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For every $n \geq 1$, the flat braid group FB_n is an analogue of the braid group B_n that can be described as the fundamental group of the configuration space $\{\{x_1, \dots, x_n\} \in \mathbb{R}^n/\mathrm{Sym}(n) \mid$ there exist at most two indices i, j such that $x_i = x_j\}$. Alternatively, FB_n can be described as the right-angled Coxeter group $C(P_{n-2}^{\mathrm{opp}})$, where P_{n-2}^{opp} denotes the opposite graph of the path P_{n-2} of length $n-2$. We prove that, for every $n = 7$ or ≥ 11, PFB_n is not virtually a right-angled Artin group, disproving a conjecture of Naik, Nanda, and Singh. In the opposite direction, we observe that FB_7 turns out to be commensurable to the right-angled Artin group $A(P_4)$.

1. Introduction

Recall that, given a graph Γ, the corresponding *right-angled Artin group* is

$$A(\Gamma) := \langle u \text{ vertex of } \Gamma \mid [u, v] = 1 \text{ if } u \text{ and } v \text{ are adjacent in } \Gamma \rangle,$$

and that the corresponding *right-angled Coxeter group* is

$$C(\Gamma) := \langle u \text{ vertex of } \Gamma \mid u^2 = 1 \text{ for every } u, [u, v] = 1 \text{ if } u \text{ and } v \text{ are adjacent in } \Gamma \rangle.$$

It is well-known that right-angled Artin and Coxeter groups are tightly connected. Most notably, every right-angled Artin group is isomorphic to a finite-index subgroup of some right-angled Coxeter group [Davis and Januszkiewicz 2000]. Conversely, despite the fact that every right-angled Coxeter group can be virtually described

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as a subgroup of some right-angled Artin group, right-angled Coxeter groups may not be commensurable to right-angled Artin groups. (Recall that two groups are (*abstractly*) *commensurable* whenever they contain isomorphic finite-index subgroups.) A simple example is given by $C(C_n)$, where C_n is a cycle of length $n \geq 5$. Indeed, $C(C_n)$ can be described as the reflection group associated to a right-angled n-gon in the hyperbolic plane $\mathbb{H}^2$, so $C(C_n)$ is virtually the fundamental group of a closed surface of genus ≥ 2. However, a right-angled Artin group that does not admit $\mathbb{Z}^2$ as a subgroup is automatically free, so no right-angled Artin group can be commensurable to $C(C_n)$. Consequently, one can think of the family of right-angled Coxeter groups as being strictly larger than the family of right-angled Artin groups. A natural problem, then, is to understand when a group from the bigger family belongs to the smaller family. More precisely:

Question 1.1. Given a graph Γ, when is the right-angled Coxeter group $C(\Gamma)$ commensurable to a right-angled Artin group?

Recall that two groups are (*abstractly*) *commensurable* whenever they contain two isomorphic finite-index subgroups.

Question 1.1, and its analogue where "commensurable" is replaced with "quasi-isometric", is well-known in geometric group theory. Nevertheless, it is poorly understood. Some invariants are available in order to distinguish some right-angled Artin and Coxeter groups, such as divergence and thickness [Behrstock and Charney 2012; Dani and Thomas 2015; Levcovitz 2022] or Morse boundaries [Charney and Sultan 2015; Cordes and Hume 2017; Behrstock 2019] and Morse subgroups [Genevois 2022b]. In the other directions, a few constructions are known in order to produce (finite-index) subgroups in right-angled Coxeter groups that are right-angled Artin groups. See for instance [Januszkiewicz and Świątkowski 2001], and most notably [Dani and Levcovitz 2024] (based on [LaForge 2017] and further studied in [Cashen and Edletzberger 2024]). As a concrete but very specific application, it is determined in [Dani and Levcovitz 2024] precisely when some two-dimensional one-ended right-angled Coxeter groups defined by planar graphs are commensurable to right-angled Artin groups. (These examples are not representative of the general case because they are based on the large-scale geometry of graph manifolds [Behrstock and Neumann 2008; Nguyen and Tran 2019].) Despite all these results available in the literature, no global picture seems to emerge and an answer to Question 1.1 seems currently to be out of reach in full generality.

In this article, we focus on a specific family of right-angled Coxeter groups, known as *flat braid groups*. For every $n \geq 1$, the flat braid group FB_n on n strands is the fundamental group of the configuration space

$$\big\{\{x_1, \dots, x_n\} \in \mathbb{R}^n/\mathrm{Sym}(n) \mid \text{there exist at most two indices } i,\, j \text{ such that } x_i = x_j\big\}.$$

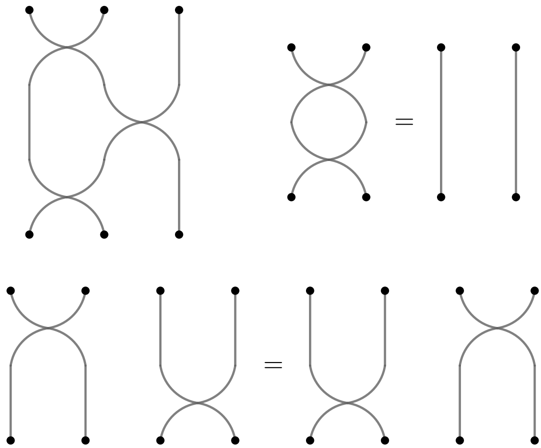


Figure 1. A flat braid from FB_3 (namely, $\sigma_1\sigma_2\sigma_1$), and the two typical relations between flat braids (namely, $\sigma_1^2 = 1$ and $\sigma_1\sigma_3 = \sigma_3\sigma_1$).

One can think of an element of FB_n as a configuration of n arcs in the infinite strip $\mathbb{R} \times [0, 1]$ connecting n marked points on each of the parallel lines $\mathbb{R} \times \{1\}$ and $\mathbb{R} \times \{0\}$ such that each arc is monotonic and no three arcs intersect at a common point. Two such configurations are considered as equivalent if one can be deformed into the other by a homotopy of such configurations in $\mathbb{R} \times [0, 1]$ keeping the end points of the arcs fixed. See Figure 1. From this description, one can define a natural morphism $\mathrm{FB}_n \to \mathrm{Sym}(n)$, encoding how a flat braid permutes the n strands. The kernel of this morphism is the *pure flat braid group* PFB_n.

If σ_i denotes the element of FB_n that twists the i-th and $(i+1)$-st strands, then it is clear that FB_n is generated by $\{\sigma_1, \dots, \sigma_{n-1}\}$. Moreover, FB_n admits

$$\langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i^2 = 1 \text{ for every } 1 \leq i \leq n-1,\ [\sigma_i, \sigma_j] = 1 \text{ whenever } |i-j| \geq 2 \rangle$$

as a presentation. Thus, FB_n coincides with the right-angled Coxeter group $C(P_{n-2}^{\mathrm{opp}})$ where P_{n-2}^{opp} denotes the opposite graph of the path P_{n-2} of length $n-2$.

Flat braid groups have been introduced in the literature under various names, often independently. For instance, one can meet *Grothendieck cartographic groups* in [Shabat and Voevodsky 1990; Voevodsky 1990]; *traid groups* in [Harshman and Knapp 2020] for applications to physics; *twin groups* in [Khovanov 1996; 1997] in connection with *doodles*; *pseudobraids groups* in [Genevois 2020] when studying *diagram groups* (see also [Farley 2021]); *planar braid groups* in [Mostovoy and Roque-Márquez 2020; Mostovoy 2020]. For notational convenience, we call our groups *flat braid groups* in reference to flat braids from [Merkov 1999].

Several articles in the literature investigate the algebraic structure of (pure) flat braid groups. See for instance [Dey and Gongopadhyay 2019] about presentations and ranks of the commutator subgroups of flat braid groups; [Bardakov et al. 2019] about similar results for pure flat braid groups; [Mostovoy and Roque-Márquez 2020] about the algebraic structures of pure flat braid groups on ≤ 6 strands; [Naik et al. 2020a] about the conjugacy problem and the structure of automorphism groups of flat braid groups; [Naik et al. 2020b] about Property R_∞ of flat braid groups; and [Bellingeri et al. 2024] about a connection with the so-called *cactus groups* (see also [Genevois 2022a]).

Motivated by the fact that pure virtual twin groups are right-angled Artin groups [Naik et al. 2023] and by the structure of pure flat braid groups on ≤ 6 strands [Mostovoy and Roque-Márquez 2020], it has been conjectured that:

Conjecture 1.2 [Naik et al. 2024]. For every $n \geq 3$, PFB_n is a right-angled Artin group.

In this article, we disprove this conjecture by showing that most pure flat braid groups are not right-angled Artin groups, even up to a finite index.

Theorem 1.3. *For every $n = 7$ or ≥ 11, the pure flat braid group PFB_n is not virtually a right-angled Artin group.*

Despite the fact that Conjecture 1.2 is false, we stress that the following (vague) question is still interesting: to which extent does a (pure) flat braid group look like a right-angled Artin group? For instance, Theorem 1.3 shows that FB_n does not contain a right-angled Artin group of finite index in PFB_n, but is there such a subgroup elsewhere in FB_n? More precisely:

Question 1.4. Is FB_n virtually a right-angled Artin group?

For instance, a natural candidate to check would be the commutator subgroup of FB_n, which is torsion-free. In case of a negative answer to Question 1.4, it would be natural to ask whether FB_n contains a finite-index subgroup that, despite not being a right-angled Artin group, turns out to be isomorphic to a finite-index subgroup of some right-angled Artin groups. In other words:

Question 1.5. Is FB_n commensurable (or at least quasi-isometric) to a right-angled Artin group?

Since, as said, flat braid groups are right-angled Coxeter groups, Question 1.5 is a particular case of Question 1.1. Perhaps surprisingly, in view on Theorem 1.3, we answer positively Question 1.5 for $n = 7$.

Theorem 1.6. *The flat braid group FB_7 is commensurable to the right-angled Artin group $A(P_4)$.*

In order to prove Theorem 1.6, we exploit the observation that FB_7 and $A(P_4)$ are both (virtually) fundamental groups of compact flip manifolds. It is known by [Behrstock and Neumann 2008] that right-angled Artin groups defined by finite trees can be described as fundamental groups of such 3-manifolds. For FB_7, we start by proving that FB_7 contains an index-8 normal subgroup isomorphic to an index-2 subgroup of $\langle a, t \mid [a, tat^{-1}] = 1\rangle$ (see Section 6.2). This group can be easily described as the fundamental group of a compact flip manifold M_2. Given another such 3-manifold M_1 whose fundamental group is isomorphic to $A(P_4)$, we construct a common finite-sheeted cover $M_0 \to M_1, M_2$ (see Section 6.3), which allows us to deduce Theorem 1.6.

About the proof of Theorem 1.3. The first step is to prove the theorem for the flat braid group on seven strands, namely:

Theorem 1.7. *The flat braid group* PFB_7 *is not virtually a right-angled Artin group.*

This is the smallest number of strands for which the statement holds (see [Naik et al. 2020a]). Theorem 1.7 is proved as follows. First, we observe that, since FB_7 does not contain a subgroup isomorphic to $\mathbb{Z}^3$, to $\mathbb{F}_2 \times \mathbb{F}_2$, or to the fundamental group of a closed surface of genus ≥ 2 (Lemma 4.2), every right-angled Artin group that appears as a subgroup of FB_7 is defined by a forest (Corollary 4.4). As a consequence, it suffices to show that PFB_7 does not contain as a finite-index subgroup a right-angled Artin group $A(T)$ defined by a finite tree T. For this, we define the *thick subgroup* $\mathrm{Thick}(G)$ of a group G as the subgroup generated by the centralisers of all the elements whose centralisers are not virtually abelian. Then, by showing that $\mathrm{Thick}(A(T)) = A(T)$ (Lemma 4.6) but that $\mathrm{Thick}(\mathrm{PFB}_7)$ has infinite index in PFB_7 (Lemma 4.7), we conclude that $A(T)$ cannot be a finite-index subgroup of PFB_7.

Once Theorem 1.7 is proved, it is not so difficult to deduce that PFB_n is not a right-angled Artin group for $n \geq 11$. The trick is that PFB_n contains a natural copy of $\mathrm{PFB}_7 \times \mathrm{PFB}_{n-7}$, which turns out to be a maximal product subgroup. But, in a right-angled Artin group, maximal product subgroups are well-understood; their factors, in particular, are also right-angled Artin groups. Since this product decomposition is unique (Corollary 3.6), it follows that, if PFB_n were a right-angled Artin group, then PBF_7 would be a right-angled Artin group as well, contradicting Theorem 1.7.

To prove that PFB_n is not virtually a right-angled Artin group, the strategy is basically the same, but some technicality is required. In Section 3, we introduce the notion of *IMC generating sets*, which is of independent interest, and we prove that, under the assumption that our groups admit IMC generating sets, factors of finite-index subgroups in products must be contained in factors of the whole product (Lemma 3.5).

2. Preliminaries on graph products

Let Γ be a graph and $\mathcal{G} = \{G_u \mid u \in \Gamma\}$ a collection of groups indexed by the vertices of Γ. The *graph product* $\Gamma\mathcal{G}$ is

$$\langle G_u \ (u \in \Gamma) \mid [G_u, G_v] = 1 \ (\{u, v\} \in E(\Gamma))\rangle$$

where $E(\Gamma)$ denotes the edge-set of Γ and where $[G_u, G_v] = 1$ is a shorthand for $[g, h] = 1$ for all $g \in G_u$, $h \in G_v$. The groups of $\mathcal{G}$ are referred to as the *vertex groups*.

Graph products of groups will allow us to state and prove results simultaneously about right-angled Artin groups (i.e., when vertex groups are infinite cyclic) and about right-angled Coxeter groups (i.e., when vertex groups are cyclic of order two).

Convention. We always assume that the groups in $\mathcal{G}$ are nontrivial. Notice that it is not a restrictive assumption, since a graph product with some trivial factors can be described as a graph product over a smaller graph all of whose factors are nontrivial.

A *word* in $\Gamma\mathcal{G}$ is a product $g_1 \cdots g_n$ where $n \geq 0$ and where, for every $1 \leq i \leq n$, $g_i \in G$ for some $G \in \mathcal{G}$; the g_i's are the *syllables* of the word, and n is the *length* of the word. Clearly, each of the following operations on a word does not modify the element of $\Gamma\mathcal{G}$ it represents:

<u>Cancellation:</u> delete the syllable g_i if $g_i = 1$.

<u>Amalgamation:</u> if $g_i, g_{i+1} \in G$ for some $G \in \mathcal{G}$, replace the two syllables g_i and g_{i+1} by the single syllable $g_i g_{i+1} \in G$.

<u>Shuffling:</u> if g_i and g_{i+1} belong to two adjacent vertex groups, switch them.

A word is *graphically reduced* if its length cannot be shortened by applying these elementary moves. Every element of $\Gamma\mathcal{G}$ can be represented by a graphically reduced word, and this word is unique up to the shuffling operation. This allows us to define the *support* of an element as the subgraph of Γ induced by the vertices labelling the syllables of a graphically reduced word representing our element. Similarly, a word is *graphically cyclically reduced* if all its cyclic permutations are graphically reduced. An element of $\Gamma\mathcal{G}$ that can be represented by a graphically cyclically reduced word is *graphically cyclically reduced*. Every element is conjugate to a graphically cyclically reduced word. One can define the *essential support* of an element as the support of a graphically cyclically reduced element that is conjugate to it. For more information on graphically reduced words, we refer to [Green 1990] (see also [Hsu and Wise 1999; Genevois 2019]).

Parabolic subgroups. Let Γ be a graph and $\mathcal{G}$ a collection of groups indexed by Γ. Given a subgraph $\Lambda \subset \Gamma$, we denote by $\langle \Lambda \rangle$ the subgroup of $\Gamma\mathcal{G}$ generated by the vertex groups labelling the vertices of Λ. A subgroup of $\Gamma\mathcal{G}$ of the form $g\langle \Phi \rangle g^{-1}$ for some element $g \in \Gamma\mathcal{G}$ and subgraph $\Phi \subset \Gamma$ is a *parabolic subgroup*. Here, we record a few basic results about parabolic subgroups. They will be useful in order to prove some preliminary results later in this section (but they will not be used in the next sections).

Lemma 2.1. *Let Γ be a graph and $\mathcal{G}$ a collection of groups indexed by Γ. The graph product $\Gamma\mathcal{G}$ is virtually cyclic if and only if one of the following conditions holds*:

- Γ *is a complete graph all of whose vertices are labelled by finite groups.*
- $\Gamma = \Xi * \{u\}$ *where Ξ is a complete graph all of whose vertices are labelled by finite groups and where u is a vertex labelled by a virtually-$\mathbb{Z}$ group.*
- $\Gamma = \Xi * \{u, v\}$ *where Ξ is a complete graph all of whose vertices are labelled by finite groups and where u, v are two nonadjacent vertices both labelled by $\mathbb{Z}_2$.*

Recall that, given two graphs Φ and Ψ, their *join* $\Phi * \Psi$ is the graph obtained from $\Phi \sqcup \Psi$ by connecting with an edge every vertex of Φ with every vertex of Ψ.

Proof of Lemma 2.1. If Γ is complete, then $\Gamma\mathcal{G}$ is the product of its vertex groups. In this case, $\Gamma\mathcal{G}$ is virtually cyclic if and only if either all its vertex groups are finite or one vertex group is virtually-$\mathbb{Z}$ and all its other vertex groups are finite. This corresponds to the first and second items of our lemma.

Now, assume that Γ is not complete. Fix two nonadjacent vertices $u, v \in \Gamma$. Since the subgroup $\langle u, v \rangle$ of $\Gamma\mathcal{G}$ decomposes as the free product $\langle u \rangle * \langle v \rangle$, the only possibility for $\Gamma\mathcal{G}$ to be virtually cyclic is that $\langle u \rangle$ and $\langle v \rangle$ are both cyclic of order two. Next, if Γ contains a vertex w that is not adjacent to both u and v, then $\langle u, v, w \rangle$ decomposes as one of the following free products: $\langle u \rangle * \langle v \rangle * \langle w \rangle$, $(\langle u \rangle \times \langle w \rangle) * \langle v \rangle$, or $\langle u \rangle * (\langle v \rangle \times \langle w \rangle)$. In any case, $\Gamma\mathcal{G}$ cannot be virtually cyclic. Therefore, Γ decomposes as a join $\Xi * \{u, v\}$. Algebraically, this implies that $\Gamma\mathcal{G}$ decomposes as the product $\langle \Xi \rangle \times \langle u, v \rangle \simeq \langle \Xi \rangle * \mathbb{D}_\infty$. Then, $\Gamma\mathcal{G}$ is virtually cyclic if and only if $\langle \Xi \rangle$ is finite, which amounts to saying that Ξ is a complete graph all of whose vertices are labelled by finite groups. This corresponds to the third item of our lemma. □

Lemma 2.2. *Let Γ be a graph and $\mathcal{G}$ a collection of groups indexed by Γ. For every subgraph $\Lambda \subset \Gamma$, $\langle \Lambda \rangle$ has finite index in $\Gamma\mathcal{G}$ if and only if $\Gamma = \Lambda * \Xi$, where Ξ is a complete graph all of whose vertices are labelled by finite groups.*

Proof. First, assume that there exist two nonadjacent vertices $u \in \Gamma \backslash \Lambda$ and $v \in \Lambda$. Fix two nontrivial elements $a \in \langle u \rangle$ and $b \in \langle v \rangle$. If there exist distinct integers $p, q \geq 1$ such that $(ba)^p$ and $(ba)^q$ belong to the same $\langle \Lambda \rangle$-coset, then we find an integer $\ell \geq 1$ such that $(ba)^\ell \in \langle \Lambda \rangle$. But this is impossible since, as a graphically reduced word, $(ba)^\ell$ cannot represent an element of $\langle \Lambda \rangle$. Consequently, $\langle \Lambda \rangle$ must have infinite index in $\Gamma\mathcal{G}$.

From now on, assume that every vertex in $\Gamma \setminus \Lambda$ is adjacent to every vertex in Λ; i.e., Γ decomposes as a join $\Lambda * \Xi$. Algebraically, $\Gamma\mathcal{G}$ decomposes as the product $\langle \Lambda \rangle * \langle \Xi \rangle$. Then, $\langle \Xi \rangle$ has finite index in $\Gamma\mathcal{G}$ if and only if $\langle \Xi \rangle$ is finite, which amounts to saying that Ξ is a complete graph all of whose vertices are labelled by finite groups. □

Lemma 2.3. *Let Γ be a graph and $\mathcal{G}$ a collection of groups indexed by Γ. For all subgraphs $\Phi, \Psi \subset \Gamma$ and element $a \in \Gamma\mathcal{G}$, the inclusion $a\langle \Phi \rangle a^{-1} \leq \langle \Psi \rangle$ holds if and only if $\Phi \subset \Psi$ and $a \in \langle \Psi \rangle \cdot \langle \mathrm{star}(\Phi) \rangle$.*

Recall that, given a subgraph Λ, its *link* $\mathrm{link}(\Lambda)$ refers to the subgraph induced by the vertices that are adjacent to all the vertices in Λ, and its *star* $\mathrm{star}(\Lambda)$ refers to the subgraph induced by $\mathrm{link}(\Lambda) \cup \Lambda$.

A proof of this lemma can be found in [Genevois and Martin 2019, Lemma 3.17]. Here is an immediate consequence:

Corollary 2.4. *Let Γ be a graph and $\mathcal{G}$ a collection of groups indexed by Γ. For all subgraph $\Phi \subset \Gamma$ and element $a \in \Gamma\mathcal{G}$, if $a\langle \Phi \rangle a^{-1} \leq \langle \Phi \rangle$ then $a\langle \Phi \rangle a^{-1} = \langle \Phi \rangle$.*

Join subgroups. Parabolic subgroups given by join subgraphs play a central role in the study of graph products of groups. For us, they will be fundamental in the proof of Theorem 5.1, allowing us to reduced a problem about flat braid groups on arbitrarily many strands to a problem about flat braid groups on seven strands. As explained by our next lemma, Join subgroups can be used in order to characterise maximal product subgroups in graph products. Here, we refer to a *maximal product subgroup* of a given group as a maximal member of the collection of the subgroups that decompose as products of two nontrivial groups.

Lemma 2.5. *Let Γ be a finite graph and $\mathcal{G}$ a collection of groups indexed by Γ. A subgroup of $\Gamma\mathcal{G}$ is a maximal product subgroup if and only if either it is conjugate to $\langle \Lambda \rangle$ for some maximal join $\Lambda \subset \Gamma$ or it is a maximal product subgroup in a conjugate of a vertex group given by an isolated vertex of Γ.*

A proof of this lemma can be found in [Genevois 2024, Proposition 2.8]. It is worth noticing that the maximality given by the previous lemma behaves nicely with respect to commensurability:

Lemma 2.6. *Let Γ be a finite graph and $\mathcal{G}$ a collection of groups indexed by Γ. A subgroup of $\Gamma\mathcal{G}$ commensurable to a product of two infinite groups is contained in a conjugate of a vertex group or in a maximal product subgroup.*

Proof. Let $H \leq \Gamma\mathcal{G}$ be a subgroup that contains a finite-index subgroup $\dot{H}$ in common with a product $A \times B$ of two infinite groups. Given an $h \in H$, there exists some $p \geq 1$ such that $h^p \in \dot{H}$. Thinking of h^p as an element of $A \times B$, it can be written as (a, b) for some $a \in A$ and $b \in B$. Fix a $q \geq 1$ such that a^q and b^q either are trivial or have infinite order. Then the centraliser of $h^{pq} = (a^q, b^q)$ in $A \times B$ is $A \times B$ if $a^q = b^q = 1$; contains $A \times \langle b \rangle \simeq A \times \mathbb{Z}$ if $a^q = 1$ and if b has infinite order; contains $\langle a \rangle \times B \simeq \mathbb{Z} \times B$ if a has infinite order and if $b^q = 1$; contains $\langle a \rangle \times \langle b \rangle \simeq \mathbb{Z}^2$ if a and b both have infinite order. In any case, the centraliser of h^{pq} in $A \times B$, and a fortiori in H is not virtually cyclic. Thus, we have proved that every element of H has a nontrivial power whose centraliser is not virtually cyclic.

Now, assume that H is not contained in a conjugate of a vertex group nor in a maximal product subgroup. We claim that H must contain an element all of whose nontrivial powers have a virtually cyclic centraliser. As a consequence of the previous observation, this will conclude the proof of our lemma. Let $g\langle\Lambda\rangle g^{-1}$ denote the smallest parabolic subgroup containing H. (Such a subgroup exists according to [Antolín and Minasyan 2015].) According to [Minasyan and Osin 2015, Corollary 6.20], there exists a tree on which $g\langle\Lambda\rangle g^{-1}$ acts such that H contains a WPD element $h \in H$. Every nontrivial power of such an element must have a virtually cyclic centralisers in $g\langle\Lambda\rangle g^{-1}$. But, as a consequence of Proposition 2.8 below, the centraliser of an element of H in $\Gamma\mathcal{G}$ decomposes as the product of the centraliser of the element in $g\langle\Lambda\rangle g^{-1}$ with $g\langle \text{link}(\Lambda)\rangle g^{-1}$. Since H cannot be contained in a join subgroup, link$(\Lambda)\rangle$ must be empty. Consequently, the centraliser of a nontrivial power of h in $\Gamma\mathcal{G}$, and a fortiori in H, is virtually cyclic. □

Finally, let us observe no two maximal product subgroups in graph products can be commensurable:

Lemma 2.7. *Let Γ be a graph and $\mathcal{G}$ a collection of groups indexed by Γ. Let $\Phi, \Psi \subset \Gamma$ be two maximal joins and let $g, h \in \Gamma\mathcal{G}$. If $g\langle\Phi\rangle g^{-1}$ has a finite-index subgroup contained in $h\langle\Psi\rangle h^{-1}$, then $g\langle\Phi\rangle g^{-1} = h\langle\Psi\rangle h^{-1}$.*

Proof. According to [Antolín and Minasyan 2015] (see also [Genevois 2022c]), an intersection of two parabolic subgroups is again a parabolic subgroup, so there exist $k \in \Gamma\mathcal{G}$ and $\Lambda \subset \Gamma$ such that

$$g\langle\Phi\rangle g^{-1} \cap h\langle\Psi\rangle h^{-1} = k\langle\Lambda\rangle k^{-1}.$$

Since $g\langle\Phi\rangle g^{-1}$ contains a finite-index subgroup that is also contained in $h\langle\Psi\rangle h^{-1}$, necessarily $k\langle\Lambda\rangle k^{-1}$ has finite index in $g\langle\Phi\rangle g^{-1}$. Notice that, as a consequence of Lemma 2.3, $\Lambda \subset \Phi$ and $g^{-1}k$ belongs to $\langle \text{star}(\Phi)\rangle$. This implies that $\langle\Lambda\rangle$ is conjugate in $\langle\Phi\rangle$ (by an element of $\langle\Phi\rangle$) to a finite-index subgroup. Since $\langle\Lambda\rangle$ then must have finite index in $\langle\Phi\rangle$, we deduce from Lemma 2.2 that Φ decomposes as a join with Λ as a factor. But Φ is by assumption a maximal join in Γ, so we must have $\Phi = \Lambda$. So far, we have proved that

$$g\langle\Phi\rangle g^{-1} \cap h\langle\Psi\rangle h^{-1} = k\langle\Phi\rangle k^{-1}.$$

From the inclusion $k\langle\Phi\rangle k^{-1} \leq g\langle\Phi\rangle g^{-1}$, we deduce from Corollary 2.4 that $k\langle\Phi\rangle k^{-1} = g\langle\Phi\rangle g^{-1}$. Then, the centred equality above amounts to saying that $g\langle\Phi\rangle g^{-1} \leq h\langle\Psi\rangle h^{-1}$. We know from Lemma 2.3 that $\Phi \subset \Psi$. But Ψ is a maximal join in Γ, so we must have $\Phi = \Psi$. Then, we conclude from Corollary 2.4 that $g\langle\Phi\rangle g^{-1} = h\langle\Psi\rangle h^{-1}$, as desired. □

Centralisers in graph products. Below, we record the structure of centralisers in graph products of groups, from which we will extract some information about stable centralisers (Definition 2.9) and virtual centres (Definition 2.12), two central ingredients in the rigidity later obtained with Lemma 3.5.

Proposition 2.8 [Barkauskas 2007]. *Let Γ be a graph and $\mathcal{G}$ a collection of groups indexed by Γ. Fix a graphically cyclically reduced element $g \in \Gamma\mathcal{G}$ and decompose its support as a join*

$$\mathrm{supp}(g) = \Phi_1 * \cdots * \Phi_r * \Psi_1 * \cdots * \Psi_s$$

such that Φ_i is reduced to a single vertex for every $1 \leq i \leq r$ and such that Ψ_i is an irreducible subgraph with at least two vertices for every $1 \leq i \leq s$. Write g as a graphically reduced word $a_1 \cdots a_r \cdot b_1 \cdots b_s$ such that $\mathrm{supp}(a_i) = \Phi_i$ for every $1 \leq i \leq r$ and $\mathrm{supp}(b_i) = \Psi_i$ for every $1 \leq i \leq s$. Then the centraliser of g in $\Gamma\mathcal{G}$ is

$$C_{\Gamma\mathcal{G}}(g) = C_{\langle\Phi_1\rangle}(a_1) \times \cdots \times C_{\langle\Phi_r\rangle}(a_r) \times \langle h_1\rangle \times \cdots \times \langle h_s\rangle \times \big\langle \mathrm{link}(\mathrm{supp}(g))\big\rangle,$$

where each h_i is a primitive element of $\langle\Psi_i\rangle$ such that $b_i \in \langle h_i\rangle$.

As a first application of this description of centralisers, we observe that, in many graph products, the centraliser of a nontrivial power of an element is pretty much the same as the centraliser of the element itself. More formally:

Definition 2.9. A group G has *stable centralisers* if, for all $g \in G$ and $k \geq 1$, $C(g)$ equals $C(g^k)$. The group has *almost stable centralisers* if, for all $g \in G$ and $k \geq 1$, $C(g)$ has finite index in $C(g^k)$.

As an easy application of Proposition 2.8:

Lemma 2.10. *Let Γ be a graph and $\mathcal{G}$ a collection of torsion-free groups indexed by Γ. If the groups in $\mathcal{G}$ have* (*almost*) *stable centralisers*, *then $\Gamma\mathcal{G}$ has* (*almost*) *stable centralisers.*

Proof. Let $g \in \Gamma\mathcal{G}$ be an element. Up to conjugating g, we assume for convenience that it is graphically cyclically reduced. Decompose $\mathrm{supp}(g)$ as a join $\Phi_1 * \cdots * \Phi_r * \Psi_1 * \cdots * \Psi_s$ and write g as a product $a_1 \cdots a_r \cdot b_1 \cdots b_s$ as in Proposition 2.8. So

$$C_{\Gamma\mathcal{G}}(g) = C_{\langle \Phi_1 \rangle}(a_1) \times \cdots \times C_{\langle \Phi_r \rangle}(a_r) \times \langle h_1 \rangle \times \cdots \times \langle h_s \rangle \times \big\langle \mathrm{link}(\mathrm{supp}(g)) \big\rangle.$$

Given a $k \geq 1$, because vertex groups are torsion-free, Proposition 2.8 applies to the decomposition $g^k = a_1^k \cdots a_r^k \cdot b_1^k \cdots b_s^k$ and shows that

$$C_{\Gamma\mathcal{G}}(g^k) = C_{\langle \Phi_1 \rangle}(a_1^k) \times \cdots \times C_{\langle \Phi_r \rangle}(a_r^k) \times \langle h_1 \rangle \times \cdots \times \langle h_s \rangle \times \big\langle \mathrm{link}(\mathrm{supp}(g)) \big\rangle.$$

If vertex groups have stable centralisers (resp. almost stable centralisers), then each $C_{\langle \Phi_i \rangle}(a_i)$ agrees with (resp. has finite index in) $C_{\langle \Phi_i \rangle}(a_i^k)$. This implies that the centraliser of g^k in $\Gamma\mathcal{G}$ agrees with (resp. has finite index in) the centraliser of g in $\Gamma\mathcal{G}$. □

Corollary 2.11. *Right-angled Artin groups, as well as their subgroups, have stable centralisers.*

There exist also many graph products that do not have almost stable centralisers. This is the case, for instance, of the product $\mathbb{D}_\infty \times \mathbb{D}_\infty$ of two infinite dihedral groups. Indeed, given an infinite-order element $a \in \mathbb{D}_\infty$ and an element of order two $b \in \mathbb{D}_\infty$, the centraliser of (a, b) is isomorphic to $\mathbb{Z} \times \mathbb{Z}_2$ but the centraliser of its square $(a^2, 1)$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Virtual centres. Motivating by the observation that the centres of a group and of its finite-index subgroups may be quite different, we introduce the notion of *virtual centre*. It will be a central ingredient in the rigidity later obtained with Lemma 3.5.

Definition 2.12. The *virtual centre* $\mathrm{VZ}(G)$ of a group G is the set of the elements that centralise some finite-index subgroups of G.

Lemma 2.13. *The virtual centre of a group is a normal subgroup.*

Proof. Let G be a group and $a, b \in G$ two elements. We make two observations:

- If a (resp. b) commutes with all the elements of a finite-index subgroup $H \leq G$ (resp. $K \leq G$), then ab^{-1} commutes with all the elements of the finite-index subgroup $H \cap K$. Hence $ab^{-1} \in \mathrm{VZ}(G)$.
- If a commutes with all the elements of a finite-index subgroup $H \leq G$, then bab^{-1} commutes with all the elements of the finite-index subgroup bHb^{-1}. Hence $bab^{-1} \in \mathrm{VZ}(G)$.

We conclude that $\mathrm{VZ}(G)$ is indeed a normal subgroup of G. □

For future use, let us describe virtual centres of graph products of groups:

Lemma 2.14. *Let Γ be a finite graph and $\mathcal{G}$ a collection of groups indexed by Γ. Decompose Γ as a join*

$$\Gamma = \{u_1\} * \cdots * \{u_r\} * \{a_1, b_1\} * \cdots * \{a_s, b_s\} * \Lambda_1 * \cdots * \Lambda_n$$

where $u_1, \ldots, u_r \in \Gamma$ are single vertices, where each $a_i, b_i \in \Gamma$ are two nonadjacent vertices labelled by $\mathbb{Z}_2$, and where each Λ_i is an irreducible subgraph containing at least two vertices not both labelled by $\mathbb{Z}_2$. Then

$$\mathrm{VZ}(\Gamma\mathcal{G}) = \mathrm{VZ}(\langle u_1 \rangle) \times \cdots \times \mathrm{VZ}(\langle u_r \rangle) \times \langle a_1 b_1 \rangle \times \cdots \times \langle a_s b_s \rangle.$$

Proof. Let $g \in \Gamma\mathcal{G}$ be an element that belongs to the virtual centre of $\Gamma\mathcal{G}$. This amounts to saying that the centraliser of g has finite index in $\Gamma\mathcal{G}$. Our goal is to prove that g belongs to the subgroup

$$V := \mathrm{VZ}(\langle u_1 \rangle) \times \cdots \times \mathrm{VZ}(\langle u_r \rangle) \times \langle a_1 b_1 \rangle \times \cdots \times \langle a_s b_s \rangle.$$

Because V clearly lies in the virtual centre of $\Gamma\mathcal{G}$, this will be sufficient to conclude the proof of our lemma. Notice also that V is a normal subgroup, so, up to conjugating g, we assume without loss of generality that g is graphically cyclically reduced. Following Proposition 2.8, decompose the support of g as a join

$$\mathrm{supp}(g) = \Phi_1 * \cdots * \Phi_p * \Psi_1 * \cdots * \Psi_q$$

such that Φ_i is reduced to a single vertex for every $1 \leq i \leq p$ and such that Ψ_i is an irreducible subgraph with at least two vertices for every $1 \leq i \leq q$. Write g as a graphically reduced word $a_1 \cdots a_p \cdot b_1 \cdots b_q$ such that $\mathrm{supp}(a_i) = \Phi_i$ for every $1 \leq i \leq p$ and $\mathrm{supp}(b_i) = \Psi_i$ for every $1 \leq i \leq q$. According to Proposition 2.8, the centraliser of g in $\Gamma\mathcal{G}$ is

$$C_{\Gamma\mathcal{G}}(g) = C_{\langle \Phi_1 \rangle}(a_1) \times \cdots \times C_{\langle \Phi_r \rangle}(a_p) \times \langle h_1 \rangle \times \cdots \times \langle h_q \rangle \times \big\langle \mathrm{link}(\mathrm{supp}(g)) \big\rangle$$

where each h_i is a primitive element of $\langle \Psi_i \rangle$ such that $b_i \in \langle h_i \rangle$. Since this centraliser has finite index in $\Gamma\mathcal{G}$, the following assertions hold:

- for every $1 \leq i \leq p$, $C_{\langle \Phi_i \rangle}(a_i)$ has finite-index in $\langle \Phi_i \rangle$, i.e., $a_i \in \mathrm{VZ}(\langle \Phi_i \rangle)$;
- $\langle h_i \rangle$ has finite index in $\langle \Psi_i \rangle$ for every $1 \leq i \leq q$;
- $\langle \Phi_1 \cup \cdots \cup \Phi_p \cup \Psi_1 \cup \cdots \cup \Psi_q \cup \mathrm{link}(\mathrm{supp}(g)) \rangle$ has finite index in $\Gamma\mathcal{G}$.

The second item implies that each $\langle \Psi_i \rangle$ is virtually cyclic. According to Lemma 2.1, the only possibility is that Ψ_i is given by two nonadjacent vertices both labelled by $\mathbb{Z}_2$. And the third item implies, according to Lemma 2.2, that

$$\Gamma = \Phi_1 * \cdots * \Phi_r * \Psi_1 * \cdots * \Psi_s * \mathrm{link}(\mathrm{supp}(g)).$$

It follows that $p = r$ and $q = s$; that $\mathrm{link}(\mathrm{supp}(g)) = \Lambda_1 * \cdots * \Lambda_n$; and that, up to reordering our subgraphs, $\Phi_i = \{u_i\}$ for every $1 \leq i \leq r$ and $\Psi_i = \{a_i, b_i\}$ for every

$1 \le i \le s$. Notice that, for every $1 \le i \le s$, b_i is an infinite-order element of the infinite dihedral group $\langle \Psi_i \rangle = \langle a_i, b_i \rangle$, so we can take $h_i = a_i b_i$. We conclude that g indeed belongs to V, as desired. □

Corollary 2.15. *For every $k \ge 4$, the FB_k and PFB_k have trivial virtual centres.*

Proof. Thinking of the flat braid group FB_k as the right-angled Coxeter group $C(P_{k-2}^{\mathrm{opp}})$, it follows immediately from Lemma 2.14 that FB_k has a trivial virtual centre as soon as $k \ge 4$. Then, notice that an element of the virtual centre of PFB_k also belongs to the virtual centre of FB_k, so it must be trivial. □

Acylindrical hyperbolicity. In order to prove Corollaries 2.17 and 3.3, we will use a few tools coming from the theory of *acylindrically hyperbolic groups*. We refer the reader to [Osin 2018] for more information on the subject. Recall that an acylindrically hyperbolic group G has a unique maximal finite normal subgroup, referred to as its *finite radical* [Dahmani et al. 2017, Theorem 2.24].

Proposition 2.16. *Let Γ be a finite graph and $\mathcal{G}$ a collection of groups indexed by Γ. If Γ is not a join and has at least two vertices, then $\Gamma\mathcal{G}$ is acylindrically hyperbolic and its finite radical is trivial.*

Proof. The acylindrical hyperbolicity of $\Gamma\mathcal{G}$ is given by [Minasyan and Osin 2015, Corollary 2.13]. It remains to verify that the finite radical R of $\Gamma\mathcal{G}$ is trivial. Let $g \in \Gamma\mathcal{G}$ be an element and $\Theta \subset \Gamma$ a subgraph such that $g\langle\Theta\rangle g^{-1}$ is the unique smallest parabolic subgroup containing R (see [Antolín and Minasyan 2015, Proposition 3.10]). As it is well-known that finite subgroups in graph products are contained in clique subgroups (see for instance [Genevois 2017, Theorem 2.115 and Corollary 8.7] for a geometric proof), we deduce from Lemma 2.3 that Θ is complete. Moreover, since $\Gamma\mathcal{G}$ normalises R, necessarily $g\langle\Theta\rangle g^{-1}$ is normalised by $\Gamma\mathcal{G}$. But, according to [Antolín and Minasyan 2015, Proposition 3.13], the normaliser of $g\langle\Theta\rangle g^{-1}$ is $g\langle \mathrm{star}(\Theta)\rangle g^{-1}$. It follows from Corollary 2.4 that $\Gamma = \mathrm{star}(\Theta)$. Since Γ is not a join and contains at least two vertices, this implies that $\Theta = \varnothing$. In other words, R must be trivial, as desired. □

Corollary 2.17. *For every $k \ge 4$, PFB_k is not virtually a product of two infinite groups.*

Proof. For $k \ge 4$, it follows from Proposition 2.16 that FB_k, thought of as the right-angled Coxeter group $C(P_{k-2}^{\mathrm{opp}})$, is acylindrically hyperbolic. As a finite-index subgroup, PFB_k must be acylindrically hyperbolic as well. This prevents PFB_k from being virtually a product of two infinite groups, for instance as a consequence of [Minasyan and Osin 2015, Lemma 6.24]. □

3. Morphisms to products

In this section, our goal is to show that a morphism between two products of groups satisfying mild assumptions has to send a factor to a factor. The main result in this direction is Lemma 3.5, using the notion of *IMC generating sets* which we now define and study.

3.1. *IMC generating sets.*

Definition 3.1. Given a group G, $S \subset G$ is an IMC generating set if it satisfies the following conditions:

Independence: for all distinct $s_1, s_2 \in S$ and all integers $p, q \geq 1$, $[s_1^p, s_2^q] \neq 1$;

Maximal centralisers: for all $s \in S$ and $g \in G$, if $C(s) \subsetneq C(g)$ then $g = 1$.

Our goal now is to show that most acylindrically hyperbolic groups admit IMC generating sets.

Proposition 3.2. *Let G be an acylindrically hyperbolic group. If the finite radical of G is trivial, then G admits an IMC generating set.*

Proof. Recall that every generalised loxodromic element $g \in G$ belongs to a unique maximal virtually cyclic subgroup of G, which we denote by $E(g)$ [Dahmani et al. 2017, Lemma 6.5]. According to [ibid., Corollary 6.6], $E(g)$ coincides with $\{h \in G \mid \exists n \geq 1, hg^nh^{-1} = g^{\pm n}\}$. As a consequence, $E(g^k) = E(g)$ for every $k \geq 1$.

Fix a nonelementary acylindrical action of G on some hyperbolic space. Following [Antolín et al. 2016], we refer to an element $g \in G$ as *special* if g is loxodromic and $E(g) = \langle g \rangle$. Let S_0 denote the set of all the special elements of G. Fix a set of representatives $S \subset S_0$ with respect to the equivalence relation: for all $r, s \in S_0$, $r \sim s$ whenever $r = s^{\pm 1}$. Let us verify that S is an IMC generating set of G.

According to [Antolín et al. 2016, Proposition 5.14], S_0 generates G, so S is a generating set of G.

Now, let $s \in S$ and $g \in G$ be two elements satisfying $C(s) \subset C(g)$. Since s and g commute, we have

$$g \in C(s) \subset E(s) = \langle s \rangle.$$

Either $g = 1$, in which case there is nothing to prove; or g is a nontrivial power of s, which implies that

$$C(s) \subset C(g) \subset E(g) = E(s) = \langle s \rangle \subset C(s),$$

hence $C(g) = C(s)$.

Next, let $r, s \in S$ and $p, q \geq 1$ be such that $[r^p, s^q] = 1$. We have

$$r^p \in C(s^q) \subset E(s^q) = E(s) = \langle s \rangle,$$

from which we deduce that

$$\langle r \rangle = E(r) = E(r^p) = E(s) = \langle s \rangle.$$

Therefore, we must have $r = s^{\pm 1}$, hence $r = s$ by definition of S. □

Corollary 3.3. *Let Γ be a finite graph and $\mathcal{G}$ a collection of groups indexed by Γ. If Γ contains at least two vertices and is not a join, then $\Gamma\mathcal{G}$ has an IMC generating set.*

Proof. According to Proposition 2.16, Proposition 3.2 applies and yields the desired conclusion. □

Lemma 3.4. *Let Γ be a finite graph and $\mathcal{G}$ a collection of groups indexed by Γ. Let Q be a maximal product subgroup of $\Gamma\mathcal{G}$ that is not contained in a conjugate of vertex group given by an isolated vertex of Γ. Then Q decomposes as $Q_1 \times \cdots \times Q_s$ where each Q_i either is conjugate to a vertex group or admits an IMC generating set.*

Proof. According to Lemma 2.5, our maximal product subgroup $Q \leq \Gamma\mathcal{G}$ can written as $g\langle \Xi \rangle g^{-1}$ for some $g \in \Gamma\mathcal{G}$ and some maximal join $\Xi \subset \Gamma$. Decompose Ξ as $\Xi_1 * \cdots * \Xi_s$ where no Ξ_i is a join. Accordingly, Q decomposes as a product $Q_1 \times \cdots \times Q_s$ where $Q_i := g\langle \Xi_i \rangle g^{-1}$ for every $1 \leq i \leq s$. For every $1 \leq i \leq s$, either Ξ_i is reduced to a single vertex, in which case Q_i is conjugate to a vertex group; or Ξ_i contains at least two vertices, in which case Q_i admits an IMC generating set according to Corollary 3.3. □

3.2. ***Some rigidity.*** Our main motivation for the introduction of IMC generating sets is the following statement, which is inspired by [Genevois 2024, Lemma 3.4] and which will be fundamental in order to deduce Theorem 5.1 from Theorem 4.1.

Lemma 3.5. *Let $H, K, A_1, \ldots, A_n$ be groups such that $H \times K$ is a finite-index subgroup of $A_1 \times \cdots \times A_n$. If $A_1, \ldots, A_n$ have almost stable centralisers and if H is a noncyclic group admitting an IMC generating set, then*

$$H \leq \mathrm{VZ}(A_1) \times \cdots \times \mathrm{VZ}(A_{i-1}) \times A_i \times \mathrm{VZ}(A_{i+1}) \times \cdots \times \mathrm{VZ}(A_n)$$

for some index $1 \leq i \leq n$.

Proof. Fix an IMC generating set $S \subset H$. Since H is not cyclic, S contains at least two elements.

Fix an $s \in S$. In $A := A_1 \times \cdots \times A_n$, we can write $s = (s_1, \ldots, s_n)$. Let $k \geq 1$ be a sufficiently large integer so that $s_i^k \in H \times K$ for every $1 \leq i \leq n$. If $s_i^k \in Z(A_i)$ for every $1 \leq i \leq n$, then s^k belongs to the centre of A, and a fortiori of H. Then, $[r, s^k] = 1$ for every $r \in S \backslash \{s\}$, contradicting the fact that S is IMC. Thus, there exists some $1 \leq i \leq n$ such that $s_i^k \notin Z(A_i)$. It follows that

$$\begin{aligned} C_{H\times K}(s) \subset C_{H\times K}(s^k) &= (H\times K)\cap C_A(s^k) \\ &= (H\times K)\cap \left(C_{A_1}(s_1^k)\times\cdots\times C_{A_n}(s_n^k)\right) \\ &\subsetneq (H\times K)\cap \left(C_{A_1}(s_1^k)\times\cdots\times C_{A_{i-1}}(s_{i-1}^k)\times A_i\times C_{A_{i+1}}(s_{i+1}^k)\times\cdots\times C_{A_n}(s_n^k)\right) \\ &\subsetneq C_{H\times K}(s_1^k\cdots s_{i-1}^k s_{i+1}^k\cdots s_n^k) \end{aligned}$$

Since S is IMC, it follows that $s_1^k\cdots s_{i-1}^k s_{i+1}^k\cdots s_n^k = 1$, which amounts to saying that $s_j^k = 1$ for every $j \neq i$, or equivalently that $s^k \in A_i$.

Notice that, for every $j \neq i$, $C_{A_j}(s_j)$ has finite index in $C_{A_j}(s_j^k) = C_{A_j}(1) = A_j$ since A_j has almost stable centralisers, which amounts to saying that s_j belongs to $\mathrm{VZ}(A_j)$.

So far, we have proved that s belongs to

$$\mathrm{VZ}(A_1)\times\cdots\times\mathrm{VZ}(A_{i-1})\times A_i\times\mathrm{VZ}(A_{i+1})\times\cdots\times\mathrm{VZ}(A_n)$$

and that $s^k \in A_i$. Given an $r \in S\backslash\{s\}$, we know similarly that there exist $\ell \geq 1$ and $1 \leq j \leq n$ such that r belongs to

$$\mathrm{VZ}(A_1)\times\cdots\times\mathrm{VZ}(A_{j-1})\times A_j\times\mathrm{VZ}(A_{j+1})\times\cdots\times\mathrm{VZ}(A_n)$$

and such that $r^\ell \in A_j$. If $i \neq j$, then clearly $[r^\ell, s^k] = 1$, which is impossible as S is IMC. Hence $i = j$. We conclude that S, and a fortiori H, is contained in

$$\mathrm{VZ}(A_1)\times\cdots\times\mathrm{VZ}(A_{i-1})\times A_i\times\mathrm{VZ}(A_{i+1})\times\cdots\times\mathrm{VZ}(A_n),$$

as desired. □

We mention a first consequence of Lemma 3.5:

Corollary 3.6. *Let Ψ_1, Ψ_2 be two finite graphs that contain at least two vertices and are not joins. If $A(\Psi_1)\times A(\Psi_2) = H_1\times H_2$ for some nontrivial subgroups $H_1, H_2 \leq A(\Psi_1)\times A(\Psi_2)$, then $H_1 = A(\Psi_1)$ and $H_2 = A(\Psi_2)$ up to switching H_1 and H_2.*

Proof. We know from Corollary 3.3 that $A(\Psi_1)$ and $A(\Psi_2)$ are not cyclic and admit an IMC generating set. Moreover, as a consequence of Corollary 2.11, H_1 and H_2 have (almost) stable centralisers. Therefore, Lemma 3.5 applies to the inclusion map $A(\Psi_1)\times A(\Psi_2) \hookrightarrow H_1\times H_2$, proving that $A(\Psi_1)$ and $A(\Psi_2)$ are contained in H_1 or H_2. Since H_1 and H_2 are nontrivial, clearly $A(\Psi_1)$ and $A(\Psi_2)$ cannot be both contained in either H_1 or H_2. Up to switching H_1 and H_2, say that $A(\Psi_1) \leq H_1$ and $A(\Psi_2) \leq H_2$. From the equality $A(\Psi_1)\times A(\Psi_2) = H_1\times H_2$, we conclude that $H_1 = A(\Psi_1)$ and $H_2 = A(\Psi_2)$. □

4. Flat braids on seven strands

This section is dedicated to the proof of the following statement, which will be the key in order to deduce that most pure flat braid groups are not virtually right-angled Artin groups.

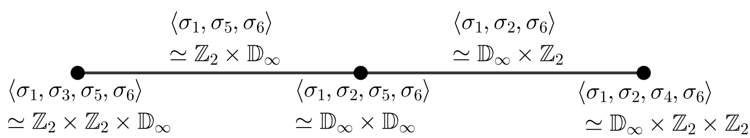
Theorem 4.1. *The group* PFB_7 *is not virtually a right-angled Artin group.*

We denote by $\sigma_1, \dots, \sigma_6$ the usual generators of FB_7, i.e., each σ_i is an elementary twist of the i-th and $(i+1)$-st strands. Equivalently, when thinking of FB_7 as the right-angled Coxeter group $C(P_5^{\mathrm{opp}})$, $\sigma_1, \dots, \sigma_6$ correspond to the generators given by the successive vertices along the path P_5.

From the presentation

$$\langle \sigma_1, \dots, \sigma_6 \mid \sigma_i^2 = 1 \ (1 \leq i \leq 6),\ [\sigma_i, \sigma_j] = 1 \ (|i-j| \geq 2) \rangle$$

of FB_7, one easily verifies that FB_7 decomposes as the following graph of groups:



We refer to this decomposition of FB_7 as its *tubular decomposition*.[1] See [Serre 1980] and [Scott and Wall 1979] for more information on graphs of groups. The key point is that vertex groups are virtually $\mathbb{Z}$ or $\mathbb{Z}^2$ and that edge groups are virtually $\mathbb{Z}$.

We use this tubular decomposition in order to find restrictions on the possible subgroups of FB_7. This will allow us to show that the only right-angled Artin groups that are subgroups of PFB_7 are of the form $A(\Gamma)$ where Γ is a forest (see Corollary 4.4).

Lemma 4.2. *The group* FB_7 *does not contain a subgroup isomorphic to* $\mathbb{Z}^3$, *to* $\mathbb{F}_2 \times \mathbb{F}_2$, *or to the fundamental group of a closed surface of genus* ≥ 2.

Proof. As a consequence of its tubular decomposition, FB_7 acts on a tree T with virtually $\mathbb{Z}$ or $\mathbb{Z}^2$ vertex-stabilisers and with virtually $\mathbb{Z}$ edge-stabilisers. If $g \in \mathrm{FB}_7$ induces a loxodromic isometry on T, then its centraliser $C(g)$ in FB_7, which must stabilise the axis of γ of g in T, is necessarily (virtually cyclic)-by-$\mathbb{Z}$. (Indeed, the action of $C(g)$ on γ by translation induces an epimorphism to $\mathbb{Z}$ whose kernel fixes γ pointwise, and consequently must be virtually cyclic since edge groups are virtually $\mathbb{Z}$.)

[1] In reference to *tubular groups*, i.e., fundamental groups of graphs of groups whose vertex groups are $\mathbb{Z}^2$ and whose edge groups are $\mathbb{Z}$.

Since every element of $\mathbb{Z}^3$ has centraliser $\mathbb{Z}^3$, it follows that, if $\mathbb{Z}^3$ is a subgroup of FB_7, then it cannot contain a loxodromic isometry. Thus, it must be elliptic in T, which is impossible since vertex-stabilisers are virtually $\mathbb{Z}$ or $\mathbb{Z}^2$. Therefore, $\mathbb{Z}^3$ cannot be a subgroup of FB_7.

If $\mathbb{F}_2 \times \mathbb{F}_2$ is a subgroup of FB_7, then an $\mathbb{F}_2$-factor cannot be elliptic in T, since vertex-stabilisers are virtually $\mathbb{Z}$ or $\mathbb{Z}^2$, so it must contain a loxodromic isometry. But the centraliser of an element of an $\mathbb{F}_2$-factor always contains a nonabelian subgroup, contracting the previous observation. Therefore, $\mathbb{F}_2 \times \mathbb{F}_2$ cannot be a subgroup of FB_7.

Finally, it remains to verify that FB_7 does not contain a subgroup isomorphic to the fundamental group of a closed surface of genus ≥ 2. More generally:

Claim 4.3. *Let G be a one-ended hyperbolic group. Then G is not isomorphic to a subgroup of* FB_7.

What we need to know about hyperbolic groups is that, for every infinite-order element $g \in G$, there exists a unique maximal virtually cyclic subgroup $E(g)$ containing $\langle g \rangle$, which we will call the *elementary closure*. (Geometrically, given a quasi-axis γ of g in G, $E(g)$ corresponds to the subgroup given by the elements $h \in G$ such that the Hausdorff distance between γ and $h\gamma$ is finite. Or equivalently, to the stabiliser of the pair of points at infinity of γ.) Assume for contradiction that G is isomorphic to a subgroup of FB_7. Fix a minimal G-invariant subtree in the Bass-Serre tree associated to the tubular decomposition of FB_7. Because G is one-ended, edge groups must be virtually $\mathbb{Z}$; and, because G does not contain $\mathbb{Z}^2$, vertex subgroups must be virtually $\mathbb{Z}$ as well. Thus, vertex groups are pairwise commensurable in G. This implies that they all have the same elementary closure E. Then, E yields a normal virtually $\mathbb{Z}$ subgroup of G. The only possibility is that G is virtually $\mathbb{Z}$ itself, which is impossible as G is supposed to be one-ended. □

Corollary 4.4. *If a right-angled Artin group $A(\Gamma)$ embeds into* FB_7, *then Γ is a forest. If Γ is disconnected, then $A(\Gamma)$ has infinite index in* FB_7.

Proof. If Γ contains an induced cycle of length three (resp. four, at least five), then $A(\Gamma)$ contains a subgroup isomorphic to $\mathbb{Z}^3$ (resp. $\mathbb{F}_2 \times \mathbb{F}_2$, the fundamental group of a closed surface of genus ≥ 2 (see for instance [Servatius et al. 1989])), which prevents FB_7 from containing $A(\Gamma)$ according to Lemma 4.2. Therefore, Γ must be a forest. If Γ is disconnected, then $A(\Gamma)$ splits as a free product of two infinite groups, which prevents FB_7 from containing $A(\Gamma)$ as a finite-index subgroup since FB_7 is one-ended (which amounts to saying, when thinking of FB_7 as the right-angled Coxeter group $C(P_5^{\mathrm{opp}})$, that no complete subgraph separates P_5^{opp}). □

In order to prove Theorem 4.1, it remains to distinguish PFB_7 from right-angled Artin groups defined by trees. For this purpose, we introduce a specific subgroup.

Definition 4.5. Let G be a group. An element $g \in G$ is *thick* if its centraliser is not virtually abelian. The *thick subgroup* $\mathrm{Thick}(G)$ is the subgroup of G generated by the centralisers of all its thick elements.

It is worth noticing that, since conjugates of thick elements are thick themselves, thick subgroups are always normal.

As a consequence of Proposition 2.8, thick elements in right-angled Artin groups defined by trees coincide with nontrivial powers of generators given by vertices that are not leaves. Thus, the tree defining our right-angled Artin group can be essentially recovered from the thick elements and how they commute with each other. In other words, the structure of the group is entirely contained in its thick elements. So it makes sense to use such elements in order to determine whether or not a group can be described as a right-angled Artin group over a tree. For us, the key observation is that thick subgroups are not proper for right-angled Artin groups defined by trees while the thick subgroup of PFB_7 is rather small (in particular, it has infinite index). This is the content of the next two lemmas.

Lemma 4.6. *For every tree Γ with at least three vertices*, $\mathrm{Thick}(A(\Gamma)) = A(\Gamma)$.

Proof. For every vertex $u \in \Gamma$ of degree ≥ 2, the centraliser of the corresponding generator of $A(\Gamma)$ is $\langle \mathrm{link}(u) \rangle$, which is a free group of rank ≥ 2. Therefore, such a generator is a thick element. Since a leaf of Γ must be adjacent to some vertex of degree ≥ 2, a generator of $A(\Gamma)$ that is not thick must belong to the centraliser of a thick centraliser. Therefore, every generator of $A(\Gamma)$ belongs to the thick subgroup, hence the desired equality. □

Lemma 4.7. *The quotient* $\mathrm{FB}_7/\mathrm{Thick}(\mathrm{PFB}_7)$ *is isomorphic to the Coxeter group* $C(\Gamma)$ *where* Γ *is the labelled graph defined as follows*:

- *the underlying graph of* Γ *is a complete graph with six vertices* $x_1, \dots, x_6$;
- *the edge connecting* x_i *and* x_j *is labelled by* 2 *whenever* $|i-j| \geq 2$;
- *the edge connecting* x_i *and* x_{i+1} *is labelled by* 3 *if* $i \neq 3$ *and* ∞ *otherwise.*

In particular, *it is infinite.*

Proof. Let $\pi : \mathrm{FB}_7 \to C(\Gamma)$ denote the morphism that sends σ_i to x_i for every $1 \leq i \leq 6$. Our goal is to prove that $\ker(\pi) = \mathrm{Thick}(\mathrm{PFB}_7)$, which will conclude the proof of our lemma. First, let us verify that $\mathrm{Thick}(\mathrm{PFB}_7)$ is contained in $\ker(\pi)$.

Claim 4.8. *An element of* FB_7 *is thick if and only if it is conjugate to a nontrivial power of* $\sigma_1\sigma_2$ *or* $\sigma_5\sigma_6$.

We think of FB_7 as the right-angled Coxeter group $C(P_5^{\mathrm{opp}})$. Let $g \in \mathrm{FB}_7$ be a thick element. Up to conjugating g, we can assume that g is graphically cyclically reduced. Because g has infinite order, necessarily $\mathrm{supp}(g)$ is not complete. And, because the centraliser of g is not virtually abelian, it follows from

Proposition 2.8 that link(supp(g)) contains at least two nonadjacent vertices but is not just a pair of two nonadjacent vertices. In P_5^{opp}, there are only two possibilities: either supp(g) $= \{\sigma_1, \sigma_2\}$ and link(supp(g)) $= \{\sigma_4, \sigma_5, \sigma_6\}$, or supp($g$) $= \{\sigma_5, \sigma_6\}$ and link(supp(g)) $= \{\sigma_1, \sigma_2, \sigma_3\}$. In the first case, g is a nontrivial power of $\sigma_1\sigma_2$; and, in the second case, g is a nontrivial power of $\sigma_5\sigma_6$. Conversely, it follows from Proposition 2.8 that the centraliser of a nontrivial power $(\sigma_1\sigma_2)^k$ (resp. $(\sigma_5\sigma_6)^k$) is $\langle \sigma_1\sigma_2\rangle \times \langle \sigma_4, \sigma_5, \sigma_6\rangle$ (resp. $\langle \sigma_1, \sigma_2, \sigma_3\rangle \times \langle \sigma_5\sigma_6\rangle$), which is isomorphic to $\mathbb{Z} \times (\mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2))$. Thus, the centraliser of $\sigma_1\sigma_2$ (resp. $\sigma_5\sigma_6$) is indeed not virtually abelian. This concludes the proof of Claim 4.8.

Since an element of PFB_7 is thick in PFB_7 if and only if it is thick in FB_7, it follows from Claim 4.8 that an element of PFB_7 is thick if and only if it is conjugate to a nontrivial power of $(\sigma_1\sigma_2)^3$ or $(\sigma_5\sigma_6)^3$. We deduce from Proposition 2.8 that the centraliser of $(\sigma_1\sigma_2)^3$ in FB_7 is $\langle \sigma_1\sigma_2\rangle \times \langle \sigma_4, \sigma_5, \sigma_6\rangle$, so the centraliser of $(\sigma_1\sigma_2)^3$ in PFB_7 is

$$\langle (\sigma_1\sigma_2)^3 \rangle \times (\mathrm{PFB}_7 \cap \langle \sigma_4, \sigma_5, \sigma_6\rangle) .$$

By noticing that π restricts on $\langle \sigma_1, \sigma_2, \sigma_3\rangle \simeq \mathrm{FB}_4$ (resp. $\langle \sigma_4, \sigma_5, \sigma_6\rangle \simeq \mathrm{FB}_4$) to the canonical map to the permutation group $\langle x_1, x_2, x_3\rangle \simeq \mathrm{Sym}(4)$ (resp. $\langle x_4, x_5, x_6\rangle \simeq \mathrm{Sym}(4)$), it follows that the centraliser above is contained in the kernel of π. Symmetrically, we show that the centraliser of $(\sigma_5\sigma_6)^3$ in PFB_7 is contained in $\ker(\pi)$. Thus, we have proved that $\ker(\pi)$ contains the centraliser in PFB_7 of every thick element of PFB_7. In other words, $\mathrm{Thick}(\mathrm{PFB}_7) \leq \ker(\pi)$, as desired.

By comparing the Coxeter presentations of FB_7 and $C(\Gamma)$, it is clear that $\ker(\pi)$ coincides with the normal closure in FB_7 of $\{(\sigma_1\sigma_2)^3, (\sigma_2\sigma_3)^3, (\sigma_4\sigma_5)^3, (\sigma_5\sigma_6)^3\}$. Because $\mathrm{Thick}(\mathrm{PFB}_7)$ is a normal subgroup of FB_7, as a consequence of Fact 4.9 below, it suffices to notice that $(\sigma_1\sigma_2)^3, (\sigma_2\sigma_3)^3, (\sigma_4\sigma_5)^3, (\sigma_5\sigma_6)^3$ all belong to $\mathrm{Thick}(\mathrm{PFB}_7)$ in order to conclude that $\ker(\pi)$ is contained in $\mathrm{Thick}(\mathrm{PFB}_7)$. But we already know that $(\sigma_1\sigma_2)^3$ and $(\sigma_5\sigma_6)^3$ are thick elements of PFB_7, and $(\sigma_2\sigma_3)^3$ (resp. $(\sigma_4\sigma_5)^3$) belongs to the centraliser of $(\sigma_5\sigma_6)^3$ (resp. of $(\sigma_1\sigma_2)^3$).

Fact 4.9. *Let G be a group. For every normal subgroup $H \lhd G$,* Thick(H) *is a normal subgroup of G.*

The action of G on H by conjugation permutes the thick elements of H and sends centralisers to centralisers. Therefore, G permutes the centralisers of the thick elements of H, proving that Thick(H) is stabilised by conjugation, or equivalently that Thick(H) is a normal subgroup of G. □

Proof of Theorem 4.1. Assume for contradiction that PFB_7 contains a right-angled Artin group $A(\Gamma)$ as a finite-index subgroup. It follows from Corollary 4.4 that Γ must be a tree (with at least three vertices since FB_7 is not virtually abelian), hence

$\mathrm{Thick}(A(\Gamma)) = A(\Gamma)$ according to Lemma 4.6. But we clearly have $\mathrm{Thick}(A(\Gamma)) \leq \mathrm{Thick}(\mathrm{PFB}_7)$, hence

$$|\mathrm{PFB}_7/\mathrm{Thick}(\mathrm{PFB}_7)| \leq |\mathrm{PFB}_7/A(\Gamma)| < \infty.$$

This contradicts Lemma 4.7, which implies that $\mathrm{PFB}_7/\mathrm{Thick}(\mathrm{PFB}_7)$ is infinite. □

5. Pure flat braid groups are not right-angled Artin groups

In this section, we prove the main result of this article:

Theorem 5.1. *For every $n = 7$ or ≥ 11, PFB_n is not virtually a right-angled Artin group.*

We start by stating and proving a general criterion that allows us to show that, under some assumptions, if a group G_1 can be realised as a finite-index subgroup in a group G_2, then every factor of a maximal product subgroup of G_2 contains as a finite-index subgroup a factor of a maximal product subgroup of G_1.

Proposition 5.2. *Let G_1 and G_2 be two torsion-free groups. Assume that*:

(i) *In both G_1 and G_2, a subgroup commensurable to a product of two infinite groups is contained in a maximal product subgroup.*

(ii) *In G_1, every maximal product subgroup Q decomposes as $Q_1 \times \cdots \times Q_s$ where each Q_i either is infinite cyclic or admits an IMC generating set.*

(iii) *In G_2, if two maximal product subgroups P_1 and P_2 are such that $P_1 \cap P_2$ has finite index in P_1, then $P_1 = P_2$.*

(iv) *G_2 has almost stable centralisers.*

If G_1 is a finite-index subgroup of G_2 and if $P := P_1 \times \cdots \times P_s$ is a maximal product subgroup of G_2 such that $\mathrm{VZ}(P) = \{1\}$ and such that no P_i is virtually a product of two infinite groups, then there exists a maximal product subgroup $R := R_1 \times \cdots \times R_s$ of G_1 such that each R_i is a finite-index subgroup of P_i.

Proof. Let $P = P_1 \times \cdots \times P_s$ be a maximal product subgroup of G_2 such that $\mathrm{VZ}(P) = \{1\}$ and such that no P_i is virtually a product of two infinite groups. It follows from (i) that $P \cap G_1$ is contained in some maximal product subgroup $R \leq G_1$. Similarly, R must be contained in some maximal product subgroup P^+ of G_2. Since $P \cap G_1$ has finite index in P, it follows from (iii) that P^+ actually coincides with P. Let $R = R_1 \times \cdots \times R_m$ denote the decomposition of R given by (ii). Notice that, since $\mathrm{VZ}(P)$ is trivial, necessarily $\mathrm{VZ}(R)$ must be trivial as well, which implies that no R_i is (virtually) cyclic.

Notice that $P \cap G_1 \leq R \leq P$ and that $P \cap G_1$ has finite index in P, so R must have finite index in P. Applying Lemma 3.5, which is possible thanks to (ii) and (iv), we deduce that, for every $1 \leq i \leq m$, there exists $1 \leq \sigma(i) \leq s$ such that $R_i \leq P_{\sigma(i)}$. Notice that

$$P/R \equiv \prod_{i=1}^{r} \Big(P_i / \prod_{j \in \sigma^{-1}(i)} R_j \Big),$$

which must be finite. Consequently, $\prod_{j \in \sigma^{-1}(i)} R_j$ must have finite index in P_i for every $1 \leq i \leq r$. But we know by assumption that no P_i is virtually a product, so σ must be bijective. Thus, up to reordering the factors of P, we have proved that $m = s$ and that R_i has finite index in P_i for every $1 \leq i \leq s$, as desired. □

Thanks to Lemmas 2.6, 3.4, 2.7, and 2.10, we have a good understanding of when a graph product satisfies the assumptions of Proposition 5.2. This applies in particular to flat braid groups, from which it is not difficult to deduce similar statements for their pure subgroups:

Lemma 5.3. *For every $n \geq 2$, the following assertions hold:*

(i) *A subgroup of* PFB_n *commensurable to a product of two infinite groups is contained in a maximal product subgroup.*

(ii) *In* PFB_n, *if two maximal product subgroups P_1 and P_2 are such that $P_1 \cap P_2$ has finite index in P_1, then $P_1 = P_2$.*

(iii) PFB_n *has almost stable centralisers.*

(iv) *The maximal product subgroups of* PFB_n *are the conjugates of*

$$\mathrm{PFB}_i \times \mathrm{PFB}_{n-i} := (\mathrm{PFB}_n \cap \langle \sigma_1, \dots, \sigma_{i-1} \rangle) \times (\mathrm{PFB}_n \cap \langle \sigma_{i+1}, \dots, \sigma_{n-1})$$

for $3 \leq i \leq n-3$.

Proof. Let P be a product in PFB_n. According to Lemma 2.6, P is contained in a maximal product subgroup Q of FB_n. Thinking of FB_n as a right-angled Coxeter group, we deduce from Lemma 2.5 that Q is conjugate to $\langle \sigma_1, \dots, \sigma_{i-1} \rangle \times \langle \sigma_{i+1}, \dots, \sigma_{n-1} \rangle$ for some $3 \leq i \leq n-3$. Thus, P is contained in a conjugate of $\mathrm{PFB}_i \times \mathrm{PFB}_{n-i}$. This proves (iv). Moreover, our argument shows the following assertion, which we will use in order to prove the rest of the lemma:

Fact 5.4. *The maximal product subgroups of* PFB_n *are the intersection with* PFB_n *of the maximal product subgroups of* FB_n.

If $H \leq \mathrm{PFB}_n$ is commensurable to a product of two infinite groups, then we know from Lemma 2.6 that H is contained in a maximal product subgroup Q of FB_n. Thus, H is contained in $Q \cap \mathrm{PFB}_n$, which is a maximal product subgroup of PFB_n according to Fact 5.4. This proves (i).

Let P_1 and P_2 be two maximal product subgroups of PFB_n such that $P_1 \cap P_2$ has finite index in P_1. According to Lemma 2.6, P_1 (resp. P_2) is contained in a maximal product subgroup P_1^+ (resp. P_2^+) of FB_n. Notice that, as a consequence of Fact 5.4, $P_1^+ \cap \mathrm{PFB}_n$ (resp. $P_2^+ \cap \mathrm{PFB}_n$) is a maximal product subgroup of PFB_n. Since it contains P_1 (resp. P_2), necessarily $P_1 = P_1^+ \cap \mathrm{PFB}_n$ (resp. $P_2 = P_2^+ \cap \mathrm{PFB}_n$). Since $P_1^+ \cap P_2^+$ contains $P_1 \cap P_2$, and since the latter has finite index in P_1, and a fortiori in P_1^+, we deduce from Lemma 2.7 that $P_1^+ = P_2^+$. Hence

$$P_1 = P_1^+ \cap \mathrm{PFB}_n = P_2^+ \cap \mathrm{PFB}_n = P_2,$$

proving assertion (ii).

Finally, assertion (iii) follows from Corollary 2.11, since pure flat braid groups embed into right-angled Artin groups (as a consequence of [Genevois 2020, Example 5.40] (see also [Farley 2021]) and [Genevois 2018, Theorem 1.2]). □

Proof of Theorem 5.1. For $n = 7$, the desired conclusion is given by Theorem 4.1. From now on, assume that $n \geq 9$. Notice that, according to Lemmas 2.6, 3.4, and 5.3, assumptions (i)–(iv) of Proposition 5.2 are satisfied for G_1 a right-angled Artin group and G_2 a pure flat braid group.

Assume for contradiction that PFB_n contains a finite-index subgroup that is a right-angled Artin group $A(\Gamma)$. According to Lemma 5.3, $P := \mathrm{PFB}_7 \times \mathrm{PFB}_{n-7}$ is a maximal product subgroup of PFB_n. We know from Corollaries 2.15 and 2.17 that PFB_k is not virtually a product of two infinite groups and has a trivial virtual centre whenever $k \geq 4$. Consequently, Proposition 5.2 applies and shows that $A(\Gamma)$ contains a maximal product subgroup $R := R_1 \times R_2$ such that R_1 (resp. R_2) has finite index in PFB_7 (resp. PFB_{n-7}). Since R_1 must be a right-angled Artin group according to Corollary 3.6, we deduce that PFB_7 is virtually a right-angled Artin group, contradicting Theorem 4.1. □

6. An instance of commensurability

In contrast with Theorem 5.1, in this section we prove the following observation:

Theorem 6.1. *The flat braid group* FB_7 *is commensurable to the right-angled Artin group* $A(P_4)$.

After preparing the ground in Section 6.1, we show in Section 6.2 that FB_7 is commensurable to a *lampraag* $A(\mathbb{L}) \rtimes \mathbb{Z}$. A key observation is that both $A(\mathbb{L}) \rtimes \mathbb{Z}$ and $A(P_4)$ can be described as fundamental groups of compact 3-manifolds. Therefore, in order to deduce that these two groups are commensurable, it suffices to construct a common finite-sheeted cover for the two corresponding manifolds. This is what we do in Section 6.3.

6.1. ***Some quasi-median geometry.*** In the next section, we will use some quasi-median geometry of graph products, as introduced in [Genevois 2017]. In this section, we recall the few definitions and results that we need.

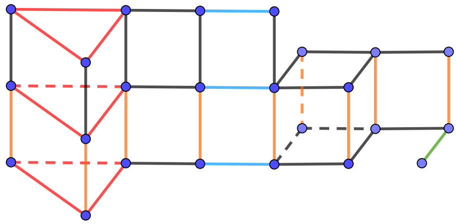


Figure 2. Some hyperplanes in a quasi-median graph.

Theorem 6.2 [Genevois 2017]. *For every graph* Γ *and every collection of groups* $\mathcal{G}$ *indexed by* Γ, *the Cayley graph*

$$\mathrm{QM}(\Gamma, \mathcal{G}) := \mathrm{Cayl}\left(\Gamma\mathcal{G}, \bigcup_{G \in \mathcal{G}} G \backslash \{1\}\right)$$

is a quasi-median graph.

Quasi-median graphs can be defined as retracts of Hamming graphs (i.e., products of complete graphs). There are many alternative characterisations of quasi-median graphs (see for instance [Bandelt et al. 1994]), but this definition is rather simple and it highlights the connection with median graphs (also known as one-skeletons of CAT(0) cube complexes; see [Genevois 2023]), which can be defined as retracts of hypercubes. It is also worth mentioning that median graphs coincide with triangle-free quasi-median graphs. As a consequence, Theorem 6.2 implies this well-known observation:

Corollary 6.3. *For every graph* Γ, *the Cayley graph* $\mathrm{Cayl}(C(\Gamma), \Gamma)$ *of the right-angled Coxeter group* $C(\Gamma)$ *is a median graph.*

Hyperplanes are the key objects in order to understand the geometry of median and quasi-median graphs. They are equivalence classes of edges with respect to the reflexive-transitive closure of the relation that identifies two edges whenever they belong to a common 3-cycle or whenever they are opposite edges in a 4-cycle. See Figure 2. Hyperplanes in quasi-median graphs of graph products are described by [Genevois 2017, Lemma 8.9 and Corollary 8.10]:

Lemma 6.4. *Let* Γ *be a graph and* $\mathcal{G}$ *a collection of groups indexed by* Γ. *Fix a vertex* $u \in \Gamma$ *and let* J_u *denote the hyperplane of* $\mathrm{QM}(\Gamma, \mathcal{G})$ *containing the edges of the clique* $\langle u \rangle$. *An edge* e *belongs to* J_u *if and only if* $e = \{g, g\ell\}$ *for some* $g \in \langle \mathrm{link}(u) \rangle$ *and* $\ell \in \langle u \rangle \backslash \{1\}$. *As a consequence, the stabiliser of* J_u *in* $\Gamma\mathcal{G}$ *is* $\langle \mathrm{star}(u) \rangle$.

As a consequence of our lemma, all the edges of a given hyperplane of $\mathrm{QM}(\Gamma, \mathcal{G})$ are naturally labelled by the same vertex of Γ. This labelling may quite useful in practice. For instance:

Lemma 6.5. *Let Γ be a graph and $\mathcal{G}$ a collection of groups indexed by Γ. Two transverse hyperplanes in $\mathrm{QM}(\Gamma, \mathcal{G})$ must be labelled by adjacent vertices of Γ.*

In order to state our last preliminary lemma, we need to introduce a couple of preliminary definitions.

Definition 6.6. Let G be a group acting on a quasi-median graph X. The *rotative-stabiliser* $\mathrm{stab}_{\circlearrowleft}(J)$ of a hyperplane J is the subgroup of $\mathrm{stab}(J)$ that stabilises each maximal complete subgraph of J.

In the sequel, given a group G acting on a median graph X and a collection of hyperplanes $\mathcal{J}$, we denote by $\mathrm{Rot}(\mathcal{J})$ the subgroup $\langle \mathrm{stab}_{\circlearrowleft}(J) \mid J \in \mathcal{J} \rangle$ of G.

Definition 6.7. Let X be a quasi-median graph and $\mathcal{J}$ a collection of hyperplanes. The *crossing graph* of $\mathcal{J}$ is the graph whose vertices are the hyperplanes of $\mathcal{J}$ and whose edges connect two hyperplanes whenever they are transverse.

Recall that two hyperplanes are *transverse* whenever they cover some 4-cycle. We also say that a hyperplane J is *tangent* to a subgraph Y whenever J contains an edge not in Y but with a vertex in Y.

Proposition 6.8. *Let G be a group acting on a quasi-median graph X. Fix a gated subgraph $Y \subset X$ and let $\mathcal{J}$ denote the collection of the hyperplanes of X tangent to Y. Assume that*

- *for every $J \in \mathcal{J}$, $\mathrm{stab}_{\circlearrowleft}(J)$ acts vertex-freely on X, and*
- *for all transverse $J, H \in \mathcal{J}$, every element of $\mathrm{stab}_{\circlearrowleft}(J)$ commutes with every element of $\mathrm{stab}_{\circlearrowleft}(H)$.*

Let Δ denote the crossing graph of $\mathcal{J}$ and let $\mathcal{G} = \{\mathrm{stab}_{\circlearrowleft}(J) \mid J \in \mathcal{J}\}$. Then

$$\langle \mathrm{Rot}(\mathcal{J}), \mathrm{stab}(Y) \rangle = \mathrm{Rot}(\mathcal{J}) \rtimes \mathrm{stab}(Y)$$

and the map $\Delta\mathcal{G} \to \mathrm{Rot}(\mathcal{J})$ that restricts on each vertex group $\mathrm{stab}_{\circlearrowleft}(J)$ to the identity is an isomorphism.

The proposition is a rather straightforward consequence of the ping-pong lemma [Genevois 2017, Proposition 8.44]. It is also a consequence of [Genevois 2017, Theorem 10.54]. For brevity, we refer the reader to [Genevois 2017] for details and we only mention that Proposition 6.8 applies to subgroups $G \leq \Gamma\mathcal{G}$ of graph products $\Gamma\mathcal{G}$, to quasi-median graphs $X = \mathrm{QM}(\Gamma, \mathcal{G})$, and to subgraphs of the form $\langle \Lambda \rangle$, where $\Lambda \subset \Gamma$ as justified by [Genevois 2017, Lemma 8.46] and [Genevois 2022c, Corollary 6.6].

6.2. ***Lampraag over*** $\mathbb{Z}$. We now introduce a family of groups we call *lampraags*, in analogy with lamplighter groups. They are particular examples of graph-wreath products [Kropholler and Martino 2016], and more generally of halo products [Genevois and Tessera 2024]. The lampraag over $\mathbb{Z}$ will provide a convenient model for FB_7 up to commensurability.

Definition 6.9. Let G be a group and $S \subset G$ a subset. The *lampraag over* (G, S) is the semidirect product

$$A(\mathrm{Cayl}(G, S)) \rtimes G,$$

where G permutes the generators of $A(\mathrm{Cayl}(G, S))$ according to its action by left-multiplication on $\mathrm{Cayl}(G, S)$.

In the definition, we do not require S to be a generating set. For instance, taking $S = \varnothing$ is allowed, in which case $A(\mathrm{Cayl}(G, \varnothing)) \rtimes G \simeq \mathbb{Z} * G$. In practice, however, we will be mainly interested in the case where $\mathrm{Cayl}(G, S)$ is connected, i.e., when S is a generating set. In fact, in the sequel, we will focus on the lampraag over $\mathbb{Z}$, where $\mathbb{Z}$ is endowed with its canonical generating set $\{1\}$. For convenience, we will denote by $\mathbb{L}$ the Cayley graph of $\mathbb{Z}$ with respect to $\{1\}$, i.e., the bi-infinite line; and by $A(\mathbb{L}) \rtimes \mathbb{Z}$ the corresponding lampraag.

Fact 6.10. *The lampraag* $A(\mathbb{L}) \rtimes \mathbb{Z}$ *admits* $\langle a, t \mid [a, tat^{-1}] = 1\rangle$ *as a presentation.*

Proof. It is clear that

$$\langle \dots, a_{-1}, a_0, a_1, \dots, t \mid ta_it^{-1} = a_{i+1} \text{ and } [a_i, a_{i+1}] = 1 \text{ for every } i \in \mathbb{Z}\rangle$$

is a presentation of our lampraag. Since $a_i = t^i a_0 t^{-i}$ for every $i \in \mathbb{Z}$, this infinite presentation can be simplified into the finite presentation given above. □

Interestingly, the lampraag over $\mathbb{Z}$ turns out to be connected to many other families of groups. For instance:

- As a consequence of Fact 6.10, this is a one-relator group. One-relator groups have been extensively studied in combinatorial group theory.
- As already mentioned, this is an example of a graph-wreath product [Kropholler and Martino 2016], and more generally of a halo product [Genevois and Tessera 2024].
- As we will see in Section 6.3, this is the fundamental group of a compact flip 3-manifold with boundary.
- Its finite-index subgroup $A(\mathbb{L}) \rtimes 2\mathbb{Z}$, which admits $\langle a, b, t \mid [a, b] = [a, tbt^{-1}] = 1\rangle$ as a presentation, is a *diagram group* [Genevois 2020, Example 5.43]. Interestingly, it is also proved in [Genevois 2020] that $A(\mathbb{L}) \rtimes 2\mathbb{Z}$ is not a right-angled Artin group.

• The lampraag $A(\mathbb{L}) \rtimes \mathbb{Z}$ is the fundamental group of a *Cartesian graph of groups* [Genevois 2017, Example 11.38].

• It is not difficult to deduce from the description of centralisers in right-angled Artin groups (see Proposition 2.8) that the lampraag $A(\mathbb{L}) \rtimes \mathbb{Z}$ cannot be a subgroup of a right-angled Artin group. However, its finite-index subgroup $A(\mathbb{L}) \rtimes 2\mathbb{Z}$ is a simple subgroup of $A(P_3)$. Indeed, if we denote by p, q, r, s the four vertices successively met along P_3, then the subgroup $\langle q, r, sp\rangle$ is isomorphic to $A(\mathbb{L}) \rtimes 2\mathbb{Z}$.

The rest of the section is dedicated to the proof of the following statement, which shows that the lampraag $A(\mathbb{L}) \rtimes \mathbb{Z}$ can be thought of as a good model of FB_7 up to commensurability.

Proposition 6.11. *The group* FB_7 *is commensurable to the lampraag* $A(\mathbb{L}) \rtimes \mathbb{Z}$.

Our argument allows us to be more explicit. To be precise, the proof of Proposition 6.11 shows that the map

$$\sigma_1\sigma_2 \mapsto a, \quad \sigma_3\sigma_4 \mapsto t^2, \quad \sigma_5\sigma_6 \mapsto tat^{-1}$$

induces an isomorphism from the index-8 normal subgroup $\langle \sigma_1\sigma_2, \sigma_3\sigma_4, \sigma_5\sigma_6\rangle$ of FB_7 to the index-2 subgroup $\langle a, tat^{-1}, t^2\rangle = A(\mathbb{L}) \rtimes 2\mathbb{Z}$ of $A(\mathbb{L}) \rtimes \mathbb{Z}$, when presented as $\langle a, t \mid [a, tat^{-1}] = 1\rangle$ (see Fact 6.10).

Proof of Proposition 6.11. As a first step, let us consider the subgroup

$$H := \langle \sigma_1\sigma_2, \sigma_3, \sigma_4, \sigma_5\sigma_6\rangle$$

of FB_7. One easily checks that this is a normal subgroup. The quotient FB_7/H is the product of two cyclic groups of order 2 generated by the images of σ_1 and σ_5. Therefore, H is a normal subgroup of index 4 in FB_7 with $\{1, \sigma_1, \sigma_5, \sigma_1\sigma_5\}$ as a set of representatives.

Now, let us investigate the structure of H. We know from Corollary 6.3 that the graph $\mathrm{M} := \mathrm{Cayl}(\mathrm{FB}_7, \{\sigma_1, \dots, \sigma_6\})$ is a median graph. Moreover, it follows from Lemma 6.5 that two hyperplanes of M labelled by σ_3 or σ_4 are never transverse; so the set $\mathcal{J}$ of all the hyperplanes of M labelled by σ_3 or σ_4 yields an arboreal structure on M, i.e., the graph $T(\mathcal{J})$ whose vertices are the connected components of the graph $\mathrm{M}\backslash\backslash\mathcal{J}$ (obtained from M by removing the hyperplanes from $\mathcal{J}$) and whose edges connect two components whenever they are separated by a single hyperplane is a tree. Make H act on $T(\mathcal{J})$.

Notice that there is a single H-orbit of vertices. Indeed, a vertex of $T(\mathcal{J})$ corresponds to a maximal subgraph of M all of whose edges are labelled by generators distinct from σ_3 and σ_4. In other words, the vertices of $T(\mathcal{J})$ correspond to the cosets of $K := \langle \sigma_1, \sigma_2, \sigma_5, \sigma_6\rangle$ in FB_7. Since we saw that $1, \sigma_1, \sigma_5, \sigma_1\sigma_5$ are representatives modulo H, it suffices to verify that K, $\sigma_1 K$, $\sigma_5 K$, and $\sigma_1\sigma_5 K$ all lie in the same H-orbit. But this is clear since

- $\sigma_1 K = \sigma_1\sigma_2 K$ with $\sigma_1\sigma_2 \in H$;
- $\sigma_5 K = \sigma_5\sigma_6 K$ with $\sigma_5\sigma_6 \in H$;
- $\sigma_1\sigma_5 K = \sigma_1\sigma_2\sigma_5\sigma_6 K$ with $\sigma_1\sigma_2, \sigma_5\sigma_6 \in H$.

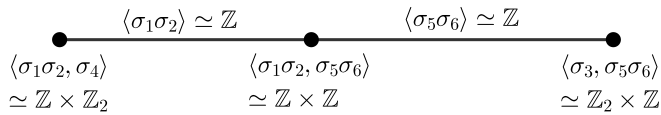
The H-stabiliser of the vertex of $T(\mathcal{J})$ given by K is $H \cap K = \langle \sigma_1\sigma_2, \sigma_5\sigma_6 \rangle$. Next, notice that there are two H-orbits of edges in $T(\mathcal{J})$. Indeed, the edges of $T(\mathcal{J})$ correspond to the hyperplanes of M labelled by σ_3 or σ_4; or equivalently, according to Lemma 6.4, to the cosets of

$$E_1 := \langle \sigma_1, \sigma_3, \sigma_5, \sigma_6 \rangle \quad \text{and} \quad E_2 := \langle \sigma_1, \sigma_2, \sigma_4, \sigma_6 \rangle$$

in FB_7. As previously, we deduce our claim from the following easy observations:

- for every $\eta \in \{\sigma_1, \sigma_5, \sigma_1\sigma_5\}$, $\eta E_1 = E_1$ since $\eta \in E_1$;
- $\sigma_1 E_2 = E_2$ since $\sigma_1 \in E_2$;
- $\sigma_5 E_2 = \sigma_5\sigma_6 E_2$ with $\sigma_5\sigma_6 \in H$;
- $\sigma_1\sigma_5 E_2 = \sigma_5 E_2 = \sigma_5\sigma_6 E_2$ with $\sigma_5\sigma_6 \in H$.

The H-stabiliser of the edge of $T(\mathcal{J})$ given by E_1 (resp. E_2) is $H \cap E_1 = \langle \sigma_3, \sigma_5\sigma_6 \rangle$ (resp. $H \cap E_2 = \langle \sigma_4, \sigma_1\sigma_2 \rangle$). We conclude from these properties satisfied by the action of H on $T(\mathcal{J})$ that H decomposes as the following graph of groups:

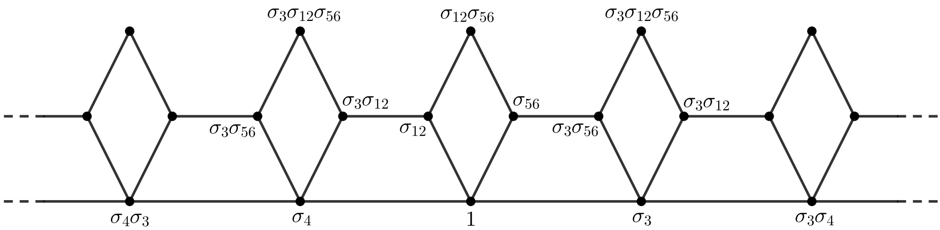


Alternatively, this amounts to describing H as the following graph product:

For convenience, we will write $\sigma_{12} := \sigma_1\sigma_2$ and $\sigma_{56} := \sigma_5\sigma_6$. Consider the action of H on the quasi-median graph QM given by this decomposition, as described in Section 6.1. Let $\mathcal{R}$ be the collection of all the hyperplanes of QM labelled by σ_{12} or σ_{56} and let Λ denote the subgraph $\langle \sigma_3, \sigma_4 \rangle \subset \mathrm{QM}$. Notice that Λ is a connected component of $\mathrm{QM}\backslash\backslash\mathcal{R}$. Of course, since $\langle \sigma_3, \sigma_4 \rangle$ is an infinite dihedral group, Λ is just a bi-infinite line; namely

$$\ldots,\ \sigma_4,\ \sigma_3\sigma_4,\ \sigma_4\sigma_3,\ \sigma_4,\ 1,\ \sigma_3,\ \sigma_3\sigma_4,\ \sigma_3\sigma_4\sigma_3,\ \ldots$$

The hyperplanes of $\mathcal{R}$ tangent to Λ are the $\langle \sigma_3, \sigma_4 \rangle$-translates of $J_{\sigma_{12}}$ and $J_{\sigma_{56}}$. The configuration in QM is the following:



Thus, the crossing graph of the hyperplanes of $\mathcal{R}$ tangent to Λ is also a bi-infinite line, and $\langle \sigma_3, \sigma_4 \rangle$ acts on it through two reflections fixing two adjacent vertices. Applying Proposition 6.8, we obtain the decomposition:

$$H = \langle \text{conjugates of } \sigma_1\sigma_2 \text{ and } \sigma_5\sigma_6 \rangle \rtimes \langle \sigma_3, \sigma_4 \rangle \simeq A(\mathbb{L}) \rtimes \mathbb{D}_\infty$$

where $\mathbb{D}_\infty$ acts on $\mathbb{Z}$ via two reflections (corresponding to σ_3 and σ_4) fixing 0 and 1 respectively. We conclude that the index-2 subgroup $\langle \sigma_1\sigma_2, \sigma_3\sigma_4, \sigma_5\sigma_6 \rangle$ of H, which is therefore an index-8 subgroup of FB_7, is isomorphic to the index-2 subgroup $A(\mathbb{L}) \rtimes 2\mathbb{Z}$ of the lampraag $A(\mathbb{L}) \rtimes \mathbb{Z}$. □

6.3. ***Some flip manifolds.*** In this section, our goal is to describe $A(\mathbb{L}) \rtimes \mathbb{Z}$ and $A(P_4)$ as fundamental groups of two flip manifolds M_1 and M_2, and then to construct a third flip manifold M_0 that is a common finite-sheeted cover of M_1 and M_2. Since $A(\mathbb{L}) \rtimes \mathbb{Z}$ is commensurable to PFB_7 according to Proposition 6.11, this will allow us to deduce that PFB_7 is commensurable to $A(P_4)$. Here, by a flip manifold, we mean a 3-manifold (possibly with boundary) obtained by gluing copies of $\mathbb{S}^1 \times \mathbb{S}(b)$, where $\mathbb{S}(b)$ denotes a punctured sphere with $b \geq 1$ boundary components. Two copies of $\mathbb{S}^1 \times \mathbb{S}(b)$ will be always glued along a torus boundary component in such a way that meridians and longitudinals are switched.

Let us begin by describing $A(\mathbb{L}) \rtimes \mathbb{Z}$ as the fundamental group of the flip manifold M_1 given by Figure 3. The fundamental group of $\mathbb{S}^1 \times \mathbb{S}(3)$ can be identified with $\mathbb{Z} \times \mathbb{F}_2 = \langle a, b, c \mid [a, c] = [b, c] = 1 \rangle$ where a (resp. b) corresponds to the red (resp. blue) boundary component of $\mathbb{S}(3)$ and where c is given by the $\mathbb{S}^1$-factor. The gluing identifies c with b and a with c, so the fundamental group of M_1 admits

$$\langle a, b, c, t \mid [a, c] = [b, c] = 1,\ tct^{-1} = b,\ tat^{-1} = c \rangle$$

as a presentation, which can be simplified as $\langle a, t \mid [a, tat^{-1}] = 1 \rangle$. We conclude from Fact 6.10 that $A(\mathbb{L}) \rtimes \mathbb{Z}$ is indeed isomorphic to the fundamental group of M_1.

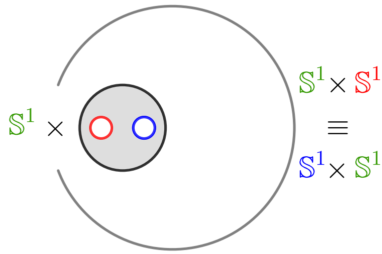


Figure 3. The flip manifold M_1 whose fundamental group is $A(\mathbb{L}) \rtimes \mathbb{Z}$.

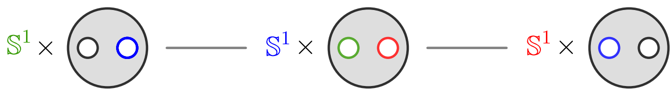


Figure 4. The flip manifold M_2 whose fundamental group is $A(P_4)$.

Next, let us describe $A(P_4)$ as the fundamental group of the flip manifold M_2 given by Figure 4. The three copies of $\mathbb{S}^1 \times \mathbb{S}(3)$ have fundamental groups isomorphic to $\mathbb{Z} \times \mathbb{F}_2$, say respectively $\langle a_i, b_i, c_i \mid [a_i, c_i] = [b_i, c_i] = 1 \rangle$ for $i = 1, 2, 3$ where a_i corresponds to the left (resp. right) inner boundary component of $\mathbb{S}(3)$ and where c_i is given by the $\mathbb{S}^1$-factor. The first gluing identifies c_1 with a_2 and b_1 with c_2, and the second gluing identifies c_2 with a_3 and b_2 with c_3. Therefore, the fundamental group of M_2 admits

$$\left\langle \begin{array}{c} a_1, a_2, a_3, \\ b_1, b_2, b_3, \\ c_1, c_2, c_3 \end{array} \middle| \begin{array}{l} [a_1, c_1] = [b_1, c_1] = [a_2, c_2] = [b_2, c_2] = [a_3, c_3] = [b_3, c_3] = 1, \\ c_1 = a_2, \ b_1 = c_2, \ c_2 = a_3, \ b_2 = c_3 \end{array} \right\rangle$$

as a presentation, which can be simplified as

$$\langle a_1, a_2, b_2, c_2, b_3 \mid [a_1, a_2] = [a_2, c_2] = [c_2, b_2] = [b_2, b_3] = 1 \rangle.$$

This is clearly a presentation of $A(P_4)$, concluding the proof that $A(P_4)$ is indeed isomorphic to the fundamental group of M_2.

Finally, let us construct a common finite cover of M_1 and M_2. Our flip manifold M_0 is described by Figure 5. In order to describe the covering maps $M_0 \to M_1, M_2$, we need two specific covering maps $\alpha, \beta : \mathbb{S}(6) \to \mathbb{S}(3)$. They are respectively described by Figures 6 and 7. It is worth noticing that the restriction of α to each boundary component is a 2-sheeted cover; and that the restriction of β to a red (resp. blue) boundary component is a 4-sheeted (resp. 1-sheeted) cover.

The covering map $\mu : M_0 \to M_1$ is defined as follows. The two pieces $\mathbb{S}^1 \times \mathbb{S}(6)$ of M_0 are sent to the piece $\mathbb{S}^1 \times \mathbb{S}(3)$ of M_1 through $(1, \alpha)$. And the four pieces $\mathbb{S}^1 \times \mathbb{S}(3)$ of M_0 are sent to the piece $\mathbb{S}^1 \times \mathbb{S}(3)$ of M_1 through $(2, \mathrm{id})$. By

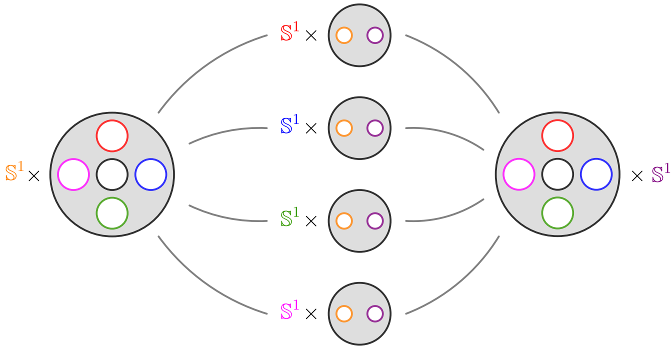


Figure 5. The flip manifold M_0, a common cover of M_1 and M_2.

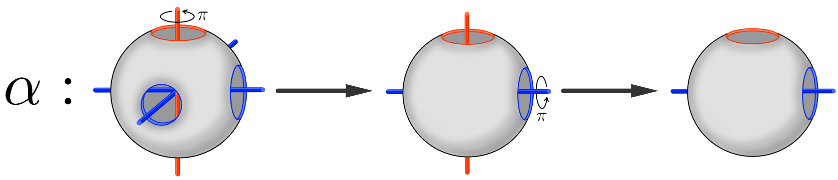


Figure 6. The 4-sheeted covering map $\alpha : \mathbb{S}(6) \to \mathbb{S}(3)$.

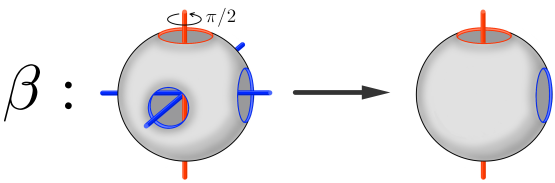


Figure 7. The 4-sheeted covering map $\beta : \mathbb{S}(6) \to \mathbb{S}(3)$.

construction, these isolated maps are compatible with the gluings defining M_0, and we get a 4-sheeted covering map $\mu : M_0 \to M_1$.

The covering map $\nu : M_0 \to M_2$ is defined as follows. The two pieces $\mathbb{S}^1 \times \mathbb{S}(6)$ of M_0 are sent to the left and right pieces $\mathbb{S}^1 \times \mathbb{S}(3)$ of M_2 through $(1, \beta)$. And the four pieces $\mathbb{S}^1 \times \mathbb{S}(3)$ of M_0 are sent to the middle piece $\mathbb{S}^1 \times \mathbb{S}(3)$ of M_2 through $(4, \beta)$. By construction, these isolated maps are compatible with the gluings defining M_0, and we get a 4-sheeted covering map $\nu : M_0 \to M_2$.

Since we have constructed 4-sheeted covering maps from M_0 to M_1 and M_2, the fundamental groups of M_1 and M_2 share isomorphic index-4 subgroups. Hence:

Proposition 6.12. *The lampraag $A(\mathbb{L}) \rtimes \mathbb{Z}$ and the right-angled Artin group $A(P_4)$ share an isomorphic index-4 subgroup.*

Combined with Proposition 6.11, we conclude that Theorem 6.1 holds. □

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