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The Braverman–Kazhdan program, later refined by Ngô, aims to understand Langlands L-functions attached to a reductive group G and a representation $\rho : {}^LG \to \mathrm{GL}_{V_\rho}$ of its L-group, plus certain additional desiderata. Such pairs (G, ρ) are called Braverman–Kazhdan–Ngô (BKN) pairs. We explain in this paper how it is enough to consider BKN pairs (G, ρ), in order to understand general Langlands L-functions. A key tool in the approach of Braverman and Kazhdan is a certain reductive monoid attached to ρ. There are two methods of constructing such a reductive monoid in the literature. We prove that the two methods yield the same monoid when (G, ρ) is a BKN pair.

1. Introduction

Let G be a split reductive algebraic group defined over a number field k, and $G^\vee$ be the dual group over $\mathbb{C}$ with the root datum dual to G. For an irreducible finite-dimensional algebraic representation (ρ, V_ρ) of $G^\vee$ and an irreducible cuspidal automorphic representation π of $G(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of k, R. Langlands introduced in [9] the automorphic L-function $L(s, \pi, \rho)$ associated with (π, ρ) and proved that $L(s, \pi, \rho)$ can be defined as an Euler product of local L-factors

$$L(s, \pi, \rho) = \prod_{\nu \in |k|} L_\nu(s, \pi_\nu, \rho)$$

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when $s \in \mathbb{C}$ with $\mathrm{Re}(s)$ sufficiently positive, where $|k|$ is the set of all local places of k, $\pi = \otimes_\nu \pi_\nu$ (the restricted tensor product decomposition), and the local L-factors $L_\nu(s, \pi_\nu, \rho)$ can be defined through the local Langlands conjecture. It is well known that the local L-factors are well-defined when ν is archimedean or any finite local place at which π_ν is unramified. The Langlands conjecture predicts that $L(s, \pi, \rho)$ admits a meromorphic continuation to $s \in \mathbb{C}$ and satisfies the standard global functional equation. One of the central problems in the theory of automorphic forms and in the Langlands program is to prove the Langlands conjecture for general automorphic L-functions $L(s, \pi, \rho)$.

The Langlands–Shahidi method and the Rankin–Selberg method established the Langlands conjecture for long lists of cases. However, the general situation of the Langlands conjecture is still widely open. Around 2000, A. Braverman and D. Kazhdan [4] proposed a framework to establish the Langlands conjecture for general L-functions $L(s, \pi, \rho)$, when (G, ρ) is a Braverman–Kazhdan–Ngô pair as defined below, via local and global harmonic analysis on G. It generalizes the method of J. Tate's thesis [28] and the work of R. Godement and H. Jacquet [6].

Definition 1.1 (Braverman–Kazhdan–Ngô pair). Let G be a split reductive group over a field of characteristic zero and let $\rho : G^\vee \to \mathrm{GL}_{V_\rho}$ be a representation of its dual group. The pair (G, ρ) is a BKN pair if it satisfies the following conditions.

(1) The kernel of $\rho : G^\vee \to \mathrm{GL}_{d_\rho}$ is trivial, where d_ρ is the dimension of ρ.

(2) There exists a character $\mathfrak{d} : G \to \mathbb{G}_m$, such that the exact sequence

$$1 \to G_0 \to G \xrightarrow{\mathfrak{d}} \mathbb{G}_m \to 1$$

holds with $G_0 = [G, G]$ the derived group of G.

(3) The derived group $G_0 = [G, G]$ is simply connected.

(4) The dual morphism $\mathfrak{d}^\vee : \mathbb{G}_m \to G^\vee$ composed with the given representation ρ is a scalar multiplication of $\mathbb{G}_m$ on the vector space V_ρ, i.e., the identity $\rho(\mathfrak{d}^\vee(a)) = a \cdot \mathrm{I}_{V_\rho}$ holds for any $a \in \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$.

Note that in Definition 1.1, we take any field of characteristic zero as the ground field, which puts the notion in algebraic terms, and keep in mind that the notion of the BKN pairs may have a connection to the relative Langland duality of D. Ben-Zvi, Y. Sakellaridis and A. Venkatesh [5].

The local conjecture in the Braverman–Kazhdan proposal for any local field F of characteristic zero can be found in [13, Section 4] and [11, Section 2.1]. The assumptions for a BKN pair are natural but restrictive. For the Langlands conjecture on automorphic L-function, we may consider the following pairs (G, ρ), without loss of generality.

Definition 1.2 (L-pair). Let G be a reductive group that splits over a number field k and let $\rho : G^\vee \to \mathrm{GL}_{V_\rho}$ be a representation of its dual group. We say (G, ρ) is an *L-pair* if the restriction of ρ to any essentially simple component of the derived group of $G^\vee$ is nontrivial.

Theorem 1.3. *Let F be a local field of characteristic zero, let G be a split reductive F-group, and let (G, ρ) be an L-pair. Then there is a canonical BKN pair $(H_\rho, \rho_{\mathrm{fp}})$ attached to (G, ρ) with the following property: Assume that the local Langlands correspondence is known for $G(F)$ and $H_\rho(F)$. If π is an irreducible admissible representation of $G(F)$, then there is an irreducible admissible representation σ of $H_\rho(F)$ such that the local L-factors and local ϵ-factors*

$$L(s, \pi, \rho) = L(s, \sigma, \rho_{\mathrm{fp}}) \quad \textit{and} \quad \epsilon(s, \pi, \rho, \psi) = \epsilon(s, \sigma, \rho_{\mathrm{fp}}, \psi)$$

hold as meromorphic functions of s for all nontrivial characters $\psi : F \to \mathbb{C}^\times$.

Theorem 1.3 is one of the main results of this paper, which will be proved in Section 2. The BKN-pair assertion is proved in Section 2A (Theorem 2.4) and the assertion on local L-factors is proved in Section 2B. With the help of the Borel conjecture [1; 2; 26], one can make the representation σ more explicitly in terms of the given representation π in Theorem 1.3, which will be discussed in Section 3. Theorem 1.3 suggests that in order to study the Langlands conjecture on general automorphic L-functions via global zeta integral methods, such as the Rankin–Selberg method, the Langlands–Shahidi method, the Miller–Schmid method and the Bravermen–Kazhdan–Ngô method, the harmonic analysis can be taken over the BKN-pairs that have the extra nice structures as in Definition 1.1. This will be further explained by the examples in Section 5, and in our forthcoming work.

As explained in [3; 4; 13], the Braverman–Kazhdan–Ngô approach to the Langlands conjecture for automorphic L-functions and other related geometric problems requires a good understanding of the L-monoids associated with the given group G as its unit group. For a given BKN-pair (G, ρ), one has two different constructions of L-monoids: one is from the Vinberg method [12; 14; 29], which is denoted by $\mathcal{M}_\rho$, and the other is from the general theory of M. Putcha and L. Renner [21], which is denoted by $\mathcal{M}^\rho$. Note that the two methods construct the monoids with G as the unit groups under the different assumptions on the structure of the algebraic groups G. When G is from a BKN-pair, the two methods work and hence we have both $\mathcal{M}_\rho$ and $\mathcal{M}^\rho$. It is important to know whether $\mathcal{M}_\rho \cong \mathcal{M}^\rho$ since both have different geometric features.

Theorem 1.4. *For any given* BKN-*pair (G, ρ), let $\mathcal{M}_\rho$ be the L-monoid as constructed from the Vinberg method and let $\mathcal{M}^\rho$ be the L-monoid as constructed from the Putcha–Renner method. Then the two reductive monoids are isomorphic, i.e., $\mathcal{M}_\rho \cong \mathcal{M}^\rho$.*

The proof of Theorem 1.4 will be given in Section 4C.

Outline. Section 2 is devoted to the construction of the BKN-pair associated with any given L-pair and proves Theorem 1.3. In Section 3, we discuss the Borel conjecture on the local L-packets, which are important for us to understand the known construction of global zeta integrals for families of L-functions. Section 4 is to discuss the relation between the monoid constructed via the Putcha–Renner theory and the monoid constructed through the Vinberg universal monoid for any given BKN-pair. Finally, in Section 5, we present a series of examples of the BKN-pairs.

2. Braverman–Kazhdan–Ngô pairs

Let k be a field of characteristic zero, which can be a number field or a local field of characteristic zero — the latter being the only ones considered in this paper. Let G be a reductive algebraic group that is k-split. Take $G^\vee$ to be the k-split reductive algebraic group with its root datum dual to that of G Let ρ be a finite-dimensional k-rational representation of $G^\vee$. For a given L-pair (G, ρ), in Section 2A we show that a particular pair constructed using a fiber product is a BKN-pair. For convenience, we write $(G, \rho)_{\mathrm{BKN}}$ for a BKN-pair and $(G, \rho)_L$ for an L-pair.

For the given L-pair $(G, \rho)_L$ over k, let T be a maximal k-split torus of G and $T^\vee$ be its dual, which is a maximal k-split torus of $G^\vee$. Let $X^*(T)$ be the k-rational characters of T and $X_*(T)$ be the k-rational cocharacters of T. Then we have that $X^*(T) = X_*(T^\vee)$ and $X^*(T^\vee) = X_*(T)$. Let $\Phi = \Phi(G, T)$ be the set of roots of G with respect to T and $\Phi^\vee = \Phi(G^\vee, T^\vee)$ be the set of roots of $G^\vee$ with respect to $T^\vee$, which is dual to Φ. Let $(X, \Phi; X^\vee, \Phi^\vee)$ be the root data as in [27]. Let (B, T) be a fixed Borel pair of G that determines the subset Φ^+ of positive roots in Φ.

2A. ***BKN-pair via fiber product.*** For a given L-pair $(G, \rho)_L$ as in Definition 1.2, in order to prove Theorem 1.3, we are going to construct a BKN-pair $(G_{\mathfrak{d}}, \rho^{\mathfrak{d}})_{\mathrm{BKN}}$ via a fiber product associated to $(G, \rho)_L$. The construction is purely algebraic and we take any field k of characteristic zero as the ground field.

We first construct the dual group $(G_{\mathfrak{d}})^\vee$ by considering the fiber product diagram

$$\begin{array}{ccc} G^\vee & \xrightarrow{\ \rho\ } & \mathrm{GL}_{d_\rho} \\ \big\downarrow{\scriptstyle \mathrm{pr}} & & \big\downarrow{\scriptstyle \mathrm{pr}} \\ (G^\vee)_{\mathrm{ad}} & \xrightarrow{\ \overline{\rho}\ } & \mathrm{PGL}_{d_\rho} \end{array} \tag{2-1}$$

where d_ρ is the dimension of the representation ρ and $\overline{\rho}$ is the canonical morphism induced from ρ via the canonical projection. Let $H_\rho^\vee$ be the fiber product of GL_{d_ρ} and $(G^\vee)_{\mathrm{ad}}$ over PGL_{d_ρ} defined by the diagram (2-1), which is the universal object fits into the commutative diagram

$$\begin{tikzcd} G^\vee \arrow[dr, dashed, "\eta_{\mathrm{fp}}^\vee"] \arrow[drr, bend left, "\rho"] \arrow[ddr, bend right, "\mathrm{pr}"'] & & \\ & H_\rho^\vee \arrow[r, "\rho_{\mathrm{fp}}"] \arrow[d, "\mathrm{pr}"] & \mathrm{GL}_{d_\rho} \arrow[d, "\mathrm{pr}"] \\ & (G^\vee)_{\mathrm{ad}} \arrow[r, "\bar{\rho}"] & \mathrm{PGL}_{d_\rho} \end{tikzcd} \tag{2-2}$$

The morphisms

$$\eta_{\mathrm{fp}}^\vee : G^\vee \to H_\rho^\vee, \quad \rho_{\mathrm{fp}} : H_\rho^\vee \to \mathrm{GL}_{d_\rho}, \quad \text{and} \quad \mathrm{pr} : H_\rho^\vee \to (G^\vee)_{\mathrm{ad}}$$

are given by the universality of the fiber product, so that diagram (2-2) commutes. The fiber product fits into the following commutative diagram with exact rows:

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{G}_m \arrow[r, "\mathfrak{d}_{\mathrm{fp}}^\vee"] \arrow[d, "="] & H_\rho^\vee \arrow[r, "\mathrm{pr}"] \arrow[d, "\rho_{\mathrm{fp}}"] & (G^\vee)_{\mathrm{ad}} \arrow[r] \arrow[d, "\bar{\rho}"] & 1 \\ 1 \arrow[r] & \mathbb{G}_m \arrow[r] & \mathrm{GL}_{d_\rho} \arrow[r, "\mathrm{pr}"] & \mathrm{PGL}_{d_\rho} \arrow[r] & 1 \end{tikzcd} \tag{2-3}$$

which implies that the composition $\rho_{\mathrm{fp}} \circ \mathfrak{d}_{\mathrm{fp}}^\vee(z) = z\mathrm{I}_{d_\rho}$ for any $z \in \mathbb{G}_m$, which is Condition (4) of Definition 1.1. It is easy to see that $H_\rho^\vee$ is a k-split reductive algebraic group over k. We are going to prove that $H_\rho^\vee$ is a candidate for $(G_{\mathfrak{d}})^\vee$ and its dual $(H_\rho^\vee)^\vee$ is the candidate for the group $G_{\mathfrak{d}}$ for the BKN pair $(G_{\mathfrak{d}}, \rho^{\mathfrak{d}})$.

To make the notion clearer, we take H_ρ to be the k-split reductive algebraic group whose root datum is dual to that of $H_\rho^\vee$. In this way, we have that $H_\rho^\vee = (H_\rho)^\vee$. It is clear that H_ρ is equipped with a morphism

$$\mathfrak{d}_{\mathrm{fp}} : H_\rho \to \mathbb{G}_m, \tag{2-4}$$

which is dual to $\mathfrak{d}_{\mathrm{fp}}^\vee : \mathbb{G}_m \to H_\rho^\vee$ as in (2-3). Thus, we construct a pair which is denoted by

$$(H_\rho, \rho_{\mathrm{fp}}). \tag{2-5}$$

In order to show that the constructed pair $(H_\rho, \rho_{\mathrm{fp}})$ is a BKN-pair, it remains to show that the exact sequence

$$1 \to H_\rho^\circ \to H_\rho \xrightarrow{\mathfrak{d}_{\mathrm{fp}}} \mathbb{G}_m \to 1 \tag{2-6}$$

holds, with the kernel H_ρ° of the morphism $\mathfrak{d}_{\mathrm{fp}}$ equal to the derived group of H_ρ, i.e., $H_\rho^\circ = [H_\rho, H_\rho]$, the kernel $\ker(\rho_{\mathrm{fp}})$ being trivial, and H_ρ° simply connected according to Definition 1.1.

Proposition 2.1. *With notation as given by the fiber product diagram in* (2-2), *the kernel of the morphism* $\rho_{\mathrm{fp}} : H^\vee_\rho \to \mathrm{GL}_{d_\rho}$ *is trivial.*

Proof. Let $\lambda \in X^*(T^\vee)$ be the highest weight of ρ. Let $Z^\vee$ be the center of $G^\vee$. Then $\lambda : Z^\vee \to \mathbb{G}_m$ is a character of $Z^\vee$. We claim that

$$H^\vee_\rho \cong (\mathbb{G}_m \times G^\vee)/\{(\lambda(z)^{-1}, z) \mid z \in Z^\vee\} =: \mathbb{G}_m \underset{(\lambda^{-1},\mathrm{Id})(Z^\vee)}{\times} G^\vee. \tag{2-7}$$

In order to prove the isomorphism in (2-7), it suffices to show that the group on the right-hand side $\mathbb{G}_m \times_{(\lambda^{-1},\mathrm{Id})(Z^\vee)} G^\vee$ is also a universal object for the fiber product diagram in (2-1). First of all, we extend the representation ρ of $G^\vee$ to that of the group $\mathbb{G}_m \times G^\vee$ by defining the scalar multiplication of $\mathbb{G}_m$:

$$\rho^{\mathfrak{e}}(a, g) = a \cdot \rho(g) \quad \text{for } a \in \mathbb{G}_m \text{ and } g \in G^\vee. \tag{2-8}$$

When $(a, g) = (\lambda^{-1}(z), z) \in \mathbb{G}_m \times G^\vee$ for $z \in Z^\vee$, we have

$$\rho^{\mathfrak{e}}((\lambda^{-1}(z), z)) = \lambda^{-1}(z) \cdot \rho(z) = 1.$$

The extended representation $\rho^{\mathfrak{e}}$ factors through the quotient group $\mathbb{G}_m \underset{(\lambda^{-1},\mathrm{Id})(Z^\vee)}{\times} G^\vee$:

$$\mathbb{G}_m \underset{(\lambda^{-1},\mathrm{Id})(Z^\vee)}{\times} G^\vee \xrightarrow{\rho^{\mathfrak{e}}} \mathrm{GL}_{d_\rho} : [(a, g)] \mapsto a \cdot \rho(g). \tag{2-9}$$

On the other hand, we have the natural projection

$$\mathbb{G}_m \underset{(\lambda^{-1},\mathrm{Id})(Z^\vee)}{\times} G^\vee \xrightarrow{\mathrm{pr}} (G^\vee)_{\mathrm{ad}} : [(a, g)] \mapsto [g]. \tag{2-10}$$

It is clear that the above construction fits into the fiber product diagram as in (2-1):

$$\begin{array}{ccc} \mathbb{G}_m \times_{(\lambda^{-1},\mathrm{Id})(Z^\vee)} G^\vee & \xrightarrow{\rho^{\mathfrak{e}}} & \mathrm{GL}_{d_\rho} \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}} \\ (G^\vee)_{\mathrm{ad}} & \xrightarrow{\bar\rho} & \mathrm{PGL}_{d_\rho} \end{array} \tag{2-11}$$

It remains to show the universality of the group $\mathbb{G}_m \times_{(\lambda^{-1},\mathrm{Id})(Z^\vee)} G^\vee$ with respect to the fiber product diagram in (2-11).

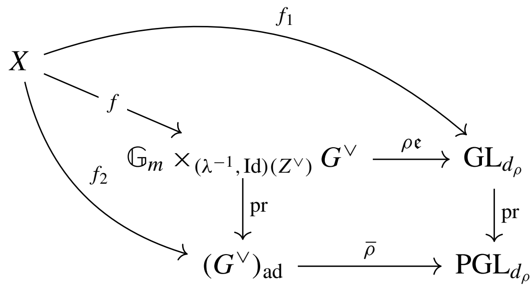
Let X be any variety fitting into the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f_1} & \mathrm{GL}_{d_\rho} \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle \mathrm{pr}} \\ (G^\vee)_{\mathrm{ad}} & \xrightarrow{\bar\rho} & \mathrm{PGL}_{d_\rho} \end{array} \tag{2-12}$$

We have to show that there exists a unique morphism $f : X \to \mathbb{G}_m \times_{(\lambda^{-1},\mathrm{Id})(Z^\vee)} G^\vee$

such that the following diagram commutes:

(2-13)



We first construct the map f and then prove that it is an algebraic morphism.

We first consider the morphism $f_2 : X \to (G^\vee)_{\mathrm{ad}}$ and the short exact sequence

$$1 \to Z^\vee \to G^\vee \xrightarrow{\mathrm{pr}} (G^\vee)_{\mathrm{ad}} \to 1.$$

For any $x \in X$, we take a local section $\mathfrak{s}_x$ of the projection pr at $f_2(x)$, namely, take an open affine neighborhood U_x of $f_2(x)$ together with a morphism $\mathfrak{s}_x : U_x \to V_x \subset G^\vee$, where V_x is some open affine subvariety of $G^\vee$, such that $(\mathrm{pr} \circ \mathfrak{s}_x)(y) = y$ for any $y \in U_x$. Let W_x be an open affine neighborhood of x in X such that $f_2(W_x) \subset U_x$ since f_2 is a morphism.

Write $g_x = \mathfrak{s}_x(f_2(x)) \in G^\vee$. According to the commutative diagram in (2-13), there is some $a_x \in \mathbb{G}_m$ such that

$$f_1(x) = a_x\rho(g_x) = \rho^{\mathfrak{e}}(a_x, g_x), \quad \text{with } (a_x, g_x) \in \mathbb{G}_m \times G^\vee. \tag{2-14}$$

Since the local section is not unique, if we take another local section $\mathfrak{s}'_x$ at $f_2(x)$ with similar open affine neighborhoods, then we have that $g'_x = \mathfrak{s}'_x(x)$ and

$$f_1(x) = a'_x\rho(g'_x) = \rho^{\mathfrak{e}}(a'_x, g'_x), \quad \text{with } (a'_x, g'_x) \in \mathbb{G}_m \times G^\vee.$$

Hence

$$f_1(x) = a_x\rho(g_x) = a'_x\rho(g'_x),$$

which can be rewritten as

$$\rho(g_x(g'_x)^{-1}) = a_x^{-1}a'_x.$$

Since (G, ρ) is an L-pair as in Definition 1.2, and $\mathrm{pr} \circ \rho = \bar{\rho} \circ \mathrm{pr}$ by (2-1), there is some $z \in Z^\vee$ such that $g_x(g'_x)^{-1} = z$, which implies that

$$(a_x, g_x) = (a'_x, g'_x)(\lambda(z)^{-1}, z).$$

Hence the image of $(a_x, g_x) \in \mathbb{G}_m \times G^\vee$ in the quotient group $\mathbb{G}_m \times_{(\lambda^{-1},\mathrm{Id})(Z^\vee)} G^\vee$, which is denoted by $[a_x, g_x]$, does not depend on the choice of the local sections and is uniquely determined by the given $x \in X$. Therefore, we construct a map

$$x \mapsto f(x) = [a_x, g_x] \tag{2-15}$$

from X to the quotient group $\mathbb{G}_m \times_{(\lambda^{-1},\mathrm{Id})(Z^\vee)} G^\vee$. It is clear from the construction that the map f fits into the commutative diagram (2-13). From the construction of the map f, we have

$$g_x = \mathfrak{s}_x(f_2(x)) \quad \text{and} \quad a_x = f_1(x) \cdot \rho(g_x)^{-1} = f_1(x) \cdot \rho(\mathfrak{s}_x(x))^{-1}$$

according to (2-14). Hence the map f is locally explicitly given by

$$x \mapsto f(x) = [a_x, g_x] = [f_1(x) \cdot \rho(\mathfrak{s}_x(x))^{-1}, \mathfrak{s}_x(f_2(x))]$$

which is clearly algebraic and thus, a morphism.

It remains to prove the uniqueness of the morphism $f : X \to \mathbb{G}_m \times_{(\lambda^{-1},\mathrm{Id})(Z^\vee)} G^\vee$ that fits into the commutative diagram in (2-13). Suppose that

$$X \xrightarrow{f'} \mathbb{G}_m \underset{(\lambda^{-1},\mathrm{Id})(Z^\vee)}{\times} G^\vee : x \mapsto [b_x, h_x]$$

is another morphism fitting into the commutative diagram in (2-13), where $b_x \in \mathbb{G}_m$ and $h_x \in G^\vee$. For any $x \in X$, since $\mathrm{pr}(h_x) = [h_x] = f_2(x) \in (G^\vee)_{\mathrm{ad}}$, $h_x \in G^\vee$ is a pre-image of $f_2(x)$ (with a choice of a section). With h_x chosen, b_x must satisfy the equation

$$b_x \rho(h_x) = \rho^{\mathfrak{c}}(b_x, h_x) = f_1(x)$$

according to the commutative diagram in (2-13). From the discussion above, the morphism f is independent of the choice of the local sections. Hence we must have that $f = f'$, which proves the uniqueness of the morphism f.

Finally, it is easy to verify that the kernel of the morphism $\rho_{\mathrm{fp}} : H_\rho^\vee \to \mathrm{GL}_{d_\rho}$ is trivial from the commutative diagram

$$\begin{array}{ccccc}
H_\rho^\vee & & \xrightarrow{\quad\rho_{\mathrm{fp}}\quad} & & \\
& \searrow^{\cong} & & & \\
& & \mathbb{G}_m \times_{(\lambda^{-1},\mathrm{Id})(Z^\vee)} G^\vee & \xrightarrow{\rho^{\mathfrak{c}}} & \mathrm{GL}_{d_\rho} \\
& {\scriptstyle \mathrm{pr}} & \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}} \\
& & (G^\vee)_{\mathrm{ad}} & \xrightarrow{\quad\bar{\rho}\quad} & \mathrm{PGL}_{d_\rho}
\end{array} \tag{2-16}$$

In fact, if $h \in H_\rho^\vee$ belongs to the kernel $\ker(\rho_{\mathrm{fp}})$, then $f(h) = [a, g]$ belongs to the $\ker(\rho^{\mathfrak{c}})$. This means that $1 = \rho^{\mathfrak{c}}(a, g) = a\rho(g)$ and $\rho(g) = a^{-1} \in \mathbb{G}_m$. Since (G, ρ) is an L-pair, we must obtain that $g = z \in Z^\vee$, which implies that $a = \rho(z)^{-1} = \lambda(z)^{-1}$ and $[a, g] = [\lambda(z)^{-1}, z]$ must be the identity element of

$\mathbb{G}_m \times_{(\lambda^{-1},\mathrm{Id})(Z^\vee)} G^\vee$. Therefore, the given element h must be the identity element in $H_\rho^\vee$. We are done with $(H_\rho, \rho_{\mathrm{fp}}) = (G_{\mathfrak{d}}, \rho^{\mathfrak{d}})$ being the candidate for the canonical BKN pair associated with the given L-pair $(G, \rho)_L$. □

The proof of Proposition 2.1 yields the following structure of the group $H_\rho^\vee$.

Corollary 2.2. *The isomorphism* $H_\rho^\vee \overset{\varsigma}{\cong} \mathbb{G}_m \times_{(\lambda^{-1},\mathrm{Id})(Z^\vee)} G^\vee$ *as in the commutative diagram in* (2-16) *is explicitly given by*

$$h \in H_\rho^\vee \mapsto \varsigma(h) = [\rho_{\mathrm{fp}}(h)\cdot \rho(\mathfrak{s}_h(h))^{-1}, \mathfrak{s}_h(\mathrm{pr}(h))] \in \mathbb{G}_m \underset{(\lambda^{-1},\mathrm{Id})(Z^\vee)}{\times} G^\vee,$$

which is independent of the choice of the local section $\mathfrak{s}_h$ *near* $\mathrm{pr}(h)$ *from* $(G^\vee)_{\mathrm{ad}}$ *to* $G^\vee$.

Proposition 2.3. (1) *The group* H_ρ° *defined via* (2-6) *is the derived group of* H_ρ, *i.e.*, $H_\rho^\circ = [H_\rho, H_\rho]$.

(2) *The group* H_ρ, *which is dual to* $H_\rho^\vee$ *as in* (2-2), *has the following properties*:

(i) $H_\rho \cong \mathbb{G}_m \ltimes_\lambda G_0^{\mathrm{sc}}$, *whose multiplication is given as in* (2-17).

(ii) *Its derived group* $H_\rho^\circ = [H_\rho, H_\rho] \cong G_0^{\mathrm{sc}}$ *is simply connected.*

Proof. Under the isomorphism of Corollary 2.2, we may identify $\mathbb{G}_m \times_{(\lambda^{-1},\mathrm{Id})(Z^\vee)} T^\vee$ with a maximal torus of $H_\rho^\vee$. One has an isomorphism

$$\mathbb{G}_m \underset{(\lambda^{-1},\mathrm{Id})(Z^\vee)}{\times} T^\vee \cong \mathbb{G}_m \times (T^\vee)_{\mathrm{ad}} : [a, t] \mapsto (\lambda(t)a, [t]).$$

Let $(X, \Phi, X^\vee, \Phi^\vee)$ be the root datum of G and denote the root lattice, the coroot lattice, and the weight lattice by Q, $Q^\vee$ and P respectively. We may denote by $\alpha \mapsto \bar{\alpha}$ the map $X \to P$, which is dual to $Q^\vee \hookrightarrow X^\vee$, and identify a maximal split torus of $H_\rho^\vee$ with $\mathbb{G}_m \times (T^\vee)_{\mathrm{ad}}$. Then the root datum of $H_\rho^\vee$ is given by

$$(\mathbb{Z} \oplus Q^\vee, \{0\} \oplus \Phi^\vee, \mathbb{Z} \oplus P, (\{\langle \alpha, \lambda\rangle, \alpha) \mid \alpha \in \Phi\}).$$

Associated with the natural morphism $G^\vee \to H_\rho^\vee$, one has the map between the root data given by

$$\mathbb{Z} \oplus Q^\vee \to X^\vee : (n, \alpha^\vee) \mapsto n\lambda + \alpha^\vee,$$

with its dual

$$X \to \mathbb{Z} \oplus P : \alpha \mapsto (\langle \alpha, \lambda\rangle, \bar{\alpha}).$$

On the other hand, let G_0^{sc} be the simply connected cover of G_0. We consider the semi-direct product $\mathbb{G}_m \ltimes_\lambda G_0^{\mathrm{sc}}$ with the multiplication given by

(2-17)
$$(a_1, g_1)(a_2, g_2) = (a_1 a_2, \lambda(a_2)^{-1} g_1 \lambda(a_2) g_2) \quad \text{for } a_1, a_2 \in \mathbb{G}_m, g_1, g_2 \in G_0^{\mathrm{sc}}.$$

Here the highest weight $\lambda \in X^*(T^\vee)$ is used as a cocharacter $\lambda \in X_*(T)$ of T. We may use the same λ to denote its image in $X_*(T_{\mathrm{ad}})$ via the projection from T to T_{ad}.

Since G and G_0^{sc} share the same adjoint group, we have $T_0^{\mathrm{sc}}/Z_0^{\mathrm{sc}} = T_{\mathrm{ad}}$, where T_0^{sc} is the maximal torus of G_0^{sc} and Z_0^{sc} is the center of G_0^{sc}. Hence the formula for the multiplication in (2-17) is well-defined. Take the maximal split torus $\mathbb{G}_m \ltimes_\lambda T_0^{\mathrm{sc}}$ of $\mathbb{G}_m \ltimes_\lambda G_0^{\mathrm{sc}}$. Since $\lambda(\mathbb{G}_m)$ commutes with T_0^{sc}, we have $\mathbb{G}_m \ltimes_\lambda T_0^{\mathrm{sc}} = \mathbb{G}_m \times T_0^{\mathrm{sc}}$. Thus, the root datum of $\mathbb{G}_m \ltimes_\lambda G_0^{\mathrm{sc}}$ with respect to $\mathbb{G}_m \times T_0^{\mathrm{sc}}$ is

$$\big(\mathbb{Z} \oplus P, \{(\langle \alpha, \lambda\rangle, \alpha) \mid \alpha \in \Phi\}, \mathbb{Z} \oplus Q^\vee, \{0\} \oplus \Phi^\vee\big).$$

Denote the covering $G_0^{\mathrm{sc}} \to G_0$ by $g \mapsto \bar{g}$, and consider the homomorphism

$$\mathbb{G}_m \ltimes_\lambda G_0^{\mathrm{sc}} \to G : (a, g) \mapsto \lambda(a)\bar{g}. \tag{2-18}$$

Then the associated map on the root data is given by

$$X \to \mathbb{Z} \oplus P : \alpha \mapsto (\langle \alpha, \lambda \rangle, \bar{\alpha}),$$

with its dual

$$\mathbb{Z} \oplus Q^\vee \to X^\vee : (n, \alpha^\vee) \mapsto n\lambda + \alpha^\vee.$$

This proves that $H_\rho \cong \mathbb{G}_m \ltimes_\lambda G_0^{\mathrm{sc}}$. Hence the derived subgroup $[H_\rho, H_\rho]$ is isomorphic to the derived subgroup of $\mathbb{G}_m \ltimes_\lambda G_0^{\mathrm{sc}}$, which is clearly $\{1\} \times G_0^{\mathrm{sc}}$.

Finally, from the above discussion, the projection

$$\mathbb{G}_m \underset{(\lambda^{-1}, \mathrm{Id})(Z^\vee)}{\times} G^\vee = \mathbb{G}_m \times G^\vee / \{(\lambda(z)^{-1}, z) \mid z \in Z^\vee\} \to (G^\vee)_{\mathrm{ad}} : [a, g] \mapsto [g]$$

is the dual of the embedding

$$G_0^{\mathrm{sc}} \to \mathbb{G}_m \ltimes_\lambda G_0^{\mathrm{sc}} : g \mapsto (1, g),$$

which implies the kernel H_ρ° is also the subgroup $\{1\} \times G_0^{\mathrm{sc}}$. Therefore we prove that $H_\rho^\circ \cong [H_\rho, H_\rho]$ is simply connected. □

By the above discussion with Propositions 2.1 and 2.3, we obtain the following result over any field k of characteristic zero, which implies the first part of Theorem 1.3.

Theorem 2.4. *Let k be a field of characteristic zero. For any given L-pair $(G, \rho)_L$, the pair $(H_\rho, \rho_{\mathrm{fp}})$ of* (2-5) *is the canonical BKN-pair associated with $(G, \rho)_L$.*

2B. ***Local factors.*** Toward the proof of Theorem 1.3,we consider local L-factors and local ϵ-factors in the sense of Langlands over a local field F of characteristic zero.

We are going to prove a relation of the local L-factors and of the local ϵ-factors, whose definitions are given under the assumption that the local Langlands conjecture holds for the relevant group $G(F)$. More precisely, they are defined by

$$L(s, \pi, \rho) := L(s, \rho \circ \phi_\pi) \quad \text{and} \quad \epsilon(s, \pi, \rho, \psi) := \epsilon(s, \rho \circ \phi_\pi, \psi) \tag{2-19}$$

for any nontrivial character $\psi : F \to \mathbb{C}^\times$, if ϕ_π is the local L-parameter of π. Note that this definition of local L-factors is assumption-free when either π is unramified or F is archimedean. When F is non-archimedean and π is ramified, the local Langlands conjecture for (G, F) remains to be proved in general, although many cases have been proved.

To complete the proof of Theorem 1.3, we recall from the fiber product diagram in (2-2) that there is a morphism

$$\eta^{\vee}_{\mathrm{fp}} : G^{\vee} \to H^{\vee}_{\rho}. \tag{2-20}$$

Let $\mathcal{L}_F$ be the local Langlands group of F. Then the local L-parameters of $G(F)$ are the admissible homomorphisms

$$\phi : \mathcal{L}_F \to G^{\vee}(\mathbb{C})$$

up to conjugation by $G^{\vee}(\mathbb{C})$. Let $\Pi_F(G)$ be the set of equivalence classes of irreducible admissible representations of $G(F)$, which are assumed to be of the Casselman-Wallach type if F is archimedean.

By the assumption in Theorem 1.3, the local Langlands conjecture holds for the L-pair $(G, \rho)_L$ and the BKN-pair $(H_\rho, \rho_{\mathrm{fp}})_{\mathrm{BKN}}$ over F. For any $\pi \in \Pi_F(G)$, there is a local Langlands parameter ϕ_π associated with π. It is clear that the composition $\eta^{\vee}_{\mathrm{fp}} \circ \phi_\pi$ is a local Langlands parameter for the group $H_\rho(F)$. Hence by the local Langlands conjecture for $H_\rho(F)$, we obtain that $\eta^{\vee}_{\mathrm{fp}} \circ \phi_\pi$ is the local Langlands parameter for some $\sigma \in \Pi_F(H_\rho)$. In other words, we have the diagram

$$\begin{array}{ccccc} \mathcal{L}_F & \xrightarrow{\phi_\pi} & G^{\vee}(\mathbb{C}) & \xrightarrow{\rho} & \mathrm{GL}_{d_\rho}(\mathbb{C}) \\ & \searrow^{\phi_\sigma} & \downarrow{\scriptstyle \eta^{\vee}_{\mathrm{fp}}} & \nearrow_{\rho_{\mathrm{fp}}} & \\ & & H^{\vee}_{\rho}(\mathbb{C}) & & \end{array} \tag{2-21}$$

with $\phi_\sigma = \eta^{\vee}_{\mathrm{fp}} \circ \phi_\pi$. From the definition of local L-factors in (2-19), we obtain

$$L(s, \pi, \rho) = L(s, \rho \circ \phi_\pi) = L(s, \rho_{\mathrm{fp}} \circ \eta^{\vee}_{\mathrm{fp}} \circ \phi_\pi) = L(s, \rho_{\mathrm{fp}} \circ \phi_\sigma) = L(s, \sigma, \rho_{\mathrm{fp}}).$$

Similarly, we obtain $\epsilon(s, \pi, \rho, \psi) = \epsilon(s, \sigma, \rho_{\mathrm{fp}}, \psi)$ for any nontrivial character $\psi : F \to \mathbb{C}^{\times}$. Combining with Theorem 2.4, this completes the proof of Theorem 1.3.

3. Borel's conjecture

To provide more explicit information about the representation σ for the given π in Theorem 1.3, we recall the following conjecture by A. Borel in [2, 10.3.5].

Conjecture 3.1 (Borel). *Let H be a connected reductive group over a local field F and $\eta : H \to G$ a homomorphism with commutative kernel and cokernel. Denote the induced map on the L-parameters by $\Phi(\eta)$. Then for any $\pi \in \Pi_F(G)$ with an L-parameter ϕ, the pullback $\eta^{*}\pi$ of π is a finite direct sum of irreducible admissible representations $H(F)$ with L-parameter $\Phi(\eta)(\phi)$.*

This conjecture is refined and proved for several cases in [26]. The refined conjecture requires the kernel of $d\eta : \mathrm{Lie}(H) \to \mathrm{Lie}(G)$, the corresponding Lie algebra of H and G, respectively, is central and the cokernel of η is commutative. We intend to show that the morphism

$$\eta_{\mathrm{fp}} : H_\rho \to G \tag{3-1}$$

which is dual to the morphism $\eta_{\mathrm{fp}}^\vee : G^\vee \to H_\rho^\vee$ as defined in (2-2), satisfies this condition.

Lemma 3.2. *The morphism $\eta_{\mathrm{fp}} : H_\rho \to G$ in* (3-1) *enjoys the following properties.*

(1) *The kernel of* $d\eta_{\mathrm{fp}}$ *is central.*

(2) *The cokernel of η_{fp} is a commutative group defined over F.*

Proof. According to the proof of Proposition 2.3, the morphism in (2-18) and Proposition 2.3, the morphism η_{fp} is explicitly given by

$$H_\rho \cong \mathbb{G}_m \ltimes_\lambda G_0^{\mathrm{sc}} \to G : (a, g) \mapsto \lambda(a)\bar{g}.$$

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}_0 := [\mathfrak{g}, \mathfrak{g}]$, which is the Lie algebra of G_0^{sc}. Let $\mathfrak{g}_a$ be the Lie algebra of $\mathbb{G}_m$, then the morphism $d\eta$ is given by

$$d\eta_{\mathrm{fp}} : \mathfrak{g}_a \oplus \mathfrak{g}_0 \to \mathfrak{g} : (A, X) \mapsto d\lambda(A) + X,$$

where $d\lambda : \mathfrak{g}_a \to \mathfrak{g}$ is the differential of $\lambda : \mathbb{G}_m \to G$. Hence we have the kernel of $d\eta$ given by

$$\{(A, -d\lambda(A)) \mid A \in \mathfrak{g}_a\}.$$

According to the multiplication of H_ρ given in Proposition 2.3, the Lie bracket in $\mathfrak{g}_a \oplus \mathfrak{g}_0$ is given as

$$[(A_1, X_1), (A_2, X_2)]_{H_\rho} = (0, -[[d\lambda(A_2), X_1], X_2]),$$

where the brackets on the right-hand side of the equality is the Lie bracket in $\mathfrak{g}$. Then, for any $A \in \mathfrak{g}_a$ and $(A', X') \in \mathfrak{g}_a \oplus \mathfrak{g}_0$, we have

$$[(A, -d\lambda(A)), (A', X')]_{H_\rho} = (0, [[d\lambda(A'), d\lambda(A)], X']) = 0,$$

since $[d\lambda(A'), d\lambda(A)] = d\lambda([A, A']) = 0$, where $[A, A']$ denotes the trivial Lie bracket in $\mathfrak{g}_a$. This proves part (1).

As for part (2), note that $G/\eta_{\mathrm{fp}}(H_\rho) = G/\lambda(\mathbb{G}_m)G_0$, and since G/G_0 is already abelian, we obtain $G/\eta_{\mathrm{fp}}(H_\rho)$ is also abelian and defined over F since every group is defined over F. We are done. □

Assume that the general Borel conjecture holds for the morphism $\eta_{\mathrm{fp}} : H_\rho \to G$, which under the above Condition of the Lemma 3.2 is proved for many cases in [26, Theorem 3]. We deduce that the σ appearing in Theorem 1.3 occurs as a direct summand in the direct sum decomposition of the pull-back of π under the morphism η_{fp}, with expected multiplicity [26, Conjecture 2]. Hence Theorem 1.3 can be improved:

Theorem 3.3. *Assume that the local Langlands conjecture over F holds for a given L-pair $(G, \rho)_L$ and its associated* BKN*-pair $(H_\rho, \rho_{\mathrm{fp}})_{\mathrm{BKN}}$. Assume that the Borel conjecture holds for the morphism $\eta_{\mathrm{fp}} : H_\rho \to G$ as in* (3-1). *Then for any $\pi \in \Pi_F(G)$, there exists a $\sigma \in \Pi_F(H_\rho)$, which occurs in the direct decomposition of the pull-back of π via the morphism η_{fp}, such that*

$$L(s, \pi, \rho) = L(s, \sigma, \rho_{\mathrm{fp}}) \quad \textit{and} \quad \epsilon(s, \pi, \rho, \psi) = \epsilon(s, \sigma, \rho_{\mathrm{fp}}, \psi)$$

for any nontrivial character $\psi : F \to \mathbb{C}^\times$.

4. L-monoids and BKN-pairs

By Theorem 1.3, for any given L-pair $(G, \rho)_L$, in order to understand the Langlands L-function associated with $(G, \rho)_L$, locally and globally, it is enough to work with the associated BKN-pair $(H_\rho, \rho_{\mathrm{fp}})_{\mathrm{BKN}}$, which is the pair given by Definition 1.1. In this section, we may take any BKN-pair $(G, \rho)_{\mathrm{BKN}}$ from Definition 1.1.

As explained in [13], the Schwartz space in the Braverman–Kazhdan proposal may depend on the geometry of the L-monoid $\mathcal{M}$ that contains G as the unit group, which is the open subset of all invertible elements. From [13, Proposition 5.1], one can deduce the existence of such L-monoid $\mathcal{M}^\rho$, following the general theory of M. Putcha and L. Renner [21]. For the BKN-pair $(G, \rho)_{\mathrm{BKN}}$, since the derived group $G_0 = [G, G]$ is simply connected, one may also construct a monoid $\mathcal{M}_\rho$ from the construction of the Vinberg universal monoid [12; 14; 29] that also contains G as the unit group. The goal of this section is to prove Theorem 1.4, which states that these two constructions yield the same monoid.

4A. ***Construction via the Vinberg method.*** We now explain how to construct the Vinberg L-monoid $\mathcal{M}_\rho$ of a BKN pair $(G, \rho)_{\mathrm{BKN}}$.

Take k to be a field of characteristic zero and fix an k-Borel pair (B, T) of G. By Definition 1.1, the derived group $G_0 = [G, G]$ is simply connected. Let $(B_0, T_0) = (B \cap G_0, T \cap G_0)$ be the corresponding k-Borel pair of G_0. Let Z be the center of G and Z_0 the center of G_0. Then we have the adjoint group $G_{\mathrm{ad}} = G/Z = G_0/Z_0$, and in particular, $T_{\mathrm{ad}} = T_0/Z_0 = T/Z$.

Associated with (G_0, B_0, T_0), we take Δ to be the set of simple roots and $\hat{\Delta}$ to be the set of the associated fundamental weights. For each fundamental weight $\omega \in \hat{\Delta}$, denote by ρ_ω the associated irreducible rational representation of G_0 with highest weight ω. Denote by V_ω the underlying space of ρ_ω. Define $G_0^+ := (T_0 \times G_0)/Z_0^\Delta$, where Z_0^Δ denotes the diagonal embedding of the center Z_0 of G_0 into $T_0 \times G_0$.

We may extend the representation ρ_ω of G_0 to G_0^+ by

$$\rho_\omega^+(t, g) = \omega(w_0(t^{-1}))\rho_\omega(g)' for (t, g) \in T_0 \times G_0, \tag{4-1}$$

where w_0 is the longest Weyl element in the Weyl group $W = W(G_0, T_0)$. Each root $\alpha \in \Delta$ can also be extended to G_0^+ via $\alpha^+(t, g) := \alpha(t)$.

Following [29], we construct the Vinberg universal monoid. Define a map ι by composing the representation

$$\Big(\prod_{\alpha\in\Delta} \alpha^+\Big)\times\Big(\prod_{\omega\in\hat{\Delta}} \rho_\omega^+\Big)\colon G_0^+ \to \Big(\prod_{\alpha\in\Delta} \mathbb{G}_m\Big)\times\Big(\prod_{\omega\in\hat{\Delta}} \mathrm{GL}(V_\omega)\Big) \tag{4-2}$$

with the embedding

$$\Big(\prod_{\alpha\in\Delta} \mathbb{G}_m\Big)\times\Big(\prod_{\omega\in\hat{\Delta}} \mathrm{GL}(V_\omega)\Big)\to \Big(\prod_{\alpha\in\Delta} \mathbb{A}^1\Big)\times\Big(\prod_{\omega\in\hat{\Delta}} \mathrm{End}(V_\omega)\Big).$$

Take $\mathcal{M}^+$ to be the normalization of the closure of the image $\iota(G_0^+)$ of G_0^+, which is the *Vinberg universal monoid* associated with the given reductive group G from $(G, \rho)_{\mathrm{BKN}}$. Denote by G^+ the unit group of the monoid $\mathcal{M}^+$. It is clear that $G^+ \times G^+$ acts on $\mathcal{M}^+$ by

$$m(g_1, g_2) = g_1^{-1} m g_2 \quad \text{for } m \in \mathcal{M}^+, (g_1, g_2) \in G^+ \times G^+.$$

Let $G^+_{\mathrm{der}} = [G^+, G^+]$ be the derived group of the unit group G^+. The abelianization morphism from $\mathcal{M}^+$ to the GIT quotient $\mathcal{M}^+/\!\!/(G^+_{\mathrm{der}} \times G^+_{\mathrm{der}})$ by $G^+_{\mathrm{der}} \times G^+_{\mathrm{der}}$ is denoted by

$$\pi^+ : \mathcal{M}^+ \to \mathcal{M}^+/\!\!/(G^+_{\mathrm{der}} \times G^+_{\mathrm{der}}) = \prod_{\alpha\in\Delta} \mathbb{A}^1. \tag{4-3}$$

It is a flat morphism as algebraic varieties with equidimensional reduced fibers [12; 14; 29].

Since the derived group G_0 of G is split and simply connected, its dual group $G_0^\vee$ is of adjoint type. Let $G_0^{\vee,\mathrm{sc}}$ be the simply connected cover of $G_0^\vee$. Then $G_0^{\vee,\mathrm{sc}}$ is the dual group of the adjoint group G_{ad} of G. Let $T_0^\vee$ be the maximal torus of $G_0^\vee$ that is dual to T_0 and $T_0^{\vee,\mathrm{sc}}$ be the maximal torus of $G_0^{\vee,\mathrm{sc}}$ that is dual to the torus T_{ad}. It is clear that $T_{\mathrm{ad}} = T_0/Z_0 = T/Z$ and we have the following exact sequences

$$1 \to Z_0 \to T_0 \to T_0/Z_0 \to 1 \quad \text{and} \quad 1 \to Z_0^{\vee,\mathrm{sc}} \to T_0^{\vee,\mathrm{sc}} \to T_0^\vee \to 1,$$

where $Z_0^{\vee,\mathrm{sc}}$ is the center of $G_0^{\vee,\mathrm{sc}}$ and is the Cartier dual of Z_0. Let $T^\vee$ be the maximal torus of $G^\vee$ that is dual to T. Then $X^*(T) = X_*(T^\vee)$ and $X^*(T^\vee) = X_*(T)$.

Let λ be the highest weight of ρ. In this case, we may write $\rho = \rho_\lambda$. This dominant weight λ can be viewed as a dominant cocharacter in $X_*(T)$ of T. By composing with the canonical map $T \to T/Z = T_0/Z_0 = T_{0,\mathrm{ad}}$, we regard such

$\lambda \in X_*(T)$ as an element in $X_*(T_{0,\mathrm{ad}})$, which is still denoted by λ if it does not cause any confusion. Then we obtain its composition

$$\mathbb{G}_m \to T_0/Z_0 \to \prod_{\alpha\in\Delta} \mathbb{G}_m, \tag{4-4}$$

which extends to $\mathbb{A}^1 \to \prod_{\alpha\in\Delta} \mathbb{A}^1$ and is still denoted by λ. The monoid $\mathcal{M}_\lambda$ associated with λ or the representation ρ is defined to be the fiber product

$$\begin{array}{ccc} \mathcal{M}_\lambda & \longrightarrow & \mathcal{M}^+ \\ \downarrow & & \downarrow{\scriptstyle \pi^+} \\ \mathbb{A}^1 & \xrightarrow{\lambda} & \prod_{\alpha\in\Delta} \mathbb{A}^1 \end{array} \tag{4-5}$$

where the abelianization morphism π^+ is as given in (4-3). The normality of $\mathcal{M}_\lambda$ follows from [22, Proposition 2, Theorem 9].

Let G_λ be the unit group of $\mathcal{M}_\lambda$, which fits into an exact sequence

$$1 \to G_0 \to G_\lambda \overset{m_\lambda}{\to} \mathbb{G}_m \to 1 \tag{4-6}$$

where m_λ is the abelianization morphism of G_λ. More precisely, by the definition of G_0^+ and by (4-2), (4-4) and (4-5), the unit group G_λ fits into the fiber product

$$\begin{array}{ccc} G_\lambda & \longrightarrow & G_0^+ = (T_0 \times G_0)/Z_0^\Delta \\ \downarrow & & \downarrow{\scriptstyle \mathrm{pr}_1} \\ \mathbb{G}_m & \xrightarrow{\lambda} & T_{\mathrm{ad}} = T_0/Z_0 \end{array} \tag{4-7}$$

and the multiplication of the group G_λ is induced from these of $\mathbb{G}_m$ and G_0^+.

Lemma 4.1. *We may write $G_\lambda = \mathbb{G}_m \ltimes_\lambda G_0$, with the multiplication given by*

$$(a_1, g_1)(a_2, g_2) = (a_1a_2, \lambda(a_2)^{-1}g_1\lambda(a_2)g_2) \quad \textit{for } a_1, a_2 \in \mathbb{G}_m,\ g_1, g_2 \in G_0. \tag{4-8}$$

Since $\lambda(a_1) \in T_0/Z_0$, the product $\lambda(a_2)g_1\lambda(a_2)^{-1}$ is well-defined.

Proof. We have an isomorphism of varieties

$$(T_0 \times G_0)/Z_0^\Delta \cong (T_0/Z_0) \times G_0 : [(t, g)] \mapsto ([t], t^{-1}g)$$

with inverse given by $(T_0/Z_0) \times G_0 \ni ([t], g) \mapsto [(t, tg)] \in (T_0 \times G_0)/Z_0^\Delta$. Then the projection $\mathrm{pr}_1 : (T_0 \times G_0)/Z_0^\Delta$ transfers to the canonical projection

$$(T_0/Z_0) \times G_0 \to T_0/Z_0.$$

Then $\mathbb{G}_m \times G_0$ is the fiber product of $(T_0/Z_0) \times G_0$ with $\mathbb{G}_m$ over T_0/Z_0 according to the transitivity of the product (see the second to last paragraph in [7, p. 89], for instance):

(4-9)
$$\begin{array}{ccc} \mathbb{G}_m \times G_0 & \xrightarrow{\lambda\times\mathrm{Id}} & (T_0/Z_0)\times G_0 \\ \big\downarrow & & \big\downarrow \mathrm{pr}_1 \\ \mathbb{G}_m & \xrightarrow{\lambda} & T_{\mathrm{ad}} = T_0/Z_0 \end{array}$$

with morphisms

$$q_1' = \lambda \times \mathrm{Id} : \mathbb{G}_m \times G_0 \to (T_0/Z_0)\times G_0 : (a, g) \mapsto (\lambda(a), g)$$

and

$$q_2 = \mathrm{pr}_1 : \mathbb{G}_m \times G_0 \to \mathbb{G}_m : (a, g) \mapsto a.$$

Moreover, for any variety X with $f_1 = (f_1^{(1)}, f_1^{(2)}) : X \to (T_0/Z_0) \times G_0$ and $f_2 : X \to \mathbb{G}_m$ such that $\mathrm{pr}_1 \circ f_1 = \lambda \circ f_2$, there is a unique morphism $f : X \to \mathbb{G}_m \times G_0$ such that $(\lambda \times \mathrm{Id}) \circ f = f_1$ and $q_2 \circ f = f_2$, which is given by $x \mapsto (f_2(x), f_1^{(2)}(x))$.

Under the isomorphism $(T_0/Z_0)\times G_0 \cong (T_0 \times G_0)/Z_0^{\Delta}$, the morphism q_1' induces the corresponding morphism $q_1 : \mathbb{G}_m \times G_0 \to (T_0 \times G_0)/Z_0^{\Delta}$ is given by $(a, g) \mapsto [(\lambda(a), \lambda(a)g)]$. Then we obtain the fiber product diagram

(4-10)
$$\begin{array}{ccc} \mathbb{G}_m \times G_0 & \xrightarrow{q_1} & (T_0\times G_0)/Z_0^{\Delta} \\ \big\downarrow q_2 & & \big\downarrow \mathrm{pr}_1 \\ \mathbb{G}_m & \xrightarrow{\lambda} & T_{\mathrm{ad}} = T_0/Z_0 \end{array}$$

with $\mathbb{G}_m \times G_0$ as the fiber product. Moreover, using the above identification, for any variety X with $f_1 = (f_1^{(1)}, f_1^{(2)}) : X \to (T_0 \times G_0)/Z_0^{\Delta}$ and $f_2 : X \to \mathbb{G}_m$ such that $\mathrm{pr}_1 \circ f_1 = \lambda \circ f_2$, there is a unique morphism $f : X \to \mathbb{G}_m \times G_0$ such that $q_1 \circ f = f_1$ and $q_2 \circ f = f_2$ and is given by $x \mapsto (f_2(x), f_1^{(1)}(x)^{-1} \cdot f_1^{(2)}(x))$. It remains to determine the multiplication structure explicitly.

For given $(a_1, g_1), (a_2, g_2) \in \mathbb{G}_m \times G_0$, we have

$$q_2((a_1, g_1)) \cdot q_2((a_2, g_2)) = a_1 a_2$$

and

$$\begin{aligned} q_1((a_1, g_1)) \cdot q_1((a_2, g_2)) &= [(\lambda(a_1), \lambda(a_1)g_1)] \cdot [(\lambda(a_2), \lambda(a_2)g_2)] \\ &= [(\lambda(a_1)\lambda(a_2), \lambda(a_1)g_1\lambda(a_2)g_2)], \end{aligned}$$

where the first multiplication is in $\mathbb{G}_m$ and the second multiplication is in $(T_0 \times G_0)/Z_0^{\Delta}$. In this way, we get the morphisms

$$(\mathbb{G}_m \times G_0) \times (\mathbb{G}_m \times G_0) \to \mathbb{G}_m : ((a_1, g_1), (a_2, g_2)) \mapsto a_1 a_2$$

and

$$\begin{aligned} &(\mathbb{G}_m \times G_0) \times (\mathbb{G}_m \times G_0) \to (T_0 \times G_0)/Z_0^{\Delta} \\ &\qquad ((a_1, g_1), (a_2, g_2)) \mapsto [(\lambda(a_1)\lambda(a_2), \lambda(a_1)g_1\lambda(a_2)g_2)], \end{aligned}$$

whose corresponding morphism $(\mathbb{G}_m \times G_0) \times (\mathbb{G}_m \times G_0) \to (\mathbb{G}_m \times G_0)$ is given by

$$(a_1, g_1)\cdot(a_2, g_2) = (a_1a_2, \lambda(a_1a_2)^{-1}\lambda(a_1)g_1\lambda(a_2)g_2) = (a_1a_2, \lambda(a_2)^{-1}g_1\lambda(a_2)g_2) \tag{4-11}$$

according to the last paragraph.

As the fiber product of the fiber product diagram in (4-10), the group $\mathbb{G}_m \times G_0$ with the multiplication given in (4-11) is isomorphic to the group G_λ because of the universality. □

From the proof of Lemma 4.1, it is easy to obtain that when the cocharacter $\lambda : \mathbb{G}_m \to T_0/Z_0$ factors through $\mathbb{G}_m \to T_0 \to T_0/Z_0$, or equivalently, the representation ρ_λ can factor through the adjoint group $G^\vee_{\mathrm{ad}}$, the unit group G_λ is isomorphic to the direct product $\mathbb{G}_m \times G_0$, which is explicitly given by

$$G_\lambda \cong \mathbb{G}_m \times G_0, \quad \text{with } (a, g) \mapsto (a, \lambda(a)g). \tag{4-12}$$

For convenience, we state it as a corollary.

Corollary 4.2. *If the representation ρ_λ can factor through the adjoint group $(G^\vee)_{\mathrm{ad}}$, then the unit group G_λ is isomorphic to the direct product $\mathbb{G}_m \times G_0$, which is explicitly given as in* (4-12).

For any BKN-pair $(G, \rho)_{\mathrm{BKN}}$ with $\rho = \rho_\lambda$, one checks easily that the natural homomorphism

$$G^\vee \to H^\vee_\rho \cong (\mathbb{G}_m \times G^\vee)/\{(\lambda(z)^{-1}, z) \mid z \in Z^\vee\} : g \mapsto [(1, g)]$$

is an isomorphism. By Proposition 2.3 and Lemma 4.1, we obtain the following

Corollary 4.3. *For a given* BKN*-pair $(G, \rho)_{\mathrm{BKN}}$ with $\rho = \rho_\lambda$, the group G is isomorphic to the unit group $G_\lambda = G_\rho = G(\mathcal{M}_\rho)$, where $\mathcal{M}_\rho = \mathcal{M}_\lambda$ is the L-monoid as constructed via the Vinberg method in* (4-5).

4B. ***Construction via the Putcha–Renner method.*** Given a BKN-pair $(G, \rho)_{\mathrm{BKN}}$ with $\rho = \rho_\lambda$ as before, we are going to recall the construction of the monoid $\mathcal{M}^\rho$ via the Putcha–Renner method and show that $G = G(\mathcal{M}^\rho)$, the unit group of the monoid $\mathcal{M}^\rho$.

A series of papers by Putcha [15; 16] and Renner [17; 18; 19; 20] classified the reductive monoid characterized by the toric embedding of maximal tori. The survey [21] is also a good reference. An explicit construction over not necessarily algebraically closed fields is given in [14].

Following the notation in [14], we denote by $\Omega(\rho)$ the set of weights of ρ and by $\xi(\rho)$ the convex cone generated by $\Omega(\rho)$. According to [13, Proposition 5.1], since the central character of ρ is the scalar multiplication, it follows that $\xi(\rho)$ is strictly convex. Let $\xi(\rho)^\vee$ be the sub-monoid of $X^*(T)$ consisting of $\alpha \in X^*(T)$ such that the restriction of α to $\Omega(\rho)$ takes non-negative values. Let $\alpha_1, \dots, \alpha_k \in X^*(T)$ such that their orbits under the Weyl group W generate $\xi(\rho)^\vee$. By replacing α_i by a W-conjugate if necessary, we may assume that each α_i lies in the positive Weyl chamber. Let $\omega_{\alpha_i} : G \to \mathrm{GL}(V_{d_i})$ be the representation of G with α_i being the highest weight, where d_i is the dimension of the representation. Then the monoid $\mathcal{M}^\rho$ is defined as the normalization of the closure of the image G under

$$\prod_{i=1}^k \omega_{\alpha_i} : G \to \prod_{i=1}^k \mathrm{End}(V_{d_i}).$$

According to [14, Proposition 5.1], the monoid $\mathcal{M}^\rho$ is independent of the choice of the generators $\alpha_1, \dots, \alpha_k$ and is characterized by the property that for every maximal torus T of G, the normalization of its closure in $\mathcal{M}^\rho$ corresponds to the strictly convex cone generated by the set of weights $\Omega(\rho)$ in the sense that the ring of regular functions on this toric embedding is $k[\xi(\rho)^\vee]$.

4C. ***Proof of Theorem 1.4.*** For a given BKN-pair $(G, \rho)_{\mathrm{BKN}}$ with $\rho = \rho_\lambda$ as before, we constructed in Section 4A a monoid $\mathcal{M}_\rho$ with $G = G(\mathcal{M}_\rho)$, the unit group of $\mathcal{M}_\rho$, and in Section 4B a monoid $\mathcal{M}^\rho$ with $G = G(\mathcal{M}^\rho)$, the unit group of $\mathcal{M}^\rho$. To prove Theorem 1.4, it is enough to prove that $\mathcal{M}_\rho \cong \mathcal{M}^\rho$.

We regard λ as a dominant element in $X_*(T_{0,\mathrm{ad}}) = X^*(T_0^{\vee,\mathrm{sc}})$ as in Section 4A. Recall from Corollary 4.3 that $G \cong G_\lambda = \mathbb{G}_m \ltimes_\lambda G_0$. When $\lambda = 0$, it is clear that ρ does not satisfy (4) in Definition 1.1. Hence we are able to the condition that $\lambda \neq 0$.

Take the maximal split torus $T_\lambda = \mathbb{G}_m \times T_0$ and recall the commutative diagram

$$\begin{array}{ccccccc}
T_\lambda = \mathbb{G}_m \times T_0 & \longrightarrow & G_\lambda = \mathbb{G}_m \ltimes_\lambda G_0 & \longrightarrow & \mathcal{M}_\rho & \longrightarrow & \mathbb{A}^1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\lambda} \\
T_0^+ = (T_0 \times T_0)/Z_0^\Delta & \longrightarrow & G_0^+ & \longrightarrow & \mathcal{M}^+ & \xrightarrow[\pi^+]{} & \prod_{\alpha\in\Delta} \mathbb{A}^1
\end{array}$$

where every square is Cartesian and the left vertical arrow is given by

$$\mathbb{G}_m \times T_0 \to (T_0 \times G_0)/Z_0^\Delta : (a, t) \mapsto [\lambda(a), \lambda(a)t].$$

Recall $\mathcal{M}^+$ is the normalization of the image of the following map

$$\begin{aligned}\iota : G_0^+ = (T_0 \times G_0)/Z_0^\Delta &\to \Big(\prod_{\alpha\in\Delta} \mathbb{A}^1\Big) \times \Big(\prod_{\omega\in\hat{\Delta}} \mathrm{End}(V_\omega)\Big), \\ [t, g] &\longmapsto \Big(\prod_{\alpha\in\Delta} \alpha(t)\Big) \times \Big(\prod_{\omega\in\hat{\Delta}} \omega(w_0(t^{-1}))\rho_\omega(g)\Big).\end{aligned}$$

Regard ι as a representation of G_0^+ and denote the convex cone generated by all of the weights in ι by $\xi^+ \subset X^*(T_0^+) \otimes \mathbb{R}$. Let $\overline{T}_0^+$ denote the Zariski closure of T_0^+ in $\mathcal{M}^+$. According to [21, Theorem 5.4], $\overline{T}_0^+$ is a normal affine toric variety. It follows from [21, Section 3.3] that $\overline{T}_0^+ = \operatorname{Spec} k[X(\overline{T}_0^+)]$, where the character lattice $X(\overline{T}_0^+)$ of $\overline{T}_0^+$ is given by

$$X(\overline{T}_0^+) = \xi^+ \cap X^*(T_0^+).$$

Let $\overline{T}_\lambda$ denote the Zariski closure of T_λ in $\mathcal{M}_\rho$. Note $\lambda \neq 0$. Then the commutative diagram

$$\begin{array}{ccc} \overline{T}_\lambda & \longrightarrow & \mathbb{A}^1 \\ \big\downarrow & & \big\downarrow{\scriptstyle\lambda} \\ \overline{T}_0^+ & \longrightarrow & \prod_{\alpha\in\Delta} \mathbb{A}^1 \end{array}$$

is Cartesian.

Identify the character lattice $X^*(T_\lambda)$ of $T_\lambda = \mathbb{G}_m \times T_0$ with $\mathbb{Z} \oplus X^*(T_0)$. Denote by $\xi_\lambda \subset X^*(T_\lambda) \otimes_{\mathbb{Z}} \mathbb{R}$ the convex cone generated by the characters

$$(1,0) \quad \text{and} \quad (\langle \omega' - w_0\omega, \lambda\rangle, \omega'), \ \ \omega \in \hat{\Delta}.$$

Here ω' denotes any weight in ρ_ω and $\langle\ ,\ \rangle$ denotes the pairing between $X^*(T_{0,\mathrm{ad}})$ and $X_*(T_{0,\mathrm{ad}})$. Note $\omega' - w_0\omega \in X^*(T_{0,\mathrm{ad}})$, so the pairing is well-defined.

By the Cartesian diagram above, one calculates directly that $\overline{T}_\lambda = \operatorname{Spec} k[X(\overline{T}_\lambda)]$, where the character lattice of $\overline{T}_\lambda$ is given by

$$X(\overline{T}_\lambda) = \xi_\lambda \cap X^*(T_\lambda).$$

On the other hand, by the same arguments as in Proposition 2.3, the Langlands dual group of $G \cong G_\lambda$ is

$$G_\lambda^\vee = \mathbb{G}_m \times_{(\lambda^{-1},\mathrm{Id})(Z_0^{\vee,\mathrm{sc}})} G_0^{\vee,\mathrm{sc}} \cong G^\vee.$$

The isomorphism is induced from $\mathfrak{d}^\vee : \mathbb{G}_m \to G^\vee$ and the homomorphism $G_0^{\vee,\mathrm{sc}} \to G^\vee$, which is dual to the quotient $G \to G_{\mathrm{ad}} = G_{0,\mathrm{ad}}$. Under this isomorphism, we may write the representation ρ of $G^\vee$ as the representation of $G_\lambda^\vee$ given by

$$G_\lambda^\vee = \mathbb{G}_m \times_{(\lambda^{-1},\mathrm{Id})(Z_0^{\vee,\mathrm{sc}})} G_0^{\vee,\mathrm{sc}} \to \mathrm{GL}(V_\rho) : (a,g) \mapsto a\mathrm{I}_{V_\rho} \cdot \rho'_\lambda(g),$$

where ρ'_λ denotes the highest weight representation of $G_0^{\vee,\mathrm{sc}}$ associated with $\lambda \in X^*(T_0^{\vee,\mathrm{sc}})$.

Take the maximal torus $\mathbb{G}_m \times_{(\lambda^{-1},\mathrm{Id})(Z_0^{\vee,\mathrm{sc}})} T_0^{\vee,\mathrm{sc}}$, which has the isomorphism

$$\mathbb{G}_m \times_{(\lambda^{-1},\mathrm{Id})(Z_0^{\vee,\mathrm{sc}})} T_0^{\vee,\mathrm{sc}} \cong \mathbb{G}_m \times (T_0^{\vee,\mathrm{sc}})_{\mathrm{ad}} = \mathbb{G}_m \times T_0^\vee = T_\lambda^\vee :$$
$$[a,t] \mapsto (\lambda(t)a, [t]).$$

Identify the character lattice $X^*(\mathbb{G}_m \times_{(\lambda^{-1},\mathrm{Id})(Z_0^{\vee,\mathrm{sc}})} T_0^{\vee,\mathrm{sc}})$ with $\mathbb{Z} \oplus X^*(T_0^\vee)$. Then the weights of ρ_λ are given by

$$(1, \lambda' - \lambda)$$

where $\lambda' \in X^*(T_0^{\vee,\mathrm{sc}})$ denotes any weight ρ'_λ. Note $\lambda' - \lambda \in X^*(T_0^\vee)$. So $\xi(\rho) \subset X_*(T_\lambda) \otimes_{\mathbb{Z}} \mathbb{R}$ is equal to the convex cone generated by the weights above. Let $\xi(\lambda)^\vee$ be its dual cone.

Let $\overline{T}^\lambda$ denote the Zariski closure of T_λ in $\mathcal{M}^\rho$. Similarly $\overline{T}^\lambda$ is the normal affine toric variety Spec $k[X(\overline{T}^\lambda)]$, where

$$X(\overline{T}^\lambda) = \xi(\lambda)^\vee \cap X^*(T_\lambda) = \xi(\rho)^\vee.$$

We next prove that $\xi_\lambda = \xi(\lambda)^\vee$, which implies that $\overline{T}_\lambda \cong \overline{T}^\lambda$. Then by [21, Theorem 5.4] this isomorphism extends to an isomorphism $\mathcal{M}_\rho \cong \mathcal{M}^\rho$.

Proposition 4.4. *With notation as above, the cones coincide*: $\xi_\lambda = \xi(\lambda)^\vee$.

For the proof, we notice that the isogeny $T_0 \to T_{0,\mathrm{ad}}$ induces isomorphisms

$$X^*(T_{0,\mathrm{ad}}) \otimes_{\mathbb{Z}} \mathbb{R} \cong X^*(T_0) \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{and} \quad X_*(T_0) \otimes_{\mathbb{Z}} \mathbb{R} \cong X_*(T_{0,\mathrm{ad}}) \otimes_{\mathbb{Z}} \mathbb{R}.$$

So we can use the same notation $\langle\ ,\ \rangle$ to denote the pairing between $X^*(T_{0,\mathrm{ad}}) \otimes_{\mathbb{Z}} \mathbb{R} \cong X^*(T_0) \otimes_{\mathbb{Z}} \mathbb{R}$ and $X_*(T_0) \otimes_{\mathbb{Z}} \mathbb{R} \cong X_*(T_{0,\mathrm{ad}}) \otimes_{\mathbb{Z}} \mathbb{R}$. If the context is clear, we may also use $\langle\ ,\ \rangle$ to denote the pairing between

$$X^*(T_\lambda) \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R} \oplus X^*(T_0) \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{and} \quad X_*(T_\lambda) \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R} \oplus X_*(T_0) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Proof. We first prove that $\xi_\lambda \subset \xi(\lambda)^\vee$. By definition,

$$\langle (1,0), (1, \lambda' - \lambda) \rangle = 1 > 0,$$

and

$$\begin{aligned} \big\langle (\langle \omega' - w_0\omega, \lambda\rangle, \omega'), (1, \lambda' - \lambda) \big\rangle &= \langle \omega' - w_0\omega, \lambda\rangle + \langle \omega', \lambda' - \lambda\rangle \\ &= \langle \omega', \lambda'\rangle - \langle w_0\omega, \lambda\rangle. \end{aligned}$$

Any weight in the highest weight representation associated with λ lies in the convex hull of $\{W\lambda\}$. It follows that for a fixed ω', one has

$$\langle \omega', \lambda' \rangle \geq \langle \omega', w\lambda \rangle = \langle w^{-1}\omega', \lambda \rangle$$

for certain $w \in W$, which implies that

$$\big\langle (\langle \omega' - w_0\omega, \lambda\rangle, \omega'), (1, \lambda' - \lambda) \big\rangle \geq \langle w_0(w_0 w^{-1}\omega' - \omega), \lambda \rangle.$$

Since ω is the highest weight in ρ_ω, we have

$$\omega - w_0 w^{-1}\omega' = \sum_{\alpha\in\Delta} n_\alpha \alpha, \quad n_\alpha \geq 0,$$

and

$$w_0(w_0w^{-1}\omega' - \omega) = \sum_{\alpha\in\Delta} n_\alpha(-w_0\alpha).$$

Since $-w_0\Delta = \Delta$ and λ is dominant we obtain $\langle w_0(w_0w^{-1}\omega' - \omega), \lambda\rangle \geq 0$, which implies that $\xi_\lambda \subset \xi(\lambda)^\vee$.

It remains to prove that $\xi(\lambda)^\vee \subset \xi_\lambda$. Suppose that $(a, \mu') \in \xi(\lambda)^\vee$. Then $\mu' = w\mu$ for certain $w \in W$ and dominant $\mu \in X_*(T_0) \otimes_{\mathbb{Z}} \mathbb{R}$. It follows that

$$\langle (a, \mu'), (1, \lambda' - \lambda)\rangle = a + \langle w\mu, \lambda' - \lambda\rangle = a + \langle \mu, w^{-1}\lambda'\rangle - \langle w\mu, \lambda\rangle \geq 0.$$

By taking $\lambda' = ww_0\lambda$, we obtain

$$a \geq \langle w\mu, \lambda\rangle - \langle \mu, w_0\lambda\rangle = \langle w\mu - w_0\mu, \lambda\rangle.$$

Since μ is dominant, we write $\mu = \sum_{\omega\in\hat{\Delta}} m_\omega\omega$ for $m_\omega \geq 0$. This implies

$$(a, \mu') = (a - \langle w\mu - w_0\mu, \lambda\rangle)(1, 0) + \sum_{\omega\in\hat{\Delta}} m_\omega(\langle w\omega - w_0\omega, \lambda\rangle, w\omega) \in \xi_\lambda. \quad \square$$

5. Examples

We provide the explicit BKN-pairs $(H_\rho, \rho_{\mathrm{fp}})_{\mathrm{BKN}}$ for some examples of L-pairs $(G, \rho)_L$ in order to illustrate the discussions above.

5A. ***Symmetric powers of* GL_2.** We take the L-pair $(G, \rho)_L$ to be $(\mathrm{GL}_2, \mathrm{sym}^n)_L$. Since $\rho = \mathrm{sym}^n$, the symmetric n-th power of GL_2, we must have that the highest weight $\lambda_n = n\omega_1$, where ω_1 is the first fundamental weight of $\mathrm{GL}_2(\mathbb{C})$.

In the case $G = \mathrm{GL}_2$, we have $G_0 = \mathrm{SL}_2$. We take T to be the maximal torus of GL_2 consisting of all diagonal matrices, then $T_0 = T \cap \mathrm{SL}_2$ consisting of all diagonal matrices in SL_2, and then $Z_0 = \{\pm \mathrm{I}_2\} \subset T_0$. Since the cocharacter is given by

$$\lambda = \lambda_n : \mathbb{G}_m \to T_0/Z_0 \cong \mathbb{G}_m, \quad x \mapsto x^n.$$

We have $H_\rho = \mathbb{G}_m \times_\lambda \mathrm{SL}_2$. We conclude that $H_\rho \cong \mathbb{G}_m \times \mathrm{SL}_2$ when n is even and $H_\rho \cong \mathrm{GL}_2$ when n is odd, which can be verified as follows.

When n is even, since λ could lift to a cocharacter

$$\lambda : \mathbb{G}_m \to T_0 : x \mapsto \begin{pmatrix} x^{n/2} & 0 \\ 0 & x^{-n/2} \end{pmatrix},$$

by Corollary 4.2, which is a special case of Lemma 4.1, we have $G_\lambda \cong \mathbb{G}_m \times \mathrm{SL}_2$.

When n is odd, there is an algebraic group isomorphism

$$\mathbb{G}_m \ltimes_\lambda \mathrm{SL}_2 \to \mathrm{GL}_2 : (a, g) \mapsto \begin{pmatrix} a^{(n+1)/2} & 0 \\ 0 & a^{(1-n)/2} \end{pmatrix} g$$

with inverse given by

$$g \mapsto \left(\det g, \begin{pmatrix} \det^{(-n-1)/2} g & 0 \\ 0 & \det^{(n-1)/2} g \end{pmatrix} g\right).$$

On the dual side, by Corollary 2.2, we have

$$G_\lambda^\vee = \mathbb{G}_m \times \mathrm{SL}_2/\{(\pm 1)^n, \pm \mathrm{I}_2\}.$$

When n is even, we have $(\pm 1)^n = 1$ and hence $G_\lambda^\vee$ is isomorphic to $\mathbb{G}_m \times \mathrm{PGL}_2$. When n is odd, we have the following isomorphism of algebraic groups:

$$G_\lambda^\vee \cong \mathrm{GL}_2 : (a, g) \mapsto a\mathrm{I}_2 \cdot g. \tag{5-1}$$

Finally, we consider the representation ρ_{fp}. When n is even, the representation is given by

$$\rho_{\mathrm{fp}} = \mathrm{st}_{\mathbb{G}_m} \otimes \mathrm{sym}_n^*, \tag{5-2}$$

where $\mathrm{st}_{\mathbb{G}_m}$ is the standard representation of $\mathbb{G}_m$ acting by scalar, and sym_n^* is the induced symmetric n-th power on PGL_2. When n is odd, according to the isomorphism (5-1), the representation ρ_{fp} is given as follows. For any $g \in \mathrm{GL}_2$, choose $a \in \mathbb{G}_m$ such that $a^2 = \det g$, which implies that $a^{-1}g \in \mathrm{SL}_2$. Hence

$$\rho_{\mathrm{fp}}(g) = a \cdot \mathrm{sym}^n(a^{-1}g). \tag{5-3}$$

Note that this is not the symmetric n-th power of GL_2. We summarize the discussion as a proposition.

Proposition 5.1. *For any integer $n \geq 1$, let $\lambda = n\omega_1$ be the dominant weight for the symmetric n-th power representation ρ_λ of* GL_2. *Then the* BKN*-pair* $(H_\rho, \rho_{\mathrm{fp}})_{\mathrm{BKN}}$ *associated to the given L-pair* $(\mathrm{GL}_2, \rho_\lambda)_L$ *is explicitly given as follows.*

(1) *The group H_ρ is given by*

$$H_\rho = \mathbb{G}_m \ltimes_\lambda \mathrm{SL}_2 \cong \begin{cases} \mathbb{G}_m \times \mathrm{SL}_2 & \text{if } n \text{ is even,} \\ \mathrm{GL}_2 & \text{if } n \text{ is odd.} \end{cases}$$

(2) *The dual group $H_\rho^\vee$ is given by*

$$H_\rho^\vee \cong \begin{cases} \mathbb{G}_m \times \mathrm{PGL}_2 & \text{if } n \text{ is even,} \\ \mathrm{GL}_2 & \text{if } n \text{ is odd.} \end{cases}$$

(3) *The representation ρ_{fp} is given by*

$$\rho_{\mathrm{fp}} = \begin{cases} \mathrm{st}_{\mathbb{G}_m} \otimes \mathrm{sym}_n^* \text{ as in (5-2)} & \text{if } n \text{ is even,} \\ \rho_{\mathrm{fp}}(g) = a \cdot \mathrm{sym}^n(a^{-1}g) \text{ as in (5-3)} & \text{if } n \text{ is odd.} \end{cases}$$

(4) *The generalized determinant* $\eth_{\mathrm{fp}} : H_\rho \to \mathbb{G}_m$ *is given by*

$$H_\rho \cong \begin{cases} \mathbb{G}_m \times \mathrm{SL}_2 : (a, g) \mapsto a & \text{if } n \text{ is even,} \\ \mathrm{GL}_2 : g \mapsto \det g & \text{if } n \text{ is odd.} \end{cases}$$

We also refer to [10; 24; 25] for discussions of this case.

5B. ***The standard representation of*** $\mathbf{GL}_n$**.** In this case, the L-pair is $(\mathrm{GL}_n, \rho_\lambda)_L$, with ρ_λ being the standard representation of GL_n associated to the first fundamental weight ω_1. To explicate the associated BKN-pair $(H_\rho, \rho_{\mathrm{fp}})_{\mathrm{BKN}}$, we take T to be the maximal torus of GL_n consisting of all diagonal matrices, and Z be the center of GL_n consisting of all scalar matrices. Then $T_0 = T \cap \mathrm{SL}_n$ and $Z_0 = Z \cap \mathrm{SL}_n = \mu_n$, where μ_n is the group of the n-th roots of unity. Consider the isomorphism

$$T_0/Z_0 \cong T/Z \cong \mathbb{G}_m^{n-1} : \mathrm{diag}(t_1, \dots, t_n) \mapsto \left(\frac{t_1}{t_n}, \dots, \frac{t_{n-1}}{t_n}\right).$$

The highest weight $\lambda = \omega_1$ of the standard representation defines a cocharacter

$$\lambda : \mathbb{G}_m \to T/Z \cong \mathbb{G}_m^{n-1}, \quad t \mapsto \mathrm{diag}(t, 1, \dots, 1) \mapsto (t, 1, \dots, 1).$$

Then $H_\rho = \mathbb{G}_m \ltimes_\lambda \mathrm{SL}_n$, with group law given by

$$(\mathbb{G}_m \ltimes_\lambda \mathrm{SL}_n) \times (\mathbb{G}_m \ltimes_\lambda \mathrm{SL}_n) \to \mathbb{G}_m \ltimes_\lambda \mathrm{SL}_n :$$
$$((a, g), (a', g')) \mapsto (aa', \mathrm{diag}((a')^{-1}, 1, \dots, 1) g \cdot \mathrm{diag}(a', 1, \dots, 1) \cdot g').$$

As algebraic groups, we have the isomorphism

$$\mathbb{G}_m \ltimes_\lambda \mathrm{SL}_n \cong \mathrm{GL}_n : (a, g) \mapsto \mathrm{diag}(a, 1, \dots, 1) \cdot g,$$

with the inverse given by

$$\mathrm{GL}_n \ni g \mapsto (\det g, \mathrm{diag}(\det{}^{-1} g, 1, \dots, 1) \cdot g) \in \mathbb{G}_m \ltimes_\lambda \mathrm{SL}_n.$$

On the dual side, we have

$$\begin{aligned} G_\lambda^\vee &= \mathbb{G}_m \times \mathrm{SL}_n / \{(\omega_\lambda(z)^{-1}, z) \mid z \in \mu_n \subset \mathrm{SL}_n\} \\ &= \mathbb{G}_m \times \mathrm{SL}_n / \{(z^{-1}, z) \mid z \in \mu_n \subset \mathrm{SL}_n\}, \end{aligned}$$

which is isomorphic to GL_n via the map

$$G_\lambda^\vee \to \mathrm{GL}_n : [(a, g)] \mapsto ag.$$

Finally, according to the construction of ρ_{fp} in this case, it is still the standard representation of GL_n.

To summarize the discussion:

Proposition 5.2. *Let λ be the highest weight for the standard representation ρ_λ of GL_n. Then the* BKN*-pair $(H_\rho, \rho_{\mathrm{fp}})_{\mathrm{BKN}}$ associated to the L-pair $(\mathrm{GL}_n, \rho_\lambda)_L$ is explicitly given as follows.*

(1) *The group H_ρ is given by*

$$H_\rho = \mathbb{G}_m \ltimes_\lambda \mathrm{SL}_n \cong \mathrm{GL}_n.$$

(2) *The dual group $H_\rho^\vee$ is given by*

$$H_\rho^\vee \cong \mathrm{GL}_n.$$

(3) *The representation ρ_{fp} is given by*

$$\rho_{\mathrm{fp}} = \rho_\lambda = \mathrm{st}_{\mathrm{GL}_n},$$

the standard representation of GL_n.

(4) *The generalized determinant $\mathfrak{d}_{\mathrm{fp}} : H_\rho \to \mathbb{G}_m$ is given by*

$$H_\rho \cong \mathrm{GL}_n \to \mathbb{G}_m : g \mapsto \det(g).$$

5C. ***The adjoint representation of $\mathbf{GL}_n$.*** Let $G = \mathrm{GL}_n$ and ρ_λ be the adjoint representation of $\mathrm{GL}_n(\mathbb{C})$ and λ be its highest weight. In this case, we have $G_0 = \mathrm{SL}_n$. Take T and T_0 as in Section 5B.

Since the character λ can be lifted to a cocharacter

$$\mathbb{G}_m \to T_0 : a \mapsto \mathrm{diag}(a, 1, \dots, 1, a^{-1}),$$

by Corollary 4.2, which is a special case of Lemma 4.1, we obtain $H_\rho = \mathbb{G}_m \times \mathrm{SL}_n$.

On the dual side, we have

$$H_\rho^\vee = \mathbb{G}_m \times \mathrm{SL}_n / \{(\omega_\lambda(z))^{-1}, z) \mid z \in \mu_n \subset \mathrm{SL}_n\} \cong \mathbb{G}_m \times (\mathrm{SL}_n/\mu_n),$$

where μ_n is the finite group scheme of n-th roots of unity, and the last equality is because $\omega_\lambda(z) = 1$ for all $z \in \mu_n$. As algebraic groups, we obtain $G_\lambda^\vee = \mathbb{G}_m \times \mathrm{PGL}_n$.

Finally the corresponding representation is $\rho_{\mathrm{fp}} = \mathrm{st}_{\mathbb{G}_m} \otimes \mathrm{Ad}$, where Ad is the adjoint representation of PGL_n induced from that of SL_n. To summarize:

Proposition 5.3. *Let λ be the highest weight for the adjoint representation ρ_λ of GL_n. Then the* BKN*-pair $(H_\rho, \rho_{\mathrm{fp}})_{\mathrm{BKN}}$ associated to the L-pair $(\mathrm{GL}_n, \rho_\lambda)_L$ is explicitly given as follows.*

(1) *The group H_ρ is given by*

$$H_\rho = \mathbb{G}_m \ltimes_\lambda \mathrm{SL}_n \cong \mathbb{G}_m \times \mathrm{SL}_n.$$

(2) *The dual group* $H_\rho^\vee$ *is given by*

$$H_\rho^\vee \cong \mathbb{G}_m \times \mathrm{PGL}_n.$$

(3) *The representation* ρ_{fp} *is given by*

$$\rho_{\mathrm{fp}} = \mathrm{st}_{\mathbb{G}_m} \otimes \mathrm{Ad},$$

where Ad *is the adjoint representation of* PGL_n *induced from that of* SL_n.

(4) *The generalized determinant* $\mathfrak{d}_{\mathrm{fp}} : H_\rho \to \mathbb{G}_m$ *is given by*

$$\mathbb{G}_m \times \mathrm{SL}_n \to \mathbb{G}_m : (a, g) \mapsto a.$$

5D. ***The symmetric square representation of*** $\mathbf{GL_n}$**.** In this case, we consider the L-pair $(\mathrm{GL}_n, \rho_\lambda)_L$ with $G = \mathrm{GL}_n$ and ρ_λ being the symmetric square representation $\mathrm{GL}_n \to \mathrm{GL}_{n(n+1)/2}$, associated to the highest weight $2\omega_1$. Take T and T_0 as in Section 5B.

The highest weight $\lambda = 2\omega_1 \in X^*(T^\vee) = X_*(T)$ of the symmetric square representation of $G^\vee$ gives a cocharacter

$$\lambda : \mathbb{G}_m \to T, \quad a \mapsto \mathrm{diag}(a^2, 1, \dots, 1).$$

Similar to the previous case, we can write H_ρ as the image of the following homomorphism

$$\mathbb{G}_m \ltimes_\lambda \mathrm{SL}_n \to \mathbb{G}_m \times \mathrm{GL}_n : (a, g) \mapsto (a, \mathrm{diag}(a^2, 1, \dots, 1) \cdot g)$$

which is clearly equal to

$$\{(a, g) \in \mathbb{G}_m \times \mathrm{GL}_n \mid \det(g) = a^2\}.$$

More specifically, when $n = 2l$, we have the following isomorphism

$$\mathbb{G}_m \times \mathrm{SL}_n/\{(z^{-1}, z\mathrm{I}_n \mid z \in \mu_l\} \cong H_\rho, \quad [(a, g)] \mapsto (a^l, g \cdot a\mathrm{I}_n),$$

which is the same as in [23].

On the dual side, we have

$$H_\rho^\vee = \mathbb{G}_m \times \mathrm{SL}_n/\{(z^{-2}, z\mathrm{I}_n) \mid z \in \mu_n\},$$

with the isomorphisms

$$\mathbb{G}_m \times \mathrm{SL}_n/\{(z^{-2}, z\mathrm{I}_n) \mid z \in \mu_n\} \cong \mathbb{G}_m \times \mathrm{GL}_n/\{(z^{-2}, z\mathrm{I}_n) \mid z \in \mathbb{G}_m\} : [(a, g)] \mapsto [(a, g)]$$

and

$$\mathrm{GL}_n/\{\pm \mathrm{I}_n\} \cong \mathbb{G}_m \times \mathrm{GL}_n/\{(z^{-2}, z\mathrm{I}_n) \mid z \in \mathbb{G}_m\} : [g] \mapsto [(1, g)].$$

Then the unique morphism $\mathrm{GL}_n \to H_\rho^\vee$ is given by the central isogeny

$$\mathrm{GL}_n \to \mathrm{GL}_n/\{\pm \mathrm{I}_n\} : g \mapsto [g].$$

Since the symmetric square representation has its kernel equal to $\{\pm \mathrm{I}_n\}$, the given representation ρ_λ descents to a representation ρ^{n} of $\mathrm{GL}_n/\{\pm \mathrm{I}_n\}$.

Dual to the morphism $G^\vee \to G_\lambda^\vee$, we have the morphism $G_\lambda \to G$, which is explicitly given by

$$\{(a, g) \in \mathbb{G}_m \times \mathrm{GL}_n \mid \det(g) = a^2\} \to \mathrm{GL}_n : (a, g) \mapsto g.$$

To summarize:

Proposition 5.4. *Let $\lambda = 2\omega_1$ be the highest weight for the symmetric square representation ρ_λ of* GL_n. *Then the* BKN*-pair $(H_\rho, \rho_{\mathrm{fp}})_{\mathrm{BKN}}$ associated to the given L-pair $(\mathrm{GL}_n, \rho_\lambda)_L$ is explicitly given as follows.*

(1) *The group H_ρ is given by*

$$H_\rho = \mathbb{G}_m \ltimes_\lambda \mathrm{SL}_n \cong \{(a, g) \in \mathbb{G}_m \times \mathrm{GL}_n \mid \det(g) = a^2\}$$

(2) *The dual group $H_\rho^\vee$ is given by*

$$H_\rho^\vee \cong \mathrm{GL}_n/\{\pm \mathrm{I}_n\}$$

(3) *The representation ρ_{fp} is given by the descending of ρ_λ from* GL_n *to $H_\rho^\vee$.*

(4) *The generalized determinant $\mathfrak{d}_{\mathrm{fp}} : H_\rho \to \mathbb{G}_m$ is given by*

$$H_\rho \cong \{(a, g) \in \mathbb{G}_m \times \mathrm{GL}_n \mid \det(g) = a^2\} \to \mathbb{G}_m : (a, g) \mapsto a.$$

5E. ***The doubling case of*** $\mathbf{Sp_{2n}}$**.** In this case, the given L-pair is $(\mathrm{Sp}_{2n}, \rho_\lambda)_L$ with $G = \mathrm{Sp}_{2n} = G_0$. Let $T = T_0$ be a maximal torus of G. Let ρ_λ be the standard representation of SO_{2n+1} with λ its highest weight. Since $T = T_0$ and λ is already a cocharacter of T_0, by Corollary 4.2, which is a special case of Lemma 4.1, we have in this case $H_\rho = \mathbb{G}_m \times \mathrm{Sp}_{2n}$. On the dual side,

$$H_\rho^\vee = \mathbb{G}_m \times \mathrm{Spin}_{2n+1}/\{(1, \pm \mathrm{I}_{2n+1})\} \cong \mathbb{G}_m \times \mathrm{SO}_{2n+1}.$$

Finally due to the construction of ρ_{fp}, we see in this case that $\rho_{\mathrm{fp}} = \mathrm{st}_{\mathbb{G}_m} \otimes \mathrm{st}_{\mathrm{SO}_{2n+1}}$, where $\mathrm{st}_{\mathbb{G}_m}$ is the standard representation of $\mathbb{G}_m$ and $\mathrm{st}_{\mathrm{SO}_{2n+1}}$ is the standard presentation of SO_{2n+1}. In summary:

Proposition 5.5. *Let λ be the highest weight for the standard representation ρ_λ of* SO_{2n+1}. *Then the* BKN*-pair $(H_\rho, \rho_{\mathrm{fp}})_{\mathrm{BKN}}$ associated to the L-pair $(\mathrm{Sp}_{2n}, \rho_\lambda)_L$ is explicitly given as follows.*

(1) *The group H_ρ is given by*

$$H_\rho = \mathbb{G}_m \ltimes_\lambda \mathrm{Sp}_{2n} \cong \mathbb{G}_m \times \mathrm{Sp}_{2n}.$$

(2) *The dual group $H_\rho^\vee$ is given by*

$$H_\rho^\vee \cong \mathbb{G}_m \times \mathrm{SO}_{2n+1}.$$

(3) *The representation ρ_{fp} is given by*

$$\rho_{\mathrm{fp}} = \mathrm{st}_{\mathbb{G}_m} \otimes \mathrm{st}_{\mathrm{SO}_{2n+1}},$$

where $\mathrm{st}_{\mathbb{G}_m}$ is the standard representation of $\mathbb{G}_m$ and $\mathrm{st}_{\mathrm{SO}_{2n+1}}$ is the standard presentation of SO_{2n+1}.

(4) *The generalized determinant $\mathfrak{d}_{\mathrm{fp}} : H_\rho \to \mathbb{G}_m$ is given by*

$$H_\rho \cong \mathbb{G}_m \times \mathrm{Sp}_{2n} \to \mathbb{G}_m : (a, g) \mapsto a.$$

This example explains that in [8], the authors must take the group $\mathbb{G}_m \times \mathrm{Sp}_{2n}$ for the local theory of the Braverman–Kazhdan–Ngô proposal in the case of the standard L-functions of Sp_{2n}.

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