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Let (π, V) be a generic irreducible representation of a general linear group over a p-adic field. Jacquet, Piatetski-Shapiro, and Shalika gave an open compact subgroup K, so that the subspace V^K consisting of $v \in V$ fixed by K is one-dimensional. If π has a Shalika model Λ, then we call vectors in $\Lambda(V)$ the Shalika forms of π, and those in $\Lambda(V^K)$ the Shalika newforms. In this article, in the case where π is supercuspidal, we show the nonvanishing of Shalika newforms at a minimal point in a sense. This point is not the identity, and the Shalika newform vanishes at the identity if the character defining the Shalika model is ramified. In view of this result, in this case, we give another Shalika form with nice properties.

1. Main results

Let $n = 2m$ be an even integer. The so-called Shalika period for an irreducible cuspidal automorphic representation of GL_n over a number field was introduced by Jacquet and Shalika [8] to characterize its pole at $1 \in \mathbb{C}$ of the partial exterior square L-function. The local analogue of the Shalika period, which we call the Shalika form, is defined as follows.

Let F be a p-adic field, and S the Shalika subgroup of $G_n = \mathrm{GL}_n(F)$ consisting of matrices

$$s = \begin{bmatrix} a & b \\ & a \end{bmatrix}.$$

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Let $\psi : F \to \mathbb{C}^\times$ be a nontrivial continuous additive character. Let $\chi : F^\times \to \mathbb{C}^\times$ be a continuous character. Let π be an irreducible smooth representation of G_n. If π is realized in a subspace $\mathbb{S}_\pi(\chi)$ of the space consisting of continuous functions $J : G_n \to \mathbb{C}$ such that

$$J(sg) = \chi \circ \det(a)\psi(\mathrm{tr}(a^{-1}b))J(g),$$

then we call $J \in \mathbb{S}_\pi(\chi)$ a Shalika form of π relevant to χ. If a globally generic cuspidal automorphic representation $\Pi = \bigotimes_v \Pi_v$ of GL_n over a number field has a Shalika period relevant to a global character $\prod_v \xi_v$, then each Π_v has a realization in $\mathbb{S}_{\Pi_v}(\xi_v)$, i.e., Shalika model relevant to ξ_v, where v indicate the places of the number field, and Π_v, ξ_v indicate the v-components. Shalika forms and models for archimedean places are defined similarly. In the archimedean case, the uniqueness of Shalika models relevant to the trivial character was showed by Aizenbud, Gourevitch, Jacquet [1]. In the nonarchimedean case, that relevant to the trivial one was showed by Jacquet and Rallis [7], and those relevant to nontrivial ones by Chen and Sun [3]. Further, in this case, for generic square-integrable π, the Shalika model relevant to the trivial character characterizes the pole at $0 \in \mathbb{C}$ of the local exterior square L-function defined in [8], which is a local analogue of the original work.

Now let us dive into the heart of the matter. For various arithmetic applications, it is important to discover good Shalika forms and study their properties. In the case where π is unramified, the $\mathrm{GL}_n(\mathfrak{o})$-invariant Shalika form, unique up to a scalars, will play the expected roles, and its values are computed by Sakellaridis [15]. Our concern is the ramified case. At first, we consider the specific Shalika form — we call the Shalika newform, determined by the following newform theory due to Jacquet, Piatetski-Shapiro, and Shalika [10]. Suppose that π is generic with representation space V, conductor $\mathfrak{c}_\pi (\geq 0)$ and central character ω_π. Then, there exists a unique $v^{\mathrm{new}} \in V$ up to a scalar such that

$$\pi(k)v^{\mathrm{new}} = \omega_\pi(k_{n,n})v^{\mathrm{new}}$$

for k lying in the open compact subgroup

$$\begin{aligned}\Gamma_n(\mathfrak{c}_\pi) = \Gamma(\mathfrak{c}_\pi) &:= \mathrm{GL}_n(\mathfrak{o}) \cap \begin{bmatrix} 1_{n-1} & \\ & \varpi^{\mathfrak{c}_\pi} \end{bmatrix} \mathrm{GL}_n(\mathfrak{o}) \begin{bmatrix} 1_{n-1} & \\ & \varpi^{\mathfrak{c}_\pi} \end{bmatrix}^{-1} \\ &= \{k \in \mathrm{GL}_n(\mathfrak{o}) \mid k_{n,1}, \dots, k_{n,n-1} \in \mathfrak{p}^{\mathfrak{c}_\pi}\}\end{aligned}$$

where $\mathfrak{o}$ indicates the ring of integers of F, $\mathfrak{p} = \varpi\mathfrak{o}$ the prime ideal, and 1_{n-1} the unit matrix of G_{n-1}. With setting V as $\mathbb{S}_\pi(\chi)$, we call v^{new} the Shalika newform of π relevant to a fixed χ, and denote by J^{new}.

To describe our main result, we make the following assumption on generic irreducible (π, V) with $V = \mathbb{S}_\pi(\chi)$, since our preparation is not yet sufficient to deal with the general case. Let $P_n \subset G_n$ denote the mirabolic subgroup. It is known that, taking Bernstein and Zelevinsky's derivatives of V repeatedly, a smooth irreducible P_n-submodule $V_l \subset V$ (the nondegenerate part of V) is obtained finally. The assumption is:

$$\text{there exists a } J \in V_l \text{ such that } J(1_n) \neq 0. \tag{1-1}$$

If π is supercuspidal (and admits a Shalika model relevant to χ), then $V = V_l$, and this condition is empty. Let e denote the conductor of χ. Our main result is this:

Theorem 1.1. *Suppose that $\psi(\mathfrak{o}) = \{1\} \neq \psi(\mathfrak{p}^{-1})$, and keep assumption (1-1).*

(i) *If $e = 0$, J^{new} does not vanish at the identity.*

(ii) *If $e > 0$, J^{new} does not vanish at*

$$g_n := \left[\begin{array}{ccccc|c} \varpi^{e} & & & & 1 & \\ & \varpi^{3e} & & & \varpi^{e} & \\ & & \ddots & & \vdots & \\ & & & \varpi^{(n-3)e} & \varpi^{(m-2)e} & \\ & & & & \varpi^{(m-1)e} & \\ \hline & & & & & 1_m \end{array}\right].$$

Remark 1.2. Although we do not give a proof in this article, it holds that

$$\operatorname{supp}(J^{\text{new}}|_{P_n}) \begin{cases} \supsetneq (S \cap P_n)P_n(\mathfrak{o}) & \text{if } e = 0, \\ = (S \cap P_n)g_n P_n(\mathfrak{o}) & \text{if } e > 0, \end{cases}$$

when the standard L-function of π equals 1, where $P_n(\mathfrak{o}) = P_n \cap \mathrm{GL}_n(\mathfrak{o})$, and supp indicates the support.

In the case of $e > 0$, an elementary argument shows that J^{new} vanishes at 1_n. In the case of $e = 0$ (resp. $e > 0$), 1_n (resp. g_n) is minimal in a certain order among the points at which $P_n(\mathfrak{o})$-invariant Shalika forms do not necessarily vanish (cf. Lemmas 4.7 and 4.9).

Apart from our work, Grobner and Matringe [5] showed the nonvanishing at 1_n, when π is unitary and $e = 0$ without the assumption (1-1). However, our concern is beyond the unitary case, and since there exist many non-unitary representations satisfying (1-1), the nonvanishing at the point is expected in general.

But here a problem arises. From our aesthetic and various application perspectives, we think it is not desirable that J^{new} vanishes at 1_n in the case of $e > 0$, contrary to the nonvanishing of the Whittaker newform at 1_n. Further, from some results on representations (τ, V) of connected reductive groups over the algebraic closure of F, in the case where an L-function $L(s, \tau)$ is defined as a generator of a fractional ideal of the principal ideal domain $\mathbb{C}[q^s, q^{-s}]$, $s \in \mathbb{C}$ (q is the cardinality of $\mathfrak{o}/\mathfrak{p}$) spanned by some zeta integrals, and a functional equation between $L(s, \tau)$

and $L(1-s, \tau^\vee)$ for the contragredient $\tau^\vee$ holds, we infer that the desirable newform theory should satisfy the following:

(i) For a suitable open compact subgroup K, and a homomorphism $\Omega : K \to \mathbb{C}^\times$, the subspace $V_\Omega := \{v \in V \mid \tau(k)v = \Omega(k)v\}$ is spanned by a vector v_τ.

(ii) The open compact subgroup K depends on the choice of the realization V, and that of L-function (there are several L-functions and zeta integrals corresponding to the representations of the L-group.).

(iii) If V consists of functionals on G of a certain type, then v_τ does not vanish at the identity.

(iv) The zeta integral of v_τ, or its "suitable" arrangement, coincides with $L(s, \tau)$, and its suitable conjugate coincides with the product of $L(s, \tau^\vee)$ and the root number appearing in the functional equation.

For these reasons, in the case of $e > 0$, we construct a Shalika form J_π from J^{new} via some translations and integrations assuming that

- J^{new} does not vanish at g_n in the case of $e > 0$, and
- $\mathfrak{c}_\pi \geq me$.

In the case of $e = 0$, assuming $J^{\text{new}}(1_n) \neq 0$, we set $J_\pi = J^{\text{new}}$. We will describe some properties of J_π below. We set

$$l = \mathfrak{c}_\pi - (m-1)e$$

and let $\mathcal{O}_r$ be the ring of $r \times r$ matrices with entries in $\mathfrak{o}$. For $t \in \mathbb{Z}$, we have

$$\mathcal{O}_r(t) = \begin{bmatrix} 1_{r-1} & \\ & \varpi^t \end{bmatrix} \mathcal{O}_r \begin{bmatrix} 1_{r-1} & \\ & \varpi^t \end{bmatrix}^{-1}, \quad \mathcal{R}_r(t) = \mathcal{O}_r \cap \mathcal{O}_r(t).$$

We define the ring

$$R_{\mathfrak{c}_\pi} = \mathcal{R}_n(l) \cap \begin{bmatrix} 1_m & \\ & \varpi^e 1_m \end{bmatrix} \mathcal{O}_n \begin{bmatrix} 1_m & \\ & \varpi^e 1_m \end{bmatrix}^{-1},$$

and denote by $\mathbb{K}(\mathfrak{c}_\pi)$ the group of units of $R_{\mathfrak{c}_\pi}$, which is an open compact subgroup of G_n and equals $\Gamma(\mathfrak{c}_\pi)$ in the case of $e = 0$. Explicitly, $R_{\mathfrak{c}_\pi}$ consists of matrices of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \text{with } D \in \mathcal{R}_m(l) \text{ and } A, B, \begin{bmatrix} \varpi^e 1_{m-1} & \\ & \varpi^l \end{bmatrix}^{-1} C \in \mathcal{O}_m. \tag{1-2}$$

In the case of $n = 4$,

$$R_{\mathfrak{c}_\pi} = \begin{bmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{o} & \mathfrak{o} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^e & \mathfrak{p}^e & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^l & \mathfrak{p}^l & \mathfrak{p}^l & \mathfrak{o} \end{bmatrix}.$$

The first property of J_π is that

$$\pi(k)J_\pi = \chi \circ \det(d_k)J_\pi, \quad k \in \mathbb{K}(\mathfrak{c}_\pi),$$

where d_k indicates the $m \times m$ block matrix of k in the lower right corner. For $i \in \mathbb{Z}$, let

$$B_{m,i} = \{b \in B_m \cap \mathcal{O}_m \mid \det(b) \in \varpi^i \mathfrak{o}^\times\},$$

where B_m indicates the Borel subgroup of G_m. The second property of J_π is

$$L(s,\pi) = \sum_{i=0}^{\infty} c_i q^{i(-s+1/2)}, \quad c_i = \sum_{b \in B_{m,i}/B_{m,0}} J_\pi\left(\begin{bmatrix} b & \\ & 1_m \end{bmatrix}\right).$$

Here c_i has another expression:

$$c_i = \int_{\{g \in \mathrm{GL}_m(F) \mid \det(g) = \varpi^i \mathfrak{o}^\times\}} J_\pi\left(\begin{bmatrix} g & \\ & 1_m \end{bmatrix}\right) \mathrm{d}g \tag{1-3}$$

(cf. Proposition 5.3), where dg indicates the Haar measure on $\mathrm{GL}_m(F)$ normalized so that $\mathrm{vol}(\mathrm{GL}_m(\mathfrak{o})) = 1$. Observe that $J_\pi(1_n) = 1$. Set

$$\upsilon_{\mathfrak{c}_\pi} = \begin{bmatrix} \varpi^l & & \\ & \varpi^e 1_{m-1} & \\ & & w_m \end{bmatrix} \in G_n,$$

where w_m indicates the standard antidiagonal Weyl element in G_m. Define $J_\pi^* \in \mathbb{S}_{\pi^\vee}(\chi^{-1})$ by

$$J_\pi^*(g) = J_\pi(w_n {}^t g^{-1} \upsilon_{\mathfrak{c}_\pi}). \tag{1-4}$$

Let $\mathbb{K}(\mathfrak{c}_\pi)^* = \upsilon_{\mathfrak{c}_\pi}^{-1} {}^t\mathbb{K}(\mathfrak{c}_\pi)\upsilon_{\mathfrak{c}_\pi}$, which is the units group of the ring $R^*_{\mathfrak{c}_\pi}$ consisting of matrices of the form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \text{with}$$
$$D \in w_m {}^t\mathcal{R}_m(l) w_m, \quad C \in \mathcal{O}_m \begin{bmatrix} \varpi^l & \\ & \varpi^e 1_{m-1} \end{bmatrix}, \quad A, B \in w_m {}^t\mathcal{O}_m(l') w_m, \tag{1-5}$$

where $l' := l - e$. In the case of $n = 4$,

$$R^*_{\mathfrak{c}_\pi} = \begin{bmatrix} \mathfrak{o} & \mathfrak{p}^{-l'} & \mathfrak{o} & \mathfrak{p}^{-l'} \\ \mathfrak{p}^{l'} & \mathfrak{o} & \mathfrak{p}^{l'} & \mathfrak{o} \\ \mathfrak{p}^{l} & \mathfrak{p}^{e} & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^{l} & \mathfrak{p}^{e} & \mathfrak{p}^{l} & \mathfrak{o} \end{bmatrix}.$$

From the first property of J_π, it follows that

$$\pi(k)J_\pi^* = \chi \circ \det(d_k)^{-1} J_\pi^*, \quad k \in \mathbb{K}(\mathfrak{c}_\pi)^*.$$

For $i \in \mathbb{Z}$, let

$$B^e_{m,i} = \{b \in B_m \mid \det(b) \in \varpi^i \mathfrak{o}^\times, b_{11} \in \mathfrak{o}, b_{1j} \in \mathfrak{p}^{-l'}; j > 1, b_{hj} \in \mathfrak{o}; h > 1\}.$$

Then,

$$\varepsilon_\pi L(s, \pi^\vee) = \sum_{i=0}^{\infty} c_i^* q^{i(-s+1/2)}, \; c_i^* = \sum_{b \in B^e_{m,i}/B^e_{m,0}} J^*_\pi\left(\begin{bmatrix} b & \\ & 1_m \end{bmatrix}\right),$$

where c_i^* also has other expressions:

$$c_i^* = \int_{\{g \in \mathrm{GL}_m(F) \mid \det(g) = \varpi^i \mathfrak{o}^\times\}} J^*_\pi\left(\begin{bmatrix} g & \\ & 1_m \end{bmatrix}\right) \mathrm{d}g = \int_{\{g \in \mathrm{GL}_m(F) \mid \det(g) = \varpi^{i-\mathfrak{c}_\pi} \mathfrak{o}^\times\}} J_\pi\left(\begin{bmatrix} & 1_m \\ g^{-1} & \end{bmatrix}\right) \mathrm{d}g. \tag{1-6}$$

Observe that $J^*_\pi(1_n)$ equals ε_π, the root number of π. So, our J_π and $\mathbb{K}(\mathfrak{c}_\pi)$ satisfy the above conditions except for (i) in the case of $e > 0$, the one-dimensionality. This problem will be discussed in a future work.

We remark that for a cohomological cuspidal automorphic representation Π of GL_n over a number field, if one chooses local Shalika forms J_{Π_v} and $K(\mathfrak{c}_{\Pi_v})$ at bad places v, then the corresponding period in Theorem 6.7.1 of [6] becomes just $L(1/2, \Pi)/\omega^{\varepsilon_0}(\Pi_f)\omega(\Pi_\infty)$.

Another motivation of the above construction is a theta lift from G_4 to $\mathrm{GSp}_4(F)$. It is known that, in most cases, an irreducible, admissible representations τ of $\mathrm{GSp}_4(F)$ is a theta lift from a generic, irreducible smooth representation π with a Shalika model relevant to the central character of τ. In a forthcoming paper, we will construct a Whittaker "newform" for a generic, irreducible, admissible representation τ of $\mathrm{GSp}_4(F)$ using the above J_π, and show that the inequality $\mathfrak{c}_\pi \geq 2e$ holds for any generic π with a Shalika model. We think that this supports the validity of our Shalika form J_π and open compact subgroup $\mathbb{K}(\mathfrak{c}_\pi)$.

Outline. In Section 2, we introduce fundamental terminologies used throughout. In Section 3, we show the uniqueness of pre-Shalika models of V_l. In Section 4, we define an equivalence relation and ordering for the set consisting of elements in P_n at which $P_n(\mathfrak{o})$-invariant Shalika forms do not vanish necessarily (this set changes depending on e). According to this order, 1_n (resp. g_n) is minimal in the case of $e = 0$ (resp. $e > 0$). Using Hecke operators, we show that all $P_n(\mathfrak{o})$-invariant Shalika forms vanish on P_n, assuming that J^{new} vanishes at the minimal point. Theorem 1.1 follows from this result and the uniqueness of pre-Shalika models. The above J_π is constructed in Section 5.

Notation. Throughout, F denotes a p-adic field with ring of integers $\mathfrak{o}$. Let $\mathfrak{p} = \varpi\mathfrak{o}$ denote the prime ideal of $\mathfrak{o}$ and q the cardinality of the residue field. We fix an additive character $\psi : F \to \mathbb{C}^\times$ such that

$$\psi(\mathfrak{o}) = \{1\} \neq \psi(\mathfrak{p}^{-1}),$$

where $\mathbb{C}^\times$ indicates the multiplicative group of the complex number field $\mathbb{C}$. Let $\chi : F^\times \to \mathbb{C}^\times$ denote a continuous homomorphism, and e its (order of) conductor. Let $o(x)$ denote the p-adic order of $x \in F$, and $\nu(x)$ the p-adic norm: $\nu(x) = q^{-o(x)}$.

If G is a group, h, g are elements in G and H is a subgroup of G, then we write h^g and H^g for ghg^{-1} and $\{h^g \mid h \in H\}$, respectively. We use this notation for the groups:

$$\begin{aligned}
G_r &= \mathrm{GL}_r(F),\\
B_r &= \text{the standard Borel subgroup of } G_r,\\
D_r &= \text{the diagonal matrices in } G_r,\\
N_r &= \text{the unipotent matrices in } B_r,\\
P_r &= \{(p_{ij}) \in G_r \mid p_{rj} = 0 \text{ for } j < r,\ p_{rr} = 1\},\\
U_r &= \{(u_{ij}) \in N_r \mid u_{ij} = \delta_{ij} \text{ for } j < r\},\\
K_r &= \mathrm{GL}_r(\mathfrak{o}),\\
\mathfrak{S}_r &= \text{the symmetric group of degree } r.
\end{aligned}$$

For a permutation w of the set $\{1, \dots, r\}$, we also denote by w the permutation matrix in G_r defined by

$$w[g_{i,j}]w^{-1} = [g_{w(i),w(j)}],$$

and identify $\mathfrak{S}_r$ with a subgroup of G_r naturally. When a positive integer l is clear from the context, let $\acute{*} : G_r \to G_{r+l}$ denote the embedding

$$g \mapsto \acute{g} = \begin{bmatrix} g & \\ & 1_l \end{bmatrix},$$

and for a function f on a subset of G_{r+l} containing $\acute{G}_r$, let $\acute{f}$ denote the function on G_r by the pullback, where $1_l \in G_l$ indicates the identity. For a set X and a subset $A \subset X$, let $\mathrm{Ch}(x; A)$, $x \in X$, denote the characteristic (indicator) function of A.

2. Preparation

Let r be an integer larger than 1. Denote also by ψ the homomorphism

$$N = N_r \ni n \mapsto \prod_{1 \le i \le r-1} \psi(n_{i,i+1}) \in \mathbb{C}^\times. \tag{2-1}$$

Let π be an irreducible representation of $G = G_r$. It is known that $\mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G \psi)$, the $\mathbb{C}$-space of Whittaker models, is at most one-dimensional, where Ind indicates the induction functor. If π has a Whittaker model, then π is called *generic*. In this case, let $\mathbb{W}_\pi$ denote the image of π under it, and call $W \in \mathbb{W}_\pi$ *Whittaker forms of* π. For a positive integer m, let $\Gamma_r(m) = \Gamma(m) \subset G_r$ be the open compact subgroup

$$\{k \in K_r \mid k_{r,1}, \ldots, k_{r,r-1} \in \mathfrak{p}^m\}. \tag{2-2}$$

By the work of Jacquet, Piatetski-Shapiro, and Shalika [10], there exists a unique $W^{\mathrm{new}} \in \mathbb{W}_\pi$ such that

$$W^{\mathrm{new}}(1_r) = 1, \qquad \pi(k)W = \omega_\pi(k_{rr})W, \quad k \in \Gamma(\mathfrak{c}_\pi),$$

where $\mathfrak{c}_\pi$ and ω_π indicate the conductor and central character of π, respectively. We call W^{new} the *canonical Whittaker newform* of π.

If r is even, then *the Shalika subgroup* $S_r = S \subset G_r$ is defined to be the subgroup consisting of the matrices

$$s = \begin{bmatrix} a & \\ & a \end{bmatrix} \begin{bmatrix} 1_{r/2} & b \\ & 1_{r/2} \end{bmatrix}.$$

For χ, let $\chi_\psi : S \to \mathbb{C}^\times$ denote the homomorphism

$$s \mapsto \chi \circ \det(a)\psi(\mathrm{tr}(b)). \tag{2-3}$$

F. Chen and B. Sun [3] showed that $\mathrm{Hom}_G(\pi, \mathrm{Ind}_S^G(\chi_\psi))$, the $\mathbb{C}$-space of Shalika models, is at most one-dimensional. If π has a Shalika model, we denote by $\mathbb{S}_\pi(\chi)$ the image of π under it, and call $J \in \mathbb{S}_\pi(\chi)$ *Shalika forms of* π *relevant to* χ. By definition,

$$J(sg) = \chi_\psi(s)J(g). \tag{2-4}$$

By the above Whittaker newform theory, there exists a unique $J \in \mathbb{S}_\pi(\chi)$ up to a scalar such that

$$\pi(k)J = \omega_\pi(k_{rr})J, \quad k \in \Gamma(\mathfrak{c}_\pi),$$

which are called *the Shalika newforms of* π (*relevant to* χ).

Let

$$S_r^\circ = S_r \cap P_r.$$

For an irreducible smooth representation τ of P_r, we call the $\mathbb{C}$-space

$$\mathrm{Hom}_{P_r}(\tau, \mathrm{Ind}_{S_r^\circ}^{P_r}(\chi_\psi)) \simeq \mathrm{Hom}_{S_r^\circ}(\tau, \chi_\psi)$$

the pre-Shalika models of τ *relevant to* χ, and will show that it is at most one-dimensional in the next section. If τ has a pre-Shalika model, we denote by $\mathbb{I}_\tau(\chi)$ its image of τ, and call vectors in $\mathbb{I}_\tau(\chi)$ *pre-Shalika forms of* τ *relevant to* χ.

3. Pre-Shalika model

In this section, firstly we show that, for an arbitrary χ, the irreducible smooth representation

$$\psi_n := \text{c-Ind}_{N_n}^{P_n}(\psi) \tag{3-1}$$

has a unique pre-Shalika model up to a scalar, where c-Ind indicates the compact induction functor. By this uniqueness, the restriction to P_n of a Shalika form of a supercuspidal representation coincides with a pre-Shalika form of ψ_n, and vice versa (cf. Propositions 3.10 and 3.11). Secondly, we compute the support of a $P_n(\mathfrak{o})$-invariant pre-Shalika form, playing the essentially important role in this article, of ψ_n constructed by the pre-Shalika model, where

$$P_n(\mathfrak{o}) := \{p \in P_n \mid p_{ij} \in \mathfrak{o}\}.$$

For an l-group G (cf. [2]), we denote by $\mathscr{A}(G)$ (resp. $\mathscr{I}(G)$) the category of smooth (resp. smooth irreducible) $\mathbb{C}[G]$-modules. For $\tau \in \mathscr{A}(P_r) \cup \mathscr{A}(G_r)$, we call *the lift of τ to P_l* with $l > r$ the representation

$$\tau_l := \begin{cases} \Psi^{l-r}(\tau) & \text{if } \tau \in \mathscr{A}(P_r) \\ \Psi^{l-r-1} \circ \Upsilon(\tau) & \text{if } \tau \in \mathscr{A}(G_r) \end{cases}$$

where $\Psi : \mathscr{A}(P_s) \to \mathscr{A}(P_{s+1})$ and $\Upsilon : \mathscr{A}(G_s) \to \mathscr{A}(P_{s+1})$ are the functors defined by

$$\Psi : \tau \mapsto \text{c-Ind}_{\acute{P}_s U_{s+1}}^{P_{s+1}}(\tau \ltimes \psi), \quad \Upsilon : \tau \mapsto \tau \ltimes \mathbf{1}_{U_{s+1}}.$$

Here $\tau \ltimes \psi$ (resp. $\tau \ltimes \mathbf{1}_{U_{s+1}}$) indicates the representation sending $\acute{p}u \in \acute{P}_s U_{s+1}$ to $\psi(u)\tau(p)$ (resp. $\tau(p)$). Abbreviate $\text{Ind}_{S_n^\circ}^{P_n}(\chi_\psi)$ as $\mathbb{I}^n(\chi)$.

Proposition 3.1. *Let $\tau \in \mathscr{I}(G_r)$. For even $n > r$, we have*:

(i) *If r is odd, then τ_n has no pre-Shalika model.*

(ii) *If r is even, then*

$$\dim \text{Hom}_{P_n}(\tau_n, \mathbb{I}^n(\chi)) = \dim \text{Hom}_{G_r}\big(\tau, \text{Ind}_{S_r}^{G_r}(\nu^{(n-r)/2}\chi_\psi)\big).$$

This follows from induction on n and the following three lemmas.

Lemma 3.2. *If r is odd, then any lift τ_{r+1} of $\tau \in \mathscr{A}(G_r)$ does not have a pre-Shalika model relevant to any χ.*

Proof. Let $\lambda \in \text{Hom}_{P_{r+1}}(\tau_{r+1}, \mathbb{I}^{r+1}(\chi))$, and $f \in \tau_{r+1}$. By the definition of Υ and that of the pre-Shalika space,

$$\lambda(f) = \lambda(\tau_{r+1}(u)f) = \psi(u_{(r+1)/2,r+1})\lambda(f), \quad u \in U_{r+1}.$$

Hence $\lambda(f)$ vanishes, and the assertion follows. □

Lemma 3.3. *For $\tau \in \mathscr{A}(G_n)$ with n even, and an arbitrary χ,*

$$\dim \operatorname{Hom}_{P_{n+2}}(\tau_{n+2}, \mathbb{I}^{n+2}(\chi)) = \dim \operatorname{Hom}_{G_n}(\tau, \operatorname{Ind}_{S_n}^{G_n}(\nu\chi_\psi)).$$

Lemma 3.4. *For $\tau \in \mathscr{A}(P_n)$ with n even, and an arbitrary χ,*

$$\dim \operatorname{Hom}_{P_{n+2}}(\tau_{n+2}, \mathbb{I}^{n+2}(\chi)) = \dim \operatorname{Hom}_{P_n}(\tau, \mathbb{I}^{n}(\nu\chi)).$$

To show the last two lemmas, we use the distributional technique of [loc. cit.] for l-spaces X. For a $\mathbb{C}$-vector space V, let $\mathscr{S}(X, V)$ denote the space of all Schwartz functions on X with values in V. Linear functionals T on $\mathscr{S}(X, V)$ are called V-distributions on X. Additionally, if X is an l-group, and T is right (resp. left) invariant under an open compact subgroup, we say T is *locally constant on the right* (resp. *left*).

Proposition 3.5. *Let G be an l-group, and V a vector space over $\mathbb{C}$. Let T be a V-distribution on G.*

(i) *If T is right (resp. left) invariant, then T is the product of a right (resp. left) Haar measure on G and a linear functional on V.*

(ii) *If T is locally constant on the right (resp. left), then T is the product of a right (resp. left) Haar measure on G, and a V^*-valued continuous function on G, where V^* indicates the full-dual of V.*

Proof. Suppose that T is right invariant. Let $\mathrm{d}_r g$ denote a right Haar measure on G. Let $\{N_\alpha \mid \alpha \in \mathfrak{A}\}$ be a fundamental system of neighborhoods of the identity consisting of open compact subgroups of G. For $v \in V$, define $\varphi_v^\alpha \in \mathscr{S}(G, V)$ by

$$\varphi_v^\alpha(g) = \operatorname{Ch}(g; N_\alpha)v.$$

For each $\alpha \in \mathfrak{A}$, define $v_\alpha^* \in V^*$ by

$$\langle v_\alpha^*, v\rangle = \operatorname{vol}(N_\alpha)^{-1} T(\varphi_v^\alpha), \quad v \in V.$$

Therefore $v_\alpha^* = v_\beta^*$ if N_α contains N_β, since $T(\varphi_v^\alpha)$ equals $[N_\alpha : N_\beta]T(\varphi_v^\beta)$ by the right invariance property of T. This implies that v_α^* is independent of the choice of α. Since each $\varphi \in \mathscr{S}(G, V)$ is right invariant under some N_α, we may express φ as a finite sum of right translations of $\varphi_{v_i}^\alpha$ with some v_i's. By the right invariance property of T again, we have

$$\begin{aligned} T(\varphi) &= T\Big(\sum_i \varphi_{v_i}^\alpha\Big) = \sum_i \operatorname{vol}(N_\alpha)\langle v_\alpha^*, v_i\rangle \\ &= \sum_i \int_G \langle v_\alpha^*, \varphi_{v_i}^\alpha(g)\rangle \, \mathrm{d}_r g = \int_G \langle v_\alpha^*, \varphi(g)\rangle \, \mathrm{d}_r g. \end{aligned}$$

This is the first assertion. The second then follows from the proof of Proposition 1.28 of [loc. cit.]. □

Let G be an l-group. Let Q_0 and U be its closed subgroups such that $Q_0 \cap U = \{1\}$, and Q_0 normalizes U. Set

$$Q = Q_0 U,$$

which is a closed subgroup of G. Let $\xi : U \to \mathbb{C}^\times$ be a continuous homomorphism stabilized by Q_0. Let $H \subset G$ be a closed subgroup, and $\rho : H \to \mathbb{C}^\times$ a continuous homomorphism. Assume that

$$QH = \left\{ g \in G \mid \xi(h^g) = \rho(h) \text{ for all } h \in H \cap U^{g^{-1}} \right\}. \tag{3-2}$$

Observe the last set is left Q-, right H-invariant, closed, and an l-space in the induced topology. Additionally, assume that

$$Q \cap H = (Q_0 \cap H)(U \cap H). \tag{3-3}$$

Proposition 3.6. *With the above assumptions, for an arbitrary $\pi \in \mathscr{A}(Q_0)$, we have*

$$\dim \operatorname{Hom}_H(\text{c-Ind}_Q^G(\pi \ltimes \xi), \rho) = \dim \operatorname{Hom}_{Q_0 \cap H}(\pi, \Delta_{Q\cap H} \Delta_H^{-1} \rho),$$

where $\Delta_{Q\cap H}$, Δ_H indicate the modular characters on the groups.

Proof. Our proof is a modification of that of Proposition 1 of [12]. Abbreviate $\pi \ltimes \xi$ and $\Delta_{Q\cap H}\Delta_H^{-1}$ as π_ξ and Δ, respectively. We will construct a linear map from the $\mathbb{C}$-space $\operatorname{Hom}_H(\text{c-Ind}_Q^G(\pi_\xi), \rho)$ to $\operatorname{Hom}_{Q_0\cap H}(\pi, \Delta\rho)$. Denote the representation space of π by V_π, on which Q also acts by π_ξ. For $\phi \in \mathscr{S}(G, V_\pi)$, define $f_\phi \in \text{c-Ind}_Q^G(\pi_\xi)$ by

$$f_\phi(g) = \int_Q \pi_\xi(q^{-1})\phi(qg)\, \mathrm{d}_r q.$$

The linear map $\phi \mapsto f_\phi$ is bijective. For $\mu \in \operatorname{Hom}_H(\text{c-Ind}_Q^G(\pi_\xi), \rho)$, define a V_π-distribution T_μ on G by $T_\mu(\phi) = \mu(f_\phi)$. Observe that $T = T_\mu$ satisfies

$$\begin{aligned} T \circ R(h) &= \rho(h) T && (h \in H), \\ T \circ L(q) &= \Delta_Q(q) T \circ \pi_\xi(q^{-1}) && (q \in Q). \end{aligned} \tag{3-4}$$

Consider the double coset space $Q\backslash G/H$. By (3-2), if g does not lie in QH, then $\rho(h) \neq \xi(h^g)$ for some $h \in H$, and $T(g)$ is zero by (3-4). Therefore, the support of T is contained in QH, and we may regard T as a V_π-distribution on the closed subset QH (an l-space by (3-2)).

Let $\varphi \in \mathscr{S}(Q \times H, V_\pi)$. Define $\overline{\varphi} \in \mathscr{S}(QH, V_\pi)$ by

$$\overline{\varphi}(q^{-1}h) = \int_{Q\cap H} \Delta_Q(aq)\pi_\xi((aq)^{-1})\rho(ah)\varphi(aq, ah)\, \mathrm{d}_r a.$$

The linear map $\varphi \mapsto \overline{\varphi}$ is bijective. Therefore any V_π-distribution T' on $Q \times H$ is derived from a V_π-distribution T on QH satisfying (3-4) by setting $T'(\varphi) = T(\overline{\varphi})$.

For $q' \in Q$, $h' \in H$, we compute

$$\begin{aligned}\overline{R(q',h')\varphi}(q^{-1}h) &= \int_{Q\cap H} \Delta_Q(aq)\pi_\xi((aq)^{-1})\rho(ah)\varphi(aqq',ahh')\,\mathrm{d}_r a\\ &= \frac{\pi_\xi(q')}{\Delta_Q(q')\rho(h')}\int_{Q\cap H}\Delta_Q(aqq')\pi_\xi((aqq')^{-1})\rho(ahh')\varphi(aqq',ahh')\,\mathrm{d}_r a\\ &= \frac{\pi_\xi(q')}{\Delta_Q(q')\rho(h')}L(q')R(h')\overline{\varphi}(q^{-1}h).\end{aligned}$$

Therefore T' is right invariant by (3-4). By Proposition 3.5, there exists a linear functional μ' on V_π such that

$$T'(\varphi) = \int_H\int_Q \langle \mu', \varphi(q,h)\rangle\,\mathrm{d}_r q\,\mathrm{d}_r h. \tag{3-5}$$

For $b \in Q\cap H$, we compute

$$\begin{aligned}\overline{L(b,b)\varphi}(q^{-1}h) &= \int_{Q\cap H}\Delta_Q(aq)\pi_\xi((aq)^{-1})\rho(ah)\varphi(b^{-1}aq, b^{-1}ah)\,\mathrm{d}_r a\\ &= \Delta_{Q\cap H}(b)\int_{Q\cap H}\Delta_Q(baq)\pi_\xi((aq)^{-1}b^{-1})\rho(bah)\varphi(aq,ah)\,\mathrm{d}_r a\\ &= \Delta_{Q\cap H}(b)\Delta_Q(b)\rho(b)\overline{\pi_\xi(b^{-1})\varphi}(q^{-1}h).\end{aligned}$$

Thus

$$T'\circ L(b,b) = \Delta_{Q\cap H}(b)\Delta_Q(b)\rho(b)T'\circ\pi_\xi(b^{-1}).$$

Now from (3-5),

$$\Delta_H(b)\langle\mu',\varphi(q,h)\rangle = \Delta_{Q\cap H}(b)\rho(b)\langle\mu',\pi_\xi(b^{-1})\varphi(q,h)\rangle.$$

This means that μ' lies in $\mathrm{Hom}_{Q\cap H}(\pi_\xi, \Delta\rho)$. By restricting to Q_0, we obtain a $\mu'' \in \mathrm{Hom}_{Q_0\cap H}(\pi, \Delta\rho)$. The linear map $\mu' \mapsto \mu''$ is bijective by (3-3). We have constructed the desired map $\mu \mapsto \mu''$. One can reverse the above steps, and easily find that this map is bijective. □

Remark 3.7. The spaces $\mathrm{Hom}_H(\text{c-Ind}_Q^G(\pi_\xi), \rho)$ and $\mathrm{Hom}_{Q\cap H}(\pi_\xi, \Delta\rho)$ are both zero if $\Delta|_{U\cap H} \not\equiv \mathbf{1}$.

Let $n = 2m$ be an even integer. By Frobenius duality,

$$\mathrm{Hom}_{P_{n+2}}(\tau_{n+2}, \mathbb{I}^{n+2}(\chi)) \simeq \mathrm{Hom}_{S^\circ_{n+2}}(\tau_{n+2}, \chi_\psi),$$
$$\mathrm{Hom}_{P_n}(\tau, \mathbb{I}^n(\nu\chi)) \simeq \mathrm{Hom}_{S^\circ_n}(\tau, \nu\chi_\psi).$$

We will prove Lemma 3.4 by showing the spaces in the right hands are same-dimensional. By the previous proposition, it suffices to check (3-2), (3-3) in the situation considered. Set $G = P_{n+2}$. Let $h_{n+1} \in \mathfrak{S}_{n+1}$ be the permutation

(3-6) $$\begin{pmatrix} 1 & \cdots & m & m+1 & m+2 & m+3 & \cdots & n+1 \\ 1 & \cdots & m & 2m+1 & m+1 & m+2 & \cdots & n \end{pmatrix}.$$

Abbreviate

(3-7) $$\eta = \acute{h}_{n+1} \in P_{n+2}.$$

Set $H = \eta S^{\circ}_{n+2} \eta^{-1}$. Let $\rho : H \to \mathbb{C}^{\times}$ be the homomorphism

$$h \mapsto \chi_{\psi}(\eta^{-1} h \eta).$$

If we write a typical $s \in S^{\circ}_{n+2}$ as

$$\begin{bmatrix} A & {}^t\alpha & X & {}^t\beta \\ & 1 & \gamma & y \\ & & A & {}^t\alpha \\ & & & 1 \end{bmatrix},$$

where A, X are $m \times m$ matrices, α, β, γ are m-dimensional row vectors, and y is an element of F, then

$$s^{\eta} = \begin{bmatrix} A & X & {}^t\alpha & {}^t\beta \\ & A & & {}^t\alpha \\ & \gamma & 1 & y \\ & & & 1 \end{bmatrix}, \qquad \rho(s^{\eta}) = \chi \circ \det(A) \psi(y + \mathrm{tr}(A^{-1}X - A^{-1\,t}\alpha\gamma)).$$

Set $Q_0 = \acute{P}_n \subset G_{n+2}$, and $U = U_{n+1} U_{n+2}$. Set $\xi = \psi|_U$. Observe that

(a) $\tau_{n+2} = \text{c-Ind}_Q^G(\tau \ltimes \xi)$,

(b) $\rho(\acute{t}) = \chi_{\psi}(t)$ for $t \in S_n$,

(c) $\acute{S}^{\circ}_n = Q_0 \cap H$,

(d) $\Delta_{Q \cap H} \Delta_H^{-1}(\acute{t}) = |\det(A)|$ for $t = \begin{bmatrix} A & * \\ & A \end{bmatrix} \in S^{\circ}_n$.

It is easy to check (3-3). For (3-2), it suffices to see that any matrix p in the RHS of (3-2) satisfies:

$$p_{n,1} = \cdots = p_{n,m} = p_{n+1,1} = \cdots = p_{n+1,m} = 0.$$

Let $E_{i,j}$ denote the i-th row and j-th column matrix unit (having all entries 0 apart from 1 in position (i, j)). If $p_{n+1,i} \neq 0$ for $i \in \{1, \ldots, m\}$, then for some $h = 1_{n+2} + x E_{i,n+2} \in H$,

$$\rho(h) = 1 \neq \psi(x p_{n+1,i}) = \xi(h^p).$$

Hence, $p_{n+1,1} = \cdots = p_{n+1,m} = 0$. Now we may assume that p lies in $\acute{P}_{n+1} U_{n+2}$, since the RHS of (3-2) is right H-invariant. If $p_{n,i} \neq 0$ for $i \in \{1, \ldots, m\}$, then for some $h' = 1_{n+2} + y E_{i,n+1} + y E_{i+m,n+2} \in H$, we have $\rho(h') = 1 \neq \psi(y p_{n,i}) = \xi(h'^p)$. Hence, $p_{n,1} = \cdots = p_{n,m} = 0$. Now (3-2) is checked, and the proof of the lemma is completed.

For Lemma 3.3, reset $Q_0 = \acute{G}_n$. Then (a) and (b) above hold. If one replaces S_n° with S_n at (c) and (d), then they hold. It is also easy to check (3-3), and the same argument for (3-2) works. This means that $\mathrm{Hom}_{S_{n+2}^\circ}(\tau_{n+2}, \chi_\psi)$ and $\mathrm{Hom}_{S_n}(\tau, \nu\chi_\psi)$ have the same dimension for $\tau \in \mathscr{A}(G_n)$.

Similar to Proposition 3.1, the following holds.

Proposition 3.8. *For an arbitrary χ, ψ_n has a unique pre-Shalika model relevant to χ, up to a scalar.*

Proof. Since ψ_n equals $\Psi^{n-2}(\psi_2)$, the assertion is reduced to the evaluation of the dimension of $\mathrm{Hom}_{P_2}(\psi_2, \mathbb{I}^2(\chi))$ by Lemma 3.4. Consider the space of corresponding $\mathbb{C}$-distributions on P_2, and the corresponding double coset space $N_2\backslash P_2/N_2$. As a complete system of its representatives, using $\{\acute{t} \in P_2 \mid t \in F^\times\}$, we find that all supports of the distributions are contained in N_2. □

Since any irreducible smooth representation of P_n is equivalent to ψ_n, or a lift from $\mathscr{I}(G_r)$, $r < n$ (cf. [2]), we obtain:

Theorem 3.9. *An irreducible smooth representation of P_n has no or a unique pre-Shalika model relevant to χ, up to a scalar.*

From now, let $(\pi, V) \in \mathscr{I}(G_n)$ be generic. There exists a Jordan–Hölder sequence of smooth P_n-modules $V_l \subset \cdots \subset V_0 = V$ with the following properties (cf. [2]):

- V_l is equivalent to ψ_n.
- Each V_i/V_{i+1} is equivalent to some lift from $\mathscr{I}(G_r)$, $r < n$.
- π is supercuspidal if and only if $V = V_l$.

Proposition 3.10. *If (1-1) is assumed, then, for any pre-Shalika form of ψ_n relevant to χ, there exists a Shalika form in V_l whose restriction to P_n coincides with it.*

Proof. By the assumption, for a Shalika form in V_l, its restriction to P_n is nontrivial, and can be regarded as a pre-Shalika form of ψ_n. The assertion follows from the irreducibility of $V_l \simeq \psi_n \simeq \mathbb{I}_{\psi_n}(\chi)$. □

Proposition 3.11. *Assume that each V_i/V_{i+1} has no pre-Shalika model relevant to χ. Then, (1-1) is satisfied. Further, for a Shalika form J, the following statements are equivalent.*

(i) *J does not vanish on P_n.*

(ii) *J lies in V_l.*

In this case, $J|_{P_n}$ coincides with a pre-Shalika form of ψ_n.

Proof. Let V_i^0 be the P_n-submodule

$$\{J \in V_i \mid J|_{P_n} \equiv 0\} \subset V_i.$$

Take the minimal V_r containing a Shalika form that does not vanish at the identity. By definition, if $r \neq l$, then we have $V_{r+1} = V^0_{r+1} \subset V^0_r$ and V_r / V_{r+1} has a pre-Shalika model, conflicting with the assumption. Hence the first assertion follows. This argument also implies the equivalence of (i) and (ii). The last assertion follows from the proof of Proposition 3.10. □

The following is an easy sufficient condition for the lack of Shalika models.

Lemma 3.12. *Let $n = 2m$ be an even integer, and $\pi \in \mathscr{I}(G_n)$ be unramified. If χ is ramified, then π has no Shalika model relevant to χ.*

Proof. Assume that π has a Shalika model relevant to a ramified χ. Then there exists a nontrivial K_n-invariant $J \in \mathbb{S}_\pi(\chi)$. Let $\acute{J}$ denote the restriction to G_m. By the Iwasawa decomposition of G_n, $J|_{B_n} \not\equiv 0$. By (2-4), $\acute{J}|_{B_m} \not\equiv 0$. By (2-4) and the Cartan decomposition of G_m, there exists a $d \in D_m$ such that $\acute{J}(d) \neq 0$. By (2-4) and the K_n-invariance property of J,

$$\chi(t)\acute{J}(d) = \acute{J}(\mathrm{diag}(t, \overbrace{1, \dots, 1}^{m-1})\, d) = \acute{J}(d\, \mathrm{diag}(t, \overbrace{1, \dots, 1}^{m-1})) = \acute{J}(d), \quad t \in \mathfrak{o}^\times.$$

Since χ is ramified, it follows that $\acute{J}(d) = 0$, a contradiction. □

We are also interested in the question whether the lift of $\tau \in \mathscr{I}(G_r)$ to P_n, $n > r$ has a $P_n(\mathfrak{o})$-invariant vector, or not.

Lemma 3.13. *If $\tau \in \mathscr{I}(G_r)$ is ramified, then the lift τ_n, $n > r$ to P_n has no nontrivial $P_n(\mathfrak{o})$-invariant vector.*

Proof. Let $f \in \tau_n$ be $P_n(\mathfrak{o})$-invariant. Define the subgroups T, $N^r_n \subset P_n$ by

$$T = \{\mathrm{diag}(t_1, \dots, t_{n-1}, 1) \mid t_1 = \dots = t_r = 1\},$$
$$N^r_n = \{u \in N_n \mid u_{ij} = 0, i < j \le r\}.$$

By the inclusion $P_n \subset N^r_n \acute{G}_r T P_n(\mathfrak{o})$ and the definition of τ_n, it suffices to show that f vanishes on $\acute{G}_r T$. The restriction to G_r of the right translation of f by $t \in T$ is K_r-invariant. However, it is identically zero since τ is ramified. □

Combining these lemmas and Proposition 3.1, we have:

Proposition 3.14. *Any lift of $\tau \in \mathscr{I}(G_r)$ has no $P_n(\mathfrak{o})$-invariant pre-Shalika form relevant to χ if one of τ, χ is ramified and another is unramified.*

4. A vanishing theorem for Shalika forms

Let $n = 2m$ be an even integer. Let $\pi \in \mathscr{I}(G_n)$. For a nonnegative integer M, let $\Gamma(M) \subset G_n$ be the open compact subgroup defined in Section 1, and define

$$\Gamma^\circ = \Gamma^\circ(M) = \{k \in \Gamma(M) \mid k_{n,n} - 1 \in \mathfrak{p}^M\}.$$

For $f = (f_1, \dots, f_{n-1}) \in \mathbb{Z}^{n-1}$, let

$$\varpi^f = \operatorname{diag}(\varpi^{f_1}, \dots, \varpi^{f_{n-1}}),$$

and let T_f denote the Hecke operator that acts on right Γ°-invariant vectors ξ in π by

$$T_f \xi = \int_{G_n} \operatorname{Ch}(k; \Gamma^\circ \acute{\varpi}^f \Gamma^\circ) \pi(k) \xi \, \mathrm{d}k,$$

where $\mathrm{d}k$ indicates the Haar measure on G_n normalized so that $\operatorname{vol}(\Gamma^\circ) = 1$. In this section, we show:

Theorem 4.1. *Let χ be a character of F with conductor e. Let $\pi \in \mathscr{I}(G_n)$ be generic with a Shalika model relevant to χ. Let $J \in \mathbb{S}_\pi(\chi)$ be $\Gamma^\circ(M)$-invariant for an M. If J is a* (*simultaneous*) *eigenvector for all T_f, $f \in \mathbb{Z}^{n-1}$, and vanishes at 1_n* (*resp. g_n*) *in the case of $e = 0$* (*resp. $e > 0$*), *then J vanishes on $S_n P_n \Gamma(M)$.*

To show the vanishing of J on $S_n P_n \Gamma(M)$, it suffices to show it on $\acute{G}_m$. We use the notation

$$u(x) = u_x = \begin{bmatrix} 1_r & {}^t x \\ & 1 \end{bmatrix} \in U_{r+1}, \quad x \in F^r \tag{4-1}$$

and introduce reduced elements of G_m. Let $\beta \in F^{m-1}$ and $d = \operatorname{diag}(d_1, \dots, d_m) \in D_m$. Set

$$c = o(d_m), \qquad a_i = o(d_i),\ b_i = -o(\beta_i), \quad i \in \{1, \dots, m-1\}.$$

If $a_1 \le \dots \le a_{m-1}$, then we say d is *aligned*, and set $j_1, \dots, j_r$ by

$$a_{j_1} = a_1 = \dots = a_{j_2 - 1} < a_{j_2} = \dots = a_{j_r - 1} < a_{j_r} = \dots = a_{m-1}. \tag{4-2}$$

Understand $j_{r-1} = 0$ if $r = 1$. If $j_s \le i < j_{s+1}$, $s < r$ (resp. $j_r \le i \le m-1$), then let $s(i)$ denote s (resp. r). Let $S(b) = \{s \mid \beta_{j_s} \ne 0\}$. We say du_β is *reduced* if, in addition, the following conditions are satisfied:

(a) $\beta_i = 0$ for all $i \notin \{j_1, \dots, j_r\}$.

(b) $\beta_{j_s} = 0$ for all $s \in \{1, \dots, r\}$ such that $\beta_{j_s} \in \mathfrak{o}$.

(c) For $s, t \in S(b)$, if $s < t$, then $b_{j_s} < b_{j_t}$.

(d) For $s, t \in S(b)$, if $s < t$, then $a_{j_s} - b_{j_s} < a_{j_t} - b_{j_t}$.

For $g_1, g_2 \in G_m$, if $\acute{K}^1_{m-1} g_1 K_m = \acute{K}^1_{m-1} g_2 K_m$ (resp. $\acute{K}_{m-1} g_1 K_m = \acute{K}_{m-1} g_2 K_m$), where $K^1_{m-1} = \mathrm{SL}_{m-1}(\mathfrak{o})$, then we say g_1 is equivalent (resp. quasi-equivalent) to g_2, and denote

$$g_1 \approx g_2 \ \text{ (resp. } g_1 \sim g_2).$$

A reason why we define the above relations for elements in G_m is the following.

Lemma 4.2. *Any $(S_n^\circ, P_n(\mathfrak{o}))$-double coset in P_n contains the embedding of an element of the form $du_\beta \in G_m$.*

Proof. Since P_n is isomorphic to $G_{n-1} \ltimes U_n$, any left coset of $P_n(\mathfrak{o})$ intersects with B_n by the Iwasawa decomposition of G_{n-1}. Therefore, any $(S_n^\circ, P_n(\mathfrak{o}))$-double coset intersects with $\acute{B}_m$. The assertion follows from the Cartan decomposition of G_m. □

For a while, in preparation for the proof of the theorem, we will discuss the above concepts. In case of $m > 2$, define $w'_{m-1} \in K^1_{m-1}$ by

$$w'_{m-1} = \begin{cases} w_{m-1} & \text{if } m \in 1 + 4\mathbb{Z}, \\ \operatorname{diag}(-1, \overbrace{1, \dots, 1}^{m-2}) w_{m-1} & \text{otherwise,} \end{cases}$$

where w_{m-1} indicates the standard antidiagonal Weyl element in G_{m-1}, and for $g \in G_m$, denote

$$g^\iota = g^{\acute{w}'_{m-1}}.$$

Observe that $g \sim g^\iota$, and that, if $g = du_\beta$, then

$$g^\iota = \operatorname{diag}(d_{m-1}, d_{m-2}, \dots, d_1, d_m) \times \begin{cases} u(\beta_{m-1}, \beta_{m-2}, \dots, \beta_1) & \text{if } m \in 1 + 4\mathbb{Z}, \\ u(-\beta_{m-1}, \beta_{m-2}, \dots, \beta_1) & \text{otherwise.} \end{cases}$$

Proposition 4.3. *We keep the notation from above.*

(i) *Let $d \in D_m$ be aligned. For an arbitrary $\beta \in F^{m-1}$, there exist $d' \in D_m$, $\gamma \in F^{m-1}$ with the following properties.*
  - *$o(d'_i) = o(d_i)$ for all i.*
  - *$d'u_\gamma$ is equivalent to du_β and reduced.*

(ii) *Any equivalence class of G_m contains a reduced element.*

Proof. (i) There exists a Weyl element $w \in K^1_{m-1}$ such that

$$o((\acute{w} d \acute{w}^{-1})_i) = o(d_i), \quad o((\beta^t w)_{j_{s(i)}}) \le o((\beta^t w)_i), \quad i \in \{1, \dots, m-1\}.$$

Set $d = \acute{w} d \acute{w}^{-1}$ and $\beta' = \beta^t w$. Then,

$$du_\beta \approx \acute{w} du_\beta \acute{w}^{-1} = \acute{w} d \acute{w}^{-1} \acute{w} u_\beta \acute{w}^{-1} = d' u_{\beta'}.$$

Preserving the equivalence class, we will translate β' step by step to satisfy conditions (a)–(d), and attain γ. Assume that there exists an $i \notin \{j_1, \dots, j_r\}$ such that $\beta'_i \neq 0$. Since $o(\beta'_{j_{s(i)}}) \le o(\beta'_i)$, we can choose a $v \in {}^t N_{m-1}(\mathfrak{o})$ such that

$$(\beta'^t v)_k = \begin{cases} 0 & \text{if } k = i, \\ \beta'_k & \text{otherwise.} \end{cases}$$

By the lemma below, $d'u(\beta'^t v) \approx d'u_{\beta'}$. Iterating such translations, we attain a β'' satisfying $d'u_{\beta''} \approx d'u_{\beta'}$ and condition (a). Similarly, if there exist $s, t \in S(b'')$ (b'' is defined by $b''_i = -o(\beta''_i)$) such that $s < t$ and $b''_{j_s} \geq b''_{j_t}$, then we can choose a $v' \in {}^t N_{m-1}(\mathfrak{o})$ such that

$$(\beta''^t v')_k = \begin{cases} 0 & \text{if } k = j_t, \\ \beta'_k & \text{otherwise.} \end{cases}$$

Iterating such translations, we attain β''' satisfying $d'u_{\beta'''} \approx d'u_{\beta''}$ and also condition (c). Set $\beta'''' \in F^{m-1}$ by

$$\beta''''_i = \begin{cases} 0 & \text{if } \beta'''_i \in \mathfrak{o}, \\ \beta''_i & \text{otherwise.} \end{cases}$$

Then, $d'u_{\beta''''} = d'u_{\beta'''}u(\beta'''' - \beta''') \approx d'u_{\beta'''}$, and β'''' satisfies also condition (b). Assume that some β''''_{j_s} does not lie in $\bigcup_{k=s+1}^{r} (d'_{j_k}\beta''''_{j_k}/d'_{j_s})\mathfrak{o}$. Let $d'^\circ = \operatorname{diag}(d'_1, \ldots, d'_{m-1})$. Then, we can choose a $v'' \in (d'^\circ)^{-1} N_{m-1}(\mathfrak{o}) d'^\circ (\subset N_{m-1(\mathfrak{o})})$ such that

$$(\beta''''^t v'')_k = \begin{cases} 0 & \text{if } k = j_s, \\ \beta''''_k & \text{otherwise.} \end{cases}$$

Observe that

$$d'u(\beta''''^t v'') = d'\acute{v}''u(\beta'''')\acute{v}''^{-1} = d'\acute{v}''d'^{-1}d'u(\beta'''')\acute{v}''^{-1} \approx d'u(\beta'''').$$

Iterating such translations, we attain the desired γ.

(ii) By the proof of Lemma 4.2, any equivalence class contains an element of the form du_β. We may choose a Weyl element w in K^1_{m-1} so that $\acute{w}d\acute{w}^{-1}$ is aligned. Now the assertion follows from (i). □

Lemma 4.4. *Let $d \in D_m$ be aligned. For $\beta \in F^{m-1}$, $v \in N_{m-1}(\mathfrak{o})$, it holds that*

$$du(\beta v) \approx du(\beta), \quad d^t u(\beta^t v) \approx d^t u(\beta).$$

Proof. Since d is aligned, $({}^t\acute{v})^d \in \acute{K}^1_{m-1}$, and $du(\beta v) = d^t\acute{v}u_\beta{}^t\acute{v}^{-1} = ({}^t\acute{v})^d du_\beta \acute{v}^{-1} \approx du_\beta$. Another equivalence is proved similarly. □

Let

$$\widetilde{P}_m = \{g \in G_m \mid g_{mi} = 0, i \in \{1, \ldots, m-1\}\}.$$

For

$$p = \begin{bmatrix} h & * \\ & * \end{bmatrix} \in \widetilde{P}_m, \quad h \in G_{m-1},$$

we call $o(\det(h))$ the *weight* of p, and denote it by $\mathrm{wt}(p)$. Observe that

$$\mathrm{wt}(p_1 p_2) = \mathrm{wt}(p_1) + \mathrm{wt}(p_2), \quad p_1, p_2 \in \widetilde{P}_m. \tag{4-3}$$

Lemma 4.5. *Let $p, p' \in \widetilde{P}_m$. If $p \sim p'$, then* $\mathrm{wt}(p) = \mathrm{wt}(p')$.

Proof. Assume $p' = \acute{h}pk$ for some $h \in K^1_{m-1}$, $k \in K_m$. Write

$$p = \begin{bmatrix} g & {}^t\beta \\ & t \end{bmatrix}, \quad p' = \begin{bmatrix} g' & {}^t\beta' \\ & t' \end{bmatrix}, \quad k = \begin{bmatrix} x & {}^t y \\ z & w \end{bmatrix} \in K_m$$

by $\beta, \beta', y, z \in \mathfrak{o}^{m-1}$, $w, t, t' \in \mathfrak{o}$ and $m-1$ by $m-1$ matrices g, g', x with entries in $\mathfrak{o}$. Then we have

$$\begin{bmatrix} g' & {}^t\beta' \\ & t' \end{bmatrix} = \begin{bmatrix} h(gx + {}^t\beta z) & h(g^t y + {}^t\beta w) \\ tz & tw \end{bmatrix}.$$

It follows that $z = 0$, and $x \in K_{m-1}$, and

$$\mathrm{wt}(p') = o(\det(g')) = o(\det(h(gx + {}^t\beta z))) = o(\det(gx)) = \mathrm{wt}(p).$$

This completes the proof. □

For $(a, b, c) \in \mathbb{Z}^{m-1} \times \mathbb{Z}^{m-1} \times \mathbb{Z}$, let

$$p(a, b, c) = \varpi^{(a,c)} u(\varpi^{-b_1}, \ldots, \varpi^{-b_{m-1}}) \in \widetilde{P}_m.$$

Proposition 4.6. *We keep the notation from above.*

(i) *Any quasi-equivalence class of G_m contains a $p(a, b, c)$.*

(ii) *If both $p(a, b, c)$ and $p(a', b', c')$ are reduced, then*

$$p(a, b, c) \sim p(a', b', c') \;(\Longleftrightarrow)\; (a, b, c) = (a', b', c').$$

Proof. (i) By Proposition 4.3, any equivalence class contains a reduced du_β. Set $c = o(d_m)$, and $a_i = o(d_i)$, $b_i = -o(\beta_i)$ for $i < m$. Then, $p(a, b, c) \sim du_\beta$.

(ii) We only show that $(a, b, c) = (a', b', c')$ if there exist $h \in K_{m-1}$ and $k \in K_m$ such that $\acute{h}p(a, b, c)k = p(a', b', c')$. From Lemma 4.5, $c = c'$ is derived. Set $\beta = (\varpi^{-b_1}, \ldots, \varpi^{-b_m})$, $\beta' = (\varpi^{-b'_1}, \ldots \varpi^{-b'_m})$. Write

$$k = \begin{bmatrix} x & {}^t y \\ z & w \end{bmatrix}$$

by $y, z \in \mathfrak{o}^{m-1}$, $w \in \mathfrak{o}$ and $m-1$ by $m-1$ matrix x with entries in $\mathfrak{o}$. Then,

$$\begin{bmatrix} \varpi^{a'} & \varpi^{a'} {}^t\beta' \\ & \varpi^c \end{bmatrix} = \begin{bmatrix} h\varpi^a(x + {}^t\beta z) & h\varpi^a({}^t y + {}^t\beta w) \\ \varpi^c z & \varpi^c w \end{bmatrix}.$$

Therefore, $w = 1$ and $z = 0$. Since k lies in K_m, x lies in K_{m-1}. Therefore, $\varpi^{a'} = h\varpi^a x$, and $a = a'$ follows from the Cartan decomposition of G_{m-1}. Now, set $j_1, \ldots, j_r$ by (4-2). Since both $p(a, b, c)$ and $p(a, b', c)$ are reduced,

$$\beta_i = \beta'_i = 0, \quad i \notin \{j_1, \ldots, j_r\}. \tag{4-4}$$

The remained task is to show $\beta_{j_s}=\beta'_{j_s}$, $s\in\{1,\dots,r\}$. By the symmetry argument, it suffices to show $\beta_{j_s}=\beta'_{j_s}$ for each $s\in S(b')$. Since $x=\varpi^{-a}(h^{-1})\varpi^{a}$ lies in $K_{m-1}\cap\varpi^{-a}K_{m-1}\varpi^{a}$, we have

$$x_{ik}\in\begin{cases}\mathfrak{p}^{a_{j_{s(k)}}-a_{j_{s(i)}}} & \text{if } i<k,\\ \mathfrak{o} & \text{otherwise.}\end{cases}\tag{4-5}$$

Therefore, we have

$$(x_{ik})_{j_r\le i,k<m}\in K_{m-j_r},\qquad (x_{ik})_{j_s\le i,k<j_{s+1}}\in K_{j_{s+1}-j_s}\qquad (s<r).\tag{4-6}$$

From the identity $\varpi^{a\,t}\beta'=h\varpi^{a}({}^t y+{}^t\beta)$, we obtain $\beta-\beta'^t x\in\mathfrak{o}^{m-1}$, that is,

$$\beta_i\in\mathfrak{o}+\sum_{k=1}^{m-1}x_{ik}\beta'_k=\mathfrak{o}+\sum_{t\in S(b')}x_{ij_t}\beta'_{j_t}.$$

Assume that $x_{j_sj_s}\notin\mathfrak{o}^{\times}$. Then, by (4-6), there exists an $l\in\{j_s+1,\dots,j_{s+1}-1\}$ such that $x_{lj_s}\in\mathfrak{o}^{\times}$. By (4-5), β_l lies in

$$\begin{aligned}\mathfrak{o}+\sum_{t\in S(b')}x_{lj_t}\beta'_{j_t}&=\mathfrak{o}+x_{lj_s}\beta'_{lj_s}+\sum_{s>t\in S(b')}x_{lj_t}\beta'_{j_t}+\sum_{s<t\in S(b')}x_{lj_t}\beta'_{j_t}\\&\in\mathfrak{o}+\varpi^{-b'_{j_s}}\mathfrak{o}^{\times}+\mathfrak{p}^{1-b'_{j_s}}+\sum_{s<t\in S(b')}\mathfrak{p}^{a_{j_t}-a_{j_s}-b'_{j_t}}\\&=\varpi^{-b'_{j_s}}\mathfrak{o}^{\times},\end{aligned}$$

conflicting with (4-4). Hence, $x_{j_sj_s}\in\mathfrak{o}^{\times}$, and

$$\beta_{j_s}\in\mathfrak{o}+\sum_{t\in S(b')}x_{j_sj_t}\beta'_{j_t}=\varpi^{-b'_{j_s}}\mathfrak{o}^{\times}.$$

This completes the proof. □

Now, let J be a $P_n(\mathfrak{o})$-invariant Shalika form relevant to a character χ. In order to know the value $\acute{J}(g)$, $g\in G_m$, it suffices to know that at $p(a,b,c)$ quasi-equivalent to g by the identity

$$\acute{J}(\acute{h}gk)=\chi(\det(h))\acute{J}(g),\qquad h\in K_{m-1},k\in K_m.\tag{4-7}$$

A necessary condition for $\acute{J}(p(a,b,c))\neq 0$ is as follows.

Lemma 4.7. *With notation as above, assume that* $\acute{J}(p(a,b,c))\neq 0$.

(i) $c,a_i,a_i-b_i\ge 0$ *for all* i.

(ii) *If* $p(a,b,c)$ *is reduced, then for* $S(b)=\{s_1<\cdots<s_l\}$,

$$0\le a_{j_{s_1}}-b_{j_{s_1}}\le\cdots\le a_{j_{s_l}}-b_{j_{s_l}}.$$

Proof. This follows from the next (obvious) lemma and the identity

$$(4\text{-}8)\qquad J\left(\begin{bmatrix} p & \\ & 1_m\end{bmatrix}\begin{bmatrix} 1_m & X \\ & 1_m\end{bmatrix}\right) = \psi\left(\varpi^c x_{mm} + \sum_{i=1}^{m-1} \varpi^{a_i}(x_{ii} + \varpi^{-b_i} x_{mi})\right)\acute{J}(p)$$

where $p = p(a, b, c)$. □

Lemma 4.8. *Let Ω be a field. Let G be a group, and H, K be its subgroups. Let ξ and ω be homomorphisms into $\Omega^\times$ of H and K, respectively. Let J be a Ω-valued function on G such that*

$$J(hgk) = \xi(h)\omega(k)J(g), \quad h \in H,\ g \in G,\ k \in K.$$

Then $J(g_0)$ vanishes at $g_0 \in G$ if there exists an $h \in H$ such that $h^{g_0} \in K$ and $\xi(h) \neq \omega(g_0^{-1}hg_0)$. □

In the case where χ is ramified, the following stronger statement can be made. Let $e > 0$ be the conductor of χ. Let $p_e(a, b, c) = p(a^+, b^+, c^+)$ with

$$\begin{aligned} a^+ &= (a_1 + e, a_2 + 3e, \ldots, a_{m-1} + (2m-3)e), \\ b^+ &= (b_1 + e, \ldots, b_{m-1} + (m-1)e), \\ c^+ &= c + (m-1)e. \end{aligned}$$

Lemma 4.9. *With notation as above, assume that $\acute{J}(p_e(a, b, c)) \neq 0$. Then*:

(i) $0 \le b_1 \le \cdots \le b_{m-1}$.

(ii) $0 \le a_1 - b_1 \le \cdots \le a_{m-1} - b_{m-1} \le c$ (*therefore*, $0 \le a_1 \le \cdots \le a_{m-1}$).

In particular, $p_e(a, b, c) = p(a^+, b^+, c^+)$ is reduced and $j_1 = 1, \ldots, j_{m-1} = m-1$, where $j_1, \ldots, j_{m-1}$ are defined for a^+.

Proof. Let $d = \varpi^{(a,c)}$, and $\beta = (\varpi^{-b_1^+}, \ldots, \varpi^{-b_{m-1}^+})$. For $s = \mathrm{diag}(s_1, \ldots, s_{m-1}, 1)$ with $s_j \in \mathfrak{o}^\times$, it holds that

$$\acute{J}(du(s_1\beta_1, \ldots, s_{m-1}\beta_{m-1})) = \acute{J}(dsu_\beta s^{-1}) = \chi(\det(s))\acute{J}(du_\beta).$$

By Lemma 4.8, $b_i^+ \ge e$ for all i. For $x \in \mathfrak{o}$ and $i \le m-2$, there exists a $v \in {}^tN_{m-1}(\mathfrak{o})$ such that

$$(\beta^t v)_j = \begin{cases} \beta_{i+1}(1 + x\beta_i/\beta_{i+1}) & \text{if } j = i+1, \\ \beta_j & \text{otherwise.} \end{cases}$$

By (4-7) and Lemma 4.4, $\acute{J}(du(\beta)) = \acute{J}(du(\beta^t v))$. Therefore, if $b_i > b_{i+1}$ for some $i \le m-2$, then $o(\beta_i/\beta_{i+1})$ is less than e, and $\acute{J}(du_\beta)$ vanishes by Lemma 4.8. Hence (i) follows. If $a_1 < b_1$, then $\acute{J}(du_\beta) = 0$ by (4-8) and Lemma 4.8. Hence, $a_1 \ge b_1$. Set $\gamma = (d_1\beta_1/d_m, \ldots, d_{m-1}\beta_{m-1}/d_m)$. Similar to (i), noting that

$$\acute{J}(du_\beta) = \chi(\det(d))J\left(\begin{bmatrix} 1_m & \\ & d^{-1}u_\gamma\end{bmatrix}\right),$$

we can show that $\acute{J}(du_\beta) = 0$ assuming $o(\gamma_{m-1}) > -e$ (which is equivalent to $a_{m-1} - b_{m-1} - c > 0$) or $o(\gamma_{i+1}/\gamma_i) < e$ (which is equivalent to $a_{i+1} - b_{i+1} < a_i - b_i$), for $i \le m-2$. Hence, (ii) follows. □

Now we begin the proof of the theorem dividing the cases whether e, the conductor of χ, is zero or not. Let J be a Shalika form as in the theorem. In the case $e = 0$, by (4-7) and Proposition 4.6, it suffices to show $\acute{J}(p(a, b, c)) = 0$ for all reduced $p(a, b, c)$ satisfying the conditions in Lemma 4.7. Equipping the set of these elements with a suitable linear order so that the set is well-ordered, and $p(0, 0, 0)$ is minimal, we prove $\acute{J}(p(a, b, c)) = 0$ by transfinite induction, assuming $\acute{J}(p(0, 0, 0)) = 0$.

Our proof for the case of $e > 0$ is similar, but the set consists of $p_e(a, b, c)$ satisfying the conditions in Lemma 4.9, and is equipped with another order so that $p_e(0, 0, 0)$ is minimal. To this end, we use the Hecke operators T_f, $f \in \mathbb{Z}^{n-1}$. Choose a p' less than p among the set. Since p' is less than p and J is a Hecke eigenvector, $\acute{J}(p') = 0$ by the induction hypothesis and $T_f \acute{J}(p') = 0$. An elementary computation shows that

$$\begin{aligned} T_f \acute{J}(p') &= \sum_{\sigma \in \mathfrak{S}_{n-1}/\mathfrak{S}(f)} \sum_{v \in \mathcal{N}(f^\sigma)} J\left(\begin{bmatrix} p' & \\ & 1_m \end{bmatrix} v \acute{\varpi}^{f^\sigma}\right) \\ &= \sum_{\sigma \in \mathfrak{S}_{n-1}/\mathfrak{S}(f)} \sum_{v \in \mathcal{N}(f^\sigma)} \psi(\mathrm{tr}(p' X_v)) J\left(\begin{bmatrix} p' & \\ & 1_m \end{bmatrix} \begin{bmatrix} v_+ \varpi^{f_+^\sigma} & \\ & v_- \acute{\varpi}^{f_-^\sigma} \end{bmatrix}\right), \end{aligned}$$

where

$$\begin{aligned} \mathfrak{S}(f) &= \{\sigma \in \mathfrak{S}_{n-1} \mid f^\sigma = f\}, \\ f_i^\sigma &= f_{\sigma(i)}, \\ f_+^\sigma &= (f_1^\sigma, \dots, f_m^\sigma), \\ f_-^\sigma &= (f_{(m+1)}^\sigma, \dots, f_{(n-1)}^\sigma), \\ \mathcal{N}(f^\sigma) &= \left\{v \in N_n \mid v_{in} \in \mathfrak{o}/\mathfrak{p}^{\max\{f_i^\sigma, 0\}},\ v_{ij} \in \mathfrak{o}/\mathfrak{p}^{\max\{f_i^\sigma - f_j^\sigma, 0\}}\ (j \le n-1)\right\}, \\ v &= \begin{bmatrix} 1_m & X_v \\ & 1_m \end{bmatrix} \begin{bmatrix} v_+ & \\ & v_- \end{bmatrix}, \quad v_+, v_- \in N_m(\mathfrak{o}). \end{aligned}$$

Here, X_v is an m by m matrix with entries in $\mathfrak{o}$. Since p' satisfies the conditions in Lemma 4.7, $\psi(\mathrm{tr}(p' X_v)) = 1$, and thus each term of the above expansion is of the form

$$\chi(\det(\varpi^{f_-^\sigma})) \acute{J}(\acute{\varpi}^{-f_-^\sigma} v_-^{-1} p' v_+ \varpi^{f_+^\sigma}). \tag{4-9}$$

We prove $\acute{J}(p) = 0$ by showing that this term equals $c\acute{J}(p)$ for a nonzero constant c independent of the choice of σ, or the element $\acute{\varpi}^{-f_-^\sigma} v_-^{-1} p' v_+ \varpi^{f_+^\sigma}$ is quasi-equivalent to a p'' which is less than p or does not satisfy the conditions in

Lemma 4.7, 4.9. We will use the identity

$$\mathrm{wt}(\acute{\varpi}^{-f_-^\sigma} v_-^{-1} p' v_+ \varpi^{f_+^\sigma}) = \mathrm{wt}(p') + \sum_{i=1}^{m-1} f_i^\sigma - \sum_{i=m+1}^{n-1} f_i^\sigma (\leq \mathrm{wt}(p)), \tag{4-10}$$

derived from (4-3), repeatedly. For positive integers $s < t$, let

$$e_s^t = (\overbrace{0, \dots, 0}^{s-1}, 1, 0, \dots, 0) \in \mathbb{Z}^t.$$

<u>Case of $e = 0$:</u> In this case, the quasi-equivalence preserves the value of J by (4-7). The order for the set of reduced $p(a, b, c)$ satisfying the conditions in Lemma 4.7 is as follows. Let

$$A = \{a \in \mathbb{Z}_{\geq 0}^{m-1} \mid a_1 \leq \cdots \leq a_{m-1}\}. \tag{4-11}$$

We equip A with the following linear order. Set $a < a'$ if one of the following conditions holds:

(i) $\sum_{i=1}^{m-1} a_i < \sum_{i=1}^{m-1} a_i'$.

(ii) $\sum_{i=1}^{m-1} a_i = \sum_{i=1}^{m-1} a_i'$ and $a_j < a_j'$ for the last j such that $a_j \neq a_j'$.

Let

$$B = \left\{b \in \mathbb{Z}_{\geq 0}^{m-1} \mid b_i < b_j \text{ if } i < j \text{ and } b_i b_j \neq 0\right\}.$$

We equip B with a linear order by the following rule. For $b, b' \in B$, let $S(b) = \{i_1 < \cdots < i_t\}$ and $S(b') = \{j_1 < \cdots < j_u\}$ be the total sets of indices at which the entries of b and b' are nonzero, respectively. Let k be the maximal number such that

$$i_{t-k+1} = j_{u-k+1}, \dots, i_t = j_u, \quad b_{i_{t-k+1}} = b'_{j_{u-k+1}}, \dots, b_{i_t} = b'_{j_u}.$$

Understand $k = 0$ if $(i_t, b_{i_t}) \neq (j_u, b'_{j_u})$, or, at least one of $S(b)$, $S(b')$ is empty. By definition, $(i_{t-k}, b_{i_{t-k}}) \neq (j_{u-k}, b'_{j_{u-k}})$. Set $b < b'$ if one of the following conditions holds:

(i) $b_{i_{t-k}} < b'_{j_{u-k}}$.

(ii) $b_{i_{t-k}} = b'_{j_{u-k}}$, and $j_{u-k} < i_{t-k}$.

We equip $A \times B \times \mathbb{Z}_{\geq 0}$ with a linear order by setting $(a, b, c) < (a', b', c')$ if $a < a'$, or $a = a'$ and $b < b'$, or $a = a'$, $b = b'$ and $c < c'$. Obviously, $(A, <)$, $(B, <)$ are well-ordered sets, and so is $(A \times B \times \mathbb{Z}_{\geq 0}, <)$. A coordinate (a, b, c) satisfying the conditions in the lemma is identified with the quasi-equivalence class of the reduced $p(a, b, c)$ by Proposition 4.6, and such coordinates consist a subset of $A \times B \times \mathbb{Z}_{\geq 0}$, which is also well-ordered.

To begin with, we will show

$$\acute{J}(p(0,0,c)) = 0, \quad c \in \mathbb{Z}. \tag{4-12}$$

By Lemma 4.7, it suffices to show this for $c > 0$. Set

$$f = (\varpi^c, \dots, \varpi^c), \quad p' = p(0,0,0).$$

Then, $\mathfrak{S}(f) = \mathfrak{S}_{n-1}$, and (4-9) is of the form

$$\chi(\varpi)^{c(m-1)} \acute{J}(u_x \varpi^{ce^m_m})$$

where $x \in \mathfrak{o}^{m-1}$, and equals $\chi(\varpi)^{c(m-1)} \acute{J}(p(0,0,c))$. Thus (4-12) follows.

Now, we start the induction. By (4-12), we may assume that $a_{j_r} > 0$. By Lemma 4.7, $a_{j_r} \geq b_{j_r}$.

First, suppose that $b_{j_r} = 0$. Set

$$f = (0, \dots, 0, \overbrace{-1, \dots, -1}^{m-j_r}),$$
$$p' = du_\beta = p(a', b, c), \quad a' = a' = a - (0, \dots, 0, \overbrace{1, \dots, 1}^{m-j_r}).$$

Observe that p' is reduced, and satisfies the conditions in Lemma 4.7. By the induction hypothesis and (4-10) we may assume that

$$f^\sigma_+ = 0, \quad v_+ = 1_m$$

since otherwise $\acute{\varpi}^{-f^\sigma_-} v_-^{-1} p' v_+ \varpi^{f^\sigma_+}$ has the weight less than that of p, and is less than p by the definition of the order. Write

$$v_-^{-1} = \begin{bmatrix} v' & \\ & 1 \end{bmatrix}, \quad v' \in N_{m-1}(\mathfrak{o}).$$

Let $d^\circ = \operatorname{diag}(d_1, \dots, d_{m-1})$. Since d is aligned, we have

$$v'' := d^{\circ -1} v' d^\circ \in N_{m-1}(\mathfrak{o}), \quad d^{-1} v_-^{-1} d = (\acute{v''}) \in N_m(\mathfrak{o}).$$

Then,

$$\acute{\varpi}^{-f^\sigma_-} v_-^{-1} p' = \acute{\varpi}^{-f^\sigma_-} d d^{-1} v_-^{-1} d u_\beta = \acute{\varpi}^{-f^\sigma_-} d(\acute{v''}) u_\beta \approx \acute{\varpi}^{-f^\sigma_-} d u(\beta^t v'').$$

Put $\gamma = \beta^t v''$. Let $E_\sigma = \{i \in \{1, \dots, m-1\} \mid f^\sigma_{m+i} = -1\}$. By the induction hypothesis again, we may assume that

$$f^\sigma_i = 0, \quad i \leq m + j_{r-1}, \tag{4-13}$$

since otherwise, $\acute{\varpi}^{-f^\sigma_-} v_-^{-1} p' (\approx \acute{\varpi}^{-f^\sigma_-} d u_\gamma)$ is quasi-equivalent to an element less than p by Proposition 4.3, 4.6 (compare the A-part). Therefore:

- $E_\sigma = \{j_r, \dots, m-1\}$ if $a_{j_r} - a_{j_{r-1}} > 1$.
- $E_\sigma \subset \{j_{r-1}, \dots, m-1\}$ if $a_{j_r} - a_{j_{r-1}} = 1$.

If $i < k$, then we have

$$v'_{ik} \in \begin{cases} \mathfrak{o}/\mathfrak{p} & \text{if } i \notin E_\sigma \ni k, \\ \{0\} & \text{otherwise,} \end{cases}$$
$$v''_{ik} \in \begin{cases} \varpi^{a'_k - a'_i}(\mathfrak{o}/\mathfrak{p}) & \text{if } i \notin E_\sigma \ni k, \\ \{0\} & \text{otherwise.} \end{cases}$$

Since $E_\sigma \subset \{j_{r-1}, \dots, m-1\}$, and $\beta_k = 0$ for $k > j_{r-1}$ (recall that p is reduced, and $b_{j_r} = 0$ is assumed), we have

$$\gamma_i = \beta_i + \begin{cases} \sum_{k \in E_\sigma} v''_{ik}\beta_k & \text{if } i \notin E_\sigma, \\ 0 & \text{otherwise,} \end{cases} \tag{4-14}$$
$$= \beta_i + \begin{cases} v''_{i,j_{r-1}}\beta_{j_{r-1}} & \text{if } i < j_{r-1} \in E_\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\gamma_{j_{r-1}} = \beta_{j_{r-1}}, \gamma_{j_{r-1}+1} = \cdots = \gamma_{m-1} = 0.$$

If $i < j_{r-1} \in E_\sigma$ and $b_{j_{s(i)}} > 0$, then by (4-14),

$$\gamma_i = \beta_i + v''_{i,j_{r-1}}\beta_{j_{r-1}} \in \mathfrak{p}^{-b_{j_{s(i)}}} + \mathfrak{p}^{a_{j_{r-1}} - a_{j_{s(i)}} - b_{j_{r-1}}} \subset \mathfrak{p}^{-b_{j_{s(i)}}},$$
$$\gamma_{j_{s(i)}} = \beta_{j_{s(i)}} + v''_{j_{s(i)},j_{r-1}}\beta_{j_{r-1}} \in \varpi^{-b_{j_{s(i)}}}\mathfrak{o}^\times + \mathfrak{p}^{a_{j_{r-1}} - a_{j_{s(i)}} - b_{j_{r-1}}}$$
$$\subset \varpi^{-b_{j_{s(i)}}}\mathfrak{o}^\times + \mathfrak{p}^{1-b_{j_{s(i)}}} = \varpi^{-b_{j_{s(i)}}}\mathfrak{o}^\times$$

(use Lemma 4.7, and the fact $a'_k = a_k$ for $k < j_r$). If $i < j_{r-1} \in E_\sigma$ and $b_{s(i)} = 0$ (i.e., $\beta_{s(i)} = 0$), then $\beta_i = 0$ and

$$\gamma_i = \begin{cases} v''_{i,j_{r-1}}\beta_{j_{r-1}} & \text{if } \beta_{j_{r-1}} \neq 0 \text{ and } v'_{i,j_{r-1}} \in \mathfrak{o}^\times, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\acute{\varpi}^{-f^\sigma_-} du_\gamma$ is quasi-equivalent to p if $\beta_{j_{r-1}} = 0$. From now, assume that $\beta_{j_{r-1}} \neq 0$, i.e., $b_{j_{r-1}} > 0$. From the above argument, we conclude that $\acute{\varpi}^{-f^\sigma_-} du_\gamma$ is quasi-equivalent to $p(a, b', c)$ with

$$b'_i = 0, \quad i \notin \{j_1, \dots, j_r\},$$
$$b'_{j_{s(<r-1)}} = \begin{cases} b_{j_{r-1}} + a_{j_s} - a_{j_{r-1}} \text{ or } 0 & \text{if } b_{j_s} = 0 \text{ and } j_{r-1} \in E_\sigma, \\ b_{j_s} & \text{otherwise,} \end{cases}$$
$$(b'_{j_{r-1}}, b'_{j_r}) = \begin{cases} (0, b_{j_{r-1}}) & \text{if } j_{r-1} \in E_\sigma, \\ (b_{j_{r-1}}, 0) & \text{otherwise.} \end{cases}$$

Therefore, if $j_{r-1} \in E_\sigma$, then b' is of the form

$$(b'_1, \dots, b'_{j_{r-1}-1}, \overbrace{0, \dots, 0}^{j_r - j_{r-1}}, \overbrace{b_{j_{r-1}}, \dots, 0}^{m-j_r}), \quad b'_{j_{s(<r-1)}} \le b_{j_s} (< b_{j_{r-1}})$$

and, by the proof of Proposition 4.3(i), $p(a, b', c)$ is quasi-equivalent to the reduced

$$p(a, (\dots, \overbrace{0, \dots, 0}^{j_r - j_{r-1}}, \overbrace{b_{j_{r-1}}, \dots, 0}^{m-j_r}), c),$$

which is less than p. Otherwise, $p(a, b', c) = p$. This settles the case of $b_{j_r} = 0$.

Next, suppose that $b_{j_r} > 0$. Set

$$f = (\overbrace{1, \dots, 1}^{m-j_r}, 0, \dots, 0), \qquad p' = du_\beta = p(a', b', c)^\iota,$$
$$a' = a - (\overbrace{0, \dots, 0}^{j_r-1}, 1, \dots, 1), \quad b' = b - (\overbrace{0, \dots, 0}^{j_r-1}, 1, 0, \dots, 0).$$

Observe that d^ι is aligned, and that $p(a', b', c)$ is reduced element satisfying the conditions in Lemma 4.7. Write

$$v_+ = \acute{v}_0 u_x, \quad v_0 \in N_{m-1}(\mathfrak{o}).$$

Let $E_\sigma = \{i \mid f_i^\sigma = 1\}$. By (4-10) and the induction hypothesis, we may assume that $E_\sigma \subset \{1, \dots, m-1\}$. Therefore, $v_- = 1_m$, and

$$\acute{\varpi}^{-f_-^\sigma} v_-^{-1} p' v_+ \varpi^{f_+^\sigma} = du_\beta \acute{v}_0 u_x \varpi^{f_+^\sigma} = d\acute{v}_0 u(\beta^t (v_0)^{-1}) u_x \varpi^{f_+^\sigma}.$$

Set $\beta' = \beta^t (v_0)^{-1}$. Let $y_{ik} (\in \mathfrak{o})$ denote the i, k entry of v_0^{-1}. Then,

$$\beta'_i = \beta_i + \begin{cases} \sum_{k=i+1}^{m-1} y_{ik}\beta_k & \text{if } i \in E_\sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Since d^ι is aligned, $d\acute{v}_0 d^{-1} \in K_{m-1}^1$, and

$$d\acute{v}_0 u_{\beta'} u_x \varpi^{f_+^\sigma} \approx du(\beta' + x)\varpi^{f_+^\sigma} = d\varpi^{f_+^\sigma} u((\beta' + x)\varpi^{-f_+^\sigma}).$$

Set $\gamma = (\beta' + x)\varpi^{-f_+^\sigma}$. Now by the induction hypothesis, we may assume

$$E_\sigma \subset \{1, \dots, m - j_{r-1}\},$$

and have

$$\gamma_i = \begin{cases} \varpi^{-1}\bigl(x_i + \beta_i + \sum_{k=i+1}^{m-1} y_{ik}\beta_k\bigr) & \text{if } i \in E_\sigma, \\ \beta_i & \text{otherwise.} \end{cases} \tag{4-15}$$

There are three cases:

(i) $b_{j_r} = 1$; (ii) $a_{j_r} - a_{j_{r-1}} > 1$ and $b_{j_r} \ge 2$; (iii) $a_{j_r} - a_{j_{r-1}} = 1$ and $b_{j_r} \ge 2$.

In case (i), all the β_i are zero, and

$$d\varpi^{f_+^\sigma} u_\gamma \sim \begin{cases} p & \text{if } x_i \in \mathfrak{o}^\times \text{ for some } i \in E_\sigma, \\ p(a,0,c)^\iota (\sim p(a,0,c)) & \text{otherwise.} \end{cases}$$

Of course, $p(a,0,c)$ is less than p.

In case (ii), since $a_{j_r} - a_{j_{r-1}} > 1$, $E_\sigma = \{1,\dots,m-j_r\}$ by the induction hypothesis. From (4-15), it follows that

$$\begin{aligned} \gamma_i &= \beta_i && (i > m-j_r), \\ o(\gamma_i) &= o\Big(x_i + \beta_i + \sum_{k=i+1}^{m-1} y_{ik}\beta_k\Big) - 1 \geq -b_{j_r} && (i \leq m-j_r), \\ o(\gamma_{m-j_r}) &= -b_{j_r}. \end{aligned}$$

Therefore, $d\varpi^{f_+^\sigma} u_\gamma$ is quasi-equivalent to p.

In case (iii), we may assume that $E_\sigma \subset \{1,\dots,m-j_{r-1}\}$ by the induction hypothesis. From (4-15),

$$\begin{aligned} \gamma_i &= \beta_i && (i > m-j_{r-1}), \\ o(\gamma_i) &\geq -b_{j_r} && (i \in E_\sigma), \\ o(\gamma_i) &\in \{0, -b_{j_{r-1}}, 1-b_{j_r}\} && (i \in \{1,\dots,m-j_{r-1}\} \setminus E_\sigma). \end{aligned}$$

By the proof of Proposition 4.3, $d\varpi^{f_+^\sigma} u_\gamma$ is quasi-equivalent to $p(a,b',c)^\iota$ with b' of the form

$$(b_1,\dots,b_{j_{r-1}-1}, \overbrace{b'_{j_{r-1}},0,\dots,0}^{j_r-j_{r-1}}, \overbrace{b'_{j_r},0,\dots,0}^{m-j_r}), \quad b'_{j_{r-1}} \in \{0, b_{j_{r-1}}, b_{j_r}-1\}, \quad b'_{j_r} \leq b_{j_r}.$$

If $b'_{j_{r-1}} = b_{j_r} - 1 (\geq 1)$, then

$$b'_{j_r} \leq b'_{j_{r-1}} \quad \text{or} \quad b'_{j_r} - b'_{j_{r-1}} = 1 (= a_{j_r} - a_{j_{r-1}}),$$

and, by the proof of Proposition 4.3 again, $p(a,b',c)^\iota$ is equivalent to

$$p(a, (b_1,\dots,b_{j_{r-1}-1}, \overbrace{b_{j_r}-1,0,\dots,0}^{m-j_{r-1}}), c)^\iota \quad (< p),$$

or

$$p(a, (b_1,\dots,b_{j_{r-1}-1}, \overbrace{0,\dots,0}^{j_r-j_{r-1}}, \overbrace{b'_{j_r},0,\dots,0}^{m-j_r}), c)^\iota \quad (\leq p).$$

Otherwise, obviously, $p^\iota(a,b',c) \leq p$. We get the desired conclusion in the case of $e = 0$.

Case of $e>0$: The order for the set of reduced $p_e(a,b,c)$ satisfying the conditions in Lemma 4.9 is as follows. Let

(4-16) $$A^+=B^+=\{a\in\mathbb{Z}_{\geq 0}^{m-1}\mid a_1\leq\cdots\leq a_{m-1}\}.$$

equipped with the linear order defined by setting $a<a'$ if one of the following conditions holds:

(i) $\sum_{i=1}^{m-1}a_i<\sum_{i=1}^{m-1}a'_i$.

(ii) $\sum_{i=1}^{m-1}a_i=\sum_{i=1}^{m-1}a'_i$, and $a_j>a'_j$(!) for the last j such that $a_j\neq a'_j$.

We equip $A^+\times B^+\times\mathbb{Z}_{\geq 0}$ with a linear order by setting $(a,b,c)<(a',b',c')$ if $a<a'$, or $a=a'$ and $b<b'$, or $a=a'$, $b=b'$ and $c<c'$. Obviously, $(A^+\times B^+\times\mathbb{Z}_{\geq 0},<)$ is also an well-ordered set. A coordinate (a,b,c) satisfying the conditions in the lemma is identified with the quasi-equivalence class of the reduced $p_e(a,b,c)$, and such coordinates consist a subset of $A^+\times B^+\times\mathbb{Z}_{\geq 0}$.

To begin with, we will show

(4-17) $$\acute{J}(p_e(0,0,c))=0,\quad c\in\mathbb{Z}.$$

By Lemma 4.9, it suffices to show this for $c>0$. Set

$$f=(\varpi^c,\dots,\varpi^c),\quad p'=du_\beta=p_e(0,0,0).$$

Then, $\mathfrak{S}(f)=\mathfrak{S}_{n-1}$, and it is easy to see that (4-9) is of the form

$$\chi(\varpi)^{c(m-1)}\acute{J}(\varpi^{ce_m^m}u_x du_\beta),$$

where $x\in\mathfrak{o}^{m-1}$. Define x' by $u_{x'}=d^{-1}u_xd$. From (4-7) it follows that

$$\acute{J}(\varpi^{ce_m^m}u_xdu_\beta)=\acute{J}(\varpi^{ce_m}du(x'+\beta))=\acute{J}(t\varpi^{ce_m}du(\beta)t^{-1})=\acute{J}(\varpi^{ce_m}du_\beta)=\acute{J}(p)$$

for

$$t:=\mathrm{diag}(1+x'_1/\beta_1,\dots,1+x'_{m-1}/\beta_{m-1},1),$$

since $o(\beta_i)=-ie$ and $o(x'_i)\geq(m-2i)e$ for $i\in\{1,\dots,m-1\}$. Therefore, (4-17) follows.

Now, we start the induction. For positive integers $r<s$, and $x\in F^r$, let

$$v_r^s(x)=\begin{bmatrix}1_{s-r-1}&&\\&1&x\\&&1_r\end{bmatrix}.$$

By (4-17), we may assume that $a_{m-1}>0$. By Lemma 4.9, $a_l\geq b_l$. Let l be the first number such that $a_l>0$.

First, suppose that $a_l = b_l$. Set

$$h = a_l = b_l, \quad f = he_1^n, \quad i = \sigma(1),$$
$$p' = du_\beta = p\big((\overbrace{0,\dots,0}^{l}, a_{l+1},\dots,a_{m-1}),\ (\overbrace{0,\dots,0}^{l}, b_{l+1},\dots,b_{m-1}), c\big)^\iota.$$

Observe that d^ι is aligned, and p' is reduced. The expansion of $T_f\acute{J}(p')$ is

$$\sum_{1\le i\le n-1}\ \sum_{x\in(\mathfrak{o}/\mathfrak{p}^h)^{n-i}} J\Big(\begin{bmatrix} p' & \\ & 1_m\end{bmatrix} v_{n-i}^n(x)\varpi^{he_i^n}\Big).$$

In case of $i \ge m$, the weight of $\acute{\varpi}^{-f_-^\sigma} v_-^{-1} p' v_+ \varpi^{f_+^\sigma}$ is less than $\mathrm{wt}(p)$. In case of $i \le m-1$, we have $v_- = 1_m$, and (4-9) equals

$$\acute{J}(du_\beta v_{m-i}^m(x')\varpi^{he_i^m}),$$

where $x' = (x_1,\dots,x_{m-i})$. We compute

$$u_\beta v_{m-i}^m(x') = \acute{v}_{m-i-1}^{m-1}(x_1,\dots,x_{m-i-1})u_{\beta'},$$
$$\beta'_j = \begin{cases}\beta_i + (x_{m-i} - x_1\beta_{i+1} - \cdots - x_{m-i-1}\beta_{m-1}) & \text{if } j = i,\\ \beta_j & \text{otherwise.}\end{cases}$$

Here since β_k/β_j lies in $\mathfrak{p}^e$ if $k > j$ by Lemma 4.9, and x_i lies in $\mathfrak{o}$, we have

$$\frac{\beta'_j}{\beta_j} \in 1+\mathfrak{p}^e, \quad j\in\{1,\dots,m-1\}. \tag{4-18}$$

Since d^ι is aligned, $d\acute{v}_{m-i-1}^{m-1}(x_1,\dots,x_{m-i-1})d^{-1}$ lies in $N_m(\mathfrak{o})$, and therefore,

$$\begin{aligned} du_\beta v_{m-i}^m(x')\varpi^{he_i^m} &= d\acute{v}_{m-i-1}^{m-1}(x_1,\dots,x_{m-i-1})u_{\beta'}\varpi^{he_i^m}\\ &\approx du_{\beta'}\varpi^{he_i^m} = d\varpi^{he_i^m}u(\beta''),\end{aligned}$$

where

$$\beta'' = \beta'\varpi^{-he_i^m}, \qquad \beta''_j = \begin{cases}\varpi^{-h}\beta'_i & \text{if } j = i,\\ \beta'_j & \text{otherwise.}\end{cases}$$

If $i < l$, then $d\varpi^{he_i^m}u(\beta'')$ is quasi-equivalent to an element $p_e(a', b', c)$ with $a' < a$ by the proof of Proposition 4.3. If $i > l$, then $d\varpi^{he_i^m}u(\beta'')$ is quasi-equivalent to an element which does not satisfy Lemma 4.9(ii). If $i = l$, then

$$\acute{J}(d\varpi^{he_l^m}u(\beta'')) = \acute{J}(d\varpi^{he_l^m}u(\beta'\varpi^{he_l^m})) = \acute{J}(d\varpi^{he_l^m}u(\beta\varpi^{he_l^m})) = \acute{J}(p)$$

by (4-18). This settles the case of $a_l = b_l$.

Next, suppose that $a_l > b_l$. Set

$$h = a_l - b_l, \quad f = -he_1^n, \quad i = \sigma(1),$$
$$p' = du_\beta = p\big((0,\dots,0,b_l,a_{l+1},\dots,a_{m-1}),\ (0,\dots,0,b_l,b_{l+1},\dots,b_{m-1}),c\big).$$

The expansion of $T\acute{J}(p')$ is

$$\sum_{1\le i\le n-1}\ \sum_{x\in(\mathfrak{o}/\mathfrak{p}^h)^{i-1}} J\left(\begin{bmatrix} p' & \\ & 1_m\end{bmatrix}\acute{u}(x)\varpi^{-he_i^n}\right).$$

In case of $i \le m$, the weight of $\acute{\varpi}^{-f_-^\sigma} v_-^{-1} p' v_+ \varpi^{f_+^\sigma}$ is less than $\mathrm{wt}(p)$. In case of $i > m$, $f_+^\sigma = 0$, $v_+ = 1_m$, and (4-9) equals

$$\chi(\varpi)^{-h}\acute{J}(\varpi^{he_{i-m}^m}\acute{u}(-x')p'),$$

where $x' = (x_{m+1},\dots,x_{i-1})$. Since d is aligned, $d^{-1}\acute{u}(-x')d$ lies in $N_m(\mathfrak{o})$, and

$$\acute{u}(-x')p' = \acute{u}(-x')du_\beta = d(d^{-1}\acute{u}(-x')d)u_\beta = du(\beta')d^{-1}\acute{u}(-x')d \approx du(\beta'),$$

with

$$\beta'_j = \begin{cases}\beta_j(1 + x_{m+j}\varpi^{(i-m-j)e+(a_{i-m}-b_{i-m})-(a_j-b_j)}) & \text{if } j < i-m,\\ \beta_j & \text{if } j \ge i-m.\end{cases}$$

We have $\acute{J}(\varpi^{he_{i-m}^m}u(-x')p') = \acute{J}(\varpi^{he_{i-m}^m}du(\beta'))$. By the induction hypothesis and Lemma 4.9(ii) again, we may assume that $i = m+l$, and

$$\beta'_j = \beta_j(1 + x_{m+j}\varpi^{(l-j)e+(a_l-b_l)-(a_j-b_j)}) \in \beta_j(1+\mathfrak{p}^e)$$

for $j < i-m = l$. Therefore,

$$\acute{J}(\varpi^{he_l^m}du(\beta')) = \acute{J}(\varpi^{he_l^m}du(\beta)) = \acute{J}(p).$$

This completes the proof of the theorem. □

Now, according to Reeder's oldform theory [14], any $P_n(\mathfrak{o})$-invariant vector in a generic π is a linear combination of $T'_f v^{\mathrm{new}}$, $f \in \mathbb{Z}_{\ge 0}^{n-1}$, where v^{new} is the newvector and T'_f is the Hecke operator defined by

$$T'_f\xi = \int_{G_{n-1}} \mathrm{Ch}(k;\, \acute{K}_{n-1}\acute{\varpi}^f\acute{K}_{n-1})\pi(k)\xi\,\mathrm{d}k$$

for $P_n(\mathfrak{o})$-invariant $\xi \in \pi$, where $\mathrm{d}k$ indicates the Haar measure on G_{n-1} normalized so that $\mathrm{vol}(\acute{K}_{n-1}) = 1$. An elementary computation shows that

$$\pi(g)T'_f\xi = \sum_{\sigma\in\mathfrak{S}_{n-1}/\mathfrak{S}(f)}\ \sum_{v\in\mathcal{N}'(f^\sigma)} \pi(gv\varpi^{f^\sigma})\xi,$$

where

$$\mathcal{N}'(f^\sigma) = \{v \in N_n \mid v_{ij} \in \mathfrak{o}/\mathfrak{p}^{\max\{f_i^\sigma - f_j^\sigma, 0\}},\ j \le n-1,\ v_{in} = 0\}.$$

Therefore, if J is the Shalika newform, then $T'_f \acute{J}(p)$, $p \in \widetilde{P}_m$ is a linear combination of $\acute{J}(g)$, $g \in G_m$. Therefore, from Theorem 4.1, the following conclusion is deduced:

If we assume that the Shalika newform in $\mathbb{S}_\pi(\chi)$ vanishes at the minimal point, then

$$J|_{S_n P_n \Gamma(\mathfrak{c}_\pi)} \equiv 0$$

for any $P_n(\mathfrak{o})$-invariant Shalika form $J \in \mathbb{S}_\pi(\chi)$, where $\mathfrak{c}_\pi$ indicates the conductor of the generic π.

4.1. ***Proof for Theorem 1.1.*** There exists a nontrivial $P_n(\mathfrak{o})$-invariant vector in ψ_n. For example, we can define the $P_n(\mathfrak{o})$-invariant $\xi_n \in \psi_n$ by

$$\xi_n(\acute{\varpi}^f) = \begin{cases} 1 & \text{if } f = 0 \in \mathbb{Z}^{n-1}, \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 3.8, ψ_n has a nontrivial $P_n(\mathfrak{o})$-invariant pre-Shalika form. By the assumption (1-1) and Proposition 3.10, there exists a $P_n(\mathfrak{o})$-invariant Shalika form whose restriction to P_n is nontrivial. Now the assertion follows from the above conclusion.

5. Zeta integrals

Let $n = 2m$ be an even integer, and $\pi \in \mathscr{I}(G_n)$ be generic with a Shalika model relevant to χ. Let J^{new} be a Shalika newform (recall this is unique up to a scalar). Assume that J^{new} does not vanish at g_n of Theorem 1.1. This condition is empty if π is supercuspidal. Assume that

$$\mathfrak{c}_\pi \ge me.$$

Put

$$l = \mathfrak{c}_\pi - (m-1)e \quad (\ge e).$$

Let $\mathbb{K}(\mathfrak{c}_\pi)$ be as in Section 1. First of all, we construct a $J_\pi \in \mathbb{S}_\pi(\chi)$ such that

$$J_\pi(1_n) = 1, \qquad \pi(k)J_\pi = \chi \circ \det(d_k) J_\pi, \quad k \in \mathbb{K}(\mathfrak{c}_\pi),$$

where d_k indicates the $m \times m$ block matrix of k in the lower right corner. In the case where χ is unramified, there is nothing to do. Suppose that χ is ramified.

Lemma 5.1. *Let Ω be a field. Let f be a Ω-valued function on a group G such that $f(gk) = \xi(k) f(g)$ for a subgroup K and a homomorphism $\xi : K \to \Omega^\times$.*

(i) *For the right translation f^h by $h \in G$,*

$$f^h(gk) = \xi(h^{-1}kh) f^h(g), \quad k \in K^h.$$

(ii) *Let K' be a subgroup containing K. Assume that ξ is extended to $\xi' : K' \to \Omega^\times$. Define an Ω-valued function on G by*

$$f'(g) = \int_{K'/K} \xi'(k')^{-1} f(gk')\,\mathrm{d}k'.$$

Then,

$$f'(gk) = \xi'(k') f(g), \quad k' \in K'.$$

Proof. Obvious. □

Let r be a positive integer. For a set A, let $(A)^r$ denote the r-tuple product of A. Let M_r and $\mathcal{O}_r$ denote the ring of $r \times r$ matrices with entries in F and $\mathfrak{o}$, respectively. If a subgroup $K \subset G_n$ consists of matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad a \in \mathfrak{A},\ b \in \mathfrak{B},\ c \in \mathfrak{C},\ d \in \mathfrak{D}$$

for subsets $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \subset M_m$, we call K the subgroup relevant to $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$. Let

$$\delta_m = \mathrm{diag}(\varpi^e, \varpi^{3e}, \dots, \varpi^{(2m-3)e}, \varpi^{(m-1)e}) \in G_m,$$

and, for $t \in (\mathfrak{o}^\times)^{m-1}$, let

$$v_m(t) = u((\varpi^{-(m-1)e} t_1, \varpi^{-(m-2)e} t_2, \dots, \varpi^{-e} t_{m-1})), \quad \mathfrak{J}_t = \pi\left(\begin{bmatrix} 1_m & \\ & v_m(t) \end{bmatrix}\right) J^{\mathrm{new}},$$

where $u(x)$, $x \in F^{m-1}$, indicates the element in (4-1). From (2-4) and the assumption $J^{\mathrm{new}}(g_n) \neq 0$, it follows that $\mathfrak{J}_1(\acute{\delta}_m) \neq 0$. By (2-4),

$$\mathfrak{J}_t(\acute{\delta}_m) = \chi\Big(\prod_{i=1}^{m-1} t_i\Big) \mathfrak{J}_1(\acute{\delta}_m).$$

Firstly, we set

$$J_1 = \int_{(\mathfrak{o}^\times)^{m-1}} \chi(t)^{-1} \mathfrak{J}_t \,\mathrm{d}t,$$

which is not zero at $\acute{\delta}_m$. For $u = \mathrm{diag}(u_1, \dots, u_m)$ with $u_i \in \mathfrak{o}^\times$, it holds that

$$\begin{aligned} \pi\left(\begin{bmatrix} 1_m & \\ & u \end{bmatrix}\right) J_1 &= \int_{(\mathfrak{o}^\times)^{m-1}} \chi(t)^{-1} \pi\left(\begin{bmatrix} 1_m & \\ & u \end{bmatrix}\begin{bmatrix} 1_m & \\ & v_m(t) \end{bmatrix}\right) J \,\mathrm{d}t \\ &= \int_{(\mathfrak{o}^\times)^{m-1}} \chi(t)^{-1} \pi\left(\begin{bmatrix} 1_m & \\ & v_m(t') \end{bmatrix}\begin{bmatrix} 1_m & \\ & u \end{bmatrix}\right) J \,\mathrm{d}t = \chi\Big(\prod_{i=1}^{m} u_i\Big) J_1, \end{aligned}$$

where $t' = (t_1 u_1/u_m, \dots, t_{m-1} u_{m-1}/u_m)$. By (i) of the lemma, $\mathfrak{J}_t$ is invariant under the subgroup relevant to

$$\begin{gathered} \mathcal{O}_m, \quad \{b \mid b_{ij} \in \mathfrak{p}^{m-1}\}, \quad \{c \mid c_{ij} \in \mathfrak{o};\ i < m,\ c_{mj} \in \mathfrak{p}^c\}, \\ \{d \mid d_{ii} \in 1 + \mathfrak{p}^e,\ d_{ij} \in \mathfrak{o};\ i < j,\ d_{mj} \in \mathfrak{p}^c;\ j < m,\ d_{ij} \in \mathfrak{p}^{me};\ j < i < m\}. \end{gathered}$$

Therefore,

$$\pi(k)J_1 = \chi\Big(\prod_{i=1}^{m} k_{m+i,m+i}\Big) J_1$$

for k lying in the subgroup relevant to

$$\mathcal{O}_m,\quad \{b \mid b_{ij} \in \mathfrak{p}^{m-1}\},\quad \{c \mid c_{mj} \in \mathfrak{p}^{c};\, c_{ij} \in \mathfrak{o},\, i < m\},$$
$$\{d \mid d_{ij} \in \mathfrak{o};\, i \le j,\, d_{mj} \in \mathfrak{p}^{c};\, j < m,\, d_{ij} \in \mathfrak{p}^{me};\, j < i < m\}.$$

By (2-4),

$$J_1(\acute{\delta}_m) = J_1\Big(\begin{bmatrix} 1_m & \\ & \delta_m^{-1}\end{bmatrix}\Big) \ne 0$$

Secondly, we set

$$J_2 = \pi\Big(\begin{bmatrix} 1_m & \\ & \delta_m^{-1}\end{bmatrix}\Big) J_1.$$

Then $J_2(1_n) \ne 0$, and, by (i) of the lemma,

$$\pi(k)J_2 = \chi\Big(\prod_{i=1}^{m} k_{m+i,m+i}\Big) J_2$$

for k lying in the subgroup relevant to

$$\mathcal{O}_m,\quad \{b \mid b_{ij} \in \mathfrak{p}^{n-2}\},\quad \{c \mid c_{mj} \in \mathfrak{p}^{l},\, c_{ij} \in \mathfrak{o};\, i < m\},$$
$$\Big\{\begin{bmatrix} u & {}^t(\mathfrak{p}^{(m-2)e})^{m-1} \\ (\mathfrak{p}^{c_\pi})^{m-1} & * \end{bmatrix} \in K_m \mid u_{ij} \in 1 + \mathfrak{p}^{(n-4)e};\, i \ne j\Big\}.$$

Thirdly, we set

$$J_3 = \iiint \pi\left(\begin{bmatrix} 1_m & & \\ & u & {}^t x \\ & y & 1 \end{bmatrix}\right) J_2 \,\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y$$

where the integral in x is over $(\mathfrak{o}/\mathfrak{p}^{(m-2)e})^{m-1}$, that in y is over $(\mathfrak{p}^{l}/\mathfrak{p}^{c_\pi})^{m-1}$, and that in u over

$$K_{m-1}/\{u \in K_{m-1} \mid u_{ij} \in 1 + \mathfrak{p}^{(n-4)e};\, i \ne j\}$$
$$\simeq \mathrm{SL}_{m-1}(\mathfrak{o})/\{u \in \mathrm{SL}_{m-1}(\mathfrak{o}) \mid u_{ij} \in 1 + \mathfrak{p}^{(n-4)e};\, i \ne j\}.$$

By (ii) of the lemma,

$$\pi(k)J_3 = \chi \circ \det(d_k) J_3$$

for k lying in the subgroup relevant to

$$\mathcal{O}_m,\quad \{b \mid b_{ij} \in \mathfrak{p}^{n-2}\},\quad \{c \mid c_{mj} \in \mathfrak{p}^{l};\, c_{ij} \in \mathfrak{o},\, i < m\},$$
$$\Big\{d = \begin{bmatrix} u & {}^t\mathfrak{o}^{m-1} \\ (\mathfrak{p}^{l})^{m-1} & * \end{bmatrix} \in K_m \mid u \in K_{m-1}\Big\}.$$

By (2-4),

$$J_3(1_n)=\iiint J_2\left(\begin{bmatrix}1_m & & \\ & u & {}^t x\\ & y & 1\end{bmatrix}\right)\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y=\iiint J_2\left(\begin{bmatrix}u & {}^t x & \\ y & 1 & \\ & & 1_m\end{bmatrix}\right)\mathrm{d}u\,\mathrm{d}x\,\mathrm{d}y$$

is nonzero, where the integral region is the same as above. Finally, we set

$$J_\pi=c\int\pi\left(\begin{bmatrix}1 & x\\ & 1\end{bmatrix}\right)J_3\,\mathrm{d}x,$$

where the integral in x is over $\mathcal{O}_m/\{b\mid b_{ij}\in\mathfrak{p}^{n-2}\}$, and c is the nonzero constant taken so that $J_\pi(1_n)=1$. Another desired property of J_π follows from (ii) of the lemma. Let $\mathbb{K}(\mathfrak{c}_\pi)^*$ be the open compact subgroup consisting of the matrices (1-5), and $J_\pi^*\in\mathbb{S}_{\pi^\vee}(\chi^{-1})$ be the Shalika form (1-4). By (i) of the lemma,

$$\pi^\vee(k)J_\pi^*=\chi\circ\det(d_k)J_\pi^*,\ k\in\mathbb{K}(\mathfrak{c}_\pi)^*.$$

Now let us compute the Godement–Jacquet zeta integrals of these Shalika forms and specific Schwartz functions. For a finite dimensional vector space V over F, let $\mathscr{S}(V)$ denote the Schwartz space of V. Define $\varphi_{\mathfrak{c}_\pi}\in\mathscr{S}(M_m)$ as follows. In the case where χ is unramified, $\varphi_{\mathfrak{c}_\pi}(x)=\mathrm{Ch}(x;R_{\mathfrak{c}_\pi})$ (see (1-2) for the definition of $R_{\mathfrak{c}_\pi}$). Suppose that χ is ramified. Define $\chi_0\in\mathscr{S}(F)$ by

$$\chi_0(x)=\mathrm{Ch}(x;\mathfrak{o}^\times)\chi(x).$$

Define $\phi_\chi^\circ\in\mathscr{S}(M_{m-1})$ by

$$\phi_\chi^\circ(x)=\prod_{1\le i\ne j\le m-1}\mathrm{Ch}(x_{ij};\mathfrak{o})\prod_{i=1}^{m-1}\chi_0(\varpi^e x_{ii}).$$

Define $\phi_{\chi,l}\in\mathscr{S}(M_m)$ by

$$\phi_{\chi,l}\left(\begin{bmatrix}x & {}^t y\\ z & w\end{bmatrix}\right)=\frac{\mathrm{Ch}(y,z;(\mathfrak{p}^{-l})^{m-1}\times\mathfrak{o}^{m-1})\chi_0(\varpi^e w)}{\mathrm{vol}(\mathrm{SL}_{m-1}(\mathfrak{o}))}\int_{\mathrm{SL}_{m-1}(\mathfrak{o})}\phi_\chi^\circ(xu)\,\mathrm{d}u.$$

Observe that

(5-1) $$\phi_{\chi,l}(vk)=\chi\circ\det(k)\phi_{\chi,l}(v),\qquad k\in\Gamma_m(l).$$

Define $\varphi_{\mathfrak{c}_\pi}\in\mathscr{S}(M_n)$ by

$$\varphi_{\mathfrak{c}_\pi}\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right)=\mathrm{Ch}\left(\begin{bmatrix}a & \\ c & d\end{bmatrix};R_{\mathfrak{c}_\pi}\right)\phi_{\chi,l}(b).$$

We have defined $\varphi_{\mathfrak{c}_\pi}$ so that $J_\pi\varphi_{\mathfrak{c}_\pi}$ is right $\mathbb{K}(\mathfrak{c}_\pi)$-invariant.

For $J\in\mathbb{S}_\pi(\chi)$ and $\varphi\in\mathscr{S}(G_n)$, define the Godement–Jacquet zeta integral

$$Z(s,J,\varphi)=\int_{G_n}J\varphi(g)\,|\det(g)|^s\,\mathrm{d}g,$$

which is absolutely convergent when $\Re(s)$ is sufficiently large. Originally, their zeta integrals are defined for φ and matrix coefficients of π (cf. [4]), but $Z(s, J, \varphi)$ can be understood as one of them by the proof of Proposition 3.1 of [9], as follows. Take an open compact subgroup $K \subset G_n$ such that φ is left K-invariant. The linear form $J \mapsto c_K \int_K J(k)\,\mathrm{d}k$ on $\mathbb{S}_\pi(\chi)$ is smooth and thus belongs to $\pi^\vee$, where $c_K = \mathrm{vol}(K)^{-1}$. Therefore, $f_{K,J}(g) := c_K \int_K J(kg)\,\mathrm{d}k$, $g \in G_n$ is a matrix coefficient of π. When $\Re(s)$ is sufficiently large, we have

$$
\begin{aligned}
Z(s, J, \varphi) &= c_K \int_{G_n}\Bigl(\int_K \varphi(kg)\,\mathrm{d}k\Bigr) J(g)\,|\det(g)|^s\,\mathrm{d}g \\
&= c_K \int_K\Bigl(\int_{G_n} \varphi(kg) J(g)\,|\det(g)|^s\,\mathrm{d}g\Bigr)\,\mathrm{d}k \\
&= c_K \int_K\Bigl(\int_{G_n} \varphi(g) J(k^{-1}g)\,|\det(g)|^s\,\mathrm{d}g\Bigr)\,\mathrm{d}k \\
&= c_K \int_{G_n}\Bigl(\int_K J(kg)\,\mathrm{d}k\Bigr) \varphi(g)\,|\det(g)|^s\,\mathrm{d}g = Z(s, f_{K,J}, \varphi).
\end{aligned}
$$

Now, for $k \in \mathbb{Z}$, let

$$
B_{m,k} = \{b \in B_m \cap \mathcal{O}_m \mid o(\det(b)) = k\} \quad \text{and} \quad c_k = \sum_{B_{m,k}/B_{m,0}} \acute{J}_\pi(b).
$$

Proposition 5.2. *With the above notation*,

$$
Z\Bigl(s + \frac{n-1}{2}, J_\pi, \varphi_{\mathfrak{c}_\pi}\Bigr) = q^{l(m-1)} \mathrm{vol}(\mathbb{K}(\mathfrak{c}_\pi)) \sum_{i=0}^{\infty} c_i q^{i(-s+1/2)} \times \begin{cases} 1 & \textit{if } e = 0, \\ \mathfrak{g}(\chi, \psi_{\varpi^{-e}})^m & \textit{otherwise.} \end{cases}
$$

Proof. Since the proofs are similar, we only treat the case $e > 0$. We observe what $g \in G_n$ contributes to the zeta integral. Using a complete system of representatives for $\Gamma_n(\mathfrak{c}_\pi)/\mathbb{K}(\mathfrak{c}_\pi)$ (cf. Lemma 2.1 of [13]), we may assume that g is of the form

$$
\begin{bmatrix} a+bdc & bd \\ dc & d \end{bmatrix} = \begin{bmatrix} 1_m & b \\ & 1_m \end{bmatrix}\begin{bmatrix} a & \\ & d \end{bmatrix}\begin{bmatrix} 1_m & \\ c & 1_m \end{bmatrix}, \tag{5-2}
$$

where a, b, c, d are $m \times m$ block matrices. We claim that d lies in $\Gamma_m(l)$, using the identity

$$
\begin{bmatrix} 1_m & b' \\ & 1_m \end{bmatrix} g = \begin{bmatrix} (a+bdc)+b'dc & bd+b'd \\ dc & d \end{bmatrix}. \tag{5-3}
$$

If g contributes to the zeta integral, then (5-2) lies in $\mathrm{supp}(\varphi_{\mathfrak{c}_\pi})$; therefore

$$
(dc)_{mj} \in \mathfrak{p}^l, \quad (dc)_{ij} \in \mathfrak{p}^e, \qquad i \in \{1, \dots, m-1\},\ j \in \{1, \dots, m\}, \tag{5-4}
$$

and we may write

$$
d = \begin{bmatrix} d^\circ & * \\ * & d_{mm} \end{bmatrix} \in \begin{bmatrix} \mathcal{O}_{m-1} & {}^t(\mathfrak{o})^{m-1} \\ (\mathfrak{p}^l)^{m-1} & \mathfrak{o} \end{bmatrix}.
$$

By the $\mathbb{K}(\mathfrak{c}_\pi)$-invariance property of $J_\pi\varphi_{\mathfrak{c}_\pi}$, we may assume that d° is an upper triangular matrix. It suffices to show that $\det(d) \in \mathfrak{o}^\times$. Assume that $\det(d) \in \mathfrak{p}$. Then $d_{kk} \in \mathfrak{p}$ for some $k \in \{1, \dots, m\}$. Using the assumption (5-4), and that d° is an upper trianglar matrix, we find that it is possible to take a b' so that

$$b'_{kk} \in \mathfrak{p}^{-1}, \quad b'_{jj} = 0, \quad j \in \{1, \dots, m\} \setminus \{k\}, \quad b'dc \in \mathcal{O}_m, \quad b'd \in \mathfrak{o}E_{kk},$$

where E_{kk} indicates the k-th row and k-th column matrix unit. But, since (5-3) also lies in $\operatorname{supp}(\varphi_{\mathfrak{c}_\pi})$, and

$$J_\pi\varphi_{\mathfrak{c}_\pi}\left(\begin{bmatrix} 1_m & b' \\ & 1_m \end{bmatrix} g\right) = \psi(b'_{kk}) J_\pi\varphi_{\mathfrak{c}_\pi}(g)$$

by (2-4), g does not contribute. Hence, the claim. Now, it is easy to see that

$$\begin{bmatrix} & \\ c & \end{bmatrix} = \begin{bmatrix} & \\ & d^{-1} \end{bmatrix}\begin{bmatrix} & \\ dc & \end{bmatrix} \in R_{\mathfrak{c}_\pi}$$

and we may assume that

$$\begin{aligned} b_{ii} &\in \varpi^{-e}\mathfrak{o}^\times, \quad i \in \{1, \dots, m\}, \\ b_{mj} \in \mathfrak{o}, b_{jm} &\in \mathfrak{p}^{-l}, \quad j \in \{1, \dots, m-1\}, \end{aligned}$$

by (5-1). Therefore, bdc lies in $\mathcal{O}_m$, and so does a. Now the assertion follows from the argument of Lemma 4.2. □

By the way, the proof for (1-3) is as follows.

Proposition 5.3. *Set* $G_m^k = \{g \in G_m \mid o(\det(g)) = k\}$. *Then*

$$c_k = \operatorname{vol}(K_m)^{-1} \int_{G_m^k} J_\pi\left(\begin{bmatrix} g & \\ & 1_m \end{bmatrix}\right) \mathrm{d}g.$$

Proof. Let $g \in G_m^k \setminus \mathcal{O}_m$. Then, there exists an $x \in \mathcal{O}_m$ such that $\operatorname{tr}(gx) \notin \mathfrak{o}$. Therefore, $J_\pi(\acute{g}) = 0$ by Lemma 4.8, and the identity:

$$\begin{bmatrix} g & \\ & 1_m \end{bmatrix}\begin{bmatrix} 1_m & x \\ & 1_m \end{bmatrix}\begin{bmatrix} g & \\ & 1_m \end{bmatrix}^{-1} = \begin{bmatrix} 1_m & gx \\ & 1_m \end{bmatrix}.$$

Corresponding to the decomposition $G_m^k = G_m^k \cap \mathcal{O}_m \sqcup (G_m^k \setminus \mathcal{O}_m)$, the homogeneous space G_m^k/K_m decomposes into $(G_m^k \cap \mathcal{O}_m)/K_m \sqcup (G_m^k \setminus \mathcal{O}_m)/K_m$. Now the assertion follows from $B_{m,k}/B_{m,0} \simeq (G_m^k \cap \mathcal{O}_m)/K_m$ and the fact J_π is invariant under $\acute{K}_m$. □

For $\varphi \in \mathscr{S}(M_n)$, let $\varphi^\sharp$ be the Fourier transform of φ relevant to ψ

$$\varphi^\sharp(x) = \int_{G_n} \varphi(y)\psi(\operatorname{tr}(yx))\,\mathrm{d}y,$$

where $\mathrm{d}y$ indicates the self-dual Haar measure on M_n. We define

$$\varphi^*_{\mathfrak{c}_\pi}(x) = \varphi^\sharp_{\mathfrak{c}_\pi}(\upsilon_{\mathfrak{c}_\pi}^{-1t} x w_n),$$

where $\upsilon_{\mathfrak{c}_\pi}$ is the matrix defined in Section 1. For a function f on G_n, let $f^\vee$ denote the function defined by $f^\vee(g) = f(g^{-1})$. Then $J_\pi^* \varphi_{\mathfrak{c}_\pi}^*$ is right $\mathbb{K}_{\mathfrak{c}_\pi}^*$-invariant, and

$$q^{\mathfrak{c}_\pi s} Z(s, J_\pi^*, \varphi_{\mathfrak{c}_\pi}^*) = Z(s, J_\pi^\vee, \varphi_{\mathfrak{c}_\pi}^\sharp). \tag{5-5}$$

Now, in general,

$$\begin{aligned} Z(s, J^\vee, \varphi^\sharp) &= \int_{G_n} J(g^{-1}) \varphi^\sharp(g) \, |\det(g)|^s \, \mathrm{d}g \\ &= c_K \int_{G_n} J(g^{-1}) \int_K \varphi^\sharp(gk) \, \mathrm{d}k \, |\det(g)|^s \, \mathrm{d}g \\ &= c_K \int_K \int_{G_n} J(k^{-1} g^{-1}) \varphi^\sharp(g) \, |\det(g)|^s \, \mathrm{d}g \, \mathrm{d}k = Z(s, f_{K,J}^\vee, \varphi^\sharp), \end{aligned} \tag{5-6}$$

where K indicates an open compact subgroup of G_n such that φ is left K-invariant (therefore, $\varphi^\sharp$ is right K-invariant). An explicit description for $\varphi_{\mathfrak{c}_\pi}^*$ is as follows. When χ is unramified, it is given by

$$\varphi_{\mathfrak{c}_\pi}^* \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \mathrm{Ch}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}; R_{\mathfrak{c}_\pi}^* \right) \prod_{j=2}^{m} \mathrm{Ch}(b_{j1}; \mathfrak{p}^l),$$

where $R_{\mathfrak{c}_\pi}^*$ is the ring (cf. (1-5)). In the case where χ is ramified, define $\phi_\chi^{\circ *} \in \mathscr{S}(M_{m-1})$ by

$$\phi_\chi^{\circ *}(x) = \prod_{1 \le i \ne j \le m-1} \mathrm{Ch}(x_{ij}; \mathfrak{o}) \prod_{i=1}^{m-1} \chi_0^{-1}(x_{ii}).$$

Define $\phi_{\chi,l}^* \in \mathscr{S}(M_m)$ by

$$\phi_{\chi,l}^* \left(\begin{bmatrix} w & y \\ {}^t z & x \end{bmatrix} \right) = \frac{\mathrm{Ch}(y, z; \mathfrak{o}^{m-1} \times (\mathfrak{p}^l)^{m-1}) \chi_0^{-1}(w)}{\mathrm{vol}(\mathrm{SL}_{m-1}(\mathfrak{o}))} \int_{\mathrm{SL}_{m-1}(\mathfrak{o})} \phi_\chi^{\circ *}(xu) \, \mathrm{d}u.$$

Then, the explicit form of $\varphi_{\mathfrak{c}_\pi}^*$ is

$$\varphi_{\mathfrak{c}_\pi}^* \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \mathfrak{g}(\chi, \psi_{\varpi^{-e}})^m \phi_{\chi,l}^*(d) \mathrm{Ch}\left(\begin{bmatrix} a & b \\ c & \end{bmatrix}; R_{\mathfrak{c}_\pi}^* \right) \prod_{j=2}^{m} \mathrm{Ch}(b_{j1}; \mathfrak{p}^{-l}).$$

For $k \in \mathbb{Z}$, let

$$B_{m,k}^e = \{ b \in B_m \mid o(\det(b)) = k, b_{11} \in \mathfrak{o}, b_{1j} \in \mathfrak{p}^{e-l}; \, j > 1, b_{ij} \in \mathfrak{o}; \, i > 1 \},$$

and

$$c_k^* = \sum_{B_{m,k}^e / B_{m,0}^e} \acute{J}_\pi^*(b).$$

Similar to Proposition 5.3 and Proposition 5.2, we can show (1-6) and the following, respectively.

Proposition 5.4. *With the above notation,*

$$Z\Big(s+\frac{n-1}{2},\,J_\pi^*,\varphi_{\mathfrak{c}_\pi}^*\Big) = q^{l(m-1)}\mathrm{vol}(\mathbb{K}(\mathfrak{c}_\pi)^*)\sum_{i=0}^{\infty} c_i^* q^{i(-s+1/2)} \times \begin{cases} 1 & \text{if } e=0,\\ \mathfrak{g}(\chi,\psi_{\varpi^{-e}})^m & \text{otherwise.}\end{cases}$$

From (5-5), (5-6), and the functional equation in [4], it follows that

$$\frac{Z\big(\frac{n+1}{2}-s,\,J_\pi^*,\varphi_{\mathfrak{c}_\pi}^*\big)}{L(1-s,\pi^\vee)} = \varepsilon_\pi \frac{Z\big(s+\frac{n-1}{2},\,J_\pi,\varphi_{\mathfrak{c}_\pi}\big)}{L(s,\pi)}$$

where both the L-functions and the root number ε_π are same as those defined by Whittaker forms (cf. [11]). By Propositions 5.2 and 5.4, the both sides lie in $\mathbb{C}[q^{-s}]$ and $\mathbb{C}[q^s]$, respectively. Hence, both sides are nonzero constant. Thus we have:

Theorem 5.5. *With the above notation and assumptions,*

$$\sum_{i=0}^{\infty} c_i q^{i(-s+1/2)} = L(s,\pi), \quad \sum_{i=0}^{\infty} c_i^* q^{i(-s+1/2)} = \varepsilon_\pi L(s,\pi^\vee).$$

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