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CLASSIFICATION OF UNIMODAL ISOLATED COMPLETE INTERSECTION SINGULARITIES IN POSITIVE CHARACTERISTIC

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We classify unimodal isolated complete intersection singularities in arbitrary characteristic under contact equivalence. The classification over $\mathbb{C}$ has been done by A. Dimca and C. G. Gibson. We continue and generalize their work. To complete the classification, we generalized the complete transversal method into positive characteristic field, which is also useful in many other classification problem.

1. Introduction

Classification is one of the oldest topics in singularity theory. The modality of singularities for real and complex hypersurfaces was first introduced by V. I. Arnold in [2]. He also finished the classification of hypersurface singularities with small modality over $\mathbb{C}$ in [1]. G. M. Greuel and H. D. Nguyen generalized the notion of modality to the algebraic setting in [8], so that one can define modality over any algebraically closed field of arbitrary characteristic. They also classified simple (modality 0) hypersurface singularities in positive characteristic field under right equivalence.

The classification of isolated complete intersection singularities (ICIS) under contact equivalence was studied in the 1980s. Assume $(X, 0) = (f^{-1}(0), 0)$ to be a complete intersection germ with an isolated singularity defined by $f : F^n \to F^p$. We call such germs $I_{n,p}$. M. Giusti has shown that only $I_{1,1}$, $I_{2,2}$ and $I_{3,2}$ can be simple in [7]. He then classified all simple ICIS over $F = \mathbb{C}$. The classification of unimodal (i.e., modality 1) germs from plane to plane ($I_{2,2}$ case) was completed by A. Dimca and C. G. Gibson. C. T. C. Wall then classified unimodal germs of $I_{n,p}$ with $n > p$. Recently, T. H. Pham, G. Pfister, and G. M. Greuel generalized the modality of hypersurface singularities to ICIS, and classified zero-dimensional ICIS ($I_{2,2}$ case) over any algebraically closed field of arbitrary characteristic. In this

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paper, we continue their work. We use a different method to classify unimodal zero-dimensional ICIS of $I_{2,2}$ over any algebraically closed field of arbitrary characteristic. In Corollary 2.24, we show that for all zero-dimensional ICIS, only $I_{2,2}$ and $I_{3,3}$ can be unimodal. Unfortunately, the classification of $I_{3,3}$ seems very complicated and needs more new tools.

In Dimca and Gibson's previous work [5], the finite determinacy theorem works well in zero characteristic fields. T. H. Pham and G. M. Greuel generalized it to positive characteristic in [10]. Although the given boundary is sharp, it is still insufficient to deal with complicated (higher order) problems. For example, let $h = (x^3 + xy^3 + y^5, x^2y + y^4 + y^5)$ be the 5-jet of an ICIS f over an algebraically closed field F. If $\operatorname{char} F = 0$, then h is 5-determined. Therefore f is contact equivalent to $(x^3 + xy^3 + y^5, x^2y + y^4 + y^5)$, and then to $(x^3 + xy^3 + y^5, x^2y + y^4)$ after a transformation. Otherwise, we can only know that h is 7-determined in positive characteristic, which gives few idea about the form of f.

To solve the problem, we develop new tools. We generalize the complete transversal method introduced by Bruce, Kirk and du Plessis in [4] to the case of a positive characteristic field; this is useful in many classification problem (see Corollary 2.17). It can be used for semi-quasihomogeneous singularities over fields of arbitrary characteristic. For the above example, f is a semi-quasihomogeneous singularity with initial term $(x^3 + xy^3, x^2y + y^4)$. Using our method we can show that f is contact equivalent to $(x^3 + xy^3 + y^5, x^2y + y^4)$.

The main result is Theorem 8.4. Surprisingly, the classification result in positive characteristic turns out to be similar as the zero characteristic case except for some special characteristics.

In the following, we let F be an algebraically closed field with arbitrary characteristic, and write $R = F[\![x_1, \dots, x_n]\!]$, $\mathfrak{m} = \langle x_1, \dots, x_n\rangle \subset R$ and $p = \operatorname{char}(F)$.

2. Basic settings

We first talk about some basic concepts of ICIS. Then we introduce the notion of finite determinacy in positive characteristic from [10]. After that, we generalize the complete transversal method into positive characteristic. The estimation of modality is also given in this section.

2.1. *Basic concepts.*

Definition 2.1. (1) An ideal $I \subset R$ defines a complete intersection if I can be generated by $f_1, \dots, f_m$ with $f_i \in \mathfrak{m}$ for all i such that f_i is a non-zerodivisor of $R/\langle f_1, \dots, f_{i-1}\rangle$ for $i = 1, \dots, m$. Then $\dim R/I = n - m$.

(2) We call $f = (f_1, \dots, f_m)$ an isolated complete intersection singularity (ICIS) if $I = \langle f_1, \dots, f_m\rangle$ defines a complete intersection and there exists $k \in \mathbb{N}$ such that

$\mathfrak{m}^k \subset I + I_m(J(f))$, where $J(f) = (\partial f_i/\partial x_j)_{ij}$ is the $m \times n$ Jacobian matrix and $I_m(J(f))$ is the ideal generated by all $m \times m$ minors of $J(f)$. Define

$$I_{m,n} = \{ f = (f_1, \dots, f_m) \in R^m \mid f \text{ is an ICIS with codimension } n - m \}.$$

In this article, we focus on $I_{2,2}$, which denotes the zero-dimensional isolated complete intersection singularity in the plane.

Remark 2.2. A complete intersection f is isolated if and only if the corresponding Tjurina number

$$\tau(f) = \dim_F R^m \Big/ \Big(\langle f_1, \dots, f_m \rangle \cdot R^m + \Big\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \Big\rangle \Big)$$

is finite.

Definition 2.3. The contact group $\mathcal{K}$ is defined as

$$\mathcal{K} = \mathrm{GL}(m, R) \rtimes \mathrm{Aut}(R),$$

and the action of $\mathcal{K}$ on R^m is defined as

$$(U, \phi, f) \mapsto U \cdot \phi(f),$$

with $U \in \mathrm{GL}(m, R)$, $\phi \in \mathrm{Aut}(R)$, $f = (f_1, \dots, f_m) \in R^m$ and

$$\phi(f) = (f_1(\phi(\boldsymbol{x})), \dots, f_m(\phi(\boldsymbol{x})),$$

where $\phi(\boldsymbol{x}) = (\phi(x_1), \dots, \phi(x_n))$.

Let f and g define two isolated complete intersections of the same codimension $n - m$. f is called contact equivalent to g, denoted by $f \sim g$, if $g \in \mathcal{K} f$, that is, if there exists $U \in \mathrm{GL}(m, R)$ and $\phi \in \mathrm{Aut}(R)$ such that $g = U \cdot \phi(f)$.

2.2. *Tangent image and finite determinacy.* To classify the ICIS under contact equivalence, we need to work on jet spaces.

Definition 2.4. (1) The k-jet space of R^m is defined as $J_k = R^m/\mathfrak{m}^{k+1}R^m$. For $f \in R^m$, the k-jet of f is the image in J_k, denoted by $j_k(f)$. Let $\pi : J_l \to J_k$ be the natural projection and denote the kernel as $P_{k,l}$. If $f \in J_k$ is a k-jet, we denote the submanifold $J_l(f) = f + P_{k,l}$.

(2) We say that f is k-determined if for any $g \in R^m$ with $j_k(g) = j_k(f)$, we always have $g \sim f$.

Let $J_k = R^m/\mathfrak{m}^{k+1}R^m$ denote the k-jet space of R^m. Let

$$\mathcal{K}_k = \{ (j_k(U), j_k(\phi)) \mid U \in \mathrm{GL}(m, R), \phi \in \mathrm{Aut}(R) \}$$

be the k-jet algebraic group, with the algebraic action of $\mathcal{K}_k$ on the affine space J_k defined by

$$(j_k(U), j_k(\phi), j_k(f)) \mapsto j_k(U \cdot \phi(f)).$$

The tangent space $T_e(\mathcal{K}_k)$ of the algebraic group $\mathcal{K}_k$ has a natural Lie algebra structure (see [6, Chapter 4]). The orbit map $\pi : \mathcal{K}_k \to \mathcal{K}_k \cdot f$ induces the tangent map $d\pi : \mathrm{Lie}(\mathcal{K}_k) \to T_f(\mathcal{K}_k f)$. We denote the image of $d\pi$ by $\widetilde{T}_f(\mathcal{K}_k f)$, which coincides with $\mathrm{Lie}(\mathcal{K}_k) \cdot f$. When char $F = 0$, $d\pi$ is surjective and

$$\widetilde{T}_f(\mathcal{K}_k f) = T_f(\mathcal{K}_k f) \tag{2-1}$$

(and therefore $\widetilde{T}_f(\mathcal{K} f) = T_f(\mathcal{K} f)$). But when char $F > 0$, (2-1) may not hold. For details one can see [10, Section 2].

The tangent image is computed in [10] as follows.

Proposition 2.5. *The tangent image is identified with the submodule*

$$\widetilde{T}_f(\mathcal{K}_k f) = \Big(\langle f_1, \dots, f_m \rangle \cdot R^m + \mathfrak{m} \cdot \Big\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \Big\rangle + \mathfrak{m}^{k+1} R^m \Big) \Big/ \mathfrak{m}^{k+1} R^m.$$

The tangent image at f to the orbit $\mathcal{K} f$ is the submodule

$$\widetilde{T}_f(\mathcal{K} f) = \langle f_1, \dots, f_m \rangle \cdot R^m + \mathfrak{m} \cdot \Big\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \Big\rangle,$$

where $\langle f_1, \dots, f_m \rangle$ is regarded as an ideal of R and $\Big\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \Big\rangle$ is regarded as an ideal of R^m.

Finite determinacy is strongly related to the tangent image.

Theorem 2.6 (cf. [10, Theorem 3.2]). *Let $f = (f_1, \dots, f_m) \in R^m$. If there exists $k \in \mathbb{N}$ such that*

$$\mathfrak{m}^{k+2} \cdot R^m \subset \mathfrak{m} \cdot \widetilde{T}_f(\mathcal{K} f),$$

then f is $(2k - \mathrm{ord}(f) + 2)$-determined, where $\mathrm{ord}(f) = \min\{\mathrm{ord}(f_i) \mid i = 1, \dots, m\}$. That is, any $g \in R^m$ such that $j_{2k-\mathrm{ord}(f)+2}(g) = j_{2k-\mathrm{ord}(f)+2}(f)$ satisfies $g \sim f$.

Remark 2.7. If f is an isolated complete intersection singularity, then f is $(2\tau(f) - \mathrm{ord}(f) + 2)$-determined.

2.3. *Complete transversals.* We next introduce the complete transversal method.

Let $a = (a_1, \dots, a_n)$ be a given sequence of positive integers and $d = (d_1, \dots, d_m)$ a given sequence of nonnegative integers. Then $f = (f_1, \dots, f_m)$ is said to be weighted homogeneous of degree r (with respect to $(a; d)$) if

$$f_i(t^{a_1} x_1, \dots, t^{a_n} x_n) = t^{r+d_i} f_i(x_1, \dots, x_n)$$

for any $t \in F$ and $i = 1, 2, \dots, m$.

Let f be a k-jet in J_k, weighted homogeneous of degree 0 with respect to $(a; d)$.

Assume that

$$\text{(2-2)}\qquad \max(d_i) < (k+1)\min(a_j) \quad \text{or} \quad \min(d_i) > (k+1)\max(a_j).$$

For $l > k$, let $P_{k,l},\ J_l(f)$ be the subsets of J_l from Definition 2.4(1). We have the following useful theorem from [5].

Theorem 2.8. *For f as above, let $C \subset P_{k,l}$ be a linear subspace of $P_{k,l}$ satisfying*

$$P_{k,l} \subset C + \widetilde{T}_f(\mathcal{K}_l f) \cap P_{k,l};$$

we call C a complete transversal. Then C has the following property: every $g \in J_l(f)$ is in the same $\mathcal{K}_l$-orbit as an l-jet of the form $f + c$, for some $c \in C$.

Proof. For the case of $F = \mathbb{C}$, this is Proposition 1.3 in [5]. The original proof there carries over to positive characteristic without changes. □

Remark 2.9. Condition (2-2) is necessary. In its absence, one finds a counterexample in [3, Remark 1]: Let $f = y^2 + xy^3 \in F[[x, y]]$ with char $F = 2$. One can check that f does not satisfy (2-2). Computation shows $\widetilde{T}_f(\mathcal{K}_l f) = \mathfrak{m}^4/\mathfrak{m}^l$, which implies that the complete transversal C equals 0 for any $l > 4$ and then for any $g \in f + \mathfrak{m}^5$, $g \sim f$. But in fact $f + x^5$ is not contact equivalent to f.

2.4. ***Complete transversals and homogeneous filtrations.*** In this subsection we introduce the generalization of the complete transversal method following [4]. Then we generalize the work of [4] to any positive characteristic field.

Let $F^r_{a,d}R^m$ denote the submodule of R^m generated by the monomials of degree equal or greater than r with respect to $(a; d)$. The sequence $\{F^r_{a,d}R^m\}_{r\geq 0}$ defines a filtration of the module $F^0_{a,d}R^m$.

We introduce a filtration of the contact group $\mathcal{K}$ compatible with the weighted filtration. For details one may see [4, Section 2.3].

Definition 2.10. (i) For $r \geq 0$, define

$$F^r\mathcal{R} = (I_n + F^r_{a,a}R^n) \cap \mathcal{R}.$$

(ii) For $r \geq 0$, define

$$F^r\mathcal{C} = (I_{n+m} + F^r_{a\cup d,a\cup d}\widetilde{R}^{n+m}) \cap \mathcal{C},$$

where

$$\widetilde{R} = F[[x_1, \dots, x_n, y_1, \dots, y_m]]$$

and $a \cup d$ denotes the $(n+m)$-tuple $(a_1, \dots, a_n, d_1, \dots, d_m)$.

(iii) Since the contact group $\mathcal{K}$ equals $\mathcal{R} \rtimes \mathcal{C}$, we define

$$F^r\mathcal{K} = F^r\mathcal{R} \rtimes F^r\mathcal{C}.$$

Remark 2.11. For a survey of the standard Mather groups $\mathcal{K}, \mathcal{R}, \mathcal{C}$, one can refer to [12].

Proposition 2.12. (i) *$F^r\mathcal{K}$ respects the filtration $\{F^r_{a,d}R^m\}$; that is, for every $r, s \ge 0$, $(U,\phi) \in F^r\mathcal{K}$, $f \in F^sR^m$, $U\cdot\phi(f) \in F^sR^m$, where $F^sR^m = F^s_{a,d}R^m$ is the submodule of R^m generated by the monomials of degree equal or greater than s.*

(ii) *For $r, s, l \le 0$, the action of $F^r\mathcal{K}$ induces an action on $F^sR^m/F^{s+l}R^m$.*

(iii) *The Lie algebra action satisfies the following: for any $f - g \in F^t_{a,d}R^m$ with $f, g \in F^0_{a,d}R^m$ and $l \in \mathrm{Lie}(F^r_{a,d}R^m)$, we have $l\cdot f - l\cdot g \in F^{r+t}_{a,d}R^m$.*

After a computation of tangent spaces (similar to the one in [10, Proposition 2.5]), the tangent image of $F^r\mathcal{K}f$ can be regarded as

$$\widetilde{T}_f(F^r_{a,d}\mathcal{K}\cdot f) = F^r_{a\cup d,d}(\langle f_1,\dots,f_m\rangle\cdot R^m) + \sum_j F^r_{a,a_j}(\mathfrak{m})\cdot\frac{\partial f}{\partial x_j}, \tag{2-3}$$

and clearly

$$\widetilde{T}_f(F^r_{a,d}\mathcal{K}\cdot f) \subset T_f(F^r_{a,d}\mathcal{K}\cdot f)$$

also holds. We denote $\langle f_1,\dots,f_m\rangle\cdot R^m + \left\langle\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\right\rangle$ by $\widetilde{T}^e_f(\mathcal{K}f)$, using the same notation found in [10].

We have a similar complete transversal result for F^rR^m and $F^r\mathcal{K}$.

Proposition 2.13. *Let $f \in F^0_{a,d}R^m$. If T is a subspace of $F^r_{a,d}R^m$ satisfying*

$$F^{k+1}_{a,d}R^m \subset T + \mathrm{Lie}(F^1_{a,d}\mathcal{K})\cdot f + F^{k+2}_{a,d}R^m,$$

then any g with $g - f \in F^{k+1}_{a,d}R^m$ is contact equivalent to $f + t + \bar{f}$ for some $t \in T$ and $\bar{f} \in F^{k+2}_{a,d}R^m$.

To prove the proposition, we first need a lemma, which can be seen as Taylor series in positive characteristic. We omit the proof.

Lemma 2.14. *Let $f(x) = \sum_\alpha a_\alpha x^\alpha \in F[\![x_1,\dots,x_n]\!]$ and $\xi = (\xi_1,\dots,\xi_n) \in F^n$, then*

$$f(x+\xi) = f(x) + \sum_{i=1}^n \xi_i\frac{\partial f}{\partial x_i} + \text{terms of } f(x) \text{ with more than two } x_i \text{ replaced by } \xi_i. \tag{2-4}$$

Now we can prove Proposition 2.13.

Proof. Take the subset $W = F^{k+1}_{a,d}R^m \setminus F^{k+2}_{a,d}R^m$ of R^m, which is a finite-dimensional F-vector space. We claim that for any $f \in F^0_{a,d}R^m$ and $w \in W$, we have

$$\mathrm{Lie}(F^1\mathcal{K})\cdot(f+w) - \mathrm{Lie}(F^1\mathcal{K})\cdot f \in F^{k+2}_{a,d}R^m \tag{i}$$

and

$$f + \{\mathrm{Lie}(F^1\mathcal{K})\cdot f \cap W\} + F^{k+2}_{a,d}R^m \subset F^1\mathcal{K}\cdot f \cap \{f+W\} + F^{k+2}_{a,d}R^m. \tag{ii}$$

Inclusion (i) follows from Proposition 2.12(iii). For (ii), let $w = (w_1, \dots, w_m)$ be in $\{\mathrm{Lie}(F^1\mathcal{K}) \cdot f \cap W\}$. Then

$$w_j = \sum_{i=1}^{n} \xi_i \frac{\partial f_j}{\partial x_i} + \sum_{i=1}^{m} \lambda_{ji} f_i$$

is weighted homogeneous of degree $k+1$ with respect to $(a; d_j)$. For $r \in R$, write $r = \sum_\alpha r_\alpha x^\alpha$. Let $v_a(r) = \min_{r_\alpha \neq 0}\{v_a(x^\alpha)\}$, where $v_a(x^\alpha)$ denotes the weighted degree of the monomial x^α with respect to $(a; 0)$. Then $v_a(\xi_i) = a_i + k + 1$, $v_a(\lambda_{ji} f_i) = d_j + k + 1$. Hence $v_a(\xi_i) \geq v_a(x_i) + k + 1$ and $v_a(\lambda_{ji}) \geq d_j - d_i + k + 1$.

Let $\phi \in \mathrm{Aut}(R)$ be such that $\phi(x_1, \dots, x_n) = (x_1 - \xi_1, \dots, x_n - \xi_n)$. Set $U = (\lambda_{ji}) \in M_{m \times m}(R)$. We have

$$\phi\biggl(f + \sum_{i=1}^{n} \xi_i \frac{\partial f_j}{\partial x_i} - Uw\biggr) = f - \sum_{i=1}^{n} \xi_i \frac{\partial f_j}{\partial x_i} + h_1 + \sum_{i=1}^{n} \xi_i \frac{\partial f_j}{\partial x_i} + h_2 - Uw + h_3$$

by Lemma 2.14, where the h_i are the higher-order terms defined in (2-4). We have $h_i \in F_{a,d}^{k+2} R^m$ for $i = 1, 2, 3$ since the ξ_i appear twice in each term and $v_a(\xi_i) \geq v_a(x_i) + k + 1$. Moreover $Uw \in F_{a,d}^{k+2} R^m$ since for each j we have $v_a(\lambda_{ji} w_i) = v_a(\lambda_{ji}) + v_a(w_i) \geq d_j - d_i + k + 1 + d_i + k + 1 \geq d_j + k + 2$.

Set $\widetilde{w} = h_1 + h_2 + h_3 - Uw$. By the preceding discussion, $\widetilde{w} \in F_{a,d}^{k+2} R^m$. Without loss of generality, we can assume $v_a(f_1) \leq \cdots \leq v_a(f_m)$. Then $U(0)$ is an upper triangular matrix all of whose principal diagonal elements are zero. Hence $\mathrm{Id} - U$ must be invertible. We have $f + w \sim (\mathrm{Id} - U) \cdot \phi(f + w) = f + \widetilde{w}$. Denote $g = (\mathrm{Id} - U, \phi) \in \mathcal{K}$, we have $f + w = g^{-1}(f) + g^{-1}(\widetilde{w})$.

To finish (ii), it remains to show $g \in F^1\mathcal{K}$. Since $\xi = (\xi_1, \dots, \xi_n) \in F_{a,a}^1 R^n$ we have $\phi \in I_n + F_{a,a}^1 R^n = F^1\mathcal{R}$ by Definition 2.10. Similarly $\mathrm{Id} - U \in I_m + F_{a \cup d, a \cup d}^1 R^m \subset F^1\mathcal{C}$. Now we get $g = (\mathrm{Id} - U, \phi) \in F^1\mathcal{K}$. The claim is proved.

From inclusions (i) and (ii) we have

$$\begin{aligned}
\bigcup_{t \in T} F^1\mathcal{K} \cdot (f + t + F_{a,d}^{k+2} R^m) &\supset \bigcup_{t \in T} \{f + t + \mathrm{Lie}(F^1\mathcal{K}) \cdot (f + t) \cap W + F_{a,d}^{k+2} R^m\} \\
&= \bigcup_{t \in T} \{f + t + \mathrm{Lie}(F^1\mathcal{K}) \cdot f \cap W\} + F_{a,d}^{k+2} R^m \\
&= f + T + \mathrm{Lie}(F^1\mathcal{K}) \cdot f \cap W + F_{a,d}^{k+2} R^m \\
&= f + (T + \mathrm{Lie}(F^1\mathcal{K}) \cdot f) \cap W + F_{a,d}^{k+2} R^m \\
&= f + W + F_{a,d}^{k+2} R^m.
\end{aligned}$$

That is, for any $g = f + w + \bar{g}$ with $w \in W$ and $\bar{g} \in F_{a,d}^{k+2} R^m$, g is contact equivalent to $f + t + \bar{f}$ with $t \in T$ and $\bar{f} \in F_{a,d}^{k+2} R^m$. □

Remark 2.15. In [4], Mather's lemma is used to show that (i) ⇒ (ii). However, the proof of Mather's lemma relies on analysis in the complex field. We can complete the proof in this special case without Mather's lemma, using M. Giusti's proof of [7, Proposition 1].

Using induction, we can show:

Proposition 2.16. *Suppose that f is weighted homogeneous of weight r with respect to $(a; d)$. Take $s > r$. If T is a subspace of $F^{r+1}_{a,d} R^m$ such that*

$$F^{r+1}_{a,d} R^m \subset T + \widetilde{T}_f(F^1_{a,d}\mathcal{K} \cdot f) + F^{s+1}_{a,d} R^m, \tag{2-5}$$

then any g with $g - f \in F^{r+1}_{a,d} R^m$ is $F^1\mathcal{K}$-equivalent to $f + t + \phi$ where $t \in T$ and $\phi \in F^{s+1}_{a,d} R^m$.

Proof. See Theorem 2.28 in [4] and note that $\mathrm{Lie}(F^1_{a,d}\mathcal{K}) \cdot f = \widetilde{T}_f(F^1_{a,d}\mathcal{K} \cdot f)$ is shown above. □

For an isolated complete intersection singularity, f is always finite determined. Choosing s sufficiently large in the preceding proof, we get:

Corollary 2.17. *Suppose that f is an ICIS of weight r with respect to (a, d). Let T be a subspace of $F^{r+1}_{a,d} R^m$ such that*

$$F^{r+1}_{a,d} R^m \subset T + \widetilde{T}_f(F^1_{a,d}\mathcal{K} \cdot f).$$

Then any g with $g - f \in F^{r+1}_{a,d} R^m$ is contact equivalent to $f + t$ for some $t \in T$.

2.5. *Modality.* V. I. Arnold introduced the notion of modality in his famous [2] as follows: The modality of a point $x \in X$ under the action of a Lie group G on a manifold X is the smallest m such that a sufficiently small neighborhood of x may be covered by a finite number of m-parameter orbit families. G. M. Greuel and H. D. Nguyen generalized the notion and gave a detailed discussion in [8; 13]. For the definition of the modality of an ICIS, we refer to [11, Remark 1.13(3)].

Definition 2.18. An ICIS is called unimodal if $\mathcal{K}$-mod(f), the $\mathcal{K}$-modality of f, is equal to 1.

In this section, we give some methods to estimate a lower bound for the modality and give a criterion for non-unimodality.

Our first lower bound, from [5], uses complete transversals. It will be useful in next section. Let C be a complete transversal of f in J_l $(l > k)$. For $a \in C$, we define

$$\mathrm{cod}(f + a) = \text{codimension of } \widetilde{T}_f(\mathcal{K}_l f) \cap P_{k,l} \text{ in } P_{k,l}, \tag{2-6}$$

$$\mathrm{cod}_0(f) = \inf_{a \in C}\{\mathrm{cod}(f + a)\}. \tag{2-7}$$

Note that there exists a Zariski open subset $U \subset C$ such that $\mathrm{cod}(f + a) = \mathrm{cod}_0(f)$ if and only if $a \in U$.

Proposition 2.19. *Let $f \in J_k$ be a k-jet of weighted homogeneous type and degree* 0 *with respect to* $(a_1, \dots, a_n; d_1, \dots, d_m)$ *and satisfies condition* (2-2). *For any $a \in U$, $f + a$ has modality* $\mathrm{cod}_0(f)$ *in $J_l(f)$ under the action of the subgroup $\mathcal{K}_l(f)$ of $\mathcal{K}_l$ that stabilizes f. Thus, any jet h in $J_l(f)$ has $\mathcal{K}_l(f)$-*$\mathrm{mod}(h) \geq \mathrm{cod}_0(f)$ *in J_l.*

Proof. The main idea comes from [5, Proposition 1.4]. We rewrite the proof using tangent images instead of tangent spaces for the sake of fields with positive characteristic.

Find a subspace $\langle e_1, \dots, e_c\rangle \in P_{k,l}$ with $\langle e_1, \dots, e_c\rangle \oplus \widetilde{T}_f(\mathcal{K}_l(f+a)) \cap P_{k,l} = P_{k,l}$. Then $\mathrm{cod}(f+a) = c$. Since $\widetilde{T}_f(\mathcal{K}_l(f+b))$ varies continuously for $b \in U$, we have $\langle e_1, \dots, e_c\rangle \cap \widetilde{T}_f(\mathcal{K}_l(f+b)) = \{0\}$ for b in a Zariski open neighborhood V of a.

Consider

$$\phi : (F^c, 0) \to (J_l(f), f+a), \quad (t_1, \dots, t_c) \mapsto f + a + \sum_{i=1}^{c} t_i e_i. \tag{2-8}$$

We claim that, for any $\mathcal{K}_l(f)$-orbit X in $J_l(f)$, $\phi^{-1}(X)$ consists of finitely many points in a neighborhood of 0; hence ϕ is a minimal deformation of $f + a$ and $\mathcal{K}_l(f)$-$\mathrm{mod}(f+a) = c = \mathrm{cod}_0(f)$.

For any $g \in J_l(f)$, write $g = f + \tilde{g} = (f_1 + \tilde{g}_1, \dots, f_m + \tilde{g}_m)$, where $\tilde{g} = (\tilde{g}_1, \dots, \tilde{g}_m)$ with weighted degree of $\tilde{g}_i > d_i$. We have

$$\begin{aligned} g(x) \sim g_t(x) &= (t^{-d_1} g_1(t_1^{a_1} x_1, \dots, t_n^{a_n} x_n), \dots, t^{-d_n} g_m(t_1^{a_1} x_1, \dots, t_n^{a_n} x_n)) \\ &= (f_1 + t^{-d_1} \tilde{g}_1(t_1^{a_1} x_1, \dots, t_n^{a_n} x_n), \dots, f_m + t^{-d_n} \tilde{g}_m(t_1^{a_1} x_1, \dots, t_n^{a_n} x_n)). \end{aligned}$$

Then condition (2-2) ensures any neighborhood of f intersects $\mathcal{K}_l(f) \cdot g$; that is, any neighborhood of f intersects all $\mathcal{K}_l(f)$-orbits in $J_l(f)$. Hence for any $\mathcal{K}_l(f)$-orbit X in $J_l(f)$, there exists $b \in V$, $X = \mathcal{K}_l(f) \cdot (f+b)$. But we have

$$\widetilde{T}_{f+b}(\mathcal{K}_l(f) \cdot (f+b)) = \widetilde{T}_{f+b}(\mathcal{K}_l \cdot (f+b)) \cap P_{k,l}; \tag{2-9}$$

hence $\widetilde{T}_{f+b}(\mathcal{K}_l(f) \cdot (f+b)) \cap \langle e_1, \dots, e_c\rangle = \{0\}$, meaning that $\phi^{-1}(X)$ has only finitely many points in a neighborhood of 0. This finishes the proof. □

Remark 2.20. In fact, we can show $\mathcal{K}$-$\mathrm{mod}(f)$ is semicontinuous. That is, let $F(\boldsymbol{x}, \boldsymbol{t}) \in F[\boldsymbol{t}][[\boldsymbol{x}]]$ be such that $F_{\boldsymbol{t}_0} = F(\boldsymbol{x}, \boldsymbol{t}_0)$ is an ICIS for a $\boldsymbol{t}_0 \in F^k$. Then there is a Zariski open subset $U \in F^k$ such that $F(\boldsymbol{x}, \boldsymbol{t})$ is an ICIS for any $\boldsymbol{t} \in U$. The sets $U_i = \{\boldsymbol{t} \in U \mid \mathcal{K}\text{-mod}(F(\boldsymbol{x}, \boldsymbol{t})) \leq i\}$ are open for all $i \in \mathbb{N}$. Now let $\mathrm{mod}_{\min} = \min\{\mathcal{K}\text{-mod}(F(\boldsymbol{x}, \boldsymbol{t})) \mid \boldsymbol{t} \in F^k\}$; then $U_{\min} = \{\boldsymbol{t} \in U \mid \mathcal{K}\text{-mod}(F(\boldsymbol{x}, \boldsymbol{t})) = \mathrm{mod}_{\min}\}$ is open and dense.

We use the following facts from [13] to give a criterion for non-unimodality.

Proposition 2.21. *Let the algebraic group G act on a variety X.*

(1) *If the subvariety $X' \subset X$ is invariant under G and $x \in X'$, then*

$$G\text{-mod}(x) \text{ in } X \geq G\text{-mod}(x) \text{ in } X'.$$

(2) *Let additionally the algebraic group G' act on a variety X' and let $p : X \to X'$ be a morphism of varieties, with p open and*

$$G \cdot x \subset p^{-1}(G' \cdot p(x)),\ \forall x \in X.$$

Then

$$G\text{-mod}(x) \geq G'\text{-mod}(p(x)),\ \forall x \in X.$$

(3) *If X is irreducible, for $x \in X$, we have*

$$G\text{-mod}(x) \geq \dim X - \dim G.$$

Proposition 2.22. *Let $f \in I_{m,n}$. Then $\mathcal{K}$-mod$(f) = \mathcal{K}_k$-mod$(j_k(f))$ for k sufficiently large.*

Proof. See [13], Chapter 3. □

The following proposition is the main result of this section.

Proposition 2.23. *Let $f \in I_{n,n}$ with* $\mathrm{ord}(f) = l$ *and f unimodal. Then one of the following holds*:

(1) $n = 2$, $l \leq 3$.

(2) $n = 3$, $l = 2$.

Proof. Choose k large enough and let $X = \mathfrak{m}^l/\mathfrak{m}^{k+1}$. It follows from Propositions 2.22 and 2.21(1) that

$$1 = \mathcal{K}\text{-mod}(f) = \mathcal{K}_k\text{-mod}(f) \text{ in } J_k \geq \mathcal{K}_k\text{-mod}(f) \text{ in } X.$$

Let $X' = \mathfrak{m}^l/\mathfrak{m}^{l+1}$. The action of $\mathcal{K}_k$ on X induces the action of the algebraic group $\mathcal{K}' = \mathrm{GL}(m, F) \times \mathrm{GL}(m, F)$ on X', and it is easy to check that $p : X \to X'$ is open and $\mathcal{K}_k \cdot f \subset p^{-1}(\mathcal{K}' \cdot p(f))$. Then by Proposition 2.21(2) we have

$$\mathcal{K}_k\text{-mod}(f) \text{ in } X \geq \mathcal{K}'\text{-mod}(p(f)) \text{ in } X'.$$

It is easy to see that

$$\dim X' = n\binom{n-1+l}{l},$$

while for any $g \in X'$, $\dim(\mathcal{K}' \cdot g) \leq \dim \mathcal{K}' - 1$, since $\left\{\left(a^l E_n, \frac{1}{a} E_n\right) \mid a \in F^\times\right\} \subset \mathcal{K}'$ stabilizes g, where $F^\times = F \setminus \{0\}$ denotes the units in F.

After a small change of the proof of Proposition 2.21(3) in [13], we have

$$1 \geq \mathcal{K}'\text{-mod}(p(f)) \text{ in } X' \geq \dim X' - (\dim \mathcal{K}' - 1),$$

which is

$$1 \geq n\binom{n-1+l}{l} - (2n^2 - 1).$$

The only solutions are $n = 2$, $l \leq 3$ and $n = 3$, $l = 2$. □

Corollary 2.24. *If f is a unimodal zero-dimensional ICIS, then $f \in I_{2,2}$ or $I_{3,3}$.*

In the following we discuss the case $n = 2$. The $n = 3$ case will be presented in a later article. From now on we assume $R = F[\![x, y]\!]$, $\mathfrak{m} = \langle x, y\rangle$.

3. The classification of order 2

Some classification of ICIS of order 2 has already been discussed in [11], namely, ICIS of modality 0. Here we continue their work and finish the classification of order 2 ICIS with modality 1. In this section $f = (f_1, f_2) \in R^2$.

First, we assume char $F \neq 2$. The following two propositions are from [11].

Proposition 3.1. (i) *If some $j_2(f_i)$ is nondegenerate, then $f \sim (xy, x^n + y^m)$ for some $m, n \geq 2$, which is of modality* 0.

(ii) *If $j_2(f)$ is degenerate, then*

$$f \sim \Big(x^2 + \alpha y^s, \sum_{i\geq t} a_i y^i + x \sum_{j\geq q} b_j y^j\Big), \tag{3-1}$$

where $s \geq 3$, $\alpha \in \{0, 1\}$, $t \geq 2$, $q \geq 1$ and $a_i, b_j \in F$.

Proposition 3.2. *Let $f = (f_1, f_2) = \big(x^2 + \alpha y^s, \sum_{i\geq t} a_i y^i + x \sum_{j\geq q} b_j y^j\big)$ be an ICIS such that $s \geq 3$, $t \geq 2$, $q \geq 1$ and $\alpha \in \{0, 1\}$.*

(i) *If $a_i = 0$ for all i and $b_q \neq 0$, then $\alpha = 1$ and $f \sim (x^2 + y^s, xy^q)$.*

(ii) *If $b_j = 0$ for all j and $a_t \neq 0$, then $f \sim (x^2 + \alpha y^s, y^t)$. If $\alpha = 0$ or $t \leq s$, then $f \sim (x^2, y^t)$.*

Now assume that $a_t b_q \neq 0$.

(iii) *If $t \leq q$, then $f \sim (x^2 + \alpha y^s, y^t)$. If additionally $\alpha = 0$ or $t \leq s$, then $f \sim (x^2, y^t)$.*

(iv) *If $t > q$ and $\alpha = 0$, then $f \sim (x^2, y^t + xy^q)$.*

(v) *Let $t > q$ and $\alpha = 1$. Then $f \sim (x^2 + y^s, y^t + exy^q)$ for a suitable unit $e \in F[\![y]\!]$. If $2t - 2q - s \neq 0$ and $p = 0$ (or $p \nmid (2t - 2q - s)$), then $f \sim (x^2 + y^s, y^t + xy^q)$.*

(vi) *If $t = q + 1$ and $p \nmid t$, then $f \sim (x^2 + \alpha y^s, y^t)$.*

Next we give a criterion for modality 1:

Proposition 3.3. *Assume f is of the form* (3-1). *Then if $s \geq 5$, $t \geq 6$ and $q \geq 3$, then f is of modality at least* 2.

Proof. Let $g = j_3(f) = (x^2, 0)$. Then an open subset in $J_5'(g)$ is formed by jets equivalent to

$$h \sim (x^2 + y^5, axy^3),$$

where$J_5'(g)$ is formed by jets in $J_5(g)$ with $s \geq 5$, $t \geq 6$ and $q \geq 3$.

For all $a, b \in F$, computation shows $(y^4, 0),\ (0, y^4),\ (0, y^5) \notin \widetilde{T}_h(\mathcal{K}_5 h)$. Hence the codimension of $\widetilde{T}_h(\mathcal{K}_5 h) \cap P_{3,5}$ in $P_{3,5}$ is at least 2. By Proposition 2.19, $\mathcal{K}\text{-mod}(f) \geq \mathcal{K}_5\text{-mod}(f) \geq 2$. □

Remark 3.4. By (3-1), $h \sim (x^2 + \alpha y^5,\ y^6 + axy^3 + bxy^4 + cxy^5)$. One can show, for example, by applying $\phi(y) = y - \frac{1}{6}cx$, that $h \sim (x^2 + \alpha y^5,\ y^6 + axy^3 + bxy^4)$. This simplifies the computation of the codimension, but does not affect the result.

The following proposition is also from [11].

Proposition 3.5. *Let $p = \operatorname{char} F$. The following ICIS are the only candidates for being simple* (*i.e., of modality* 0):

(**0**) $j_2(f_i)$ *is nondegenerate*; *then* $f \sim (xy, x^s + y^m)$, $s, m \geq 2$.
(**1**) $a_i = 0$ *for all* i; *then* $f \sim (x^2 + y^3, xy^q)$, $q \geq 3$ *or* $f \sim (x^2 + y^s, xy^2)$, $s \geq 3$.
(**2**) $b_i = 0$ *for all* i, *and*
 (**2.*a***) $\alpha = 0$ *or* $t \leq s$; *then* $f \sim (x^2, y^t)$, $t = 2, 3, 4$.
 (**2.*b***) $\alpha = 1$ *and* $t > s$; *then* $f \sim (x^2 + y^3, y^t)$, $t \geq 4$.
(**3**) $a_t b_q \neq 0$ *and*
 (**3.*a***) $t \leq q$ *and*
  (**3.*a.i***) $\alpha = 0$ *or* $t \leq s$; *then* $f \sim (x^2, y^t)$, $t = 2, 3, 4$.
  (**3.*a.ii***) $\alpha = 1$ *and* $t > s$; *then* $f \sim (x^2 + y^3, y^t)$, $t \geq 4$.
 (**3.*b***) $t > q$ *and*
  (**3.*b.i***) $t = q + 1$ *and*
   (**3.*b.i.*1**) $p \nmid t$ *and*
    (**3.*b.i.*1.1**) $\alpha = 0$ *or* $t \leq s$; *then* $f \sim (x^2, y^t)$, $t = 2, 3, 4$.
    (**3.*b.i.*1.2**) $\alpha = 1$ *and* $t > s$; *then* $f \sim (x^2 + y^3, y^t)$, $t \geq 4$.
   (**3.*b.i.*2**) $p \mid t$ *and*
    (**3.*b.i.*2.1**) $\alpha = 0$; *then* $f \sim (x^2, xy^2 + y^3)$.
    (**3.*b.i.*2.2**) $\alpha = 1$ *and*
     (**3.*b.i.*2.2.1**) $s = 3$; *then* $f \sim (x^2, xy^2 + y^3)$ *for* $p = t = 3$, *or* $f \sim (x^2 + y^3, y^t + xy^{t-1})$ *for* $t \geq 4$.
     (**3.*b.i.*2.2.2**) $s > 3$, $t = 3$, $p = 3$; *then* $f \sim (x^2, y^3 + xy^2)$.
  (**3.*b.ii***) $t > q + 1$ *and*
   (**3.*b.ii.*1**) $\alpha = 0$ *and*
    (**3.*b.ii.*1.1**) $q = 1$; *then* $f \sim (xy, x^2 + y^{2t-2})$, $t \geq 2$.
    (**3.*b.ii.*1.2**) $q = 2$; *then* $f \sim (x^2 + y^{2t-4}, xy^2)$, $t \geq 4$.
   (**3.*b.ii.*2**) $\alpha = 1$ *and*

(**3.*b.ii*.2.1**) $s \geq 3$, $q \leq 2$; *then* $f \sim (xy, x^2 + y^m)$ *for some* m *and* $q = 1$, *and* $f \sim (x^2 + y^m, xy^2)$ *for some* m *and* $q = 2$.
(**3.*b.ii*.2.2**) $s = 3$, $t \geq q + 3$; *then* $f \sim (x^2 + y^3, xy^q)$.
(**3.*b.ii*.2.3**) $s = 3$, $t = q + 2$; *then* $f \sim (x^2 + y^3, xy^q + y^{q+2})$.

Next we classify ICIS of modality 1 based on Proposition 3.5.

Proposition 3.6. *The following ICIS are the only candidates of modality* 1:

symbol	*form*	*condition*
h_q	$(x^2 + y^4, xy^q)$	$q \geq 3$
i	(x^2, y^5)	
$\tilde{i}$	$(x^2, y^5 + xy^3)$	
i^5	$(x^2, y^5 + xy^4)$	$p = 5$
j_t	$(x^2 + y^4, y^t)$	$t \geq 5$
$\tilde{j}_t$	$(x^2 + y^4, y^t + xy^{t-1})$	$t \geq 5,\ p \mid t$
k_q	$(x^2 + y^4, y^{q+3} + xy^q)$	$q \geq 3$
$l_{q,\lambda}$	$(x^2 + y^4, y^{q+2} + \lambda xy^q)$	$q \geq 3, \lambda^2 \notin \{0, -1\}$
$\tilde{l}_{q,t,t'}$	$(x^2 + y^4, y^{q+2} + \lambda xy^q + uxy^t + xy^{t'})$, *where* $u = u_0 + u_1 y^p + u_2 y^{2p} + \cdots$	$\lambda^2 = -1, q \geq 3$, $t \geq q + 1, t' \geq t + 1$, $p \mid t - q, p \nmid t' - q$

Table 1. Candidate ICIS of modality 1.

Proof. (**0**) If $j_2(f_i)$ is nondegenerate, then $f \sim (xy, x^s + y^m)$, $s, m \geq 2$, which is simple.
(**1**) $\forall a_i = 0$, then $f \sim (x^2 + y^4, xy^q)$, $q \geq 3$.
(**2**) $\forall b_i = 0$,
(**2.*a***) $\alpha = 0$ or $t \leq s$, then $f \sim (x^2, y^t)$, $t = 5$.
(**2.*b***) $\alpha = 1$ and $t > s$, then $f \sim (x^2 + y^4, y^t)$, $t \geq 5$.
(**3**) $a_t b_q \neq 0$,
(**3.*a***) If $t \leq q$, then $f \sim (x^2 + \alpha y^s, y^t)$.
(**3.*a.i***) $\alpha = 0$ or $t \leq s$, then $f \sim (x^2, y^t)$, $t = 5$.
(**3.*a.ii***) $\alpha = 1$ and $t > s$, then $f \sim (x^2 + y^4, y^t)$, $t \geq 5$.
(**3.*b***) If $t > q$,
(**3.*b.i***) when $t = q + 1$,
(**3.*b.i*.1**) If $p \nmid t$, then $f \sim (x^2 + \alpha y^s, y^t)$ by Proposition 3.2(vi).
(**3.*b.i*.1.1**) $\alpha = 0$ or $t \leq s$, then $f \sim (x^2, y^t)$, $t = 5$.

(**3.*b.i.*1.2**) $\alpha = 1$ and $t > s$, then $f \sim (x^2 + y^3, y^t)$, $t \geq 4$, which is simple.

(**3.*b.i.*2**) If $p \mid t$,

(**3.*b.i.*2.1**) $\alpha = 0$, then $f \sim (x^2, y^3 + xy^2)$, which is of modality 0 by Proposition 3.5.

(**3.*b.i.*2.2**) $\alpha = 1$, then $f \sim (x^2 + y^s, y^t + exy^q)$ by Proposition 3.2(v). And we have $s = 3, 4$ or $p = t = 3, 5$ by Proposition 3.3.

(**3.*b.i.*2.2.1**) If $s = 3$, then $f \sim (x^2, xy^2 + y^3)$ for $p = t = 3$, or $f \sim (x^2 + y^3, y^t + xy^{t-1})$ for $t \geq 4$ by Proposition 3.2(v), which is of modality 0.

(**3.*b.i.*2.2.2**) If $s = 4$, then $f \sim (x^2 + y^4, y^t + xy^{t-1})$ for $t \geq 4$ by Proposition 3.2(v). If $p = t = 3$, then $f \sim (x^2 + y^4 - y(y^3 + xy^2), y^3 + xy^2) \sim (x^2, y^3 + xy^2)$ additionally, which is simple as shown in Proposition 3.5. Hence $f \sim (x^2 + y^4, y^t + xy^{t-1})$ for $p \mid t$, $t \geq 5$.

(**3.*b.i.*2.2.3**) If $p = t = 3$, $s > 4$, same as the process in [11, Proposition 2.5], $f \sim (x^2, y^3 + xy^2)$, which is simple.

(**3.*b.i.*2.2.4**) If $p = t = 5$, $s > 4$, then

$$\begin{aligned} f &\sim (x^2 + y^s, y^5 + e(y)xy^4), \text{ where } e(y) \in F[\![x, y]\!] \text{ is a unit} \\ &\sim (x^2 + e_0 y^s, y^5 + xy^4), \quad e_0 \in F \\ &\sim (x^2 + e_0 y^s - e_0 y^{s-5}(y^5 + xy^4), y^5 + xy^4) \\ &\sim (x^2 - e_0 xy^{s-1}, y^5 + xy^4) \\ &\sim \big((x - \tfrac{1}{2}e_0 y^{s-1})^2 - \tfrac{1}{4}e_0^2 y^{2s-2}, y^5 + xy^4\big). \end{aligned} \tag{3-2}$$

Using the automorphism $\phi(x) = x - \frac{1}{2}e_0 y^{s-1}$, $\phi(y) = y$, then

$$\begin{aligned} f &\sim \big(x^2 - \tfrac{1}{4}e_0^2 y^{2s-1}, y^5 + xy^4 + \tfrac{1}{2}e_0 y^{s+3}\big) \\ &\sim \big(x^2 - \tfrac{1}{4}e_0^2 y^{2s-1}, y^5 + \tilde{e}xy^4\big), \end{aligned} \tag{3-3}$$

where $\tilde{e} = 1/\big(1 + \frac{1}{2}e_0 y^{s-2}\big)$. Applying $\phi(x) = (1/\tilde{e})x$ and $\phi(y) = y$, we have $f \sim (x^2 + e_1 y^{2s-1}, y^5 + xy^4)$, where $e_1 = -\frac{1}{4}\tilde{e}^2 e_0^2$. Repeating the process, we get $f \sim (x^2, y^5 + xy^4)$ with $p = \operatorname{char} F = 5$.

(**3.*b.ii***) Now assume $t > q + 1$.

(**3.*b.ii.*1**) If $\alpha = 0$, by Proposition 3.3, we have $1 \leq q \leq 2$ or $q = 3$, $t = 5$.

(**3.*b.ii.*1.1**) If $q = 1$, then $j_2(f_2)$ is nondegenerate; hence f is simple.

(**3.*b.ii.*1.2**) If $q = 2$, then $f \sim (x^2 + y^{2t-4}, xy^2)$, $t \geq 4$, same as Proposition 3.5, which is also simple.

(**3.*b.ii.*1.3**) If $q = 3$, $t = 5$, then $f \sim (x^2, y^5 + xy^3)$.

(**3.*b.ii.*2**) If $\alpha = 1$, we have $f \sim (x^2 + y^s, y^t + exy^q)$ by Proposition 3.2(v). By Proposition 3.3, we have $s = 3, 4$ or $t = 2, 3, 4, 5$ or $q = 1, 2$.

(**3.*b.ii.*2.1**) If $q = 1, 2$ holds, then f is simple as shown in Proposition 3.5.

(**3.*b.ii*.2.2**) If $s = 3$, then $f \sim (x^2 + y^3, xy^q)$ or $(x^2 + y^3, y^{q+2} + xy^q)$ for $q \geq 3$, which is also simple.

(**3.*b.ii*.2.3**) If $s = 4$, $q > 2$,

(**3.*b.ii*.2.3.1**) If $t \geq q+4$, $f \sim (x^2+y^4, y^t+e(y)xy^q-y^{t-4}(x^2+y^4)) = (x^2+y^4, (e(y)-xy^{t-q-4})xy^q) \sim (x^2+y^4, xy^q)$, where $e(y) = b_q + b_{q+1}y + \cdots$ is a unit in $F[\![x, y]\!]$ and b_q, b_{q+1} are defined in (3-1).

(**3.*b.ii*.2.3.2**) If $t = q+3$, then $f \sim (x^2+y^4, y^{q+3}+xy^q)$ by Proposition 3.2(v), since in this case $2t-2q-s = 2$.

(**3.*b.ii*.2.3.3**) If $t = q+2$. Let $g = (x^2+y^4, y^{q+2}+\lambda xy^q)$, where $\lambda = e(0) \in F^\times$, $e(0)$ is the constant term of $e(y)$ in (**3.*b.ii*.2.3.1**). Then g is weighted homogeneous of degree 0 with respect to $(a; d)$, where $a = (2, 1)$, $d = (4, q+2)$.

(**3.*b.ii*.2.3.3.1**) If $\lambda^2 \neq -1$, we claim that $F^1_{a,d}R^2 \subset \widetilde{T}_g(F^1\mathcal{K}g)$ (the proof will be presented later). Hence by Proposition 2.16, $f \sim (x^2+y^4, y^{q+2}+\lambda xy^q)$, $\lambda \in F^\times$ and $\lambda^2+1 \neq 0$.

(**3.*b.ii*.2.3.3.2**) If $\lambda^2+1 = 0$, from the proof of (**3.*b.ii*.2.3.3.1**), we can see that $F^1_{a,d}R^2 \subset T+\widetilde{T}_g(F^1\mathcal{K}g)$, with $T = \mathrm{span}\langle(0, xy^t) \mid t > q\rangle$. Then by Proposition 2.16,

$$f \sim (x^2+y^4, y^{q+2}+\lambda xy^q+e(y)xy^t)$$

for $\lambda^2+1 = 0$, $t > q$ and $e(y)$ a unit in $F[\![y]\!]$. In fact, we can show that for some $e(y)$ and l, $f \sim (x^2+y^4, y^{q+2}+\lambda xy^q+xy^l)$, while for others, we get a family of unimodal ICIS with the form

$$(x^2+y^4, y^{q+2}+\lambda xy^q+u(y)xy^t+xy^{t'}), \tag{3-4}$$

where $u(y) = u_0+u_1y^p+u_2y^{2p}+\cdots$ is a unit, $t' > t \geq q+1$, $p \mid t-q$, $p \nmid t'-q$. The details will be shown later.

(**3.*b.ii*.2.4**) The last remaining case is $t = 5$, $q = 3$, $s > 4$. Then $f \sim (x^2+y^s, y^5+exy^3)$. Using the same method as (**3.*b.i*.2.2.4**), one can show $f \sim (x^2, y^5+xy^3)$. □

Proof of the claim in (**3.*b.ii*.2.3.3.1**). We need to show $F^1_{a,d}R^2 \subset \widetilde{T}_g(F^1\mathcal{K}g)$ for $g = (x^2+y^4, y^{q+2}+\lambda xy^q)$, where $a = (2, 1)$, $d = (4, q+2)$ and $\lambda^2 \neq -1$.

Denote $e_1 = (x^2+y^4, 0)$, $e_2 = (0, x^2+y^4)$, $e_3 = (y^{q+2}+\lambda xy^q, 0)$, $e_4 = (0, y^{q+2}+\lambda xy^q)$, $e_5 = (2x, \lambda y^q)$, $e_6 = (4y^3, (q+2)y^{q+1}+\lambda qxy^{q-1})$.

Set the weights $\mathrm{wt}(x) = 2$, $\mathrm{wt}(y) = 1$, $\mathrm{wt}(x^2+y^4) = 4$, $\mathrm{wt}(y^{q+2}+\lambda xy^q) = q+2$. Then by (2-3), $\widetilde{T}_g(F^1\mathcal{K}g)$ has the elements $x^iy^je_k$ with:

(a) $k = 1, 4$, $\mathrm{wt}(x^iy^je_k) \geq (5, q+3)$;

(b) $k = 5$, $\mathrm{wt}(x^iy^j) \geq 3$;

(c) $k = 6$, $\mathrm{wt}(x^iy^j) \geq 2$.

We have

$$xe_6 = (4xy^3, (q+2)xy^{q+1} + \lambda q x^2 y^{q-1}) \in \widetilde{T}_g(F^1\mathcal{K}g)$$

and

$$ye_4 - y^{q-1}e_2 = (0, -x^2y^{q-1} + \lambda xy^{q+1}) \in \widetilde{T}_g(F^1\mathcal{K}g).$$

Therefore

$$xe_6 + \lambda q(ye_4 - y^{q-1}e_2) = (4xy^3, (\lambda^2 q + q + 2)xy^{q+1}) \in \widetilde{T}_g(F^1\mathcal{K}g).$$

We also have

$$y^{q-1}e_2 = (0, x^2y^{q-1} + y^{q+3}) \in \widetilde{T}_g(F^1\mathcal{K}g);$$

hence,

$$xe_6 - \lambda q y^{q-1}e_2 = (4xy^3, (q+2)xu^{q+1} - \lambda q y^{q+3}) \in \widetilde{T}_g(F^1\mathcal{K}g).$$

Then

$$\begin{aligned}(q+2)&(xe_6 - \lambda q y^{q-1}e_2) + \lambda q y^2 e_6 \\ &= (4(q+2)xy^3 + 4\lambda q y^5, ((q+2)^2 + \lambda^2 q^2)xy^{q+1}) \in \widetilde{T}_g(F^1\mathcal{K}g).\end{aligned}$$

Denote this element by e_7.

Note that

$$xye_5 - 2ye_1 = (-2y^5, \lambda xy^{q+1}),\ \ y^3e_5 - \lambda ye_4 = (2xy^3, -\lambda^2 xy^{q+1}) \in \widetilde{T}_g(F^1\mathcal{K}g),$$

and thus

$$\begin{aligned}e_7 &+ 2\lambda q(xye_5 - 2ye_1) - 2(q+2)(y^3e_5 - \lambda ye_4) \\ &= (0, ((q+2)^2 + \lambda^2 q^2 + 2\lambda^2 q + 2\lambda^2(q+2))xy^{q+1}) \\ &= (0, (q+2)^2(\lambda^2+1)xy^{q+1}) \in \widetilde{T}_g(F^1\mathcal{K}g).\end{aligned}$$

Since $\lambda^2 + 1 \neq 0$, we have $(0, xy^{q+1}) \in \widetilde{T}_g(F^1\mathcal{K}g)$. The other elements in $F^1_{a,d}R^2$ follow easily. □

Proof of the claim in (**3.*b.ii.*2.3.3.2**). We have $f \sim (x^2+y^4,\ y^{q+2}+\lambda xy^q+e(y)xy^t)$. If $p \nmid t-q$, we can use an "α, β-trick" based on the implicit function theorem (see Lemma 5.10) to show that $f \sim (x^2+y^4,\ y^{q+2}+\lambda xy^q+xy^t)$ for $t \geq q+1$. See Remarks 5.11. For the case $p \mid (t-q)$, we write

$$e(y) = \sum_{i\geq 0} e_i y^{q+ip} + \sum_{\substack{j\geq 0 \\ p\nmid(t'+j)}} e'_j y^{t'+j}, \tag{3-5}$$

then $f \sim (x^2+y^4,\ y^{q+2}+\lambda xy^q(1+e_1y^p+e_2y^{2p}+\cdots)+xy^{t'}(e'_0+e'_1y+e'_2y^2+\cdots))$ with $p \nmid t'-q$. Therefore we can use α, β-trick again to reduce $e'_0+e'_1y+\cdots$. Then

$f \sim (x^2+y^4,\, y^{q+2}+\lambda xy^q+u(y)xy^t+xy^{t'})$, where $u(y)=u_0+u_1y^p+u_2y^{2p}+\cdots$ as we want. □

4. The classification of order 2 in char $F=2$

In this section, we will show:

Proposition 4.1. *A unimodal ICIS of order* 2 *in any field with characteristic equal to* 2 *must have the form in Table 2.*

symbol	*form*	*condition*
h^2_λ	$(x^2+\lambda xy^2,\, y^3)$	$\lambda \in \{0, 1\}$
i^2_k	$(x^2+y^k,\, xy^2)$	$k \geq 3$, k *is odd*
$i^2_{k,\lambda}$	$(x^2+y^k+\lambda y^{k+1},\, xy^2)$	$k \geq 3$, k *is even,* $\lambda \in \{0, 1\}$
j^2_λ	$(x^2+y^3,\, y^4+\lambda xy^3)$	$\lambda \in \{0, 1\}$
$k^2_{\lambda,\mu}$	$(x^2+\lambda xy^3,\, y^4+\mu xy^3)$	$\lambda, \mu \in \{0, 1\}$
l^2	$(x^2+xy^2,\, y^4)$	
m^2_s	$(x^2+y^s,\, xy^3)$	$s \geq 3$, s *is odd*
$m^2_{s,\lambda}$	$(x^2+\lambda x^2y+y^s,\, xy^3)$	$\lambda \in F$, $s \geq 4$, s *is even*
n^2_s	$(x^2+xy^2+y^s,\, xy^3)$	$s \geq 3$, s *is odd*
$\tilde{n}^2_{s,\lambda}$	$(x^2+xy^2+\lambda x^2y+y^s,\, xy^3)$	$\lambda \in F$, $s \geq 4$, s *is even*

Table 2. Possible ICIS of order 2 when char $F=2$.

The following result is from [11]:

Proposition 4.2. *Let* char $F=2$, $f \in I_{2,2}$ *and* $\operatorname{ord}(f)=2$. *Then one of the following cases occurs*:

(a) $f \sim (xy, g)$ *for some* $g \in \mathfrak{m}^2$. *In this case,* $f \sim (xy, x^m+y^n)$ *for some* $m, n \geq 2$, *which is simple.*

(b) $f \sim (x^2+h, g)$ *for* $h \in \mathfrak{m}^3$ *and* $g \in \mathfrak{m}^2$. *Moreover, if* $g \notin \mathfrak{m}^3$, *then* f *is simple.*

Similarly to Proposition 3.3, we can show:

Proposition 4.3. *If* $g \in \mathfrak{m}^5$, *then the modality of* $f \sim (x^2+h, g)$ *is at least* 2.

Proof. For $g \in \mathfrak{m}^5$, then $l=j_4(f)$ is of the form

$$l \sim (x^2+axy^2+by^3+cxy^3+dy^4, 0) \sim (x^2+axy^2+by^3+dy^4, 0).$$

It is easy to compute that the codimension of $\widetilde{T}_l(\mathcal{K}_4 l)$ in $P_{2,4}$ is at least 2 (note that $(0, xy^3), (0, y^4) \notin P_{2,4}$). Therefore $\mathcal{K}$-mod$(f) \geq \mathcal{K}_4$-mod$(l) \geq 2$ by Proposition 2.19. □

Therefore, we need to work on the case $f \sim (x^2 + h, g)$ with $g \in \mathfrak{m}^4 \setminus \mathfrak{m}^5$.

Proposition 4.4. *If $g \in \mathfrak{m}^3 \setminus \mathfrak{m}^4$, then $f \sim (x^2 + \lambda xy^2, y^3)$, $\lambda \in \{0, 1\}$ or $f \sim (x^2 + y^k, xy^2)$, k is odd or $f \sim (x^2 + y^k + \mu y^{k+1}, xy^2)$, k is even, $\mu \in \{0, 1\}$.*

Proof. Set $j_3(g) = ax^3 + bx^2y + cxy^2 + dy^3$.

If $d \neq 0$, let $f_0 = (x^2, dy^3) \sim (x^2, y^3)$ be the weighted 0-jet of f with respect to (a, d), where $a = (4, 3)$, $d = (8, 9)$. Then

$$F^1_{a,d} R^2 \subset \mathrm{span}\langle (xy^2, 0)\rangle + \widetilde{T}_{f_0}(F^1 \mathcal{K} f_0).$$

By Proposition 2.16, $f \sim (x^2 + cxy^2, y^3) \sim (x^2 + \lambda xy^2, y^3)$, $\lambda \in \{0, 1\}$.

If $d = 0$ and $c \neq 0$, we still let $f_0 = (x^2, cxy^2) \sim (x^2, xy^2)$ be the weighted 0-jet of f with respect to (a, d), where $a = (4, 3)$, $d = (8, 10)$. In this case

$$F^1_{a,d} R^2 \subset \mathrm{span}\langle (y^k, 0),\ k \geq 4\rangle + \widetilde{T}_{f_0}(F^1 \mathcal{K} f_0).$$

Then $f \sim (x^2 + e(y)y^k, xy^2) \sim (e(y)^{-1}x^2 + y^k, xy^2)$, $e(y) \in F[\![y]\!]$ is a unit.

If k is odd, then there exists $\tilde{e}(y)^k = e(y)$. Apply $\phi(x) = \tilde{e}(y)x$, $\phi(y) = y$, then $f \sim (x^2 + y^k, xy^2)$.

If k is even, write $e(y)^{-1} = e_0 + e_1 y + \cdots$, then

$$\begin{aligned} f &\sim (e_0x^2 + e_1x^2y + e_2x^2y^2 + \cdots + y^k, xy^2) \\ &\sim (e_0x^2 + e_1x^2y + y^k, xy^2) \\ &\sim ((e_0x^2 + e_1x^2y + y^k)(1 - (e_1/e_0)y), xy^2) \\ &\sim (e_0x^2 + y^k - (e_1/e_0)y^{k+1}, xy^2) \\ &\sim (x^2 + y^k + \lambda y^{k+1}, xy^2), \end{aligned} \tag{4-1}$$

where $\lambda \in \{0, 1\}$.

If $c = d = 0$, then $j_3(f) \sim (x^2 + h, 0)$. Then $g \in \mathfrak{m}^4$, a contradiction. □

Proposition 4.5. *If $g \in \mathfrak{m}^4$, then f is equivalent to other forms in Table 2, that is, j^2_λ, $k^2_{\lambda,\mu}$, l^2, m^2_s, m^2_{s,e_1}, n^2_s or $\tilde{n}^2_{s,e_1}$.*

Proof. Computing the complete transversal of $j_2(f) = (x^2, 0)$, we have $f \sim (x^2 + a(y)xy^r + b(y)y^s, c(y)xy^u + d(y)y^v)$ with $a(y), b(y), c(y), d(y)$ either units or 0. If $a(y)$ (resp. $b(y)$, $c(y)$, $d(y)$)$= 0$, we regard r (resp. s, u, v) as ∞. By Proposition 4.3, we have either $u = 3$ or $v = 4$.

(i) $v = 4$, $s = 3$. Then $f \sim (x^2 + a(y)xy^r + b(y)y^3, c(y)xy^u + d(y)y^4)$. Take $l = (x^2 + b_0y^3, d_0y^4) \sim (x^2 + y^3, y^4)$ to be the weighted 0-jet with respect to $a = (3, 2)$, $d = (6, 8)$. We have $F^1_{a,d} \subset \mathrm{span}\langle (0, xy^3)\rangle + \widetilde{T}_l(F^1 \mathcal{K} l)$. Therefore $f \sim (x^2 + y^3, y^4 + \lambda xy^3)$ for $\lambda \in \{0, 1\}$, which is j^2_λ in Table 2.

(ii) $v = 4$, $s > 3$. If $r \geq 3$, then we choose $l = (x^2, y^4)$ be the weighted 0-jet of f with respect to $a = (2, 1)$, $d = (4, 4)$. We have $F^1_{a,d} \subset \text{span}\langle (xy^3, 0), (0, xy^3)\rangle + \widetilde{T}_l(F^1\mathcal{K}l)$. Thus $f \sim (x^2 + \lambda xy^3, y^4 + \mu xy^3)$, $\lambda, \mu \in \{0, 1\}$ after a scaling. That is, $f \sim k^2_{\lambda,\mu}$ in Table 2. If $r = 2$, then the weighted 0-jet of f with respect to $a = (2, 1)$, $d = (4, 4)$ becomes $l = (x^2 + a_0xy^2, y^4) \sim (x^2 + xy^2, y^4)$. Computation shows that $F^1_{a,d} \subset \widetilde{T}_l(F^1\mathcal{K}l)$, then we have $f \sim (x^2 + xy^2, y^4) \sim l^2$, which is in Table 2.

(iii) $v > 4$. Then we have $s = 3$ and then $f \sim (x^2 + a(y)xy^r + b(y)y^s, xy^3 + d(y)y^v)$. Choose $l = (x^2, xy^2)$ as the weighted 0-jet of f with respect to $a = (1, 1)$, $d = (2, 4)$. We have $F^1_{a,d} \subset \text{span}\langle (xy^2, 0), (y^k, 0) \mid k \geq 3\rangle + \widetilde{T}_l(F^1\mathcal{K}l)$. Then $f \sim (x^2 + \mu xy^2 + e(y)y^s, xy^3)$, $\mu \in \{0, 1\}$, $s \geq 3$. After a scaling we can assume $e(0) = 1$.

If $\mu = 0$ and s is odd, using the α, β-trick in Remarks 5.11, we have $f \sim (x^2 + y^s, xy^3) \sim m^2_s$. If $\mu = 0$ and s is even, we have

$$\begin{aligned} f &\sim (e(y)^{-1}x^2 + y^s, xy^3) \\ &\sim ((1 + e_1y + e_2y^2 + \cdots)x^2 + y^s, xy^3) \\ &\sim (x^2 + e_1x^2y + e_2x^2y^2 + y^s, xy^3), \end{aligned} \tag{4-2}$$

where $e(y)^{-1} = e_0 + e_1y + e_2y^2 + \cdots$, and the e_3-term vanishes since $e_3x^2y^3$ is killed by the xy^3 term in the second component. We apply

$$\phi(x) = \frac{x}{1 + e_2^{1/2}y}, \phi(y) = y;$$

then $f \sim (x^2 + e_1(1 + e_2y^2)^{-1}x^2y + y^s, xy^3) \sim (x^2 + e_1x^2y + y^s, xy^3)$, $e_1 \in F$. That is, $f \sim m^2_{s,e_1}$.

If $\mu = 1$ and s is odd, we have $f \sim (x^2 + xy^2 + y^s, xy^3) \sim n^2_s$ by α, β-trick. If $\mu = 1$ and s is even, as above, we have

$$\begin{aligned} f &\sim (e(y)^{-1}(x^2 + xy^2) + y^s, xy^3) \\ &\sim ((1 + e_1y + e_2y^2 + \cdots)x^2 + y^s, xy^3) \\ &\sim (x^2 + xy^2 + e_1x^2y + e_2x^2y^2 + y^s, xy^3). \end{aligned} \tag{4-3}$$

Then applying

$$\phi(x) = \frac{x}{1 + e_2^{1/2}y}, \quad \phi(y) = y,$$

we have

$$\begin{aligned} f &\sim (x^2 + (1 + e_2^{1/2}y)^{-1}xy^2 + e_1(1 + e_2y^2)^{-1}x^2y + y^s, xy^3) \\ &\sim (x^2 + xy^2 + e_1x^2y + y^s, xy^3) \end{aligned} \tag{4-4}$$

with $e_1 \in F$. That is, $f \sim \tilde{n}^2_{s,e_1}$. □

5. The classification of order 3

In this section we assume $f = (f_1, f_2) \in F[x, y]^2$ with $\operatorname{ord}(f_1) = 3$, $\operatorname{ord}(f_2) \geq 3$. We also assume char $F = p > 3$ in this part. We begin by classifying 3-jets.

5.1. ***The classification of 3-jets.*** Choose a suitable coordinate system such that $j_3(f_1) = ax^3 + bx^2y + cxy^2 + dy^3$ with $a, b, c, d \in F$ and $a \neq 0$. Then $j_3(f_1) \sim x^3 + \frac{b}{a}x^2y + \frac{c}{a}xy^2 + \frac{d}{a}y^3 \sim (x - e_1y)(x - e_2y)(x - e_3y) \sim l_1l_2l_3$ since F is algebraically closed, where l_i, $i = 1, 2, 3$, are linear forms in R.

I. $l_1 = l_2 = l_3$. Let $\phi(x) = l_1$, $\phi(y) = y$, then $\phi(j_3(f_1)) = x^3$, i.e., $j_3(f_1) \sim x^3$. We have $j_3(f_1, f_2) \sim (x^3, ay^3 + bxy^2 + cx^2y)$.

I.1 If $a = b = c = 0$, $j_3(f) \sim (x^3, 0)$.

I.2 If $a = b = 0$, $c \neq 0$, $j_3(f) \sim (x^3, x^2y)$.

I.3 If $a = c = 0$, $b \neq 0$, $j_3(f) \sim (x^3, xy^2)$.

I.4 If $a = 0$, $b, c \neq 0$, $j_3(f) \sim (x^3, x^2y + b'xy^2)$, where $b' = \frac{b}{c}$. Let $\tilde{x} = 2b'x$, $\tilde{y} = y$, then $j_3(f) \sim (\tilde{x}^3, 2\tilde{x}^2\tilde{y} + \tilde{x}\tilde{y}^2) \sim (x^3, x^3 + 2x^2y + xy^2) \sim (x^3, x(x + y)^2)$. Let $\phi(x) = x$, $\phi(y) = y - x$, then $j_3(f) \sim (x^3, xy^2)$.

I.5 If $a \neq 0$, $b^2 \neq 3ac$, we have

$$\begin{aligned} j_3(f) &\sim (x^3, y^3 + b'xy^2 + c'x^2y) \\ &\sim \left(x^3, \left(y + \tfrac{1}{3}b'x\right)^3 + c'x^2y - \tfrac{1}{3}b'^2x^2y - \tfrac{1}{27}b'^3x^3\right) \\ &\sim \left(x^3, \left(y + \tfrac{1}{3}b'x\right)^3 + \left(c' - \tfrac{1}{3}b'^2\right)x^2\left(y + \tfrac{1}{3}b'x\right)\right), \end{aligned} \tag{5-1}$$

where $b' = b/a$, $c' = c/a$. Using the automorphism $\phi(x) = x$, $\phi(y) = y - \frac{1}{3}b'x$ we have

$$j_3(f) \sim \left(x^3, y^3 + \left(c' - \tfrac{1}{3}b'^2\right)x^2y\right).$$

Using the automorphism $\phi(x) = \left(c' - \frac{1}{3}b'^2\right)^{-1/2}x$, $\phi(y) = y$ since $c' - \frac{1}{3}b'^2 \neq 0$, we have $j_3(f) \sim (x^3, y^3 + x^2y)$.

I.6 If $a \neq 0$, $b^2 = 3ac$, as above, we have $j_3(f) \sim \left(x^3, y^3 + \left(c' - \frac{1}{3}b'^2\right)x^2y\right) \sim (x^3, y^3)$.

II. $l_1 = l_2 \neq l_3$. Let $\phi(x) = l_1$, $\phi(y) = l_2$, then $j_3(f_1) \sim x^2y$. We have $j_3(f) \sim (x^2y, ax^3 + bxy^2 + cy^3)$.

II.1 If $a = b = c = 0$, $j_3(f) \sim (x^2y, 0)$.

II.2 If $a = b = 0$, $c \neq 0$, $j_3(f) \sim (x^2y, y^3)$. Let $\phi(x) = y$, $\phi(y) = x$, we get $j_3(f) \sim (x^3, xy^2)$.

II.3 If $b = c = 0$, $a \neq 0$, $j_3(f) \sim (x^2y, x^3) \sim (x^3, x^2y)$.

II.4 If $a = c = 0$, $b \neq 0$, $j_3(f) \sim (x^2y, xy^2)$.

II.5 If $c=0,\ a,b\neq 0,\ j_3(f)\sim(x^2y, ax^3+bxy^2)$. Let $\phi(x)=x,\ \phi(y)=\sqrt{a/b}\,y$, we have $j_3(f)\sim(x^2y, x^3+xy^2)$.

II.6 If $c\neq 0,\ a\neq 0$, we have

$$j_3(f)\sim(x^2y, y^3+a'xy^2+b'x^3)\sim\bigl(x^2y, \bigl(y+\tfrac{1}{3}a'x\bigr)^3+\bigl(b'-\tfrac{1}{27}a'^3\bigr)x^3\bigr), \tag{5-2}$$

where $a'=a/c,\ b'=b/c$. Let $\phi(x)=x,\ \phi(y)=y+\frac{1}{3}a'x$, we have

$$\begin{aligned} j_3(f)&\sim\bigl(x^2\bigl(y-\tfrac{1}{3}a'x\bigr), y^3+\bigl(b'-\tfrac{1}{27}a'^3\bigr)x^3\bigr)\\ &\sim\bigl(x^2y-\tfrac{1}{3}a'x^3, y^3+\tfrac{1}{a'}3\bigl(b'-\tfrac{1}{27}a'^3\bigr)x^2y\bigr).\end{aligned} \tag{5-3}$$

Let $\phi(x)=x,\ \phi(y)=-\frac{1}{3}a'y$, we have $j_3(f)\sim(x^3+x^2y,\ y^3+\lambda x^2y)$, where $\lambda=(27b'-a'^3)/a'^3=(27bc^2-a^3)/a^3\in F$. If $\lambda=0$, we are back in case I. Hence we assume $\lambda\neq 0$ here.

II.7 If $c\neq 0,\ a=0$, as above, $j_3(f)\sim\bigl(x^2\bigl(y-\frac{1}{3}a'x\bigr), y^3+\bigl(b'-\frac{1}{27}a'^3\bigr)x^3\bigr)\sim(x^2y, y^3+b'x^3)\sim(x^2y, x^3+y^3)$.

III. $l_1\neq l_2\neq l_3$. Multiplying by a unit, one can assume $l_3=\frac{1}{2}l_1+\frac{1}{2}l_2$. Suppose $l_1=ux+vy,\ l_2=rx+sy$. Let

$$\phi(x)=\frac{u+r}{2}x+\frac{v+s}{2}y,\quad \phi(y)=\frac{u-r}{2i}x+\frac{v-s}{2i}y,$$

where $i^2=-1$. Then $\phi(l_1)=x+iy,\ \phi(l_2)=x-iy,\ \phi(l_3)=x$ and $j_3(f_1)\sim\phi(l_1l_2l_3)=x^3+xy^2$. Hence $j_3(f)\sim(x^3+xy^2, axy^2+bx^2y+cy^3)$.

III.1 If $a=b=c=0,\ j_3(f)\sim(x^3+xy^2, 0)$.

III.2 If $b=c=0,\ a\neq 0,\ j_3(f)\sim(x^3, xy^2)$.

III.3 If $c=0,\ b\neq 0,\ j_3(f)\sim(x^3+xy^2, axy^2+bx^2y)\sim(x^3+xy^2, -ax^3+bx^2y)\sim(x^3+xy^2, x^2(y-(a/b)x))$. Let $\phi(x)=x,\ \phi(y)=y+(a/b)x$; we have $j_3(f)\sim(x^2y, x^3+x(y+(a/b)x)^2)$, which goes back to II.

III.4 If $c\neq 0,\ a=b=0,\ j_3(f)\sim(x^3+xy^2, y^3)$, which goes back to I.

III.5 If $c\neq 0$, one of $a,b\neq 0$, then write $j_3(f)\sim(x^3+xy^2, y^3+uxy^2+vx^2y)$, where $u,v\in F$ and one of $u,v\neq 0$. We have $j_3(f)\sim(x^3+xy^2, y^3+uxy^2+vx^2y+\alpha(x^3+xy^2))$. Choose $\alpha,r,s\in F$ such that

$$s\neq 0,\ u+\alpha=r+2s,\ v=2rs+s^2,\ \alpha=rs^2. \tag{5-4}$$

These equations then reduce to

$$s^4+(3-v)s^2-2us-v=0,\ 2r=\frac{v}{s}-s,\ \alpha=rs^2; \tag{5-5}$$

hence such α,r,s exist. Then $j_3(f)\sim(x^3+xy^2, (y+rx)(y+sx)^2)$. Using the automorphism $\phi(x)=x+y/s,\ \phi(y)=y+rx$, we have reduced it to case I or II.

Hence:

Proposition 5.1. *Let $f \in F[x, y]^2$ with* $\operatorname{ord}(f) = 3$ *be a unimodal complete intersection singularity, then $j_3(f)$ is equivalent to one of the following*:

$$(x^3, 0),\ (x^3, x^2y),\ (x^3, xy^2),\ (x^3, y^3 + x^2y),\ (x^3, y^3),$$
$$(x^2y, 0),\ (x^2y, xy^2),\ (x^2y, x^3 + xy^2),\ (x^3 + x^2y, y^3 + \lambda x^2y)(\lambda \neq 0), \tag{5-6}$$
$$(x^2y, x^3 + y^3),\ (x^3 + xy^2, 0).$$

5.2. ***The classification of unimodals.*** We have the following classification of unimodal ICIS of order 3:

Proposition 5.2. *A unimodal ICIS of order* 3 *in any field with characteristic not equal to* 2, 3 *must have one of the forms in Table 3.*

symbol	*form*	*condition*
H	$(x^3 + x^2y, y^3 + \lambda x^2y)$	$\lambda \neq 0$
I	$(x^3, y^3 + x^2y)$	
J	(x^3, y^3)	
K_r	$(x^3 + xy^2, x^2y + y^r)$	$r \geq 4$
$L_{r,s}$	$(x^2y + y^r, xy^2 + x^s)$	$r, s \geq 4$
M_r	$(x^3 + y^r, xy^2)$	$r \geq 4$
N_λ	$(x^3 + \lambda xy^3, x^2y + y^4)$	$\lambda \notin \{1, 12\}$
$P_{r,\infty}$	$(x^3 + xy^3 + xy^r, x^2y + y^4)$	$r \geq 4$
$P_{\infty,s}$	$(x^3 + xy^3 + y^s, x^2y + y^4)$	$s \geq 5$
$P_{r,s,\lambda}$	$(x^3 + xy^3 + xy^r + \lambda y^s, x^2y + y^4)$	$r \geq 4,\ s \geq 5,$ $\lambda \in F$
$\widetilde{P}_{r,s}$	$(x^3 + xy^3 + uxy^r + vy^s, x^2y + y^4)$, *where* $u = u_0 + u_1y + \cdots,\ v = v_0 + v_1y + \cdots,$	$r \geq 4,\ s \geq 5,$ $p \mid 2r - 2s + 3$
R_t	$(x^3 + xy^3, x^2y + y^t)$	$t \geq 5$
X_λ	$(x^3 + y^4, x^2y + \lambda y^4)$	$\lambda \in \{0, 1\}$
Y_λ	$(x^3 + y^5, x^2y + \lambda y^5)$	$\lambda \in \{0, 1\}$
Z_λ	$(x^3 + 12xy^3 + \lambda y^5, x^2y + y^4)$	$\lambda \in \{0, 1\}$

Table 3. Possible ICIS of order 3 when char $F \neq 2, 3$.

We will prove Proposition 5.2 step by step:

Proposition 5.3. *If $j_3(f)$ is contact equivalent to one of the following forms*:

(5-7) $(x^3, y^3+x^2y),\ (x^3, y^3),\ (x^2y, x^3+y^3),\ (x^3+x^2y, y^3+\lambda x^2y)(\lambda \neq 0),$

then $j_3(f)$ is 3*-determined. In particular*, *f is contact equivalent to $j_3(f)$ of the above forms.*

Proof. After some computation, one can show $\mathfrak{m}^4 \subset \mathfrak{m}\cdot \widetilde{T}_f(\mathcal{K}f)$ when $j_3(f)$ has forms in 5.2. Hence by Theorem 2.6, $j_3(f)$ is 3-determined. Here we compute the case when $j_3(f) \sim (x^3, y^3+x^2y)$ as an example.

Let

$$e_1 = (x^3, 0),\ e_2 = (0, x^3),\ e_3 = (y^3+x^2y, 0),$$

$$e_4 = (0, y^3+x^2y),\ e_5 = (3x^2, 2xy),\ e_6 = (0, 3y^2+x^2).$$

Then $\widetilde{T}_f(\mathcal{K}f)$ is generated by $e_1,\ e_2,\ e_3,\ e_4,\ xe_5,\ ye_5,\ xe_6,\ ye_6$, and we have

$$\begin{aligned}
&(0, x^4) = xe_2 && \in \mathfrak{m}\widetilde{T}_f(\mathcal{K}f),\\
&(0, x^3y) = ye_2 && \in \mathfrak{m}\widetilde{T}_f(\mathcal{K}f),\\
&(0, x^2y^2) = \tfrac{1}{3}(x^2e_6 - xe_2) && \in \mathfrak{m}\widetilde{T}_f(\mathcal{K}f),\\
&(0, xy^3) = \tfrac{1}{3}(xye_6 - (0, x^3y)) && \in \mathfrak{m}\widetilde{T}_f(\mathcal{K}f),\\
&(0, y^4) = ye_4 - (0, x^2y^2) && \in \mathfrak{m}\widetilde{T}_f(\mathcal{K}f),\\
&(x^4, 0) = xe_1 && \in \mathfrak{m}\widetilde{T}_f(\mathcal{K}f),\\
&(x^3y, 0) = ye_1 && \in \mathfrak{m}\widetilde{T}_f(\mathcal{K}f),\\
&(x^2y^2, 0) = \tfrac{1}{3}(y^2e_5 - 2(0, xy^3)) && \in \mathfrak{m}\widetilde{T}_f(\mathcal{K}f),\\
&(xy^3, 0) = xe_3 - (x^3y, 0) && \in \mathfrak{m}\widetilde{T}_f(\mathcal{K}f),\\
&(y^4, 0) = ye_3 - (x^2y^2, 0) && \in \mathfrak{m}\widetilde{T}_f(\mathcal{K}f),
\end{aligned}$$

This implies $\mathfrak{m}^4\cdot R^2 \subset \mathfrak{m}\cdot \widetilde{T}_f(\mathcal{K}f)$, as claimed. □

Proposition 5.4. *If $j_3(f) \sim (x^3, 0), (x^2y, 0), (x^3+xy^2, 0)$, then $\mathcal{K}$*-mod(f) *is at least* 2*.*

Proof. We just prove the case when $j_3(f) \sim (x^2y+xy^2, 0)$. The others are similar.

In this case a complete transversal in J_4 is spanned by $(0, x^4), (0, x^2y^2), (0, y^4)$; hence $j_4(f) \sim (x^3+xy^2, ax^4+bx^2y^2+cy^4)$ by Theorem 2.8. Let $g = (x^3 + xy^2, ax^4+bx^2y^2+cy^4)$. After computation, we have $(0, xy^3), (0, y^4) \notin P_{3,4}$ for almost all a, b, c. Hence, the codimension of $\widetilde{T}_g(\mathcal{K}_4 g)$ in $P_{3,4}$ is 2, which is, the modality of $f \geq 2$ by Proposition 2.19. □

Proposition 5.5. *If $j_3(f) \sim (x^3, xy^2)$, then $f \sim (x^3+y^r, xy^2)$, $r \geq 4$.*

Proof. We can compute the complete transversal as follows: By Proposition 2.5, $\widetilde{T}_f(\mathcal{K}f)$ is generated by

$$e_1=(x^3,0),\ e_2=(0,x^3),\ e_3=(xy^2,0),\ e_4=(0,xy^2),\ xe_5,\ ye_5,\ xe_6,\ ye_6,$$

where $e_5=(3x^2,y^2)$, $e_6=(0,2xy)$. A similar computation shows that

$$(x^4,0),\ (x^3y,0),\ (x^2y^2,0),\ (xy^3,0),\ (0,x^4),\ (0,x^3y),\ (0,x^2y^2),\ (0,xy^3),\ (0,y^4)$$

lie in $\widetilde{T}_f(\mathcal{K}f)$ while $(y^4,0)\notin\widetilde{T}_f(\mathcal{K}f)$. In fact, one can easily show that $(y^l,0)\notin\widetilde{T}_f(\mathcal{K}f)$ for any $l>3$. Hence, a complete transversal is spanned by $\{(y^l,0)\mid l\geq 4\}$.

By Theorem 2.8, we have

$$f\sim\Big(x^3+\sum_{l\geq 4}b_ly^l,\ xy^2\Big)\sim(x^3+e(y)y^r,\ xy^2)\ \ (r\geq 4,\ e(y)\text{ a unit})\\ \sim(x^3+y^r,\ xy^2). \tag{5-8}$$

In the last line, we take the automorphism $\phi(x)=e(y)^{1/3}x$ and $\phi(y)=y$. □

Proposition 5.6. *If $j_3(f)\sim(x^2y,xy^2)$, then $f\sim(x^2y+y^r,xy^2+x^s)$, $r,s\geq 4$.*

Proof. Using a computation like Proposition 5.5, a complete transversal is given by $\{(y^l,0),(0,x^l)\mid l\geq 4\}$. Hence, by Theorem 2.8,

$$f\sim\Big(x^2y+\sum_{l\geq 4}a_ly^l,\ xy^2+\sum_{l\geq 4}b_lx^l\Big)\\ \sim(x^2y+a(y)y^r,\ xy^2+b(x)x^s), \tag{5-9}$$

where $a(y),b(x)$ are units in $F[\![x,y]\!]$ and $r,s\geq 4$.

Using the automorphism $\phi(x)=a(y)^{1/2}x$ and $\phi(y)=y$, we have

$$f\sim(x^2y+y^r,\ a(y)^{1/2}xy^2+a(y)^{s/2}b(a(y)^{1/2}x)x^s)\\ \sim(x^2y+y^r,\ xy^2+e(x,y)x^s), \tag{5-10}$$

where $e(x,y)$ is a unit. We write $e(x,y)=\sum_{i\geq 0}e_i(x)y^i$; then

$$f\sim\Big(x^2y+y^r,\ xy^2+\Big(\sum_{i\geq 0}e_i(x)y^i\Big)x^s\Big)\\ \sim\Big(x^2y+y^r,\ xy^2+\Big(\sum_{i\geq 0}e_i(x)y^i\Big)x^s-x^{s-2}(x^2y+y^r)\Big(\sum_{k\geq 1}e_k(x)y^{k-1}\Big)\Big)\\ \sim\Big(x^2y+y^r,\ xy^2+e_0(x)x^s-\sum_{k\geq 1}x^{s-2}e_k(x)y^{r+k-1}\Big).$$

Note that the order of x changes from s to $s-2$, and the order of y changes from 0 to r. Repeating the operation, we get $f\sim(x^2y+y^r,xy^2+e_0(x)x^s+d(x,y))$, where $d(x,y)=y^{rs/2}e'(x,y)$ or $xy^{r(s-1)/2}e'(x,y)$ depending on whether s is even or odd, and $e'(x,y)=\sum_{k\geq 1}e_k(x)y^{k-1}$, which is a unit in $F[\![x,y]\!]$. The order of d is $\geq\frac{1}{2}r(s-1)+1$.

Using the automorphism $\phi(x)=x$ and $\phi(y)=e_0(x)^{1/2}y$. Then

$$f\sim(e_0(x)^{1/2}x^2y+e_0(x)^{r/2}y^r,\,xy^2+x^s+d_1(x,y))\\ \sim(x^2y+e_0(x)^{(r-1)/2}y^r,\,xy^2+x^s+d_1(x,y)), \tag{5-11}$$

where $d_1(x,y)=d(x,e_0(x)^{1/2}y)$. Write $e_0(x)^{(r-1)/2}=\sum_{i\geq 0}u_ix^i$; then

$$\begin{aligned} f&\sim\Big(x^2y+\Big(\sum_{i\geq 0}u_ix^i\Big)y^r,xy^2+x^s+d_1(x,y)\Big)\\ &\sim\Big(x^2y+\Big(\sum_{i\geq 0}u_ix^i\Big)y^r-y^{r-2}(xy^2+x^s+d_1(x,y))\Big(\sum_{k\geq 1}u_kx^{k-1}\Big),xy^2+x^s+d_1(x,y)\Big)\\ &\sim\Big(x^2y+u_0y^r-\sum_{k\geq 1}u_ky^{r-2}x^{s+k-1}-\sum_{k\geq 1}u_ky^{r-2}d_1(x,y),xy^2+x^s+d_1(x,y)\Big).\end{aligned}$$

Repeating the operation, we get $f\sim(x^2y+u_0y^r+d_2(x,y),\,xy^2+x^s+d_1(x,y))$, where the order of $d_2(x,y)$ is $\geq \frac{1}{2}s(r-1)+1$.

Taking the automorphism $\phi(x)=\alpha x$, $\phi(y)=\beta y$, where $\alpha,\beta\in F$ satisfy $\alpha^2\beta=u_0\beta^r$, $\alpha\beta^2=\beta^s$, we have

$$f\sim(x^2y+y^r+\tilde{d}_2(x,y),\,xy^2+x^s+\tilde{d}_1(x,y)). \tag{5-12}$$

Specifically, $d_1(\tilde{x},y)$ has order $\frac{1}{2}rs$ if s is even and $\frac{1}{2}r(s-1)+1$ if s is odd, while $\tilde{d}_2(x,y)$ has order $\frac{1}{2}rs$ if r is even and $\frac{1}{2}(r-1)s+1$ if r is odd.

Now we exchange the position of x,y so that $r\geq s$. Let

$$g=j_r(f)=(x^2y+y^r,\,xy^2+x^s). \tag{5-13}$$

Using a similar computation as in the proof of (**3.*b.ii.*2.3.3.1**) in Proposition 3.6, we can show $\mathfrak{m}^{r+1}R^2\subset\mathfrak{m}\widetilde{T}_g(\mathcal{K}g)$. This means g is $(2r-3)$-determined by Theorem 2.6. Since $s\geq 4$, $\min\{\operatorname{ord}\tilde{d}_1(x,y),\operatorname{ord}\tilde{d}_2(x,y)\}>2r-3$. Therefore $f\sim j_{2r-3}(f)=(x^2y+y^r,\,xy^2+x^s)$. □

Proposition 5.7. *If $j_3(f)\sim(x^2y,\,x^3+xy^2)$, then $f\sim(x^2y+y^r,\,x^3+xy^2)$, $r\geq 4$.*

Proof. A complete transversal is given by $\{(y^l,0)\mid l\geq 4\}$; hence

$$f\sim\Big(x^2y+\sum_{l\geq 4}a_ly^l,\,x^3+xy^2\Big)\sim(x^2y+e(y)y^r,\,x^3+xy^2),$$

where $r\geq 4$.

Using the automorphism $\phi(x)=e(y)^{1/2}x$ and $\phi(y)=y$, we have

$$f\sim(x^2y+y^r,\,e(y)x^3+xy^2).$$

Write $e(y)=\sum_{i\geq 0} e_i y^i$. Then

$$f \sim \Big(x^2y+y^r,\ \Big(\sum_{i\geq 0} e_i y^i\Big)x^3+xy^2-x(x^2y+y^r)\Big(\sum_{k\geq 1} e_k y^{k-1}\Big)\Big) \tag{5-14}$$
$$\sim \Big(x^2y+y^r,\ e_0x^3+xy^2-\sum_{k\geq 1} e_k x y^{r+k-1}\Big)$$
$$\sim \Big(x^2y+y^r,\ e_0x^3+xy^2\Big(1-\sum_{k\geq 1} e_k x y^{r+k-3}\Big)\Big).$$

Set $u_0(y)=e(y),\ v_1(y)=1-\sum_{k\geq 1} e_k x y^{r+k-3}$. Then $v_1(y)\in 1+\mathfrak{m}^{r-2}$ and

$$f \sim (x^2y+y^r,\ e_0x^3+u_1(y)xy^2) \sim (x^2y+y^r,\ e_0v_1(y)^{-1}x^3+xy^2). \tag{5-15}$$

Since $v_1(y)\in 1+\mathfrak{m}^{r-2}$, we have $v_1(y)^{-1}\in 1+\mathfrak{m}^{r-2}$. Set $u_1(y)=e_0v_1(y)^{-1}$; then $u_1(y)\in e_0+\mathfrak{m}^{r-2}$ and $f\sim(x^2y+y^r,\ u_1(y)x^3+xy^2)$. Writing

$$u_1(y)=e_0+\sum_{k\geq r-2} v_k y^k, \tag{5-16}$$

we have

$$f \sim \Big(x^2y+y^r,\ \Big(e_0+\sum_{k\geq r-2} v_k y^k\Big)x^3+xy^2\Big)$$
$$\sim \Big(x^2y+y^r,\ \Big(e_0+\sum_{k\geq r-2} v_k y^k\Big)x^3+xy^2-x(x^2y+y^r)\Big(\sum_{k\geq r-2} v_k y^{k-1}\Big)\Big)$$
$$\sim \Big(x^2y+y^r,\ e_0x^3+xy^2-\sum_{k\geq r-2} v_k x y^{r+k-1}\Big)$$
$$\sim \Big(x^2y+y^r,\ e_0x^3+xy^2\Big(1-\sum_{k\geq r-2} v_k y^{r+k-3}\Big)\Big).$$

Set $v_2(y)=1-\sum_{k\geq r-2} v_k y^{r+k-3}$; then $v_2(y)\in 1+\mathfrak{m}^{2r-5}$. Setting $u_2(y)=e_0v_2(y)^{-1}\in e_0+\mathfrak{m}^{2r-5}$, we have $f\sim(x^2y+y^r,\ u_2(y)x^3+xy^2)$.

Repeating the operation, we can get a sequence of units $u_1(y), u_2(y), \dots, u_n(y), \dots$ with $u_n(y)\in e_0+\mathfrak{m}^{n(r-3)+1}$ and $f\sim(x^2y+y^r,\ u_n(y)x^3+xy^2)$.

Since $r\geq 4$, the orders of $u_n(y)-e_0$ are strictly increasing. Noting that $(x^2y+y^r,\ e_0x^3+xy^2)$ has finite Tjurina number (and hence is finite determined), we have $f\sim(x^2y+y^r,\ e_0x^3+xy^2)\sim(x^2y+y^r,\ x^3+xy^2)$ by using Remark 2.7 and applying the automorphism $\phi(x)=\alpha x,\ \phi(y)=\beta y$, where $\alpha^2\beta=\beta^r$ and $e_0\alpha^3=\alpha\beta^2$. □

The situation becomes complicated when $j_3(f)\sim(x^3, x^2y)$. A complete transversal is given by $\{(xy^{l-1},0),(y^l,0),(0,y^l)\mid l\geq 4\}$. Then

$$f \sim \Big(x^3+\sum_{i\geq 3} a_i x y^i+\sum_{j\geq 4} b_j y^j,\ x^2y+\sum_{k\geq 4} c_k y^k\Big) \tag{5-17}$$
$$\sim (x^3+a(y)xy^r+b(y)y^s,\ x^2y+c(y)y^t),$$

where $r\geq 3,\ s\geq 4,\ t\geq 4$, and $a(y), b(y), c(y)$ are units or 0.

First we have the following criterion:

Proposition 5.8. *The modality of* $f \sim (x^3 + a(y)xy^r + b(y)y^s, x^2y + c(y)y^t)$ *is at least* 2 *if* $r \geq 4$, $s \geq 6$ *and* $t \geq 5$.

Proof. If $r \geq 4$, $s \geq 6$ and $t \geq 5$, set $g = j_4(f) = (x^3, x^2y)$, then all jets in $J_5'(g)$ are equivalent to $g_{ac} := (x^3 + axy^4, x^2y + cy^5)$, where $J_5'(g)$ is formed by jets in $J_5(g)$ with $r \geq 4$, $s \geq 6$ and $t \geq 5$.

Analogously to Proposition 3.3, we can show $(y^5, 0), (0, y^5) \notin \widetilde{T}_{g_{ac}}(\mathcal{K}_5 g_{ac})$. Hence $\operatorname{cod}(g_{ac}) \geq 2$. By Propositions 2.19 and 2.21, $\mathcal{K}$-mod$(f) \geq \mathcal{K}_5$-mod$(f) \geq 2$. □

Remark 5.9. If $a(y)$ (resp. $b(y), c(y)) = 0$, we regard r (resp. $s, t) = \infty$ here.

Now we assume $r \leq 4$ or $s \leq 6$ or $t \leq 5$ and $a(y), b(y), c(y)$ are units. We can simplify (5-17) by the trick using in [11].

First we recall the implicit function theorem:

Lemma 5.10 (cf. [9, Theorem 6.2.17]). *Let* $\mathcal{K}$ *be a field and* $F \in \mathcal{K}[[x_1, \dots, x_n, y]]$ *be such that*

$$F(x_1, \dots, x_n, 0) \in \langle x_1, \dots, x_n \rangle, \quad \frac{\partial F}{\partial y}(x_1, \dots, x_n, 0) \notin \langle x_1, \dots, x_n \rangle. \tag{5-18}$$

Then there exists a unique $y(x_1, \dots, x_n) \in \langle x_1, \dots, x_n \rangle \mathcal{K}[[x_1, \dots, x_n]]$ *such that*

$$F(x_1, \dots, x_n, y(x_1, \dots, x_n)) = 0.$$

Back to f in (5-17): if

$$3r - 2s \neq 0 \quad \text{and} \quad p \nmid 3r - 2s, \tag{5-19}$$

applying $\phi(x) = a(y)^{1/2}x$ and $\phi(y) = y$, we get

$$f \sim (x^3 + xy^r + y^s \tilde{b}(y), x^2y + y^t c_1(y)),$$

where $\tilde{b}(y) = b(y)/a(y)$ and $c_1(y) = c(y)/a(y)$.

Write $\tilde{b}(y) = \sum_{i \geq 0} b_i y^i$. Consider the function

$$F(z) = z^{2s-3r} \sum_{i \geq 0} b_i y^i z^{2i} - b_0.$$

We have $F(1) \in \langle y \rangle F[[y]]$, and

$$F'(1) = (2s - 3r) \sum_{i \geq 0} b_i y^i - 2 \sum_{i \geq 1} i b_i y^i$$

is a unit by (5-19). Applying Lemma 5.10 to the function $G(z) = F(z+1)$, we find $\tilde{z}(y)$ with $G(\tilde{z}(y)) = 0$. Let $z(y) = \tilde{z}(y) + 1$; then $z(y)$ is a unit and $F(z(y)) = 0$.

Using the automorphism $\phi(x) = z(y)^r x$ and $\phi(y) = z(y)^2 x$, we have $f \sim (x^3 + xy^r + b_0 y^s, x^2y + y^t c_2(y))$. Then apply $\xi(x) = \alpha x$, $\xi(y) = \beta(y)$ with

$\alpha,\ \beta \in F$ satisfying $\alpha^3 = \alpha\beta^r,\ \alpha\beta^r = b_0\beta^s$ (such $\alpha,\ \beta$ exist since $3r - 2s \neq 0$). We have

$$f \sim (x^3 + xy^r + y^s, x^2y + y^t\tilde{c}(y)), \tag{5-20}$$

where $\tilde{c}(y)$ is the image of $c_2(y)$ under the automorphism ξ.

Similarly, if

$$2s - 3t + 3 \neq 0 \quad \text{and} \quad p \nmid 2s - 3t + 3, \tag{5-21}$$

we have

$$f \sim (x^3 + \tilde{a}(y)xy^r + y^s, x^2y + y^t), \tag{5-22}$$

where $\tilde{a}(y)$ is the image of $a(y)$ under the similar automorphism.

If

$$r + 1 - t \neq 0 \quad \text{and} \quad p \nmid r + 1 - t, \tag{5-23}$$

we have

$$f \sim (x^3 + xy^r + \tilde{b}(y)y^s, x^2y + y^t). \tag{5-24}$$

If (5-19), (5-21), (5-23) all fail, then r, s, t satisfy

$$3r - 2s = ap, \quad 2s - 3t + 3 = bp, \quad r + 1 - t = cp. \tag{5-25}$$

But the minimal solution of (5-25) with $r \geq 3,\ s \geq 4,\ t \geq 4$ and $p \geq 5$ is exactly $r = 4,\ s = 6,\ t = 5$, in which case f is not unimodal.

Remarks 5.11. (i) We call the technique we use here the α, β-trick, since we can easily apply $\phi(x) = \alpha x,\ \phi(y) = \beta y$ in (5-17) and get the result.

For example, when (5-19) holds, choose α, β as the simultaneous solution of

$$\alpha^3 = \alpha\beta^r a(\beta y), \quad \alpha\beta^r a(\beta y) = \beta^s b(\beta y); \tag{5-26}$$

then apply ϕ on (5-17), we can get (5-20). The trick was shown in [5]. The implicit function theorem provides a complete proof with the same result. But it is useless in some special characteristics, e.g., if $p \mid 3r - 2s$ here.

(ii) We call the technique used in Propositions 5.6 and 5.7 the r, s-trick. It has no restriction on characteristic, although the process is a little tedious. Later we will use the r, s-trick again but omit the process.

Proposition 5.12. *If* $s = 4$, *then* $f \sim (x^3 + y^4, x^2y + \lambda y^4) \sim X_\lambda$ *in Table 3*, *where* $\lambda = 0$ or 1.

Proof. Let $h = (x^3 + y^4, x^2 y)$ be the weighted 0-jet of f with respect to (a, d), where $a = (4, 3)$, $d = (12, 11)$.

We choose $T = \operatorname{span}\langle(0, y^4)\rangle$ as a complete transversal. Then we have $F^1_{a,d} \subset T + \widetilde{T}_h(F^1_{a,d}\mathcal{K}h)$. By Proposition 2.16, $f \sim (x^3 + y^4, x^2 y + ay^4)$ for some $a \in F$. After an obvious scaling, $f \sim (x^3 + y^4, x^2 y + y^4)$ or $(x^3 + y^4, x^2 y)$. □

Proposition 5.13. *If $s = 5$ and $r = 3$, then $f \sim N_\lambda, R_t, P_{r,s}, Z_\lambda$ in Table 3.*

Proof. In this case $p \nmid 3r - 2s$. By (5-20), we have $j_4(f) \sim (x^3 + xy^3, x^2 y + \lambda_0 y^4)$, which is weighted homogeneous of degree 0 with respect to $(a; d)$, where $a = (3, 2)$, $d = (9, 8)$. Write $g = j_4(f)$. Next we find $T \subset F^1_{a,d} R^2 \setminus F^r_{a,d} R^2$ such that $F^1_{a,d} R^2 \subset T + \widetilde{T}_g(F^1\mathcal{K}g)$. Then by Proposition 2.16, $f \sim g + t$ with $t \in T$.

After some computation (easily done by hand), we know that for $\lambda_0 \neq 0, 1, \frac{1}{12}$, $F^1_{a,d} R^2 \subset \widetilde{T}_g(F^1\mathcal{K}g)$ (note that char $F \neq 2, 3$, hence $\frac{1}{12}$ is well-defined). In that case $f \sim g = (x^3 + xy^3, x^2 y + \lambda_0 y^4) \sim (x^3 + \lambda xy^3, x^2 y + y^4)$, where $\lambda = 1/\lambda_0$.

If $\lambda_0 = 0$, then $F^1_{a,d} R^2 \subset T + \widetilde{T}_g(\mathcal{K}\cdot g)$ where T is spanned by $\{(0, y^5), (0, y^6), \dots\}$. Hence $f \sim (x^3 + xy^3, x^2 y + ey^t)$ with $t \geq 5$ or $f \sim (x^3 + xy^3, x^2 y)$ (which is not an ICIS). Using the r, s-trick we can show $f \sim (x^3 + xy^3, x^2 y + y^t)$.

If $\lambda_0 = 1$ or $\lambda_0 = \frac{1}{12}$, we write $g \sim (x^3 + \lambda xy^3, x^2 y + y^4)$ with $\lambda = 1, 12$. The process will be shown later in Proposition 5.15. □

Proposition 5.14. *If $s = 5$ and $r \geq 4$, then $f \sim N_0$ or Y_λ in Table 3, where $\lambda \in \{0, 1\}$.*

Proof. We have $f \sim (x^3 + a(y)xy^r + b(y)y^5, x^2 y + c(y)y^t)$.

If $t = 4$, let $g = j_4(f) = (x^3, x^2 y + cy^4) \sim (x^3, x^2 y + y^4)$ be the 4-jet of f. Then g is weighted homogeneous of degree 0 with respect to $(a; d)$, where $a = (3, 2)$, $d = (9, 8)$ as in Proposition 5.13. After the same computation, we have $F^1_{a,d} R^2 \subset \widetilde{T}_g(F^1\mathcal{K}g)$. That is, $f \sim (x^3, x^2 y + y^4)$.

If $t \geq 5$, let $h = (x^3 + by^5, x^2 y) \sim (x^3 + y^5, x^2 y)$ be the weighted 0-jet with respect to (a, d), where $a = (5, 3)$, $d = (15, 13)$. Computation shows that $F^1_{a,d} R^2 \subset T + \widetilde{T}_g(F^1\mathcal{K}g)$ for $T = \operatorname{span}\langle(0, y^5)\rangle$ (whether $p = 5$ or not). Hence $f \sim (x^3 + y^5, x^2 y + ay^5)$. An obvious scaling shows $f \sim (x^3 + y^5, x^2 y + \lambda y^5)$ for $\lambda \in \{0, 1\}$. □

Proposition 5.15. *If $s \geq 6$ and $t = 4$, then $f \sim N_\lambda, P_{r,\infty}, P_{\infty,s}, P_{r,s,\lambda}, \widetilde{P}_{r,s}$ or Z_λ in Table 3.*

Proof. Consider the 4-jet $j_4(f) = (x^3 + axy^3, x^2 y + cy^4) \sim (x^3 + \lambda xy^3, x^2 y + y^4)$, where $\lambda = a/c$. If $\lambda \neq 1, 12$, a computation similar to that of Proposition 5.13 shows $j_4(f) \sim (x^3 + \lambda xy^3, x^2 y + y^4)$ and furthermore $f \sim (x^3 + \lambda xy^3, x^2 y + y^4)$. We have

$$F^1_{a,d} R^2 \subset \operatorname{span}\langle(y^5, 0)\rangle + \widetilde{T}_g(F^1\mathcal{K}g).$$

Then, by Proposition 2.16, $f \sim (x^3 + 12xy^3, x^2 y + y^4)$ or $(x^3 + 12xy^3 + y^5, x^2 y + y^4)$.

If $\lambda_0 = 1$, we write $g \sim (x^3 + xy^3, x^2y + y^4)$. Then $F^1_{a,d}R^2 \subset T + \widetilde{T}_g(F^1\mathcal{K}g)$ with $T = \operatorname{span}\langle (y^l, 0), (xy^j, 0) \mid l \geq 5,\ j \geq 4\rangle$. Hence

$$f \sim (x^3 + xy^3 + u(y)xy^r + v(y)y^s, x^2y + y^4)$$

with $r \geq 5$, $s \geq 4$.

If $v(y) = 0$, write $u(y) = u_0 + u_1y + \cdots$. Through the process

$$\begin{aligned}(5\text{-}27)\quad f &\sim (x^3+xy^3+u(y)xy^r, x^2y+y^4)\\ &\sim \Big((x^3+xy^3+u(y)xy^r)\Big(1-\frac{u_1}{u_0}y\Big), x^2y+y^4\Big)\\ &\sim \Big(x^3+xy^3+\Big(u_0+\Big(u_2-\frac{u_1^2}{u_0}\Big)y^2+\cdots\Big)xy^r-\frac{u_1}{u_0}x(x^2y+y^4), x^2y+y^4\Big)\\ &\sim \Big(x^3+xy^3+\Big(u_0+\Big(u_2-\frac{u_1^2}{u_0}\Big)y^2+\cdots\Big)xy^r, x^2y+y^4\Big),\end{aligned}$$

we reduce u_1 to 0. Repeating the process, we can reduce $u(y)$ to $u_0 \in F$ and finally to 1. Then $f \sim (x^3 + xy^3 + xy^r, x^2y + y^4) \sim P_{r,\infty}$.

If $u(y) = 0$, similarly we can get $f \sim (x^3 + xy^3 + y^s, x^2y + y^r) \sim P_{\infty,s}$.

If $u(y), v(y) \neq 0$, apply $\phi(x) = \alpha(y)^{-3}$, $\phi(y) = \alpha(y)^{-2}$. This leads to $f \sim (x^3+xy^3+u(y)\alpha(y)^{2r-6}xy^r+v(y)\alpha(y)^{2s-9}y^s, x^2y+y^s)$. Using the α, β-trick, if $p \nmid 2r-2s-3$, there exists $\alpha(y)$ such that $\alpha(y)2r-6u(y) = \alpha(y)^{2s-9}v(y)$. By the same process as for (5-27), we have $f \sim (x^3 + xy^3 + u_0xy^r + u_0y^s, x^2y + y^4) \sim (x^3 + xy^3 + xy^r + \lambda y^s, x^2y + y^4) \sim P_{r,s,\lambda}$.

If $p \mid 2r-2s-3$, we get a family $\widetilde{P}_{r,s}$. □

Proposition 5.16. *If $r = 3$, $s \geq 6$, $t \geq 5$, then $f \sim R_t$ in Table 3.*

Proof. In this case, let $g = j_4(f) = (x^3 + xy^3, x^2y)$. An ordinary complete transversal is given by $T = \operatorname{span}\langle (0, y^5), (0, y^6), \ldots\rangle$. Hence $f \sim (x^3 + xy^3, x^2y + e(y)y^t)$. Using the r, s-trick we get $f \sim (x^3 + xy^3, x^2y + y^t)$, $t \geq 5$. □

The above propositions finish the proof of Proposition 5.2.

6. The classification of order 3 when char $F = 2$

The process of classification in the field of characteristic 2 is quite similar to that of other characteristics. We first classify 3-jets and then classify all germs.

6.1. ***The classification of 3-jets.*** Same as in Section 5, for $f = (f_1, f_2)$ with $\operatorname{ord}(f_1) = 3$, we have $j_3(f_1) \sim (x - e_1y)(x - e_2y)(x - e_3y) \sim l_1l_2l_3$ since F is algebraically closed, where e_1, e_2, e_3 are the roots of $j_3(f_1(x, 1))$, and l_i, $i = 1, 2, 3$, are linear forms in R. Here we repeat the discussion at the beginning of Section 5.

I. $l_1 = l_2 = l_3$. Let $\phi(x) = l_1$, $\phi(y) = y$, then $\phi(j_3(f_1)) = x^3$, i.e., $j_3(f_1) \sim x^3$. We have $j_3(f_1, f_2) \sim (x^3, ay^3 + bxy^2 + cx^2y)$.

I.1 If $a=b=c=0,\ j_3(f)\sim(x^3,0)$.

I.2 If $a=b=0,\ c\neq 0,\ j_3(f)\sim(x^3,x^2y)$.

I.3 If $a=c=0,\ b\neq 0,\ j_3(f)\sim(x^3,xy^2)$.

I.4 If $a=0,\ b,c\neq 0,\ j_3(f)\sim(x^3,bxy^2+cx^2y)\sim(x^3,xy^2+x^2y)$.

I.5 If $a\neq 0$ and $b^2\neq ac$, then we have $j_3(f)\sim(x^3,\ y^3+b'xy^2+c'x^2y)\sim(x^3,\ (y+b'x)^3+(c'-b'^2)x^2y)$, where $b'=b/a,\ c'=c/a$. Using the automorphism $\phi(x)=x,\ \phi(y)=y-b'x$, we have $j_3(f)\sim(x^3,\ y^3+(c'-b'^2)x^2(y-b'x))\sim(x^3,\ y^3+x^2y)$ since $b^2\neq ac$.

I.6 If $a\neq 0,\ b^2=3ac$, as above, we have $j_3(f)\sim(x^3,\ y^3+(c'-b'^2)x^2(y-b'x))\sim(x^3,\ y^3)$.

II. $l_1=l_2\neq l_3$. Let $\phi(x)=l_1,\ \phi(y)=l_2$, then $j_3(f_1)\sim x^2y$. We have $j_3(f)\sim(x^2y,ax^3+bxy^2+cy^3)$

II.1 If $a=b=c=0,\ j_3(f)\sim(x^2y,0)$.

II.2 If $a=b=0,\ c\neq 0,\ j_3(f)\sim(x^2y,\ y^3)$. Let $\phi(x)=y,\ \phi(y)=x$, we get $j_3(f)\sim(x^3,\ xy^2)$.

II.3 If $b=c=0,\ a\neq 0,\ j_3(f)\sim(x^2y,x^3)\sim(x^3,x^2y)$.

II.4 If $a=c=0,\ b\neq 0,\ j_3(f)\sim(x^2y,xy^2)$.

II.5 If $c=0,\ a,b\neq 0,\ j_3(f)\sim(x^2y,ax^3+bxy^2)$. Let $\phi(x)=x,\ \phi(y)=\sqrt{a/b}y$, we have $j_3(f)\sim(x^2y,x^3+xy^2)$.

II.6 If $c\neq 0,\ b\neq 0$, then $j_3(f)\sim(x^2y,y^3+b'xy^2+a'x^3)\sim(x^2y,(y+b'x)^3+(a'-b'^3)x^3)$, where $a'=a/c,\ b'=b/c$. Letting $\phi(x)=x,\ \phi(y)=y-b'x$, we have $j_3(f)\sim(x^2(y-b'x),\ y^3+(a'-b'^3)x^3)\sim(x^2y-b'x^3,\ y^3+((a'-b'^3)/b')x^2y)$. Letting $\phi(x)=x,\ \phi(y)=-b'y$, we have $j_3(f)\sim(x^3+x^2y,\ y^3+\lambda x^2y)$, where $\lambda=(a'-b'^3)/b'^3=(ac^2-b^3)/b^3\in F$. We still assume $\lambda\neq 0$ here.

II.7 If $c\neq 0,\ b=0$, as above, $j_3(f)\sim(x^2(y-b'x),\ y^3+(a'-b'^3)x^3)\sim(x^2y,\ y^3+a'x^3)\sim(x^2y,\ x^3+y^3)$.

III. $l_1\neq l_2\neq l_3$. Multiplying by a unit, one can assume $l_3=l_1+l_2$. Suppose $l_1=x$, $l_2=y$; then $j_3(f_1)\sim x^2y+xy^2$. Hence $j_3(f)\sim(x^2y+xy^2,ax^3+bx^2y+cy^3)$.

III.1 If $a=b=c=0$, then $j_3(f)\sim(x^2y+xy^2,0)$.

III.2 If $a=c=0,\ b\neq 0$, then $j_3(f)\sim(x^2y+xy^2,x^2y)\sim(x^2y,xy^2)$.

III.3 If exactly one of a,c is equal to 0, assume $a\neq 0$ and $c=0$. Then $j_3(f)\sim(x^2y+xy^2,x^3+b'x^2y)\sim(x^2y+xy^2,x^2(x+b'y))$, where $b'=b/a$. Using the automorphism $\phi(x)=x,\ \phi(y)=(y-x)/b'$, we reduce the situation to I or II depending on whether $b'=0$.

III.4 If $a, c \neq 0$ and $b = 0$, then $j_3(f) \sim (x^2y + xy^2, x^3 + c'y^3) \sim (x^2y + xy^2, x^3+c'^{1/2}x^2y+c'^{1/2}xy^2+c'y^3) \sim (x^2y+xy^2, (x+cy)(x+c'^{1/2}y)^2)$, where $c' = c/a$. It's easily to see that this reduces to I or II depending on whether $c' = 1$.

III.5 If $a, b, c \neq 0$, $j_3(f) \sim (x^2y + xy^2, x^3 + b'x^2y + c'xy^2)$, where $b' = b/a$, $c' = c/a$. Let α be the root of $\alpha^2 + b'\alpha = c'$. Then

$$\begin{aligned} f &\sim (x^2y + xy^2, x^3 + (\alpha + b')x^2y + \alpha xy^2 + c'y^3 \\ &\sim (x^2y + xy^2, x^2(x + \lambda y) + \alpha y^2(x + \lambda y)) \\ &\sim (x^2y + xy^2, (x + \lambda y)(x + \alpha^{1/2}y)^2), \end{aligned}$$

where $\lambda = \alpha + b = c/\alpha$. This reduces it to case I or II.

Hence we get the result:

Proposition 6.1. *Let $f \in F[\![x, y]\!]^2$ be a unimodal complete intersection singularity with* $\operatorname{ord}(f) = 3$ *in a field F with characteristic* 2, *then $j_3(f)$ is contact equivalent to one of the following*:

$$\begin{aligned} &(x^3, 0),\ (x^3, x^2y),\ (x^3, xy^2),\ (x^3, x^2y + xy^2),\ (x^3, y^3 + x^2y),\ (x^3, y^3), \\ &(x^2y, 0),\ (x^2y, xy^2),\ (x^2y, x^3 + xy^2),\ (x^3 + x^2y, y^3 + \lambda x^2y)\ (\lambda \neq 0), \\ &(x^2y, x^3 + y^3),\ (x^2y + xy^2, 0). \end{aligned} \tag{6-1}$$

6.2. ***The classification of unimodals.***

Proposition 6.2. *A unimodal ICIS of order* 3 *in any field with characteristic* 2 *must have one of the forms in Table 5* (on this page and the next).

symbol	*form*	*condition*
H_λ	$(x^3 + x^2y, y^3 + \lambda x^2y)$	$\lambda \neq 0$
I	$(x^3, y^3 + x^2y)$	
J	(x^3, y^3)	
K_r	$(x^3 + xy^2, x^2y + y^r)$	$r \geq 4$
$L_{r,s}$	$(x^2y + y^r, xy^2 + x^s)$	$r, s \geq 4$
M^2	$(x^3 + y^3, x^2y)$	
M_r	$(x^3 + y^r, xy^2)$	$r \geq 4$
M_r^2	$(x^3 + y^r, x^2y + xy^2)$	$r \geq 4$, *r is even*
$\widetilde{M}_r^2$	$(x^3 + y^r + ey^l, x^2y + xy^2)$, *where* $e = e_0 + e_1y^2 + e_2y^4 + \cdots$	$r \geq 4$, *r is odd,* $l > r$, *l is even*
N_λ	$(x^3 + \lambda xy^3, x^2y + y^4)$	$\lambda \neq 1$

symbol	*form*	*condition*
N^2_λ	$(x^3, x^2y+y^4+\lambda xy^3)$	$\lambda \in F$
$P_{r,\infty}$	$(x^3+xy^3+xy^r, x^2y+y^4)$	$r \geq 4$
$P_{\infty,s}$	$(x^3+xy^3+y^s, x^2y+y^4)$	$s \geq 5$
$P_{r,s,\lambda}$	$(x^3+xy^3+xy^r+\lambda y^s, x^2y+y^4)$	$r \geq 4,\ s \geq 5,\ \lambda \in F$
R_t	(x^3+xy^3, x^2y+y^t)	$t \geq 5$
X^2_μ	$(x^3+y^4, x^2y+\mu xy^3)$	$\mu \in \{0, 1\}$
$\widetilde{X}^2_\lambda$	$(x^3+y^4+xy^3, x^2y+\lambda xy^3)$	$\lambda \in F$
Y_λ	$(x^3+y^5, x^2y+\lambda y^5)$	$\lambda \in \{0, 1\}$

Table 5. Unimodal ICIS of order 3 when char $F = 2$ (starts on previous page).

The remaining propositions in this section finish the proof.

Proposition 6.3. *If $j_3(f)$ is contact equivalent to one of the following forms*:

(6-2) $(x^3, y^3+x^2y),\ (x^3, y^3),\ (x^3+x^2y, y^3+\lambda x^2y)\ (\lambda \neq 1),\ (x^2y, x^3+y^3),$

then $j_3(f)$ is 3-determined, and so f is contact equivalent to the $j_3(f)$ in (6-2).

Proof. If g is one of the first three germs, we can show $\mathfrak{m}^4 \subset \mathfrak{m} \cdot \widetilde{T}_g(\mathcal{K}g)$ as before; hence g is 3-determined.

For $g = j_3(f) = (x^2y, x^3+y^3)$, $\widetilde{T}_g(\mathcal{K}g)$ is spanned by $\{(x^2y, 0), (0, x^2y), (x^3+y^3, 0), (0, x^3+y^3), (0, x^2), (x^2, y^2)\}$, and a complete transversal is spanned by $\{(x^r, 0) \mid r \geq 4\}$. Hence $f \sim (x^2y+e(x)x^r, x^3+y^3)$ by Theorem 2.8 where $e(x) \in F[\![x]\!]$ is a unit and $r \geq 4$. Applying the automorphism $\phi(x) = x, \phi(y) = y - e(x)x^{r-2}$, $f \sim (x^2y, x^3+(y+e(x)x^{r-2})^3) \sim (x^2y, x^3+y^3+e(x)^3x^{3r-6}) \sim (x^2y, u(x)x^3+y^3)$, where $u(x) = 1+e(x)^3x^{3r-9}$ is a unit in $F[\![x]\!]$. Applying $\phi(x) = x, \phi(y) = u(x)^{1/3}y$, then $f \sim (x^2y, x^3+y^3)$. □

Proposition 6.4. *If $j_3(f) \sim (x^3, xy^2)$, then $f \sim (x^3+y^r, xy^2) \sim M_r$, $r \geq 4$ in Table 5.*

Proof. The complete transversal is still given by $\{(y^r, 0) \mid r \geq 4\}$. Then the process is same as Proposition 5.5. □

Proposition 6.5. *If $j_3(f) \sim (x^3, x^2y+xy^2)$, then $f \sim M^2_r$ or $\widetilde{M}^2_r$ in Table 5.*

Proof. A complete transversal is $T = \mathrm{span}\langle (y^r, 0),\ r \geq 4\rangle$; then $f \sim (x^3+u(y)y^r, x^2y+xy^2)$.

If r is even, using the α, β-trick we have $f \sim (x^3 + y^r, x^2y + xy^2)$ since $2 \nmid r - 3$. If r is odd, write

$$f \sim (x^3 + e_0 y^{2k+1} + e_1 y^{2k+2} + \cdots, x^2y + xy^2)$$
$$= (x^3 + (e_0 + e_2 y^2 + e_4 y^4 + \cdots) y^{2k+1} + (e_1 + e_3 y^2 + \cdots) y^{2k+2}, x^2y + xy^2).$$

There exists $e(x)^2 = e_0 + e_2 y^2 + \cdots$ that allows us to use the α, β-trick again. Resetting the symbols, we get a family $f \sim (x^3 + y^r + (e_0 + e_1 y^2 + \cdots) y^l, x^2y + xy^2)$. □

Proposition 6.6. *If $j_3(f) \sim (x^2y, xy^2)$, then $f \sim (x^2y + y^r, xy^2 + y^s) \sim L_{r,s}$, $r, s \geq 4$ in Table 5.*

Proof. The complete transversal is the same as Proposition 5.6, and the later process also follows from Proposition 5.6. □

Proposition 6.7. *If $j_3(f) \sim (x^2y, x^3 + xy^2)$, then $f \sim (x^2y + y^r, x^3 + xy^2) \sim K_r$ in Table 5, $r \geq 4$.*

Proof. Same as Proposition 5.7. □

If $j_3(f) \sim (x^3, x^2y)$, a complete transversal is given by

$$\{(xy^r, 0), (y^s, 0), (0, xy^u), (0, y^v) \mid r, u \geq 3,\ s, v \geq 4\}.$$

Then

$$f \sim (x^3 + a(y)xy^r + b(y)y^s, x^2y + c(y)xy^u + d(y)y^v), \tag{6-3}$$

with $r, u \geq 3,\ s, v \geq 4$

Proposition 6.8. *If f is of the form* (6-3) *and $r \geq 4,\ s \geq 6,\ v \geq 5$, then f has modality at least* 2.

Proof. For $r \geq 4,\ s \geq 6,\ v \geq 5$, we write $g = j_4(f) = (x^3, x^2y)$, and any 5-jet in an open dense subset of $J_5'(g)$ is of the form $g_{ac} = (x^3 + axy^4, x^2y + cxy^3 + dy^5)$ with $a, c, d \in F$, where $J_5'(g)$ is formed by jets in $J_5(g)$ with $r \geq 4,\ s \geq 6,\ v \geq 5$. Computation, perhaps in a program such as Singular, shows $(y^5, 0), (0, y^5) \notin P_{4,5}$ in each case; hence $\operatorname{cod}(g_{ac}) = 2$ for all $a, c \in F$. By Propositions 2.19 and 2.21, $\mathcal{K}$-mod$(f) \geq \mathcal{K}_5$-mod$(f) \geq \inf\{\operatorname{cod}(g_{ac})\} \geq 2$. □

Proposition 6.9. *If f is of the form* (6-3) *with $s = 4$, then f is contact equivalent to the form $\sim (x^3 + y^4, x^2y + \mu xy^3)$ with $\mu = 0, 1$ or $(x^3 + y^4 + xy^3, x^2y + \lambda xy^3)$ with $\lambda \in F$. That is, $f \sim X_\mu^2$ or $\widetilde{X}_\lambda^2$ in Table 5.*

Proof. In this case $f \sim (x^3 + y^4 + a(y)xy^r, x^2y + c(y)xy^u + d(y)y^v)$ after a scalar transform. Let $g = (x^3 + y^4, x^2y)$, then g is weighted homogeneous of degree 0 with respect to (a, d), where $a = (4, 3)$, $d = (12, 11)$. Let $T = \operatorname{span}\langle (xy^3, 0), (0, xy^3) \rangle$, then we check that $F_{a,d}^1 R^2 \subset T + \widetilde{T}_g(F^1\mathcal{K}g)$, which means $f \sim (x^3 + y^4 + axy^3,$

$x^2y + bxy^3)$ for $a, b \in F$. Take a scalar transform; we have $f \sim (x^3 + y^4, x^2y)$, $(x^3 + y^4, x^2y + xy^3)$ or $(x^3 + y^4 + xy^3, x^2y + \lambda xy^3)$ with $\lambda \in F$. $\square$

Proposition 6.10. *If f is of the form* (6-3) *with $v = 4$, $s > 4$, then $f \sim N_\lambda$, N_λ^2 or $P_{r,\infty}$, $P_{\infty,s}$, $P_{r,s,\lambda}$ in Table 5.*

Proof. If $r \geq 4$, let $g = (x^3, x^2y + y^4)$ be the weighted 0-jet of f with respect to (a, d), where $a = (3, 2)$, $d = (9, 8)$. Let $T = \text{span}\langle(0, xy^3)\rangle$; then $F^1_{a,d}R^2 \subset T + \widetilde{T}_g(F^1\mathcal{K}g)$, which means $f \sim (x^3, x^2y + y^4 + \lambda xy^3)$ for $\lambda \in F$.

If $r = 3$, let $g = (x^3 + a(0)xy^3, x^2y + y^4)$, $a(0) \in F^\times$ be the weighted 0-jet of f with respect to (a, d), where $a = (3, 2)$, $d = (9, 8)$. If $a(0) \neq 1$, then $F^1_{a,d}R^2 \subset \widetilde{T}_g(F^1\mathcal{K}g)$ automatically holds; hence $f \sim (x^3 + \lambda xy^3, x^2y + y^4)$, $\lambda \in F^\times$.

In the case $a(0) = 1$, T is spanned by $\{(xy^r, 0), (y^s, 0) \mid r \geq 4, s \geq 5\}$, and $f \sim (x^3 + xy^3 + u(y)xy^r + v(y)y^s, x^2y + y^4)$. Similar to Proposition 5.15, for $v(y) = 0$ (resp. $u(y) = 0$), we have $f \sim P_{r,\infty}$ (resp. $P_{\infty,s}$). Otherwise, since $p = 2$, we have $p \nmid 2r - 2s - 3$ for any r, s. Then $f \sim (x^3 + xy^3 + e(y)xy^r + e(y)y^s, x^2y + y^4) \sim (x^3 + xy^3 + e_0xy^r + e_0y^s, x^2y + y^4) \sim (x^3 + xy^3 + xy^r + \lambda y^s, x^2y + y^4) \sim P_{r,s,\lambda}$. $\square$

Proposition 6.11. *If f is of the form* (6-3) *with $s = 5$, then $f \sim Y_\lambda$ or R_t in Table 5.*

Proof. If $r \geq 4$, let $g = (x^3 + by^5, x^2y)$ with $b \in F^\times$ be the weighted 0-jet of f with respect to (a, d), where $a = (5, 3)$, $d = (15, 13)$. As in Proposition 5.14, $f \sim (x^3 + y^5, x^2y + \lambda y^5)$, $\lambda \in \{0, 1\}$.

If $r = 3$, let $g = (x^3 + axy^3, x^2y)$ with $a \in F^\times$ be the weighted 0-jet of f with respect to (a, d), where $a = (3, 2)$, $d = (9, 8)$. As in Proposition 5.16, we have $f \sim (x^3 + xy^3, x^2y + y^t)$, $t \geq 5$. $\square$

Proposition 6.12. *If f is of the form* (6-3) *with $s \geq 6$, $v \geq 5$, then $f \sim R_t$ in Table 5.*

Proof. In this case we have $r = 3$ by Proposition 6.8. Under the assumption $s, v > 4$, we can choose $g = (x^3 + axy^3, x^2y)$ to be the weighted 0-jet of f with respect to (a, d), where $a = (3, 2)$, $d = (9, 8)$. Let $T = \text{span}\langle(0, y^l) \mid l \geq 5\rangle$; then $F^1_{a,d}R^2 \subset T + \widetilde{T}_g(F^1\mathcal{K}g)$. Therefore $f \sim (x^3 + xy^3, x^2y + e(y)y^t)$. Using the r, s-trick as in Proposition 5.16, we have $f \sim (x^3 + xy^3, x^2y + y^t)$, $t \geq 5$. $\square$

7. The classification of order 3 when char $F = 3$

Next we repeat the discussion in the case char $F = 3$.

7.1. ***The classification of 3-jets.*** As in Section 5, for $f = (f_1, f_2)$ with $\text{ord}(f_1) = 3$, we have $j_3(f_1) \sim x^3$, x^2y or $x^3 + xy^2$.

I. $j_3(f_1) \sim x^3$. We have $j_3(f_1, f_2) \sim (x^3, ay^3 + bxy^2 + cx^2y)$.

I.1 If $c=0$, then $j_3(f)\sim(x^3,x^2y)$ or (x^3,xy^2) or (x^3,y^3+xy^2) or $(x^3,0)$, depending on whether $a,b=0$.

I.2 If $c\neq 0$, $b=0$, then $j_3(f)\sim(x^3,y^3+x^2y)$.

I.3 If $b,c\neq 0$, we can write $j_3(f)\sim\big(x^3,y\big(x+\frac{1}{2}by\big)^2+\big(a-\frac{1}{4}b^2\big)y^3\big)\sim\big(x^3,\frac{1}{4}b^2y\big(y+\frac{1}{b}2x\big)^2+\big(a-\frac{1}{4}b^2\big)(y+(2/b)x)^3\big)$. Using $\phi(x)=x$, $\phi(y)=y+(2/b)x$, we then get $j_3(f)\sim\big(x^3,\frac{1}{4}b^2(y-(2/b)x)y^2+\big(a-\frac{1}{4}b^2\big)y^3\big)\sim\big(x^3,ay^3-\frac{1}{2}bxy^2\big)\sim(x^3,xy^2)$ or (x^3,y^3+xy^2) depending on whether $a=0$.

II. $j_3(f_1)\sim(x^2y,0)$. We have $j_3(f)\sim(x^2y,ax^3+bxy^2+cy^3)$.

II.1 If $c=0$, then $j_3(f)\sim(x^2y,ax^3+bxy^2)\sim(x^2y,x^3)$ or (x^2y,xy^2) or (x^2y,x^3+xy^2) or $(x^2y,0)$ depending on whether a,b equal 0.

II.2 If $c\neq 0$, $b=0$, then $j_3(f)\sim(x^2y,ax^3+cy^3)\sim(x^2,(a^{1/3}x+c^{1/3}y)^3)$, which is back to case I.

II.3 If $b,c\neq 0$, $a=0$, then $j_3(f)\sim(x^2y,y^3+xy^2)$.

II.4 If $a,b,c\neq 0$, we change notation, obtaining $j_3(f)\sim(x^2y,y^3+ax^3+bxy^2)$. Applying the automorphism $\phi(x)=x$, $\phi(y)=y+\alpha x$, where α is a nonzero root of $\alpha^3-b\alpha^2+a=0$, we have

$$\begin{aligned}j_3(f)&\sim(x^2y+\alpha x^3,\,y^3+(\alpha^3+b\alpha^2+a)x^3+bxy^2+2b\alpha x^2y)\\&\sim(x^2y+\alpha x^3,\,y^3+bxy^2+2b\alpha(x^2y+\alpha x^3))\\&\sim(x^2y+\alpha x^3,\,y^3+bxy^2)\sim(x^3+x^2y,\,y^3+\lambda xy^2),\end{aligned}$$

with $\lambda\neq 1$; here we used that $\alpha\neq b$ (since $a\neq 0$).

III. $j_3(f_1)\sim x^3+xy^2$. We have $j_3(f)\sim(x^3+xy^2,axy^2+bx^2y+cy^3)$. The argument is the same as in Section 5, except for the characteristic being 3, with no other changes.

In conclusion:

Proposition 7.1. *Let $f\in F[\![x,y]\!]^2$ be a unimodal complete intersection singularity with* $\operatorname{ord}(f)=3$ *in a field F with characteristic* 3, *then $j_3(f)$ is contact equivalent to one of the following, where $\lambda\neq 1$:*

$(x^3,0)$, (x^3,x^2y), (x^3,xy^2), (x^3,y^3+xy^2), (x^3,y^3+x^2y), (x^3,y^3), $(x^2y,0)$, (x^2y,xy^2), (x^2y,x^3+xy^2), (x^2y,y^3+xy^2), $(x^3+x^2y,y^3+\lambda xy^2)$, $(x^3+xy^2,0)$.

7.2. *The classification of unimodals.*

Proposition 7.2. *A unimodal ICIS of order* 3 *in any field with characteristic* 3 *must have one of the forms in Table 6.*

symbol	*form*	*condition*
H	$(x^3+x^2y,\ y^3+\lambda x^2y)$	$\lambda\neq 0$
I^3_λ	$(x^3+\lambda y^4,\ y^3+x^2y)$	$\lambda\in\{0,1\}$
$\tilde{I}^3_\lambda$	$(x^3+\lambda y^4,\ y^3+xy^2)$	$\lambda\in\{0,1\}$
$J^3_{\lambda,\mu}$	$(x^3+\lambda x^2y^2,\ y^3+\mu x^2y^2)$	$\lambda,\mu\in\{0,1\}$
K_r	$(x^3+xy^2,\ x^2y+y^r)$	$r\geq 4$
K^3_r	$(y^3+xy^2+x^r,\ x^2y)$	$r\geq 4$
$\widetilde{K}^3_{r,\lambda}$	$(y^3+xy^2+\lambda xy^3+x^r,\ x^2y)$	$r\geq 4,\ \lambda\in F,\ 3\mid r$
$L_{r,s}$	$(x^2y+y^r,\ xy^2+x^s)$	$r,s\geq 4$
M^3_r	$(x^3+y^r,\ xy^2)$	$r\geq 4$
$\widetilde{M}^3_{r,\lambda}$	$(x^3+\lambda x^3y+y^r,\ xy^2)$	$r\geq 4,\ \lambda\in F,\ 3\mid r$
N^3_λ	$(x^3+\lambda xy^3,\ x^2y+y^4)$	$\lambda\neq 1$
$\widetilde{N}^3_s$	$(x^3+y^s,\ x^2y+y^4)$	$s\in\{5,6\}$
$P_{r,\infty}$	$(x^3+xy^3+xy^r,\ x^2y+y^4)$	$r\geq 4$
$P_{\infty,s}$	$(x^3+xy^3+y^s,\ x^2y+y^4)$	$s\geq 5$
$P_{r,s,\lambda}$	$(x^3+xy^3+xy^r+\lambda y^s,\ x^2y+y^4)$	$r\geq 4,\ s\geq 5,\ \lambda\in F$
$\widetilde{P}_{r,s}$	$(x^3+xy^3+uxy^r+vy^s,\ x^2y+y^4)$, *where* $u=u_0+u_1y+\cdots,\ v=v_0+v_1y+\cdots,$	$r\geq 4,\ s\geq 5,$ $3\mid 2r-2s$
R_t	$(x^3+xy^3,\ x^2y+y^t)$	$t\geq 5$
X_λ	$(x^3+y^4,\ x^2y+\lambda y^4)$	$\lambda\in\{0,1\}$
Y_λ	$(x^3+y^5,\ x^2y+\lambda y^5)$	$\lambda\in\{0,1\}$

Table 6. Unimodal ICIS of order 3 when char $F=3$.

As for the assignment of cases, most of the discussion is the same as before, so we just make a table to present the results. See Table 7 on the next page.

When $j_3(f)\sim(x^3,xy^2)$, $(x^2y,\ y^3+xy^2)$, (x^3,x^2y), we need further discussion.

Proposition 7.3. *If* $j_3(f)\sim(x^3,xy^2)$, *then* $f\sim M^3_r$ *or* $\widetilde{M}^3_{r,\lambda}$.

Proof. We have $f\sim(x^3+e(y)y^r,\ xy^2)$ with $r\geq 4$, $e(y)\in F[\![y]\!]$ is a unit.

If $3\nmid r$, using α,β-trick we have $f\sim(x^3+y^r,\ xy^2)$.

If $3\mid r$, reset notations, write $f\sim(e(y)x^3+y^r,\ xy^2)\sim(x^3+e_1x^3y+y^r,\ xy^2)$, where $e_1\in F$. Then $f\sim\widetilde{M}^3_{r,e_1}$. □

$j_3(f)$	*complete transversal*	f
(x^3, xy^2)	$(y^r, 0), r \geq 4$	$(x^3 + e(y)y^r, xy^2), r \geq 4$
$(x^3, y^3 + xy^2)$	$(y^4, 0)$	$(x^3 + \lambda y^4, y^3 + xy^2), \lambda \in \{0, 1\}$
$(x^3, y^3 + x^2y)$	$(y^4, 0)$	$(x^3 + \lambda y^4, y^3 + x^2y), \lambda \in \{0, 1\}$
(x^3, y^3)	$(x^2y^2, 0), (0, x^2y^2)$	$(x^3 + \lambda x^2y^2, y^3 + \mu x^2y^2)$, $\lambda, \mu \in \{0, 1\}$
(x^2y, xy^2)	$(y^r, 0), (0, x^s), r, s \geq 4$	$(x^2y + y^r, xy^2 + x^s), r, s \geq 4$
$(x^2y, x^3 + xy^2)$	$(y^r, 0), r \geq 4$	$(x^2y + y^r, x^3 + xy^2), r \geq 4$
$(x^2y, y^3 + xy^2)$	$(0, x^r), r \geq 4$	$(x^2y, y^3 + xy^2 + e(x)x^r), r \geq 4$
$(x^3{+}x^2y, y^3{+}\lambda xy^2)$	3-*determined*	$(x^3 + x^2y, y^3 + \lambda xy^2)$
(x^3, x^2y)	$(y^r, 0), (xy^s, 0), (0, y^t)$	$(x^3 + a(y)y^r + b(y)y^s, xy^2 + c(y)y^t)$

Table 7. Determination of cases for unimodal ICIS of order 3 when char $F = 3$.

Proposition 7.4. *If $j_3(f) \sim (x^2y, y^3 + xy^2)$, then $f \sim K^3_r$ or $\widetilde{K}^3_{r,\lambda}$.*

Proof. We have $f \sim (x^2y, y^3 + xy^2 + e(x)x^r)$.

If $3 \nmid r$, using α, β-trick we have $f \sim (x^2y, y^3 + xy^2 + x^r) \sim K^3_r$.

If $3 \mid r$, similar as above, $f \sim (y^3 + xy^2 + \lambda xy^3 + x^r, x^2y) \sim \widetilde{K}^3_{r,\lambda}$ with $\lambda \in F$. □

When $j_3(f) \sim (x^3, x^2y)$, the following result is the same as the case char $F > 3$.

Proposition 7.5. *If $j_3(f) \sim (x^3, x^2y)$, then $f \sim (x^3 + a(y)y^r + b(y)y^s, xy^2 + c(y)y^t)$ with $r, t \geq 4$, $s \geq 3$, $a(y), b(y), c(y)$ are units. And when $r \geq 4$, $s \geq 6$, $t \geq 5$, f is not unimodal.*

We omit the discussion, which is similar to that of Proposition 5.16, and give the result directly in Table 8. This finishes the proof of Proposition 7.2.

8. Checking the modality

Let $T^{1,\mathrm{sec}}(f) = R^m \big/ \big(\langle f_1, \ldots, f_m\rangle \cdot R^m + \mathfrak{m}\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n\rangle\big)$. Choose an F-basis $g_1, \ldots, g_d$ of $T^{1,\mathrm{sec}}(f)$. T. H. Pham, G. Pfister, and G. M. Greuel have shown in [11] that $F_{\boldsymbol{t}}(x) = F(x, t_1, \ldots, t_d) = f + t_1g_1 + \cdots + t_dg_d$ represents a formally semiuniversal deformation of f, where $\boldsymbol{t} = (t_1, \ldots, t_d)$. If $F(x, t)$ is equivalent to a family of ICIS of at most one parameter for $t \in F^d$, then f is unimodal. Here we check $f \sim l_{q,\lambda} \sim (x^2 + y^4, y^{q+2} + \lambda xy^q)$, $q \geq 3$, $\lambda^2 \notin \{0, -1\}$ in Table 1 as an example.

	weighted jet	*weight*	*complete transversal*	*form*
$s=4$	$(x^3+y^4,\, x^2y)$	$(4,3;12,11)$	$(0,\, y^4)$	$(x^3+y^4,\, x^2y+\mu y^4)$, $\mu\in\{0,1\}$
$s=5$, $r=3$	$(x^3+xy^3,\, x^2y+\lambda y^4)$ $\lambda\notin\{0,1\}$	$(3,2;9,8)$	0	$(x^3+xy^3,\, x^2y+\lambda y^4)$, $\lambda\notin\{0,1\}$
$s=5$, $r=3$	$(x^3+xy^3,\, x^2y+\lambda y^4)$ $\lambda=0$	$(3,2;9,8)$	$(0,\, y^t)$, $t\geq 5$	$(x^3+xy^3,\, x^2y+y^t)$, $t\geq 5$
$s=5$, $r=3$	$(x^3+xy^3,\, x^2y+\lambda y^4)$ $\lambda=1$	$(3,2;9,8)$	$(xy^r,0)$, $(y^s,0)$	$(x^3+xy^3+u(y)xy^r$ $+v(y)y^s,\, x^2y+y^4)$ *where* $u(y)$, $v(y)$ *are units in* $F[\![x,y]\!]$, $r\geq 4$, $s\geq 5$
$s=5$, $r\geq 4$, $t=4$	$(x^3,\, x^2y+y^4)$	$(3,2;9,8)$	$(y^5,0)$, $(y^6,0)$	$(x^3+\lambda y^s,\, x^2y+y^4)$, $s=5,6$, $\lambda=0,1$
$s=5$, $r\geq 4$, $t\geq 5$	$(x^3+y^5,\, x^2y)$	$(5,3;15,13)$	$(0,\, y^5)$	$(x^3+y^5,\, x^2y+\mu y^5)$, $\mu\in\{0,1\}$
$s\geq 6$, $t=4$	$(x^3+\lambda xy^3,\, x^2y+y^4)$ $\lambda\notin\{0,1\}$	$(3,2;9,8)$	0	$(x^3+\lambda xy^3,\, x^2y+y^4)$, $\lambda\notin\{0,1\}$
$s\geq 6$, $t=4$	$(x^3+\lambda xy^3,\, x^2y+y^4)$ $\lambda=0$	$(3,2;9,8)$	$(y^5,0)$, $(y^6,0)$	$(x^3+\lambda y^s,\, x^2y+y^4)$, $s=5,6$, $\lambda=0,1$
$s\geq 6$, $t=4$	$(x^3+\lambda xy^3,\, x^2y+y^4)$	$(3,2;9,8)$	$(xy^r,0)$, $(y^s,0)$	$(x^3+xy^3+u(y)xy^r$ $+v(y)y^s,\, x^2y+y^4)$ *where* $u(y)$, $v(y)$ *are units in* $F[\![x,y]\!]$, $r\geq 4$, $s\geq 5$
$r=3$, $s\geq 6$, $t\geq 5$	$(x^3+xy^3,\, x^2y)$	$(3,2;9,8)$	$(0,\, y^t)$, $t\geq 5$	$(x^3+xy^3,\, x^2y+y^t)$, $t\geq 5$

Table 8. Summary of omitted arguments completing the proof of Proposition 7.2.

First we choose generators

$$(y,0),\ (y^2,0),\ (y^3,0),\ (0,y),\ (0,y^2),\ \dots,\ (0,y^{q+2})\in T^{1,\mathrm{sec}}(f).$$

Note that $T^{1,\mathrm{sec}}(f)=(\mathfrak{m}R^2)/\widetilde{T}_f(\mathcal{K}f)$. In the proof of (**3.*b.ii.*2.3.3.1**) in Proposition 3.6, we have shown that $(0,xy^{q+1})\in\widetilde{T}_f(\mathcal{K}f)$, so $(0,y^{q+3})\in\widetilde{T}_f(\mathcal{K}f)$. Also $(0,y^{q+2})$ generates $(0,xy^q)$, while $(y^3,0)$ and $(0,y^{q+1})$ generate $(0,xy^{q-1})$. Then we add $(0, x)$, $(0, xy)$, ..., $(0, xy^{q-2})$ as generators. These generators form a basis of $T^{1,\sec}(f)$.

Let

$$g_1 = (y^2, 0),\ g_2 = (y^3, 0),\ g_3 = (0, y^2),\ \ldots,\ g_{q+3} = (0, y^{q+2}),$$

$$g_{q+4} = (0, xy),\ \ldots,\ g_{2q+1} = (0, xy^{q-2}).$$

We consider $F(x, \boldsymbol{t}) = f + t_1 g_1 + \cdots + t_{2q+1} g_{2q+1}$, where $\boldsymbol{t} = (t_1, \ldots, t_{2q+1})$. We write $F(x, \boldsymbol{t}) = (G_1, G_2) \in R^2$.

If $t_1 \neq 0$ or $t_3 \neq 0$ or $t_{q+4} \neq 0$, then $j_2(G_1)$ is nondegenerate, which means G is simple by Proposition 3.5. From now we assume $t_1 = t_3 = t_{q+4} = 0$.

If $t_2 \neq 0$ or $t_4 \neq 0$, then $j_2(G_1) \sim (x^2 + y^3)$. By Proposition 3.5, G is simple. From now we assume $t_2 = t_4 = 0$.

Now G is of the form

$$G \sim (x^2 + y^4,\ y^{q+2} + \lambda y^q + t_5 y^4 + \cdots + t_{q+3} y^{q+2} + t_{q+5} xy^2 + \cdots + t_{2q+1} xy^{q-2})$$
$$\sim \Big(x^2 + y^4,\ \sum_{i \geq u} t_{i+1} y^i + x \sum_{j \geq v} t_{j+q+3} y^j\Big),$$

where $q \geq 3$, $\lambda^2 \notin \{0, 1\}$, $u \geq 4$, $v \geq 2$. Comparing with (3-1) and Proposition 3.6, this corresponds to the case $\alpha = 1$, $s = 4$, $u = t \geq 4$, $v = q \geq 2$, where α, s, t, q are taken in the sense of (3-1).

By Proposition 3.6 (**3.*a.i***), in most cases, G is at most unimodal. The only unsure case is that there exists $q' < t < t'$ such that

$$G \sim (x^2 + y^4,\ y^{q'+2} + \lambda' xy^{q'} + u_0 xy^t + xy^{t'} + u_1 xy^{t+p})$$

with $q' \geq 3$, $t + p \leq q - 2$, $\lambda'^2 = -1$ and $p \mid t - q'$, $p \nmid t' - q'$, which is of the form $\tilde{l}_{q',t,t'}$ in Table 1 containing two parameters u_0 and u_1. Then we must have

$$p + 3 \leq p + q' \leq t \leq q - p - 2,$$

that is, $q \geq 2p + 5$. If we set $q \leq 2p + 4$, then this will not happen. That is, $l_{q,\lambda}$ is unimodal for $q \leq 2p + 4$.

Using the same method, we can give tables to show when the modality of the above class is 1.

Remark 8.1. If it's false that $(x^2 + y^4,\ y^5 + \lambda xy^3 + axy^{p+3} + xy^{4+p} + bxy^{2p+3})$ is contact equivalent to $(x^2 + y^4,\ y^5 + \lambda xy^3 + cxy^{p+3} + xy^{4+p} + dxy^{2p+3})$ for general $a, b, c, d \in F$, then we can ensure that the singularities given in Table 9 are the only unimodal ICIS. Conversely, if all singularities of the form $\tilde{l}_{q,t,t'}$ are equivalent (or at least can be presented as a one-parameter family), then all the singularities given in Table 1 are unimodal. Unfortunately, we cannot judge this equivalence yet. We post it as a conjecture.

symbol	*form*	*condition*	*when is it unimodal*
h_q	$(x^2+y^4,\, xy^q)$	$q \geq 3$	$q \leq 2p+3$
i	$(x^2,\, y^5)$		
$\tilde{i}$	$(x^2,\, y^5+xy^3)$		
i^5	$(x^2,\, y^5+xy^4)$	$p=5$	
j_t	$(x^2+y^4,\, y^t)$	$t \geq 5$	$t \leq 2p+4$
$\tilde{j}_t$	$(x^2+y^4,\, y^t+xy^{t-1})$	$t \geq 5,\ p \mid t$	$t \leq 2p+4$
k_q	$(x^2+y^4,\, y^{q+3}+xy^q)$	$q \geq 3$	$q \leq 2p+3$
$l_{q,\lambda}$	$(x^2+y^4,\, y^{q+2}+\lambda xy^q)$	$q \geq 3, \lambda^2 \notin \{0,-1\}$	$q \leq 2p+4$
$\tilde{l}_{q,t,t'}$	$(x^2+y^4,\, y^{q+2}+\lambda xy^q+uxy^t+xy^{t'})$, where $u=u_0+u_1y^p+u_2y^{2p}+\cdots$	$\lambda^2=-1, q \geq 3$, $t \geq q+1, t' \geq t+1$, $p \mid t-q,\ p \nmid t'-q$	$q \leq 2p+3$

Table 9. Unimodularity criteria (see Remark 8.1).

Conjecture 8.2. *Let F be an algebraically closed field with characteristic p. Then the isolated complete intersection singularity*

$$(x^2+y^4,\, y^5+\lambda xy^3+axy^{p+3}+xy^{4+p}+bxy^{2p+3})$$

is not contact equivalent to

$$(x^2+y^4,\, y^5+\lambda xy^3+cxy^{p+3}+xy^{4+p}+dxy^{2p+3})$$

for general $a, b, c, d \in F$.

The modality of singularities in Table 2 (i.e., of order 2 in characteristic 2 field) does not need to be checked, since every germ of the form $(x^2+h,\, g)$ with $g \in \mathfrak{m}^4 \setminus \mathfrak{m}^5$ is equivalent to a form in Table 2, which has at most one parameter.

Using the same method to check Table 3, we find the results in Table 10. Thus every singularity in Table 3 is unimodal. This is because those of modality 2 must have a deformation to $\widetilde{P}_{r,s}$. But $(xy^k, 0)$, $(y^l, 0)$ $(k \geq 4,\ l \geq 5)$ do not exist in a basis of $T^{1,\mathrm{sec}}(f)$ at the same time for every f. Then such a deformation does not exist.

For the case $\mathrm{ord}(f)=3$, $\mathrm{char}\, F=2$, see Table 11.

When an ICIS in Table 11 can deform to $\widetilde{M}_r^2$, it may have two parameters. This leads to another conjecture:

symbol	*form*	*condition*	*when is it unimodal*
H	$(x^3+x^2y,\, y^3+\lambda x^2y)$	$\lambda \neq 0$	
I	$(x^3,\, y^3+x^2y)$		
J	$(x^3,\, y^3)$		
K_r	(x^3+xy^2, x^2y+y^r)	$r \geq 4$	
$L_{r,s}$	(x^2y+y^r, xy^2+x^s)	$r, s \geq 4$	
M_r	(x^3+y^r, xy^2)	$r \geq 3$	
N_λ	$(x^3+\lambda xy^3, x^2y+y^4)$	$\lambda \notin \{1, 12\}$	
$P_{r,\infty}$	$(x^3+xy^3+xy^r, x^2y+y^4)$	$r \geq 4$	
$P_{\infty,s}$	$(x^3+xy^3+y^s, x^2y+y^4)$	$s \geq 5$	
$P_{r,s,\lambda}$	$(x^3+xy^3+xy^r+\lambda y^s, x^2y+y^4)$	$r \geq 4,\ s \geq 5,$ $\lambda \in F$	
$\widetilde{P}_{r,s}$	$(x^3+xy^3+uxy^r+vy^s, x^2y+y^4)$, *where* $u=u_0+u_1y+\cdots,\ v=v_0+v_1y+\cdots,$	$r \geq 4,\ s \geq 5$ $p \mid 2r-2s+3$	
R_t	(x^3+xy^3, x^2y+y^t)	$t \geq 5$	
X_λ	$(x^3+y^4, x^2y+\lambda y^4)$	$\lambda \in \{0, 1\}$	
Y_λ	$(x^3+y^5, x^2y+\lambda y^5)$	$\lambda \in \{0, 1\}$	
Z_λ	$(x^3+12xy^3+\lambda y^5, x^2y+y^4)$	$\lambda \in \{0, 1\}$	

Table 10. Unimodularity criteria when $\operatorname{ord}(f) = 3$, char $F \neq 2, 3$.

Conjecture 8.3. *Let F be an algebraically closed field with characteristic 2. Then the isolated complete intersection singularity $(x^3+y^5+ay^6+by^8, x^2y+xy^2)$ is not contact equivalent to $(x^3+y^5+cy^6+dy^8, x^2y+xy^2)$ for general $a, b, c, d \in F$.*

If the conjecture holds, then all unimodal ICIS of order 3 in a characteristic 2 field are presented in Table 11.

For the case of $\operatorname{ord}(f) = 3$, char $F = 3$, we can check that every singularity in Table 6 is unimodal.

In conclusion, we get the following classification theorem:

Theorem 8.4. *Let F be an algebraically closed field with arbitrary characteristic. Then every unimodal isolated complete intersection singularity (ICIS) in $F[\![x, y]\!]$ has the form in Tables 1, 2, 3, 5, 6. Besides Tables 1, 5, every form in the other tables is unimodal. If additionally Conjecture 8.2 (resp. Conjecture 8.3) holds, then all the unimodal ICIS in Table 1 (resp. Table 5) are presented in Table 9 (resp. Table 11).*

symbol	*form*	*condition*	*when is it unimodal*
H	$(x^3+x^2y,\ y^3+\lambda x^2y)$	$\lambda \neq 0$	
I	$(x^3,\ y^3+x^2y)$		
J	$(x^3,\ y^3)$		
K_r	(x^3+xy^2, x^2y+y^r)	$r \geq 4$	
$L_{r,s}$	(x^2y+y^r, xy^2+x^s)	$r, s \geq 4$	
M^2	(x^3+y^3, x^2y)		
M_r	(x^3+y^r, xy^2)	$r \geq 4$	$r \leq 8$
M_r^2	(x^3+y^r, x^2y+xy^2)	$r \geq 4$, r *is even*	$r \leq 8$
$\widetilde{M}_r^2$	$(x^3+y^r+ey^l, x^2y+xy^2)$, *where* $e=e_0+e_1y^2+e_2y^4+\cdots$	$r \geq 4$, r *is odd*, $l > r$, l *is even*	$l \leq 7$
N_λ	$(x^3+\lambda xy^3, x^2y+y^4)$	$\lambda \neq 1$	
N_λ^2	$(x^3, x^2y+y^4+\lambda xy^3)$	$\lambda \in F$	
$P_{r,\infty}$	$(x^3+xy^3+xy^r, x^2y+y^4)$	$r \geq 4$	$r \leq 7$
$P_{\infty,s}$	$(x^3+xy^3+y^s, x^2y+y^4)$	$s \geq 5$	$s \leq 8$
$P_{r,s,\lambda}$	$(x^3+xy^3+xy^r+\lambda y^s, x^2y+y^4)$	$r \geq 4$, $s \geq 5$, $\lambda \in F$	$r \leq s+1$, $r \leq 7$ *or* $s \leq r$, $s \leq 8$
R_t	(x^3+xy^3, x^2y+y^t)	$t \geq 5$	
X_μ^2	$(x^3+y^4, x^2y+\mu xy^3)$	$\mu \in \{0, 1\}$	
$\widetilde{X}_\lambda^2$	$(x^3+y^4+xy^3, x^2y+\lambda xy^3)$	$\lambda \in F$	
Y_λ	$(x^3+y^5, x^2y+\lambda y^5)$	$\lambda \in \{0, 1\}$	

Table 11. Unimodularity criteria when $\operatorname{ord}(f)=3$, $\operatorname{char} F=2$.

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