] L. Bayle and A. Beauville, "Birational involutions of $\mathbb{P}^2$", *Asian J. Math* **4** (2000), 11–18.

[2] E. Bertini, "Ricerche sulle trasformazioni univoche involutorie nel piano", *Ann. Mat. Pura Appl.* **8** (1877), 244–286.

[3] J. Blanc, "Elements and cyclic subgroups of finite order of the Cremona group", *Comment. Math. Helv.* **86**:2 (2011), 469–497. MR

[4] S. Cantat, "Dynamique des automorphismes des surfaces K3", *Acta Math.* **187** (2001), 1–57.

[5] W. Chen, T.-J. Li, and W. Wu, "Symplectic rational G-surfaces and equivariant symplectic cones", *J. Differential Geom.* **119**:2 (2021), 221, 260.

[6] T. D. Cochran and N. Habegger, "On the homotopy theory of simply connected four manifolds", *Topology* **29**:4 (1990), 419–440.

[7] J. Diller and C. Favre, "Dynamics of bimeromorphic maps of surfaces", *Am. J. Math.* **123** (2001), 1135–1169.

[8] I. V. Dolgachev and V. A. Iskovskikh, "Finite subgroups of the plane Cremona group", pp. 443–548 in *Algebra*, *arithmetic*, *and geometry*: *in honor of Yu. I. Manin*, vol. 1, edited by Y. Tschinkel and Y. Zarhin, Birkhäuser, 2009.

[9] B. Farb and E. Looijenga, "Moduli spaces and period mappings of genus one fibered K3 surfaces", preprint, 2023. arXiv 2303.08214

[10] M. H. Freedman, "The topology of four-dimensional manifolds", *J. Differ. Geom.* **17** (1982), 357–453.

[11] D. Gabai, D. T. Gay, D. Hartman, V. Krushkal, and M. Powell, "Pseudo-isotopies of simply connected 4-manifolds", preprint, 2023. arXiv 2311.11196

[12] M. H. Gizatullin, "Rational G-surfaces", *Math. USSR Izv.* **16** (1981), 103–134.

[13] R. E. Gompf, "Locally holomorphic maps yield symplectic structures", *Comm. Anal. Geom.* **13** (2005), 511–525.

[14] R. E. Gompf and A. I. Stipsicz, *4-manifolds and Kirby calculus*, American Mathematical Society, 1999.

[15] B.-H. Li and T.-J. Li, "Symplectic genus, minimal genus and diffeomorphisms", *Asian J. Math.* **6** (2002), 123–144.

[16] L. J. Mordell, "The definite quadratic forms in eight variables with determinant unity", *J. Math. Pures Appl.* **17** (1938), 41–46.

[17] P. Oszváth and Z. Szabó, "The symplectic Thom conjecture", *Ann. Math.* **151** (2000), 93–124.

[18] B. Perron, "Pseudo-isotopies et isotopies en dimension quatre dans la catégorie topologique", *Topology* **25**:4 (1986), 381–397. MR

[19] F. Quinn, "Isotopy of 4-manifolds", *J. Differ. Geom.* **24** (1986), 343–372.

[20] W. P. Thurston, *Three-dimensional geometry and topology*, vol. 1, Princeton University Press, 1997.

[21] C. Wall, "Diffeomorphisms of 4-manifolds", *J. London Math. Soc.* **s1-39**:1 (1964), 131–140.

Received July 6, 2023. Revised September 10, 2025.

SERAPHINA EUN BI LEE
DEPARTMENT OF MATHEMATICS
HARVARD UNIVERSITY
UNITED STATES
slee@math.harvard.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 339, No. 2, 2025
https://doi.org/10.2140/pjm.2025.339.309

LOWER BOUNDS FOR FRACTIONAL ORLICZ-TYPE EIGENVALUES

ARIEL SALORT

We establish precise lower bounds for the eigenvalues and critical values associated with the fractional A-Laplacian operator, where A is a Young function. The obtained bounds are expressed in terms of the domain geometry and the growth properties of the function A. We do not assume that A or its complementary function satisfies the Δ_2 condition.

1. Introduction

One of the central problems in the analysis of p-Laplacian type operators is the study of its eigenvalues, which are closely related to the structure of the underlying domain and the boundary conditions imposed. In particular, the first eigenvalue

$$\lambda_1 = \inf\left\{ \int_\Omega |\nabla u|^p \, dx \text{ for } u \in C_c^\infty(\Omega) \text{ such that } \int_\Omega \omega(x)|u|^p \, dx = 1 \right\}$$

related to the nonlinear problem defined for $p > 1$ as

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda \omega |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

has been extensively studied, as it provides important information about the behavior of solutions to the geometry of the domain. Here ω is a suitable weight function and $\Omega \subset \mathbb{R}^n$ denotes an open and bounded set. See for instance [29; 30].

While upper bounds for eigenvalues have been established in a variety of settings, obtaining sharp lower bounds remains a challenging and active area of research. Lower bounds are of particular importance because they offer insights into the stability and regularity of solutions, as well as estimates for the oscillatory behavior of eigenfunctions.

In the one-dimensional case with a weight function, lower bounds were obtained in [17; 28; 34; 35; 40]. When $\Omega \subset \mathbb{R}^n$, $n \geq 2$ several results are known. In [6; 16], lower bounds in terms of the measure of the domain were obtained. Indeed, when $\omega \in L^\theta(\Omega)$ then we have

MSC2020: 35P20, 46E30, 35R11, 47J10.

Keywords: eigenvalues, nonlocal, fractional Orlicz–Sobolev, lower bound.

$$\frac{C}{|\Omega|^{\frac{p}{n}-\frac{1}{\theta}}\|\omega\|_{L^\theta(\Omega)}} \leq \lambda_1 \quad \text{where } \theta = \begin{cases} \theta_0 \in \left(\frac{n}{p}, \infty\right] & \text{when } 1 < p < n \\ 1 & \text{when } n < p, \end{cases}$$

where $C = C(n, p) > 0$. In addition, in [33; 35], more accurate bounds involving the inner radius r_Ω of Ω are obtained as a consequence of Lyapunov type inequalities:

$$\text{(1-2)} \qquad \frac{C}{r_\Omega^{p-\frac{n}{\theta}}\|\omega\|_{L^\theta(\Omega)}} \leq \lambda_1 \quad \text{where } \theta = \begin{cases} \theta_0 \in \left(\frac{n}{p}, \infty\right] & \text{when } 1 < p < n, \\ 1 & \text{when } n < p, \end{cases}$$

where $C = C(n, p) > 0$ and $r_\Omega := \max\{\operatorname{dist}(x, \partial\Omega) : x \in \Omega\}$ is the inradius of Ω. In [26] these results were extended to the nonlocal case, obtaining that when $\omega \in L^\theta(\Omega)$

$$\text{(1-3)} \qquad \frac{C}{r_\Omega^{sp-\frac{n}{\theta}}\|\omega\|_{L^\theta(\Omega)}} \leq \lambda_1^s \quad \text{where } \theta = \begin{cases} \theta_0 \in \left(\frac{n}{sp}, \infty\right] & \text{when } 1 < sp < n, \\ 1 & \text{when } n < sp, \end{cases}$$

where λ_1^s is the first eigenvalue related to the fractional p-Laplacian operator of order $s \in (0, 1)$.

When operators follow a growth more general than a power law, the concept of eigenvalue becomes highly dependent on the normalization of the eigenfunction due to the potential lack of homogeneity. More precisely, equation (1-1) can be generalized by replacing the power p with a so-called Young function: given a Young function A, and a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, consider the problem

$$\text{(1-4)} \qquad \begin{cases} -\operatorname{div}\Big(a(|\nabla u|)\dfrac{\nabla u}{|\nabla u|}\Big) = \lambda\omega a(|u|)\dfrac{u}{|u|} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda \in \mathbb{R}$ is the eigenvalue parameter, ω is a suitable weight function and $a(t) = A'(t)$, $t > 0$. Observe that (1-4) boils down to (1-1) when $A(t) = t^p$, $p > 1$.

In this case, in [23; 25] it is proved that given $\alpha > 0$ there exists a critical value $\lambda_{1,\alpha} > 0$ and a function u_α such that $\int_\Omega \omega A(|u_\alpha|)\,dx = \alpha$ and $\frac{1}{\alpha}\int_\Omega A(|\nabla u_\alpha|)\,dx = \lambda_{1,\alpha}$. From this, one can deduce the existence of an eigenvalue $\Lambda_{1,\alpha}$ with corresponding eigenfunction u_α in the sense that pair $(\Lambda_{1,\alpha}, u_\alpha)$ satisfies (1-4) in the weak sense. The quantities Λ_1 and λ_1 are in general different and coincide only when A is homogeneous.

A first result concerning the lower bounds of (1-4) can be found in [32]. In the one-dimensional case, assuming that A satisfies the Δ_2 condition (that is, there exists $c \geq 1$ such that $A(2t) \leq cA(t)$ for any $t > 0$), the authors establish that for any $\alpha > 0$,

$$\frac{C_p}{\|\omega\|_{L^1(a,b)}} \leq \lambda_{1,\alpha}$$

where $\Omega = (a, b) \subset \mathbb{R}$, and $p > 1$ is defined as $\lim_{r\to\infty} A(rt)/A(r) = t^{p-1}$. Similar bounds in the one-dimensional case were found in [39] when A is a submultiplicative

Young function (that is, there exists $c \geq 1$ such that $A(rt) \leq cA(r)A(t)$ for any $r, t \geq 0$).

When $\Omega \subset \mathbb{R}^n$, $n \geq 1$ and A is a submultiplicative Young function, in Theorems 4.4 and 4.2 of [37] it is proved that given $\alpha > 0$ there exists a computable constant $C > 0$, independent of α and depending only on A and n, such that

$$\frac{1}{\alpha}\left[A\left(\frac{C\sigma(r_\Omega)}{A^{-1}\big(\|\omega\|^{-1}_{L^1(\Omega)}\big)}\right)\right]^{-1} \leq \lambda_{1,\alpha} \quad \text{when } \omega \in L^1(\Omega) \text{ and } \sigma(1) < \infty,$$

$$\frac{1}{\alpha}\left[A\left(\frac{C}{A^{-1}\big(\|\omega\|^{-1}_{L^\infty(\Omega)}/\tau_A(\Omega)\big)}\right)\right]^{-1} \leq \lambda_{1,\alpha} \quad \text{when } \omega \in L^\infty(\Omega) \text{ and } \sigma(1) = \infty,$$

where

$$\sigma(t) = \int_{t^{-n}}^{\infty} A^{-1}(r) r^{-(1+\frac{1}{n})}\, dr \quad \text{and} \quad \tau_A(\Omega) := |\Omega|(\tilde{A})^{-1}(|\Omega|^{-1}),$$

$\tilde{A}$ being the complementary function of A. These inequalities, in the case $A(t) = t^p$, $p > 1$, recover the corresponding inequalities in (1-2).

In the last years nonlocal operators with nonstandard growth have received an increasing amount of attention and an active community is currently working on problems involving operators defined in terms of a Young function $A(t) = \int_0^t a(\tau)\, d\tau$ having the form

$$(-\Delta_a)^s u(x) = \text{p.v.} \int_{\mathbb{R}^n} a(|D^s u|) \frac{D^s u}{|D^s u|} \frac{dy}{|x-y|^n},$$

where $s \in (0,1)$, $D^s u(x, y) = (u(x) - u(y))/|x-y|^s$ and p.v. stands for "principal value". This nonhomogeneous operator is a generalization of the fractional p-Laplacian of order $s \in (0,1)$. See also [1; 2; 3; 4; 8; 12; 13; 14; 19; 20; 18; 21; 22; 36; 37; 38]. In particular, the nonlocal version of problem (1-4) takes the form

$$\begin{cases} (-\Delta_a)^s u = \lambda\omega \dfrac{a(|u|)}{|u|} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1-5}$$

where $\lambda \in \mathbb{R}$ is the eigenvalue parameter and ω is a suitable weight function. We refer to [7; 12; 15; 19; 22; 36; 37; 38] for properties and results related with the nonlocal nonstandard growth eigenvalue problem (1-5).

As in the local case, the nonhomogeneity of the problem makes the eigenvalue highly dependent on the normalization of the eigenfunction. More precisely, in [36; 38] it is proved that given $\alpha > 0$ there exists a critical value $\lambda^s_{1,\alpha} > 0$ and u^s_α such that $\int_\Omega \omega A(|u^s_\alpha|)\, dx = \alpha$ and $\frac{1}{\alpha}\int_\Omega A(|D^s u^s_\alpha|)\, dx = \lambda^s_{1,\alpha}$. More precisely, we consider

$$\lambda^s_{1,\alpha} = \inf\left\{ \frac{1}{\alpha}\iint_{\mathbb{R}^{2n}} A(|D^s u|)\,d\nu_n : u \in C^\infty_c(\Omega),\ \int_\Omega \omega A(|u|)\,dx = \alpha \right\}, \tag{1-6}$$

where $d\nu_n = |x-y|^{-n}\,dx\,dy$ and ω is a suitable weight function.

From this can be deduced the existence of an eigenvalue $\Lambda^s_{1,\alpha}$ with corresponding eigenfunction u^s_α in the sense that pair $(\Lambda^s_{1,\alpha}, u^s_\alpha)$ satisfies (1-5) in the weak sense, being Λ^s_1 and λ^s_1 different when A is inhomogeneous, but comparable each other when A satisfies the doubling condition.

To the best of our knowledge, estimates for $\Lambda^s_{1,\alpha}$ and $\lambda^s_{1,\alpha}$ have not been previously studied in the literature. The goal of this article is to establish lower bounds for these quantities.

An important aspect of analyzing (1-5) is whether the Young function A satisfies a so-called *doubling condition*. This condition is crucial for controlling constants within the function:

- A satisfies the *doubling condition near infinity* (denoted as $A \in \Delta_2^\infty$) if there exists $C_\infty \ge 2$ such that $A(2t) \le C_\infty A(t)$ for all $t \ge T_\infty$,
- A satisfies the *doubling condition near zero* (denoted as $A \in \Delta_2^0$) if there exists $C_0 \ge 2$ such that $A(2t) \le C_0 A(t)$ for all $t \le T_0$.
- A satisfies the *global doubling condition* (denoted as $A \in \Delta_2$) if the previous condition and fulfilled, and it is denoted $\Delta_2 = \Delta_2^0 \cap \Delta_2^\infty$.

Assuming or relaxing the doubling condition introduces significant technical challenges in the analysis, such as the potential loss of reflexivity in the associated fractional Orlicz–Sobolev spaces. Moreover, imposing this condition on either the function A or its conjugate $\tilde{A}$ is known to yield both upper and lower bounds for the corresponding Young function in terms of power functions. For further details, see Section 2.1.

To characterize the growth of a general Young function A (which may not satisfy the doubling condition), we use the Matuszewska–Orlicz functions associated with A, along with the corresponding indexes, defined as follows:

$$M_A(t) = \sup_{\alpha>0} \frac{A(\alpha t)}{A(\alpha)}, \quad M_0(t, A) = \liminf_{\alpha\to 0^+} \frac{A(\alpha t)}{A(\alpha)}, \quad M_\infty(t, A) = \liminf_{\alpha\to\infty} \frac{A(\alpha t)}{A(\alpha)},$$

$$i(A) = \lim_{t\to\infty} \frac{\log M_A(t)}{\log t}, \quad i_0(A) = \lim_{t\to\infty} \frac{\log M_0(t, A)}{\log t}, \quad i_\infty(A) = \lim_{t\to\infty} \frac{\log M_\infty(t, A)}{\log t}.$$

See Section 2.2 for details and precise definitions.

Main results. We emphasize that, unless explicitly stated otherwise, we do not assume the Δ_2 condition on A or its complementary function $\tilde{A}$.

Theorem 1.1. *Let $s \in (0,1)$ and let A be a Young function satisfying (2-4). Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with inner radius r_Ω. Given $\omega \in L^1(\mathbb{R}^n)$ and $\alpha > 0$, consider the critical value $\lambda^s_{1,\alpha}$ defined in (1-6).*

(i) *There exists a unique $\alpha_0 > 0$ satisfying the equation*

$$\alpha_0 \lambda^s_{1,\alpha_0} = r^n_\Omega.$$

(ii) *Assume that $i_0(A) > \frac{n}{s}$ when $\alpha \le \alpha_0$ or $i_\infty(A) > \frac{n}{s}$ when $\alpha > \alpha_0$. Then, there exists a positive, computable constant $C = C(n, s, A)$ such that*

$$\frac{C}{\|\omega\|_{L^1(\Omega)}} \frac{r^n_\Omega}{M_A(r^s_\Omega)} \le \lambda^s_{1,\alpha}.$$

In particular, this holds when $i_0(A) > \frac{n}{s}$ if $\alpha \ll 1$ or when $i_\infty(A) > \frac{n}{s}$ if $\alpha \gg 1$.

In Section 2.3, we compute the Matuszewska–Orlicz functions and indices for several notable Young functions. With this, we state Theorem 1.1 for some interesting cases:

(i) When $A(t) = t^p$, $p > 1$ the eigenvalue problem becomes homogeneous. Then, for any $\alpha > 0$, when $sp > n$, we have

$$\frac{1}{\|\omega\|_{L^1(\Omega)}} \frac{C}{r^{sp-n}_\Omega} \le \lambda^s_{1,\alpha}$$

which in some extent recovers (1-3).

(ii) Given $1 < p < q < \infty$, consider $A(t) = \frac{t^p}{p} + \frac{t^q}{q}$. Then $A \in \Delta_2$. This gives the eigenvalue problem for the fractional p-q-Laplacian (see for instance [5]). When $\alpha \ll 1$ and $sp > n$, or $\alpha \gg 1$ and $sq > n$, we have

$$\frac{1}{\|\omega\|_{L^1(\Omega)}} \frac{C}{\max\{r^{sp-n}_\Omega, r^{sq-n}_\Omega\}} \le \lambda^s_{1,\alpha}.$$

(iii) Given $p, q, r \ge 1$, consider $A(t) = t^p \ln^r(1+t^q)$. Then $A \in \Delta_2$. When $\alpha \ll 1$ and $s(p+qr) > n$, or $\alpha \gg 1$ and $sp > n$, then

$$\frac{1}{\|\omega\|_{L^1(\Omega)}} \frac{C}{\max\{r^{s(p-qr)-n}_\Omega, r^{sp-n}_\Omega\}} \le \lambda^s_{1,\alpha}.$$

(iv) For $k \in \mathbb{N}$ define $A(t) = e^t - \sum_{j=0}^{k-1} t^j/j!$. Then $A \in \Delta^0_2$ but $A \notin \Delta^\infty_2$. When $\alpha \ll 1$ and $sk > n$, when $r_\Omega \le 1$ we have

$$\frac{1}{\|\omega\|_{L^1(\Omega)}} \frac{C}{r^{sk-n}_\Omega} \le \lambda^s_{1,\alpha}.$$

(v) Consider the function $A(t) = e^{e^t} - e$. Then $A \in \Delta_2^0$ but $A \notin \Delta_2^\infty$. When $\alpha \gg 1$ we have

$$\frac{1}{\|\omega\|_{L^1(\Omega)}} \frac{C}{r_\Omega^{s-n}} \leq \lambda_{1,\alpha}^s.$$

In Theorem 4.1, we also derive a lower bound for the minimizer using the inverse of the Young function instead of the Matuszewska–Orlicz function. Moreover, in Corollary 4.2 we prove that the same lower bounds hold for the eigenvalue $\Lambda_{1,\alpha}^s$ when $A \in \Delta_2$.

Theorem 1.2. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with diameter d_Ω containing the origin. Let $s \in (0,1)$ and let A be a Young function satisfying conditions (2-6) and $i(A) < \frac{n}{s}$. Given $\omega \in L^\infty(\mathbb{R}^n)$ and $\alpha > 0$, consider $\lambda_{1,\alpha}^s$ as in (1-6). Then, there exists a positive constant $C = C(n, s, A)$ such that*

$$\frac{C}{\|\omega\|_{L^\infty(\Omega)} M_A(d_\Omega^s)} \leq \lambda_{1,\alpha}^s.$$

To apply Theorem 1.2, there is an implicit growth condition: the condition $i(A) < \frac{n}{s}$ is not satisfied when $A \notin \Delta_2^0$ or $A \notin \Delta_2^\infty$ (see Lemma 2.2 for details). Therefore, this result can be applied only when $A \in \Delta_2$.

Under the assumption of the Δ_2 condition on A, we improve Theorem 1.2 by replacing the diameter with the inner radius.

Theorem 1.3. *Let $s \in (0,1)$ and let $A \in \Delta_2$ be a Young function satisfying (2-7). Assume that $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain with inner radius r_Ω. Given $\omega \in L^\infty(\mathbb{R}^n)$ and $\alpha > 0$, consider $\lambda_{1,\alpha}^s$ as in (1-6). Then, there exists a positive constant $C = C(n, s, A)$ such that*

$$\frac{C}{\|\omega\|_{L^\infty(\Omega)} M_A(r_\Omega^s)} \leq \lambda_{1,\alpha}^s.$$

Here we state some notable examples derived from Theorem 1.3.

(i) When $A(t) = t^p$, $p > 1$, this gives the eigenvalue problem for the fractional p-Laplacian, which is homogeneous. Then for any $\alpha > 0$, when $sp < n$:

$$\frac{1}{\|\omega\|_{L^\infty(\Omega)}} \frac{C}{r_\Omega^{sp}} \leq \lambda_{1,\alpha}^s$$

which in some extent recovers (1-3).

(ii) Given $1 < p < q < \infty$, consider $A(t) = \frac{t^p}{p} + \frac{t^q}{q}$. Then $A \in \Delta_2$. This gives the eigenvalue problem for the fractional $p-q$-Laplacian (see for instance [5]). Then, for $\alpha > 0$, when $sq < n$, we have

$$\frac{1}{\|\omega\|_{L^\infty(\Omega)}} \frac{C}{\max\{r_\Omega^{sq}, r_\Omega^{sp}\}} \leq \lambda_{1,\alpha}^s.$$

(iii) Given $p, q, r \geq 1$, consider $A(t) = t^p \ln^r(1+t^q)$. Then $A \in \Delta_2$. Then, for $\alpha > 0$, when $s(p+qr) < n$,

$$\frac{1}{\|\omega\|_{L^\infty(\Omega)}} \frac{C}{\max\{r_\Omega^{sp}, r_\Omega^{s(p+qr)}\}} \leq \lambda_{1,\alpha}^s.$$

In particular, Theorem 1.2 establishes these same inequalities but with the diameter in place of the inner radius under the same hypothesis on the parameters.

The same lower bounds established in Theorems 1.2 and 1.3 also hold for the eigenvalue $\Lambda_{1,\alpha}^s$, as stated in Corollary 4.3.

2. Preliminaries

2.1. *Young functions.* A function $A : [0, \infty) \to [0, \infty]$ is called a *Young function* if it is convex, nonconstant, left continuous and $A(0) = 0$. A function with these properties admits the representation

$$A(t) = \int_0^t a(\tau)\, d\tau \qquad \text{for } t \geq 0,$$

for some nondecreasing, left continuous function $a : [0, \infty) \to [0, \infty]$.

The *complementary function* $\tilde{A}$ of A is the Young function defined as

$$\tilde{A}(t) = \sup\{\tau t - A(\tau) : \tau \geq 0\} \qquad \text{for } t \geq 0.$$

One has

$$t \leq A^{-1}(t)(\tilde{A})^{-1}(t) \leq 2t \qquad \text{for } t \geq 0.$$

From the convexity of the Young function it is immediate that

$$A(rt) \leq rA(t) \quad \text{for } 0 < r < 1, \qquad A(rt) \geq rA(t) \quad \text{for } r > 1. \tag{2-1}$$

From the integral representation of the Young function it follows that

$$G(2t) > tg(t), \qquad G(t) \leq tg(t).$$

2.1.1. *The doubling condition.* A Young function A lies in Δ_2^∞ (or in Δ_2^0) if and only if there exists $p > 1$ and $T_\infty > 0$ (or $T_0 > 0$) such that

$$\frac{ta(t)}{A(t)} \leq p \quad \text{for all } t \geq T_\infty \quad (\text{or } 0 < t \leq T_0). \tag{2-2}$$

It is easy to see that

- $A \in \Delta_2^\infty$ if there exists $C_\infty \geq 2$ such that $A(2t) \leq C_\infty A(t)$ for all $t \geq T_\infty$, and
- $A \in \Delta_2^0$ if there exists $C_0 \geq 2$ such that $A(2t) \leq C_0 A(t)$ for all $t \leq T_0$.

We define $\Delta_2 = \Delta_2^0 \cap \Delta_2^\infty$. The following statements are equivalent:

(i) $A \in \Delta_2$.

(ii) There exists $p > 1$ such that $ta(t)/A(t) \leq p$ for all $t > 0$.

(iii) There exists $C \geq 2$ such that $A(2t) \leq CA(t)$ for all $t > 0$.

Proposition 2.1. *Let A be a Young function such that $A \in \Delta_2^0$ and let $p > 1$ be the number defined in (2-2). Then*

$$\tau^p A(t) \leq A(t\tau) \leq \tau A(t) \quad \textit{for } 0 < \tau < 1 \textit{ and } t < T_0.$$

Similarly, if $A \in \Delta_2^\infty$ we have

$$\tau A(t) \leq A(t\tau) \leq \tau^p A(t) \quad \textit{for } \tau > 1 \textit{ and } t > T_\infty.$$

Hence, if $A \in \Delta_2$, then

$$\min\{\tau, \tau^p\} A(t) \leq A(t\tau) \leq \max\{\tau, \tau^p\} A(t) \quad \textit{for } \tau > 0.$$

2.1.2. *Ordering of functions.* A Young function A dominates another Young function B near infinity if there exists a positive constant c and t_0 such that

$$B(t) \leq A(ct) \qquad \text{for } t \geq t_0.$$

The functions A and B are called equivalent near infinity if they dominate each other in the respective range of values of their arguments; in this case we write $A \simeq B$.

$A \approx B$ means that A and B are bounded by each other, up to a multiplicative constant.

2.2. *Matuszewska indexes.* The Matuszewska–Orlicz functions associated to the Young function A are defined by

$$M_A(t) = \sup_{\alpha > 0} \frac{A(\alpha t)}{A(\alpha)}, \quad M_0(t, A) = \liminf_{\alpha \to 0^+} \frac{A(\alpha t)}{A(\alpha)}, \quad M_\infty(t, A) = \liminf_{\alpha \to \infty} \frac{A(\alpha t)}{A(\alpha)}.$$

They are nondecreasing, submultiplicative in the variable t and equal to 1 at $t = 1$. We also consider the Matuszewska–Orlicz indices at zero and infinity, defined as

$$i(A) = \lim_{t \to \infty} \frac{\log M_A(t)}{\log t}, \quad i_0(A) = \lim_{t \to \infty} \frac{\log M_0(t, A)}{\log t}, \quad i_\infty(A) = \lim_{t \to \infty} \frac{\log M_\infty(t, A)}{\log t}.$$

When there is no confusion, we will remove the dependence on A.

It is easy to see that the Matuszewska function can be bounded in terms of powers if and only if $A, \tilde{A} \in \Delta_2$, that is,

$$\min\{t^{p_A^+}, t^{p_A^-}\} \leq M(t, A) \leq \max\{t^{p_A^-}, t^{p_A^+}\} \tag{2-3}$$

where $p_A^+ = \sup_{t>0} a(t)t/A(t)$ and $p_A^- = \inf_{t>0} a(t)t/A(t)$.

For a comprehensive approach on these functions and indices we refer to the monograph [31].

2.3. ***Examples of Young functions.*** Here we provide for some examples of Young functions and compute their corresponding Matuszewska functions and indexes. For further examples we refer to [31].

Example 1. Let $p > 1$, and assume that

$$A(t) \simeq t^p \qquad \text{when } t \ll 1.$$

Then $A \in \Delta_2^0$. In this case we have $M_0(t) = t^p$ and $i_0(A) = p$. If we assume that

$$A(t) \simeq t^p \qquad \text{when } t \gg 1,$$

then $A \in \Delta_2^\infty$, $M_\infty(t) = t^p$ and $i_\infty(A) = p$.
In particular, when $A(t) = t^p$, $M(t, A) = M_0(t, A) = M_\infty(t, A) = t^p$ and $i(A) = i_0(A) = i_\infty(A) = p$. As a special case, if $1 < p < q < \infty$,

$$A(t) = \frac{t^p}{p} + \frac{t^q}{q};$$

then

$$M_0(t, A) = t^p, \qquad M_\infty(t, A) = t^q, \qquad M(t, A) = \max\{t^p, t^q\},$$

and $i_0(A) = p$, $i_\infty(A) = i(A) = q$.

Example 2. Let $r \geq 0$ and $p \geq 1$. Then if

$$A(t) \simeq t^p \ln^r t \qquad \text{when } t \ll 1$$

then $A \in \Delta_2^0$ and in this case, $M_0(t) = t^p$ and $i_0(A) = p$. If

$$A(t) \simeq t^p \ln^r t \qquad \text{when } t \gg 1,$$

then $M_\infty(t) = t^p$ and $i_\infty(A) = p$.

As a special case, if $p, q, r \geq 0$ and $A(t) = t^p \ln^r(1 + t^q)$ then

$$M_0(t, A) = t^{p+qr}, \qquad M_\infty(t, A) = t^p, \qquad M(t, A) = \max\{t^p, t^{p+qr}\}$$

and $i(A) = i_0(A) = p + qr$, $i_\infty(A) = p$.

Example 3. For $k \in \mathbb{N}$ define $A(t) = e^t - \sum_{j=0}^{k-1} t^j / j!$. Then $A \in \Delta_2^0$ but $A \notin \Delta_2^\infty$, and

$$M_0(t) = t^k, \qquad M_\infty(t) = \begin{cases} 0 & \text{if } 0 < t < 1, \\ \infty & \text{if } t > 1, \end{cases} \qquad M(t) = \begin{cases} t^k & \text{if } 0 < t \leq 1, \\ \infty & \text{if } t > 1, \end{cases}$$

$i_0(A) = k$, $i(A) = i_\infty(A) = \infty$.

Example 4. For $r > 0$, assume that $A(t) \simeq e^{-t^{-r}}$ for $t \ll 1$. Then $A \notin \Delta_2^0$ and

$$M_0(t) = \begin{cases} 0 & \text{if } 0 < t < 1, \\ 1 & \text{if } t = 1, \\ \infty & \text{if } t > 1. \end{cases}$$

In particular, when $A(t) = e^{-t^{-r}}$, we have

$$M_\infty(A) = 1, \qquad M_0(t) = \begin{cases} 0 & \text{if } 0 < t < 1, \\ 1 & \text{if } t = 1, \\ \infty & \text{if } t > 1, \end{cases} \qquad M(t) = \begin{cases} 1 & \text{if } 0 < t \le 1, \\ \infty & \text{if } t > 1. \end{cases}$$

and $i_\infty(A) = i(A) = 0$, $i_0(A) = \infty$.

Example 5. Assume that $A(t) \simeq e^{e^t}$ for $t \gg 1$. Then $A \notin \Delta_2^\infty$ and

$$M_\infty(t) = \begin{cases} 0 & \text{if } 0 < t < 1, \\ 1 & \text{if } t = 1, \\ \infty & \text{if } t > 1. \end{cases}$$

In particular, when $A(t) = e^{e^t} - e$, since $A(t) \simeq et$ when $t \ll 1$,

$$M_0(t) = t, \qquad M_\infty(t) = \begin{cases} 0 & \text{if } 0 < t < 1, \\ 1 & \text{if } t = 1, \\ \infty & \text{if } t > 1, \end{cases} \qquad M(t) = \begin{cases} t & \text{if } 0 < t \le 1, \\ \infty & \text{if } t > 1, \end{cases}$$

and $i_0(A) = 1$, $i(A) = i_\infty(A) = \infty$.

Lemma 2.2. *Let A be a Young function such that $A \notin \Delta_2^k$ for $k = 0$ or $k = \infty$. Then*

$$M(t) = \begin{cases} 1 & \text{if } t = 1, \\ \infty & \text{if } t > 1. \end{cases}$$

When $0 < t < 1$ we have $M(t) \le t$. Moreover, $i(A) = \infty$.

Proof. First, observe that from (2-1), we have $M(t) \le t \quad$ for $0 < t < 1$.

In light of [11, Proposition 2.1], if $A \notin \Delta_2^0$ or $A \notin \Delta_2^\infty$ then

$$M_k(t) = \begin{cases} 1 & \text{if } t = 1, \\ \infty & \text{if } t > 1, \end{cases}$$

$k = 0, \infty$, respectively. By definition, $M_0(t) \le M(t)$ and $M_\infty(t) \le M(t)$ for any $t > 0$. This gives immediately that when $A \notin \Delta_2^k$ for $k = 0$ or $k = \infty$, one has $M(1) = 1$ and $M(t) = \infty$ when $t > 1$.

Since $M(t) = \infty$ for $t > 1$, this gives that $i(A) = \infty$. □

Lemma 2.3. *If $A \in \Delta_2$ then there exists $p \ge 1$ such that $M(t) = t^p$.*

Proof. Since $A \in \Delta_2$, there exists $q > 1$ such that $A(rt) \le \max\{t, t^q\} A(r)$ for any $t, r \ge 0$. Then M is finite for any $t > 0$:

$$M(t) = \sup_{\alpha>0} \frac{A(\alpha t)}{A(\alpha)} \le \max\{t, t^q\}.$$

Moreover, observe that

$$M(rt) = \sup_{\alpha>0} \frac{A(tr\alpha)}{A(r\alpha)} \frac{A(r\alpha)}{A(\alpha)} \le \sup_{\alpha>0} M(t) \frac{A(r\alpha)}{A(\alpha)} \le M(t)M(s)$$

and $M(1) = 1$, that is, $M(t)$ is submultiplicative.

Define $v(t) = \ln(M(e^t))$. This function is additive, that is, $v(r+t) = v(r) + v(t)$ for any $s, t \in \mathbb{R}$. It is well known that measurable additive functions are linear, therefore, there exists $p \in \mathbb{R}$ such that $v(t) = pt$ from there $M(t) = t^p$. Finally, from (2-1),

$$M(t) \le t \quad \text{for } 0 < t < 1 \quad \text{and} \quad M(t) \ge t \quad \text{for } t > 1.$$

which implies that $p \ge 1$. □

2.4. *Some useful inequalities.* Given $s \in (0, 1)$ and a Young function A such that

$$\int^{\infty} \left(\frac{t}{A(t)} \right)^{\frac{s}{n-s}} dt < \infty \tag{2-4}$$

consider the Young function E given by

$$E(t) = t^{\frac{n}{n-s}} \int_t^{\infty} \frac{\tilde{A}(\tau)}{\tau^{1+\frac{n}{n-s}}} \, d\tau \quad \text{for } t \ge 0. \tag{2-5}$$

Consider also $\Psi_s : (0, \infty) \to (0, \infty)$ defined by

$$\Psi_s(r) = \frac{1}{r^{n-s} E^{-1}(r^{-n})}, \qquad \text{for } r > 0.$$

Lemma 2.4 [3, Proposition 2.1]. *Let $s \in (0, 1)$ and let A be a Young function. Assume (2-4). Then*:

(i) *The function Ψ_s is nondecreasing.*

(ii) *Define the Young function $B = \tilde{E}$. Then $\Psi_s(r) \approx r^s B^{-1}(r^{-n})$ for $r > 0$.*

(iii) *If $i_\infty(A) > \frac{n}{s}$ then $B \simeq A$ near infinity, and $\Psi_s(r) \approx r^s A^{-1}(r^{-n})$ for $0 < r \le 1$.*

(iv) *If $i_0(A) > \frac{n}{s}$ then $B \simeq A$ near 0, and $\Psi_s(r) \approx r^s A^{-1}(r^{-n})$ for $r \ge 1$.*

The following modular Morrey-type inequality is proved in [3, Remark 4.3]:

Proposition 2.5. *Let $s \in (0,1)$ and let A be a Young function satisfying (2-4). Then, $W^s L^A(\mathbb{R}^n) \subset C^{\Psi_s(\cdot)}(\mathbb{R}^n)$. Moreover, for any $u \in W^s L^A(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$,*

$$|u(x) - u(y)| \le C_M |x-y|^s B^{-1}\left(\frac{1}{|x-y|^n} \iint_{\mathbb{R}^{2n}} A(D^s u(z,w))\, d\nu_n(z,w) \right)$$

for some constant C_M depending on n and s, where the Young function B is given by $B(t) = \tilde{E}(t)$, being E the Young function defined in (2-5).

The following Hardy type inequality is proved in Theorem 5.1 and Proposition C in [1].

Proposition 2.6. *Let $s \in (0,1)$ and let A be a Young function satisfying $i(A) < \frac{n}{s}$ and the conditions*

$$\int^{\infty} \left(\frac{t}{A(t)} \right)^{\frac{s}{n-s}} dt = \infty, \qquad \int_0 \left(\frac{t}{A(t)} \right)^{\frac{s}{n-s}} dt < \infty. \tag{2-6}$$

Then for all $u \in C_c^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} A\left(C_{H_1} \frac{|u(x)|}{|x|^s} \right) dx \le (1-s) \iint_{\mathbb{R}^{2n}} A(C_{H_2} |D^s u|)\, d\nu_n$$

for positive constants C_{H_1} and C_{H_2} depending only on n and s.

Given a bounded domain $\Omega \subset \mathbb{R}^n$, we denote $\delta_\Omega(x) := \inf\{|x-y| : y \in \Omega^c\}$ the distance from x to $\partial\Omega$.

The following Hardy type inequality is proved in [10, Theorem 1.5].

Proposition 2.7. *Let $s \in (0,1)$ and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. If $A \in \Delta_2$ and*

$$\lim_{k\to\infty} \sup_{t \ge 0} \frac{A(kt)}{k^{\frac{n}{s}} A(t)} = 0 = \lim_{k \to 0^+} \sup_{t \ge 0} \frac{A(kt)}{k^{\frac{1}{s}} A(t)}, \tag{2-7}$$

there exists a positive constant C_{H_3} such that for all $u \in C_c^\infty(\Omega)$

$$\int_\Omega A\left(\frac{|u(x)|}{\delta_\Omega(x)} \right) dx \le C_{H_3} \iint_{\mathbb{R}^{2n}} A(|D^s u|)\, d\nu_n.$$

2.5. ***Orlicz and Orlicz–Sobolev spaces.*** The main reference for Orlicz spaces is the book [27]. For Orlicz–Sobolev spaces, the reader can consult [24], for instance. Fractional-order Orlicz–Sobolev spaces, as we will use them here, were introduced in [18] and then further analyze by several authors. The results used in this paper can be found in [1; 2; 19].

2.5.1. *Orlicz spaces.* Given a bounded domain $\Omega \subset \mathbb{R}^n$ and a Young function A, the *Orlicz class* is defined as

$$\mathscr{L}^A(\Omega) := \left\{ u \in L^1_{\mathrm{loc}}(\Omega) : \int_\Omega A(|u|)\, dx < \infty \right\}.$$

The *Orlicz space* $L^A(\Omega)$ is defined as the linear hull of $\mathscr{L}^A(\Omega)$ and is characterized as

$$L^A(\Omega) = \left\{ u \in L^1_{\mathrm{loc}}(\Omega) : \text{ there exists } k > 0 \text{ such that } \int_\Omega A\left(\frac{|u|}{k}\right) dx < \infty \right\}.$$

In general the Orlicz class is strictly smaller than the Orlicz space, and $\mathscr{L}^A(\Omega) = L^A(\Omega)$ if and only if $A \in \Delta_2^\infty$. The space $L^A(\Omega)$ is a Banach space when it is endowed, for instance, with the *Luxemburg norm*, i.e.,

$$\|u\|_{L^A(\Omega)} = \|u\|_A := \inf \left\{ k > 0 : \int_\Omega A\left(\frac{|u|}{k}\right) dx \le 1 \right\}.$$

This space $L^A(\Omega)$ turns out to be separable if and only if $A \in \Delta_2^\infty$.

An important subspace of $L^A(\Omega)$ is $E^A(\Omega)$ that it is defined as the closure of the functions in $L^A(\Omega)$ that are bounded. This space is characterized as

$$E^A(\Omega) = \left\{ u \in L^1_{\mathrm{loc}}(\Omega) : \int_\Omega A\left(\frac{|u|}{k}\right) dx < \infty \text{ for all } k > 0 \right\}.$$

This subspace $E^A(\Omega)$ is separable, and we have the inclusions

$$E^A(\Omega) \subset \mathscr{L}^A(\Omega) \subset L^A(\Omega)$$

with equalities if and only if $A \in \Delta_2^\infty$. Moreover, the following duality relation holds

$$(E^A(\Omega))^* = L^{\tilde{A}}(\Omega),$$

where the equality is understood via the standard duality pairing. This automatically implies that $L^A(\Omega)$ is reflexive if and only if $A, \tilde{A} \in \Delta_2^\infty$.

2.5.2. *Fractional Orlicz–Sobolev spaces.* Given a fractional parameter $s \in (0, 1)$, we define the *Hölder quotient* of a function $u \in L^A(\Omega)$ as

$$D^s u(x, y) = \frac{u(x) - u(y)}{|x - y|^s}.$$

Then, the *fractional Orlicz–Sobolev space* of order s is defined as

$$W^s L^A(\mathbb{R}^n) := \{u \in L^A(\mathbb{R}^n) : D^s u \in L^A(\mathbb{R}^{2n}, d\nu_n)\},$$

where $d\nu_n = |x - y|^{-n}\, dx\, dy$ and

$$W^s E^A(\mathbb{R}^n) := \{u \in E^A(\mathbb{R}^n) : D^s u \in E^A(\mathbb{R}^{2n}, d\nu_n)\}.$$

When $A \in \Delta_2$, these spaces coincide and we write

$$W^{s,A}(\mathbb{R}^n) = W^s L^A(\mathbb{R}^n) = W^s E^A(\mathbb{R}^n).$$

The space $W^s L^A(\mathbb{R}^n)$ is reflexive if and only if $A, \tilde{A} \in \Delta_2$.

In these spaces the norm considered is

$$\|u\|_{W^s L^A(\mathbb{R}^n)} = \|u\|_{s,A} = \|u\|_A + [u]_{s,A,\mathbb{R}^n}$$

with

$$[u]_{s,A,\mathbb{R}^n} = \inf\left\{k > 0 : \iint_{\mathbb{R}^{2n}} A\left(\frac{|D^s u(x,y)|}{k}\right) d\nu_n \le 1\right\}.$$

Again, with this norm, $W^s L^A(\mathbb{R}^n)$ is a Banach space and $W^s E^A(\mathbb{R}^n)$ is a closed subspace. The space $W_0^s L^A(\Omega)$ is then defined as the closure of $C_c^\infty(\Omega)$ with respect to the topology $\sigma(W^s L^A(\mathbb{R}^n), W^s E^{\tilde{A}}(\mathbb{R}^n))$ and $W_0^s E^A(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in norm topology.

3. Eigenvalues and critical points

Let A be a Young function, and let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. For a fixed normalization parameter $\alpha > 0$, we define the *critical point* $\lambda_{1,\alpha}^s$ as

$$\lambda_{1,\alpha}^s = \inf\left\{\frac{1}{\alpha}\iint_{\mathbb{R}^{2n}} A(|D^s u|)\, d\nu_n : u \in C_c^\infty(\Omega), \int_\Omega \omega A(|u|)\, dx = \alpha\right\}. \tag{3-1}$$

Here, ω is a suitable positive weight function. We assume that $\omega \in L^1(\mathbb{R}^n)$ when (2-4) holds, and $\omega \in L^\infty(\mathbb{R}^n)$ when (2-6) holds.

In [38] (see also [36] when $A \in \Delta_2$) it is proved that (3-1) is solvable, that is, there exists a *minimizer* $u_\alpha^s \in W_0^s L^A(\Omega)$ such that $\int_\Omega \omega A(|u_\alpha^s|)\, dx = \alpha$ and

$$\iint_{\mathbb{R}^{2n}} A(|D^s u_\alpha^s|)\, d\nu_n = \lambda_{1,\alpha}^s \int_\Omega \omega A(|u_\alpha^s|)\, dx. \tag{3-2}$$

By applying an appropriate version of the Lagrange multipliers theorem, we can establish the existence of an eigenvalue $\Lambda_{1,\alpha}^s$ with corresponding eigenfunction u_α^s. More precisely, u_α is a weak solution to the following equation, with $\Lambda = \Lambda_{1,\alpha}^s$:

$$\begin{cases} (-\Delta_a)^s u = \Lambda \omega \dfrac{a(|u|)}{|u|} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3-3}$$

with $a(t) = A'(t)$ for $t > 0$, and where the fractional a-Laplacian of order $s \in (0, 1)$ is the nonlocal and nonstandard growth operator defined as

$$(-\Delta_a)^s u(x) = \text{p.v.} \int_{\mathbb{R}^n} a(|D^s u|) \frac{D^s u}{|D^s u|}\, d\nu_n,$$

that is, for all $v \in C_c^\infty(\Omega)$

$$\iint_{\mathbb{R}^{2n}} a(|D^s u_\alpha^s|) \frac{D^s u_\alpha^s D^s v}{|D^s u_\alpha^s|} \, d\nu_n = \Lambda_{1,\alpha}^s \int_\Omega \omega a(|u_\alpha^s|) \frac{u_\alpha^s v}{|u_\alpha^s|} \, dx.$$

Refer also to [9; 19] for the existence of higher-order eigenvalues.

Lemma 3.1. *Let A be a Young function such that $A'(t) = a(t)$ for any $t \geq 0$. Then*

$$\frac{1}{p_A} \Lambda_{1,\alpha}^s \leq \lambda_{1,\alpha}^s \leq p_A \Lambda_{1,\alpha}^s.$$

The number $p_A := \sup_{\beta > 0} a(\beta)\beta / A(\beta)$ is finite if and only if $A \in \Delta_2$.

Proof. Given $\alpha > 0$, consider the critical point $\lambda_{1,\alpha}^s$ associated to the minimizing function u_α^s such that $\int_\Omega A(|u_\alpha|) \, dx = \alpha$. Observe that

$$\text{(3-4)} \quad \int_\Omega \omega A(|u_\alpha^s|) \, dx \geq \inf_{\beta > 0} \frac{A(\beta)}{a(\beta)\beta} \int_\Omega \omega a(|u_\alpha^s|)|u_\alpha^s| \, dx = \frac{1}{p_A} \int_\Omega \omega a(|u_\alpha^s|)|u_\alpha| \, dx.$$

Moreover, since $a(t) = A'(t)$ is increasing, $A(t) = \int_0^t a(\tau) \, d\tau \leq a(t)t$ for any $t > 0$. This fact, together with (3-4) gives that

$$\lambda_{1,\alpha}^s = \frac{\int_\Omega A(|D^s u_\alpha^s|) \, d\nu_n}{\int_\Omega \omega A(|u_\alpha^s|) \, d\nu_n} \leq \frac{\int_\Omega a(|D^s u_\alpha^s|)|D^s u_\alpha^s| \, d\nu_n}{\frac{1}{p_A} \int_\Omega \omega a(|u_\alpha^s|)|u_\alpha^s| \, dx} = p_A \Lambda_{1,\alpha}^s.$$

The other bound is analogous. Finally, note that $p_A < \infty$ if and only if $A \in \Delta_2$. □

For example, the number p_A as defined in Lemma 3.1, takes the following form for the following notable Young functions $A \in \Delta_2$.

(i) Let $p > 1$ and $A(t) = t^p$ then $p_A = p$.

(ii) Let $p, q > 1$, $r \geq 0$ and consider $A(t) = t^p/p + t^q/q$. Then $p_A = q$ when $t \geq 1$ and $p_A = p$ when $t < 1$.

(iii) Let $p, q > 1$, $r \geq 0$ and consider $A(t) = (t^p/p) \ln^r(1 + t^q)$. Then $p_A = p + qr$.

Lemma 3.2. *For $\alpha > 0$ define the function $E(\alpha) := \alpha\lambda_{1,\alpha}^s$. Then E is strictly positive, strictly increasing, $E(0) = 0$, $E(\infty) = \infty$ and E is a Lipschitz function for $\alpha > 0$.*

In particular, that $\lambda_{1,1} \leq \alpha\lambda_{1,\alpha}^s$ when $\alpha > 1$ and $\alpha\lambda_{1,\alpha}^s \leq \lambda_{1,1}$ when $\alpha < 1$.

Proof. Given $\beta > 0$ and a fixed function $u \in C_c^\infty(\Omega)$ such that $\int_\Omega A(|u_\beta|) \, dx = \beta$, since A is continuous and nondecreasing, if we define the function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\phi(r) = \int_\Omega A(r|u_\beta|) \, dx,$$

it follows that ϕ is continuous, nondecreasing, $\phi(0)=0$, $\phi(1)=\beta$ and $\phi(\infty)=\infty$. Hence, for any $\alpha>0$ there exists $r_\alpha>0$ such that $\phi(r_\alpha)=\alpha$ and in particular

$$r_\alpha<1 \text{ when } \alpha<\beta, \qquad r_\alpha>1 \text{ when } \alpha>\beta, \tag{3-5}$$

$$r_\alpha\to 0 \text{ when } \alpha\to 0, \qquad r_\alpha\to\infty \text{ when } \alpha\to\infty. \tag{3-6}$$

Let us see that E is strictly increasing. Let $0<\alpha<\beta$ and in light of (3-2), let $u_\beta\in W_0^sE^A(\Omega)$ be such that

$$\int_\Omega A(|u_\beta|)\,dx=\beta, \qquad \lambda^s_{1,\beta}=\frac{1}{\beta}\iint_{\mathbb{R}^{2n}} A(|D^su_\beta|)\,d\nu_n.$$

By (3-5) there exists $r_\alpha<1$ such that $\int_\Omega A(r_\alpha|u_\beta|)\,dx=\alpha$. Therefore, using the convexity of A we obtain the desired inequality:

$$\alpha\lambda^s_{1,\alpha}\le\iint_{\mathbb{R}^{2n}} A(r_\alpha|D^su_\beta|)\,d\nu_n\le r_\alpha\iint_{\mathbb{R}^{2n}} A(|D^su_\beta|)\,d\nu_n<\beta\lambda^s_{1,\beta}.$$

Moreover, from the previous inequality together with (3-6) we get that

$$0\le\lim_{\alpha\to 0^+}E(\alpha)\le\lim_{\alpha\to 0^+}r_\alpha\iint_{\mathbb{R}^{2n}} A(|D^su_\beta|)\,d\nu_n=E(\beta)\lim_{\alpha\to 0^+}r_\alpha=0,$$

from where $E(0)=0$. $E(\alpha)$ is lower semicontinuous by [11, Lemma 4.3], and then $\liminf_{\alpha\to\infty}E(\alpha)\ge\infty$. Finally, E is Lipschitz continuous by Theorem 4.5 in [11]. □

Proposition 3.3. *Let A be a Young function and let $\Omega\subset\mathbb{R}^n$ be open and bounded, let $B_1\subset\mathbb{R}^n$ be a ball such that $|\Omega|=|B_1|$ and let $B_2\subset\Omega$ be a ball. Then*

$$\lambda^s_{1,\alpha}(B_1)\le\lambda^s_{1,\alpha}(\Omega)\le\lambda^s_{1,\alpha}(B_2).$$

Proof. Let $u\in W_0^sL^A(\Omega)$ be such that $\int_\Omega A(|u|)\,dx=\alpha$. Denote by u^* the symmetric rearrangement of u. Thus, u^* is radially decreasing about 0 and is equidistributed with u. Using the Pólya–Szegő principle stated in [2, Theorem 3.1] we get

$$\iint_{\mathbb{R}^{2n}} A(|D^su^*|)\,d\nu_n\le\iint_{\mathbb{R}^{2n}} A(|D^su|)\,d\nu_n.$$

Hence, if B_1 is a ball of same measure of Ω, since $\int_{B_1} A(|u^*|)\,dx=\alpha$, we get

$$\lambda^s_{1,\alpha}(B_1)\le\lambda^s_{1,\alpha}(\Omega).$$

On the other hand, consider a ball $B_2\subset\Omega$ and the function $u_{B_2}\in W_0^sL^A(B_2)$ such that $\int_B A(|u_{B_2}|)\,dx=\alpha$ to be a minimizer for $\lambda^s_{1,\alpha}(B_2)$. Define $v\in W_0^{1,A}(\Omega)$ defined as the extension of u_{B_2} by zero outside Ω. Then $\int_\Omega A(|v|)\,dx=\int_\Omega A(|u_{B_2}|)\,dx=\alpha$

and therefore v is admissible in the variational characterization of $\lambda^s_{1,\alpha}(\Omega)$. Hence

$$\lambda^s_{1,\alpha}(\Omega) \le \frac{1}{\alpha} \iint_{\mathbb{R}^{2n}} A(|D^s v|)\, d\nu_n = \lambda^s_{1,\alpha}(B_2),$$

which concludes the proof. □

4. Lower bounds of critical values and eigenvalues

Proof of Theorem 1.1. Fix $\alpha > 0$ and let $u^s_\alpha \in W^s_0 L^A(\Omega)$ be a minimizer of (3-1) such that $\int_\Omega \omega A(|u^s_\alpha|)\, dx = \alpha$, i.e., the pair $(u^s_\alpha, \lambda^s_{1,\alpha})$ satisfies equation (3-2), where $\lambda^s_{1,\alpha}$ is defined in (3-1). Since $s \in (0, 1)$ is fixed, for simplicity we will drop the dependence on s.

In light of Proposition 2.5, u_α is continuous and so there exists $x_0 \in \Omega$ such that

$$|u_\alpha(x_0)| = \max\{|u_\alpha(x)| : x \in \mathbb{R}^n\} > 0.$$

From Proposition 2.5, for any $x, y \in \mathbb{R}^n$ we have

$$|u_\alpha(x) - u_\alpha(y)| \le C_M |x - y|^s B^{-1}\left(\frac{1}{|x-y|^n} \iint_{\mathbb{R}^{2n}} A(|D^s u_\alpha|)\, d\nu_n\right),$$

where the Young function B complementary to the Young function defined in (2-5). We take $x = x_0$, $y \in \partial\Omega$; the previous expression becomes

$$|u_\alpha(x_0)| \le C_M |x_0 - y|^s B^{-1}\left(\frac{1}{|x_0-y|^n} \iint_{\mathbb{R}^{2n}} A(|D^s u_\alpha|)\, d\nu_n\right).$$

Using expression (3-2) and item (ii) of Lemma 2.4, since $\Psi_s(r) \approx r^s B^{-1}(r^{-n})$ for all $r > 0$, there exists $c_1 > 0$ depending only on n and s such that

$$\begin{aligned} |u_\alpha(x_0)| &\le C_M |x_0 - y|^s B^{-1}\left(\frac{\lambda_{1,\alpha}}{|x_0-y|^n} \int_\Omega \omega A(|u_\alpha|)\, dx\right) \\ &= C_M |x_0 - y|^s B^{-1}\left(\frac{\alpha\lambda_{1,\alpha}}{|x_0-y|^n}\right) \\ &= C_M (\alpha\lambda_{1,\alpha})^{\frac{s}{n}} \big((\alpha\lambda_{1,\alpha})^{-\frac{1}{n}}|x_0-y|\big)^s B^{-1}\left(\left((\alpha\lambda_{1,\alpha}^{-\frac{1}{n}}|x_0-y|\right)^{-n}\right) \\ &\le c_1 C_M (\alpha\lambda_{1,\alpha})^{\frac{s}{n}} \Psi_s\big(|x_0-y|(\alpha\lambda^s_{1,\alpha})^{-\frac{1}{n}}\big). \end{aligned} \tag{4-1}$$

Moreover, by definition of inner radius we get that

$$|x_0 - y| \le \max\{d(x, \partial\Omega) : x \in \Omega\} = r_\Omega.$$

Hence, since Ψ_s is nondecreasing in light of Lemma 2.4, inequality (4-1) yields

$$|u_\alpha(x_0)| \le c_1 C_M (\alpha\lambda_{1,\alpha})^{\frac{s}{n}} \Psi_s\big(r_\Omega(\alpha\lambda_{1,\alpha})^{-\frac{1}{n}}\big). \tag{4-2}$$

From Lemma 3.2, the function $E(\alpha) := \alpha\lambda^s_{1,\alpha}$ is strictly increasing, positive and Lipschitz continuous for $\alpha > 0$, and satisfies that $E(0) = 0$, $E(\infty) = \infty$. Hence, defining $f(\alpha) := r_\Omega E(\alpha)^{-\frac{1}{n}}$, $\alpha > 0$, we get that f is a strictly decreasing continuous function such that $f(0) := \lim_{\alpha\to 0^+} f(\alpha) = \infty$ and $f(\infty) := \lim_{\alpha\to\infty} f(\alpha) = 0$. From these properties there exists $\alpha_0 > 0$ such that $f(\alpha_0) = 1$ and

$$f(\alpha) > 1 \quad \text{when } \alpha < \alpha_0, \qquad f(\alpha) < 1 \quad \text{when } \alpha > \alpha_0.$$

In particular, $f(\alpha) > 1$ when $\alpha \ll 1$ and $f(\alpha) < 1$ when $\alpha \gg 1$.

<u>Case $\alpha > \alpha_0$.</u> Since $f(\alpha) < 1$, assuming that $i_\infty(A) > \frac{n}{s}$, by of Lemma 2.4 (iii) we get

$$\Psi_s(f(\alpha)) \approx f(\alpha)^s A^{-1}(f(\alpha)^{-n}).$$

Then, there exists $c_2 > 0$ depending only on n and s, and (4-2) gives

$$|u_\alpha(x_0)| \le c_1 C_M (\alpha\lambda_{1,\alpha})^{\frac{s}{n}} \Psi_s\big(r_\Omega(\alpha\lambda_{1,\alpha})^{-\frac{1}{n}}\big) \le C r^s_\Omega A^{-1}\left(r_\Omega^{-n}\alpha\lambda_{1,\alpha}\right)$$

with $C = c_1 c_2 C_M$. Moreover, since A is nondecreasing, the previous expression yields

$$A(C^{-1} r_\Omega^{-s}|u_\alpha(x_0)|) \le r_\Omega^{-n}\alpha\lambda_{1,\alpha} = r_\Omega^{-n}\lambda_{1,\alpha}\int_\Omega \omega A(|u_\alpha(x)|)\,dx \le r_\Omega^{-n}\lambda_{1,\alpha} A(|u_\alpha(x_0)|)\|\omega\|_{L^1(\Omega)}. \tag{4-3}$$

As a consequence, equation (4-3) yields

$$\frac{r^n_\Omega}{\|\omega\|_{L^1(\Omega)}} \le \lambda_{1,\alpha}\frac{A(|u_\alpha(x_0)|)}{A(C^{-1}r_\Omega^{-s}|u_\alpha(x_0)|)} \le \lambda_{1,\alpha}\sup_{t>0}\frac{A(t)}{A(C^{-1}r_\Omega^{-s}t)} = \lambda_{1,\alpha}\sup_{\tau>0}\frac{A(Cr^s_\Omega\tau)}{A(\tau)} = \lambda_{1,\alpha} M_A(Cr^s_\Omega),$$

where M_A is the Matuszewska–Orlicz function associated to A defined in Section 2.2. Since M is submultiplicative, there is $c > 0$ depending on A such that $M_A(Cr^s_\Omega) \le cM_A(C)M_A(r^s_\Omega)$, and the inequality above leads to the following lower bound for $\lambda_{1,\alpha}$:

$$\frac{r^n_\Omega}{cM_A(C)|\omega\|_{L^1(\Omega)}}\frac{1}{M_A(r^s_\Omega)} \le \lambda_{1,\alpha}.$$

<u>Case $\alpha < \alpha_0$.</u> Here $f(\alpha) > 1$. Then, assuming $i_0(A) > \frac{n}{s}$, by Lemma 2.4(iv) we get

$$\Psi_s(f(\alpha)) \approx f(\alpha)^s A^{-1}(f(\alpha)^{-n}).$$

Proceeding analogously as in the previous case we get the result. □

A similar argument to the proof of Theorem 1.1 yields a lower bound for the critical value, involving the inverse of A instead of the Matuszewska–Orlicz function M_A.

Theorem 4.1. *Let $s \in (0,1)$, $\alpha > 0$ and let A be a Young function satisfying (2-4). Given $\omega \in L^1(\Omega)$ consider the critical value $\lambda^s_{1,\alpha}$ defined in (3-1). Then*

(i) *There exists a unique $\alpha_0 > 0$ satisfying the equation*

$$\alpha_0 \lambda^s_{1,\alpha_0} = r^n_\Omega.$$

(ii) *Assume that $i_0(A) > \frac{n}{s}$ when $\alpha \le \alpha_0$, or $i_\infty(A) > \frac{n}{s}$ when $\alpha > \alpha_0$. Then, there exists a $C > 0$ depending only on s, n and A such that*

$$\frac{r^n_\Omega}{\alpha} A\left(\frac{1}{C r^s_\Omega} A^{-1}\left(\frac{\alpha}{\|\omega\|_{L^1(\Omega)}}\right)\right) \le \lambda^s_{1,\alpha}. \tag{4-4}$$

In particular, this holds when $i_0(A) > \frac{n}{s}$ if $\alpha \ll 1$ and when $i_\infty(A) > \frac{n}{s}$ if $\alpha \gg 1$.

Proof. Fix $\alpha > 0$ and let $u^s_\alpha \in W^s_0 L^A(\Omega)$ be a minimizer of (3-1) such that $\int_\Omega \omega A(|u^s_\alpha|)\,dx = \alpha$, i.e., the pair $(u^s_\alpha, \lambda^s_{1,\alpha})$ satisfies equation (3-2), where $\lambda^s_{1,\alpha}$ is defined in (3-1). Since $s \in (0,1)$ is fixed, for simplicity we will drop the dependence on s.

In light of Proposition 2.5 u_α is continuous and hence there exists $x_0 \in \Omega$ such that $|u_\alpha(x_0)| = \max\{|u_\alpha(x)| : x \in \mathbb{R}^n\} > 0$. Since

$$\alpha = \int_\Omega \omega A(|u_\alpha|)\,dx \le A(|u_\alpha(x_0)|)\|\omega\|_{L^1(\Omega)},$$

using (4-1) and the fact the A is nondecreasing, we get

$$\alpha \le A\big(c_1 C_M (\alpha\lambda_{1,\alpha})^{\frac{s}{n}} \Psi_s\big(|x_0 - y|(\alpha\lambda_{1,\alpha})^{-\frac{1}{n}}\big)\big)\|\omega\|_{L^1(\Omega)}$$

for any $y \in \partial\Omega$, and therefore, denoting by r_Ω the inner radius of Ω, we get

$$\alpha\|\omega\|^{-1}_{L^1(\Omega)} \le A\big(c_1 C_M (\alpha\lambda_{1,\alpha})^{\frac{s}{n}} \Psi_s\big(r_\Omega(\alpha\lambda_{1,\alpha})^{-\frac{1}{n}}\big)\big). \tag{4-5}$$

As in the proof of Theorem 1.1, there exists $\alpha_0 > 0$ such that $r^n_\Omega \le \alpha\lambda_{1,\alpha}$ when $\alpha > \alpha_0$ and $r^n_\Omega \ge \alpha\lambda_{1,\alpha}$ when $\alpha < \alpha_0$.

Case $\alpha > \alpha_0$. Assuming $i_\infty(A) > \frac{n}{s}$, by Lemma 2.4(iii) we get $\Psi_s(t) \approx t^s A^{-1}(t^{-n})$ for any $t > 0$. Then, there exists $c_2 = c_2(n,s) > 0$ for which (4-5) yields

$$\alpha\|\omega\|^{-1}_{L^1(\Omega)} \le A\left(C r^s_\Omega A^{-1}\left(r^{-n}_\Omega \alpha\lambda_{1,\alpha}\right)\right)$$

where $C = c_1 c_2 C_M$. Since A is nondecreasing, the previous expression gives

$$\frac{r^n_\Omega}{\alpha} A\left(\frac{1}{C r^s_\Omega} A^{-1}\left(\frac{\alpha}{\|\omega\|_{L^1(\Omega)}}\right)\right) \le \lambda_{1,\alpha}.$$

Case $\alpha < \alpha_0$. Assuming that $i_0(A) > \frac{n}{s}$, the bound follows analogously. □

As a direct consequence of Theorem 1.1 and Lemma 3.1 we get the following.

Corollary 4.2. *Under the assumptions and notation of Theorem 1.1, if additionally $A \in \Delta_2$, then*

$$\frac{C}{p_A\|\omega\|_{L^1(\Omega)}}\frac{r_\Omega^n}{M(r_\Omega^s)} \le \Lambda_{1,\alpha}^s$$

where $p_A = \sup_{\beta>0} a(\beta)\beta/A(\beta)$.

Proof. It is direct from Theorem 1.1 by using Lemma 3.1. □

Proof of Theorem 1.2. Fix $\alpha > 0$ and let $u_\alpha^s \in W_0^s L^A(\Omega)$ be a minimizer of (3-1) such that $\int_\Omega \omega A(|u_\alpha^s|)\,dx = \alpha$, that is, the pair $(u_\alpha^s, \lambda_{1,\alpha}^s)$ satisfies (3-2), where $\lambda_{1,\alpha}^s$ is defined in (3-1).

Denote by d_Ω the diameter of Ω. The Hardy inequality given in Proposition 2.6 together with (3-2) and the monotonicity of A gives

$$\begin{aligned}\text{(4-6)}\quad \int_{\mathbb{R}^n} A\left(\frac{c_1|u_\alpha(x)|}{\mathrm{d}_\Omega^s}\right)dx \le \int_\Omega A\left(\frac{c_1|u_\alpha(x)|}{|x|^s}\right)dx \le (1-s)\iint_{\mathbb{R}^{2n}} A(|D^s u_\alpha|)\,d\nu_n \\ = (1-s)\lambda_{1,\alpha}^s \int_\Omega \omega A(|u_\alpha|)\,dx\end{aligned}$$

where $c_1 = C_{H_1}C_{H_2}^{-1}$, and $C_{H_1}, C_{H_2} > 0$ are the constants given in Proposition 2.6, which depend only on n and s. Now, we compute the following inequality:

$$\begin{aligned}\text{(4-7)}\qquad \int_\Omega A(|u_\alpha|)\,dx &= \int_\Omega \frac{A(|u_\alpha|)}{A(c_1\mathrm{d}_\Omega^{-s}|u_\alpha|)}A(c_1\mathrm{d}_\Omega^{-s}|u_\alpha|)\,dx \\ &\le \sup_{t\in(0,\|u\|_\infty)} \frac{A(t)}{A(c_1\mathrm{d}_\Omega^{-s}t)}\int_\Omega A(c_1\mathrm{d}_\Omega^{-s}|u_\alpha|)\,dx \\ &\le \sup_{\tau>0}\frac{A(c^{-1}d_\Omega^s\tau)}{A(\tau)}\int_\Omega A(c_1\mathrm{d}_\Omega^{-s}|u_\alpha|)\,dx \\ &= M_A(c^{-1}d_\Omega^s)\int_\Omega A\left(\frac{c_1|u_\alpha|}{d_\Omega^s}\right)dx.\end{aligned}$$

From (4-6), (4-7) and the fact that M_A is submultiplicative, we get

$$1 \le (1-s)\lambda_{1,\alpha}^s\|\omega\|_{L^\infty(\Omega)}M_A(c^{-1}d_\Omega^s) \le (1-s)\lambda_{1,\alpha}^s\|\omega\|_{L^\infty(\Omega)}\tilde{c}M_A(c^{-1})M_A(d_\Omega^s),$$

with $\tilde{c} = \tilde{c}(c, A)$, which concludes the proof. □

As a direct consequence of Theorem 1.2 and Lemma 3.1 we get the following.

Corollary 4.3. *Under the assumptions and notation of Theorem 1.2, if additionally $A \in \Delta_2$, then*

$$\frac{C}{p_A\|\omega\|_{L^\infty(\Omega)}M_A(Cd_\Omega^s)} \le \Lambda_{1,\alpha}^s$$

where $p_A = \sup_{\beta>0} a(\beta)\beta/A(\beta)$.

When $A \in \Delta_2$ we can improve Theorem 1.2 by replacing d_Ω with r_Ω.

Proof of Theorem 1.3. The proof is analogous to that of Theorem 1.2, noting that in (4-6), the Hardy inequality stated in Proposition 2.7, together with (3-2), leads to

$$\frac{1}{C}\int_{\mathbb{R}^n} A\left(\frac{|u_\alpha(x)|}{r_\Omega^s}\right) dx \le \int_\Omega A\left(\frac{|u_\alpha(x)|}{\delta_\Omega(x)^s}\right) dx \\ \le \iint_{\mathbb{R}^{2n}} A(|D^s u_\alpha|)\, d\nu_n = \lambda^s_{1,\alpha}\int_\Omega \omega A(|u_\alpha|)\, dx,$$

where $\delta_\Omega(x)$ denotes the distance from x to $\partial\Omega$, giving the desired result. □

References

[1] A. Alberico, A. Cianchi, L. Pick, and L. Slavíková, "Fractional Orlicz–Sobolev embeddings", *J. Math. Pures Appl.* (9) **149** (2021), 216–253. MR

[2] A. Alberico, A. Cianchi, L. Pick, and L. Slavíková, "On fractional Orlicz–Sobolev spaces", *Anal. Math. Phys.* **11**:2 (2021), art. id. 84, 21 pp. Reprinted as pp. 45–65 in *Harmonic analysis and partial differential equations*: *in honor of Vladimir Maz'ya*, Birkhäuser, 2023. MR

[3] A. Alberico, A. Cianchi, L. Pick, and L. Slavíková, "On the modulus of continuity of fractional Orlicz–Sobolev functions", *Math. Ann.* **391**:2 (2025), 2429–2477. MR

[4] C. O. Alves, S. Bahrouni, and M. L. M. Carvalho, "Multiple solutions for two classes of quasilinear problems defined on a nonreflexive Orlicz–Sobolev space", *Ark. Mat.* **60**:1 (2022), 1–22. MR

[5] V. Ambrosio, "Fractional $p\&q$ Laplacian problems in $\mathbb{R}^N$ with critical growth", *Z. Anal. Anwend.* **39**:3 (2020), 289–314. MR

[6] A. Anane, "Simplicité et isolation de la première valeur propre du p-laplacien avec poids", *C. R. Acad. Sci. Paris Sér. I Math.* **305**:16 (1987), 725–728. MR

[7] E. Azroul, A. Benkirane, and M. Srati, "Nonlocal eigenvalue type problem in fractional Orlicz–Sobolev space", *Adv. Oper. Theory* **5**:4 (2020), 1599–1617. MR

[8] S. Bahrouni and A. M. Salort, "Neumann and Robin type boundary conditions in fractional Orlicz–Sobolev spaces", *ESAIM Control Optim. Calc. Var.* **27** (2021), art. id. S15, 23 pp. MR

[9] S. Bahrouni, H. Ounaies, and A. Salort, "Variational eigenvalues of the fractional g-Laplacian", *Complex Var. Elliptic Equ.* **68**:6 (2023), 1021–1044. MR

[10] K. Bal, K. Mohanta, P. Roy, and F. Sk, "Hardy and Poincaré inequalities in fractional Orlicz–Sobolev spaces", *Nonlinear Anal.* **216** (2022), art. id. 112697, 22 pp. MR

[11] J. F. Bonder and A. Salort, "On the first eigenvalue of the generalized Laplacian", preprint, 2024. arXiv 2403.05933

[12] S.-S. Byun, H. Kim, and J. Ok, "Local Hölder continuity for fractional nonlocal equations with general growth", *Math. Ann.* **387**:1-2 (2023), 807–846. MR

[13] J. Chaker, M. Kim, and M. Weidner, "Regularity for nonlocal problems with non-standard growth", *Calc. Var. Partial Differential Equations* **61**:6 (2022), art. id. 227, 31 pp. MR

[14] J. Chaker, M. Kim, and M. Weidner, "Harnack inequality for nonlocal problems with non-standard growth", *Math. Ann.* **386**:1-2 (2023), 533–550. MR

[15] Q.-H. Choi and T. Jung, "Fractional N-Laplacian boundary value problems with jumping nonlinearities in the fractional Orlicz–Sobolev spaces", *Bound. Value Probl.* (2021), art. id. 100, 27 pp. MR

[16] M. Cuesta, "Eigenvalue problems for the p-Laplacian with indefinite weights", *Electron. J. Differential Equations* (2001), art. id. 33, 9 pp. MR

[17] A. Elbert, "A half-linear second order differential equation", pp. 153–180 in *Qualitative theory of differential equations* (Szeged, 1979), vol. 30, Colloq. Math. Soc. János Bolyai, North-Holland, Amsterdam, 1981. MR

[18] J. Fernández Bonder and A. M. Salort, "Fractional order Orlicz–Sobolev spaces", *J. Funct. Anal.* **277**:2 (2019), 333–367. MR

[19] J. Fernandez Bonder and J. F. Spedaletti, "Ljusternik–Schnirelmann eigenvalues for the fractional m-Laplacian without the Δ_2 condition", *Commun. Pure Appl. Anal.* **24**:6 (2025), 1094–1117. MR

[20] J. Fernández Bonder, M. Pérez-Llanos, and A. M. Salort, "A Hölder infinity Laplacian obtained as limit of Orlicz fractional Laplacians", *Rev. Mat. Complut.* **35**:2 (2022), 447–483. MR

[21] J. Fernández Bonder, A. Salort, and H. Vivas, "Interior and up to the boundary regularity for the fractional g-Laplacian: the convex case", *Nonlinear Anal.* **223** (2022), art. id. 113060, 31 pp. MR

[22] J. Fernández Bonder, A. Salort, and H. Vivas, "Global Hölder regularity for eigenfunctions of the fractional g-Laplacian", *J. Math. Anal. Appl.* **526**:1 (2023), art. id. 127332, 15 pp. MR

[23] M. García-Huidobro, V. K. Le, R. Manásevich, and K. Schmitt, "On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz–Sobolev space setting", *NoDEA Nonlinear Differential Equations Appl.* **6**:2 (1999), 207–225. MR

[24] J.-P. Gossez, "Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients", *Trans. Amer. Math. Soc.* **190** (1974), 163–205. MR

[25] J.-P. Gossez and R. Manásevich, "On a nonlinear eigenvalue problem in Orlicz–Sobolev spaces", *Proc. Roy. Soc. Edinburgh Sect. A* **132**:4 (2002), 891–909. MR

[26] M. Jleli, M. Kirane, and B. Samet, "Lyapunov-type inequalities for fractional partial differential equations", *Appl. Math. Lett.* **66** (2017), 30–39. MR

[27] M. A. Krasnosel'skiĭ and Y. B. Rutickiĭ, *Convex functions and Orlicz spaces*, Noordhoff, Groningen, 1961. MR

[28] C.-F. Lee, C.-C. Yeh, C.-H. Hong, and R. P. Agarwal, "Lyapunov and Wirtinger inequalities", *Appl. Math. Lett.* **17**:7 (2004), 847–853. MR

[29] P. Lindqvist, "On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u)+\lambda|u|^{p-2}u=0$", *Proc. Amer. Math. Soc.* **109**:1 (1990), 157–164. MR

[30] P. Lindqvist, "A nonlinear eigenvalue problem", pp. 175–203 in *Topics in mathematical analysis*, Ser. Anal. Appl. Comput. **3**, World Scientific, Hackensack, NJ, 2008. MR

[31] L. Maligranda, *Orlicz spaces and interpolation*, Seminários de Matemática **5**, Universidade Estadual de Campinas, Campinas, SP, Brazil, 1989. MR

[32] P. L. de Nápoli and J. P. Pinasco, "A Lyapunov inequality for monotone quasilinear operators", *Differential Integral Equations* **18**:10 (2005), 1193–1200. MR

[33] P. L. de Nápoli and J. P. Pinasco, "Lyapunov-type inequalities for partial differential equations", *J. Funct. Anal.* **270**:6 (2016), 1995–2018. MR

[34] B. G. Pachpatte, "Lyapunov type integral inequalities for certain differential equations", *Georgian Math. J.* **4**:2 (1997), 139–148. MR

[35] J. P. Pinasco, "Lower bounds for eigenvalues of the one-dimensional p-Laplacian", *Abstr. Appl. Anal.* 2 (2004), 147–153. MR

[36] A. M. Salort, "Eigenvalues and minimizers for a non-standard growth non-local operator", *J. Differential Equations* **268**:9 (2020), 5413–5439. MR

[37] A. Salort, "Lower bounds for Orlicz eigenvalues", *Discrete Contin. Dyn. Syst.* **42**:3 (2022), 1415–1434. MR

[38] A. Salort and H. Vivas, "Fractional eigenvalues in Orlicz spaces with no Δ_2 condition", *J. Differential Equations* **327** (2022), 166–188. MR

[39] J. Sánchez and V. Vergara, "A Lyapunov-type inequality for a ψ-Laplacian operator", *Nonlinear Anal.* **74**:18 (2011), 7071–7077. MR

[40] I. Sim and Y.-H. Lee, "Lyapunov inequalities for one-dimensional p-Laplacian problems with a singular weight function", *J. Inequal. Appl.* (2010), art. id. 865096, 9 pp. MR

Received May 4, 2025. Revised August 20, 2025.

ARIEL SALORT
DEPARTAMENTO DE MATEMÁTICAS Y CIENCIA DE DATOS
UNIVERSIDAD SAN PABLO-CEU
URBANIZACIÓN MONTEPRÍNCIPE
28660 MADRID
SPAIN

amsalort@gmail.com

ariel.salort@ceu.es

PACIFIC JOURNAL OF MATHEMATICS
Vol. 339, No. 2, 2025
https://doi.org/10.2140/pjm.2025.339.333

GRAPH THINNESS: A LOWER BOUND AND COMPLEXITY

YAROSLAV SHITOV

The *thinness* of a simple graph $G = (V, E)$ is the smallest integer k for which there exist a total order $(V, <)$ and a partition of V into k classes $(V_1, \dots, V_k)$ such that, for all $u, v, w \in V$ with $u < v < w$, if u, v belong to the same class and $\{u, w\} \in E$, then $\{v, w\} \in E$. We prove:

- **There are n-vertex graphs of thinness $n - o(n)$, which answers a question of Bonomo-Braberman, Gonzalez, Oliveira, Sampaio, and Szwarcfiter.**
- **The computation of thinness is NP-hard, which is a solution to a long-standing open problem posed by Mannino and Oriolo.**

1. Introduction

The notion of a k-thin graph was introduced by Mannino, Oriolo, Ricci, Chandran [13; 14] and motivated by applications to frequency assignment problems. One particular result in [14] showed that the maximum weight stable set problem can be solved in polynomial time, provided that the input graph is given with the corresponding ordering and partition of its vertices, as in the definition of a k-thin graph, and where k should be bounded by a constant fixed in advance. We remark that G is 1-thin if and only if G is an interval graph, so the result of [14] came as a generalization of the earlier polynomial time solution for the maximum weight stable set problem on interval graphs [11; 12]. Nowadays, the polynomial time solutions, in cases when the inputs are restricted to the k-thin form with bounded k, are known for the maximum weight stable set [14], list matrix partition, rainbow domination [1], capacitated graph coloring [7], and several other problems [2; 5].

This article deals with the algorithmic problem corresponding to the notion of thinness. Here and in what follows, an ordered tuple $(s_1, \dots, s_k)$ is said to be a *partition* of a set V if the sets $s_1, \dots, s_k$ are pairwise disjoint and $s_1 \cup \dots \cup s_k = V$.

Problem 1 (GRAPH THINNESS).
Given: *A simple graph $G = (V, E)$, a positive integer k.*
Question: *Do there exist*

- *a total ordering $<$ of the vertex set V and*

MSC2020: 05C85, 68Q17.
Keywords: interval graph, k-thin graph, NP-completeness.

- *a partition of V into k disjoint classes $s = (s_1, \dots, s_k)$*

such that, for all $u, v, w \in V$ with

$$(u < v < w) \text{ AND } (\{u, w\} \in E) \text{ AND } (u, v \in s_i \text{ for some } i), \tag{1-1}$$

one always has $\{v, w\} \in E$?

Definition 2. If (G, k) is a yes-instance in Problem 1, then the graph G is called *k-thin*. The *thinness* of G is the smallest integer τ for which G is τ-thin.

The above mentioned algorithmic applications motivate the study of the computational complexity of k-thin graph recognition. As said above, the case $k = 1$ corresponds to interval graphs, which can be detected in polynomial time [8]. For general k, the question of the determination of the algorithmic complexity of detecting k-thin graphs was posed by Mannino and Oriolo [13] in 2002, and, until now, it remained open despite further extensive research [1; 2; 3; 4; 5; 6; 9; 10; 15]. The aim of this paper is to prove Theorems 3 and 5 below.

Theorem 3. *The problem* GRAPH THINNESS *is NP-complete.*

We refer the interested reader to [1; 2; 3; 4; 5; 6; 9; 10; 15] for further results and open problems regarding the complexities of fixed parameter versions of GRAPH THINNESS. In particular, is it hard to decide whether a given graph is 2-thin? Another question in [5] deals with the maximal value that can be taken by the thinness of a simple graph on n vertices, and the best known result was as follows.

Proposition 4 (see Lemma 16 in [1]). *For any positive integer k, there exists a simple graph with $2k$ vertices and thinness k.*

Indeed, the construction in [1, Lemma 16] is also relevant in our paper, and we revisit it in Definition 22 below. Here is the formulation of our second main result.

Theorem 5. *There exist simple graphs with n vertices and thinness $n - o(n)$.*

2. Preliminaries and overview

In what follows, the pair $(s, <)$ as in Problem 1 is to be called a *certificate* of the *k-thinness* of G, and, since the validity of such a certificate can be checked in polynomial time, Problem 1 belongs to NP. Therefore, the remaining part of Theorem 3 is the NP-hardness, and we prove it with the use of the following auxiliary problem.

Problem 6 (GRAPH THINNESS WITH A GIVEN PARTITION).

Given: *A simple graph $G=(V, E)$, an integer k, a partition $s = (s_1, \dots, s_k)$ of V.*

Question: *Does there exist a total ordering $<$ of the set V such that, for all $u, v, w \in V$ satisfying the conditions (1-1), one always has $\{v, w\} \in E$?*

Definition 7. If (G, k, s) is a yes-instance in Problem 6, then the partition s is said to *allow* a certificate of the k-thinness of G.

This problem has been considered earlier as the *ordering consistent with a given partition* in [1], and a review report suggests the *incompatibility graph coloring* as another possible name of this problem. Indeed, although there seems to be no standard terminology for the name of Problem 6, its complexity is known.

Theorem 8 (Bonomo, de Estrada [1]). *Problem 6 is NP-complete.*

Throughout our paper, all graphs are assumed to be simple. In the forthcoming Section 3, we recall some relevant notation and prove several results needed in our discussion. In Section 4, we present the polynomial reduction from GRAPH THINNESS WITH A GIVEN PARTITION to GRAPH THINNESS, and, in view of Theorem 8, this implies the validity of Theorem 3. In Section 5, we switch to Theorem 5 and discuss its motivation, and we prove this theorem with a probabilistic argument.

3. An auxiliary construction

We begin with two caveats on the use of some standard notation.

Remark 9. A *clique* of a graph $G = (V, E)$ is a subset of V that induces a complete subgraph of G. In particular, for any $u \in V$, the sets $\varnothing$ and $\{u\}$ are cliques of G.

Remark 10. We write $U \subseteq V$ for two sets U, V if every element of U is contained in V. In particular, we can write $U \subseteq V$ even if $U = V$.

We proceed with several techniques needed in our reduction.

Definition 11. Assume $G = (V, E)$ is a simple graph, and let $U \subseteq V$. We define $\mathcal{B}(G, U)$ as the graph

- with the vertex set $V \cup \{\alpha, \beta\}$, where $\alpha, \beta \notin V$ are new vertex labels,
- with all edges in E, and, apart from these, with an edge from α to every vertex in $V \setminus U$, and with an edge from β to every vertex in $V \setminus U$.

Our next result may look similar to Lemma 16 in [1], which essentially states that thinness$(\mathcal{B}(G, \varnothing))$ = thinness$(G) + 1$ if G is not a clique.

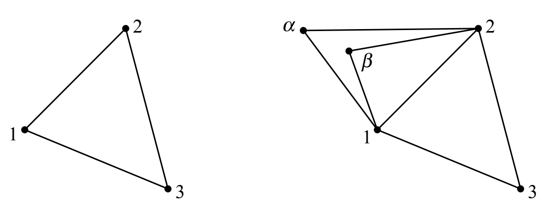


Figure 1. An example of G and $\mathcal{B}(G, \{3\})$ as in Definition 11.

Lemma 12. *Let G, U be as in Definition 11. Suppose that, for some integer k, a partition $s = (s_1, \dots, s_k)$ allows a k-thinness certificate of $\mathcal{B}(G, U)$. Then for some*

- *label $i \in \{1, \dots, k\}$ and*
- *subset $C \subseteq V \setminus U$ that is a clique of G,*

one has $s_i \subseteq U \cup C \cup \{\alpha, \beta\}$.

Proof. Let $<$ be an ordering of $V \cup \{\alpha, \beta\}$ that is compatible with the partition s, so that $(s, <)$ is a certificate of the k-thinness of $\mathcal{B}(G, U)$. Since the labels α, β can be swapped without changing the graph $\mathcal{B}(G, U)$, we can assume without loss of generality that $\alpha < \beta$, and then we take a label i such that $\alpha \in s_i$.

Step 1. Let $w \in V \setminus U$ be a vertex with $w < \alpha$. If $w \in s_i$, then we have

$$w < \alpha < \beta \text{ and } w, \alpha \in s_i. \tag{3-1}$$

However, Definition 11 implies that w and β are adjacent in $\mathcal{B}(G, U)$, but at the same time α, β are not adjacent. Therefore, we arrived at a contradiction with the fact that $(s, <)$ is a k-thinness certificate, and hence we cannot have $w \in s_i$.

Step 2. Now let $w', w'' \in V \setminus U$ be two vertices such that

$$\alpha < w' < w'' \text{ and } w' \in s_i. \tag{3-2}$$

Also, by Definition 11, the vertices α, w'' are adjacent in $\mathcal{B}(G, U)$, and, since $(s, <)$ is a k-thinness certificate, we get $\{w', w''\} \in E$.

In Step 1, we showed that every vertex

$$w \in s_i \cap (V \setminus U) \tag{3-3}$$

should satisfy $\alpha < w$. Using Step 2, we see that, if there are two such vertices w', w'', then they should be adjacent in E. In other words, the set of all vertices w as in (3-3) should be a clique of G. □

Now we explain how to extend a k-thinness certificate of G to $\mathcal{B}(G, U)$.

Lemma 13. *Let G, U be as in Definition 11. Suppose that, for some integer k, a partition $s = (s_1, \dots, s_k)$ and ordering $<$ certify the k-thinness of G, where*

$$s_1 = U. \tag{3-4}$$

Also, we define $\sigma_1 = s_1 \cup \{\alpha, \beta\}$ and extend the ordering $<$ by adding the relations

$$v < \alpha, \ v < \beta, \ \alpha < \beta$$

for all $v \in V$. Then the partition $s' = (\sigma_1, s_2, \dots, s_k)$ and the extended ordering $<$ are a k-thinness certificate of $\mathcal{B}(G, U)$.

Proof. In order to apply the definition of the k-thinness, we take $u, v, w \in V \cup \{\alpha, \beta\}$ such that $u < v < w$ and u, w are adjacent in $\mathcal{B}(G, U)$, and

(3-5) $$u, v \text{ belong to the same class of the partition } s'.$$

We need to check that v, w are adjacent in $\mathcal{B}(G, U)$.

Step 1. If $u, v, w \in V$, the conclusion follows because the purported certificate of the k-thinness of $\mathcal{B}(G, U)$ extends the initial certificate for G.

Step 2. If exactly one of the vertices u, v, w is in $\{\alpha, \beta\}$, then we assume without loss of generality that $w = \beta$. By Definition 11, since β and u are adjacent, we get $u \notin U$, and, according to the condition (3-4), this implies $u \notin s_1$. Now we apply (3-5) to get $v \notin s_1$, and then we use (3-4) to get $v \notin U$, from which, by Definition 11, we get a desired conclusion that v and β are adjacent in $\mathcal{B}(G, U)$.

Step 3. If both α, β appear among u, v, w, then $v = \alpha$, $w = \beta$. Similarly to Step 2, we get $u \notin s_1$ and hence $u \notin \sigma_1$. Since $v = \alpha \in \sigma_1$ by the assumptions of the lemma, the condition (3-5) violates, so there is nothing to prove in Step 3.

Since Steps 1, 2, 3 cover all possibilities, the proof is complete. □

4. The reduction

The following is the main construction in our reduction.

Definition 14. Assume that $G = (V, E)$ is a simple graph, k is a positive integer, and $s = (s_1, \dots, s_k)$ is a partition of V. We define the graph $\mathcal{G}(G, k, s)$ as

$$\mathcal{B}(\mathcal{B}(\dots\mathcal{B}(G, s_1), \dots), s_{k-1}), s_k),$$

that is, in other words, $\mathcal{G}(G, k, s)$ is the k-fold application of the construction in Definition 11, in which the i-th application is

(4-1) $$\mathcal{G}_i := \mathcal{B}\left(\mathcal{G}_{i-1}, s_i\right),$$

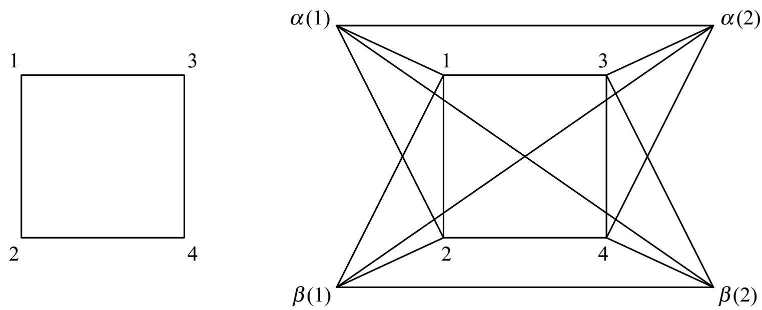


Figure 2. An example of a cycle graph C with vertices $\{1, 2, 3, 4\}$ and the graph $\mathcal{G}(C,\ 2,\ (\{3, 4\},\ \{1, 2\}))$ as in Definition 14.

where $\mathcal{G}_0 = G$, and $\mathcal{G}_{i-1}$ is the graph obtained at the $(i-1)$-st iteration.

For yes-instances of Problem 6, the desired outcome is straightforward.

Lemma 15. *If (G, k, s) is a yes-instance of Problem 6, then $\mathcal{G}(G, k, s)$ is k-thin.*

Proof. Let α_i, β_i be the vertices added to the graph at the i-th application of Definition 11. Then, for every $t \in \{0, \dots, k\}$, the partition

$$(s_1 \cup \{\alpha_1, \beta_1\}, \dots, s_t \cup \{\alpha_t, \beta_t\}, s_{t+1}, \dots, s_k) \tag{4-2}$$

allows a k-thinness certificate of $\mathcal{G}_t$ because, in fact, for $t = 0$, this is true as (G, k, s) is a yes-instance, and, for $t > 0$, this follows from Lemma 13 by the induction. In particular, the $t = k$ version of (4-2) certifies the k-thinness of $\mathcal{G}_k = \mathcal{G}(G, k, s)$. □

A converse direction of Lemma 15 requires some further work.

Definition 16. If $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ are simple graphs with $V_1 \cap V_2 = \varnothing$, then we define $G_1 \oplus G_2$ as the graph $(V_1 \cup V_2, E_1 \cup E_2)$.

Definition 17. If $s = (s_1, \dots, s_k)$ is a partition of a set S, and P is a subset of S, then the partition $(s_1 \cap P, \dots, s_k \cap P)$ is called the *restriction* of s on P.

Lemma 18. *Let $G^1, \dots, G^{k+1}$ be nonempty simple graphs. We consider the graph*

$$G = G^1 \oplus \dots \oplus G^{k+1}$$

and a partition $s = (s_1, \dots, s_k)$ of the vertex set of G. If, for every $q \in \{1, \dots, k+1\}$,

(N1) *the graph G^q is not $(k-1)$-thin,*

(N2) *the triple (G^q, k, ψ^q) is a no-instance for Problem 6, where ψ^q is the restriction of s on the vertex set of G^q,*

then the graph $\mathcal{G}(G, k, s)$ is not k-thin.

Proof. We argue by contradiction, so we assume that $\mathcal{G}(G, k, s)$ is k-thin. Therefore, some partition $\sigma = (\sigma_1, \dots, \sigma_k)$ allows a k-thinness certificate of $\mathcal{G}(G, k, s)$, and then, for all $t \in \{0, \dots, k\}$ and $j \in \{1, \dots, k\}$, we define

$$\sigma_{jt} := \sigma_j \cap V_t,$$

where V_t is the vertex set of the graph $\mathcal{G}_t$ as in Definition 14. Since the k-thinness certificates remain valid at the restrictions to induced subgraphs, the partition

$$(\sigma_{1t}, \dots, \sigma_{kt})$$

certifies the k-thinness of $\mathcal{G}_t$. We recall that $\mathcal{G}_t = \mathcal{B}(\mathcal{G}_{t-1}, s_t)$ by the condition (4-1), and an application of Lemma 12 to this graph $\mathcal{G}_t$ allows us to find a clique $\mathcal{C}_t$ in $\mathcal{G}_{t-1}$ such that the set $V_t \setminus (s_t \cup \mathcal{C}_t)$ lies in the union of at most $(k-1)$ classes in

$(\sigma_{1t}, \ldots, \sigma_{kt})$. A restriction of the latter statement to the vertices of G gives us a clique C_t of G and an index r_t such that

$$V \setminus (s_t \cup C_t) \text{ is a subset of } V \setminus \sigma_{0r_t} \tag{4-3}$$

for all $t \in \{1, \ldots, k\}$. Since we had $G = G^1 \oplus \ldots \oplus G^{k+1}$ by the initial assumption, there exists an index $q \in \{1, \ldots, k+1\}$ for which

$$G^q \text{ does not intersect } C_1 \cup \ldots \cup C_k. \tag{4-4}$$

Now we take the partition $\psi^q = (\psi_{q1}, \ldots, \psi_{qk})$ as in (N2), that is, ψ^q is the restriction of s on the vertex set U^q of G^q, which means that we have

$$\psi_{qj} = s_j \cap U^q \tag{4-5}$$

for all $j \in \{1, \ldots, k\}$. Also, we define another partition $\tau = (\tau_1, \ldots, \tau_k)$ of U^q as the restriction of σ, that is, we get

$$\tau_j = \sigma_{0j} \cap U^q \tag{4-6}$$

for all $j \in \{1, \ldots, k\}$. Now we use the conditions (4-5) and (4-6) to restrict the formula (4-3) to the set U^q. In view of (4-4), we get that

$$U^q \setminus \psi_{qt} \text{ is a subset of } U^q \setminus \tau_{r_t}$$

for all $t \in \{1, \ldots, k\}$, and hence

$$\tau_{r_t} \text{ is a subset of } \psi_{qt} \tag{4-7}$$

for all $t \in \{1, \ldots, k\}$. If $t \to r_t$ is a permutation, then (4-7) implies $\tau_{r_t} = \psi_{qt}$ for all t, which shows that τ is a permutation of ψ^q, and hence ψ^q allows a k-thinness certificate of G^q. This contradicts to the condition (N2), and hence, in fact, the mapping $t \to r_t$ cannot be injective. Therefore, there exists a label h such that

$$h = r_{t_1} = r_{t_2} \text{ for some } t_1 \neq t_2,$$

and then τ_h is a subset of both ψ_{qt_1} and ψ_{qt_2}. Since ψ^q is a partition, this implies that τ_h is an empty set, so the graph G^q admits a k-thinness certificate in which the empty set appears in the corresponding partition of the vertices. This means that G^q is $(k-1)$-thin, so we obtain a contradiction to (N1) and complete the proof. □

We need to generalize the notation from Definition 16.

Definition 19. Let V_1, V_2 be disjoint sets. For some k, let $s^1 = (s_{11}, \ldots, s_{1k})$, $s^2 = (s_{21}, \ldots, s_{2k})$ be partitions of V_1, V_2, respectively. Then we define

$$s^1 \oplus s^2 = (s_{11} \cup s_{21}, \ldots, s_{1k} \cup s_{2k}).$$

Definition 20. Let V_1, V_2 be disjoint sets, and let $<_1$ and $<_2$ be total orderings on V_1, V_2, respectively. Then a total ordering $<$ on $V_1 \cup V_2$ is denoted $<_1 \oplus <_2$ if

- $<_1$ is contained in $<$,
- $<_2$ is contained in $<$,
- one has $v_1 < v_2$, for all $v_1 \in V_1$, $v_2 \in V_2$.

Observation 21. Assume $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be simple graphs with disjoint vertex sets. Let k be an integer, and assume that

- $<_1$ is a total ordering of V_1,
- $<_2$ is a total ordering of V_2,
- s^1 is a partition of V_1 into k classes,
- s^2 is a partition of V_2 into k classes,

then the following are equivalent:

- $(s^1 \oplus s^2, <_1 \oplus <_2)$ is a k-thinness certificate of $G_1 \oplus G_2$,
- $(s^i, <_i)$ is a k-thinness certificate of G_i with both $i = 1, 2$.

Our further arguments require the graph that appears as an iterative application of the construction in [1, Lemma 16] (see also the discussion of Proposition 4 above).

Definition 22. We consider the graph H_k with vertices $\{\alpha_1, \beta_1, \dots, \alpha_k, \beta_k\}$ in which $*_i$ and $\star_j$ are adjacent if and only if $i \neq j$, for any $*, \star$ in $\{\alpha, \beta\}$.

Remark 23. According to Lemma 16 in [1], the thinness of H_k equals k.

Definition 24. We write $(\psi_k, <_k)$ for the k-thinness certificate of H_k defined by

- $\psi_k = (\{\alpha_1, \beta_1\}, \dots, \{\alpha_k, \beta_k\})$,
- $*_i < \star_j$ if $i < j$ and $*, \star \in \{\alpha, \beta\}$,
- $\alpha_i < \beta_i$, for any $i \in \{1, \dots, k\}$.

Now we are ready to proceed with the reduction. To this end, we recall that the $\oplus$ construction is the one introduced in Definitions 16 and 19 above.

Definition 25. Let (G, k, s) be an instance of Problem 6. We create $k+1$ copies each of G and H_k, labeling them $G_1, \dots, G_{k+1}$ and $H_{k1}, \dots, H_{k\,k+1}$, so that the vertex sets of all these copies are pairwise disjoint. We define

$$\overline{G} = (G_1 \oplus H_{k1}) \oplus \dots \oplus (G_{k+1} \oplus H_{k\,k+1}),$$

and, assuming that s_j and ψ_{kj} denote the copies of the corresponding partitions of the vertex sets of G and H_k, we take $\bar{s} = (s_1 \oplus \psi_{k1}) \oplus \dots \oplus (s_{k+1} \oplus \psi_{k\,k+1})$, and then we define the graph $\Gamma(G, k, s) := \mathcal{G}(\overline{G}, k, \bar{s})$, where $\mathcal{G}$ stands for the construction in Definition 14.

Theorem 26. *An instance (G, k, s) of Problem 6 is a 'yes' if and only if the graph $\Gamma(G, k, s) = \mathcal{G}(\overline{G}, k, \bar{s})$ is k-thin.*

Proof. If (G, k, s) is a yes-instance in Problem 6, then $(\overline{G}, k, \bar{s})$ is also a yes-instance by Observation 21. In this case, the graph $\Gamma(G, k, s)$ is k-thin by Lemma 15.

If (G, k, s) is a no-instance in Problem 6, then we apply Lemma 18 to the graph $\overline{G}$. Then the corresponding condition (N1) is true by Remark 23, and we get the validity of (N2) from Observation 21. Therefore, the assertion of Lemma 18 is applicable, and hence the graph $\Gamma(G, k, s)$ is not k-thin. □

Theorem 26 gives a polynomial reduction to GRAPH THINNESS from Problem 6, which is known to be NP-complete [1]. This implies Theorem 3.

5. Graphs with large thinness

As said above, the notion of thinness is being extensively studied for almost two decades, but there are still many open questions on the behavior of this function and its relations to other graph invariants [1; 2; 5; 6; 13; 14]. One particular natural problem concerns the largest value of the thinness of an n-vertex graph.

Problem 27 (Section 5 in [5]). *Is there an n-vertex graph G with thinness $> n/2$?*

This section is devoted to the proof of Theorem 5, which gives an affirmative solution to Problem 27, and, in fact, this theorem determines the largest value of the thinness in the asymptotic sense. As we will see, our proof is probabilistic.

Definition 28. A graph $G = (V, E)$ with $|V| = 3m$ is called *m-obstructive* if one can enumerate its vertices as $(u_1, \ldots, u_m, v_1, \ldots, v_m, w_1, \ldots, w_m)$ so that, for every i and j in $\{1, \ldots, m\}$, one has either $\{u_i, w_j\} \notin E$ or $\{v_i, w_j\} \in E$.

Lemma 29. *If m, n are positive integers with $m > 11 \ln n$, then there exists a graph with n vertices which has no m-obstructive induced subgraphs.*

Proof. We consider the random graph $G = (V, E)$ with $|V| = n$ such that the edges of G appear independently with probability $1/2$ each. For every fixed nonrepeating sequence $\alpha = (u_1, \ldots, u_m, v_1, \ldots, v_m, w_1, \ldots, w_m)$ of vertices in V, the probability that α certifies the m-obstruction is $(3/4)^{m^2}$ because there are m^2 independent choices of (i, j) as in Definition 28, and each of the corresponding events $\{u_i, w_j\} \notin E$ or $\{v_i, w_j\} \in E$ happens with probability $1/2$ (which implies that their union occurs with probability $3/4$ by the independence). Since there are a total of at most n^{3m} ways to choose α, the expected total number of all those choices of α which give the m-obstruction certificates is at most

$$n^{3m} \cdot \left(\tfrac{3}{4}\right)^{m^2} = \exp\left(3m \ln n - m^2 \ln \tfrac{4}{3}\right) < 1,$$

and hence some choices of G do not admit m-obstructions at all. □

Now we are ready to complete the proof of Theorem 5.

Theorem 30. *For any positive integer n, there exists a graph with n vertices whose thinness is at least $n - 72\ln n$.*

Proof. Using Lemma 29, we take a graph $G = (V, E)$ with $|V| = n$ such that

(5-1) G has no induced m-obstructive subgraphs with any $m > 11\ln n$.

We are going to complete the proof by showing that the thinness of G is at least $n - 72\ln n$ as desired. If this was not the case, there would exist an ordering $(V, <)$ and a partition of V into at most $n - 72\ln n$ classes as in the definition of the thinness, and then, for some integer c satisfying

$$c \geqslant 72\ln n/3 = 24\ln n, \tag{5-2}$$

we should be able to find c disjoint pairs in each of which both vertices are in the same class (the bound (5-2) follows because the worst case scenario is when every class has either 1 or 3 vertices). We enumerate these pairs as follows:

$$(u_1, v_1), \ \ldots, \ (u_c, v_c) \ \text{ with } \ v_1 < v_2 < \ldots < v_c \ \text{ and } \ u_i < v_i \ \text{ for all } \ i.$$

By the thinness, for any $i \in \{1, \ldots, c\}$, it never occurs that

$$(\{u_i, x\} \in E) \quad \text{AND} \quad (\{v_i, x\} \notin E) \quad \text{with} \ x \in \{v_{i+1}, \ldots, v_c\}. \tag{5-3}$$

Now we define $m = \lfloor c/2 \rfloor$ and note that the sequence

$$\alpha = (u_1, \ldots, u_m, v_1, \ldots, v_m, v_{m+1}, \ldots, v_{2m})$$

induces an m-obstruction because the condition (5-3) never occurs. By (5-1), we get $m \leqslant 11\ln n$ and hence $c \leqslant 22\ln n + 1$. A comparison to (5-2) implies $24\ln n \leqslant 22\ln n + 1$, which is a contradiction unless $n = 1$. □

In other words, our result proves $\theta(n) > n - 72\ln n$ whenever $\theta(n)$ is the largest possible thinness of any graph with n vertices. A review report suggests a modification of our argument that implies $\theta(n) > n - 48\ln n$, and, although our research does not immediately lead to any bound stronger than $\theta(n) > n - O(\ln n)$, a more detailed analysis can lead to further minor improvements in the coefficient of $\ln n$.

Acknowledgement

I thank the reviewers for helpful comments.

References

[1] F. Bonomo and D. de Estrada, "On the thinness and proper thinness of a graph", *Discrete Appl. Math.* **261** (2019), 78–92. MR

[2] F. Bonomo-Braberman and G. A. Brito, "Intersection models and forbidden pattern characterizations for 2-thin and proper 2-thin graphs", *Discrete Appl. Math.* **339** (2023), 53–77. MR

[3] F. Bonomo-Braberman, F. S. Oliveira, M. S. Sampaio, Jr., and J. L. Szwarcfiter, "Reconhecimento de grafos fino [sic] de precedência", in *Anais do III Encontro de Teoria da Computa cão*, SBC, 2019.

[4] F. Bonomo-Braberman, F. S. Oliveira, M. S. Sampaio, Jr., and J. L. Szwarcfiter, "A complexidade do reconhecimento de grafos k-fino próprio de precedência", in *Anais do V Encontro de Teoria da Computa cão*, SBC, 2020.

[5] F. Bonomo-Braberman, C. L. Gonzalez, F. S. Oliveira, M. S. Sampaio, Jr., and J. L. Szwarcfiter, "Thinness of product graphs", *Discrete Appl. Math.* **312** (2022), 52–71. MR

[6] F. Bonomo-Braberman, F. S. Oliveira, M. S. Sampaio, Jr., and J. L. Szwarcfiter, "Precedence thinness in graphs", *Discrete Appl. Math.* **323** (2022), 76–95. MR

[7] F. Bonomo, S. Mattia, and G. Oriolo, "Bounded coloring of co-comparability graphs and the pickup and delivery tour combination problem", *Theoret. Comput. Sci.* **412**:45 (2011), 6261–6268. MR

[8] K. S. Booth and G. S. Lueker, "Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms", *J. Comput. System Sci.* **13**:3 (1976), 335–379. MR

[9] G. A. Brito, *Sobre la thinness en un grafo*, Ph.D. thesis, Universidad de Buenos Aires, 2020.

[10] D. de Estrada, *On the thinness and proper thinness of a graph*, Ph.D. thesis, Universidad de Buenos Aires, 2020.

[11] U. I. Gupta, D. T. Lee, and J. Y.-T. Leung, "Efficient algorithms for interval graphs and circular-arc graphs", *Networks* **12**:4 (1982), 459–467. MR

[12] J. Y. Hsiao, C. Y. Tang, and R. S. Chang, "An efficient algorithm for finding a maximum weight 2-independent set on interval graphs", *Inform. Process. Lett.* **43**:5 (1992), 229–235. MR

[13] C. Mannino and G. Oriolo, "Solving stability problems on a superclass of interval graphs", technical report 511, Centro Vito Volterra, 2002.

[14] C. Mannino, G. Oriolo, F. Ricci, and S. Chandran, "The stable set problem and the thinness of a graph", *Oper. Res. Lett.* **35**:1 (2007), 1–9. MR

[15] M. S. Sampaio, Jr., F. S. Oliveira, and J. L. Szwarcfiter, "Sobre finura própria de grafos", in *Anais do III Encontro de Teoria da Computa cão*, SBC, 2018.

Received February 5, 2025. Revised August 18, 2025.

YAROSLAV SHITOV
MOSCOW, RUSSIA
yaroslav-shitov@yandex.ru

PACIFIC JOURNAL OF MATHEMATICS
Vol. 339, No. 2, 2025
https://doi.org/10.2140/pjm.2025.339.345

CLASSIFYING PREAISLES OF DERIVED CATEGORIES OF COMPLETE INTERSECTIONS

RYO TAKAHASHI

Let R be a commutative noetherian ring. Denote by $\mathsf{mod}\, R$ the category of finitely generated R-modules, by $\mathsf{D^b}(R)$ the bounded derived category of $\mathsf{mod}\, R$, and by $\mathsf{D_{sg}}(R)$ the singularity category of R. The main result of this paper provides, when R is a complete intersection, a complete classification of the preaisles of $\mathsf{D^b}(R)$ containing R and closed under direct summands, which includes as restrictions the classification of thick subcategories of $\mathsf{D_{sg}}(R)$ due to Stevenson, and the classification of resolving subcategories of $\mathsf{mod}\, R$ due to Dao and Takahashi.

The author was partly supported by JSPS Grant-in-Aid for Scientific Research 23K03070.
MSC2020: primary 13D09; secondary 13C60.
Keywords: derived category, complete intersection, (pre)(co)aisle, module category, resolving subcategory, thick subcategory, singularity category, hypersurface, perfect complex, maximal Cohen–Macaulay module/complex, t-structure, Koszul complex, projective dimension, G-dimension, filtration by supports (sp-filtration).

1. Introduction

A *thick subcategory* of a triangulated category is a full triangulated subcategory closed under direct summands. Classifying thick subcategories of a triangulated category has been one of the most central subjects shared by many areas of mathematics including representation theory, homotopy theory, algebraic geometry and commutative/noncommutative algebra; see [11; 15; 16; 17; 26; 29; 31; 32; 38; 39; 46; 45; 50; 53; 54; 56] for instance. A significant work in commutative algebra is a classification of thick subcategories of derived categories of complete intersections by Stevenson [46]. We introduce a setup to explain Stevenson's theorem.

Setup 1.1. Let (R, V) be a pair, where R and V satisfy either of the following two conditions.

(1) R is a commutative noetherian ring which is locally a hypersurface, and V is the singular locus of R.

(2) R is a quotient ring of the form $S/(\boldsymbol{a})$ where S is a regular ring of finite Krull dimension and $\boldsymbol{a} = a_1, \dots, a_c$ is a regular sequence, and V is the singular locus of the zero subscheme of $a_1x_1 + \cdots + a_cx_c \in \Gamma(X, \mathcal{O}_X(1))$ where $X = \mathbb{P}_S^{c-1} = \operatorname{Proj}(S[x_1, \dots, x_c])$.

Note that in both situations of this setup R is locally a complete intersection. Under this setup, Stevenson [46] proved the following classification theorem of thick subcategories.

Theorem 1.2 (Stevenson). *Let (R, V) be as in Setup 1.1. Then there are one-to-one correspondences*

$$\left\{\begin{matrix}\text{thick subcategories}\\ \text{of } \mathsf{D}_{\mathsf{sg}}(R)\end{matrix}\right\} \cong \left\{\begin{matrix}\text{thick subcategories}\\ \text{of } \mathsf{D}^{\mathsf{b}}(R) \text{ containing } R\end{matrix}\right\} \overset{\text{(a)}}{\cong} \left\{\begin{matrix}\text{specialization-closed}\\ \text{subsets of } V\end{matrix}\right\}.$$

Here, $\mathsf{D}^{\mathsf{b}}(R)$ denotes the bounded derived category of the category $\operatorname{mod} R$ of finitely generated R-modules, and $\mathsf{D}_{\mathsf{sg}}(R)$ stands for the *singularity category* of R, that is to say, the Verdier quotient of $\mathsf{D}^{\mathsf{b}}(R)$ by the full subcategory $\mathsf{D}^{\mathsf{perf}}(R)$ of perfect complexes, i.e., $\mathsf{D}_{\mathsf{sg}}(R) = \mathsf{D}^{\mathsf{b}}(R)/\mathsf{D}^{\mathsf{perf}}(R)$.

A *resolving subcategory* of an abelian category with enough projective objects is a full subcategory containing projectives and closed under direct summands, extensions and syzygies. This notion has been studied in various approaches so far; see [4; 3; 23; 25; 24; 30; 33; 36; 37; 43; 49; 50; 51; 52] for instance. In commutative algebra, Dao and Takahashi [24] gave a complete classification of the resolving subcategories of $\operatorname{mod} R$ under the setup introduced above.

Theorem 1.3 (Dao–Takahashi). *Let (R, V) be as in Setup 1.1. Then there is a one-to-one correspondence*

$$\left\{\begin{matrix}\text{resolving subcategories}\\ \text{of } \mathsf{mod}\, R\end{matrix}\right\} \overset{\text{(b)}}{\cong} \left\{\begin{matrix}\text{grade-consistent}\\ \text{functions on Spec } R\end{matrix}\right\} \times \left\{\begin{matrix}\text{specialization-closed}\\ \text{subsets of } V\end{matrix}\right\}.$$

Here, a *grade-consistent function* on $\operatorname{Spec} R$ is an order-preserving map $f : \operatorname{Spec} R \to \mathbb{N}$ which satisfies the inequality $f(\mathfrak{p}) \leqslant \operatorname{grade}(\mathfrak{p})$ for every $\mathfrak{p} \in \operatorname{Spec} R$.

The notion of a *t-structure* in a triangulated category has been introduced by Beĭlinson, Bernstein and Deligne [14] in the 1980s. As with classifying thick subcategories and resolving subcategories mentioned above, classifying t-structures in a given triangulated category $\mathcal{T}$, which is equivalent to classifying *aisles* of $\mathcal{T}$, has been an important fundamental problem. Actually, this problem has almost been settled for $\mathsf{D}^{\mathsf{b}}(R)$ for a commutative noetherian ring R. Indeed, if R has a dualizing complex, then the aisles of $\mathsf{D}^{\mathsf{b}}(R)$ were completely classified by Alonso Tarrío, Jeremías López and Saorín [2] in terms of the filtrations by supports that satisfy the weak Cousin condition. Recently, this has been extended by Takahashi [55] to the case where R has finite Krull dimension such that $\operatorname{Spec} R$ is a CM-excellent scheme in the sense of Česnavičius [20].

Now that classifying aisles of $\mathsf{D}^{\mathsf{b}}(R)$ has almost been completed, what we should consider next is classifying *preaisles* of $\mathsf{D}^{\mathsf{b}}(R)$, which are defined as full subcategories closed under extensions and positive shifts. An aisle is none other than a preaisle whose inclusion functor has a right adjoint, but there exists a big difference between being an aisle and being a preaisle. Classifying preaisles is thus much harder than classifying aisles, and so it would be reasonable to impose some appropriate assumptions on the preaisles we try to classify.

The main result of this paper is the following theorem. This theorem provides a classification of preaisles of $\mathsf{D}^{\mathsf{b}}(R)$ that satisfy some mild and natural conditions. Also, the theorem includes both the classification of thick subcategories by Stevenson and the classification of resolving subcategories by Dao and Takahashi.

Theorem 1.4. *Let (R, V) be a pair as in Setup 1.1. Then there are one-to-one correspondences*

$$\left\{\begin{matrix}\text{preaisles of } \mathsf{D}^{\mathsf{b}}(R)\\ \text{containing } R \text{ and closed}\\ \text{under direct summands}\end{matrix}\right\} \cong \left\{\begin{matrix}\text{resolving}\\ \text{subcategories}\\ \text{of } \mathsf{D}^{\mathsf{b}}(R)\end{matrix}\right\} \overset{(*)}{\cong} \left\{\begin{matrix}\text{order-preserving}\\ \text{maps from Spec } R\\ \text{to } \mathbb{N}\cup\{\infty\}\end{matrix}\right\} \times \left\{\begin{matrix}\text{specialization-}\\ \text{closed subsets}\\ \text{of } V\end{matrix}\right\}.$$

The restriction of $(*)$ *to the thick subcategories of* $\mathsf{D}^{\mathsf{b}}(R)$ *containing R is identified with* (a) *in Theorem 1.2. Each resolving subcategory $\mathcal{X}$ of* $\mathsf{mod}\, R$ *equals the restriction to* $\mathsf{mod}\, R$ *of the smallest resolving subcategory $\widetilde{\mathcal{X}}$ of* $\mathsf{D}^{\mathsf{b}}(R)$ *containing $\mathcal{X}$. The composition of* $(*)$ *with the map $\mathcal{X} \mapsto \widetilde{\mathcal{X}}$ coincides with* (b) *in Theorem 1.3.*

Here, a *resolving subcategory* of $\mathsf{D}^{\mathsf{b}}(R)$ is a full subcategory containing R and closed under direct summands, extensions and negative shifts, which we shall newly introduce in this paper. We adopt this name because it is viewed as a triangulated category version of a resolving subcategory of an abelian category stated above.

This paper is organized as follows. In Section 2, together with several preliminaries for later sections, we give the precise definition of a resolving subcategory of $\mathsf{D}^{\mathsf{b}}(R)$ and states its basic properties. In Section 3, we introduce the key notion of NE-loci in $\mathsf{D}^{\mathsf{b}}(R)$, which are regarded as extensions of nonfree loci in $\operatorname{mod} R$. We find out several fundamental properties of NE-loci. In Section 4, applying some results in the literature based on techniques of unbounded derived categories, we classify the preaisles of $\mathsf{D}^{\mathsf{perf}}(R)$ containing R and closed under direct summands, which enables us to get a complete classification of the resolving subcategories of $\mathsf{D}^{\mathsf{perf}}(R)$ in terms of order-preserving maps from $\operatorname{Spec} R$ to $\mathbb{N} \cup \{\infty\}$. We also compare our results with a classification theorem of aisles given in [55].

From Section 5 to 7 we mainly handle locally complete intersection rings. In Section 5, we prove that the resolving subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ bijectively correspond to the direct product of the resolving subcategories of perfect complexes and the resolving subcategories of maximal Cohen–Macaulay complexes. In Section 6, applying the result obtained in Section 5, we provide complete classifications of the resolving subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ and the preaisles of $\mathsf{D}^{\mathsf{b}}(R)$ containing R and closed under direct summands. We also observe that this classification restricts to the classification of thick subcategories of $\mathsf{D}_{\mathsf{sg}}(R)$ given in Theorem 1.2. In Section 7, we realize the classification of resolving subcategories of $\operatorname{mod} R$ given in Theorem 1.3 as a restriction of the classification of resolving subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ given in Section 6.

In Appendices A and B, we mainly deal with perfect complexes. In Appendix A, we give other proofs of the classification theorems of resolving subcategories and preaisles of $\mathsf{D}^{\mathsf{perf}}(R)$ given in Section 4 without using techniques of unbounded derived categories. Instead, the use of Koszul complexes and NE-loci is crucial here. In Appendix B, we realize results of Dao and Takahashi [24] about modules of finite projective dimension, as restrictions of our results about perfect complexes which are obtained in Appendix A.

Finally, we should emphasize that some of our methods to investigate resolving subcategories of $\mathsf{D}^{\mathsf{b}}(R)$ are similar to methods given in the literature to investigate resolving subcategories of $\operatorname{mod} R$, but we do need to invent and develop a lot of new techniques to obtain our results. We should also emphasize that the proof of our main result, Theorem 1.4, is completed at the end of this paper (before Appendices), by using results given in all the previous sections.

2. Resolving subcategories of triangulated categories

In this section, we state basic definitions which are used throughout this paper. Mimicking the definition of a resolving subcategory of an abelian category, we define a resolving subcategory of a triangulated category. We also explore fundamental properties of resolving subcategories. We begin with giving our convention.

Convention 2.1. All subcategories are assumed to be strictly full. An object X of a category $\mathcal{C}$ is identified with the subcategory of $\mathcal{C}$ consisting of X. An exact triangle $A \to B \to C \to A[1]$ is often abbreviated to $A \to B \to C \rightsquigarrow$. Let R be a commutative noetherian ring with identity. For a prime ideal $\mathfrak{p}$ of R, we denote by $\kappa(\mathfrak{p})$ the residue field of the local ring $R_{\mathfrak{p}}$, that is, $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Subscripts and superscripts may be omitted if there is no danger of confusion.

In the next two definitions, we explain basic closedness conditions in an additive/abelian/triangulated category, and introduce certain subcategories determined by a given subcategory.

Definition 2.2. Let $\mathcal{C}$ be an additive category, and let $\mathcal{X}$ be a subcategory of $\mathcal{C}$.

(1) We say that $\mathcal{X}$ is *closed under finite direct sums* provided that for any finite number of objects $X_1, \dots, X_n$ in $\mathcal{X}$ the direct sum $X_1 \oplus \cdots \oplus X_n$ is also in $\mathcal{X}$. This is equivalent to saying that the direct sum of any two objects in $\mathcal{X}$ also belongs to $\mathcal{X}$.

(2) We say that $\mathcal{X}$ is *closed under direct summands* provided that if X is an object in $\mathcal{X}$ and Y is a direct summand of X in $\mathcal{A}$, then Y is also in X.

(3) We denote by $\mathsf{add}_{\mathcal{C}}\,\mathcal{X}$ the *additive closure* of $\mathcal{X}$, that is, the smallest subcategory of $\mathcal{C}$ containing $\mathcal{X}$ and closed under finite direct sums and direct summands.

(4) Assume $\mathcal{C}$ is abelian (resp. triangulated). We say that $\mathcal{X}$ is *closed under extensions* provided for an exact sequence $0 \to L \to M \to N \to 0$ (resp. exact triangle $L \to M \to N \rightsquigarrow$) in $\mathcal{C}$, if $L, N \in \mathcal{X}$, then $M \in \mathcal{X}$.

(5) Suppose that $\mathcal{C}$ is either abelian or triangulated. The *extension closure* $\mathsf{ext}_{\mathcal{C}}\,\mathcal{X}$ of $\mathcal{X}$ is defined as the smallest subcategory of $\mathcal{C}$ containing $\mathcal{X}$ and closed under direct summands and extensions.

Definition 2.3. Let $\mathcal{T}$ be a triangulated category, and let $\mathcal{X}$ be a subcategory of $\mathcal{T}$.

(1) For any $n \in \mathbb{Z}$ denote by $\mathcal{X}[n]$ the subcategory of $\mathcal{T}$ consisting of objects of the form $X[n]$ with $X \in \mathcal{X}$.

(2) We say that $\mathcal{X}$ is *closed under positive shifts* (resp. *closed under negative shifts*) if $\mathcal{X}[n]$ is contained in $\mathcal{X}$ for all $n > 0$ (resp. $n < 0$), which is equivalent to saying that $\mathcal{X}[1]$ (resp. $\mathcal{X}[-1]$) is contained in $\mathcal{X}$.

(3) We say that $\mathcal{X}$ is *thick* if $\mathcal{X}$ is a nonempty triangulated subcategory of $\mathcal{T}$ closed under direct summands. We denote by $\mathsf{thick}_{\mathcal{T}}\,\mathcal{X}$ the *thick closure* of $\mathcal{X}$, namely, the smallest thick subcategory of $\mathcal{T}$ containing $\mathcal{X}$.

In the next proposition, we compare closedness under positive/negative shifts with other conditions regarding subcategories of a triangulated category. The proof of the proposition is left as an easy exercise to the reader.

Proposition 2.4. *Let $\mathcal{T}$ be a triangulated category. Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$.*

(1) *Suppose $\mathcal{X}$ is closed under extensions and contains the zero object of $\mathcal{T}$. Then the following are equivalent.*
   - (a) *The subcategory $\mathcal{X}$ of $\mathcal{T}$ is closed under positive* (*resp. negative*) *shifts.*
   - (b) *If $A \to B \to C \rightsquigarrow$ is an exact triangle with $A, B \in \mathcal{X}$* (*resp. $B, C \in \mathcal{X}$*), *then C* (*resp. A*) *is in $\mathcal{X}$.*

(2) *If $\mathcal{X}$ is nonempty, then $\mathcal{X}$ is a thick subcategory of $\mathcal{T}$ if and only if $\mathcal{X}$ is closed under direct summands, extensions, positive shifts and negative shifts.*

We introduce categories and subcategories which we basically use in this paper.

Definition 2.5. We denote by $\mathsf{mod}\,R$ the category of finitely generated R-modules, by $\mathsf{proj}\,R$ the subcategory of $\mathsf{mod}\,R$ consisting of projective modules, and by $\mathsf{fpd}\,R$ the subcategory of $\mathsf{mod}\,R$ consisting of modules of finite projective dimension. We denote by $\mathsf{D}(R)$ the bounded derived category of $\mathsf{mod}\,R$ which is denoted by $\mathsf{D}^{\mathsf{b}}(R)$ in Section 1, and by $\mathsf{K}(R)$ the bounded homotopy category of $\mathsf{proj}\,R$. Via the natural fully faithful functors, we regard $\mathsf{mod}\,R$ and $\mathsf{K}(R)$ as (strictly full) subcategories of $\mathsf{D}(R)$. Thus we identify $\mathsf{K}(R)$ with the derived category $\mathsf{D}^{\mathsf{perf}}(R)$ of perfect R-complexes that appears in Section 1. Here, a *perfect* complex is defined to be a bounded complex of finitely generated projective modules. We thus have inclusions

$$\mathsf{proj}\,R \subseteq \mathsf{fpd}\,R = \mathsf{K}(R) \cap \mathsf{mod}\,R, \quad \mathsf{fpd}\,R \subseteq \mathsf{K}(R) \subseteq \mathsf{D}(R), \quad \mathsf{fpd}\,R \subseteq \mathsf{mod}\,R \subseteq \mathsf{D}(R).$$

Definition 2.6 (resolving subcategories of module categories). Let $\mathcal{X}$ be a subcategory of $\mathsf{mod}\,R$.

(1) We say that $\mathcal{X}$ is *resolving* if it satisfies the following four conditions.

   (i) $\mathcal{X}$ contains R. (ii) $\mathcal{X}$ is closed under direct summands.
   (iii) $\mathcal{X}$ is closed under extensions.
   (iv) For an exact sequence $0 \to A \to B \to C \to 0$ in $\mathsf{mod}\,R$ with $B, C \in \mathcal{X}$, one has $A \in \mathcal{X}$.

   Conditions (i) and (ii) imply that a resolving subcategory of $\mathsf{mod}\,R$ contains the zero object 0 of $\mathsf{mod}\,R$. Condition (i) can be replaced with the condition that $\mathcal{X}$ contains $\mathsf{proj}\,R$. Condition (iv) can be replaced with the condition that $\mathcal{X}$ is closed under syzygies; see [49, Remark 2.3].

(2) The *resolving closure* $\mathsf{res}_{\mathsf{mod}\,R}\,\mathcal{X}$ of $\mathcal{X}$ is the smallest resolving subcategory of $\mathsf{mod}\,R$ containing $\mathcal{X}$.

Definition 2.7 (resolving subcategories of triangulated categories). Let $\mathcal{T}$ be a triangulated subcategory of $\mathsf{D}(R)$ containing R. Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$.

(1) We say that $\mathcal{X}$ is *resolving* if it satisfies the following four conditions.

(i) $\mathcal{X}$ contains R. (ii) $\mathcal{X}$ is closed under direct summands.

(iii) $\mathcal{X}$ is closed under extensions.

(iv) For an exact triangle $A \to B \to C \rightsquigarrow$ in $\mathcal{T}$ with $B, C \in \mathcal{X}$, one has $A \in \mathcal{X}$.

Conditions (i) and (ii) imply that a resolving subcategory of $\mathcal{T}$ contains the zero object 0 of $\mathcal{T}$. Condition (i) can be replaced with the condition that $\mathcal{X}$ contains $\mathsf{proj}\,R$. Condition (iv) can be replaced with the condition that $\mathcal{X}$ is closed under negative shifts; see Proposition 2.4(1).

(2) The *resolving closure* $\mathsf{res}_{\mathcal{T}}\,\mathcal{X}$ of $\mathcal{X}$ is defined to be the smallest resolving subcategory of $\mathcal{T}$ containing $\mathcal{X}$.

In the next proposition we explore the relationship between resolving closures and shifts. It turns out that compatibility of taking the resolving closure and taking a shift is subtle; see also Remark 2.16 given later.

Proposition 2.8. *Let $\mathcal{T}$ be a triangulated subcategory of $\mathsf{D}(R)$ containing R.*

(1) *For each object X of $\mathcal{T}$ and each integer n, there is an equality*

$$\mathsf{res}_{\mathcal{T}}\{X[i] \mid i \in \mathbb{Z}\} = \mathsf{res}_{\mathcal{T}}\{X[i] \mid i \geqslant n\}.$$

(2) *Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$, and let n be an integer.*

(a) *Let $n \leqslant 0$. Then there is an inclusion $(\mathsf{res}_{\mathcal{T}}\,\mathcal{X})[n] \subseteq \mathsf{res}_{\mathcal{T}}(\mathcal{X}[n])$.*

(b) *Let $n \geqslant 0$. If $\mathcal{X}$ is resolving, then so is $\mathcal{X}[n]$. More generally, $(\mathsf{res}_{\mathcal{T}}\,\mathcal{X})[n] = \mathsf{res}_{\mathcal{T}}(\mathcal{X}[n] \cup \{R[n]\})$.*

Proof. (1) Set $\mathcal{X} = \mathsf{res}_{\mathcal{T}}\{X[i] \mid i \geqslant n\}$. Fix $j \in \mathbb{Z}$. If $j \geqslant n$, then $X[j]$ is clearly in $\mathcal{X}$. If $j < n$, then $j - n < 0$ and one has $X[j] = (X[n])[j-n] \in \mathsf{res}(X[n]) \subseteq \mathcal{X}$. Hence $X[j] \in \mathcal{X}$ for all $j \in \mathbb{Z}$, and the assertion follows.

(2a) Consider the subcategory $\mathcal{Y} = \{Y \in \mathcal{T} \mid Y[n] \in \mathsf{res}_{\mathcal{T}}(\mathcal{X}[n])\}$ of $\mathcal{T}$. Since $\mathsf{res}_{\mathcal{T}}(\mathcal{X}[n])$ is resolving, it contains R. As $n \leqslant 0$, we have $R[n] \in \mathsf{res}_{\mathcal{T}}(\mathcal{X}[n])$. Hence R belongs to $\mathcal{Y}$. Let Y be an object in $\mathcal{Y}$ and Z a direct summand of Y. Then $Y[n]$ is in $\mathsf{res}_{\mathcal{T}}(\mathcal{X}[n])$ and $Z[n]$ is a direct summand of $Y[n]$. Hence $Z[n]$ is in $\mathsf{res}_{\mathcal{T}}(\mathcal{X}[n])$, which implies $Z \in \mathcal{Y}$. Let $A \to B \to C \rightsquigarrow$ be an exact triangle in $\mathcal{T}$ with $C \in \mathcal{Y}$. Then there is an exact triangle $A[n] \to B[n] \to C[n] \rightsquigarrow$ and $C[n]$ is in $\mathsf{res}_{\mathcal{T}}(\mathcal{X}[n])$. Hence $A[n] \in \mathsf{res}_{\mathcal{T}}(\mathcal{X}[n])$ if and only if $B[n] \in \mathsf{res}_{\mathcal{T}}(\mathcal{X}[n])$. Therefore, $A \in \mathcal{Y}$ if and only if $B \in \mathcal{Y}$. Consequently, $\mathcal{Y}$ is a resolving subcategory of $\mathcal{T}$. Since $\mathcal{Y}$

contains $\mathcal{X}$, we see that $\mathcal{Y}$ contains $\operatorname{res}_{\mathcal{T}} \mathcal{X}$. It follows that $(\operatorname{res}_{\mathcal{T}} \mathcal{X})[n] \subseteq \operatorname{res}_{\mathcal{T}}(\mathcal{X}[n])$.

(2b) To show the first assertion, suppose that $\mathcal{X}$ is resolving. As $\mathcal{X}$ is closed under negative shifts, we have that $\mathcal{X}[-1] \subseteq \mathcal{X}$, and that $R[-n] \in \mathcal{X}$ since $R \in \mathcal{X}$ and $-n \leqslant 0$. Hence $(\mathcal{X}[n])[-1] = (\mathcal{X}[-1])[n] \subseteq \mathcal{X}[n]$ and $R = (R[-n])[n] \in \mathcal{X}[n]$, that is, $\mathcal{X}[n]$ is closed under negative shifts and contains R. Let $A \to B \to C \rightsquigarrow$ be an exact triangle in $\mathcal{T}$ with $A, C \in \mathcal{X}[n]$. Then there is an exact triangle $A[-n] \to B[-n] \to C[-n] \rightsquigarrow$ in $\mathcal{T}$ and $A[-n], C[-n] \in \mathcal{X}$. Since $\mathcal{X}$ is closed under extensions, it contains $B[-n]$. Hence $B = (B[-n])[n]$ belongs to $\mathcal{X}[n]$, and therefore $\mathcal{X}[n]$ is closed under extensions. Let K be an object in $\mathcal{X}[n]$ and L a direct summand of K. Then $K[-n]$ is in $\mathcal{X}$ and $L[-n]$ is a direct summand of $K[-n]$. As $\mathcal{X}$ is closed under direct summands, $L[-n]$ is in $\mathcal{X}$. Hence L belongs to $\mathcal{X}[n]$, and therefore $\mathcal{X}[n]$ is closed under direct summands. Consequently, $\mathcal{X}[n]$ is a resolving subcategory of $\mathcal{T}$.

Now we prove the second assertion. Replacing $\mathcal{X}$ with $\mathcal{X} \cup \{R\}$, we may assume $R \in \mathcal{X}$. We want to deduce $(\operatorname{res}_{\mathcal{T}} \mathcal{X})[n] = \operatorname{res}_{\mathcal{T}}(\mathcal{X}[n])$. As $\operatorname{res}_{\mathcal{T}} \mathcal{X} \supseteq \mathcal{X}$, we have $(\operatorname{res}_{\mathcal{T}} \mathcal{X})[n] \supseteq \mathcal{X}[n]$. As $(\operatorname{res}_{\mathcal{T}} \mathcal{X})[n]$ is resolving by the first assertion, $(\operatorname{res}_{\mathcal{T}} \mathcal{X})[n]$ contains $\operatorname{res}_{\mathcal{T}}(\mathcal{X}[n])$. To show the opposite inclusion, consider the subcategory $\mathcal{Y} = \{Y \in \mathcal{T} \mid Y[n] \in \operatorname{res}_{\mathcal{T}}(\mathcal{X}[n])\}$ of $\mathcal{T}$. Since $R \in \mathcal{X}$, we get $R[n] \in \mathcal{X}[n] \subseteq \operatorname{res}_{\mathcal{T}}(\mathcal{X}[n])$, which implies $R \in \mathcal{Y}$. An analogous argument as in the proof of (1) shows $\mathcal{Y}$ is a resolving subcategory of $\mathcal{T}$. Since $\mathcal{Y}$ contains $\mathcal{X}$, it contains $\operatorname{res}_{\mathcal{T}} \mathcal{X}$. Thus $(\operatorname{res}_{\mathcal{T}} \mathcal{X})[n] \subseteq \operatorname{res}_{\mathcal{T}}(\mathcal{X}[n])$. We now obtain $(\operatorname{res}_{\mathcal{T}} \mathcal{X})[n] = \operatorname{res}_{\mathcal{T}}(\mathcal{X}[n])$. $\square$

The next notion plays a crucial role in the proofs of our main results.

Definition 2.9 (minimum resolving subcategories). Let $\mathcal{T}$ be a triangulated subcategory of $\mathsf{D}(R)$ containing R. We set $\mathcal{E}_{\mathcal{T}} = \operatorname{res}_{\mathcal{T}} 0$ and call it the *minimum resolving subcategory* of $\mathcal{T}$. It is minimum in the sense that every resolving subcategory of $\mathcal{T}$ contains $\mathcal{E}_{\mathcal{T}}$. We simply write $\mathcal{E}_R = \mathcal{E}_{\mathsf{D}(R)}$.

The resolving closure $\operatorname{res}_{\mathcal{T}} \mathcal{X}$ of a subcategory $\mathcal{X}$ of $\mathcal{T}$, particularly the minimum resolving subcategory $\mathcal{E}_{\mathcal{T}} = \operatorname{res}_{\mathcal{T}} 0$ of $\mathcal{T}$, depends on which triangulated subcategory $\mathcal{T}$ of $\mathsf{D}(R)$ is taken as the ambient category. The proposition below collects properties of resolving subcategories, the second and third of which produce sufficient conditions for $\mathcal{T}$ to satisfy $\operatorname{res}_{\mathcal{T}} \mathcal{X} = \operatorname{res}_{\mathsf{D}(R)} \mathcal{X}$ for every subcategory $\mathcal{X}$ of $\mathcal{T}$.

Proposition 2.10. (1) *If $\mathcal{T}$ is a triangulated subcategory of $\mathsf{D}(R)$ containing R, then $\mathcal{E}_{\mathcal{T}} = \operatorname{res}_{\mathcal{T}} R$. If $\mathcal{T}$ is a thick subcategory of $\mathsf{D}(R)$ containing R, then $\mathcal{E}_{\mathcal{T}} = \operatorname{res}_{\mathcal{T}}(\operatorname{proj} R)$.*

(2) *Let $\mathcal{T}$ be a thick subcategory of $\mathsf{D}(R)$ containing R. Then the resolving subcategories of $\mathcal{T}$ are the resolving subcategories of $\mathsf{D}(R)$ contained in $\mathcal{T}$. Hence $\operatorname{res}_{\mathcal{T}} \mathcal{X} = \operatorname{res}_{\mathsf{D}(R)} \mathcal{X}$ for any subcategory $\mathcal{X}$ of $\mathcal{T}$.*

(3) *The equality* $\mathsf{K}(R) = \mathsf{thick}_{\mathsf{D}(R)} R$ *holds. Hence, there is an equality* $\mathsf{res}_{\mathsf{K}(R)} \mathcal{X} = \mathsf{res}_{\mathsf{D}(R)} \mathcal{X}$ *for any subcategory* $\mathcal{X}$ *of* $\mathsf{K}(R)$. *In particular,* $\mathcal{E}_{\mathsf{K}(R)} = \mathcal{E}_R$.

(4) *If* $\mathcal{X}$ *is a resolving subcategory of* $\mathsf{D}(R)$, *then* $\mathcal{X} \cap \mathsf{mod}\, R$ *is a resolving subcategory of* $\mathsf{mod}\, R$. *If* $\mathcal{X}$ *is a resolving subcategory of* $\mathsf{K}(R)$, *then* $\mathcal{X} \cap \mathsf{mod}\, R$ *is a resolving subcategory of* $\mathsf{mod}\, R$ *contained in* $\mathsf{fpd}\, R$.

Proof. (1) The first assertion holds since $R \in \mathcal{E}_{\mathcal{T}} = \mathsf{res}_{\mathcal{T}}\, 0 \subseteq \mathsf{res}_{\mathcal{T}}\, R$. In the situation of the second assertion, $\mathcal{T}$ contains $\mathsf{proj}\, R$. Hence the assertion follows from the inclusions $\mathsf{proj}\, R \subseteq \mathcal{E}_{\mathcal{T}} = \mathsf{res}_{\mathcal{T}}\, 0 \subseteq \mathsf{res}_{\mathcal{T}}\, R \subseteq \mathsf{res}_{\mathcal{T}}(\mathsf{proj}\, R)$.

(2) Let $\mathcal{X}$ be a subcategory of $\mathcal{T}$ with $R \in \mathcal{X}$ and $\mathcal{X}[-1] \subseteq \mathcal{X}$. Since $\mathcal{T}$ is closed under extensions as a subcategory of $\mathsf{D}(R)$, we see that $\mathcal{X}$ is closed under extensions as a subcategory of $\mathcal{T}$ if and only if $\mathcal{X}$ is closed under extensions as a subcategory of $\mathsf{D}(R)$. If A, B are objects of $\mathsf{D}(R)$ with $A \oplus B \in \mathcal{X}$, then $A \oplus B \in \mathcal{T}$, which implies $A, B \in \mathcal{T}$ since $\mathcal{T}$ is thick. It is seen that $\mathcal{X}$ is closed under direct summands as a subcategory of $\mathcal{T}$ if and only if $\mathcal{X}$ is closed under direct summands as a subcategory of $\mathsf{D}(R)$. The first assertion follows.

Since $\mathsf{res}_{\mathcal{T}}\, \mathcal{X}$ is a resolving subcategory of $\mathcal{T}$ containing $\mathcal{X}$, by the first assertion it is a resolving subcategory of $\mathsf{D}(R)$ contained in $\mathcal{T}$ and containing $\mathcal{X}$. Hence $\mathcal{T} \supseteq \mathsf{res}_{\mathcal{T}}\, \mathcal{X} \supseteq \mathsf{res}_{\mathsf{D}(R)}\, \mathcal{X}$. Therefore, $\mathsf{res}_{\mathsf{D}(R)}\, \mathcal{X}$ is a resolving subcategory of $\mathsf{D}(R)$ contained in $\mathcal{T}$ and containing $\mathcal{X}$, so that it is a resolving subcategory of $\mathcal{T}$ containing $\mathcal{X}$ by the first assertion again. This implies that $\mathsf{res}_{\mathsf{D}(R)}\, \mathcal{X} \supseteq \mathsf{res}_{\mathcal{T}}\, \mathcal{X}$. The second assertion now follows.

(3) It is a well-known fact that $\mathsf{K}(R) = \mathsf{thick}_{\mathsf{D}(R)}\, R$; see [38, Proposition 1.4(2)] for instance. In particular, $\mathsf{K}(R)$ is a thick subcategory of $\mathsf{D}(R)$ containing R. It follows from (2) that $\mathsf{res}_{\mathsf{K}(R)}\, \mathcal{X} = \mathsf{res}_{\mathsf{D}(R)}\, \mathcal{X}$ for any subcategory $\mathcal{X}$ of $\mathsf{K}(R)$. We get the equalities $\mathcal{E}_{\mathsf{K}(R)} = \mathsf{res}_{\mathsf{K}(R)}\, 0 = \mathsf{res}_{\mathsf{D}(R)}\, 0 = \mathcal{E}_{\mathsf{D}(R)} = \mathcal{E}_R$.

(4) Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{D}(R)$. Then R belongs to $\mathcal{X} \cap \mathsf{mod}\, R$. If $M \in \mathcal{X} \cap \mathsf{mod}\, R$ and N is a direct summand in $\mathsf{mod}\, R$ of M, then M is in $\mathcal{X}$ and N is a direct summand in $\mathsf{D}(R)$ of M, so that N is in $\mathcal{X}$ and hence $N \in \mathcal{X} \cap \mathsf{mod}\, R$. Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathsf{mod}\, R$ with $C \in \mathcal{X} \cap \mathsf{mod}\, R$. Then there is an exact triangle $A \to B \to C \rightsquigarrow$ in $\mathsf{D}(R)$ and $C \in \mathcal{X}$. Hence $A \in \mathcal{X}$ if and only if $B \in \mathcal{X}$, so that $A \in \mathcal{X} \cap \mathsf{mod}\, R$ if and only if $B \in \mathcal{X} \cap \mathsf{mod}\, R$. Thus, $\mathcal{X} \cap \mathsf{mod}\, R$ is a resolving subcategory of $\mathsf{mod}\, R$.

Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{K}(R)$. By (2) and (3), $\mathcal{X}$ is a resolving subcategory of $\mathsf{D}(R)$ contained in $\mathsf{K}(R)$. Hence $\mathcal{X} \cap \mathsf{mod}\, R$ is a resolving subcategory of $\mathsf{mod}\, R$ contained in $\mathsf{K}(R) \cap \mathsf{mod}\, R = \mathsf{fpd}\, R$. □

In the proposition below, we state simple observations about representing each closure as an intersection of subcategories. We leave the proof of the proposition as an easy exercise to the reader.

Proposition 2.11. *Let $\mathcal{C}$ be an additive category, and let $\mathcal{X}$ be a subcategory of $\mathcal{C}$.*

(1) *The additive closure $\mathsf{add}_{\mathcal{C}}\,\mathcal{X}$ is equal to the intersection of all subcategories of $\mathcal{C}$ that contain $\mathcal{X}$ and are closed under finite direct sums and direct summands.*

(2) *Assume that $\mathcal{C}$ is either abelian or triangulated. Then the extension closure $\mathsf{ext}_{\mathcal{C}}\,\mathcal{X}$ is equal to the intersection of all subcategories of $\mathcal{C}$ that contain $\mathcal{X}$ and are closed under direct summands and extensions.*

(3) *Assume that $\mathcal{C}$ is triangulated. Then the thick closure $\mathsf{thick}_{\mathcal{C}}\,\mathcal{X}$ is equal to the intersection of all thick subcategories of $\mathcal{C}$ containing $\mathcal{X}$.*

(4) *Assume that $\mathcal{C}$ is a triangulated subcategory of $\mathsf{D}(R)$ containing R. Then the resolving closure $\mathsf{res}_{\mathcal{C}}\,\mathcal{X}$ is equal to the intersection of all resolving subcategories of $\mathcal{C}$ containing $\mathcal{X}$. In particular, $\mathcal{E}_{\mathcal{C}}$ coincides with the intersection of all resolving subcategories of $\mathcal{C}$.*

Next we recall the definitions of projective dimension and depth for complexes, and of Koszul complexes.

Definition 2.12. (1) The *supremum* $\sup X$ and *infimum* $\inf X$ of an object $X \in \mathsf{D}(R)$ is defined by $\sup X = \sup\{i \in \mathbb{Z} \mid \mathrm{H}^i X \neq 0\}$ and $\inf X = \inf\{i \in \mathbb{Z} \mid \mathrm{H}^i X \neq 0\}$.

(2) The *projective dimension* $\mathrm{pd}_R X$ of an object $X \in \mathsf{D}(R)$ is the infimum of integers n such that $X \cong P$ in $\mathsf{D}(R)$ for some perfect R-complex P with $P^{-i} = 0$ for all integers $i > n$. One has $\mathrm{pd}\, X \in \mathbb{Z} \cup \{\pm\infty\}$ and $\mathrm{pd}\, X \geqslant -\inf X$; note that $\mathrm{pd}\, X = -\infty$ if and only if $X \cong 0$ in $\mathsf{D}(R)$. Also, $\mathrm{pd}\, X < \infty$ if and only if $X \in \mathsf{K}(R)$. One does not necessarily have $\mathrm{pd}\, X \leqslant n$ even if $X \cong P$ in $\mathsf{D}(R)$ for some complex P of finitely generated projective modules with $P^{-i} = 0$ for all $i > n$; see [9, 2.6.P]. We refer to [9, 1.2.P, 1.7, 2.3.P, 2.4.P and 2.7.P] for details of projective dimension.

(3) For each integer n, we denote by $\mathsf{K}^n(R)$ the subcategory of $\mathsf{K}(R)$ consisting of perfect complexes having projective dimension at most n.

(4) For a sequence $\boldsymbol{x} = x_1, \ldots, x_n$ we denote by $\mathrm{K}(\boldsymbol{x}, R)$ the Koszul complex of $\boldsymbol{x}$ over R; we refer the reader to [18, §1.6] for the definition and details of Koszul complexes. When the ambient ring R is clear, we simply write $\mathrm{K}(\boldsymbol{x})$.

(5) Let R be a local ring with residue field k. For an object X of $\mathsf{D}(R)$, we denote by $\mathrm{depth}_R X$ the *depth* of X, which is defined by the equality $\mathrm{depth}_R X = \inf \mathbf{R}\mathrm{Hom}_R(k, X)$. Note that if X belongs to $\mathsf{mod}\, R$, then $\mathrm{depth}_R X = \inf\{i \in \mathbb{N} \mid \mathrm{Ext}^i_R(k, X) \neq 0\}$, so it coincides with the classical definition of the depth of a finitely generated module over a local ring.

We make a collection of basic properties of projective dimension and depth which are frequently used later.

Proposition 2.13. (1) *Let X be an object of* $\mathsf{D}(R)$, *and let r be an integer. Then* $\operatorname{pd}_R(X[r]) = \operatorname{pd}_R X + r$. *When the ring R is local, the equality* $\operatorname{depth}_R(X[r]) = \operatorname{depth}_R X - r$ *holds.*

(2) *Let R be a local ring with residue field k. Let $X \in \mathsf{D}(R)$. One has the equality* $\operatorname{pd}_R X = -\inf(X \otimes_R^{\mathbf{L}} k)$. *Also, the* ***Auslander–Buchsbaum formula*** *holds: If* $\operatorname{pd}_R X < \infty$, *then* $\operatorname{pd}_R X = \operatorname{depth} R - \operatorname{depth}_R X$.

(3) *Let $A \to B \to C \rightsquigarrow$ be an exact triangle in $\mathsf{D}(R)$. Then the following inequalities hold true, where for the latter ones we assume that the ring R is local.*

$$\begin{cases} \operatorname{pd}_R B \leqslant \sup\{\operatorname{pd}_R A, \operatorname{pd}_R C\}, \\ \operatorname{pd}_R A \leqslant \sup\{\operatorname{pd}_R B, \operatorname{pd}_R C - 1\}, \\ \operatorname{pd}_R C \leqslant \sup\{\operatorname{pd}_R B, \operatorname{pd}_R A + 1\}, \end{cases} \qquad \begin{cases} \operatorname{depth}_R B \geqslant \inf\{\operatorname{depth}_R A, \operatorname{depth}_R C\}, \\ \operatorname{depth}_R A \geqslant \inf\{\operatorname{depth}_R B, \operatorname{depth}_R C + 1\}, \\ \operatorname{depth}_R C \geqslant \inf\{\operatorname{depth}_R B, \operatorname{depth}_R A - 1\} \end{cases}$$

(4) *Let X and Y be objects of $\mathsf{D}(R)$. Then* $\operatorname{pd}_R(X \oplus Y) = \sup\{\operatorname{pd}_R X, \operatorname{pd}_R Y\}$. *When R is local,* $\operatorname{depth}_R(X \oplus Y) = \inf\{\operatorname{depth}_R X, \operatorname{depth}_R Y\}$.

(5) *For all nonnegative integers n, the subcategory $\mathsf{K}^n(R)$ of $\mathsf{K}(R)$ is resolving.*

(6) *There is an equality $\mathsf{K}^0(R) = \mathcal{E}_R$. In particular, the equality $\mathcal{E}_R \cap \operatorname{mod} R = \operatorname{proj} R$ holds.*

(7) *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let $\boldsymbol{x} = x_1, \ldots, x_n$ be a sequence of elements of R. If $x_i \in \mathfrak{m}$ for all i, then* $\operatorname{pd}_R \mathrm{K}(\boldsymbol{x}) = n$. *If $x_i \notin \mathfrak{m}$ for some i, then* $\mathrm{K}(\boldsymbol{x}) \cong 0$ *in* $\mathsf{K}(R)$ *and* $\operatorname{pd}_R \mathrm{K}(\boldsymbol{x}) = -\infty$.

Proof. (1) We easily deduce the assertion from the definitions of projective dimension and depth.

(2) The first assertion follows from [21, (A.5.7.2)]. The second assertion is stated in [22, (1.5)] for example.

(3) Suppose that R is a local ring with residue field k. The two exact triangles

$$\mathbf{R}\mathrm{Hom}_R(k, A) \to \mathbf{R}\mathrm{Hom}_R(k, B) \to \mathbf{R}\mathrm{Hom}_R(k, C) \rightsquigarrow, \qquad A \otimes_R^{\mathbf{L}} k \to B \otimes_R^{\mathbf{L}} k \to C \otimes_R^{\mathbf{L}} k \rightsquigarrow$$

give rise to the inequalities

$$\inf \mathbf{R}\mathrm{Hom}_R(k, B) \geqslant \inf\{\inf \mathbf{R}\mathrm{Hom}_R(k, A), \inf \mathbf{R}\mathrm{Hom}_R(k, C)\}$$

and $\inf(B \otimes_R^{\mathbf{L}} k) \geqslant \inf\{\inf(A \otimes_R^{\mathbf{L}} k), \inf(C \otimes_R^{\mathbf{L}} k)\}$. Therefore we have $\operatorname{depth} B \geqslant \inf\{\operatorname{depth} A, \operatorname{depth} C\}$, and by (2) we get

$$\begin{aligned} \operatorname{pd}_R B &= -\inf(B \otimes_R^{\mathbf{L}} k) \leqslant -\inf\{\inf(A \otimes_R^{\mathbf{L}} k), \inf(C \otimes_R^{\mathbf{L}} k)\} \\ &= \sup\{-\inf(A \otimes_R^{\mathbf{L}} k), -\inf(C \otimes_R^{\mathbf{L}} k)\} = \sup\{\operatorname{pd}_R A, \operatorname{pd}_R C\}. \end{aligned}$$

Now we consider the case where R is nonlocal. Using [9, Proposition 5.3.P] and the local case, we get

$$\begin{aligned}\operatorname{pd}_R B &= \sup_{\mathfrak{p}\in\operatorname{Spec} R}\{\operatorname{pd}_{R_\mathfrak{p}} B_\mathfrak{p}\}\\ &\leqslant \sup_{\mathfrak{p}\in\operatorname{Spec} R}\{\sup\{\operatorname{pd}_{R_\mathfrak{p}} A_\mathfrak{p}, \operatorname{pd}_{R_\mathfrak{p}} C_\mathfrak{p}\}\} \leqslant \sup\{\operatorname{pd}_R A, \operatorname{pd}_R C\}.\end{aligned}$$

Applying the argument given so far to the exact triangles $C[-1] \to A \to B \rightsquigarrow$ and $B \to C \to A[1] \rightsquigarrow$ and using (1), we obtain the remaining four inequalities.

(4) Suppose that the ring R is local, and let k be the residue field of R. Using (2) for the former, we have

$$\begin{aligned}\operatorname{pd}_R(X\oplus Y) &= -\inf((X\oplus Y)\otimes^{\mathbf{L}}_R k)\\ &= -\inf((X\otimes^{\mathbf{L}}_R k)\oplus(Y\otimes^{\mathbf{L}}_R k)) = -\inf\{\inf(X\otimes^{\mathbf{L}}_R k), \inf(Y\otimes^{\mathbf{L}}_R k)\}\\ &= \sup\{-\inf(X\otimes^{\mathbf{L}}_R k), -\inf(Y\otimes^{\mathbf{L}}_R k)\} = \sup\{\operatorname{pd}_R X, \operatorname{pd}_R Y\},\\ \operatorname{depth}_R(X\oplus Y) &= \inf \mathbf{R}\mathrm{Hom}_R(k, X\oplus Y) = \inf(\mathbf{R}\mathrm{Hom}_R(k, X)\oplus \mathbf{R}\mathrm{Hom}_R(k, Y))\\ &= \inf\{\inf \mathbf{R}\mathrm{Hom}_R(k, X), \inf \mathbf{R}\mathrm{Hom}_R(k, Y)\}\\ &= \inf\{\operatorname{depth}_R X, \operatorname{depth}_R Y\}.\end{aligned}$$

Now let the ring R be nonlocal. Applying (3) to the exact triangle

$$X \to X\oplus Y \to Y \rightsquigarrow$$

gives $\operatorname{pd}_R(X\oplus Y) \leqslant \sup\{\operatorname{pd}_R X, \operatorname{pd}_R Y\}$. Assume $\operatorname{pd}_R(X\oplus Y) < \sup\{\operatorname{pd}_R X, \operatorname{pd}_R Y\}$. We may assume $\operatorname{pd}_R X \geqslant \operatorname{pd}_R Y$.

We claim that if $\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p} < \infty$ for all prime ideals $\mathfrak{p}$ of R, then $\operatorname{pd}_R X < \infty$. Indeed, putting $t = \inf X$ and $s = \sup X$, we find a complex $P = (\cdots \to P^t \to \cdots \to P^s \to 0)$ of finitely generated projective R-modules such that $P \cong X$ in $\mathsf{D}(R)$. Let C be the cokernel of the map $P^{t-1} \to P^t$. Let $Q = (0 \to P^{t+1} \to \cdots \to P^s \to 0)$ be the truncation of P, which is a perfect complex. There is an exact triangle $Q \to P \to C[-t] \rightsquigarrow$. For each $\mathfrak{p} \in \operatorname{Spec} R$ we have $\operatorname{pd}_{R_\mathfrak{p}} Q_\mathfrak{p} < \infty$ and $\operatorname{pd}_{R_\mathfrak{p}} P_\mathfrak{p} = \operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p} < \infty$, so that $\operatorname{pd}_{R_\mathfrak{p}} C_\mathfrak{p} < \infty$. It follows from [13, Lemma 4.5] that $\operatorname{pd}_R C < \infty$. As $\operatorname{pd}_R Q < \infty$, we get $\operatorname{pd}_R X = \operatorname{pd}_R P < \infty$. The claim thus follows.

The claim and [9, Proposition 5.3.P] produce a prime ideal $\mathfrak{p}$ such that $\operatorname{pd}_R X = \operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p} \leqslant \infty$. We have

$$\begin{aligned}\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p} &\leqslant \sup\{\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p}, \operatorname{pd}_{R_\mathfrak{p}} Y_\mathfrak{p}\} = \operatorname{pd}_{R_\mathfrak{p}}(X_\mathfrak{p}\oplus Y_\mathfrak{p})\\ &\leqslant \operatorname{pd}_R(X\oplus Y) < \sup\{\operatorname{pd}_R X, \operatorname{pd}_R Y\} = \operatorname{pd}_R X = \operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p},\end{aligned}$$

where the first equality holds since the ring $R_\mathfrak{p}$ is local. We now get a contradiction, and therefore, the equality $\operatorname{pd}_R(X\oplus Y) = \sup\{\operatorname{pd}_R X, \operatorname{pd}_R Y\}$ holds.

(5) As n is nonnegative, R belongs to $\mathsf{K}^n(R)$. The assertion is shown to hold by using (3) and (4).

(6) As $\mathcal{E}_R$ contains R and is closed under negative shifts, it contains $R[i]$ for all $i \leqslant 0$. Hence we will get the required equality $\mathsf{K}^0(R) = \mathcal{E}_R$ once we prove that the following inclusions hold.

$$\mathsf{K}^0(R) \subseteq \mathsf{ext}_{\mathsf{K}(R)}\{R[i] \mid i \leqslant 0\} \subseteq \mathcal{E}_R \subseteq \mathsf{K}^0(R).$$

The second inclusion holds since $\mathcal{E}_R$ is closed under extensions, while the last inclusion comes from the fact that $\mathsf{K}^0(R)$ is a resolving subcategory and $\mathcal{E}_R$ is a minimum resolving subcategory. To show the first inclusion, pick an object P in $\mathsf{K}^0(R)$. We may assume that $P = (0 \to P^0 \to P^1 \to \cdots \to P^s \to 0)$. Then P belongs to $\mathsf{ext}_{\mathsf{K}(R)}\{P^s[-s], \dots, P^1[-1], P^0\}$, which is contained in $\mathsf{ext}_{\mathsf{K}(R)}\{R[i] \mid i \leqslant 0\}$. Thus the first inclusion follows.

(7) If $x_i \in \mathfrak{m}$ for all i, then $\operatorname{pd}_R \mathrm{K}(\boldsymbol{x}) = n$ by (2). If $x_i \notin \mathfrak{m}$ for some i, then $\mathrm{K}(x_i) \cong 0$ in $\mathsf{D}(R)$, and hence $\mathrm{K}(\boldsymbol{x}) \cong \mathrm{K}(x_1) \otimes_R^{\mathbf{L}} \cdots \otimes_R^{\mathbf{L}} \mathrm{K}(x_i) \otimes_R^{\mathbf{L}} \cdots \otimes_R^{\mathbf{L}} \mathrm{K}(x_n) \cong 0$ in $\mathsf{D}(R)$. Hence $\mathrm{K}(\boldsymbol{x}) \cong 0$ in $\mathsf{K}(R)$. □

Here, let us present an application of the above proposition. The corollary below is thought of as a derived category version of [50, Proposition 1.12(2)].

Corollary 2.14. *Let R be a local ring. Let X and Y be complexes that belong to $\mathsf{D}(R)$. Suppose that X is in the resolving closure $\mathsf{res}_{\mathsf{D}(R)}\, Y$. Then there is an inequality* $\operatorname{depth}_R X \geqslant \inf\{\operatorname{depth}_R Y, \operatorname{depth} R\}$.

Proof. Let $\mathcal{Z}$ be the subcategory of $\mathsf{D}(R)$ consisting of objects Z such that $\operatorname{depth} Z \geqslant \inf\{\operatorname{depth} Y, \operatorname{depth} R\}$. It is evident that $\mathcal{Z}$ contains Y and R. Using the depth equality in Proposition 2.13(4), we see that $\mathcal{Z}$ is closed under direct summands. Also, the first depth inequality in Proposition 2.13(3) shows that $\mathcal{Z}$ is closed under extensions. By the depth equality in Proposition 2.13(1), it follows that $\mathcal{Z}$ is closed under negative shifts. Consequently, $\mathcal{Z}$ is a resolving subcategory of $\mathsf{D}(R)$ containing Y. Hence $\mathcal{Z}$ contains $\mathsf{res}_{\mathsf{D}(R)}\, Y$, and therefore X belongs to $\mathcal{Z}$. Now the assertion of the corollary follows. □

By definition, a thick subcategory of $\mathsf{D}(R)$ containing R is a resolving subcategory of $\mathsf{D}(R)$. The converse of this statement is not necessarily true. Actually, we state and prove the following proposition, which gives rise to an example of a resolving subcategory of $\mathsf{D}(R)$ that is not a thick subcategory of $\mathsf{D}(R)$.

Proposition 2.15. *The equality* $\mathsf{res}_{\mathsf{D}(R)}(\mathsf{mod}\, R) = \{X \in \mathsf{D}(R) \mid \mathrm{H}^{<0} X = 0\}$ *of subcategories of* $\mathsf{D}(R)$ *holds. Thus, the resolving subcategory* $\mathsf{res}_{\mathsf{D}(R)}(\mathsf{mod}\, R)$ *of* $\mathsf{D}(R)$ *is not thick*; *it is not closed under positive shifts.*

Proof. Let $\mathcal{X}$ be the subcategory of $\mathsf{D}(R)$ consisting of complexes X with $\mathrm{H}^{<0} X = 0$. Evidently, $\mathcal{X}$ contains $\mathsf{mod}\, R$. In particular, $\mathcal{X}$ contains R. It is straightforward to verify that $\mathcal{X}$ is closed under direct summands, extensions, and negative shifts. Hence $\mathcal{X}$ is a resolving subcategory of $\mathsf{D}(R)$ containing $\mathsf{mod}\, R$. Therefore, $\mathcal{X}$

contains $\mathsf{res}_{\mathsf{D}(R)}(\mathsf{mod}\,R)$. Conversely, pick $X \in \mathcal{X}$. Since $\mathrm{H}^{<0}X = 0$, we may assume $X = (0 \to X^0 \to X^1 \to \cdots \to X^n \to 0)$; see [21, (A.1.14)]. There is a series $\{X^i[-i] \to X_i \to X_{i-1} \rightsquigarrow\}_{i=0}^n$ of exact triangles in $\mathsf{D}(R)$ with $X_n = X$ and $X_{-1} = 0$. The object $X^i[-i]$ is in $\mathsf{res}_{\mathsf{D}(R)}(\mathsf{mod}\,R)$ for all $0 \leqslant i \leqslant n$, since $X^i \in \mathsf{mod}\,R$ and $-i \leqslant 0$. It is observed that X belongs to $\mathsf{res}_{\mathsf{D}(R)}(\mathsf{mod}\,R)$. Therefore, $\mathcal{X}$ is contained in $\mathsf{res}_{\mathsf{D}(R)}(\mathsf{mod}\,R)$.

As for the last assertion of the proposition, we have $R \in \mathcal{X}$, but $R[1] \notin \mathcal{X}$ since $\mathrm{H}^{-1}(R[1]) = R \neq 0$. □

We close the section by stating a remark on the second assertion of Proposition 2.8.

Remark 2.16. Let $\mathcal{T}$ be a triangulated subcategory of $\mathsf{D}(R)$ containing R. Let X and Y be objects of $\mathcal{T}$. Assume that X belongs to $\mathsf{res}\,Y$. Then $X[n]$ belongs to $\mathsf{res}(Y[n])$ if $n \leqslant 0$ by Proposition 2.8(2a). However, $X[n]$ does not necessarily belong to $\mathsf{res}(Y[n])$, if $n > 0$. In fact, we have the following observations.

(1) Let $X = R$ and $Y = R[-1]$. Then $X \in \mathsf{res}_{\mathsf{D}(R)}\,Y$, but $X[1] = R[1] \notin \mathcal{E}_R = \mathsf{res}_{\mathsf{D}(R)}\,R = \mathsf{res}_{\mathsf{D}(R)}(Y[1])$; see Proposition 2.10(1). Indeed, we have $\operatorname{pd} R[1] = 1$ and $R[1] \notin \mathsf{K}^0(R) = \mathcal{E}_R$ by Proposition 2.13(1)(6).

(2) Suppose that there exists an exact triangle $X \to E \to Y \rightsquigarrow$ in $\mathcal{T}$ such that $E \in \mathcal{E}_{\mathcal{T}}$. Then X belongs to $\mathsf{res}_{\mathcal{T}}\,Y$. An exact triangle $X[1] \to E[1] \to Y[1] \rightsquigarrow$ in $\mathcal{T}$ is induced. If $E[1]$ is in $\mathsf{res}_{\mathcal{T}}(Y[1])$, then $X[1]$ is in $\mathsf{res}_{\mathcal{T}}(Y[1])$. However, as we have seen in (1), the object $E[1]$ may not belong to $\mathsf{res}_{\mathcal{T}}(Y[1])$.

3. NE-loci of objects and subcategories of $\mathsf{D}(R)$

Recall that the *nonfree locus* $\mathrm{NF}(M)$ of each object M of $\mathsf{mod}\,R$ is by definition the set of prime ideals $\mathfrak{p}$ of R such that the localization $M_{\mathfrak{p}}$ is nonfree as an $R_{\mathfrak{p}}$-module. Also, recall that the *nonfree locus* $\mathrm{NF}(\mathcal{X})$ of a subcategory $\mathcal{X}$ of $\mathsf{mod}\,R$ is defined as the union of $\mathrm{NF}(M)$ where M runs through the objects of $\mathcal{X}$. In this section, we introduce and study NE-loci $\mathrm{NE}(-)$, which extend nonfree loci $\mathrm{NF}(-)$ to objects and subcategories of $\mathsf{D}(R)$.

We begin with stating the definitions of an R-linear additive category and a quotient of such a category by an ideal.

Definition 3.1. Let $\mathcal{C}$ be an *R-linear* additive category, that is, an additive category whose hom-sets are R-modules and composition of morphisms is R-bilinear.

(1) An *ideal* $\mathcal{I}$ of $\mathcal{C}$ is by definition a family $\{\mathcal{I}(X, Y)\}_{X,Y\in\mathcal{C}}$ of R-submodules of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ such that $bfa \in \mathcal{I}(W, Z)$ for all $a \in \mathrm{Hom}_{\mathcal{C}}(W, X)$, $f \in \mathcal{I}(X, Y)$, $b \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ and $W, X, Y, Z \in \mathcal{C}$. The *quotient* $\mathcal{C}/\mathcal{I}$ of $\mathcal{C}$ by the ideal $\mathcal{I}$ is by definition the category whose objects are those of $\mathcal{C}$ and morphisms are given by $\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)/\mathcal{I}(X, Y)$ for $X, Y \in \mathcal{C}$. Note that $\mathcal{C}/\mathcal{I}$ is an R-linear additive category.

(2) Let $\mathcal{D}$ be a subcategory of $\mathcal{C}$. For two objects X, Y of $\mathcal{C}$, let $[\mathcal{D}](X, Y)$ be the subset of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ consisting of morphisms that factor through some finite direct sums of objects in $\mathcal{D}$. Then it is easy to observe that $[\mathcal{D}]$ is an ideal of $\mathcal{C}$, and hence the quotient $\mathcal{C}/[\mathcal{D}]$ is defined.

The category $\underline{\mathsf{D}}(R)$ that we now define plays an important role in this section.

Definition 3.2. We denote by $\underline{\mathsf{D}}(R)$ the quotient $\mathsf{D}(R)/[\mathcal{E}_R]$. The hom-set

$$\mathrm{Hom}_{\underline{\mathsf{D}}(R)}(X, Y)$$

is a finitely generated R-module for all $X, Y \in \underline{\mathsf{D}}(R)$, as it is a factor of the finitely generated R-module $\mathrm{Hom}_{\mathsf{D}(R)}(X, Y)$.

The proposition below concerns when an object and a morphism in $\mathsf{D}(R)$ are zero in the category $\underline{\mathsf{D}}(R)$. The proof is standard, so we omit it.

Proposition 3.3. (1) *A morphism in $\mathsf{D}(R)$ is zero in $\underline{\mathsf{D}}(R)$ if and only if it factors through an object in $\mathcal{E}_R$.*

(2) *Let $X \in \mathsf{D}(R)$. The following are equivalent*:

(a) $X \cong 0$ *in* $\underline{\mathsf{D}}(R)$. (b) $\mathrm{Hom}_{\underline{\mathsf{D}}(R)}(X, X) = 0$. (c) $X \in \mathcal{E}_R$.

We define the localization of a given subcategory of $\mathsf{D}(R)$ by a multiplicatively closed subset of R.

Definition 3.4. Let $\mathcal{X}$ be a subcategory of $\mathsf{D}(R)$. For a multiplicatively closed subset S of R, we define the subcategory $\mathcal{X}_S$ of $\mathsf{D}(R_S)$ by $\mathcal{X}_S = \{X_S \mid X \in \mathcal{X}\}$. When $S = R \setminus \mathfrak{p}$ with $\mathfrak{p} \in \operatorname{Spec} R$, we set $\mathcal{X}_{\mathfrak{p}} = \mathcal{X}_S$.

In the lemma below we study the structure of localizations of morphisms in the category $\underline{\mathsf{D}}(R)$.

Lemma 3.5. *Let $\mathfrak{p}$ be a prime ideal of R. Taking the localization of a morphism in $\mathsf{D}(R)$ at $\mathfrak{p}$ induces an isomorphism $\mathrm{Hom}_{\underline{\mathsf{D}}(R)}(X, Y)_{\mathfrak{p}} \to \mathrm{Hom}_{\underline{\mathsf{D}}(R_{\mathfrak{p}})}(X_{\mathfrak{p}}, Y_{\mathfrak{p}})$ of $R_{\mathfrak{p}}$-modules for all objects $X, Y \in \mathsf{D}(R)$.*

Proof. Localization at $\mathfrak{p}$ gives an isomorphism

$$\phi : \mathrm{Hom}_{\mathsf{D}(R)}(X, Y)_{\mathfrak{p}} \to \mathrm{Hom}_{\mathsf{D}(R_{\mathfrak{p}})}(X_{\mathfrak{p}}, Y_{\mathfrak{p}});$$

see [21, Lemma (A.4.5)]. If $P = (0 \to P^0 \to \cdots \to P^n \to 0)$ is a perfect R-complex, then $P_{\mathfrak{p}} = (0 \to P^0_{\mathfrak{p}} \to \cdots \to P^n_{\mathfrak{p}} \to 0)$ is a perfect $R_{\mathfrak{p}}$-complex. By Proposition 2.13(6), the subcategory $(\mathcal{E}_R)_{\mathfrak{p}}$ of $\mathsf{D}(R_{\mathfrak{p}})$ is contained in $\mathcal{E}_{R_{\mathfrak{p}}}$. Thus ϕ restricts to an injection $\psi : ([\mathcal{E}_R](X, Y))_{\mathfrak{p}} \to [\mathcal{E}_{R_{\mathfrak{p}}}](X_{\mathfrak{p}}, Y_{\mathfrak{p}})$. Let

$$Q = (0 \to R_{\mathfrak{p}}^{\oplus r_0} \to \cdots \to R_{\mathfrak{p}}^{\oplus r_n} \to 0)$$

be a perfect $R_\mathfrak{p}$-complex with $Q^i = R_\mathfrak{p}^{\oplus r_i}$ for each $i \in \mathbb{Z}$. Then we easily find a perfect R-complex $P = (0 \to R^{\oplus r_0} \to \cdots \to R^{\oplus r_n} \to 0)$ such that $P_\mathfrak{p}$ is isomorphic to Q as an $R_\mathfrak{p}$-complex; see [1, Lemma 4.2(1) and its proof]. Hence P belongs to $\mathcal{E}_R$, and the equality $(\mathcal{E}_R)_\mathfrak{p} = \mathcal{E}_{R_\mathfrak{p}}$ follows. Consider the decomposition

$$(X_\mathfrak{p} \xrightarrow{\frac{f}{s}} Y_\mathfrak{p}) = (X_\mathfrak{p} \xrightarrow{\frac{g}{t}} E_\mathfrak{p} \xrightarrow{\frac{h}{u}} Y_\mathfrak{p})$$

of a morphism $\frac{f}{s} : X_\mathfrak{p} \to Y_\mathfrak{p}$ in $\mathsf{D}(R_\mathfrak{p})$, where f, g, h are morphisms in $\mathsf{D}(R)$, $s, t, u \in R \setminus \mathfrak{p}$ and $E \in \mathcal{E}_R$. Then $vutf = vshg$ for some $v \in R \setminus \mathfrak{p}$. We have $\frac{f}{s} = \frac{vutf}{vuts}$ and

$$(X \xrightarrow{vutf} Y) = (X \xrightarrow{g} E \xrightarrow{vsh} Y).$$

This shows that the injection ψ is surjective. Consequently, ϕ induces an isomorphism $\operatorname{Hom}_{\underline{\mathsf{D}}(R)}(X, Y)_\mathfrak{p} \to \operatorname{Hom}_{\underline{\mathsf{D}}(R_\mathfrak{p})}(X_\mathfrak{p}, Y_\mathfrak{p})$. □

Definition 3.6 (NE-loci of objects of $\mathsf{D}(R)$). Let X be an object of $\mathsf{D}(R)$. We denote by $\mathrm{NE}(X)$ the set of prime ideals $\mathfrak{p}$ of R such that $X_\mathfrak{p} \notin \mathcal{E}_{R_\mathfrak{p}}$, and call it the *NE-locus* of X. According to Proposition 2.13(6), this is equal to the set of prime ideals $\mathfrak{p}$ of R such that the $R_\mathfrak{p}$-complex $X_\mathfrak{p}$ has positive (possibly infinite) projective dimension. Thus we may also call $\mathrm{NE}(X)$ the *positive projective dimension locus* of X. Clearly, the equality $\mathrm{NE}(M) = \mathrm{NF}(M)$ holds for each finitely generated R-module. Note that $\mathrm{NE}(X)$ is contained in $\operatorname{Supp}(X)$, where the latter set is the *support* of X, which is defined by the equality $\operatorname{Supp}(X) = \{\mathfrak{p} \in \operatorname{Spec} R \mid X_\mathfrak{p} \ncong 0 \text{ in } \mathsf{D}(R_\mathfrak{p})\}$.

We state a basic fact on free resolutions and truncations of complexes, which is frequently used later.

Remark 3.7. Let $X \in \mathsf{D}(R)$ be a complex. Put $t = \inf X$ and $s = \sup X$. Then there exists a complex

$$F = (\cdots \xrightarrow{\partial^{t-1}} F^t \xrightarrow{\partial^t} F^{t+1} \xrightarrow{\partial^{t+1}} \cdots \xrightarrow{\partial^{s-2}} F^{s-1} \xrightarrow{\partial^{s-1}} F^s \to 0)$$

of finitely generated free R-modules such that $X \cong F$ in $\mathsf{D}(R)$; see [21, (A.3.2)] for instance.

(1) Let C be the cokernel of the differential map ∂_{t-1}, and let $P = (0 \to F^{t+1} \to \cdots \to F^s \to 0)$ be a truncation of F. Then P is a perfect complex and one has an exact triangle $P \to X \to C[-t] \rightsquigarrow$ in $\mathsf{D}(R)$.

(2) Let $P = (0 \to F^1 \to \cdots \to F^s \to 0)$ and $Y = (\cdots \to F^{-1} \to F^0 \to 0)$ be truncations of F. Then Y is an object of $\mathsf{D}(R)$ with $\sup Y \leqslant 0$. There is an exact triangle $P \to X \to Y \rightsquigarrow$ in $\mathsf{D}(R)$, which induces an exact triangle $X \to Y \to P[1] \rightsquigarrow$ in $\mathsf{D}(R)$. Proposition 2.13(6) implies that P and $P[1]$ belong to $\mathcal{E}_R$. It is seen that Y belongs to $\mathsf{res}_{\mathsf{D}(R)}\, X$.

Here are several basic properties of the NE-loci of objects of the derived category $\mathsf{D}(R)$.

Lemma 3.8. (1) *Let X be an object of $\mathsf{D}(R)$. Then, the set* $\mathrm{NE}(X)$ *is empty if and only if X belongs to $\mathcal{E}_R$.*

(2) *For any objects $X_1, \dots, X_n$ of $\mathsf{D}(R)$ one has the equality* $\mathrm{NE}(\bigoplus_{i=1}^n X_i) = \bigcup_{i=1}^n \mathrm{NE}(X_i)$.

(3) *For every $X \in \mathsf{D}(R)$ there exists $Y \in \mathsf{D}(R)$ with* $\sup Y \leqslant 0$, $\mathsf{res}_{\mathsf{D}(R)} X = \mathsf{res}_{\mathsf{D}(R)} Y$ *and* $\mathrm{NE}(X) = \mathrm{NE}(Y)$.

(4) *For an exact triangle $X \to Y \to Z \rightsquigarrow$ one has* $\mathrm{NE}(X) \subseteq \mathrm{NE}(Y) \cup \mathrm{NE}(Z)$ *and* $\mathrm{NE}(Y) \subseteq \mathrm{NE}(X) \cup \mathrm{NE}(Z)$.

Proof. (1) By [9, Proposition 5.3.P] and Proposition 2.13(6), we get $\mathrm{NE}(X) = \varnothing$ if and only if $X_\mathfrak{p} \in \mathcal{E}_{R_\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} R$, if and only if $\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p} \leqslant 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$, if and only if $\operatorname{pd}_R X \leqslant 0$, if and only if $X \in \mathcal{E}_R$.

(2) Let $\mathfrak{p}$ be a prime ideal of R. Since $\mathcal{E}_{R_\mathfrak{p}}$ is a resolving subcategory of $\mathsf{D}(R_\mathfrak{p})$, we have $\bigoplus_{i=1}^n (X_i)_\mathfrak{p} = \left(\bigoplus_{i=1}^n X_i\right)_\mathfrak{p} \in \mathcal{E}_{R_\mathfrak{p}}$ if and only if $(X_i)_\mathfrak{p} \in \mathcal{E}_{R_\mathfrak{p}}$ for all $1 \leqslant i \leqslant n$. The assertion follows from the contrapositive.

(3) According to Remark 3.7(2), there exists an exact triangle $X \to Y \to Z \rightsquigarrow$ in $\mathsf{D}(R)$ such that $\sup Y \leqslant 0$ and $Z \in \mathcal{E}_R$. For each resolving subcategory $\mathcal{X}$ of $\mathsf{D}(R)$ we have $X \in \mathcal{X}$ if and only if $Y \in \mathcal{X}$. In particular, $\mathsf{res}_{\mathsf{D}(R)} X = \mathsf{res}_{\mathsf{D}(R)} Y$. For every $\mathfrak{p} \in \operatorname{Spec} R$ there exists an exact triangle $X_\mathfrak{p} \to Y_\mathfrak{p} \to Z_\mathfrak{p} \rightsquigarrow$ in $\mathsf{D}(R_\mathfrak{p})$. Since $Z_\mathfrak{p}$ belongs to $\mathcal{E}_{R_\mathfrak{p}}$, we again have $X_\mathfrak{p} \in \mathcal{Y}$ if and only if $Y_\mathfrak{p} \in \mathcal{Y}$ for each resolving subcategory $\mathcal{Y}$ of $\mathsf{D}(R_\mathfrak{p})$. In particular, $X_\mathfrak{p} \notin \mathcal{E}_{R_\mathfrak{p}}$ if and only if $Y_\mathfrak{p} \notin \mathcal{E}_{R_\mathfrak{p}}$. Hence the equality $\mathrm{NE}(X) = \mathrm{NE}(Y)$ holds.

(4) Fix $\mathfrak{p} \in \operatorname{Spec} R$. By Proposition 2.13(3), if $\operatorname{pd} Y_\mathfrak{p} \leqslant 0$ and $\operatorname{pd} Z_\mathfrak{p} \leqslant 0$, then $\operatorname{pd} X_\mathfrak{p} \leqslant \sup\{\operatorname{pd} Y_\mathfrak{p}, \operatorname{pd} Z_\mathfrak{p} - 1\} \leqslant 0$. Also, if $\operatorname{pd} X_\mathfrak{p} \leqslant 0$ and $\operatorname{pd} Z_\mathfrak{p} \leqslant 0$, then $\operatorname{pd} Y_\mathfrak{p} \leqslant \sup\{\operatorname{pd} X_\mathfrak{p}, \operatorname{pd} Z_\mathfrak{p}\} \leqslant 0$. The assertion follows. □

Remark 3.9. In view of Lemma 3.8(4), we may wonder if the inclusion $\mathrm{NE}(Z) \subseteq \mathrm{NE}(X) \cup \mathrm{NE}(Y)$ holds for every exact triangle $X \to Y \to Z \rightsquigarrow$ in $\mathsf{D}(R)$. This is not true in general. In fact, the exact triangle $R \xrightarrow{=} R \to 0 \rightsquigarrow$ induces an exact triangle $R \to 0 \to R[1] \rightsquigarrow$. Then $\mathrm{NE}(R[1]) = \operatorname{Spec} R$ because $\operatorname{pd}(R[1])_\mathfrak{p} = \operatorname{pd} R_\mathfrak{p}[1] = \operatorname{pd} R_\mathfrak{p} + 1 = 1$ for each $\mathfrak{p} \in \operatorname{Spec} R$ by Proposition 2.13(1), while $\mathrm{NE}(R) \cup \mathrm{NE}(0)$ is the empty set.

We provide a generalization (or a derived category version) of [49, Proposition 2.10 and Corollary 2.11(1)].

Proposition 3.10. *For every object X of $\mathsf{D}(R)$ there is an equality* $\mathrm{NE}(X) = \operatorname{Supp}_R(\operatorname{Hom}_{\underline{\mathsf{D}}(R)}(X, X))$. *In particular, the NE-loci of objects of $\mathsf{D}(R)$ are closed subsets of* $\operatorname{Spec} R$ *in the Zariski topology.*

Proof. A prime ideal $\mathfrak{p}$ of R does not belong to the support of the R-module $\operatorname{Hom}_{\underline{\mathsf{D}}(R)}(X, X)$ if and only if $\operatorname{Hom}_{\underline{\mathsf{D}}(R)}(X, X)_\mathfrak{p} = 0$, and this happens if and only if $\operatorname{Hom}_{\underline{\mathsf{D}}(R_\mathfrak{p})}(X_\mathfrak{p}, X_\mathfrak{p}) = 0$ by Lemma 3.5, if and only if $X_\mathfrak{p}$ belongs to $\mathcal{E}_{R_\mathfrak{p}}$ by Proposition 3.3(2), if and only if $\mathfrak{p}$ is not in $\mathrm{NE}(X)$. It follows that $\mathrm{NE}(X) = \operatorname{Supp}(\operatorname{Hom}_{\underline{\mathsf{D}}(R)}(X, X))$. □

Definition 3.11 (NE-loci of subcategories of $\mathsf{D}(R)$). For a subcategory $\mathcal{X}$ of $\mathsf{D}(R)$, we set $\mathrm{NE}(\mathcal{X}) = \bigcup_{X\in\mathcal{X}} \mathrm{NE}(X)$ and call it the *NE-locus* of $\mathcal{X}$. Since each $\mathrm{NE}(X)$ in the union is a Zariski-closed subset of $\operatorname{Spec} R$ by Proposition 3.10, the subset $\mathrm{NE}(\mathcal{X})$ of $\operatorname{Spec} R$ is specialization-closed. (This is a generalization of [49, Corollary 2.11(2)].)

The following proposition is regarded as a derived category version of [49, Corollary 3.6].

Proposition 3.12. *For every subcategory $\mathcal{X}$ of $\mathsf{D}(R)$ the equality $\mathrm{NE}(\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X}) = \mathrm{NE}(\mathcal{X})$ holds.*

Proof. Since $\mathcal{X}$ is contained in $\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X}$, we see that $\mathrm{NE}(\mathcal{X})$ is contained in $\mathrm{NE}(\mathsf{res}\,\mathcal{X})$. Let $\mathfrak{p}$ be a prime ideal of R with $\mathfrak{p} \notin \mathrm{NE}(\mathcal{X})$. We have $\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p} \leqslant 0$ for every $X \in \mathcal{X}$, so that $\mathcal{X}$ is contained in the subcategory $\mathcal{Y}$ of $\mathsf{D}(R)$ consisting of complexes Y such that $\operatorname{pd}_{R_\mathfrak{p}} Y_\mathfrak{p} \leqslant 0$. Clearly, $\mathcal{Y}$ contains R. By the projective dimension equality in Proposition 2.13(4), we see that $\mathcal{Y}$ is closed under direct summands. The first projective dimension inequality in Proposition 2.13(3) shows that $\mathcal{Y}$ is closed under extensions. The projective dimension equality in Proposition 2.13(1) implies that $\mathcal{Y}$ is closed under negative shifts. Thus, $\mathcal{Y}$ is a resolving subcategory of $\mathsf{D}(R)$ containing $\mathcal{X}$, so that it contains $\mathsf{res}\,\mathcal{X}$. It follows that $\mathfrak{p} \notin \mathrm{NE}(\mathsf{res}\,\mathcal{X})$. Thus, $\mathrm{NE}(\mathcal{X}) = \mathrm{NE}(\mathsf{res}\,\mathcal{X})$. □

Recall that a finitely generated R-module M is called a *maximal Cohen–Macaulay module* provided that it satisfies the inequality $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geqslant \dim R_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of R. Now, extending this, we introduce the notion of a maximal Cohen–Macaulay complex. This plays an important role in the rest of this paper.

Definition 3.13. (1) We call an object X of $\mathsf{D}(R)$ a *maximal Cohen–Macaulay complex* if $\operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} \geqslant \dim R_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$. By definition, a finitely generated R-module M is maximal Cohen–Macaulay if and only if the complex $(0 \to M \to 0)$ concentrated in degree zero is maximal Cohen–Macaulay.

(2) We denote by $\mathsf{C}(R)$ the subcategory of $\mathsf{D}(R)$ consisting of all maximal Cohen–Macaulay R-complexes.

Recall that $\operatorname{Sing} R$ stands for the *singular locus* of R, that is to say, the set of prime ideals $\mathfrak{p}$ of R such that the local ring $R_\mathfrak{p}$ is not regular. In the proposition below, we state some properties of maximal Cohen–Macaulay complexes, whose module category version can be found in [49, Example 2.9].

Proposition 3.14. (1) *Let $X \in \mathsf{D}(R)$ be maximal Cohen–Macaulay. Then* $\mathrm{NE}(X)$ *is contained in* $\operatorname{Sing} R$.

(2) *The subcategory* $\mathsf{C}(R)$ *of* $\mathsf{D}(R)$ *is closed under direct summands, extensions and negative shifts. If R is a Cohen–Macaulay ring, then* $\mathsf{C}(R)$ *is a resolving subcategory of* $\mathsf{D}(R)$ *and vice versa.*

Proof. (1) Let $\mathfrak{p}$ be a prime ideal of R with $\mathfrak{p} \notin \operatorname{Sing} R$. Then $R_\mathfrak{p}$ is a regular local ring, so that $\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p} < \infty$. By Proposition 2.13(2), we have $\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p} - \operatorname{depth} X_\mathfrak{p} = \dim R_\mathfrak{p} - \operatorname{depth} X_\mathfrak{p} \leqslant 0$. Thus $\mathfrak{p} \notin \operatorname{NE}(X)$.

(2) The depth equality in Proposition 2.13(4) shows $\mathsf{C}(R)$ is closed under direct summands. Using the first inequality in Proposition 2.13(3), we observe that $\mathsf{C}(R)$ is closed under extensions. It is seen from the depth equality in Proposition 2.13(1) that $\mathsf{C}(R)$ is closed under negative shifts. The ring R is Cohen–Macaulay if and only if R belongs to $\mathsf{C}(R)$, if and only if $\mathsf{C}(R)$ is a resolving subcategory of $\mathsf{D}(R)$. □

Recall that a *thick* subcategory $\mathcal{X}$ of $\mathsf{CM}(R)$ is by definition a subcategory of $\mathsf{CM}(R)$ closed under direct summands and such that for every short exact sequence $0 \to L \to M \to N \to 0$ of maximal Cohen–Macaulay R-modules, if two of L, M, N belong to $\mathcal{X}$, then so does the third. Also, for each set Φ of prime ideals of R, the subcategory $\operatorname{NF}^{-1}(\Phi)$ is defined as the subcategory of $\operatorname{mod} R$ consisting of modules whose nonfree loci are contained in Φ, and $\operatorname{NF}^{-1}_{\mathsf{CM}}(\Phi)$ is defined to be the intersection of $\operatorname{NF}^{-1}(\Phi)$ with $\mathsf{CM}(R)$. These three subcategories play important roles in [50; 52; 53]. Now we introduce their derived category versions.

Definition 3.15. (1) We say that a subcategory $\mathcal{X}$ of $\mathsf{C}(R)$ is *thick* provided that $\mathcal{X}$ is closed under direct summands in the additive category $\mathsf{C}(R)$, and that for each exact triangle $A \to B \to C \rightsquigarrow$ in $\mathsf{D}(R)$ such that $A, B, C \in \mathsf{C}(R)$, if two of A, B, C belong to $\mathcal{X}$, then so does the third. (We should be careful not to confuse a thick subcategory of $\mathsf{C}(R)$ with a thick subcategory of $\mathsf{D}(R)$ in the sense of Definition 2.3.)

(2) For a subset Φ of $\operatorname{Spec} R$, we denote by $\operatorname{NE}^{-1}(\Phi)$ the subcategory of $\mathsf{D}(R)$ consisting of complexes whose NE-loci are contained in Φ. We define the subcategory $\operatorname{NE}^{-1}_{\mathsf{C}}(\Phi)$ of $\mathsf{C}(R)$ by $\operatorname{NE}^{-1}_{\mathsf{C}}(\Phi) = \operatorname{NE}^{-1}(\Phi) \cap \mathsf{C}(R)$.

The following proposition includes a derived category version of [50, Propositions 1.15(3), 4.2 and Theorem 4.10(3)]. Compare this proposition with Theorem 6.1 stated later.

Proposition 3.16. (1) *Every thick subcategory of* $\mathsf{D}(R)$ *contained in* $\mathsf{C}(R)$ *is a thick subcategory of* $\mathsf{C}(R)$. *Every thick subcategory of* $\mathsf{C}(R)$ *containing R is a resolving subcategory of* $\mathsf{D}(R)$ *contained in* $\mathsf{C}(R)$.

(2) *For* $\Phi \subseteq \operatorname{Spec} R$ *the subcategory* $\operatorname{NE}^{-1}(\Phi)$ *of* $\mathsf{D}(R)$ *is resolving. If R is Cohen–Macaulay, then* $\operatorname{NE}^{-1}_{\mathsf{C}}(\Phi)$ *is a thick subcategory of* $\mathsf{C}(R)$ *containing R, and a resolving subcategory of* $\mathsf{D}(R)$ *contained in* $\mathsf{C}(R)$.

Proof. (1) First of all, note that since $\mathsf{C}(R)$ is closed under direct summands as a subcategory of $\mathsf{D}(R)$ by Proposition 3.14(2), being closed under direct summands as a subcategory of $\mathsf{C}(R)$ implies being closed under direct summands as a subcategory of $\mathsf{D}(R)$. The first assertion now follows. To show the second, let $\mathcal{X}$ be a thick subcategory of $\mathsf{C}(R)$ containing R. Then $\mathcal{X}$ is closed under direct summands as a subcategory of $\mathsf{D}(R)$. Let $A \to B \to C \rightsquigarrow$ be an exact triangle in $\mathsf{D}(R)$ with $C \in \mathcal{X}$. Then C is in $\mathsf{C}(R)$, and we observe from Propositions 3.14(2) and 2.4(1) that $A \in \mathsf{C}(R)$ if and only if $B \in \mathsf{C}(R)$. Since $\mathcal{X}$ is a thick subcategory of $\mathsf{C}(R)$, it is easy to verify that $A \in \mathcal{X}$ if and only if $B \in \mathcal{X}$. Thus, $\mathcal{X}$ is a resolving subcategory of $\mathsf{D}(R)$.

(2) Since $\mathrm{NE}(R) = \varnothing \subseteq \Phi$, we have $R \in \mathrm{NE}^{-1}(\Phi)$. Using Lemma 3.8(2), we see that $\mathrm{NE}^{-1}(\Phi)$ is closed under direct summands. Let $X \to Y \to Z \rightsquigarrow$ be an exact triangle in $\mathsf{D}(R)$ with $Z \in \mathrm{NE}^{-1}(\Phi)$. Then Φ contains $\mathrm{NE}(Z)$. It follows from Lemma 3.8(4) that Φ contains $\mathrm{NE}(X)$ if and only if Φ contains $\mathrm{NE}(Y)$. This means that $X \in \mathrm{NE}^{-1}(\Phi)$ if and only if $Y \in \mathrm{NE}^{-1}(\Phi)$. Thus, $\mathrm{NE}^{-1}(\Phi)$ is a resolving subcategory of $\mathsf{D}(R)$.

Let R be Cohen–Macaulay. The first assertion of (2) shows $\mathrm{NE}^{-1}(\Phi)$ is a resolving subcategory of $\mathsf{D}(R)$, and so is $\mathsf{C}(R)$ by Proposition 3.14(2), whence so is $\mathrm{NE}^{-1}(\Phi) \cap \mathsf{C}(R) = \mathrm{NE}_{\mathsf{C}}^{-1}(\Phi)$. Let $A \to B \to C \rightsquigarrow$ be an exact triangle in $\mathsf{D}(R)$ with $A, B \in \mathrm{NE}_{\mathsf{C}}^{-1}(\Phi)$ and $C \in \mathsf{C}(R)$. Assume $C \notin \mathrm{NE}_{\mathsf{C}}^{-1}(\Phi)$. Then we find $\mathfrak{p} \in \mathrm{NE}(C)$ such that $\mathfrak{p} \notin \Phi$. Hence $\mathfrak{p}$ does not belong to $\mathrm{NE}(A)$ or $\mathrm{NE}(B)$, which means that $\operatorname{pd} A_{\mathfrak{p}} \leqslant 0$ and $\operatorname{pd} B_{\mathfrak{p}} \leqslant 0$. The exact triangle $A_{\mathfrak{p}} \to B_{\mathfrak{p}} \to C_{\mathfrak{p}} \rightsquigarrow$ in $\mathsf{D}(R_{\mathfrak{p}})$ and Proposition 2.13(3) show $\operatorname{pd} C_{\mathfrak{p}} \leqslant 1 < \infty$, and we get

$$\operatorname{pd} C_{\mathfrak{p}} = \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} C_{\mathfrak{p}} = \dim R_{\mathfrak{p}} - \operatorname{depth} C_{\mathfrak{p}} \leqslant 0,$$

where the first equality comes from Proposition 2.13(2), the second equality holds since the local ring $R_{\mathfrak{p}}$ is Cohen–Macaulay, and the inequality holds as C is maximal Cohen–Macaulay. This is a contradiction because $\mathfrak{p} \in \mathrm{NE}(C)$. It follows that $C \in \mathrm{NE}_{\mathsf{C}}^{-1}(\Phi)$. Thus $\mathrm{NE}_{\mathsf{C}}^{-1}(\Phi)$ is a thick subcategory of $\mathsf{C}(R)$ (containing R). □

Remark 3.17. The converse of the first assertion of Proposition 3.16(1) does not necessarily hold true. In fact, $\mathsf{C}(R)$ is itself a thick subcategory of $\mathsf{C}(R)$, but it is not necessarily a thick subcategory of $\mathsf{D}(R)$. For example, let R be a Cohen–Macaulay local ring of positive Krull dimension. Then there exists a non-zerodivisor x in the maximal ideal of R, which gives rise to an exact sequence $0 \to R \xrightarrow{x} R \to R/(x) \to 0$ in $\operatorname{mod} R$, which induces an exact triangle $R \xrightarrow{x} R \to R/(x) \rightsquigarrow$ in $\mathsf{D}(R)$. We have $R \in \mathsf{C}(R)$ but $R/(x) \notin \mathsf{C}(R)$. Therefore, $\mathsf{C}(R)$ is not a thick subcategory of $\mathsf{D}(R)$. This argument also shows the module category version: a thick subcategory of $\mathsf{CM}(R)$ is not necessarily a thick subcategory of $\operatorname{mod} R$ contained in $\mathsf{CM}(R)$.

4. Classification of certain preaisles of $\mathsf{K}(R)$

We will now consider classifying certain preaisles of the triangulated category $\mathsf{K}(R)$. We begin by recalling the definitions of preaisles and several related notions.

Definition 4.1 [2, §1.1; 14, §1.3; 35, §1.1]. Let $\mathcal{T}$ be a triangulated category. A *preaisle* (resp. *precoaisle*) of $\mathcal{T}$ is by definition a subcategory of $\mathcal{T}$ closed under extensions and positive (resp. negative) shifts. A preaisle (resp. precoaisle) $\mathcal{X}$ of $\mathcal{T}$ is called an *aisle* (resp. a *coaisle*) if the inclusion functor $\mathcal{X} \hookrightarrow \mathcal{T}$ has a right (resp. left) adjoint. For an aisle $\mathcal{X}$ and a coaisle $\mathcal{Y}$ of $\mathcal{T}$, the pair $(\mathcal{X}, \mathcal{Y}[1])$ is called a *t-structure* of $\mathcal{T}$ if $\mathrm{Hom}_{\mathcal{T}}(X, Y) = 0$ for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$.

Denote by $(-)^*$ the R-dual functor $\mathrm{Hom}_R(-, R)$. The assignment $P \mapsto P^*$ gives a duality of $\mathsf{K}(R)$, which sends an exact triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

to the exact triangle

$$C^* \xrightarrow{g^*} B^* \xrightarrow{f^*} A^* \xrightarrow{h^*[1]} C^*[1].$$

For a subcategory $\mathcal{X}$ of $\mathsf{K}(R)$, we denote by $\mathcal{X}^*$ the subcategory of $\mathsf{K}(R)$ consisting of complexes of the form X^* with $X \in \mathcal{X}$. The following lemma is straightforward from the definitions of preaisles and precoaisles.

Lemma 4.2. *The assignment $\mathcal{X} \mapsto \mathcal{X}^*$ produces a one-to-one correspondence*

$$\left\{\begin{array}{c}\text{preaisles of } \mathsf{K}(R) \text{ containing } R\\ \text{and closed under direct summands}\end{array}\right\} \cong \left\{\begin{array}{c}\text{precoaisles of } \mathsf{K}(R) \text{ containing } R\\ \text{and closed under direct summands}\end{array}\right\}.$$

Remark 4.3. In [27] a preaisle closed under direct summands is called a *thick preaisle*.

Let us recall the definition of a certain fundamental filtration of subsets of Spec R.

Definition 4.4. A *filtration by supports* or *sp-filtration* of Spec R is by definition an order-reversing map $\phi : \mathbb{Z} \to 2^{\mathrm{Spec}\, R}$ such that for each $i \in \mathbb{Z}$ the subset $\phi(i)$ of Spec R is specialization-closed.

Here we need to introduce some notation. Let f be a map $\mathrm{Spec}\, R \to \mathbb{Z} \cup \{\pm\infty\}$.

$\mathsf{P}(f)(i) = \{\mathfrak{p} \in \mathrm{Spec}\, R \mid f(\mathfrak{p}) > i\}$ for $i \in \mathbb{Z}$.

$\mathsf{E}(f) = \{X \in \mathsf{K}(R) \mid \mathrm{pd}\, X_{\mathfrak{p}}^* \leqslant f(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Spec}\, R\}$.

$\mathsf{F}(\phi)(\mathfrak{p}) = \sup\{j \in \mathbb{Z} \mid \mathfrak{p} \in \phi(j)\} + 1$ for a map $\phi : \mathbb{Z} \to 2^{\mathrm{Spec}\, R}$ and $\mathfrak{p} \in \mathrm{Spec}\, R$.

$\mathsf{Q}(\mathcal{X})(\mathfrak{p}) = \sup\{\mathrm{pd}\, X_{\mathfrak{p}}^* \mid X \in \mathcal{X}\}$ for a subcategory $\mathcal{X}$ of $\mathsf{K}(R)$ and $\mathfrak{p} \in \mathrm{Spec}\, R$.

The following theorem classifies certain preaisles of $\mathsf{K}(R)$, which can be shown by using techniques of the unbounded derived category $\mathrm{D}(\mathrm{Mod}\, R)$ of the category $\mathrm{Mod}\, R$ of all (possibly infinitely generated) R-modules.

Theorem 4.5. *There are one-to-one correspondences*

$$\left\{\begin{matrix}\text{preaisles of } \mathsf{K}(R)\\ \text{containing } R\\ \text{and closed under}\\ \text{direct summands}\end{matrix}\right\} \underset{\mathsf{E}}{\overset{\mathsf{Q}}{\rightleftarrows}} \left\{\begin{matrix}\text{order-preserving maps}\\ \operatorname{Spec} R \to \mathbb{N}\cup\{\infty\}\end{matrix}\right\} \underset{\mathsf{F}}{\overset{\mathsf{P}}{\rightleftarrows}} \left\{\begin{matrix}\text{sp-filtrations } \phi\\ \text{of } \operatorname{Spec} R \text{ with}\\ \phi(-1)=\operatorname{Spec} R\end{matrix}\right\}.$$

Proof. If ϕ is an sp-filtration of $\operatorname{Spec} R$ with $\phi(-1)=\operatorname{Spec} R$, then $\mathsf{F}(\phi)(\mathfrak{p}) = \sup\{i\in\mathbb{Z}\mid \mathfrak{p}\in\phi(i)\}+1 \geqslant (-1)+1=0$ for each prime ideal $\mathfrak{p}$ of R, and hence $\mathsf{F}(\phi)$ is regarded as a map from $\operatorname{Spec} R$ to $\mathbb{N}\cup\{\infty\}$. If $f:\operatorname{Spec} R\to\mathbb{N}\cup\{\infty\}$ is an order-preserving map, then $\mathsf{P}(f)(-1)=\{\mathfrak{p}\in\operatorname{Spec} R\mid f(\mathfrak{p})\geqslant 0\}=\operatorname{Spec} R$. Thus, it follows from [55, Proposition 4.3] that the maps (P,F) appearing in the assertion are mutually inverse bijections.

Let A be the set of preaisles of $\mathsf{K}(R)$ closed under direct summands, B the set of aisles of compactly generated t-structures of $\mathsf{D}(\operatorname{Mod} R)$, and C the set of sp-filtrations of $\operatorname{Spec} R$. Then, [47, Theorem 4.10] or the combination of [41, Proposition 1.9(ii)] with [34, Theorem A.7] implies that the map $f:A\to B$ is bijective, which sends each $\mathcal{X}\in A$ to the aisle of $\mathsf{D}(\operatorname{Mod} R)$ generated by $\mathcal{X}$. In [2, Theorem 3.11] it is proved that the map $g:B\to C$ is bijective, which sends each $\mathcal{Y}\in B$ to the map $\phi:\mathbb{Z}\to 2^{\operatorname{Spec} R}$ given by $\phi(n)=\{\mathfrak{p}\in\operatorname{Spec} R\mid (R/\mathfrak{p})[-n]\in\mathcal{Y}\}$ for each $n\in\mathbb{Z}$. We see that the maps (Q,E) appearing in the assertion are mutually inverse bijections. □

Here we define assignments between subcategories of $\mathsf{D}(R)$ and maps from $\operatorname{Spec} R$ to $\mathbb{Z}\cup\{\pm\infty\}$, and state a couple of properties.

Definition 4.6. For a subcategory $\mathcal{X}$ of $\mathsf{D}(R)$, we define the map $\Phi(\mathcal{X}):\operatorname{Spec} R\to\mathbb{Z}\cup\{\pm\infty\}$ by $\Phi(\mathcal{X})(\mathfrak{p})=\sup_{X\in\mathcal{X}}\{\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p}\}$ for $\mathfrak{p}\in\operatorname{Spec} R$. For a map $f:\operatorname{Spec} R\to\mathbb{Z}\cup\{\pm\infty\}$, we denote by $\Psi(f)$ the subcategory of $\mathsf{D}(R)$ consisting of objects X with $\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p}\leqslant f(\mathfrak{p})$ for all $\mathfrak{p}\in\operatorname{Spec} R$. We equip the sets $\operatorname{Spec} R$ and $\mathbb{Z}\cup\{\pm\infty\}$ with the partial orders given by the inclusion relation $(\subseteq)$ and the inequality relation $(\leqslant)$, respectively.

Lemma 4.7. (1) *Let $\mathcal{X}$ be a subcategory of $\mathsf{D}(R)$ which contains R. Then $\Phi(\mathcal{X})$ defines an order-preserving map from $\operatorname{Spec} R$ to $\mathbb{N}\cup\{\infty\}$.*

(2) *Let $f:\operatorname{Spec} R\to\mathbb{N}\cup\{\infty\}$ be a map. Then $\Psi(f)$ is a resolving subcategory of $\mathsf{D}(R)$.*

Proof. (1) Fix a prime ideal $\mathfrak{p}$ of R. Then $\sup_{X\in\mathcal{X}}\{\operatorname{pd} X_\mathfrak{p}\}\geqslant \operatorname{pd} R_\mathfrak{p}=0$, since R belongs to $\mathcal{X}$. Thus, $\Phi(\mathcal{X})(\mathfrak{p})$ is an element of $\mathbb{N}\cup\{\infty\}$. If $\mathfrak{q}$ is a prime ideal of R that contains $\mathfrak{p}$, then we have $\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p}=\operatorname{pd}_{(R_\mathfrak{q})_{\mathfrak{p}R_\mathfrak{q}}}(X_\mathfrak{q})_{\mathfrak{p}R_\mathfrak{q}}\leqslant \operatorname{pd}_{R_\mathfrak{q}} X_\mathfrak{q}$ for each $X\in\mathsf{D}(R)$ (see [9, Proposition 5.1(P)]), whence $\Phi(\mathcal{X})(\mathfrak{p})\leqslant\Phi(\mathcal{X})(\mathfrak{q})$.

(2) We have $\operatorname{pd} R_{\mathfrak{p}} = 0 \leqslant f(\mathfrak{p})$ for every prime ideal $\mathfrak{p}$ of R, which shows that $\Psi(f)$ contains R. If X is an object of $\Psi(f)$ and Y is a direct summand of X in $\mathsf{D}(R)$, then $\operatorname{pd} Y_{\mathfrak{p}} \leqslant \operatorname{pd} X_{\mathfrak{p}} \leqslant f(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$ by Proposition 2.13(4), which shows that Y belongs to $\Psi(f)$. Let $X \to Y \to Z \rightsquigarrow$ be an exact triangle in $\mathsf{D}(R)$ with $Z \in \Psi(f)$. Then for each $\mathfrak{p} \in \operatorname{Spec} R$ there is an exact triangle $X_{\mathfrak{p}} \to Y_{\mathfrak{p}} \to Z_{\mathfrak{p}} \rightsquigarrow$ in $\mathsf{D}(R_{\mathfrak{p}})$ and $\operatorname{pd} Z_{\mathfrak{p}} \leqslant f(\mathfrak{p})$. It is seen from Proposition 2.13(3) that $\operatorname{pd} X_{\mathfrak{p}} \leqslant f(\mathfrak{p})$ if and only if $\operatorname{pd} Y_{\mathfrak{p}} \leqslant f(\mathfrak{p})$. Therefore, X is in $\Psi(f)$ if and only if Y is in $\Psi(f)$. We now conclude that $\Psi(f)$ is a resolving subcategory of $\mathsf{D}(R)$. □

Note that the resolving subcategories of $\mathsf{K}(R)$ are precisely the precoaisles of $\mathsf{K}(R)$ that contain R and are closed under direct summands. Therefore, applying Lemmas 4.2, 4.7 and Theorem 4.5, we immediately get the following theorem, which provides a complete classification of the resolving subcategories of $\mathsf{K}(R)$.

Theorem 4.8. *The assignments $\mathcal{X} \mapsto \Phi(\mathcal{X})$ and $f \mapsto \Psi(f) \cap \mathsf{K}(R)$ give mutually inverse bijections between the resolving subcategories of* $\mathsf{K}(R)$, *and the order-preserving maps from* $\operatorname{Spec} R$ *to* $\mathbb{N} \cup \{\infty\}$.

Remark 4.9. (1) The proof of Theorem 4.5 given above actually classifies the preaisles of $\mathsf{K}(R)$ closed under direct summands, which do not necessarily contain R. For our purpose, the assertion of Theorem 4.5 suffices.

(2) There are other proofs of Theorems 4.5 and 4.8 without using techniques of unbounded derived categories. The proofs are longer than the ones given above, but instead elementary enough for those who are unfamiliar with unbounded derived categories to understand easily. They will be given in Appendix A.

To compare Theorem 4.5 with classification of aisles of $\mathsf{D}(R)$, we need to recall some notions.

Definition 4.10. (1) A map $f : \operatorname{Spec} R \to \mathbb{Z} \cup \{\pm\infty\}$ is called a *t-function* on $\operatorname{Spec} R$ if for each inclusion $\mathfrak{p} \subseteq \mathfrak{q}$ in $\operatorname{Spec} R$ there are inequalities $f(\mathfrak{p}) \leqslant f(\mathfrak{q}) \leqslant f(\mathfrak{p}) + \operatorname{ht} \mathfrak{q}/\mathfrak{p}$.

(2) An sp-filtration ϕ is said to satisfy the *weak Cousin condition* provided that for all integers i and for all saturated inclusions $\mathfrak{p} \subsetneq \mathfrak{q}$ in $\operatorname{Spec} R$, if $\mathfrak{q}$ belongs to $\phi(i)$, then $\mathfrak{p}$ belongs to $\phi(i-1)$.

(3) We say that R is *CM-excellent* if R is universally catenary, the formal fibers of the localizations of R are Cohen–Macaulay, and the Cohen–Macaulay locus of each finitely generated R-algebra is Zariski-open.

Takahashi [55, Theorem 5.5] proved the following, which yields a complete classification of the t-structures of $\mathsf{D}(R)$ when R is a CM-excellent ring of finite Krull dimension. We set

$$\mathsf{H}(f) = \{X \in \mathsf{D}(R) \mid \mathrm{H}^{\geqslant f(\mathfrak{p})}(X_{\mathfrak{p}}) = 0 \text{ for all } \mathfrak{p} \in \operatorname{Spec} R\}$$

for a map $f : \operatorname{Spec} R \to \mathbb{Z} \cup \{\pm\infty\}$, and

$$\mathsf{R}(\mathcal{X})(\mathfrak{p}) = \sup\{i \in \mathbb{Z} \mid (R/\mathfrak{p})[-i] \in \mathcal{X}\} + 1$$

for a subcategory $\mathcal{X}$ of $\mathsf{D}(R)$ and $\mathfrak{p} \in \operatorname{Spec} R$.

Theorem 4.11 (Takahashi). *When R is CM-excellent and* $\dim R < \infty$, *there are one-to-one correspondences*

$$\left\{\begin{matrix}\text{aisles}\\ \text{of } \mathsf{D}(R)\end{matrix}\right\} \underset{\mathsf{H}}{\overset{\mathsf{R}}{\rightleftarrows}} \left\{\begin{matrix}t\text{-functions}\\ \text{on } \operatorname{Spec} R\end{matrix}\right\} \underset{\mathsf{F}}{\overset{\mathsf{P}}{\rightleftarrows}} \left\{\begin{matrix}\text{sp-filtrations of } \operatorname{Spec} R\\ \text{satisfying the weak Cousin condition}\end{matrix}\right\}.$$

The next proposition records a relationship between Theorem 4.5 and the restriction of Theorem 4.11 to $\mathsf{K}(R)$. Note that the intersection of the set of order-preserving maps from $\operatorname{Spec} R$ to $\mathbb{N} \cup \{\infty\}$ and the set of t-functions on $\operatorname{Spec} R$ consists of the t-functions whose images are contained in $\mathbb{N} \cup \{\infty\}$.

Proposition 4.12. *Let R be a CM-excellent ring of finite Krull dimension. Let f be a t-function on* $\operatorname{Spec} R$ *whose image is contained in* $\mathbb{N} \cup \{\infty\}$. *Then there is an equality* $\mathsf{E}(f)[1] = \mathsf{H}(f) \cap \mathsf{K}(R)$.

Proof. We claim that if R is local, $X \in \mathsf{K}(R)$ and $n \in \mathbb{Z}$, then $\operatorname{pd} X^* \leqslant n$ if and only if $\mathrm{H}^{>n}(X) = 0$. In fact, letting k be the residue field of R, we have that $\operatorname{pd} X^* = \sup \mathbf{R}\mathrm{Hom}_R(X^*, k)$, that $\mathbf{R}\mathrm{Hom}_R(X^*, k) \cong X \otimes_R^{\mathbf{L}} k$ and that $\sup(X \otimes_R^{\mathbf{L}} k) = \sup X$ by [21, (A.5.7.3), (A.4.24) and (A.6.3.2)], respectively.

Let X be an object of $\mathsf{K}(R)$. Using the above claim, we see that $X \in \mathsf{E}(f)[1]$ if and only if $X[-1] \in \mathsf{E}(f)$, if and only if $\operatorname{pd}(X[-1])_{\mathfrak{p}}^* \leqslant f(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$, if and only if $\operatorname{pd} X_{\mathfrak{p}}^* \leqslant f(\mathfrak{p}) - 1$ for all $\mathfrak{p} \in \operatorname{Spec} R$, if and only if $\mathrm{H}^{\geqslant f(\mathfrak{p})}(X_{\mathfrak{p}}) = 0$, if and only if $X \in \mathsf{H}(f)$. It follows that $\mathsf{E}(f)[1] = \mathsf{H}(f) \cap \mathsf{K}(R)$. □

Question 4.13. Let R be a CM-excellent ring of finite Krull dimension. Is there any relationship between Theorem 4.5 and the restriction of Theorem 4.11 to $\mathsf{K}(R)$, other than the one shown in Proposition 4.12?

Remark 4.14. In view of what we have stated so far, it is quite natural to ask if the aisles of $\mathsf{K}(R)$ can be classified. If R is regular, then $\mathsf{K}(R)$ coincides with $\mathsf{D}(R)$, and Theorem 4.11 gives an answer. In case R is singular, it is known that $\mathsf{K}(R)$ possesses only trivial aisles under mild assumptions: Smith [44, Theorems 1.2 and 1.3] proved that if R has finite Krull dimension, then $\mathsf{K}(R)$ has no bounded t-structure, and if moreover R is irreducible, then 0 and $\mathsf{K}(R)$ are the only aisles of $\mathsf{K}(R)$. The former statement has recently been extended to schemes by Neeman [42, Theorem 0.1], which resolves a conjecture of Antieau, Gepner and Heller [5].

5. Separating the resolving subcategories of $\mathsf{D}(R)$

In this section, for a complete intersection R we separate the resolving subcategories of $\mathsf{D}(R)$ into the resolving subcategories contained in $\mathsf{K}(R)$ and the resolving subcategories contained in $\mathsf{C}(R)$. For this, we need several preparations. We start by recalling the definition of the cosyzygies of a finitely generated module.

Definition 5.1. Let M be a finitely generated R-module and $n \geqslant 1$ an integer. We denote by $\Omega_R^{-n} M$ the *nth cosyzygy* of M. This is defined inductively as follows. Let $\Omega_R^{-1} M$ be the cokernel of a *left* $(\operatorname{proj} R)$*-approximation* of M, namely, a homomorphism $f : M \to P$ such that P is a finitely generated projective R-module and the map $\operatorname{Hom}_R(f, R) : \operatorname{Hom}_R(P, R) \to \operatorname{Hom}_R(M, R)$ is surjective. For $n \geqslant 2$ we set $\Omega_R^{-n} M = \Omega_R^{-1}(\Omega_R^{-(n-1)} M)$. The nth cosyzygy of M is uniquely determined by M and n up to projective summands. For details, see [48, Sections 2 and 7] for instance.

Next we recall the definition of a certain numerical invariant for complexes.

Definition 5.2. The (*large*) *restricted flat dimension* $\operatorname{Rfd}_R X$ of an R-complex $X \in \mathsf{D}(R)$ is defined by

$$\operatorname{Rfd}_R X = \sup_{\mathfrak{p} \in \operatorname{Spec} R}\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} X_{\mathfrak{p}}\}.$$

One has inequalities $-\inf X \leqslant \operatorname{Rfd}_R X < \infty$; see [10, Theorem 1.1] and [22, Proposition (2.2) and Theorem (2.4)]. Also, note that if R is Cohen–Macaulay, then X is maximal Cohen–Macaulay if and only if $\operatorname{Rfd}_R X \leqslant 0$.

For a complex $X \in \mathsf{D}(R)$, we denote by $\operatorname{Gdim}_R X$ the *Gorenstein dimension* (*G-dimension* for short) of X. Recall that a *totally reflexive module* is defined to be a finitely generated module of G-dimension at most zero. For the details of G-dimension and totally reflexive modules, we refer the reader to [21]. In the following lemma, we make a list of properties of G-dimension we need to use later. Assertions (2) and (3) of the lemma correspond to assertions (1) and (3) of Proposition 2.13 concerning projective dimension.

Lemma 5.3. (1) *Let* $Y, Z \in \mathsf{D}(R)$. *If* $\operatorname{pd}_R Y < \infty$ *and* $\operatorname{Gdim}_R Z \leqslant 0$, *then* $\operatorname{Ext}_R^i(Z, Y) = 0$ *for all* $i > \sup Y$.

(2) *For every* $X \in \mathsf{D}(R)$ *and every* $n \in \mathbb{Z}$ *the equality* $\operatorname{Gdim}_R(X[n]) = \operatorname{Gdim}_R X + n$ *holds.*

(3) *Let* $X \to Y \to Z \rightsquigarrow$ *be an exact triangle in* $\mathsf{D}(R)$. *Then*

$$\operatorname{Gdim}_R X \leqslant \sup\{\operatorname{Gdim}_R Y, \operatorname{Gdim}_R Z - 1\},$$
$$\operatorname{Gdim}_R Y \leqslant \sup\{\operatorname{Gdim}_R X, \operatorname{Gdim}_R Z\},$$
$$\operatorname{Gdim}_R Z \leqslant \sup\{\operatorname{Gdim}_R X + 1, \operatorname{Gdim}_R Y\}.$$

(4) *An object $X \in \mathsf{D}(R)$ is isomorphic to a totally reflexive module if and only if* $\operatorname{Gdim}_R X \leqslant 0$ *and* $\sup X \leqslant 0$.

(5) *Suppose that the ring R is Gorenstein. Then every complex X of $\mathsf{D}(R)$ satisfies* $\operatorname{Gdim}_R X = \operatorname{Rfd}_R X < \infty$. *In particular, X is a maximal Cohen–Macaulay R-complex if and only if one has* $\operatorname{Gdim}_R X \leqslant 0$.

Proof. In what follows, [21, (2.3.8)] is our fundamental tool. Also, we set

$$(-)^\star = \mathbf{R}\mathrm{Hom}_R(-, R),$$

and for each complex $C \in \mathsf{D}(R)$ such that $C^\star \in \mathsf{D}(R)$, let $f_C : C \to C^{\star\star}$ stand for the natural morphism.

(1) If R is local, then $\sup \mathbf{R}\mathrm{Hom}(Z, Y) \leqslant \operatorname{Gdim} Z + \sup Y \leqslant \sup Y$ by [21, (2.4.1)], and the assertion is deduced. Suppose R is nonlocal, and fix $\mathfrak{p} \in \operatorname{Spec} R$. Then $\operatorname{pd}_{R_\mathfrak{p}} Y_\mathfrak{p} < \infty$, $Z_\mathfrak{p} \in \mathsf{D}(R_\mathfrak{p})$, and $\operatorname{Gdim}_{R_\mathfrak{p}} Z_\mathfrak{p} \leqslant 0$ by [21, (2.3.11)]. The assertion in the local case shows $\operatorname{Ext}^i_{R_\mathfrak{p}}(Z_\mathfrak{p}, Y_\mathfrak{p}) = 0$ for all $i > \sup Y_\mathfrak{p}$. As $\sup Y_\mathfrak{p} \leqslant \sup Y$, we have $\operatorname{Ext}^i_R(Z, Y)_\mathfrak{p} = \operatorname{Ext}^i_{R_\mathfrak{p}}(Z_\mathfrak{p}, Y_\mathfrak{p}) = 0$ for all $i > \sup Y$. Therefore, $\operatorname{Ext}^i_R(Z, Y) = 0$ for all $i > \sup Y$.

(2) The assertion is straightforward (from the definition of G-dimension or from [21, (2.3.8)]).

(3) We have only to verify $\operatorname{Gdim} Y \leqslant \sup\{\operatorname{Gdim} X, \operatorname{Gdim} Z\}$, because once it is done, applying it to the exact triangles $Y \to Z \to X[1] \rightsquigarrow$ and $Z[-1] \to X \to Y \rightsquigarrow$ and using (2) will give the inequalities $\operatorname{Gdim} Z \leqslant \sup\{\operatorname{Gdim} Y, \operatorname{Gdim} X + 1\}$ and $\operatorname{Gdim} X \leqslant \sup\{\operatorname{Gdim} Z - 1, \operatorname{Gdim} Y\}$. The inequality is obvious if either $\operatorname{Gdim} X$ or $\operatorname{Gdim} Z$ is infinite. We may assume $\operatorname{Gdim} X$ and $\operatorname{Gdim} Z$ are both finite. Then $X^\star$, $Z^\star$ belong to $\mathsf{D}(R)$, and f_X, f_Z are isomorphisms. The induced exact triangle $Z^\star \to Y^\star \to X^\star \rightsquigarrow$ (in the derived category of left-bounded R-complexes) shows that $Y^\star$ is in $\mathsf{D}(R)$. There is a commutative diagram

$$\begin{array}{ccccccc}
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\
\downarrow{\scriptstyle f_X} & & \downarrow{\scriptstyle f_Y} & & \downarrow{\scriptstyle f_Z} & & \downarrow{\scriptstyle f_{X[1]}} \\
X^{\star\star} & \longrightarrow & Y^{\star\star} & \longrightarrow & Z^{\star\star} & \longrightarrow & X^{\star\star}[1]
\end{array}$$

of exact triangles in $\mathsf{D}(R)$. Since f_X and f_Z are isomorphisms, so is f_Y. We conclude that $\operatorname{Gdim} Y$ is finite, and get $\operatorname{Gdim} Y = \sup \mathbf{R}\mathrm{Hom}(Y, R) = \sup Y^\star \leqslant \sup\{\sup X^\star, \sup Z^\star\} = \sup\{\operatorname{Gdim} X, \operatorname{Gdim} Z\}$.

(4) The "only if" part is obvious. By [21, (2.3.3)] we have $\operatorname{Gdim} X \geqslant -\inf X$. Suppose that $\operatorname{Gdim} X \leqslant 0$ and $\sup X \leqslant 0$. We then have $\sup X \leqslant 0 \leqslant -\operatorname{Gdim} X \leqslant \inf X$, which implies $\sup X = \inf X = 0$ or $X \cong 0$ in $\mathsf{D}(R)$. Hence, X is isomorphic in $\mathsf{D}(R)$ to a totally reflexive module. Thus the "if" part follows.

(5) The second assertion follows from the first and the fact that for a Cohen–Macaulay ring R one has $X \in \mathsf{C}(R)$ if and only if $\operatorname{Rfd}_R X \leqslant 0$. To show the first assertion, put $r = \operatorname{Rfd}_R X$. Fix $\mathfrak{p} \in \operatorname{Spec} R$. As $R_\mathfrak{p}$ is a Gorenstein local ring, $\operatorname{Gdim}_{R_\mathfrak{p}} X_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p} - \operatorname{depth} X_\mathfrak{p} \leqslant r$; see [21, (2.3.13) and (2.3.14)]. Hence $\operatorname{Ext}^i_R(X, R)_\mathfrak{p} = \operatorname{Ext}^i_{R_\mathfrak{p}}(X_\mathfrak{p}, R_\mathfrak{p}) = 0$ for all $i > r$. Therefore, $\operatorname{Ext}^i_R(X, R) = 0$ for all $i > r$. We see that $(f_X)_\mathfrak{p} = f_{X_\mathfrak{p}}$ is an isomorphism for all $\mathfrak{p} \in \operatorname{Spec} R$, so that f_X is an isomorphism (see [21, (A.4.5) and (A.8.4.1)]). Hence $\operatorname{Gdim}_R X = \sup X^\star \leqslant r$. If $\operatorname{Gdim}_R X \leqslant r - 1$, then $\operatorname{depth} R_\mathfrak{p} - \operatorname{depth} X_\mathfrak{p} = \operatorname{Gdim}_{R_\mathfrak{p}} X_\mathfrak{p} \leqslant r - 1$ for all $\mathfrak{p} \in \operatorname{Spec} R$ by [21, (2.3.11) and (2.3.13)], and $\operatorname{Rfd}_R X \leqslant r - 1$, a contradiction. Thus $\operatorname{Gdim}_R X = r$. □

In the lemma below we study the structure of a resolving subcategory of a certain form. The idea of the proof comes from the proof of [24, Lemma 7.2].

Lemma 5.4. *Let $\mathcal{Y}$ and $\mathcal{Z}$ be resolving subcategories of $\mathsf{D}(R)$. Let $\mathcal{X}$ be the subcategory of $\mathsf{D}(R)$ consisting of complexes X which fits into an exact triangle $Z \to X \oplus X' \to Y \rightsquigarrow$ in $\mathsf{D}(R)$ with $Z \in \mathcal{Z}$ and $Y \in \mathcal{Y}$.*

(1) *There is an inclusion $\mathcal{X} \subseteq \mathsf{res}_{\mathsf{D}(R)}(\mathcal{Y} \cup \mathcal{Z})$ of subcategories of $\mathsf{D}(R)$.*

(2) *The subcategory $\mathcal{X}$ coincides with the subcategory of $\mathsf{D}(R)$ consisting of objects X which fits into an exact triangle $Z \to X \oplus X' \to Y \rightsquigarrow$ in $\mathsf{D}(R)$ with $Z \in \mathcal{Z}$, $Y \in \mathcal{Y}$ and $\sup Y \leqslant 0$.*

(3) *Suppose that $\operatorname{pd}_R Y < \infty$ for each $Y \in \mathcal{Y}$ and $\operatorname{Gdim}_R Z \leqslant 0$ for each $Z \in \mathcal{Z}$. Then $\mathcal{X} = \mathsf{res}_{\mathsf{D}(R)}(\mathcal{Y} \cup \mathcal{Z})$.*

Proof. (1) Let $Z \to X \oplus X' \to Y \rightsquigarrow$ be an exact triangle in $\mathsf{D}(R)$ such that $Z \in \mathcal{Z}$ and $Y \in \mathcal{Y}$. As $\mathsf{res}(\mathcal{Y} \cup \mathcal{Z})$ contains $\mathcal{Z}$ and $\mathcal{Y}$, the objects Z, Y are in $\mathsf{res}(\mathcal{Y} \cup \mathcal{Z})$. The triangle implies that X belongs to $\mathsf{res}(\mathcal{Y} \cup \mathcal{Z})$.

(2) Let $Z \to X \oplus X' \to Y \rightsquigarrow$ be an exact triangle with $Z \in \mathcal{Z}$ and $Y \in \mathcal{Y}$. Remark 3.7(2) gives an exact triangle $P \to Y \to Y' \rightsquigarrow$ with $P \in \mathcal{E}_R$, $Y' \in \mathcal{Y}$ and $\sup Y' \leqslant 0$. The octahedral axiom yields a commutative diagram

$$
\begin{array}{ccccccc}
X\oplus X' & \longrightarrow & Y & \longrightarrow & Z[1] & \longrightarrow & (X\oplus X')[1]\\
\| & & \downarrow & & \downarrow & & \|\\
X\oplus X' & \longrightarrow & Y' & \longrightarrow & Z'[1] & \longrightarrow & (X\oplus X')[1]\\
\downarrow & & \| & & \downarrow & & \downarrow\\
Y & \longrightarrow & Y' & \longrightarrow & P[1] & \longrightarrow & Y[1]\\
\downarrow & & \downarrow & & \| & & \downarrow\\
Z[1] & \longrightarrow & Z'[1] & \longrightarrow & P[1] & \longrightarrow & Z[2]
\end{array}
$$

of exact triangles, and the bottom row induces an exact triangle $Z \to Z' \to P \rightsquigarrow$. As Z and P are in $\mathcal{Z}$, so is Z'. An exact triangle $Z' \to X \oplus X' \to Y' \rightsquigarrow$ is induced from the second row. Now the assertion follows.

(3) The inclusion ($\subseteq$) is shown in (1). We prove the opposite inclusion ($\supseteq$). For $Y \in \mathcal{Y}$ and $Z \in \mathcal{Z}$ there are exact triangles $0 \to Y \oplus 0 \to Y \rightsquigarrow$ and $Z \to Z \oplus 0 \to 0 \rightsquigarrow$. This shows that $\mathcal{Y} \cup \mathcal{Z} \subseteq \mathcal{X}$. We will be done once we prove that $\mathcal{X}$ is a resolving subcategory of $\mathsf{D}(R)$. As R belongs to $\mathcal{Y}$ (and $\mathcal{Z}$), it belongs to $\mathcal{X}$.

Let X be an object of $\mathcal{X}$, and let W be a direct summand of X in $\mathsf{D}(R)$. Then there is an exact triangle $Z \to X \oplus X' \to Y \rightsquigarrow$ in $\mathsf{D}(R)$ such that $Z \in \mathcal{Z}$ and $Y \in \mathcal{Y}$, and also $X = W \oplus V$ for some $V \in \mathsf{D}(R)$. Setting $W' = V \oplus X'$, we have an exact triangle $Z \to W \oplus W' \to Y \rightsquigarrow$. Hence $\mathcal{X}$ is closed under direct summands.

Every exact triangle $Z \to X \oplus X' \to Y \rightsquigarrow$ with $Z \in \mathcal{Z}$ and $Y \in \mathcal{Y}$ induces an exact triangle $Z[-1] \to X[-1] \oplus X'[-1] \to Y[-1] \rightsquigarrow$, and we have $Z[-1] \in \mathcal{Z}$ and $Y[-1] \in \mathcal{Y}$. Thus $\mathcal{X}$ is closed under negative shifts.

It remains to prove that $\mathcal{X}$ is closed under extensions. Let $L \xrightarrow{f} M \xrightarrow{g} N \rightsquigarrow$ be an exact triangle in $\mathsf{D}(R)$ with $L, N \in \mathcal{X}$. Then there exist exact triangles $Z_1 \to L \oplus L' \to Y_1 \rightsquigarrow$ and $Z_2 \to N \oplus N' \to Y_2 \rightsquigarrow$ in $\mathsf{D}(R)$ such that $Z_1, Z_2 \in \mathcal{Z}$ and $Y_1, Y_2 \in \mathcal{Y}$. In view of (2), we may assume that $\sup Y_1 \leqslant 0$. The octahedral axiom yields the following commutative diagrams of exact triangles in $\mathsf{D}(R)$.

$$
\begin{array}{ccccccc}
Z_1 & \longrightarrow & L \oplus L' & \longrightarrow & Y_1 & \longrightarrow & Z_1[1] \\
\| & & \downarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right)} & & \downarrow & & \| \\
Z_1 & \longrightarrow & M \oplus L' & \xrightarrow[(p\ q)]{} & A & \xrightarrow[r]{} & Z_1[1] \\
\downarrow & & \| & & \downarrow{h} & & \downarrow \\
L \oplus L' & \xrightarrow{\left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right)} & M \oplus L' & \xrightarrow{(g\ 0)} & N & \longrightarrow & (L \oplus L')[1] \\
\downarrow & & \downarrow & & \| & & \downarrow \\
Y_1 & \longrightarrow & A & \xrightarrow{h} & N & \xrightarrow{k} & Y_1[1]
\end{array}
$$

$$
\begin{array}{ccccccc}
A \oplus N' & \xrightarrow{\left(\begin{smallmatrix} h & 0 \\ 0 & 1 \end{smallmatrix}\right)} & N \oplus N' & \xrightarrow{(k\ 0)} & Y_1[1] & \longrightarrow & (A \oplus N')[1] \\
\| & & \downarrow & & \downarrow & & \| \\
A \oplus N' & \longrightarrow & Y_2 & \longrightarrow & B[1] & \longrightarrow & (A \oplus N')[1] \\
\downarrow & & \| & & \downarrow & & \downarrow \\
N \oplus N' & \longrightarrow & Y_2 & \longrightarrow & Z_2[1] & \longrightarrow & (N \oplus N')[1] \\
\downarrow & & \downarrow & & \| & & \downarrow \\
Y_1[1] & \longrightarrow & B[1] & \longrightarrow & Z_2[1] & \xrightarrow{\delta[1]} & Y_1[2]
\end{array}
$$

The bottom row of the second diagram induces an exact triangle

$$\sigma : Y_1 \to B \to Z_2 \xrightarrow{\delta} Y_1[1]$$

in $\mathsf{D}(R)$. We have $\delta \in \mathrm{Hom}_{\mathsf{D}(R)}(Z_2, Y_1[1]) \cong \mathrm{Ext}^1_R(Z_2, Y_1) = 0$ by Lemma 5.3(1). Hence σ splits (see [40, Corollary 1.2.7]), which gives an isomorphism $B \cong Y_1 \oplus Z_2$. An exact triangle $Y_1 \oplus Z_2 \to A \oplus N' \to Y_2 \rightsquigarrow$ is induced from the second row of the second diagram. Applying the octahedral axiom again, we obtain commutative diagrams

$$
\begin{array}{ccccccc}
Z_2 & \xrightarrow{\binom{0}{1}} & Y_1 \oplus Z_2 & \xrightarrow{(1\ 0)} & Y_1 & \rightsquigarrow & \\
\| & & \downarrow & & \downarrow & & \\
Z_2 & \longrightarrow & A \oplus N' & \longrightarrow & Y & \rightsquigarrow & \\
\downarrow & & \| & & \downarrow & & \\
Y_1 \oplus Z_2 & \longrightarrow & A \oplus N' & \longrightarrow & Y_2 & \rightsquigarrow & \\
\downarrow & & \downarrow & & \| & & \\
Y_1 & \longrightarrow & Y & \longrightarrow & Y_2 & \rightsquigarrow &
\end{array}
$$

$$
\begin{array}{ccccccc}
M \oplus L' \oplus N' & \xrightarrow{\left(\begin{smallmatrix} p & q & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right)} & A \oplus N' & \xrightarrow{(r\ 0)} & Z_1[1] & \rightsquigarrow & \\
\| & & \downarrow & & \downarrow & & \\
M \oplus L' \oplus N' & \longrightarrow & Y & \longrightarrow & Z[1] & \rightsquigarrow & \\
\downarrow & & \| & & \downarrow & & \\
A \oplus N' & \longrightarrow & Y & \longrightarrow & Z_2[1] & \rightsquigarrow & \\
\downarrow & & \downarrow & & \| & & \\
Z_1[1] & \longrightarrow & Z[1] & \longrightarrow & Z_2[1] & \rightsquigarrow &
\end{array}
$$

of exact triangles. We obtain exact triangles $Y_1 \to Y \to Y_2 \rightsquigarrow$ and $Z_1 \to Z \to Z_2 \rightsquigarrow$, which imply $Y \in \mathcal{Y}$ and $Z \in \mathcal{Z}$. We also have an exact triangle $Z \to M \oplus L' \oplus N' \to Y \rightsquigarrow$, which shows that M belongs to $\mathcal{X}$. □

In the following lemma, we investigate syzygies and cosyzygies for complexes.

Lemma 5.5. *Let X be an object of $\mathsf{D}(R)$.*

(1) *There exists an exact triangle $Y \to E \to X \rightsquigarrow$ in $\mathsf{D}(R)$ such that $E \in \mathcal{E}_R$, $\sup E \leqslant \sup\{\sup X, 0\}$ and $\sup Y \leqslant 0$. If $\sup X \leqslant 0$, then E is isomorphic in $\mathsf{D}(R)$ to a projective R-module.*

(2) *Suppose that $\operatorname{Gdim}_R X \leqslant 0$. Then there exists an exact triangle $X \to P \to Y \rightsquigarrow$ in $\mathsf{D}(R)$ such that P is a projective R-module and $\operatorname{Gdim}_R Y \leqslant 0$.*

Proof. (1) Put $u = \sup\{\sup X, 0\}$. Thanks to [21, (A.3.2)], we can choose a complex $F = (\cdots \to F^{u-1} \to F^u \to 0)$ of finitely generated projective R-modules which is isomorphic to X in $\mathsf{D}(R)$. (In the case $\sup X < 0$, we can put $F^i = 0$ for all integers i such that $\sup X + 1 \leqslant i \leqslant 0 = u$.) As $u \geqslant 0$, we can take the truncation $E = (0 \to F^0 \to \cdots \to F^u \to 0)$ of F. We have $\sup E \leqslant u$, and $E \in \mathcal{E}_R$ by Proposition 2.13(6). Taking the truncation $X' = (\cdots \to F^{-2} \to F^{-1} \to 0)$ of F, we get an exact triangle $E \to X \to X' \rightsquigarrow$ in $\mathsf{D}(R)$. Setting $Y = X'[-1]$, we get an exact triangle $Y \to E \to X \rightsquigarrow$ in $\mathsf{D}(R)$, and it holds that $\sup Y \leqslant 0$.

Suppose that $\sup X \leqslant 0$. Then $u = \sup\{\sup X, 0\} = 0$. Therefore, we have $E = (0 \to F^0 \to 0)$, which is isomorphic in $\mathsf{D}(R)$ to the finitely generated projective R-module F^0.

(2) In view of Remark 3.7(2), we can take an exact triangle $X \to X' \to E \rightsquigarrow$ in $\mathsf{D}(R)$ such that $\sup X' \leqslant 0$ and $E \in \mathcal{E}_R$. It is observed from Lemma 5.3(3) and [21, (2.3.10)]

that $\operatorname{Gdim} X' \leqslant 0$. By Lemma 5.3(4), we may assume that X' is a totally reflexive R-module. Hence, there is an exact sequence $0 \to X' \to P \to \Omega^{-1}X' \to 0$ in $\operatorname{mod} R$ such that P is projective, which induces an exact triangle $X' \to P \to \Omega^{-1}X' \rightsquigarrow$ in $\mathsf{D}(R)$. By the octahedral axiom, we obtain a commutative diagram

$$\begin{array}{ccccccc}
X & \longrightarrow & X' & \longrightarrow & E & \longrightarrow & X[1] \\
\| & & \downarrow & & \downarrow & & \| \\
X & \longrightarrow & P & \longrightarrow & Y & \longrightarrow & X[1] \\
\downarrow & & \| & & \downarrow & & \downarrow \\
X' & \longrightarrow & P & \longrightarrow & \Omega^{-1}X' & \longrightarrow & X'[1] \\
\downarrow & & \downarrow & & \| & & \downarrow \\
E & \longrightarrow & Y & \longrightarrow & \Omega^{-1}X' & \longrightarrow & E[1]
\end{array}$$

of exact triangles in $\mathsf{D}(R)$. Using Lemma 5.3(3) and [21, (2.3.10)] again, we see from the bottom row that $\operatorname{Gdim} Y \leqslant 0$. Thus the second row in the above diagram is such an exact triangle as in the assertion. $\square$

To show our next proposition, we need to prepare one more lemma.

Lemma 5.6. *Let $Z \to X \oplus W \to Y \rightsquigarrow$ be an exact triangle in $\mathsf{D}(R)$. Then there exists an exact triangle $Z \to X' \oplus W \to Y' \rightsquigarrow$ in $\mathsf{D}(R)$ such that $\operatorname{res}_{\mathsf{D}(R)} X = \operatorname{res}_{\mathsf{D}(R)} X'$, $\operatorname{res}_{\mathsf{D}(R)} Y = \operatorname{res}_{\mathsf{D}(R)} Y'$ and $\sup X' \leqslant 0$.*

Proof. By Remark 3.7(2), there exists an exact triangle $X \xrightarrow{a} X' \xrightarrow{b} E \rightsquigarrow$ in $\mathsf{D}(R)$ such that $\sup X' \leqslant 0$ and $E \in \mathcal{E}_R$. Note then that $\operatorname{res} X = \operatorname{res} X'$. The octahedral axiom gives rise to a commutative diagram

$$\begin{array}{ccccccc}
Z & \longrightarrow & X \oplus W & \longrightarrow & Y & \longrightarrow & Z[1] \\
\| & & \downarrow & & \downarrow & & \| \\
Z & \longrightarrow & X' \oplus W & \longrightarrow & Y' & \longrightarrow & Z[1] \\
\downarrow & & \| & & \downarrow & & \downarrow \\
X \oplus W & \xrightarrow{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)} & X' \oplus W & \xrightarrow{(b\ 0)} & E & \longrightarrow & (X \oplus W)[1] \\
\downarrow & & \downarrow & & \| & & \downarrow \\
Y & \longrightarrow & Y' & \longrightarrow & E & \longrightarrow & Y[1]
\end{array}$$

of exact triangles in $\mathsf{D}(R)$. From the bottom row in the commutative diagram we observe that $\operatorname{res} Y = \operatorname{res} Y'$. Thus the second row in the commutative diagram is an exact triangle as in the assertion of the lemma. $\square$

We denote by $\mathsf{G}(R)$ the subcategory of $\mathsf{D}(R)$ consisting of objects X satisfying the inequality $\operatorname{Gdim}_R X \leqslant 0$. We can now prove the proposition below, which includes a derived category version of [24, Proposition 7.3].

Proposition 5.7. *Let $\mathcal{Y}$ and $\mathcal{Z}$ be resolving subcategories of $\mathsf{D}(R)$ such that $\mathcal{Y} \subseteq \mathsf{K}(R)$ and $\mathcal{Z} \subseteq \mathsf{G}(R)$. Then*

$$\mathcal{Y} = \operatorname{res}_{\mathsf{D}(R)}(\mathcal{Y} \cup \mathcal{Z}) \cap \mathsf{K}(R), \qquad \mathcal{Z} = \operatorname{res}_{\mathsf{D}(R)}(\mathcal{Y} \cup \mathcal{Z}) \cap \mathsf{G}(R).$$

Proof. Let us begin with the first equality. The inclusion ($\subseteq$) is clear. To show ($\supseteq$), pick $X \in \mathsf{res}(\mathcal{Y} \cup \mathcal{Z}) \cap \mathsf{K}(R)$. According to Lemma 5.4(3), there is an exact triangle $Z \to X \oplus W \to Y \rightsquigarrow$ in $\mathsf{D}(R)$ with $Z \in \mathcal{Z}$ and $Y \in \mathcal{Y}$. What we want to show is that X is in $\mathcal{Y}$. For this purpose, thanks to Lemma 5.6, we may assume $\sup X \leqslant 0$. Lemma 5.5(2) yields an exact triangle $Z \to P \to V \rightsquigarrow$ in $\mathsf{D}(R)$ such that P is a projective module and $\operatorname{Gdim} V \leqslant 0$. The octahedral axiom gives the following commutative diagrams of exact triangles in $\mathsf{D}(R)$.

$$
\begin{array}{ccccccc}
Y[-1] & \longrightarrow & Z & \longrightarrow & X \oplus W & \longrightarrow & Y \\
\| & & \downarrow & & \downarrow{\scriptstyle (f\ g)} & & \| \\
Y[-1] & \longrightarrow & P & \longrightarrow & Y' & \longrightarrow & Y \\
\downarrow & & \| & & \downarrow & & \downarrow \\
Z & \longrightarrow & P & \longrightarrow & V & \longrightarrow & Z[1] \\
\downarrow & & \downarrow & & \| & & \downarrow \\
X \oplus W & \xrightarrow{(f\ g)} & Y' & \longrightarrow & V & \longrightarrow & X[1]
\end{array}
$$

$$
\begin{array}{ccccccc}
W & \xrightarrow{\binom{0}{1}} & X \oplus W & \xrightarrow{(1\ 0)} & X & \longrightarrow & W[1] \\
\| & & \downarrow{\scriptstyle (f\ g)} & & \downarrow{\scriptstyle l} & & \| \\
W & \longrightarrow & Y' & \xrightarrow{h} & U & \longrightarrow & W[1] \\
\downarrow & & \| & & \downarrow & & \downarrow \\
X \oplus W & \xrightarrow{(f\ g)} & Y' & \longrightarrow & V & \longrightarrow & (X \oplus W)[1] \\
\downarrow & & \downarrow & & \| & & \downarrow \\
X & \xrightarrow{l} & U & \longrightarrow & V & \xrightarrow{\delta} & X[1]
\end{array}
$$

From the exact triangle $P \to Y' \to Y \rightsquigarrow$ we see that Y' is in $\mathcal{Y}$. The equality $h(f\ g) = l(1\ 0)$ implies $l = hf$. Also, by Lemma 5.3(1) we have $\delta \in \operatorname{Hom}_{\mathsf{D}(R)}(V, X[1]) = \operatorname{Ext}^1_R(V, X) = 0$. There are commutative diagrams

$$
\begin{array}{ccccccc}
X & \xrightarrow{l} & U & \longrightarrow & V & \xrightarrow{0} & X[1] \\
\| & & \cong\downarrow{\scriptstyle \binom{s}{t}} & & \| & & \| \\
X & \xrightarrow{\binom{1}{0}} & X \oplus V & \xrightarrow{(0\ 1)} & V & \longrightarrow & X[1]
\end{array}
\qquad
\begin{array}{ccccccc}
X & \xrightarrow{f} & Y' & \xrightarrow{k} & C & \longrightarrow & X[1] \\
\| & & \cong\downarrow{\scriptstyle \binom{sh}{k}} & & \| & & \| \\
X & \xrightarrow{\binom{1}{0}} & X \oplus C & \xrightarrow{(0\ 1)} & C & \longrightarrow & X[1]
\end{array}
$$

of exact triangles in $\mathsf{D}(R)$. Indeed, we get the first diagram by [40, proof of Corollary 1.2.7], while the equality $\binom{1}{0} = \binom{s}{t} l$ coming from the commutativity of the first diagram shows $1 = sl = shf$, so that the second diagram is obtained by [40, Remark 1.2.9]. The isomorphism $X \oplus C \cong Y'$ shows that X belongs to $\mathcal{Y}$, as desired.

Next, we prove the second equality (the proof has the same stream as that of the first equality, but there are actually various different places). It is evident that ($\subseteq$) holds. To prove ($\supseteq$), let X be an object in the subcategory $\mathsf{res}(\mathcal{Y} \cup \mathcal{Z}) \cap \mathsf{G}(R)$. By Lemma 5.4(2)(3), there is an exact triangle $Z \to X \oplus W \to Y \rightsquigarrow$ in $\mathsf{D}(R)$

with $Z \in \mathcal{Z}$, $Y \in \mathcal{Y}$ and $\sup Y \leqslant 0$. We want to show that X belongs to $\mathcal{Z}$. Using Lemma 5.5(1), we get an exact triangle $Y' \to P \to Y \rightsquigarrow$ in $\mathsf{D}(R)$ such that P is a projective module and $\sup Y' \leqslant 0$. Note then that Y' is in $\mathcal{Y}$. The octahedral axiom gives the following commutative diagrams of exact triangles in $\mathsf{D}(R)$.

$$\begin{CD}
X \oplus W @>>> Y @>>> Z[1] @>>> (X \oplus W)[1] \\
@| @VVV @VVV @| \\
X \oplus W @>{(f\ g)}>> Y'[1] @>>> Z'[1] @>>> (X \oplus W)[1] \\
@VVV @| @VVV @VVV \\
Y @>>> Y'[1] @>>> P[1] @>>> Y[1] \\
@VVV @VVV @| @VVV \\
Z[1] @>>> Z'[1] @>>> P[1] @>>> Z[2]
\end{CD}$$

$$\begin{CD}
X @>{\binom{1}{0}}>> X \oplus W @>{(0\ 1)}>> W @>>> X[1] \\
@| @VV{(f\ g)}V @VVV @| \\
X @>{f}>> Y'[1] @>>> V @>{h}>> X[1] \\
@VV{\binom{1}{0}}V @| @VV{k}V @VV{\binom{1}{0}}V \\
X \oplus W @>{(f\ g)}>> Y'[1] @>>> Z'[1] @>{\binom{p}{q}}>> X[1] \oplus W[1] \\
@VVV @VVV @| @VVV \\
W @>>> V @>{k}>> Z'[1] @>>> W[1]
\end{CD}$$

The induced exact triangle $Z \to Z' \to P \rightsquigarrow$ shows that the object Z' belongs to $\mathcal{Z}$. Lemma 5.3(1) implies $f \in \mathrm{Hom}_{\mathsf{D}(R)}(X, Y'[1]) = \mathrm{Ext}^1_R(X, Y') = 0$. By [40, proof of Corollary 1.2.7] there is a commutative diagram

$$\begin{CD}
X @>>> Y'[1] @>{\binom{1}{0}}>> Y'[1] \oplus X[1] @>{(0\ 1)}>> X[1] \\
@| @| @V{(s\ t)}V{\cong}V @| \\
X @>{0}>> Y'[1] @>>> V @>{h}>> X[1]
\end{CD}$$

of exact triangles in $\mathsf{D}(R)$, which gives $ht = 1$. The equality $\binom{p}{q}k = \binom{1}{0}h$ implies $h = pk$. Hence $pkt = 1$, and it follows from [40, Lemma 1.2.8] that $X[1]$ is isomorphic in $\mathsf{D}(R)$ to a direct summand of $Z'[1]$. This implies that X is isomorphic in $\mathsf{D}(R)$ to a direct summand of Z'. Since Z' belongs to $\mathcal{Z}$, so does X. □

We say that R is a *complete intersection* if the completion of the local ring $R_{\mathfrak{p}}$ is isomorphic to a quotient of a regular local ring by a regular sequence for each prime ideal $\mathfrak{p}$ of R. The exact triangle appearing in the third assertion of the following proposition is regarded as a derived category version of a *finite projective hull* in the sense of Auslander and Buchweitz [7].

Proposition 5.8. *Let R be a complete intersection.*

(1) *Let M be a maximal Cohen–Macaulay R-module. Then the cosyzygy $\Omega_R^{-1}M$ belongs to $\mathsf{res}_{\mathsf{mod}\,R}\, M$.*

(2) *For every maximal Cohen–Macaulay complex $X \in \mathsf{D}(R)$, there exists an exact triangle $X \to P \to Y \rightsquigarrow$ in $\mathsf{D}(R)$ such that P is a projective module and Y belongs to the resolving closure $\mathsf{res}_{\mathsf{D}(R)}\, X$.*

(3) *Each $X \in \mathsf{D}(R)$ admits an exact triangle $X \to P \to Y \rightsquigarrow$ with $P \in \mathsf{K}(R)$ and $Y \in (\mathsf{res}_{\mathsf{D}(R)}\, X) \cap \mathsf{C}(R)$.*

Proof. (1) Fix a prime ideal $\mathfrak{p}$ of R. Then $M_\mathfrak{p}$ is a maximal Cohen–Macaulay $R_\mathfrak{p}$-module. In $\mathsf{mod}\,R_\mathfrak{p}$ we have

$$(\Omega_R^{-1}M)_\mathfrak{p} \approx \Omega_{R_\mathfrak{p}}^{-1}(M_\mathfrak{p}) \in \mathsf{res}_{\mathsf{mod}\,R_\mathfrak{p}}\, M_\mathfrak{p} \subseteq \mathsf{add}_{\mathsf{mod}\,R_\mathfrak{p}}(\mathsf{res}_{\mathsf{mod}\,R}\, M)_\mathfrak{p}.$$

Here, by $A \approx B$ we mean $A \cong B$ up to free summands. The containment and the inclusion follow from [23, Theorem 4.15] and [24, Lemma 3.2(1)], respectively. By [24, Proposition 3.3], we get $\Omega_R^{-1}M \in \mathsf{res}_{\mathsf{mod}\,R}\, M$.

(2) By Remark 3.7(2) there exists an exact triangle $X \to X' \to E \rightsquigarrow$ in $\mathsf{D}(R)$ such that $\sup X' \leqslant 0$ and $E \in \mathcal{E}_R$. As X belongs to $\mathsf{C}(R)$, so does X' by Proposition 3.14(2). By Lemma 5.3(4)(5), we may assume that X' is a totally reflexive module. There is an exact sequence $0 \to X' \to P \to \Omega^{-1}X' \to 0$ in $\mathsf{mod}\,R$ with P projective, and $\Omega^{-1}X'$ belongs to $\mathsf{res}_{\mathsf{mod}\,R}\, X'$ by (1). The octahedral axiom gives a commutative diagram

$$\begin{array}{ccccccc}
X & \longrightarrow & X' & \longrightarrow & E & \longrightarrow & X[1] \\
\| & & \downarrow & & \downarrow & & \| \\
X & \longrightarrow & P & \longrightarrow & Y & \longrightarrow & X[1] \\
\downarrow & & \| & & \downarrow & & \downarrow \\
X' & \longrightarrow & P & \longrightarrow & \Omega^{-1}X' & \longrightarrow & X'[1] \\
\downarrow & & \downarrow & & \| & & \downarrow \\
E & \longrightarrow & Y & \longrightarrow & \Omega^{-1}X' & \longrightarrow & E[1]
\end{array}$$

of exact triangles in $\mathsf{D}(R)$. It is seen from the bottom row that Y belongs to $\mathsf{res}_{\mathsf{D}(R)}\, X'$, which coincides with $\mathsf{res}_{\mathsf{D}(R)}\, X$ (by the first row). Thus, the second row provides such an exact triangle as we want.

(3) We may assume that $\sup X \leqslant 0$. In fact, by Remark 3.7(2) there exists an exact triangle $X \to X' \to E \rightsquigarrow$ in $\mathsf{D}(R)$ such that $\sup X' \leqslant 0$ and $E \in \mathcal{E}_R$. Suppose that we have got an exact triangle $X' \to P \to C \rightsquigarrow$ in $\mathsf{D}(R)$ such that $P \in \mathsf{K}(R)$ and $C \in (\mathsf{res}\, X') \cap \mathsf{C}(R)$. Then we have $C \in (\mathsf{res}\, X) \cap \mathsf{C}(R)$ as $\mathsf{res}\, X' = \mathsf{res}\, X$. The

octahedral axiom gives rise to a commutative diagram of exact triangles in $\mathsf{D}(R)$:

$$\begin{array}{ccccccc}
X & \longrightarrow & X' & \longrightarrow & E & \longrightarrow & X[1] \\
\| & & \downarrow & & \downarrow & & \| \\
X & \longrightarrow & P & \longrightarrow & Y & \longrightarrow & X[1] \\
\downarrow & & \| & & \downarrow & & \downarrow \\
X' & \longrightarrow & P & \longrightarrow & C & \longrightarrow & X'[1] \\
\downarrow & & \downarrow & & \| & & \downarrow \\
E & \longrightarrow & Y & \longrightarrow & C & \longrightarrow & E[1]
\end{array}$$

The bottom row in the above diagram shows that Y belongs to $(\operatorname{res} X) \cap \mathsf{C}(R)$ by Proposition 3.14(2). Consequently, the second row in the above diagram is such an exact triangle as in the assertion.

Since the ring R is a complete intersection, it is Gorenstein. By Lemma 5.3(5), the number $n := \operatorname{Gdim}_R X$ is finite. We use induction on n. Let $n \leqslant 0$. Then X is a maximal Cohen–Macaulay complex; see Lemma 5.3(5). Thus the assertion follows from (2); note that $\operatorname{res} X = (\operatorname{res} X) \cap \mathsf{C}(R)$. Let $n > 0$. Since $\sup X \leqslant 0$, by Lemma 5.5(1) there is an exact triangle $Y \to P \to X \rightsquigarrow$ in $\mathsf{D}(R)$ such that P is a projective module and $\sup Y \leqslant 0$. Note then that $Y \in \operatorname{res} X$, so that $\operatorname{res} Y \subseteq \operatorname{res} X$. As $n-1 \geqslant 0$, Lemma 5.3(3) implies $\operatorname{Gdim} Y \leqslant \sup\{\operatorname{Gdim} P, \operatorname{Gdim} X - 1\} = n-1$. The induction hypothesis yields an exact triangle $Y \to K \to C \rightsquigarrow$ in $\mathsf{D}(R)$ such that $K \in \mathsf{K}(R)$ and $C \in (\operatorname{res} Y) \cap \mathsf{C}(R)$. By the octahedral axiom, we get the commutative diagram (a) of exact triangles in $\mathsf{D}(R)$. The second row in (a) shows $C' \in \mathsf{C}(R)$ and $\operatorname{res} C' \subseteq \operatorname{res} C$. By (2) there is an exact triangle $C' \to Q \to C'' \to C'[1]$ in $\mathsf{D}(R)$ such that Q is a projective module and $C'' \in \operatorname{res} C'$. Applying the octahedral axiom again, we obtain the commutative diagram (b) of exact triangles in $\mathsf{D}(R)$.

$$\text{(a)}: \begin{array}{ccccccc}
C[-1] & \longrightarrow & Y & \longrightarrow & K & \longrightarrow & C \\
\| & & \downarrow & & \downarrow & & \| \\
C[-1] & \longrightarrow & P & \longrightarrow & C' & \longrightarrow & C \\
\downarrow & & \| & & \downarrow & & \downarrow \\
Y & \longrightarrow & P & \longrightarrow & X & \longrightarrow & Y[1] \\
\downarrow & & \downarrow & & \| & & \downarrow \\
K & \longrightarrow & C' & \longrightarrow & X & \longrightarrow & K[1]
\end{array}
\qquad
\text{(b)}: \begin{array}{ccccccc}
K & \longrightarrow & C' & \longrightarrow & X & \longrightarrow & K[1] \\
\| & & \downarrow & & \downarrow & & \| \\
K & \longrightarrow & Q & \longrightarrow & K' & \longrightarrow & K[1] \\
\downarrow & & \| & & \downarrow & & \downarrow \\
C' & \longrightarrow & Q & \longrightarrow & C'' & \longrightarrow & C'[1] \\
\downarrow & & \downarrow & & \| & & \downarrow \\
X & \longrightarrow & K' & \longrightarrow & C'' & \longrightarrow & X[1]
\end{array}$$

The second row in (b) shows K' is in $\mathsf{K}(R)$. We have $C'' \in \operatorname{res} C' \subseteq \operatorname{res} C \subseteq (\operatorname{res} Y) \cap \mathsf{C}(R) \subseteq (\operatorname{res} X) \cap \mathsf{C}(R)$. Consequently, the bottom row in (b) provides such an exact triangle as in the assertion. □

We record a direct consequence of the second assertion of the above proposition.

Corollary 5.9. *Let R be a complete intersection. Let $X \in \mathsf{D}(R)$ be a maximal Cohen–Macaulay complex. Then X belongs to the resolving closure $\operatorname{res}_{\mathsf{D}(R)}(X[-i])$ for every nonnegative integer i.*

Proof. Proposition 5.8(2) gives rise to an exact triangle $X \to P \to Y \rightsquigarrow$ in $\mathsf{D}(R)$ such that P is a projective module and $Y \in \operatorname{res} X$. An exact triangle $Y[-1] \to X \to P \rightsquigarrow$ is induced, which shows that X is in $\operatorname{res}(Y[-1])$. Proposition 2.8(2a) implies $Y[-1] \in (\operatorname{res} X)[-1] \subseteq \operatorname{res}(X[-1])$. Hence $X \in \operatorname{res}(X[-1])$. If $X \in \operatorname{res}(X[-j])$ for an integer j, then

$$X[-1] \in (\operatorname{res}(X[-j]))[-1] \subseteq \operatorname{res}(X[-j][-1]) = \operatorname{res}(X[-j-1])$$

by Proposition 2.8(2a) again, and we get $X \in \operatorname{res}(X[-1]) \subseteq \operatorname{res}(X[-j-1])$. It follows that X belongs to $\operatorname{res}(X[-i])$ for all $i \geqslant 0$. $\square$

The main goal of this section is the following theorem, which is viewed as a derived category version of [24, Theorem 7.4].

Theorem 5.10. *Suppose that R is a complete intersection. Then there are mutually inverse bijections*

$$\left\{\begin{array}{c}\text{resolving}\\ \text{subcategories}\\ \text{of } \mathsf{D}(R)\end{array}\right\} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \left\{\begin{array}{c}\text{resolving}\\ \text{subcategories of } \mathsf{D}(R)\\ \text{contained in } \mathsf{K}(R)\end{array}\right\} \times \left\{\begin{array}{c}\text{resolving}\\ \text{subcategories of } \mathsf{D}(R)\\ \text{contained in } \mathsf{C}(R)\end{array}\right\},$$

where the maps ϕ, ψ are given by $\phi(\mathcal{X}) = (\mathcal{X} \cap \mathsf{K}(R), \mathcal{X} \cap \mathsf{C}(R))$ and $\psi(\mathcal{Y}, \mathcal{Z}) =$ $\operatorname{res}_{\mathsf{D}(R)}(\mathcal{Y} \cup \mathcal{Z})$.

Proof. Clearly, the maps ϕ, ψ are well-defined. Lemma 5.3(5) implies $\mathsf{G}(R) = \mathsf{C}(R)$. Proposition 5.7 says $\phi\psi = \mathrm{id}$. Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{D}(R)$. Then $\psi\phi(\mathcal{X}) = \operatorname{res}((\mathcal{X} \cap \mathsf{K}(R)) \cup (\mathcal{X} \cap \mathsf{C}(R)))$ is clearly contained in $\mathcal{X}$. Let X be any object in $\mathcal{X}$. It follows from Proposition 5.8(3) that there is an exact triangle $X \to P \to Y \rightsquigarrow$ in $\mathsf{D}(R)$ such that $P \in \mathsf{K}(R)$ and $Y \in (\operatorname{res} X) \cap \mathsf{C}(R) \subseteq \mathcal{X} \cap \mathsf{C}(R)$. We see that P is in $\mathcal{X} \cap \mathsf{K}(R)$, so that X is in $\operatorname{res}((\mathcal{X} \cap \mathsf{K}(R)) \cup (\mathcal{X} \cap \mathsf{C}(R)))$. Thus X belongs to $\psi\phi(\mathcal{X})$, and we obtain $\psi\phi = \mathrm{id}$. $\square$

6. Classification of resolving subcategories and certain preaisles of $\mathsf{D}(R)$

The main goal of this section is to give a complete classification of resolving subcategories of $\mathsf{D}(R)$ and preaisles of $\mathsf{D}(R)$ containing R and closed under direct summands, in the case where R belongs to a certain class of complete intersection rings. First of all, applying the main result of the previous section, we prove the following theorem. The bijections given in the theorem say that classifying the resolving subcategories of maximal Cohen–Macaulay complexes is equivalent to classifying the thick subcategories containing R. The equality given in the theorem is a derived category version of [23, Corollary 4.16].

Theorem 6.1. *Let R be a complete intersection. There are mutually inverse bijections and an equality*

$$\left\{\begin{matrix}\text{thick}\\ \text{subcategories of}\\ \mathsf{D}(R)\text{ containing }R\end{matrix}\right\} \underset{\mathsf{thick}_{\mathsf{D}(R)}(-)}{\overset{(-)\cap\mathsf{C}(R)}{\rightleftarrows}} \left\{\begin{matrix}\text{resolving}\\ \text{subcategories of }\mathsf{D}(R)\\ \text{contained in }\mathsf{C}(R)\end{matrix}\right\} = \left\{\begin{matrix}\text{thick}\\ \text{subcategories of }\mathsf{C}(R)\\ \text{containing }R\end{matrix}\right\}.$$

Proof. We start by proving the equality, using the bijections. By Proposition 3.16(1), it suffices to show that each resolving subcategory $\mathcal{X}$ of $\mathsf{D}(R)$ contained in $\mathsf{C}(R)$ is a thick subcategory of $\mathsf{C}(R)$, and for this it is enough to verify that for each exact triangle $A \to B \to C \rightsquigarrow$ in $\mathsf{D}(R)$ with $A, B, C \in \mathsf{C}(R)$, if A and B belong to $\mathcal{X}$, then so does C. By the first assertion of the theorem we have $\mathcal{X} = (\mathsf{thick}_{\mathsf{D}(R)}\,\mathcal{X}) \cap \mathsf{C}(R)$. Hence A and B are in $\mathsf{thick}_{\mathsf{D}(R)}\,\mathcal{X}$, and so is C. It follows that C belongs to $(\mathsf{thick}_{\mathsf{D}(R)}\,\mathcal{X}) \cap \mathsf{C}(R) = \mathcal{X}$, and we are done.

We proceed with showing the bijections. Clearly, the two maps are well-defined. Fix a thick subcategory $\mathcal{X}$ of $\mathsf{D}(R)$ containing R and a resolving subcategory $\mathcal{Z}$ of $\mathsf{D}(R)$ contained in $\mathsf{C}(R)$. Then $\mathcal{X}$ is a resolving subcategory of $\mathsf{D}(R)$, so that Theorem 5.10 shows $\mathcal{X} = \psi\phi(\mathcal{X}) = \mathsf{res}((\mathcal{X} \cap \mathsf{K}(R)) \cup (\mathcal{X} \cap \mathsf{C}(R)))$. Since $\mathcal{X}$ is thick and contains R, it contains $\mathsf{K}(R) = \mathsf{thick}\,R$; see Proposition 2.10(3). Hence $\mathcal{X} \cap \mathsf{K}(R) = \mathsf{K}(R)$, and

$$\mathcal{X} = \mathsf{res}(\mathsf{K}(R) \cup (\mathcal{X} \cap \mathsf{C}(R))) \subseteq \mathsf{thick}(\mathsf{K}(R) \cup (\mathcal{X} \cap \mathsf{C}(R))) = \mathsf{thick}(\mathcal{X} \cap \mathsf{C}(R)) \subseteq \mathcal{X}.$$

Therefore, $\mathcal{X} = \mathsf{thick}(\mathcal{X} \cap \mathsf{C}(R))$. On the other hand, applying Theorem 5.10 again, we have

$$(\mathsf{K}(R), \mathcal{Z}) = \phi\psi(\mathsf{K}(R), \mathcal{Z}) = (\mathsf{res}(\mathsf{K}(R) \cup \mathcal{Z}) \cap \mathsf{K}(R), \mathsf{res}(\mathsf{K}(R) \cup \mathcal{Z}) \cap \mathsf{C}(R)),$$

which gives us the equality $\mathcal{Z} = \mathsf{res}(\mathsf{K}(R) \cup \mathcal{Z}) \cap \mathsf{C}(R)$.

We claim that $\mathsf{res}(\mathsf{K}(R) \cup \mathcal{Z})$ is a thick subcategory of $\mathsf{D}(R)$. Indeed, it suffices to verify that $\mathsf{res}(\mathsf{K}(R) \cup \mathcal{Z})$ is closed under positive shifts. Using Proposition 2.8(2b), we get equalities

$$\begin{aligned}(\mathsf{res}(\mathsf{K}(R) \cup \mathcal{Z}))[1] &= \mathsf{res}((\mathsf{K}(R) \cup \mathcal{Z})[1] \cup \{R[1]\}) \\ &= \mathsf{res}((\mathsf{K}(R) \cup \mathcal{Z})[1]) = \mathsf{res}(\mathsf{K}(R) \cup \mathcal{Z}[1]).\end{aligned} \tag{6.1.1}$$

Pick $Z \in \mathcal{Z}$. Then Z is maximal Cohen–Macaulay, and Corollary 5.9 implies $Z \in \mathsf{res}(Z[-1])$. We obtain

$$Z[1] \in (\mathsf{res}(Z[-1]))[1] = \mathsf{res}\{Z, R[1]\} \subseteq \mathsf{res}(\mathsf{K}(R) \cup \mathcal{Z}),$$

where for the equality we apply Proposition 2.8(2b) again. It follows that $\mathcal{Z}[1]$ is contained in $\mathsf{res}(\mathsf{K}(R) \cup \mathcal{Z})$. This and (6.1.1) yield that $(\mathsf{res}(\mathsf{K}(R) \cup \mathcal{Z}))[1]$ is contained in $\mathsf{res}(\mathsf{K}(R) \cup \mathcal{Z})$. Thus the claim follows.

The above claim guarantees that $\mathsf{res}(\mathsf{K}(R) \cup \mathcal{Z}) = \mathsf{thick}(\mathsf{K}(R) \cup \mathcal{Z}) = \mathsf{thick}\,\mathcal{Z}$, and we obtain an equality $\mathcal{Z} = (\mathsf{thick}\,\mathcal{Z}) \cap \mathsf{C}(R)$. Now we conclude that the two maps in the assertion are mutually inverse bijections. □

Denote by $\mathsf{S}(R)$ the *singularity category* $\mathsf{D}_{\mathsf{sg}}(R)$ of R, that is, the Verdier quotient of $\mathsf{D}(R)$ by $\mathsf{K}(R)$. The following lemma enables us to obtain a classification of preaisles in the next theorem.

Lemma 6.2. (1) *There is a natural one-to-one correspondence*

$$\left\{\begin{array}{c}\text{thick subcategories}\\ \text{of } \mathsf{S}(R)\end{array}\right\} \cong \left\{\begin{array}{c}\text{thick subcategories}\\ \text{of } \mathsf{D}(R) \text{ containing } R\end{array}\right\}.$$

(2) Suppose that R is Gorenstein. Assigning to each subcategory $\mathcal{X}$ of $\mathsf{D}(R)$ the subcategory $\mathbf{R}\mathrm{Hom}_R(\mathcal{X}, R)$ of $\mathsf{D}(R)$ consisting of objects of the form $\mathbf{R}\mathrm{Hom}_R(X, R)$ with $X \in \mathcal{X}$, one gets a one-to-one correspondence

$$\left\{\begin{array}{c}\text{preaisles of } \mathsf{D}(R)\\ \text{containing } R \text{ and closed}\\ \text{under direct summands}\end{array}\right\} \cong \left\{\begin{array}{c}\text{precoaisles of } \mathsf{D}(R)\\ \text{containing } R \text{ and closed}\\ \text{under direct summands}\end{array}\right\} = \left\{\begin{array}{c}\text{resolving}\\ \text{subcategories}\\ \text{of } \mathsf{D}(R)\end{array}\right\}.$$

Proof. (1) The assertion comes from a general fact on Verdier quotients; see [57, Chapitre II, Proposition 2.3.1], [53, Lemma 3.1] and [54, Lemma 10.5].

(2) The equality follows by definition. As R is Gorenstein, for each $C \in \mathsf{D}(R)$ the complex $\mathbf{R}\mathrm{Hom}(C, R)$ is bounded, so that it is in $\mathsf{D}(R)$; see [21, (2.3.8)] and Lemma 5.3(5). Thus, the contravariant exact (additive) functor $\mathbf{R}\mathrm{Hom}(-, R)$ gives a duality of $\mathsf{D}(R)$. Since $\mathbf{R}\mathrm{Hom}(R, R) = R$, we can easily get the bijection. □

Combining Theorems 4.5, 5.10, 6.1 and Lemma 6.2, we obtain the theorem below. Thanks to this theorem, to classify the resolving subcategories of $\mathsf{D}(R)$ we have only to classify the thick subcategories of $\mathsf{S}(R)$.

Theorem 6.3. *Let R be a complete intersection. Then there are one-to-one correspondences*

$$\left\{\begin{array}{c}\text{preaisles of } \mathsf{D}(R)\\ \text{containing } R\\ \text{and closed under}\\ \text{direct summands}\end{array}\right\} \cong \left\{\begin{array}{c}\text{resolving}\\ \text{subcategories}\\ \text{of } \mathsf{D}(R)\end{array}\right\} \cong \left\{\begin{array}{c}\text{order-preserving}\\ \text{maps from Spec } R\\ \text{to } \mathbb{N} \cup \{\infty\}\end{array}\right\} \times \left\{\begin{array}{c}\text{thick}\\ \text{subcategories}\\ \text{of } \mathsf{S}(R)\end{array}\right\}.$$

We recall the definition of a hypersurface, related notions and basic properties.

Definition 6.4. (1) Let $(R, \mathfrak{m}, k)$ be a local ring. We denote by $\operatorname{edim} R$ the *embedding dimension* of R, that is, the number of elements in a minimal system of generators of $\mathfrak{m}$, which is equal to the dimension of the k-vector space $\mathfrak{m} \otimes_R k = \mathfrak{m}/\mathfrak{m}^2$.

(2) Let R be a local ring. We denote by $\operatorname{codim} R$ and $\operatorname{codepth} R$ the *codimension* and the *codepth* of R, respectively, that is to say, $\operatorname{codim} R = \operatorname{edim} R - \dim R$ and $\operatorname{codepth} R = \operatorname{edim} R - \operatorname{depth} R$.

(3) For a local ring R, the following three conditions are equivalent; see [8, §5.1].
(a) There is an inequality $\operatorname{codepth} R \leqslant 1$. (b) The local ring R is Cohen–Macaulay and $\operatorname{codim} R \leqslant 1$.
(c) The completion of R is isomorphic to the residue ring of a regular local ring by a single element.
When one of these equivalent conditions holds, the local ring R is called a *hypersurface*. By [8, Corollary 7.4.6], if a local ring R is a hypersurface, then so is the local ring $R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of R.

(4) We say that R is a *hypersurface* if the local ring $R_{\mathfrak{p}}$ is a hypersurface (in the sense of (2)) for every prime ideal $\mathfrak{p}$ of R.

To state theorems of Stevenson, Dao and Takahashi, and ours, we establish the following setup.

Setup 6.5. Let (R, V) be a pair that satisfies either of the following two conditions.

(1) R is a hypersurface and $V = \operatorname{Sing} R$.

(2) $R = S/(\boldsymbol{a})$ where S is a regular ring of finite Krull dimension and $\boldsymbol{a} = a_1, \dots, a_c$ is an S-regular sequence, and $V = \operatorname{Sing} Y = \{y \in Y \mid \mathcal{O}_{Y,y} \text{ is not regular}\}$ where $X = \mathbb{P}^{c-1}_S = \operatorname{Proj}(S[x_1, \dots, x_c])$ and Y is the zero subscheme of $a_1x_1 + \cdots + a_cx_c \in \Gamma(X, \mathcal{O}_X(1))$.

Remark 6.6. In view of [19, Theorem 2.10], [46, Corollary 7.9 and the beginning of Section 10], Setup 6.5(2) is equivalent to the following condition.

(2′) $R = S/(\boldsymbol{a})$ where S is a regular ring of finite Krull dimension and $\boldsymbol{a} = a_1, \dots, a_c$ is an S-regular sequence, and $V = \operatorname{Sing} Y$ where $Y = \operatorname{Proj} G$ and $G = S[x_1, \dots, x_c]/(f)$ is the *generic hypersurface*, that is, the homogeneous S-algebra ($\deg(s) = 0$ for $s \in S$ and $\deg(x_i) = 1$ for $i = 1, \dots, c$) defined as the quotient ring of the polynomial ring over S in c variables $x_1, \dots, x_c$ by the polynomial $f = a_1x_1 + \cdots + a_cx_c$.

The following is the theorem of Stevenson [46]. Its assertion for Setup 6.5(1) is shown in [46, Theorem 6.13], whose local case is [53, Theorem 3.13(1)]. Its assertion for Setup 6.5(2) is shown in [46, Theorem 8.8]. The first one-to-one correspondence in the theorem is the one given in Lemma 6.2(1).

Theorem 6.7 (Stevenson). *Let (R, V) be as in Setup 6.5. Then there is a one-to-one correspondence*

$$\left\{\begin{matrix}\text{thick subcategories}\\ \text{of } \mathsf{S}(R)\end{matrix}\right\} \cong \left\{\begin{matrix}\text{thick subcategories}\\ \text{of } \mathsf{D}(R) \text{ containing } R\end{matrix}\right\} \overset{\text{(a)}}{\cong} \left\{\begin{matrix}\text{specialization-closed}\\ \text{subsets of } V\end{matrix}\right\}.$$

We obtain the following bijections by applying Theorems 6.3 and 6.7.

Corollary 6.8. *Let (R, V) be as in Setup 6.5. Then there are one-to-one correspondences*

$$\left\{\begin{array}{c}\text{preaisles of } \mathsf{D}(R)\\ \text{containing } R\\ \text{and closed under}\\ \text{direct summands}\end{array}\right\} \cong \left\{\begin{array}{c}\text{resolving}\\ \text{subcategories}\\ \text{of } \mathsf{D}(R)\end{array}\right\} \overset{\text{(b)}}{\cong} \left\{\begin{array}{c}\text{order-preserving}\\ \text{maps from}\\ \text{Spec } R \text{ to } \mathbb{N}\cup\{\infty\}\end{array}\right\} \times \left\{\begin{array}{c}\text{specialization-}\\ \text{closed}\\ \text{subsets of } V\end{array}\right\}.$$

Remark 6.9. (1) By Proposition 3.16(1), thick subcategories of $\mathsf{D}(R)$ containing R are resolving subcategories of $\mathsf{D}(R)$. Restricting the bijection (b) in Corollary 6.8 to the thick subcategories of $\mathsf{D}(R)$ containing R, one recovers the bijection (a) in Theorem 6.7. In fact, let $\mathcal{X}$ be a thick subcategory of $\mathsf{D}(R)$ containing R. Then $\mathcal{X}$ contains $\mathsf{thick}_{\mathsf{D}(R)} R = \mathsf{K}(R)$ by Proposition 2.10(3). Hence $\mathcal{X} \cap \mathsf{K}(R) = \mathsf{K}(R)$, and $\sup_{X \in \mathcal{X} \cap \mathsf{K}(R)}\{\operatorname{pd} X_{\mathfrak{p}}\} = \infty$ for each prime ideal $\mathfrak{p}$ of R. Note that this actually holds for $\mathcal{X} := \mathsf{K}(R)$. Define the map $\xi : \operatorname{Spec} R \to \mathbb{N} \cup \{\infty\}$ by $\xi(\mathfrak{p}) = \infty$ for every $\mathfrak{p} \in \operatorname{Spec} R$. It is observed along the way to get Corollary 6.8 that the bijection (b) in Corollary 6.8 restricts to the bijection below, which can be identified with the bijection (a) in Theorem 6.7.

$$\{\text{thick subcategories of } \mathsf{D}(R) \text{ containing } R\} \cong \{\xi\} \times \{\text{specialization-closed subsets of } V\}.$$

(2) Taking Remark 4.9(1) into account, we may wonder if it is possible to classify the preaisles of $\mathsf{D}(R)$ closed under direct summands, including those ones which do not contain R. There would be no straightforward modifications of our techniques to achieve this. In fact, the condition of containing R is used in a lot of places of this paper. For example, Theorem 5.10 is one of our key results which played an essential role in the proof of Corollary 6.8. To extend Theorem 5.10 directly to the precoaisles of $\mathsf{D}(R)$ closed under direct summands, one has to generalize the lemmas and propositions given in Section 5 to the setting where the condition of containing R is not assumed. It should already be a big obstruction here that those subcategories to be classified are not necessarily closed under syzygies of modules. On the other hand, in [45, Theorem 4.9] all the thick subcategories of $\mathsf{D}(R)$ are classified when (R, V) is as in Setup 6.5. Not assuming the thick subcategories contain R, a compatibility condition between the specialization closed subsets of V and $\operatorname{Spec} R$ shows up. This would also say that our results cannot be straightforwardly extended to encompass all preaisles closed under direct summands.

It may be interesting to consider the following question, which is similar to Question 4.13.

Question 6.10. Let R be as in Corollary 6.8. Hence R is Cohen–Macaulay, so it is CM-excellent. Assume that R has finite Krull dimension. Then, the aisles of $\mathsf{D}(R)$ containing R are classified by both Theorem 4.11 and Corollary 6.8. Are these two classifications (essentially) the same?

We close the section by giving, in the case of a hypersurface, an explicit description in terms of NE-loci of the restriction of the one-to-one correspondence (b) in Corollary 6.8 to the resolving subcategories of maximal Cohen–Macaulay complexes. For a subcategory $\mathcal{C}$ of $\mathsf{D}(R)$, denote by $\mathrm{IPD}(\mathcal{C})$ the set of prime ideals $\mathfrak{p}$ of R with $\mathrm{pd}_{R_\mathfrak{p}} X_\mathfrak{p} = \infty$ for some $X \in \mathcal{C}$. For a set Φ of prime ideals of R, denote by $\mathrm{IPD}^{-1}(\Phi)$ the subcategory of $\mathsf{D}(R)$ consisting of complexes X such that every prime ideal $\mathfrak{p}$ of R with $\mathrm{pd}_{R_\mathfrak{p}} X_\mathfrak{p} = \infty$ belongs to Φ.

Proposition 6.11. *Let R be a hypersurface. One then has the following mutually inverse bijections.*

$$\left\{\begin{matrix}\text{resolving subcategories of } \mathsf{D}(R)\\ \text{contained in } \mathsf{C}(R)\end{matrix}\right\} \underset{\mathrm{NE}_{\mathsf{C}}^{-1}(-)}{\overset{\mathrm{NE}(-)}{\rightleftarrows}} \left\{\begin{matrix}\text{specialization-closed subsets}\\ \text{of } \mathrm{Sing}\, R\end{matrix}\right\}$$

Proof. Fix a resolving subcategory $\mathcal{X}$ of $\mathsf{D}(R)$ contained in $\mathsf{C}(R)$ and a specialization-closed subset W of $\mathrm{Sing}\, R$. By Proposition 2.13(2) and [54, Remark 10.2(8)], we get $\mathrm{IPD}(\mathsf{thick}_{\mathsf{D}(R)}\, \mathcal{X}) = \mathrm{NE}(\mathcal{X})$ and $\mathrm{IPD}^{-1}(W) \cap \mathsf{C}(R) = \mathrm{NE}_{\mathsf{C}}^{-1}(W)$. The assertion follows by combining this with Theorem 6.1 and [53, Theorem 3.13(1)]. □

Remark 6.12. Another way in the case where R is a hypersurface to deduce the equality given in Theorem 6.1 is obtained by the combination of Propositions 6.11 and 3.16.

7. Restricting the classification of resolving subcategories of $\mathsf{D}(R)$

In this section, restricting the classification theorem of resolving subcategories of $\mathsf{D}(R)$ obtained in the previous section, we consider the resolving subcategories of $\mathsf{mod}\, R$. We begin with establishing a lemma.

Lemma 7.1. *Let R be a complete intersection. Let $\mathcal{X}$ be a resolving subcategory of* $\mathsf{mod}\, R$.

(1) *Let $\mathfrak{p}$ be a prime ideal of R. One has the equality* $\sup_{X\in\mathcal{X}}\{\mathrm{depth}\, R_\mathfrak{p} - \mathrm{depth}\, X_\mathfrak{p}\} = \sup_{Y\in\mathcal{X}\cap\mathsf{fpd}\, R}\{\mathrm{pd}\, Y_\mathfrak{p}\}$.

(2) *There is an equality* $\mathsf{thick}_{\mathsf{D}(R)}\, \mathcal{X} = \mathsf{thick}_{\mathsf{D}(R)}(\mathcal{X} \cap \mathsf{CM}(R))$ *of thick closures in* $\mathsf{D}(R)$.

Proof. (1) The inequality ($\geqslant$) holds by the Auslander–Buchsbaum formula. To show the opposite inequality ($\leqslant$), put $t = \sup_{X\in\mathcal{X}}\{\mathrm{depth}\, R_\mathfrak{p} - \mathrm{depth}\, X_\mathfrak{p}\}$. As $X_\mathfrak{p}$ is an $R_\mathfrak{p}$-module, we have $\mathrm{depth}\, X_\mathfrak{p} \in \mathbb{N} \cup \{\infty\}$. Hence $t \leqslant \mathrm{depth}\, R_\mathfrak{p} < \infty$. We have $t = \operatorname{depth} R_\mathfrak{p} - \operatorname{depth} X_\mathfrak{p}$ for some $X \in \mathcal{X}$. Set $n = \operatorname{Rfd}_R X$. We see that $\Omega^n X$ is a maximal Cohen–Macaulay R-module. By [24, proof of Theorem 7.4] there is an exact sequence $0 \to X \to L \to D \to 0$ in $\operatorname{mod} R$ such that L has finite projective dimension and $D = \Omega^{-n-1}\Omega^n X$ is maximal Cohen–Macaulay. Applying Proposition 5.8(1) to $\Omega^n X \in \mathcal{X}$, we get $D \in \mathcal{X}$, and hence $L \in \mathcal{X} \cap \operatorname{fpd} R$. As $D_\mathfrak{p}$ is a maximal Cohen–Macaulay $R_\mathfrak{p}$-module, we have $\operatorname{depth} D_\mathfrak{p} \geqslant \operatorname{ht}\mathfrak{p}$. The depth lemma implies

$$\operatorname{depth} X_\mathfrak{p} \geqslant \inf\{\operatorname{depth} L_\mathfrak{p}, \operatorname{depth} D_\mathfrak{p} + 1\} \geqslant \inf\{\operatorname{depth} L_\mathfrak{p}, \operatorname{ht}\mathfrak{p} + 1\} = \operatorname{depth} L_\mathfrak{p}.$$

Hence $\operatorname{pd} L_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p} - \operatorname{depth} L_\mathfrak{p} \geqslant \operatorname{depth} R_\mathfrak{p} - \operatorname{depth} X_\mathfrak{p} = t$. We obtain $\sup_{Y \in \mathcal{X} \cap \operatorname{fpd} R}\{\operatorname{pd} Y_\mathfrak{p}\} \geqslant \operatorname{pd} L_\mathfrak{p} \geqslant t$.

(2) It suffices to show that $\mathcal{X}$ is contained in $\operatorname{thick}_{\mathsf{D}(R)}(\mathcal{X} \cap \mathsf{CM}(R))$. Fix an R-module $X \in \mathcal{X}$, and put $n = \operatorname{Rfd}_R X$. Since R is Gorenstein, there is an exact sequence $0 \to P \to \Omega^{-n}\Omega^n X \to X \to 0$ in $\operatorname{mod} R$ such that P has finite projective dimension; see [6, (2.21) and (4.22)]. The R-module $\Omega^n X$ is maximal Cohen–Macaulay. Proposition 5.8(1) implies $\Omega^{-n}\Omega^n X \in \operatorname{res}_{\operatorname{mod} R}(\Omega^n X) \subseteq \mathcal{X}$. Hence $\Omega^{-n}\Omega^n X$ is in $\mathcal{X} \cap \mathsf{CM}(R)$. As $P \in \operatorname{thick}_{\mathsf{D}(R)} R$ and $R \in \mathcal{X} \cap \mathsf{CM}(R)$, both P and $\Omega^{-n}\Omega^n X$ are in $\operatorname{thick}_{\mathsf{D}(R)}(\mathcal{X} \cap \mathsf{CM}(R))$, and so is X. □

Using the above lemma and results in the previous sections, we can show that for each resolving subcategory of $\operatorname{mod} R$, taking the resolving closure in $\mathsf{D}(R)$ commutes with taking the restriction to $\mathsf{K}(R)$ and $\mathsf{C}(R)$.

Proposition 7.2. *Suppose that R is a complete intersection. Let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$. Then*:

(1) $\operatorname{res}_{\mathsf{D}(R)}(\mathcal{X} \cap \mathsf{K}(R)) = \operatorname{res}_{\mathsf{D}(R)}(\mathcal{X} \cap \operatorname{fpd} R) = (\operatorname{res}_{\mathsf{D}(R)} \mathcal{X}) \cap \mathsf{K}(R)$.

(2) $\operatorname{res}_{\mathsf{D}(R)}(\mathcal{X} \cap \mathsf{C}(R)) = \operatorname{res}_{\mathsf{D}(R)}(\mathcal{X} \cap \mathsf{CM}(R)) = (\operatorname{res}_{\mathsf{D}(R)} \mathcal{X}) \cap \mathsf{C}(R)$.

Proof. The first equalities in the two assertions hold since $\mathcal{X} \cap \mathsf{K}(R) = \mathcal{X} \cap \operatorname{fpd} R$ and $\mathcal{X} \cap \mathsf{C}(R) = \mathcal{X} \cap \mathsf{CM}(R)$. In what follows, we show the second equalities.

(1) Since $\mathsf{K}(R)$ is a resolving subcategory of $\mathsf{D}(R)$ by Proposition 2.10(3), we see that both $\operatorname{res}_{\mathsf{D}(R)}(\mathcal{X} \cap \operatorname{fpd} R)$ and $(\operatorname{res}_{\mathsf{D}(R)} \mathcal{X}) \cap \mathsf{K}(R)$ are resolving subcategories of $\mathsf{D}(R)$ contained in $\mathsf{K}(R)$. Put

$$a = \sup_{X \in \operatorname{res}_{\mathsf{D}(R)}(\mathcal{X} \cap \operatorname{fpd} R)}\{\operatorname{pd} X_\mathfrak{p}\},$$
$$b = \sup_{X \in (\operatorname{res}_{\mathsf{D}(R)} \mathcal{X}) \cap \mathsf{K}(R)}\{\operatorname{pd} X_\mathfrak{p}\},$$
$$c = \sup_{X \in \mathcal{X} \cap \operatorname{fpd} R}\{\operatorname{pd} X_\mathfrak{p}\}.$$

By Theorem 4.8, it is enough to verify that $a=b$. Since

$$\mathcal{X}\cap \mathsf{fpd}\, R \subseteq \mathsf{res}_{\mathsf{D}(R)}(\mathcal{X}\cap \mathsf{fpd}\, R) \subseteq (\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X})\cap \mathsf{K}(R),$$

we have $c \leqslant a \leqslant b$. Thus it suffices to show that $b \leqslant c$, which we do as follows.

$$\begin{aligned} b &\overset{\text{(i)}}{=} \sup_{X\in(\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X})\cap\mathsf{K}(R)}\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} X_{\mathfrak{p}}\} \\ &\overset{\text{(ii)}}{\leqslant} \sup_{X\in\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X}}\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} X_{\mathfrak{p}}\} \\ &\overset{\text{(iii)}}{=} \sup_{X\in\mathcal{X}}\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} X_{\mathfrak{p}}\} \\ &\overset{\text{(iv)}}{=} c. \end{aligned}$$

Here, (i) follows from Proposition 2.13(2) and (iv) from Lemma 7.1(1). The inclusion $(\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X})\cap\mathsf{K}(R)\subseteq\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X}$ implies (ii). As for (iii), the inequality ($\geqslant$) holds since $\mathcal{X}$ is contained in $\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X}$. It is observed from Proposition 2.13(2)(3) that the subcategory $\mathcal{Y}$ of $\mathsf{D}(R)$ consisting of objects Y such that

$$\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} Y_{\mathfrak{p}} \leqslant \sup_{X\in\mathcal{X}}\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} X_{\mathfrak{p}}\}$$

is resolving and contains $\mathcal{X}$. Therefore, the subcategory $\mathcal{Y}$ contains $\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X}$. Thus ($\leqslant$) follows.

(2) By Proposition 3.14(2), $\mathsf{res}_{\mathsf{D}(R)}(\mathcal{X}\cap\mathsf{CM}(R))$ and $(\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X})\cap\mathsf{C}(R)$ are resolving subcategories of $\mathsf{D}(R)$ contained in $\mathsf{C}(R)$. By virtue of Theorem 6.1, it is enough to show that

$$\mathsf{thick}_{\mathsf{D}(R)}(\mathsf{res}_{\mathsf{D}(R)}(\mathcal{X}\cap\mathsf{CM}(R))) \quad\text{and}\quad \mathsf{thick}_{\mathsf{D}(R)}((\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X})\cap\mathsf{C}(R))$$

coincide. We have

$$\begin{aligned} \mathsf{thick}_{\mathsf{D}(R)}(\mathsf{res}_{\mathsf{D}(R)}(\mathcal{X}\cap\mathsf{CM}(R))) &= \mathsf{thick}_{\mathsf{D}(R)}(\mathcal{X}\cap\mathsf{CM}(R)) \\ &= \mathsf{thick}_{\mathsf{D}(R)}\,\mathcal{X} \\ &= \mathsf{thick}_{\mathsf{D}(R)}(\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X}) \\ &\supseteq \mathsf{thick}_{\mathsf{D}(R)}((\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X})\cap\mathsf{C}(R)) \\ &\supseteq \mathsf{thick}_{\mathsf{D}(R)}(\mathcal{X}\cap\mathsf{CM}(R)), \end{aligned}$$

where the first and third equalities and the inclusions are clear, while the second equality follows from Lemma 7.1(2). Thus those two inclusions are equalities, and we obtain the desired equality of thick closures. □

In the next result, $\underline{\mathsf{CM}}(R)$ denotes the *stable category* of $\mathsf{CM}(R)$ (the definition of a thick subcategory of $\mathsf{CM}(R)$ is given in Section 3). This proposition particularly says that, over a complete intersection, the resolving subcategories of maximal Cohen–Macaulay modules bijectively and naturally correspond to the resolving subcategories of maximal Cohen–Macaulay complexes.

Proposition 7.3. *Let R be a complete intersection. Then there are natural one-to-one correspondences*

$$\left\{\begin{matrix}\text{resolving}\\ \text{subcategories of } \mathsf{mod}\, R\\ \text{contained in } \mathsf{CM}(R)\end{matrix}\right\} = \left\{\begin{matrix}\text{thick subcategories}\\ \text{of } \mathsf{CM}(R)\\ \text{containing } R\end{matrix}\right\} \cong \left\{\begin{matrix}\text{thick subcategories}\\ \text{of } \underline{\mathsf{CM}}(R)\end{matrix}\right\}$$

$$\cong \left\{\begin{matrix}\text{thick subcategories}\\ \text{of } \mathsf{S}(R)\end{matrix}\right\} \cong \left\{\begin{matrix}\text{thick subcategories}\\ \text{of } \mathsf{D}(R)\\ \text{containing } R\end{matrix}\right\}$$

$$\cong \left\{\begin{matrix}\text{thick subcategories}\\ \text{of } \mathsf{C}(R)\\ \text{containing } R\end{matrix}\right\} = \left\{\begin{matrix}\text{resolving}\\ \text{subcategories of } \mathsf{D}(R)\\ \text{contained in } \mathsf{C}(R)\end{matrix}\right\}.$$

In particular, *one has the following one-to-one correspondence.*

$$\left\{\begin{matrix}\text{resolving subcategories}\\ \text{of } \mathsf{mod}\, R \text{ contained in } \mathsf{CM}(R)\end{matrix}\right\} \underset{(-)\cap\mathsf{CM}(R)}{\overset{\mathsf{res}_{\mathsf{D}(R)}(-)}{\rightleftarrows}} \left\{\begin{matrix}\text{resolving subcategories}\\ \text{of } \mathsf{D}(R) \text{ contained in } \mathsf{C}(R)\end{matrix}\right\}.$$

Proof. We start by showing the first assertion. The first equality can be obtained by [23, Corollary 4.16], where the ring is assumed to be local, but the argument works if we replace [23, Theorem 4.15(1)] used there with Proposition 5.8(1). The first bijection follows from [50, Proposition 6.2], where the ring is again assumed to be local but it is not used. Since R is Gorenstein, the assignment $M \mapsto M$ gives a triangle equivalence

$$\eta : \underline{\mathsf{CM}}(R) = \underline{\mathsf{GP}}(R) \xrightarrow{\cong} \mathsf{S}(R),$$

where $\underline{\mathsf{GP}}(R)$ denotes the stable category of the category $\mathsf{GP}(R)$ of totally reflexive R-modules; see [12, 1.3]. The second bijection in the assertion is induced from the equivalence η. The third bijection is given in Lemma 6.2(1). The last bijection and the last equality follow from Theorem 6.1.

From now on, we give a proof of the last assertion of the proposition. There is a commutative diagram

$$\begin{array}{ccc} \mathsf{CM}(R) & \xrightarrow{\quad\text{inc}\quad} & \mathsf{D}(R) \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\pi} \\ \underline{\mathsf{CM}}(R) & \xrightarrow{\quad\eta\quad} & \mathsf{S}(R) \end{array}$$

where inc is the inclusion functor, η is the triangle equivalence, and ε, π are the canonical quotient functors.

Fix a resolving subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ contained in $\mathsf{CM}(R)$. The resolving subcategory of $\mathsf{D}(R)$ contained in $\mathsf{C}(R)$ that corresponds to $\mathcal{X}$ is $\pi^{-1}\eta\varepsilon(\mathcal{X}) \cap \mathsf{C}(R)$, which coincides with $\pi^{-1}\pi(\mathcal{X}) \cap \mathsf{C}(R)$. As this is a resolving subcategory of $\mathsf{D}(R)$

containing $\mathcal{X}$, it contains $\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X}$ as well. Pick an object $C \in \pi^{-1}\pi(\mathcal{X}) \cap \mathsf{C}(R)$. Then $\pi(C)$ is in $\pi(\mathcal{X})$, and $\pi(C)$ is isomorphic to $\pi(X)$ for some $X \in \mathcal{X}$. There are exact triangles $\sigma : E \to C \to A \rightsquigarrow$ and $\tau : E \to X \to B \rightsquigarrow$ in $\mathsf{D}(R)$ with $A, B \in \mathsf{K}(R)$; see [40, Proposition 2.1.35]. We see from τ that E is in $\mathsf{thick}_{\mathsf{D}(R)}\,\mathcal{X}$, and from σ that C is in $\mathsf{thick}_{\mathsf{D}(R)}\,\mathcal{X}$. Hence $\pi^{-1}\pi(\mathcal{X}) \cap \mathsf{C}(R) \subseteq (\mathsf{thick}_{\mathsf{D}(R)}\,\mathcal{X}) \cap \mathsf{C}(R) = \mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X}$, where the equality follows from Theorem 6.1. We now conclude that $\pi^{-1}\pi(\mathcal{X}) \cap \mathsf{C}(R) = \mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X}$.

Fix a resolving subcategory $\mathcal{X}$ of $\mathsf{D}(R)$ contained in $\mathsf{C}(R)$. The resolving subcategory of $\mathsf{mod}\,R$ contained in $\mathsf{CM}(R)$ that corresponds to $\mathcal{X}$ is $\varepsilon^{-1}\eta^{-1}\pi(\mathsf{thick}_{\mathsf{D}(R)}\,\mathcal{X})$. Note that the equality $\pi^{-1}\pi(\mathcal{Y}) = \mathcal{Y}$ holds for each thick subcategory $\mathcal{Y}$ of $\mathsf{D}(R)$ containing R. We get the following equalities of subcategories of $\mathsf{CM}(R)$.

$$\begin{aligned}\varepsilon^{-1}\eta^{-1}\pi(\mathsf{thick}_{\mathsf{D}(R)}\,\mathcal{X}) &= \pi^{-1}\pi(\mathsf{thick}_{\mathsf{D}(R)}\,\mathcal{X}) \cap \mathsf{CM}(R) = (\mathsf{thick}_{\mathsf{D}(R)}\,\mathcal{X}) \cap \mathsf{CM}(R)\\ &= (\mathsf{thick}_{\mathsf{D}(R)}\,\mathcal{X}) \cap \mathsf{C}(R) \cap \mathsf{CM}(R) = \mathcal{X} \cap \mathsf{CM}(R).\end{aligned}$$

Here, the last equality follows from Theorem 6.1. Now we obtain the mutually inverse bijections in the last statement of the proposition. □

Proposition 7.3 says that when R is a complete intersection, the equality $\mathcal{X} = \mathsf{res}_{\mathsf{D}(R)}(\mathcal{X} \cap \mathsf{CM}(R))$ holds for every resolving subcategory $\mathcal{X}$ of $\mathsf{D}(R)$ contained in $\mathsf{C}(R)$. This equality holds in a more general setting.

Proposition 7.4. *The equality* $\mathcal{X} = \mathsf{res}_{\mathsf{D}(R)}(\mathcal{X} \cap \mathsf{GP}(R))$ *holds for every resolving subcategory* $\mathcal{X}$ *of* $\mathsf{D}(R)$ *contained in* $\mathsf{G}(R)$. *In particular, if the ring* R *is Gorenstein, then the equality* $\mathcal{X} = \mathsf{res}_{\mathsf{D}(R)}(\mathcal{X} \cap \mathsf{CM}(R))$ *holds for every resolving subcategory* $\mathcal{X}$ *of* $\mathsf{D}(R)$ *contained in* $\mathsf{C}(R)$.

Proof. The last assertion follows from the first and Lemma 5.3(5). To show the first assertion, let $\mathcal{X}$ be a resolving subcategory of $\mathsf{D}(R)$ contained in $\mathsf{G}(R)$. It clearly holds that $\mathcal{X}$ contains $\mathsf{res}_{\mathsf{D}(R)}(\mathcal{X} \cap \mathsf{GP}(R))$. Pick any $X \in \mathcal{X}$. Remark 3.7(2) gives an exact triangle $X \to Y \to E \rightsquigarrow$ in $\mathsf{D}(R)$ with $\sup Y \leqslant 0$ and $E \in \mathcal{E}_R$. By Lemma 5.3(3)(4) there exists a totally reflexive R-module T such that $Y \cong T$ in $\mathsf{D}(R)$. Since Y is in $\mathcal{X}$, we have $T \in \mathcal{X} \cap \mathsf{GP}(R)$. Hence Y is in $\mathsf{res}_{\mathsf{D}(R)}(\mathcal{X} \cap \mathsf{GP}(R))$, and so is X. Thus the first assertion follows. □

In view of Propositions 7.3 and 7.4, it is quite natural to ask the following question. Proposition 7.3 guarantees that the question has an affirmative answer in the case where R is a complete intersection.

Question 7.5. Suppose that the ring R is Gorenstein. Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{mod}\,R$ contained in $\mathsf{CM}(R)$. Then, does the equality $\mathcal{X} = (\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X}) \cap \mathsf{CM}(R)$ hold?

To show our next result, we establish a lemma on projective dimension.

Lemma 7.6. *Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$ *contained in* $\operatorname{fpd} R$*. Let Y be an object in* $\operatorname{res}_{\mathsf{D}(R)} \mathcal{X}$*, and let $\mathfrak{p}$ be a prime ideal of R. Then one has the inequality* $\operatorname{pd}_{R_\mathfrak{p}} Y_\mathfrak{p} \leqslant \operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p}$ *for some object $X \in \mathcal{X}$.*

Proof. Let $\mathcal{Z}$ be the subcategory of $\mathsf{D}(R)$ consisting of complexes Z such that $\operatorname{pd} Z_\mathfrak{p} \leqslant \operatorname{pd} X_\mathfrak{p}$ for some $X \in \mathcal{X}$. Clearly, $\mathcal{X}$ is contained in $\mathcal{Z}$, and in particular, R is in $\mathcal{Z}$. If Z is an object in $\mathcal{Z}$ and W is a direct summand of Z, then $\operatorname{pd} W_\mathfrak{p} \leqslant \operatorname{pd} Z_\mathfrak{p} \leqslant \operatorname{pd} X_\mathfrak{p}$ for some $X \in \mathcal{X}$ by Proposition 2.13(4), and hence W is also in $\mathcal{Z}$. Let $A \to B \to C \rightsquigarrow$ be an exact triangle in $\mathsf{D}(R)$ with $C \in \mathcal{Z}$. Then $\operatorname{pd} C_\mathfrak{p} \leqslant \operatorname{pd} X_\mathfrak{p}$ for some $X \in \mathcal{X}$. If $\operatorname{pd} A_\mathfrak{p}$ (resp. $\operatorname{pd} B_\mathfrak{p}$) is at most $\operatorname{pd} X'_\mathfrak{p}$ for some $X' \in \mathcal{X}$, then $\operatorname{pd} B_\mathfrak{p}$ (resp. $\operatorname{pd} A_\mathfrak{p}$) is at most $\sup\{\operatorname{pd} A_\mathfrak{p}, \operatorname{pd} C_\mathfrak{p}\}$ (resp. $\sup\{\operatorname{pd} B_\mathfrak{p}, \operatorname{pd} C_\mathfrak{p} - 1\}$) by Proposition 2.13(3), which is at most $\operatorname{pd} X''_\mathfrak{p}$ where $X'' = X \oplus X' \in \mathcal{X}$ by Proposition 2.13(4). Hence $A \in \mathcal{Z}$ if and only if $B \in \mathcal{Z}$. Thus, $\mathcal{Z}$ is a resolving subcategory of $\mathsf{D}(R)$ containing $\mathcal{X}$. Then $\mathcal{Z}$ contains $\operatorname{res}_{\mathsf{D}(R)} \mathcal{X}$, and we get $Y \in \mathcal{Z}$. We conclude $\operatorname{pd} Y_\mathfrak{p} \leqslant \operatorname{pd} X_\mathfrak{p}$ for some $X \in \mathcal{X}$. □

Now we find out a close relationship of each resolving subcategory of $\operatorname{mod} R$ with its resolving closure in $\mathsf{D}(R)$ when R is a complete intersection. In the proof we use the map Φ which is defined in Definition 4.6.

Proposition 7.7. *Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$*. Suppose either that $\mathcal{X}$ is contained in* $\operatorname{fpd} R$ *or that R is a complete intersection. Then the equality* $\mathcal{X} = (\operatorname{res}_{\mathsf{D}(R)} \mathcal{X}) \cap \operatorname{mod} R$ *holds true.*

Proof. We set up three steps, and in each step we prove the equality given in the proposition.

(1) Assume that $\mathcal{X}$ is contained in $\operatorname{fpd} R$. Then $\mathcal{X}$ is contained in $\mathsf{K}(R)$, and so is $\operatorname{res}_{\mathsf{D}(R)} \mathcal{X}$ by Proposition 2.10(3). Proposition 2.10(4) says that $(\operatorname{res}_{\mathsf{D}(R)} \mathcal{X}) \cap \operatorname{mod} R$ is a resolving subcategory of $\operatorname{mod} R$ contained in $\operatorname{fpd} R$. There are inclusions $\mathcal{X} \subseteq \operatorname{res}_{\mathsf{D}(R)} \mathcal{X} \cap \operatorname{mod} R \subseteq \operatorname{res}_{\mathsf{D}(R)} \mathcal{X}$, which induce the inequalities $\Phi(\mathcal{X})(\mathfrak{p}) \leqslant \Phi((\operatorname{res}_{\mathsf{D}(R)} \mathcal{X}) \cap \operatorname{mod} R)(\mathfrak{p}) \leqslant \Phi(\operatorname{res}_{\mathsf{D}(R)} \mathcal{X})(\mathfrak{p})$ for each prime ideal $\mathfrak{p}$ of R. Then Lemma 7.6 yields

$$\Phi(\operatorname{res}_{\mathsf{D}(R)} \mathcal{X})(\mathfrak{p}) = \sup_{Y \in \operatorname{res}_{\mathsf{D}(R)} \mathcal{X}} \{\operatorname{pd} Y_\mathfrak{p}\} \leqslant \sup_{X \in \mathcal{X}} \{\operatorname{pd} X_\mathfrak{p}\} = \Phi(\mathcal{X})(\mathfrak{p}),$$

and therefore $\Phi(\mathcal{X})(\mathfrak{p}) = \Phi((\operatorname{res}_{\mathsf{D}(R)} \mathcal{X}) \cap \operatorname{mod} R)(\mathfrak{p}) = \Phi(\operatorname{res}_{\mathsf{D}(R)} \mathcal{X})(\mathfrak{p})$. This shows that $\Phi(\mathcal{X})$ coincides with $\Phi((\operatorname{res}_{\mathsf{D}(R)} \mathcal{X}) \cap \operatorname{mod} R)$. By [24, Theorem 1.2], we obtain $\mathcal{X} = (\operatorname{res}_{\mathsf{D}(R)} \mathcal{X}) \cap \operatorname{mod} R$.

(2) Assume that $\mathcal{X}$ is contained in $\mathsf{CM}(R)$ and that R is a complete intersection. Proposition 7.3 implies $\mathcal{X} = (\operatorname{res}_{\mathsf{D}(R)} \mathcal{X}) \cap \mathsf{CM}(R)$. As $\mathsf{C}(R)$ is a resolving subcategory of $\mathsf{D}(R)$ by Proposition 3.14(2), it contains $\operatorname{res}_{\mathsf{D}(R)} \mathcal{X}$. We obtain $\mathcal{X} = (\operatorname{res}_{\mathsf{D}(R)} \mathcal{X}) \cap \mathsf{CM}(R) = (\operatorname{res}_{\mathsf{D}(R)} \mathcal{X}) \cap \mathsf{C}(R) \cap \operatorname{mod} R = (\operatorname{res}_{\mathsf{D}(R)} \mathcal{X}) \cap \operatorname{mod} R$.

(3) Suppose that R is a complete intersection. Put $\mathcal{Y} = (\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X}) \cap \mathsf{mod}\,R$. We want to prove $\mathcal{X} = \mathcal{Y}$. By [24, Theorem 7.4], it suffices to show $\mathcal{X} \cap \mathsf{fpd}\,R = \mathcal{Y} \cap \mathsf{fpd}\,R$ and $\mathcal{X} \cap \mathsf{CM}(R) = \mathcal{Y} \cap \mathsf{CM}(R)$. We have

$$\begin{aligned}\mathcal{Y} \cap \mathsf{fpd}\,R = (\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X}) \cap \mathsf{fpd}\,R &= (\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X}) \cap \mathsf{K}(R) \cap \mathsf{mod}\,R \\ &= \mathsf{res}_{\mathsf{D}(R)}(\mathcal{X} \cap \mathsf{fpd}\,R) \cap \mathsf{mod}\,R = \mathcal{X} \cap \mathsf{fpd}\,R,\end{aligned}$$

where the fourth equality follows by (1) and the third one by Proposition 7.2(1). Similarly, we have

$$\begin{aligned}\mathcal{Y} \cap \mathsf{CM}(R) = (\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X}) \cap \mathsf{CM}(R) &= (\mathsf{res}_{\mathsf{D}(R)}\,\mathcal{X}) \cap \mathsf{C}(R) \cap \mathsf{mod}\,R \\ &= \mathsf{res}_{\mathsf{D}(R)}(\mathcal{X} \cap \mathsf{CM}(R)) \cap \mathsf{mod}\,R = \mathcal{X} \cap \mathsf{CM}(R),\end{aligned}$$

where the fourth equality follows from (2) and the third from Proposition 7.2(2). □

Let $f : A \to B$ and $g : B \to A$ be maps. We call (f, g) a *section-retraction pair* (resp. *bijection pair*) if gf is an identity map (resp. gf, fg are identity maps). In this case, we denote it by $f \dashv g$ (resp. $f \sim g$). Now we can state and prove the following theorem, which describes a natural relationship between the resolving subcategories of $\mathsf{D}(R)$ and the resolving subcategories of $\mathsf{mod}\,R$ in the case where R is a complete intersection.

Theorem 7.8. *Let R be a complete intersection. Then there is a diagram*

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{resolving}\\ \text{subcategories}\\ \text{of } \mathsf{D}(R)\end{array}\right\} & \underset{\mathsf{res}_{\mathsf{D}(R)}(-\cup\cdots)}{\overset{((-)\cap\mathsf{K}(R),(-)\cap\mathsf{C}(R))}{\rightleftarrows}} \ (\sim) & \left\{\begin{array}{c}\text{resolving}\\ \text{subcategories}\\ \text{of } \mathsf{D}(R)\\ \text{contained in } \mathsf{K}(R)\end{array}\right\} \times \left\{\begin{array}{c}\text{resolving}\\ \text{subcategories}\\ \text{of } \mathsf{D}(R)\\ \text{contained in } \mathsf{C}(R)\end{array}\right\} \\
\mathsf{res}_{\mathsf{D}(R)}(-)\uparrow \dashv \downarrow(-)\cap\mathsf{mod}\,R & & \mathsf{res}_{\mathsf{D}(R)}(-)\times\mathsf{res}_{\mathsf{D}(R)}(-)\uparrow \dashv \downarrow((-)\cap\mathsf{mod}\,R)\times((-)\cap\mathsf{mod}\,R) \\
\left\{\begin{array}{c}\text{resolving}\\ \text{subcategories}\\ \text{of } \mathsf{mod}\,R\end{array}\right\} & \underset{\mathsf{res}_{\mathsf{mod}\,R}(-\cup\cdots)}{\overset{((-)\cap\mathsf{fpd}\,R,(-)\cap\mathsf{CM}(R))}{\rightleftarrows}} \ (\sim) & \left\{\begin{array}{c}\text{resolving}\\ \text{subcategories}\\ \text{of } \mathsf{mod}\,R\\ \text{contained in } \mathsf{fpd}\,R\end{array}\right\} \times \left\{\begin{array}{c}\text{resolving}\\ \text{subcategories}\\ \text{of } \mathsf{mod}\,R\\ \text{contained in } \mathsf{CM}(R)\end{array}\right\}
\end{array}$$

The pairs of top (*resp. bottom*) *horizontal arrows are bijection pairs given in Theorem 5.10* (*resp.* [24, Theorem 7.4]). *The pairs of vertical arrows are section-retraction pairs. The diagram with vertical arrows from the bottom* (*resp. top*) *to the top* (*resp. bottom*) *is commutative.*

Proof. It follows from Proposition 7.7 that $(\mathsf{res}_{\mathsf{D}(R)}(-), (-) \cap \mathsf{mod}\,R)$ and

$$\big(\mathsf{res}_{\mathsf{D}(R)}(-) \times \mathsf{res}_{\mathsf{D}(R)}(-),\ ((-) \cap \mathsf{mod}\,R) \times ((-) \cap \mathsf{mod}\,R)\big)$$

are section-retraction pairs of maps. Also, it follows from Proposition 7.2 that

$$((-)\cap \mathsf{K}(R),\,(-)\cap \mathsf{C}(R))\circ \mathsf{res}_{\mathsf{D}(R)}(-)$$
$$= (\mathsf{res}_{\mathsf{D}(R)}(-)\times \mathsf{res}_{\mathsf{D}(R)}(-))\circ((-)\cap \mathsf{fpd}\,R,\,(-)\cap \mathsf{CM}(R));$$

the equality

$$((-)\cap \mathsf{fpd}\,R,\,(-)\cap \mathsf{CM}(R))\circ((-)\cap \mathsf{mod}\,R)$$
$$= (((-)\cap \mathsf{mod}\,R)\times((-)\cap \mathsf{mod}\,R))\circ((-)\cap \mathsf{K}(R),\,(-)\cap \mathsf{C}(R))$$

is also straightforward to verify. □

Remark 7.9. The section-retraction pair $(\mathsf{res}_{\mathsf{D}(R)}(-),\ (-)\cap \mathsf{mod}\,R)$ in Theorem 7.8 is *never* a bijection pair. Indeed, if so, then $\mathsf{res}_{\mathsf{D}(R)}(\mathcal{X}\cap \mathsf{mod}\,R)=\mathcal{X}$ for every resolving subcategory $\mathcal{X}$ of $\mathsf{D}(R)$. However, this equality does not hold even for $\mathcal{X}=\mathsf{D}(R)$, because in this case we have the following equalities

$$\mathsf{res}_{\mathsf{D}(R)}(\mathcal{X}\cap \mathsf{mod}\,R)=\mathsf{res}_{\mathsf{D}(R)}(\mathsf{mod}\,R)=\{X\in \mathsf{D}(R)\mid \mathrm{H}^{<0}X=0\}$$

by Proposition 2.15, which is strictly contained in $\mathcal{X}=\mathsf{D}(R)$.

The corollary below is an immediate consequence of Theorem 7.8, Corollary 6.8 and [24, Theorem 1.5]. This corollary says that the classification of resolving subcategories of $\mathsf{mod}\,R$ due to Dao and Takahashi [24] is a restriction of our classification of resolving subcategories of $\mathsf{D}(R)$.

Corollary 7.10. *Let (R, V) be as in Setup 6.5. Then there is a commutative diagram*

$$\begin{array}{ccccc}
\left\{\begin{matrix}\text{resolving}\\ \text{subcategories}\\ \text{of } \mathsf{D}(R)\end{matrix}\right\} & \xleftrightarrow[(\alpha)]{\cong} & \left\{\begin{matrix}\text{order-preserving maps}\\ \text{from Spec } R \text{ to } \mathbb{N}\cup\{\infty\}\end{matrix}\right\} & \times & \left\{\begin{matrix}\text{specialization-closed}\\ \text{subsets of } V\end{matrix}\right\} \\
\Big\uparrow{\scriptstyle \mathsf{res}_{\mathsf{D}(R)}(-)} & & \Big\uparrow{\scriptstyle \mathrm{inc}\times\mathrm{id}} & & \\
\left\{\begin{matrix}\text{resolving}\\ \text{subcategories}\\ \text{of } \mathsf{mod}\,R\end{matrix}\right\} & \xleftrightarrow[(\beta)]{\cong} & \left\{\begin{matrix}\text{grade-consistent}\\ \text{functions on Spec } R\end{matrix}\right\} & \times & \left\{\begin{matrix}\text{specialization-closed}\\ \text{subsets of } V\end{matrix}\right\},
\end{array}$$

where (α) is the bijection from Corollary 6.8 and (β) the one from [24, Theorem 1.5].

Finally, we give a proof of our main result stated in the Introduction.

Proof of Theorem 1.4. The assertion follows from Corollaries 6.8 and 7.10, Proposition 7.7 and Remark 6.9(1). □

Appendix A. Classification of resolving subcategories of $\mathsf{K}(R)$ with no use of $\mathbf{D}(\mathbf{Mod}\,R)$

The purpose of this appendix is to classify the resolving subcategories of $\mathsf{K}(R)$ without using methods of unbounded derived categories; we shall give longer but more elementary proofs of Theorems 4.5 and 4.8. We will also obtain derived category versions of various results on the module category in the literature.

Definition A.1. Let R be a local ring with maximal ideal $\mathfrak{m}$.

(1) For a minimal system of generators $\boldsymbol{x}$ of $\mathfrak{m}$, we set $\mathrm{K}_R = \mathrm{K}(\boldsymbol{x}, R)$ and call it the *Koszul complex* of R.

(2) We denote by $\mathsf{D}_0(R)$ the subcategory of $\mathsf{D}(R)$ consisting of complexes X with $X_\mathfrak{p} \in \mathcal{E}_{R_\mathfrak{p}}$ (or in other words, $\mathrm{pd}_{R_\mathfrak{p}} X_\mathfrak{p} \leqslant 0$ by Proposition 2.13(6)) for all prime ideals $\mathfrak{p}$ of R such that $\mathfrak{p} \neq \mathfrak{m}$. We set

$$\mathsf{K}_0(R) = \mathsf{K}(R) \cap \mathsf{D}_0(R),$$
$$\mathsf{K}_0^n(R) = \mathsf{K}^n(R) \cap \mathsf{K}_0(R) = \mathsf{K}^n(R) \cap \mathsf{D}_0(R) \quad \text{for } n \in \mathbb{Z}.$$

Here are a couple of statements about K_R, $\mathsf{D}_0(R)$ and $\mathsf{K}_0(R)$ for a local ring R.

Proposition A.2. *Let R be a local ring. Then the following statements are true.*

(1) *Let $X \in \mathsf{D}(R)$. If $\mathrm{H}^i X$ has finite length as an R-module for all $i \in \mathbb{Z}$, then $X[i] \in \mathsf{D}_0(R)$ for all $i \in \mathbb{Z}$.*

(2) *The Koszul complex K_R of R is uniquely determined up to complex isomorphism.*

(3) *Put $e = \mathrm{edim}\, R$ and $K = \mathrm{K}_R$. One then has that $K[i] \in \mathsf{K}_0^{e+i}(R) \setminus \mathsf{K}_0^{e+i-1}(R)$ for each integer i.*

(4) *It holds that $\mathsf{D}_0(R)$ is a resolving subcategory of $\mathsf{D}(R)$. Hence $\mathsf{K}_0(R)$ is a resolving subcategory of $\mathsf{K}(R)$, and so is $\mathsf{K}_0^n(R)$ for every nonnegative integer n.*

Proof. (1) Let $\mathfrak{p}$ be a nonmaximal prime ideal of R. Let i be an integer. Then $\mathsf{H}^j((X[i])_\mathfrak{p}) = (\mathsf{H}^{j+i} X)_\mathfrak{p} = 0$ for all $j \in \mathbb{Z}$, which means that $(X[i])_\mathfrak{p} \cong 0$ in $\mathsf{D}(R_\mathfrak{p})$. Hence $(X[i])_\mathfrak{p}$ belongs to $\mathcal{E}_{R_\mathfrak{p}}$, so that $X[i] \in \mathsf{D}_0(R)$.

(2) The assertion is shown in [18, page 52].

(3) The complex $K[i]$ is in $\mathsf{K}_0(R)$ by (1). Since $\mathrm{pd}\, K = e$, we have $\mathrm{pd}\, K[i] = e + i$ by Proposition 2.13(1).

(4) The first statement is deduced by using the fact that $\mathcal{E}_{R_\mathfrak{p}}$ is a resolving subcategory of $\mathsf{D}(R_\mathfrak{p})$ for each prime ideal $\mathfrak{p}$ of R. The second statement follows from the first statement, Propositions 2.13(5), 2.10(2)(3) and the fact that the resolving property is preserved under taking intersections. □

Here is an elementary lemma on a general triangulated category, which produces a certain exact triangle.

Lemma A.3. *Let $\mathcal{T}$ be a triangulated category.*

(1) *Suppose that there exists a commutative diagram*

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \| & & \downarrow & & \downarrow & & \| \\ A & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A[1] \end{array}$$

of exact triangles in $\mathcal{T}$. *Then there exists an exact triangle* $B \to B' \oplus C \to C' \to B[1]$ *in* $\mathcal{T}$.

(2) *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be morphisms in* $\mathcal{T}$. *Then there exists an exact triangle* $\operatorname{cone}(gf) \to \operatorname{cone}(g) \oplus X[1] \to Y[1] \to \operatorname{cone}(gf)[1]$ *in* $\mathcal{T}$.

Proof. (1) follows from [40, Lemma 1.4.3].

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms in $\mathcal{T}$. Then we have a commutative diagram of exact triangles at the lower left, which induces a commutative diagram of exact triangles at the lower right.

$$\begin{array}{ccccccc} X & \xrightarrow{gf} & Z & \longrightarrow & \operatorname{cone}(gf) & \longrightarrow & X[1] \\ {\scriptstyle f}\downarrow & & \| & & \downarrow & & \downarrow \\ Y & \xrightarrow{g} & Z & \longrightarrow & \operatorname{cone}(g) & \longrightarrow & Y[1] \end{array} \qquad \begin{array}{ccccccc} Z & \longrightarrow & \operatorname{cone}(gf) & \longrightarrow & X[1] & \longrightarrow & Z[1] \\ \| & & \downarrow & & \downarrow & & \| \\ Z & \longrightarrow & \operatorname{cone}(g) & \longrightarrow & Y[1] & \longrightarrow & Z[1] \end{array}$$

By (1), we get an exact triangle $\operatorname{cone}(gf) \to \operatorname{cone}(g) \oplus X[1] \to Y[1] \to \operatorname{cone}(gf)[1]$ in $\mathcal{T}$. □

Applying the previous lemma, we consider when a given object of the derived category $\mathsf{D}(R)$ belongs to the resolving closure of the (derived) tensor product with a Koszul complex.

Lemma A.4. (1) *For elements* $x, y \in R$ *there is an exact triangle* $\mathrm{K}(x) \to \mathrm{K}(xy) \to \mathrm{K}(y) \rightsquigarrow$ *in* $\mathsf{K}(R)$.

(2) *Let* X *be an object of* $\mathsf{D}(R)$ *and let* x *be an element of* R. *Suppose that the morphism* $X \xrightarrow{x} X$ *in* $\underline{\mathsf{D}}(R)$ *defined by multiplication by* x *is zero. Then* X *belongs to* $\operatorname{res}_{\mathsf{D}(R)}(\mathrm{K}(x) \otimes_R X[-1])$.

(3) *Suppose that* $(R, \mathfrak{m})$ *is local. Let* X *be an object in* $\mathsf{D}_0(R)$. *Let* $\boldsymbol{x} = x_1, \dots, x_n$ *be a sequence of elements in* $\mathfrak{m}$. *Then* X *belongs to* $\operatorname{res}_{\mathsf{D}(R)}(\mathrm{K}(\boldsymbol{x}) \otimes_R X[-n])$. *In particular*, X *is in* $\operatorname{res}_{\mathsf{D}(R)}(\mathrm{K}_R \otimes_R X[-\operatorname{edim} R])$.

Proof. (1) The assertion is shown by applying the octahedral axiom to $(R \xrightarrow{xy} R) = (R \xrightarrow{x} R \xrightarrow{y} R)$.

(2) There exist morphisms $X \xrightarrow{f} E \xrightarrow{g} X$ in $\mathsf{D}(R)$ such that E is in $\mathcal{E}_R$ and the composition gf is equal to the multiplication morphism $X \xrightarrow{x} X$ in $\mathsf{D}(R)$. By Lemma A.3(2) we have an exact triangle $\operatorname{cone}(gf) \to \operatorname{cone}(g) \oplus X[1] \to E[1] \rightsquigarrow$ in $\mathsf{D}(R)$. The object $\operatorname{cone}(gf)$ is the mapping cone of the morphism $X \xrightarrow{x} X$,

which is isomorphic to $\mathrm{K}(x) \otimes X$. We get an exact triangle $\mathrm{K}(x) \otimes X[-1] \to \mathrm{cone}(g)[-1] \oplus X \to E \rightsquigarrow$ in $\mathsf{D}(R)$. It follows that $X \in \mathsf{ext}_{\mathsf{D}(R)}\{\mathrm{K}(x) \otimes X[-1], E\} \subseteq \mathsf{res}_{\mathsf{D}(R)}(\mathrm{K}(x) \otimes X[-1])$.

(3) By Proposition 3.10 the R-module $\mathrm{Hom}_{\underline{\mathsf{D}}(R)}(X, X)$ has finite length, and hence it is annihilated by some power $\mathfrak{m}^r$ of $\mathfrak{m}$. Fix an element $x \in \mathfrak{m}$. Then the multiplication morphism $X \xrightarrow{x^r} X$ in $\underline{\mathsf{D}}(R)$ is zero. By (2) the object X is in $\mathsf{res}_{\mathsf{D}(R)}(\mathrm{K}(x^r) \otimes X[-1])$, which is contained in $\mathsf{res}_{\mathsf{D}(R)}(\mathrm{K}(x) \otimes X[-1])$ by (1). It follows that $\mathsf{res}_{\mathsf{D}(R)} X$ is contained in $\mathsf{res}_{\mathsf{D}(R)}(\mathrm{K}(x) \otimes X[-1])$. We observe that there is a sequence of inclusions

$$\begin{aligned}\mathsf{res}_{\mathsf{D}(R)} X &\subseteq \mathsf{res}_{\mathsf{D}(R)}(\mathrm{K}(x_1) \otimes X[-1]) \subseteq \mathsf{res}_{\mathsf{D}(R)}(\mathrm{K}(x_2) \otimes \mathrm{K}(x_1) \otimes X[-2]) \\ &\subseteq \cdots \subseteq \mathsf{res}_{\mathsf{D}(R)}(\mathrm{K}(x_n) \otimes \cdots \otimes \mathrm{K}(x_1) \otimes X[-n]) \\ &= \mathsf{res}_{\mathsf{D}(R)}(\mathrm{K}(\boldsymbol{x}) \otimes X[-n]).\end{aligned}$$

Thus, X belongs to the resolving closure $\mathsf{res}_{\mathsf{D}(R)}(\mathrm{K}(\boldsymbol{x}) \otimes_R X[-n])$. □

Using the above lemma, we prove a proposition which will play a key role for the purpose of this appendix.

Proposition A.5. *Let R be a local ring. Put $e = \mathrm{edim}\, R$ and $K = \mathrm{K}_R$.*

(1) *For every integer $n \geqslant 0$ there is an equality $\mathsf{K}_0^n(R) = \mathsf{res}_{\mathsf{K}(R)}(K[n-e])$.*

(2) *Let F be an object of $\mathsf{K}(R)$, and put $t = \mathrm{pd}_R F$. Then the object $\mathrm{K}[t-e]$ belongs to $\mathsf{res}_{\mathsf{K}(R)} F$.*

Proof. (1) We have $K[n-e] \in \mathsf{K}_0^n(R)$, so that $\mathsf{res}_{\mathsf{K}(R)}(K[n-e]) \subseteq \mathsf{K}_0^n(R)$; see (3) and (4) of Proposition A.2. Conversely, pick an object $P \in \mathsf{K}_0^n(R)$. Lemma A.4(3) and Proposition 2.10(3) imply that P belongs to $\mathsf{res}_{\mathsf{K}(R)}(K \otimes P[-e])$. We may assume that the perfect complex P has the form $(0 \to P^{-n} \to P^{-n+1} \to \cdots \to P^s \to 0)$. Then it holds that $P \in \mathsf{ext}_{\mathsf{K}(R)}\{P^s[-s], \dots, P^{-n}[n]\} \subseteq \mathsf{ext}_{\mathsf{K}(R)}\{R[i] \mid -s \leqslant i \leqslant n\}$, and hence

$$(K \otimes P)[-e] \in \mathsf{ext}_{\mathsf{K}(R)}\{K[i-e] \mid -s \leqslant i \leqslant n\} \subseteq \mathsf{res}_{\mathsf{K}(R)}(K[n-e]).$$

Therefore, the complex P belongs to $\mathsf{res}_{\mathsf{K}(R)}(K[n-e])$.

(2) We shall prove that $K[i-e] \in \mathsf{res}\, F$ for all $i \leqslant t$. For this we use induction on t. When $t \leqslant 0$, for each $i \leqslant t$ we have $\mathrm{pd}\, K[i-e] = \mathrm{pd}\, K + (i-e) = i \leqslant t \leqslant 0$ by (1) and (7) of Proposition 2.13, and hence $K[i-e] \in \mathcal{E}_R \subseteq \mathsf{res}\, F$. Let $t > 0$. We may assume that $F = (0 \to F^{-t} \xrightarrow{\partial} F^{-t+1} \to \cdots \to F^s \to 0)$, where $F^{-t}, F^{-t+1}, \dots, F^s$ are free, $F^{-t} \neq 0$, $-t+1 \leqslant s$ and $\mathrm{Im}\, \partial \subseteq \mathfrak{m} F^{-t+1}$. Since $\mathrm{pd}\, F[-1] = t-1 < t$, the induction hypothesis implies $K[j-e] \in \mathsf{res}\, F[-1]$ for all $j \leqslant t-1$. As $F[-1]$ belongs to $\mathsf{res}\, F$, we see that

(A.5.1) the object $K[j-e]$ belongs to $\mathsf{res}_{\mathsf{K}(R)} F$ for all integers $j \leqslant t-1$.

It remains to show that $K[t-e]$ is in $\operatorname{res} F$. Let $P = (0 \to F^{-t} \xrightarrow{\partial} F^{-t+1} \to 0)$ be a truncation of F. Then there is an exact triangle $F^{-t} \xrightarrow{\partial} F^{-t+1} \to P[1-t] \rightsquigarrow$ in $\mathsf{K}(R)$. Tensoring K over R gives an exact triangle

$$F^{-t} \otimes K \xrightarrow{\partial \otimes K} F^{-t+1} \otimes K \to (P \otimes K)[1-t] \rightsquigarrow \tag{A.5.2}$$

in $\mathsf{K}(R)$. Write $F^{-t} = R^{\oplus n}$ and $F^{-t+1} = R^{\oplus m}$. The inclusion $\operatorname{Im} \partial \subseteq \mathfrak{m} F^{-t+1}$ implies that the map $\partial : R^{\oplus n} \to R^{\oplus m}$ is represented by an $m \times n$ matrix (a_{ij}) with $a_{ij} \in \mathfrak{m}$. The chain map $\partial \otimes K : K^{\oplus n} \to K^{\oplus m}$ is given by the matrix (a_{ij}). The multiplication morphism $K \xrightarrow{a_{ij}} K$ is zero in $\mathsf{K}(R)$ by [38, Proposition 2.3(3)], and so is the morphism $K^{\oplus n} \xrightarrow{(a_{ij})} K^{\oplus m}$; the matrix (s_{ij}) of null-homotopies s_{ij} of $K \xrightarrow{a_{ij}} K$ is a null-homotopy of $K^{\oplus n} \xrightarrow{(a_{ij})} K^{\oplus m}$. It follows from (A.5.2) that $(P \otimes K)[1-t]$ is isomorphic to the direct sum of

$$F^{-t+1} \otimes K = K^{\oplus m} \quad \text{and} \quad (F^{-t} \otimes K)[1] = (K[1])^{\oplus n}.$$

Since $F^{-t} \neq 0$, we have $n > 0$ and the complex $K[1]$ is a direct summand of $(P \otimes K)[1-t]$ as an object of $\mathsf{K}(R)$. Applying the functor $[t-e-1]$ shows that

(A.5.3) the object $K[t-e]$ is a direct summand of the object$(P \otimes K)[-e]$ in $\mathsf{K}(R)$.

Let $Q = (0 \to F^{-t+2} \to \cdots \to F^{s} \to 0)$ be another truncation of F. There is an exact triangle $Q \to F \to P \rightsquigarrow$, which induces an exact triangle $(Q \otimes K)[-e] \to (F \otimes K)[-e] \to (P \otimes K)[-e] \rightsquigarrow$. This shows that

(A.5.4) the object $(P \otimes K)[-e]$ belongs to $\operatorname{ext}_{\mathsf{K}(R)}\{(F \otimes K)[-e], (Q \otimes K)[1-e]\}$.

The Koszul complex $K = (0 \to K^{-e} \to \cdots \to K^{0} \to 0)$ is in $\operatorname{ext}\{K^{-i}[i] \mid 0 \leqslant i \leqslant e\}$, and this extension closure is contained in $\operatorname{ext}\{R[i] \mid 0 \leqslant i \leqslant e\}$. Applying $(F \otimes -)[-e]$ shows $(F \otimes K)[-e]$ is in $\operatorname{ext}\{F[i] \mid -e \leqslant i \leqslant 0\}$, which implies

(A.5.5) the object $(F \otimes K)[-e]$ belongs to $\operatorname{res}_{\mathsf{K}(R)} F$.

The perfect complex $Q = (0 \to F^{-t+2} \to \cdots \to F^{s} \to 0)$ is in $\operatorname{ext}_{\mathsf{K}(R)}\{F^{i}[-i] \mid -t+2 \leqslant i \leqslant s\}$, which is contained in $\operatorname{ext}_{\mathsf{K}(R)}\{R[-i] \mid -t+2 \leqslant i \leqslant s\}$. Hence the object $(Q \otimes K)[1-e]$ belongs to the subcategory

$$\operatorname{ext}_{\mathsf{K}(R)}\{K[1-e-i] \mid -t+2 \leqslant i \leqslant s\} = \operatorname{ext}_{\mathsf{K}(R)}\{K[i] \mid (1-s)-e \leqslant i \leqslant (t-1)-e\}$$

of $\mathsf{K}(R)$. By (A.5.1), this extension closure is contained in the resolving closure $\operatorname{res}_{\mathsf{K}(R)} F$. Therefore,

(A.5.6) the object $(Q \otimes K)[1-e]$ belongs to $\operatorname{res}_{\mathsf{K}(R)} F$.

It follows from (A.5.3), (A.5.4), (A.5.5) and (A.5.6) that $K[t-e]$ is in $\operatorname{res}_{\mathsf{K}(R)} F$ as desired. $\square$

Remark A.6. We may wonder if the Hopkins–Neeman classification theorem [31; 39] can be applied to deduce Proposition A.5(2). Actually, the proof of the Hopkins–Neeman theorem provides a certain integer m such that $K[m]$ belongs to $\mathsf{res}_{\mathsf{K}(R)} F$. However, this is done by applying the smash nilpotence theorem [39, Theorem 1.1], which relies on the fact that R is noetherian, so that m cannot be described concretely. Note that there is no meaning for us unless $m > -e$, since we know that $K[i] \in \mathcal{E}_R \subseteq \mathsf{res}_{\mathsf{K}(R)} F$ for all $i \leqslant -e$.

The following two corollaries are direct consequences of the above proposition.

Corollary A.7. *Suppose that R is a local ring.*

(1) *There is an equality* $\mathsf{K}_0(R) = \mathsf{res}_{\mathsf{K}(R)}\{\mathrm{K}_R[i] \mid i \in \mathbb{Z}\}$ *of subcategories of* $\mathsf{K}(R)$.

(2) *Let F be an object in $\mathsf{K}_0(R)$ and assume* $\mathrm{pd}_R F = t \geqslant 0$. *Then the equality* $\mathsf{res}_{\mathsf{K}(R)} F = \mathsf{K}_0^t(R)$ *holds.*

Proof. Put $e = \mathrm{edim}\, R$, $K = \mathrm{K}_R$ and set $\mathcal{X} = \mathsf{res}\{K[i] \mid i \in \mathbb{Z}\}$. Proposition A.2(3) implies that $\mathcal{X} \subseteq \mathsf{K}_0(R)$. Fix $F \in \mathsf{K}_0(R)$ and set $t = \mathrm{pd}\, F$. If $t \leqslant 0$, then $F \in \mathcal{E}_R \subseteq \mathcal{X}$ by Proposition 2.13(6). Let $t \geqslant 0$. We have

$$F \in \mathsf{K}_0^t(R) = \mathsf{res}_{\mathsf{K}(R)} K[t-e] \subseteq \mathcal{X} \cap \mathsf{res}_{\mathsf{K}(R)} F \subseteq \mathsf{res}_{\mathsf{K}(R)} F \subseteq \mathsf{K}_0^t(R),$$

where the equality and the first inclusion follow from Proposition A.5, and the other inclusions are obvious. We thus obtain the equalities $\mathcal{X} = \mathsf{K}_0(R)$ and $\mathsf{K}_0^t(R) = \mathsf{res}_{\mathsf{K}(R)} F$. □

Corollary A.8. *Let R be a local ring. Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{K}(R)$ contained in $\mathsf{K}_0(R)$. Suppose that one has* $\sup_{X \in \mathcal{X}}\{\mathrm{pd}_R X\} = \infty$. *Then the equality* $\mathcal{X} = \mathsf{K}_0(R)$ *holds true.*

Proof. Assume that $\mathcal{X}$ is strictly contained in $\mathsf{K}_0(R)$. Then there exists an object $Y \in \mathsf{K}_0(R)$ such that $Y \notin \mathcal{X}$. Put $u = \mathrm{pd}_R Y$. As Y is a nonzero object of $\mathsf{K}(R)$, we have that $-\infty < u < \infty$. Since $\sup_{X \in \mathcal{X}}\{\mathrm{pd}\, X\} = \infty$, there exists an object $X \in \mathcal{X}$ such that $t := \mathrm{pd}\, X \geqslant \max\{u, 0\}$. Then $X \in \mathsf{K}_0(R)$, $\mathrm{pd}\, X = t \geqslant 0$ and $u \leqslant t$. Applying Corollary A.7(2), we observe $Y \in \mathsf{K}_0^u(R) \subseteq \mathsf{K}_0^t(R) = \mathsf{res}_{\mathsf{K}(R)} X \subseteq \mathcal{X}$. This gives a contradiction. □

Now we can show the following theorem. It provides an explicit description of the resolving subcategories of $\mathsf{K}(R)$ contained in $\mathsf{K}_0(R)$; in particular, they form a totally ordered set.

Theorem A.9. *Suppose that R is a local ring with $e = \mathrm{edim}\, R$ and $K = \mathrm{K}_R$. Then one has strict inclusions*

$$\text{(A.9.1)} \qquad \mathcal{E}_R = \mathsf{K}_0^0(R) \subsetneq \mathsf{K}_0^1(R) \subsetneq \cdots \subsetneq \mathsf{K}_0^n(R) \subsetneq \mathsf{K}_0^{n+1}(R) \subsetneq \cdots \subsetneq \mathsf{K}_0(R)$$

of resolving subcategories of $\mathsf{K}(R)$ *such that* $K[n-e] \in \mathsf{K}_0^n(R) \setminus \mathsf{K}_0^{n-1}(R)$ *for each* $n \geqslant 1$. *Moreover, all the resolving subcategories of* $\mathsf{K}(R)$ *contained in* $\mathsf{K}_0(R)$ *appear in the above chain of subcategories of* $\mathsf{K}(R)$.

Proof. Proposition 2.13(6) says $\mathcal{E}_R = \mathsf{K}^0(R) \supseteq \mathsf{K}_0^0(R)$. Since $\mathsf{K}_0^0(R)$ is resolving by Proposition A.2(4) and $\mathcal{E}_R$ is the minimum resolving subcategory, the equality $\mathcal{E}_R = \mathsf{K}_0^0(R)$ holds. For each $n \geqslant 1$ it is clear that $\mathsf{K}_0^{n-1}(R) \subseteq \mathsf{K}_0^n(R)$, while $K[n-e] \in \mathsf{K}_0^n(R) \setminus \mathsf{K}_0^{n-1}(R)$ by Proposition A.2(3). The first assertion now follows.

Now, let us show the second assertion. Let $\mathcal{X}$ be a resolving subcategory of $\mathsf{K}(R)$ contained in $\mathsf{K}_0(R)$. We may assume that $\mathcal{X}$ is different from $\mathsf{K}_0(R)$. Corollary A.8 says that $t := \sup_{X \in \mathcal{X}}\{\operatorname{pd} X\}$ is finite, and in particular, $\mathcal{X}$ is contained in $\mathsf{K}_0^t(R)$. Choose an object $X \in \mathcal{X}$ such that $\operatorname{pd} X = t$. We have $t \geqslant 0$ as R is in $\mathcal{X}$. Using Corollary A.7(2), we see that $\mathsf{K}_0^t(R) = \operatorname{res}_{\mathsf{K}(R)} X \subseteq \mathcal{X}$. The equality $\mathcal{X} = \mathsf{K}_0^t(R)$ follows. □

From now on, we consider classifying all the resolving subcategories of $\mathsf{K}(R)$. We start by defining, for each object of the derived category $\mathsf{D}(R)$, another object by tensoring a Koszul complex and taking a shift.

Definition A.10. Let $X \in \mathsf{D}(R)$. For $x \in R$, set $X(x) = \mathrm{K}(x) \otimes_R^{\mathbf{L}} X[-1] \in \mathsf{D}(R)$. For $\boldsymbol{x} = x_1, \dots, x_n \subseteq R$, we inductively define $X(\boldsymbol{x}) \in \mathsf{D}(R)$ by $X(x_1, \dots, x_i) = (X(x_1 \dots, x_{i-1}))(x_i)$ for each $1 \leqslant i \leqslant n$.

We make a list of basic properties of the object $X(x)$ for $X \in \mathsf{D}(R)$ and $x \in R$.

Lemma A.11. *Let X be an object of* $\mathsf{D}(R)$, *and let x be an element of R.*

(1) *If x is a unit of R, then there is an isomorphism $X(x) \cong 0$ in* $\mathsf{D}(R)$.

(2) *There exists an exact triangle* $X(x) \to X \xrightarrow{x} X \rightsquigarrow$ *in* $\mathsf{D}(R)$. *In particular, one has the containment* $X(x) \in \operatorname{res}_{\mathsf{D}(R)} X$ *and the isomorphisms* $X(x) \cong \mathbf{R}\mathrm{Hom}_R(\mathrm{K}(x), X) \cong \mathrm{Hom}_R(\mathrm{K}(x), X)$ *in* $\mathsf{D}(R)$.

(3) *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Let $x \in \mathfrak{m}$. Then one has the equalities* $\operatorname{depth}_R X(x) = \operatorname{depth}_R X$ *and* $\operatorname{pd}_R X(x) = \operatorname{pd}_R X$. *In particular,* $X \in \mathcal{E}_R$ *if and only if* $X(x) \in \mathcal{E}_R$.

(4) *The equality* $\mathrm{NE}(X(x)) = \mathrm{NE}(X) \cap \mathrm{V}(x)$ *holds, where* $\mathrm{V}(x)$ *denotes the set of prime ideals containing x.*

Proof. (1) If x is a unit of R, then $\mathrm{K}(x) \cong 0$ in $\mathsf{D}(R)$, and hence $X(x) = \mathrm{K}(x) \otimes_R^{\mathbf{L}} X[-1] \cong 0$ in $\mathsf{D}(R)$.

(2) There exists an exact triangle $e : R \xrightarrow{x} R \to \mathrm{K}(x) \rightsquigarrow$ in $\mathsf{D}(R)$. Applying the functor $- \otimes_R^{\mathbf{L}} X$ to e gives rise to an exact triangle $X \xrightarrow{x} X \to \mathrm{K}(x) \otimes_R^{\mathbf{L}} X \rightsquigarrow$ in $\mathsf{D}(R)$, which induces an exact triangle $a : X(x) \to X \xrightarrow{x} X \rightsquigarrow$ in $\mathsf{D}(R)$. Hence $X(x)$ belongs to $\operatorname{res}_{\mathsf{D}(R)} X$. Applying the functor $\mathbf{R}\mathrm{Hom}_R(-, X)$ to e yields an

exact triangle $b: \mathbf{R}\mathrm{Hom}_R(\mathrm{K}(x), X) \to X \xrightarrow{x} X \rightsquigarrow$. It follows from a and b that $X(x) \cong \mathbf{R}\mathrm{Hom}_R(\mathrm{K}(x), X)$.

(3) As x belongs to $\mathfrak{m}$, we have

$$\mathrm{K}(x) \otimes_R^{\mathbf{L}} k \cong (0 \to k \xrightarrow{x} k \to 0) = (0 \to k \xrightarrow{0} k \to 0) \cong k \oplus k[1].$$

Hence

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(k, X(x)) &\cong \mathbf{R}\mathrm{Hom}_R(k, \mathbf{R}\mathrm{Hom}_R(\mathrm{K}(x), X)) \cong \mathbf{R}\mathrm{Hom}_R(\mathrm{K}(x) \otimes_R^{\mathbf{L}} k, X)\\ &\cong \mathbf{R}\mathrm{Hom}_R(k \oplus k[1], X) \cong \mathbf{R}\mathrm{Hom}_R(k, X) \oplus \mathbf{R}\mathrm{Hom}_R(k, X)[-1]\end{aligned}$$

by (2), and

$$\begin{aligned}X(x) \otimes_R^{\mathbf{L}} k &\cong \mathrm{K}(x) \otimes_R^{\mathbf{L}} X[-1] \otimes_R^{\mathbf{L}} k \cong (\mathrm{K}(x) \otimes_R^{\mathbf{L}} k) \otimes_R^{\mathbf{L}} X[-1]\\ &\cong (k \oplus k[1]) \otimes_R^{\mathbf{L}} X[-1] \cong (X \otimes_R^{\mathbf{L}} k) \oplus (X \otimes_R^{\mathbf{L}} k)[-1].\end{aligned}$$

As $\inf Y[-1] = \inf Y + 1$ for any complex Y, we get

$$\operatorname{depth} X(x) = \inf \mathbf{R}\mathrm{Hom}_R(k, X(x)) = \inf \mathbf{R}\mathrm{Hom}_R(k, X) = \operatorname{depth} X$$

and, from Proposition 2.13(2), $\operatorname{pd} X(x) = -\inf(X(x) \otimes_R^{\mathbf{L}} k) = -\inf(X \otimes_R^{\mathbf{L}} k) = \operatorname{pd} X$. By virtue of Proposition 2.13(6), we have $X \in \mathcal{E}_R$ if and only if $X(x) \in \mathcal{E}_R$.

(4) To show ($\supseteq$), let $\mathfrak{p} \in \mathrm{NE}(X) \cap \mathrm{V}(x)$. Then $\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p} > 0$ and $\frac{x}{1} \in \mathfrak{p}R_\mathfrak{p}$. By (3) we have $\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p}\left(\frac{x}{1}\right) = \operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p} > 0$, whence $\mathfrak{p} \in \mathrm{NE}(X(x))$. To show ($\subseteq$), let $\mathfrak{p} \in \mathrm{NE}(X(x))$. Then $\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p}\left(\frac{x}{1}\right) > 0$. In particular, $X_\mathfrak{p}\left(\frac{x}{1}\right) \ncong 0$ in $\mathsf{D}(R)$. Hence $x \in \mathfrak{p}$ by (1). By (3) we get $0 < \operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p}\left(\frac{x}{1}\right) = \operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p}$, so that $\mathfrak{p} \in \mathrm{NE}(X)$. □

Assertions (1), (3) of the theorem below are viewed as derived category versions of [49, Theorem 4.3] and [23, Lemma 4.6] respectively, which concern the nonfree locus and the resolving closure of an object in $\mathsf{mod}\, R$.

Theorem A.12. *Let X be an object of $\mathsf{D}(R)$. Let W be a closed subset of* $\operatorname{Spec} R$ *contained in* $\mathrm{NE}(X)$.

(1) *There exists an object $Y \in \mathsf{res}_{\mathsf{D}(R)} X$ such that $W = \mathrm{NE}(Y)$.*

(2) *If R is local and W is nonempty, then Y can be chosen so that* $\operatorname{pd} Y = \operatorname{pd} X$ *and* $\operatorname{depth} Y = \operatorname{depth} X$.

(3) *If W is irreducible, then Y can be chosen so that* $\operatorname{pd} Y_\mathfrak{p} = \operatorname{pd} X_\mathfrak{p}$ *and* $\operatorname{depth} Y_\mathfrak{p} = \operatorname{depth} X_\mathfrak{p}$ *for all* $\mathfrak{p} \in W$.

Proof. When W is empty, we can take $Y := R$. Assume that $W \neq \varnothing$. Then there exist prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ of R such that $n > 0$ and $W = \mathrm{V}(\mathfrak{p}_1) \cup \cdots \cup \mathrm{V}(\mathfrak{p}_n)$. Each $\mathrm{V}(\mathfrak{p}_i)$ is contained in $\mathrm{NE}(X)$. If we find an object $Y_i \in \mathsf{res}_{\mathsf{D}(R)} X$ such that $\mathrm{V}(\mathfrak{p}_i) = \mathrm{NE}(Y_i)$, then $Y := Y_1 \oplus \cdots \oplus Y_n$ belongs to $\mathsf{res}\, X$ and satisfies $W = \mathrm{NE}(Y)$ by Lemma 3.8(2). If R is local, $\operatorname{pd}_R Y_i = \operatorname{pd}_R X$ and $\operatorname{depth}_R Y_i =$

$\operatorname{depth}_R X$ for all $1 \leqslant i \leqslant n$, then $\operatorname{pd}_R Y = \sup_{1\leqslant i\leqslant n}\{\operatorname{pd}_R Y_i\} = \operatorname{pd}_R X$ and $\operatorname{depth}_R Y = \inf_{1\leqslant i\leqslant n}\{\operatorname{depth}_R Y_i\} = \operatorname{depth}_R X$ by Proposition 2.13(4). Thus, it suffices to show that in the case where $W = \mathrm{V}(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ of R there exists $Y \in \operatorname{res}_{\mathsf{D}(R)} X$ such that $W = \mathrm{NE}(Y)$, $\operatorname{pd}_{R_\mathfrak{r}} Y_\mathfrak{r} = \operatorname{pd}_{R_\mathfrak{r}} X_\mathfrak{r}$ and $\operatorname{depth}_{R_\mathfrak{r}} Y_\mathfrak{r} = \operatorname{depth}_{R_\mathfrak{r}} X_\mathfrak{r}$ for all $\mathfrak{r} \in W$.

The set $\mathrm{NE}(X)$ contains $\mathrm{V}(\mathfrak{p})$. If $\mathrm{NE}(X) = \mathrm{V}(\mathfrak{p})$, then we are done by letting $Y = X$. Suppose that $\mathrm{NE}(X)$ strictly contains $\mathrm{V}(\mathfrak{p})$ and choose an element $\mathfrak{q} \in \mathrm{NE}(X) \setminus \mathrm{V}(\mathfrak{p})$. Then $\mathfrak{q}$ does not contain $\mathfrak{p}$, and we can choose an element $x \in \mathfrak{p} \setminus \mathfrak{q}$. Using Lemma A.11(4), we get $\mathfrak{p} \in \mathrm{NE}(X) \cap \mathrm{V}(x) = \mathrm{NE}(X(x))$. As $\mathrm{NE}(X(x))$ is Zariski-closed by Proposition 3.10, it contains $\mathrm{V}(\mathfrak{p})$. It follows that $\mathrm{V}(\mathfrak{p}) \subseteq \mathrm{NE}(X(x)) = \mathrm{NE}(X) \cap \mathrm{V}(\mathfrak{p}) \subseteq \mathrm{NE}(X)$ and the fact that $\mathfrak{q} \in \mathrm{NE}(X) \setminus \mathrm{V}(\mathfrak{p})$ says $\mathrm{NE}(X) \cap \mathrm{V}(\mathfrak{p}) \neq \mathrm{NE}(X)$. We conclude $\mathrm{V}(\mathfrak{p}) \subseteq \mathrm{NE}(X(x)) \subsetneq \mathrm{NE}(X)$.

By Lemma A.11(2)(3), we have that $X(x) \in \operatorname{res} X$, and that $\operatorname{pd}_{R_\mathfrak{r}} X(x)_\mathfrak{r} = \operatorname{pd}_{R_\mathfrak{r}} X_\mathfrak{r}\big(\tfrac{x}{1}\big) = \operatorname{pd}_{R_\mathfrak{r}} X_\mathfrak{r}$ and $\operatorname{depth}_{R_\mathfrak{r}} X(x)_\mathfrak{r} = \operatorname{depth}_{R_\mathfrak{r}} X_\mathfrak{r}\big(\tfrac{x}{1}\big) = \operatorname{depth}_{R_\mathfrak{r}} X_\mathfrak{r}$ for all $\mathfrak{r} \in \mathrm{V}(\mathfrak{p})$ since $x \in \mathfrak{p} \subseteq \mathfrak{r}$. If $\mathrm{NE}(X(x))$ is equal to $\mathrm{V}(\mathfrak{p})$, we are done by letting $Y = X(x)$. If $\mathrm{NE}(X(x))$ strictly contains $\mathrm{V}(\mathfrak{p})$, we apply the above argument to find $y \in \mathfrak{p}$ with $\mathrm{V}(\mathfrak{p}) \subseteq \mathrm{NE}(X(x, y)) \subsetneq \mathrm{NE}(X(x))$. Iterating this procedure, we get an ascending chain

$$\mathrm{V}(\mathfrak{p}) \subseteq \cdots \subsetneq \mathrm{NE}(x, y, z, w) \subsetneq \mathrm{NE}(x, y, z) \subsetneq \mathrm{NE}(X(x, y)) \subsetneq \mathrm{NE}(X(x)) \subsetneq \mathrm{NE}(X)$$

of subsets of $\operatorname{Spec} R$ with $x, y, z, w, \ldots \in \mathfrak{p}$. However, we can do this only finitely many times, since each NE-locus appearing in the above chain is Zariski-closed, and the topological space $\operatorname{Spec} R$ is noetherian.

We thus obtain a sequence $\boldsymbol{x} = x_1, \ldots, x_n$ in $\mathfrak{p}$ with $\mathrm{NE}(X(\boldsymbol{x})) = \mathrm{V}(\mathfrak{p})$ and $X(\boldsymbol{x}) \in \operatorname{res} X$, $\operatorname{pd}_{R_\mathfrak{r}} X(\boldsymbol{x})_\mathfrak{r} = \operatorname{pd}_{R_\mathfrak{r}} X_\mathfrak{r}$ and $\operatorname{depth}_{R_\mathfrak{r}} X(\boldsymbol{x})_\mathfrak{r} = \operatorname{depth}_{R_\mathfrak{r}} X_\mathfrak{r}$ for all $\mathfrak{r} \in \mathrm{V}(\mathfrak{p})$. The theorem follows by letting $Y = X(\boldsymbol{x})$. □

From the above theorem we can deduce the following corollary. Thanks to this result, for each object X in a fixed resolving subcategory of $\mathsf{D}(R)$, one may often assume that X belongs to $\mathsf{D}_0(R)$.

Corollary A.13. *Let R be a local ring. For every object $X \in \mathsf{D}(R)$ there exists an object $Y \in \operatorname{res}_{\mathsf{D}(R)} X \cap \mathsf{D}_0(R)$ such that* $\operatorname{pd}_R Y = \operatorname{pd}_R X$ *and* $\operatorname{depth}_R Y = \operatorname{depth}_R X$.

Proof. When X belongs to $\mathcal{E}_R$, we put $Y := X$ and are done. Let X be outside of $\mathcal{E}_R$. Then the maximal ideal $\mathfrak{m}$ of R belongs to $\mathrm{NE}(X)$ by Lemma 3.8(1) and Proposition 3.10, and hence $\mathrm{V}(\mathfrak{m})$ is contained in $\mathrm{NE}(X)$. Applying Theorem A.12 to $\mathrm{V}(\mathfrak{m})$, we find an object $Y \in \operatorname{res}_{\mathsf{D}(R)} X$ such that $\mathrm{NE}(Y) = \mathrm{V}(\mathfrak{m}) = \{\mathfrak{m}\}$, $\operatorname{pd} Y = \operatorname{pd} X$ and $\operatorname{depth} Y = \operatorname{depth} X$. The equality $\mathrm{NE}(Y) = \{\mathfrak{m}\}$ implies that Y belongs $\mathsf{D}_0(R)$. □

The following lemma can be thought of as a derived category version of [24, Lemma 3.2 and Proposition 3.3]. For a partially ordered set P we denote by $\min P$ the set of minimal elements of P.

Lemma A.14. *Let $\mathcal{X}$ be a subcategory of $\mathsf{D}(R)$.*

(1) *Let S be a multiplicatively closed subset of R. Suppose that $\mathcal{X}$ is a resolving subcategory of $\mathsf{D}(R)$. Then $\mathsf{add}_{\mathsf{D}(R_S)} \mathcal{X}_S$ is a resolving subcategory of $\mathsf{D}(R_S)$. Hence, the equality $\mathsf{add}_{\mathsf{D}(R_S)} \mathcal{X}_S = \mathsf{res}_{\mathsf{D}(R_S)} \mathcal{X}_S$ holds.*

(2) *Suppose that $\mathcal{X}$ contains R and is closed under finite direct sums. Let Z be a nonempty finite subset of* $\operatorname{Spec} R$. *Let $C \in \mathsf{D}(R)$ be such that $C_\mathfrak{p} \in \mathsf{add}_{\mathsf{D}(R_\mathfrak{p})} \mathcal{X}_\mathfrak{p}$ for all $\mathfrak{p} \in Z$. Then there exist exact triangles*

$$K \to X \to C \to K[1], \qquad L \to K \oplus C \to X \to L[1]$$

in $\mathsf{D}(R)$ such that $X \in \mathcal{X}$, that $\mathrm{NE}(L) \subseteq \mathrm{NE}(C)$, *that* $\mathrm{Supp}(L) \cap Z = \varnothing$, *and that* $\mathrm{pd}_{R_\mathfrak{p}} L_\mathfrak{p} \leqslant \mathrm{pd}_{R_\mathfrak{p}} C_\mathfrak{p}$ *and* $\mathrm{depth}_{R_\mathfrak{p}} L_\mathfrak{p} \geqslant \mathrm{depth}_{R_\mathfrak{p}} C_\mathfrak{p}$ *for all prime ideals $\mathfrak{p}$ of R.*

(3) *Assume that $\mathcal{X}$ is a resolving subcategory of $\mathsf{D}(R)$. The following are equivalent for each $C \in \mathsf{D}(R)$.*

(a) *The object C belongs to $\mathcal{X}$.*

(b) *The localization $C_\mathfrak{p}$ belongs to $\mathsf{add}_{\mathsf{D}(R_\mathfrak{p})} \mathcal{X}_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of R.*

(c) *The localization $C_\mathfrak{m}$ belongs to $\mathsf{add}_{\mathsf{D}(R_\mathfrak{m})} \mathcal{X}_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of R.*

Proof. (1) By definition, the additive closure $\mathsf{add}\,\mathcal{X}_S$ is closed under direct summands. As R is in $\mathcal{X}$, we have $R_S \in \mathcal{X}_S \subseteq \mathsf{add}\,\mathcal{X}_S$. Let $A \in \mathsf{add}\,\mathcal{X}_S$. Then $A \oplus B$ is isomorphic to X_S for some $B \in \mathsf{D}(R_S)$ and $X \in \mathcal{X}$, whence $A[-1] \oplus B[-1]$ is isomorphic to $(X[-1])_S$. As $\mathcal{X}$ is closed under negative shifts, we see that $A[-1]$ is in $\mathsf{add}\,\mathcal{X}_S$. Therefore, $\mathsf{add}\,\mathcal{X}_S$ is closed under negative shifts. Let $L \to M \to N \rightsquigarrow$ be an exact triangle in $\mathsf{D}(R_S)$ with $L, N \in \mathsf{add}\,\mathcal{X}_S$. Then $L \oplus L' \cong X_S$ and $N \oplus N' \cong Y_S$ for some $L', N' \in \mathsf{D}(R_S)$ and $X, Y \in \mathcal{X}$. Taking the direct sum with the exact triangles $L' \to L' \to 0 \rightsquigarrow$ and $0 \to N' \to N' \rightsquigarrow$, we observe that there exists an exact triangle $L' \oplus M \oplus N' \to Y_S \xrightarrow{f} X[1]_S \rightsquigarrow$ in $\mathsf{D}(R_S)$. Write $f = \frac{g}{s}$, where $g : Y \to X[1]$ is a morphism in $\mathsf{D}(R)$ and s is an element of S; see [9, Lemma 5.2(b)]. There is an exact triangle $X \to Z \to Y \xrightarrow{g} X[1]$ in $\mathsf{D}(R)$. Since $\mathcal{X}$ is closed under extensions, Z is in $\mathcal{X}$. Also, we see that Z_S is isomorphic to $L' \oplus M \oplus N'$ in $\mathsf{D}(R_S)$. It follows that M belongs to $\mathsf{add}\,\mathcal{X}_S$, which shows that $\mathsf{add}\,\mathcal{X}_S$ is closed under extensions.

(2) Write $Z = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Fix $1 \leqslant i \leqslant n$. There exists $X_i \in \mathcal{X}$ such that $C_{\mathfrak{p}_i}$ is a direct summand of $(X_i)_{\mathfrak{p}_i}$ in $\mathsf{D}(R_{\mathfrak{p}_i})$. We have a split epimorphism $f_i : (X_i)_{\mathfrak{p}_i} \to C_{\mathfrak{p}_i}$ in $\mathsf{D}(R_{\mathfrak{p}_i})$, so that there is a morphism $\alpha_i : C_{\mathfrak{p}_i} \to (X_i)_{\mathfrak{p}_i}$ in $\mathsf{D}(R_{\mathfrak{p}_i})$ with $f_i \alpha_i = \mathrm{id}_{C_{\mathfrak{p}_i}}$. Choose a morphism $g_i : X_i \to C$ in $\mathsf{D}(R)$ and an element $s_i \in R \setminus \mathfrak{p}_i$ such that $\frac{g_i}{s_i} = f_i$. Set $X = X_1 \oplus \cdots \oplus X_n \in \mathcal{X}$ and consider the morphism $g = (g_1, \dots, g_n) : X \to C$

in $\mathsf{D}(R)$. Then $\frac{g}{1} = \left(\frac{g_1}{1}, \dots, \frac{g_n}{1}\right) : X_{\mathfrak{p}_i} \to C_{\mathfrak{p}_i}$ is a split epimorphism in $\mathsf{D}(R_{\mathfrak{p}_i})$ for each i, since letting $\beta_i : C_{\mathfrak{p}_i} \to X_{\mathfrak{p}_i}$ be the transpose of $(0, \dots, 0, \frac{1}{s_i}\alpha_i, 0, \dots, 0)$, we have $\frac{g}{1}\beta_i = \mathrm{id}_{C_{\mathfrak{p}_i}}$. There is an exact triangle $K \to X \xrightarrow{g} C \xrightarrow{h} K[1]$ in $\mathsf{D}(R)$. For any integer $1 \leqslant i \leqslant n$ it holds that $h_{\mathfrak{p}_i} = \frac{h}{1} = 0$ in $\mathsf{D}(R_{\mathfrak{p}_i})$, which means that the annihilator $\mathrm{ann}_R\, h$ of $h \in \mathrm{Hom}_{\mathsf{D}(R)}(C, K[1])$ is not contained in $\mathfrak{p}_i$. By prime avoidance, we find an element $s \in \mathrm{ann}_R\, h$ such that $s \notin \mathfrak{p}$ for all $\mathfrak{p} \in Z$. The octahedral axiom gives rise to a commutative diagram

$$\begin{array}{ccccccc}
C & \xrightarrow{s} & C & \longrightarrow & \mathsf{K}(s) \otimes^{\mathbf{L}}_R C & \longrightarrow & C[1] \\
\| & & \downarrow h & & \downarrow & & \| \\
C & \xrightarrow[0]{hs} & K[1] & \longrightarrow & K[1] \oplus C[1] & \longrightarrow & C[1] \\
\downarrow & & \| & & \downarrow & & \downarrow \\
C & \xrightarrow{h} & K[1] & \longrightarrow & X[1] & \xrightarrow{g[1]} & C[1] \\
\downarrow & & \downarrow & & \| & & \downarrow \\
\mathsf{K}(s) \otimes^{\mathbf{L}}_R C & \longrightarrow & K[1] \oplus C[1] & \longrightarrow & X[1] & \longrightarrow & (\mathsf{K}(s) \otimes^{\mathbf{L}}_R C)[1]
\end{array}$$

in $\mathsf{D}(R)$ whose rows are exact triangles. The bottom row in the diagram induces an exact triangle $L \to K \oplus C \to X \to L[1]$ in $\mathsf{D}(R)$, where we put $L := C(s) = \mathsf{K}(s) \otimes^{\mathbf{L}}_R C[-1]$.

Let $\mathfrak{p} \in Z$. Then the element $\frac{s}{1}$ of $R_{\mathfrak{p}}$ is a unit, and hence it holds in $\mathsf{D}(R_{\mathfrak{p}})$ that $\mathsf{K}(s)_{\mathfrak{p}} = \mathsf{K}\left(\frac{s}{1}, R_{\mathfrak{p}}\right) \cong 0$. Thus, $L_{\mathfrak{p}} = \mathsf{K}(s)_{\mathfrak{p}} \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} C_{\mathfrak{p}}[-1] \cong 0$ in $\mathsf{D}(R_{\mathfrak{p}})$. It follows that the intersection $\mathrm{Supp}(L) \cap Z$ is the empty set.

Fix a prime ideal $\mathfrak{p}$ of R. Note that we have $L_{\mathfrak{p}} = C(s)_{\mathfrak{p}} = C_{\mathfrak{p}}\left(\frac{s}{1}\right)$. If s is in $\mathfrak{p}$, then $\mathrm{pd}\, C_{\mathfrak{p}}\left(\frac{s}{1}\right) = \mathrm{pd}\, C_{\mathfrak{p}}$ and $\mathrm{depth}\, C_{\mathfrak{p}}\left(\frac{s}{1}\right) = \mathrm{depth}\, C_{\mathfrak{p}}$ by Lemma A.11(3). If s is not in $\mathfrak{p}$, then $C_{\mathfrak{p}}\left(\frac{s}{1}\right) = 0$ by Lemma A.11(1), so that $\mathrm{pd}\, C_{\mathfrak{p}}\left(\frac{s}{1}\right) = -\infty$ and $\mathrm{depth}\, C_{\mathfrak{p}}\left(\frac{s}{1}\right) = \infty$. Thus, there are inequalities $\mathrm{pd}\, L_{\mathfrak{p}} \leqslant \mathrm{pd}\, C_{\mathfrak{p}}$ and $\mathrm{depth}\, L_{\mathfrak{p}} \geqslant \mathrm{depth}\, C_{\mathfrak{p}}$. If $\mathfrak{p}$ is not in $\mathrm{NE}(C)$, then $C_{\mathfrak{p}}$ is in $\mathcal{E}_{R_{\mathfrak{p}}}$, and so is $L_{\mathfrak{p}}$. Hence $\mathrm{NE}(L)$ is contained in $\mathrm{NE}(C)$.

(3) Localization shows the implications (a) $\Rightarrow$ (c) $\Rightarrow$ (b). Assume that (b) holds and $C \notin \mathcal{X}$. Then the set

$$A = \{\mathrm{NE}(Y) \mid Y \in \mathsf{D}(R),\ Y \notin \mathcal{X} \text{ and } Y_{\mathfrak{p}} \in \mathsf{add}_{\mathsf{D}(R_{\mathfrak{p}})} \mathcal{X}_{\mathfrak{p}} \text{ for all prime ideals } \mathfrak{p} \text{ of } R\}$$

is nonempty. Since $\mathrm{Spec}\, R$ is a noetherian space and each $\mathrm{NE}(Y)$ is Zariski-closed by Proposition 3.10, the set A contains a minimal element $\mathrm{NE}(B)$ with $B \in \mathsf{D}(R)$, $B \notin \mathcal{X}$ and $B_{\mathfrak{p}} \in \mathsf{add}\, \mathcal{X}_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ of R. If $\mathrm{NE}(B)$ is the empty set, then we have $B \in \mathcal{E}_R \subseteq \mathcal{X}$ by Lemma 3.8(1), which gives a contradiction. Thus $\mathrm{NE}(B)$ is a nonempty Zariski-closed set, which implies that $\min \mathrm{NE}(B)$ is nonempty and finite. It follows from (2) that there exist exact triangles $K \to X \to B \rightsquigarrow$ and $L \to K \oplus B \to X \rightsquigarrow$ in $\mathsf{D}(R)$ such that $X \in \mathcal{X}$, $\mathrm{NE}(L) \subseteq \mathrm{NE}(B)$ and $\mathrm{NE}(L) \cap \min \mathrm{NE}(B) = \varnothing$. In particular, $\mathrm{NE}(L)$ is strictly contained in $\mathrm{NE}(B)$.

We claim that $L_\mathfrak{p}$ is in $\operatorname{add}\mathcal{X}_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} R$. In fact, there is an exact triangle $K_\mathfrak{p} \to X_\mathfrak{p} \to B_\mathfrak{p} \rightsquigarrow$. It follows from (1) that $\operatorname{add}\mathcal{X}_\mathfrak{p}$ is a resolving subcategory of $\mathsf{D}(R_\mathfrak{p})$. Since $X_\mathfrak{p}$ and $B_\mathfrak{p}$ belong to $\operatorname{add}\mathcal{X}_\mathfrak{p}$, so does $K_\mathfrak{p}$. There is an exact triangle $L_\mathfrak{p} \to K_\mathfrak{p} \oplus B_\mathfrak{p} \to X_\mathfrak{p} \rightsquigarrow$. As $K_\mathfrak{p} \oplus B_\mathfrak{p}$ and $X_\mathfrak{p}$ are in $\operatorname{add}\mathcal{X}_\mathfrak{p}$, so is $L_\mathfrak{p}$.

Now the minimality of $\mathrm{NE}(B)$ forces L to be in $\mathcal{X}$. The exact triangle $L \to K \oplus B \to X \rightsquigarrow$ implies that B belongs to $\mathcal{X}$, which contradicts the choice of B. We thus conclude that the object C belongs to $\mathcal{X}$. □

Remark A.15. In Lemma A.14(2), the object L is taken in such a way that $\operatorname{Supp}(L) \cap Z = \varnothing$. Comparing this with the module category version of Lemma A.14(2) given in [24, Lemma 3.2], we see that the expected condition satisfied by L in Lemma A.14(2) is the weaker condition that $\mathrm{NE}(L) \cap Z = \varnothing$. It is an advantage the derived category possesses against the module category that one can get L so that $\operatorname{Supp}(L) \cap Z = \varnothing$. By the way, only for the purpose of this appendix, it suffices to have the equality $\mathrm{NE}(L) \cap Z = \varnothing$.

Now, we can provide proofs of Theorems 4.5, 4.8 that do not use methods of unbounded derived categories. We first prove Theorem 4.8, and then prove Theorem 4.5 by applying Theorem 4.8.

Alternative proof of Theorem 4.8. Fix a resolving subcategory $\mathcal{X}$ of $\mathsf{K}(R)$ and an order-preserving map $f : \operatorname{Spec} R \to \mathbb{N} \cup \{\infty\}$. Lemma 4.7 implies that $\Phi(\mathcal{X}) : \operatorname{Spec} R \to \mathbb{N} \cup \{\infty\}$ is an order-preserving map and $\Psi(f)$ is a resolving subcategory of $\mathsf{D}(R)$. Hence $\Psi(f) \cap \mathsf{K}(R)$ is a resolving subcategory of $\mathsf{K}(R)$. Fix a prime ideal $\mathfrak{p}$ of R. It is clear that

$$\begin{aligned}&\Phi(\Psi(f) \cap \mathsf{K}(R))(\mathfrak{p})\\&\qquad = \sup\{\operatorname{pd}_{R_\mathfrak{p}} P_\mathfrak{p} \mid P \in \mathsf{K}(R) \text{ and } \operatorname{pd}_{R_\mathfrak{q}} P_\mathfrak{q} \leqslant f(\mathfrak{q}) \text{ for all } \mathfrak{q} \in \operatorname{Spec} R\} \leqslant f(\mathfrak{p}).\end{aligned}$$

Let $\boldsymbol{x} = x_1, \dots, x_s$ be a system of generators of $\mathfrak{p}$, and let $\mathfrak{q}$ be a prime ideal of R. First, we consider the case where $f(\mathfrak{p}) =: n < \infty$. Set $P = \mathrm{K}(\boldsymbol{x})[n-s] \in \mathsf{K}(R)$. We have $\operatorname{pd}_{R_\mathfrak{p}} P_\mathfrak{p} = n = f(\mathfrak{p})$ by Proposition 2.13(1)(7). If $\mathfrak{p}$ is contained in $\mathfrak{q}$, then $\operatorname{pd}_{R_\mathfrak{q}} P_\mathfrak{q} = \operatorname{pd}_{R_\mathfrak{q}} \mathrm{K}(\boldsymbol{x}, R_\mathfrak{q}) + (n-s) \leqslant s + (n-s) = n = f(\mathfrak{p}) \leqslant f(\mathfrak{q})$. If $\mathfrak{p}$ is not contained in $\mathfrak{q}$, then $\operatorname{pd}_{R_\mathfrak{q}} P_\mathfrak{q} = -\infty \leqslant f(\mathfrak{q})$ by Proposition 2.13(7). Thus $\Phi(\Psi(f) \cap \mathsf{K}(R))(\mathfrak{p}) = f(\mathfrak{p})$. Next, we consider the case where $f(\mathfrak{p}) = \infty$. Then for any integer n we set $P = \mathrm{K}(\boldsymbol{x})[n-s]$ to have $\operatorname{pd}_{R_\mathfrak{p}} P_\mathfrak{p} = n$. If $\mathfrak{p}$ is contained in $\mathfrak{q}$, then $\infty = f(\mathfrak{p}) \leqslant f(\mathfrak{q})$, so that $\operatorname{pd}_{R_\mathfrak{q}} P_\mathfrak{q} \leqslant \infty = f(\mathfrak{q})$. If $\mathfrak{p}$ is not contained in $\mathfrak{q}$, then $\operatorname{pd}_{R_\mathfrak{q}} P_\mathfrak{q} = -\infty \leqslant f(\mathfrak{q})$. We get $\Phi(\Psi(f) \cap \mathsf{K}(R))(\mathfrak{p}) = \infty = f(\mathfrak{p})$. It now follows that $\Phi(\Psi(f) \cap \mathsf{K}(R)) = f$.

It remains to prove that $\Psi(\Phi(\mathcal{X})) \cap \mathsf{K}(R) = \mathcal{X}$. Note the equality and inclusion

$$\begin{aligned}\Psi(\Phi(\mathcal{X})) \cap \mathsf{K}(R) &= \{P \in \mathsf{K}(R) \mid \operatorname{pd}_{R_\mathfrak{p}} P_\mathfrak{p} \leqslant \sup_{X \in \mathcal{X}}\{\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p}\} \text{ for all } \mathfrak{p} \in \operatorname{Spec} R\}\\&\supseteq \mathcal{X}.\end{aligned}$$

Let $P \in \Psi(\Phi(\mathcal{X})) \cap \mathsf{K}(R)$. All we need to do is show that P is in $\mathcal{X}$. Fix a maximal ideal $\mathfrak{m}$ of R and a prime ideal $\mathfrak{p}$ of R contained in $\mathfrak{m}$. Then $\mathsf{add}\,\mathcal{X}_\mathfrak{m}$ is a resolving subcategory of $\mathsf{K}(R_\mathfrak{m})$ by Lemma A.14(1). We have

$$\begin{aligned} \text{(A.15.1)} \qquad \operatorname{pd}_{(R_\mathfrak{m})_{\mathfrak{p}R_\mathfrak{m}}} (P_\mathfrak{m})_{\mathfrak{p}R_\mathfrak{m}} &= \operatorname{pd}_{R_\mathfrak{p}} P_\mathfrak{p} \leqslant \sup_{X\in\mathcal{X}}\{\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p}\} \\ &= \sup_{X\in\mathcal{X}}\{\operatorname{pd}_{(R_\mathfrak{m})_{\mathfrak{p}R_\mathfrak{m}}} (X_\mathfrak{m})_{\mathfrak{p}R_\mathfrak{m}}\} \\ &\leqslant \sup_{Y\in\mathsf{add}\,\mathcal{X}_\mathfrak{m}}\{\operatorname{pd}_{(R_\mathfrak{m})_{\mathfrak{p}R_\mathfrak{m}}} Y_{\mathfrak{p}R_\mathfrak{m}}\}. \end{aligned}$$

By Lemma A.14(3), it suffices to show $P_\mathfrak{m} \in \mathsf{add}\,\mathcal{X}_\mathfrak{m}$. Thus we may assume that $(R,\mathfrak{m})$ is local. Then the dimension $n := \dim \mathrm{NE}(P)$ of the Zariski-closed set $\mathrm{NE}(P)$ is finite. We prove by induction on n that $P \in \mathcal{X}$.

When $n \leqslant 0$, the set $\mathrm{NE}(P)$ is contained in $\{\mathfrak{m}\}$, which means that P belongs to $\mathsf{K}_0(R)$. By the choice of P, we have $\operatorname{pd}_R P \leqslant \sup_{X\in\mathcal{X}}\{\operatorname{pd}_R X\}$, which implies that there is an object $X \in \mathcal{X}$ such that $\operatorname{pd}_R P \leqslant \operatorname{pd}_R X$. Corollary A.13 gives rise to an object $X' \in \mathsf{D}_0(R) \cap \mathsf{res}_{\mathsf{D}(R)} X$ with $\operatorname{pd}_R X' = \operatorname{pd}_R X$. Hence $X' \in \mathcal{X} \cap \mathsf{K}_0(R)$ and $\operatorname{pd}_R P \leqslant \operatorname{pd}_R X'$. Replacing X with X', we may assume that $X \in \mathsf{K}_0(R)$. Setting $t = \operatorname{pd}_R X$, we have the inequality $\operatorname{pd}_R P \leqslant t$, so that $P \in \mathsf{K}_0^t(R) = \mathsf{res}_{\mathsf{K}(R)} X \subseteq \mathcal{X}$, where the equality holds by Theorem A.9.

Now we consider the case $n > 0$. Then $\min \mathrm{NE}(P)$ is nonempty and finite. Write $\min \mathrm{NE}(P) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$ and fix an integer $1 \leqslant i \leqslant r$. We have $\mathrm{NE}(P_{\mathfrak{p}_i}) = \{\mathfrak{p}_i R_{\mathfrak{p}_i}\}$, whence $\dim \mathrm{NE}(P_{\mathfrak{p}_i}) = 0$. The subcategory $\mathsf{add}\,\mathcal{X}_{\mathfrak{p}_i}$ of $\mathsf{K}(R_{\mathfrak{p}_i})$ is resolving by Lemma A.14(1). Similarly as in (A.15.1), the inequality

$$\operatorname{pd}_{(R_{\mathfrak{p}_i})_{\mathfrak{q}R_{\mathfrak{p}_i}}} (X_{\mathfrak{p}_i})_{\mathfrak{q}R_{\mathfrak{p}_i}} \leqslant \sup_{Y\in\mathsf{add}\,\mathcal{X}_{\mathfrak{p}_i}} \{\operatorname{pd}_{(R_{\mathfrak{p}_i})_{\mathfrak{q}R_{\mathfrak{p}_i}}} Y_{\mathfrak{q}R_{\mathfrak{p}_i}}\}$$

holds for every prime ideal $\mathfrak{q}$ of R contained in $\mathfrak{p}_i$. The induction basis implies that $P_{\mathfrak{p}_i}$ belongs to $\mathsf{add}\,\mathcal{X}_{\mathfrak{p}_i}$. Lemma A.14(2) yields exact triangles $K \to Z \to P \rightsquigarrow$ and $L \to K \oplus P \to Z \rightsquigarrow$ in $\mathsf{D}(R)$ such that $Z \in \mathcal{X}$, $\mathrm{NE}(L) \subseteq \mathrm{NE}(P)$, $\mathrm{NE}(L) \cap \min \mathrm{NE}(P) = \varnothing$, and $\operatorname{pd}_{R_\mathfrak{p}} L_\mathfrak{p} \leqslant \operatorname{pd}_{R_\mathfrak{p}} P_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R. Since Z and P are in $\mathsf{K}(R)$, so is K, and so is L. We have $\dim \mathrm{NE}(L) < \dim \mathrm{NE}(P) = n$, while

$$\operatorname{pd}_{R_\mathfrak{p}} L_\mathfrak{p} \leqslant \operatorname{pd}_{R_\mathfrak{p}} P_\mathfrak{p} \leqslant \sup_{X\in\mathcal{X}}\{\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p}\}$$

for all $\mathfrak{p} \in \operatorname{Spec} R$. Hence we can apply the induction hypothesis to L to deduce that L is in $\mathcal{X}$. The exact triangle $L \to K \oplus P \to Z \rightsquigarrow$ shows that P is in $\mathcal{X}$. □

Alternative proof of Theorem 4.5. The resolving subcategories of $\mathsf{K}(R)$ are exactly the precoaisles of $\mathsf{K}(R)$ containing R and closed under direct summands. By Lemma 4.2 and Theorem 4.8 the maps (Q, E) are mutually inverse bijections. (The proof of (P, F) being mutually inverse bijections was given as the first paragraph of the original proof of Theorem 4.5. This is elementary, not using methods of unbounded derived categories.) □

Appendix B. Restricting the classification of resolving subcategories of $\mathsf{K}(R)$

In this appendix, we compare our results with the results of Dao and Takahashi concerning resolving subcategories of $\mathsf{mod}\, R$ contained in $\mathsf{fpd}\, R$. For this purpose, we begin with recalling some notation.

Definition B.1. Let R be a local ring. We set $\mathsf{mod}_0(R) = \mathsf{mod}\, R \cap \mathsf{D}_0(R)$. Note that $\mathsf{mod}_0(R)$ consists of the finitely generated R-modules which are locally free on the punctured spectrum of R. We also put

$$\mathsf{fpd}_0(R) = \mathsf{fpd}\, R \cap \mathsf{mod}_0(R) = \mathsf{K}_0(R) \cap \mathsf{mod}\, R = \{M \in \mathsf{mod}_0(R) \mid \mathrm{pd}_R M < \infty\},$$
$$\mathsf{fpd}_0^n(R) = \mathsf{K}_0^n(R) \cap \mathsf{mod}\, R = \mathsf{K}^n(R) \cap \mathsf{mod}_0(R) = \{M \in \mathsf{fpd}_0(R) \mid \mathrm{pd}_R M \leqslant n\} \quad \text{for } n \in \mathbb{Z}.$$

Denote by Ω and Tr the syzygy and transpose functors.

Theorem B.2 (Dao–Takahashi [24, Theorem 2.1]). *Let R be a local ring of depth t and with residue field k. Then*

$$\text{(B.2.1)} \quad \mathsf{proj}\, R = \mathsf{fpd}_0^0(R) \subsetneq \mathsf{fpd}_0^1(R) \subsetneq \cdots \subsetneq \mathsf{fpd}_0^t(R) = \mathsf{fpd}_0^{t+1}(R) = \cdots = \mathsf{fpd}_0(R)$$

such that $\mathrm{Tr}\,\Omega^{n-1}k \in \mathsf{fpd}_0^n(R) \setminus \mathsf{fpd}_0^{n-1}(R)$ *for each* $t \geqslant n \geqslant 1$. *Moreover, all the resolving subcategories of* $\mathsf{mod}\, R$ *contained in* $\mathsf{fpd}_0(R)$ *appear in the above chain.*

Remark B.3. (1) Theorem A.9 can be viewed as a derived category version of Theorem B.2.

(2) A remarkable difference between Theorems A.9 and B.2 is that the latter says that there exist only finitely many resolving subcategories of $\mathsf{mod}\, R$ contained in $\mathsf{fpd}_0(R)$, while the former says that there exist infinitely (but countably) many resolving subcategories of $\mathsf{K}(R)$ contained in $\mathsf{K}_0(R)$.

(3) Although both have similar configurations, the proof of Theorem A.9 is completely different from that of Theorem B.2. Indeed, the latter requires much more complicated arguments on modules which involve syzygies and transposes; the whole of [24, §2] is devoted to giving a proof of Theorem B.2.

(4) The restriction of (A.9.1) to $\mathsf{mod}\, R$ coincides with (B.2.1). Indeed, Proposition 2.13(6) says that $\mathcal{E}_R \cap \mathsf{mod}\, R = \mathsf{proj}\, R$, while by definition we have $\mathsf{K}_0^n(R) \cap \mathsf{mod}\, R = \mathsf{fpd}_0^n(R)$ for each integer n. The Auslander–Buchsbaum formula [18, Theorem 1.3.3] shows $\mathsf{fpd}_0^n(R) = \mathsf{fpd}_0(R)$ for all integers $n \geqslant t$.

In the proof of Theorem B.2, the resolving closure $\mathsf{res}_{\mathsf{mod}\, R}\, k$ in $\mathsf{mod}\, R$ of the residue field k of R does play a crucial role; it coincides with $\mathsf{mod}_0(R)$. Here we consider a derived category version of this fact.

Proposition B.4. *Let R be a local ring with residue field k. Let $X \in \mathsf{D}_0(R)$ and put $h = \operatorname{depth} X$. One then has $X \in \mathsf{res}_{\mathsf{D}(R)}(k[-h])$. In particular, it holds that $\mathsf{D}_0(R) = \mathsf{res}_{\mathsf{D}(R)}\{k[i] \mid i \in \mathbb{Z}\}$.*

Proof. Take a system of parameters $\boldsymbol{x} = x_1, \ldots, x_d$ of R. Set $Y = \mathrm{K}(\boldsymbol{x}) \otimes_R^{\mathbf{L}} X \in \mathsf{D}(R)$. It follows from Lemma A.4(3) that $X \in \mathsf{res}_{\mathsf{D}(R)}(Y[-d])$. Taking soft truncations of the complex Y implies that Y is in the extension closure $\mathsf{ext}_{\mathsf{D}(R)}\{\mathrm{H}^i Y[-i] \mid \inf Y \leqslant i \leqslant \sup Y\}$. Localization at nonmaximal prime ideals shows that each $\mathrm{H}^i Y$ has finite length as an R-module (see Proposition 2.13(7)), so that it is in $\mathsf{ext}_{\mathsf{D}(R)}\, k$. We have $Y \in \mathsf{ext}_{\mathsf{D}(R)}\{k[-i] \mid \inf Y \leqslant i \leqslant \sup Y\} \subseteq \mathsf{res}_{\mathsf{D}(R)}(k[-\inf Y])$, where the inclusion comes from the fact that every resolving subcategory is closed under negative shifts. Using [28, Theorem I], we get $\inf Y = h - d$, which implies $Y \in \mathsf{res}_{\mathsf{D}(R)}(k[d-h])$. Therefore, the object X belongs to $\mathsf{res}_{\mathsf{D}(R)}(k[-h])$ by Proposition 2.8(2a). □

Definition B.5 (grade-consistent functions [24]). For an ideal I of R we denote by $\operatorname{grade} I$ the *grade* of I, that is to say, the infimum of integers $i \geqslant 0$ such that $\operatorname{Ext}_R^i(R/I, R) \neq 0$. A *grade-consistent function* on $\operatorname{Spec} R$ is by definition an order-preserving map $f : \operatorname{Spec} R \to \mathbb{N}$ such that the inequality $f(\mathfrak{p}) \leqslant \operatorname{grade} \mathfrak{p}$ holds for all prime ideals $\mathfrak{p}$ of R.

The grade condition in the definition of a grade-consistent function can be changed to a depth condition.

Lemma B.6. *Let $f : \operatorname{Spec} R \to \mathbb{N} \cup \{\infty\}$ be an order-preserving map. Then f is a grade-consistent function on $\operatorname{Spec} R$ if and only if $f(\mathfrak{p}) \leqslant \operatorname{depth} R_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of R.*

Proof. Fix $\mathfrak{p} \in \operatorname{Spec} R$. The equality $\operatorname{grade} \mathfrak{p} = \inf\{\operatorname{depth} R_\mathfrak{q} \mid \mathfrak{q} \in \mathrm{V}(\mathfrak{p})\}$ holds by [18, Proposition 1.2.10(a)]. In particular, one has $\operatorname{grade} \mathfrak{p} \leqslant \operatorname{depth} R_\mathfrak{p}$, which shows the 'only if' part of the lemma. To show the 'if' part, suppose $f(\mathfrak{q}) \leqslant \operatorname{depth} R_\mathfrak{q}$ for all $\mathfrak{q} \in \operatorname{Spec} R$. Then the image of f is contained in $\mathbb{N}$. If $\mathfrak{q} \in \mathrm{V}(\mathfrak{p})$, then $\mathfrak{p} \subseteq \mathfrak{q}$ and $f(\mathfrak{p}) \leqslant f(\mathfrak{q}) \leqslant \operatorname{depth} R_\mathfrak{q}$. This shows $f(\mathfrak{p}) \leqslant \inf\{\operatorname{depth} R_\mathfrak{q} \mid \mathfrak{q} \in \mathrm{V}(\mathfrak{p})\} = \operatorname{grade} \mathfrak{p}$. Thus, we are done. □

Applying the above lemma, we can show the following result on the assignments used in Theorem 4.8.

Proposition B.7. *Let Φ and Ψ be the ones introduced in Definition 4.6.*

(1) *Let $\mathcal{X}$ be a subcategory of $\mathsf{mod}\, R$ containing R. Then $\Phi(\mathcal{X})$ is a grade-consistent function on $\operatorname{Spec} R$.*

(2) *Let $f : \operatorname{Spec} R \to \mathbb{N}$ be a map. Then the equality $\Psi(f) \cap \mathsf{mod}\, R = (\Psi(f) \cap \mathsf{K}(R)) \cap \mathsf{mod}\, R$ holds, and it is a resolving subcategory of $\mathsf{mod}\, R$ contained in $\mathsf{fpd}\, R$.*

Proof. (1) Lemma 4.7(1) implies that $\Phi(\mathcal{X})$ is an order-preserving map from Spec R to $\mathbb{N} \cup \{\infty\}$. For each $\mathfrak{p} \in \operatorname{Spec} R$ we have $\Phi(\mathcal{X})(\mathfrak{p}) = \sup_{X \in \mathcal{X}}\{\operatorname{pd} X_\mathfrak{p}\} \leqslant \operatorname{depth} R_\mathfrak{p}$ since the Auslander–Buchsbaum formula implies $\operatorname{pd} X_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p} - \operatorname{depth} X_\mathfrak{p} \leqslant \operatorname{depth} R_\mathfrak{p}$. Lemma B.6 shows $\Phi(\mathcal{X})$ is a grade-consistent function on Spec R.

(2) According to Lemma 4.7(2), the subcategory $\Psi(f)$ of $\mathsf{D}(R)$ is resolving. It follows from Proposition 2.10(4) that $\Psi(f) \cap \operatorname{mod} R$ is a resolving subcategory of $\operatorname{mod} R$, and $(\Psi(f) \cap \mathsf{K}(R)) \cap \operatorname{mod} R = \Psi(f) \cap \operatorname{fpd} R = (\Psi(f) \cap \operatorname{mod} R) \cap \operatorname{fpd} R$ is a resolving subcategory of $\operatorname{mod} R$ contained in $\operatorname{fpd} R$. Let $M \in \Psi(f) \cap \operatorname{mod} R$. Then for every prime ideal $\mathfrak{p}$ of R one has $\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant f(\mathfrak{p}) \in \mathbb{N}$, which particularly says that $\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$. By [13, Lemma 4.5], we get $\operatorname{pd}_R M < \infty$. Therefore, $\Psi(f) \cap \operatorname{mod} R$ coincides with $(\Psi(f) \cap \operatorname{mod} R) \cap \operatorname{fpd} R$. □

The following theorem is one of the main results (Theorem 1.2) of [24].

Theorem B.8 (Dao–Takahashi). *The assignments $\mathcal{X} \mapsto \Phi(\mathcal{X})$ and $f \mapsto \Psi(f) \cap \operatorname{mod} R$ define a bijective correspondence between resolving subcategories of* $\operatorname{mod} R$ *contained in* $\operatorname{fpd} R$ *and grade-consistent functions on* Spec R.

Remark B.9. Proposition B.7 says that Theorem B.8 is viewed as the restriction of Theorem 4.8 to $\operatorname{mod} R$.

Acknowlegments

The author thanks Hiroki Matsui, Tsutomu Nakamura and the referees for valuable comments. In particular, the second paragraph of the proof of Theorem 4.5 given in Section 4 (not the one in Appendix A) is due to two of the referees and Nakamura.

References

[1] T. Aihara and R. Takahashi, "Generators and dimensions of derived categories of modules", *Comm. Algebra* **43**:11 (2015), 5003–5029. MR

[2] L. Alonso Tarrío, A. Jeremías López, and M. Saorín, "Compactly generated t-structures on the derived category of a Noetherian ring", *J. Algebra* **324**:3 (2010), 313–346. MR

[3] L. Angeleri Hügel and M. Saorín, "t-structures and cotilting modules over commutative noetherian rings", *Math. Z.* **277**:3-4 (2014), 847–866. MR

[4] L. Angeleri Hügel, D. Pospíšil, J. Šťovíček, and J. Trlifaj, "Tilting, cotilting, and spectra of commutative Noetherian rings", *Trans. Amer. Math. Soc.* **366**:7 (2014), 3487–3517. MR

[5] B. Antieau, D. Gepner, and J. Heller, "K-theoretic obstructions to bounded t-structures", *Invent. Math.* **216**:1 (2019), 241–300. MR

[6] M. Auslander and M. Bridger, *Stable module theory*, Memoirs of the Amer. Math. Soc. **94**, Amer. Math. Soc., Providence, RI, 1969. MR

[7] M. Auslander and R.-O. Buchweitz, "The homological theory of maximal Cohen–Macaulay approximations", pp. 5–37 in *Colloque en l'honneur de Pierre Samuel* (Orsay, 1987), Mém. Soc. Math. France (N.S.) **38**, 1989. MR

[8] L. L. Avramov, "Infinite free resolutions", pp. 1–118 in *Six lectures on commutative algebra* (Bellaterra, 1996), Progr. Math. **166**, Birkhäuser, Basel, 1998. MR

[9] L. L. Avramov and H.-B. Foxby, "Homological dimensions of unbounded complexes", *J. Pure Appl. Algebra* **71**:2-3 (1991), 129–155. MR

[10] L. L. Avramov, S. B. Iyengar, and J. Lipman, "Reflexivity and rigidity for complexes, I: Commutative rings", *Algebra Number Theory* **4**:1 (2010), 47–86. MR

[11] P. Balmer and B. Sanders, "The spectrum of the equivariant stable homotopy category of a finite group", *Invent. Math.* **208**:1 (2017), 283–326. MR

[12] Y. Bao, X. Du, and Z. Zhao, "Gorenstein singularity categories", *J. Algebra* **428** (2015), 122–137. MR

[13] H. Bass and M. P. Murthy, "Grothendieck groups and Picard groups of abelian group rings", *Ann. of Math.* (2) **86** (1967), 16–73. MR

[14] A. A. Beilinson, J. Bernstein, and P. Deligne, "Faisceaux pervers", pp. 5–171 in *Analysis and topology on singular spaces*, *I* (Luminy, 1981), Astérisque **100**, Soc. Math. France, Paris, 1982. MR

[15] D. J. Benson, J. F. Carlson, and J. Rickard, "Thick subcategories of the stable module category", *Fund. Math.* **153**:1 (1997), 59–80. MR

[16] D. J. Benson, S. B. Iyengar, and H. Krause, "Stratifying modular representations of finite groups", *Ann. of Math.* (2) **174**:3 (2011), 1643–1684. MR

[17] D. Benson, S. B. Iyengar, H. Krause, and J. Pevtsova, "Stratification for module categories of finite group schemes", *J. Amer. Math. Soc.* **31**:1 (2018), 265–302. MR

[18] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1993. Revised 1998. MR

[19] J. Burke and M. E. Walker, "Matrix factorizations in higher codimension", *Trans. Amer. Math. Soc.* **367**:5 (2015), 3323–3370. MR

[20] K. Česnavičius, "Macaulayfication of Noetherian schemes", *Duke Math. J.* **170**:7 (2021), 1419–1455. MR

[21] L. W. Christensen, *Gorenstein dimensions*, Lecture Notes in Mathematics **1747**, Springer, 2000. MR

[22] L. W. Christensen, H.-B. Foxby, and A. Frankild, "Restricted homological dimensions and Cohen–Macaulayness", *J. Algebra* **251**:1 (2002), 479–502. MR

[23] H. Dao and R. Takahashi, "The radius of a subcategory of modules", *Algebra Number Theory* **8**:1 (2014), 141–172. MR

[24] H. Dao and R. Takahashi, "Classification of resolving subcategories and grade consistent functions", *Int. Math. Res. Not.* **2015**:1 (2015), 119–149. MR

[25] H. Dao and R. Takahashi, "The dimension of a subcategory of modules", *Forum Math. Sigma* **3** (2015), art. id. e19, 31 pp. MR

[26] E. S. Devinatz, M. J. Hopkins, and J. H. Smith, "Nilpotence and stable homotopy theory, I", *Ann. of Math.* (2) **128**:2 (1988), 207–241. MR

[27] U. V. Dubey and G. Sahoo, "Compactly generated tensor t-structures on the derived categories of Noetherian schemes", *Math. Z.* **303**:4 (2023), art. id. 100, 22 pp. MR

[28] H.-B. Foxby and S. Iyengar, "Depth and amplitude for unbounded complexes", pp. 119–137 in *Commutative algebra* (Grenoble/Lyon, 2001), Contemp. Math. **331**, Amer. Math. Soc., Providence, RI, 2003. MR

[29] E. M. Friedlander and J. Pevtsova, "Π-supports for modules for finite group schemes", *Duke Math. J.* **139**:2 (2007), 317–368. MR

[30] R. Henrard, S. Kvamme, and A.-C. van Roosmalen, "Auslander's formula and correspondence for exact categories", *Adv. Math.* **401** (2022), art. id. 108296, 65 pp. MR

[31] M. J. Hopkins, "Global methods in homotopy theory", pp. 73–96 in *Homotopy theory* (Durham, 1985), London Math. Soc. Lecture Note Ser. **117**, Cambridge Univ. Press, Cambridge, 1987. MR

[32] M. J. Hopkins and J. H. Smith, "Nilpotence and stable homotopy theory, II", *Ann. of Math.* (2) **148**:1 (1998), 1–49. MR

[33] B. Huisgen-Zimmermann, "Dualities from iterated tilting", *Israel J. Math.* **243**:1 (2021), 315–353. MR

[34] B. Keller and P. Nicolás, "Weight structures and simple dg modules for positive dg algebras", *Int. Math. Res. Not.* **2013**:5 (2013), 1028–1078. MR

[35] B. Keller and D. Vossieck, "Aisles in derived categories", *Bull. Soc. Math. Belg. Sér. A* **40**:2 (1988), 239–253. MR

[36] H. Krause and O. Solberg, "Applications of cotorsion pairs", *J. London Math. Soc.* (2) **68**:3 (2003), 631–650. MR

[37] H. Matsui and R. Takahashi, "Singularity categories and singular equivalences for resolving subcategories", *Math. Z.* **285**:1-2 (2017), 251–286. MR

[38] H. Matsui and R. Takahashi, "Thick tensor ideals of right bounded derived categories", *Algebra Number Theory* **11**:7 (2017), 1677–1738. MR

[39] A. Neeman, "The chromatic tower for $D(R)$", *Topology* **31**:3 (1992), 519–532. MR

[40] A. Neeman, *Triangulated categories*, Annals of Mathematics Studies **148**, Princeton University Press, Princeton, NJ, 2001. MR

[41] A. Neeman, "Strong generators in $\mathbf{D}^{\mathrm{perf}}(X)$ and $\mathbf{D}^{b}_{\mathrm{coh}}(X)$", *Ann. of Math.* (2) **193**:3 (2021), 689–732. MR

[42] A. Neeman, "Bounded t-structures on the category of perfect complexes", *Acta Math.* **233**:2 (2024), 239–284. MR

[43] S. Sather-Wagstaff, T. Sharif, and D. White, "Comparison of relative cohomology theories with respect to semidualizing modules", *Math. Z.* **264**:3 (2010), 571–600. MR

[44] H. Smith, "Bounded t-structures on the category of perfect complexes over a Noetherian ring of finite Krull dimension", *Adv. Math.* **399** (2022), art. id. 108241, 21 pp. MR

[45] G. Stevenson, "Duality for bounded derived categories of complete intersections", *Bull. Lond. Math. Soc.* **46**:2 (2014), 245–257. MR

[46] G. Stevenson, "Subcategories of singularity categories via tensor actions", *Compos. Math.* **150**:2 (2014), 229–272. MR

[47] J. Šťovíček and D. Pospíšil, "On compactly generated torsion pairs and the classification of co-t-structures for commutative noetherian rings", *Trans. Amer. Math. Soc.* **368**:9 (2016), 6325–6361. MR

[48] R. Takahashi, "Remarks on modules approximated by G-projective modules", *J. Algebra* **301**:2 (2006), 748–780. MR

[49] R. Takahashi, "Modules in resolving subcategories which are free on the punctured spectrum", *Pacific J. Math.* **241**:2 (2009), 347–367. MR

[50] R. Takahashi, "Classifying thick subcategories of the stable category of Cohen–Macaulay modules", *Adv. Math.* **225**:4 (2010), 2076–2116. MR

[51] R. Takahashi, "Contravariantly finite resolving subcategories over commutative rings", *Amer. J. Math.* **133**:2 (2011), 417–436. MR

[52] R. Takahashi, "Classifying resolving subcategories over a Cohen–Macaulay local ring", *Math. Z.* **273**:1-2 (2013), 569–587. MR

[53] R. Takahashi, "Thick subcategories over Gorenstein local rings that are locally hypersurfaces on the punctured spectra", *J. Math. Soc. Japan* **65**:2 (2013), 357–374. MR

[54] R. Takahashi, "Dominant local rings and subcategory classification", *Int. Math. Res. Not.* **2023**:9 (2023), 7259–7318. MR

[55] R. Takahashi, "Faltings' annihilator theorem and t-structures of derived categories", *Math. Z.* **304**:1 (2023), art. id. 10, 13 pp. MR

[56] R. W. Thomason, "The classification of triangulated subcategories", *Compositio Math.* **105**:1 (1997), 1–27. MR

[57] J.-L. Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque **239**, 1996. MR

Received August 9, 2024. Revised September 4, 2025.

RYO TAKAHASHI
GRADUATE SCHOOL OF MATHEMATICS
NAGOYA UNIVERSITY
FUROCHO, CHIKUSAKU
NAGOYA 464-8602
JAPAN

takahashi@math.nagoya-u.ac.jp

Guidelines for Authors

Authors may submit articles at msp.org/pjm/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to journals+pjm@msp.org or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

At submission time a PDF file is required, and it must be derived from a LaTeX source file in which the command `\documentclass[12pt]{amsart}` is active. No changes to horizontal or vertical margins, or line spacing, should be made. Please include in the preamble the command `\usepackage[pagewise]{lineno}\linenumbers` — this will spare referees tedious effort in line counting.

Carefully preserve all related files, such as LaTeX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to journals+pjm@msp.org.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text ("the curve looks like this:"). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as "Place Figure 1 here". The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a website in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

PACIFIC JOURNAL OF MATHEMATICS

Volume 339 No. 2 December 2025
