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ON THE PRIMITIVE IDEAL SPACE OF RADIAL REPRESENTATIONS OF FREE GROUPS

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Dedicated to the memory of Shukoh Ishikawa

Radial representations of finitely generated free groups are studied. The associated C*-algebra is located between the reduced and full group C*-algebras and its primitive ideal space is described concretely as a topological space.

Introduction

Spherical functions on finitely generated free groups have been investigated from various points of view. Among them, fundamental is the framework of Poisson boundaries, which allows us to take analogies with the case of semisimple Lie groups. In fact, studies of spherical functions go more or less around it and, as their basic properties, irreducibility and inequivalence of the associated spherical representations are established under this background.

Other than spherical representations, a series of nonirreducible representations are associated with radial functions. Within that class interesting are positive definite functions of Haagerup, which include the standard trace as a limit case. For nonirreducible representations associated with positive definite radial functions, a generalization of Plancherel formula is described in [22] via spectral decomposition of radial functions based on the radial algebra which is commutative. Notice here that, different from the semisimple case, noncommutative free groups are not type I and the uniqueness of decomposition into irreducible representations breaks down.

In other words, to obtain a Plancherel formula, we need to specify a maximal commutative subalgebra first and the above-mentioned Plancherel formula is based on the algebra of radial functions. As an instance, the standard trace belongs to the radial class and its Plancherel measure turns out to be the so-called Kesten measure, which is supported by the regular spectrum and equivalent to the Lebesgue measure.

In this paper, we look into the primitive ideal space of radial representations and show that it is given by primitive ideals of spherical representations. Furthermore,

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its hull-kernel topology is described explicitly in terms of the spectral parameter which distinguishes equivalence classes of spherical representations.

During the process of description, we also see that the kernel of the regular representation is represented by a CCR algebra under the universal radial representation and a complete description of the primitive ideal space of the radial representation is obtained in terms of primitive ideals of this CCR algebra.

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1. Background review

We shall work with the countable discrete group G generated freely by a finite set $\{s_1, \dots, s_l\}$ and representations of G by bounded linear operators on a Hilbert space $\mathcal{H}$. By freeness, such a representation is specified by assigning a finite family $(S_i)_{1\leq i\leq l}$ of bounded invertible operators on $\mathcal{H}$ so that a unitary representation corresponds to a family of unitary operators.

Let $\mathbb{C}G = \sum_{g\in G} \mathbb{C}g$ be the algebraic group algebra of G, which is naturally identified with the convolution algebra of functions on G having finite supports. The completion $\ell^1(G)$ of $\mathbb{C}G$ with respect to the ℓ^1-norm consists of summable sequences labeled by elements in G so that the convolution product makes $\ell^1(G)$ a unital Banach algebra.

The *-operation $g \mapsto g^* = g^{-1}$ on $\mathbb{C}G$ is obviously extended to $\ell^1(G)$ isometrically. Thus unitary representations of G are in one-to-one correspondence with *-representations of $\mathbb{C}G$ or $\ell^1(G)$ on Hilbert spaces. The associated universal C*-algebra is called the group C*-algebra of G and denoted by $C^*(G)$. The *parity automorphism* ϖ of $C^*(G)$ is defined by $\varpi(g) = (-1)^{|g|}g$, where $|g|$ denotes the length of $g \in G$ with respect to the generator $\{s_1, \dots, s_l\}$.

Among *-representations, the regular representation on $\ell^2(G)$ plays a central role in what follows: the left regular representation is denoted by λ. The image $\lambda(C^*(G))$, also denoted by $C^*_r(G)$, is the reduced group C*-algebra and known to be simple ([13]). Recall that the regular representation is related with the standard trace τ on $C^*(G)$ specified by $\tau(g) = \delta_{g,e}$ for $g \in \mathbb{C}G \subset C^*(G)$: Let $\tau^{1/2}$ be the GNS vector of τ. Then, for $f = \sum_{g\in G} f(g)g \in \mathbb{C}G$, $f\tau^{1/2} = \tau^{1/2} f$ is identified with the function $f \in \ell^2(G)$ so that $\lambda(f)$ is realized by left multiplication of f on the GNS representation space of τ.

The parity automorphism ϖ is implemented on $C^*_r(G)$ by the parity operator P on $\ell^2(G)$ defined by $P(g\tau^{1/2}) = \varpi(g)\tau^{1/2}$.

We shall now review (more or less well-known) relevant facts on spherical representations of free groups together with some comments.

Spherical functions on free groups are introduced and studied as counterparts of the semisimple case (see [3; 6; 9; 16] for example).

A function of $g \in G$ is said to be *radial* if it depends only through the length $|g|$. Radial functions supported by finite sets constitute a commutative *-subalgebra $\mathcal{A}$ of $\mathbb{C}G$. Let $G_n = \{g \in G; |g| = n\}$ with its indicator function denoted by 1_{G_n}. Note that the number of elements in G_n is $|G_n| = 2l(2l-1)^{n-1}$ $(n \geq 1)$. As a linear base of $\mathcal{A}$, elementary radial functions are introduced by

$$h_n = \frac{1}{|G_n|} 1_{G_n} \quad (n = 0, 1, 2, \dots),$$

which are hermitian $h_n^* = h_n$ with h_0 the unit element of $\mathbb{C}G$ and fulfill the recurrence relation

$$h_1 h_n = r h_{n-1} + (1-r) h_{n+1} \quad (n \geq 1)$$

with $r = 1/2l$ satisfying $0 < r \leq 1/2$.

If we introduce a polynomial sequence $p_n(t)$ of indeterminate t by $p_0(t) = 1$, $p_1(t) = t$ and $t p_n(t) = r p_{n-1}(t) + (1-r) p_{n+1}(t)$ $(n \geq 1)$, then $h_n = p_n(h_1)$ and

$$\sum_{n=0}^{\infty} p_n(t) z^n = \frac{1 - r - rtz}{1 - r - tz + rz^2}.$$

Consequently the correspondence $p_n(t) \mapsto h_n$ gives rise to an algebra homomorphism of $\mathbb{C}[t]$ onto $\mathcal{A}$, which is in fact an isomorphism because $\mathcal{A}$ is infinite-dimensional.

The spectrum of $\lambda(h_1)$ is exactly

$$\sigma_r = [-2\sqrt{r(1-r)}, 2\sqrt{r(1-r)}]$$

(called the *regular spectrum*) without eigenvalues inside ([8; 4]).

Let $E : \mathbb{C}G \to \mathcal{A}$ be the averaging map defined linearly by $E(g) = h_{|g|}$ $(g \in G \subset \mathbb{C}G)$, which satisfies algebraic properties of conditional expectation: $E(a) = a$, $E(f^*) = E(f)^*$ and $E(af) = aE(f)$ for $a \in \mathcal{A}$ and $f \in \mathbb{C}G$. A radial function ϕ is then in one-to-one correspondence with a linear functional φ of $\mathcal{A}$ by the relation $\phi(g) = \varphi \circ E(g)$ $(g \in G)$. The averaging map E is also characterized by the equality $\tau(f\phi) = \tau(E(f)\phi)$ for $f \in \mathbb{C}G$ and a radial function ϕ.

Given a complex number $c \in \mathbb{C}$, let δ_c be the multiplicative linear functional of $\mathcal{A}$ evaluated at c, i.e., $\delta_c(h_n) = p_n(c)$ $(n \geq 0)$. When $\varphi = \delta_c$, the associated radial function ε_c is called a *spherical function* (this being justified by a characterization in [6, Lemma 1.5]) with c referred to as the *spectral parameter*.

A multiplicative functional δ_c of spectral parameter $c \in \mathbb{C}$ is ℓ^1-bounded if and only if $c = a + ib$ $(a, b \in \mathbb{R})$ is in the elliptic disk

$$a^2 + \frac{b^2}{(1-2r)^2} \leq 1$$

(when $r = 1/2$, this shrinks to the interval $[-1, 1] \subset \mathbb{R}$) and δ_c is C*-bounded if and only if $-1 \le c \le 1$ ([3; 14]). Notice that C*-bounded multiplicative functionals are automatically *-preserving.

With this spectral property of c in hand, we see that a spherical function ε_c of spectral parameter c is positive definite if and only if $-1 \le c \le 1$.

Let A be the closure of $\mathcal{A}$ in the full group C*-algebra $C^*(G)$. Then, by Haagerup (see [20; 22]), E is extended to a conditional expectation (also denoted by E) of $C^*(G)$ onto $A \subset C^*(G)$.

Since E preserves positivity, any positive functional φ of A induces a positive functional $\varphi \circ E$ of $C^*(G)$. If we take an evaluation functional δ_t on A at $t \in [-1, 1]$ as a φ, it gives rise to a spherical state on $C^*(G)$, which is also denoted by ε_t, i.e., $\varepsilon_t = \delta_t \circ E$ and the spectrum of A is identified with $[-1, 1]$, i.e., $A \cong C([-1, 1])$ ($C(K)$ standing for the continuous function algebra of a compact set K).

Remark that the spherical state ε_t is the obvious extension of a positive definite spherical function of spectral parameter $-1 \le t \le 1$ with $\epsilon_{\pm 1}(g) = (\pm 1)^{|g|}$ multiplicative on G, i.e., the trivial/parity character (one-dimensional representation) of G.

The GNS-representation of ε_t is called a *spherical representation.*

Theorem 1.1 [6; 10; 20]. *Spherical representations are irreducible and mutually disjoint for different $t \in [-1, 1]$.*

Spherical representations of spectral parameter in σ_r appear as irreducible components of the regular representation in a form of Plancherel formula and are referred to as being *principal*, whereas ones parametrized by $[-1, 1] \setminus \sigma_r$ are said to be *complementary*.

As will be reviewed below, these series of representations (except for the residual values $t = \pm 1$) are realized as an analytic family of representations on a single Hilbert space.

We now introduce another analytic family (λ_z) of representations on $\ell^2(G)$ due to Pytlik and Szwarc. (The original notation is changed to λ_z in view of the fact that this is a deformation of λ.) Here are basic properties: (λ_z) is a family of bounded representations of G on $\ell^2(G)$ parametrized by a complex number z satisfying $z^2 \notin (1, \infty)$.

Theorem 1.2 [16]. (i) *For each $g \in G$, $\lambda_z(g)$ is a finite-rank perturbation of $\lambda(g)$: there is a finite dimensional subspace L_g of $\ell^2(G)$ satisfying $\lambda_z(g) = \lambda(g)$ on $L_g^\perp$ for each z.*

(ii) *For any $g \in G$, $\lambda_z(g)$ is continuous in z and holomorphic on*

$$\{z \in \mathbb{C} \setminus ((-\infty, -1] \sqcup [1, \infty))\}.$$

(iii) *The equality* $\lambda_z(g)^* = \lambda_{\bar{z}}(g^{-1})$ *holds for any* $g \in G$ *and any* z. *Consequently* λ_z *is unitary if* z *is real*, *i.e.*, *if* $z \in [-1, 1]$.

(iv) *For* $z^2 \notin [1, \infty)$, $\tau^{1/2}$ *is a cyclic vector of* λ_z *and satisfies*

$$(\tau^{1/2}|\lambda_z(g)\tau^{1/2}) = z^{|g|} \quad (g \in G).$$

In particular, *the function* $z^{|g|}$ *is positive definite for* $-1 \le z \le 1$ (*see* [7]) *and* $\lambda_0 = \lambda$.

(v) *Limits* $\lambda_{\pm 1} = \lim_{z \to \pm 1} \lambda_z$ *are unitary representations of* G *on* $\ell^2(G)$ *satisfying*

$$\lambda_{\pm 1}(a)g\tau^{1/2} = \begin{cases} \pm\tau^{1/2} & (g = e), \\ \mp a\tau^{1/2} & (g = a^{-1}), \\ ag\tau^{1/2} & \textit{otherwise}, \end{cases}$$

for $a \in G_1$ *and* $g \in G$.

Corollary 1.3. *According to an orthogonal decomposition* $\ell^2(G) = \mathbb{C}\tau^{1/2} \oplus \ell^2(G(1)) \oplus \cdots \oplus \ell^2(G(l))$, *where* $G(i)$ *consists of words whose right ends are in* $\{s_i, s_i^{-1}\}$, $\lambda_{\pm 1}$ *is unitarily equivalent to a direct sum of* λ *by multiplicity* l *and the parity/trivial character of* G.

Proof. Let $G^* = G \setminus \{e\}$. Since $\lambda_{-1}(g)(G^*\tau^{1/2}) = G^*\tau^{1/2}$ for $g \in G$ by (v), $\lambda_{-1}(g)$ induces a free action of G on G^* with $G^* = G(1) \sqcup \cdots \sqcup G(l)$ the decomposition of G^* into orbits. Thus λ_{-1} on $\ell^2(G^*)$ is decomposed into the direct sum of subrepresentations on $\ell^2(G(i))$ $(1 \le i \le l)$ so that the restriction of λ_{-1} on $\ell^2(G(i))$ is unitarily equivalent to the regular representation of G.

Likewise λ_1 and the parity operator P leave $\ell^2(G(i))$ invariant so that $P\lambda_1(x)P = \lambda_{-1}(\varpi(x))$ $(x \in C^*(G))$. From the decomposition of λ_{-1}, this implies that λ_1 on $\ell^2(G(i))$ is unitarily equivalent to $\lambda \circ \varpi = P\lambda P \cong \lambda$ as well. □

Remark 1. The above corollary is taken from [16] 2.4 Remark (3), where it is pointed out that λ_{-1} on $\ell^2(G^*)$ is considered by J. Cuntz to illustrate K-amenability of free groups.

The following supplements to [16] are extracted from [19; 20] with the case of critical values $\upsilon = \pm\sqrt{r/(1-r)}$ added in [22, Corollary 4.4].

Theorem 1.4. *Let* λ_υ *be a unitary representation*, *i.e.*, $\upsilon \in [-1, 1]$.

(i) *If* $|\upsilon| \le \sqrt{r/(1-r)}$, λ_υ *is unitarily equivalent to* λ.

(ii) *If* $\sqrt{r/(1-r)} < |\upsilon| < 1$, λ_υ *is unitarily equivalent to a direct sum of* λ *and the spherical representation of spectral parameter*

$$c_r(\upsilon) = \frac{r}{\upsilon} + (1-r)\upsilon$$

with the spectrum of $\lambda_\upsilon(h_1)$ *equal to* $\sigma_r \sqcup \{c_r(\upsilon)\}$.

Given a positive functional φ of A, the GNS representation space

$$\mathcal{R}_\varphi \equiv \overline{C^*(G)(\varphi \circ E)^{1/2}}$$

($(\varphi \circ E)^{1/2}$ being the GNS-cyclic vector) turns out to be a $C^*(G)$-A-bimodule: The left action of $C^*(G)$ is just the (left) GNS-representation based on $\varphi \circ E$ and the right action of A is given by $(x(\varphi \circ E)^{1/2})a = (xa)(\varphi \circ E)^{1/2}$ $(x \in C^*(G), a \in A)$.

If ψ is another positive functional of A, the space $\mathrm{Hom}(\mathcal{R}_\varphi, \mathcal{R}_\psi)$ of intertwiners is naturally isomorphic to $\mathrm{Hom}(L^2(\varphi), L^2(\psi))$ ([22]). Here $L^2(\varphi)$ denotes the GNS representation space of φ and $\mathrm{Hom}(L^2(\varphi), L^2(\psi))$ denotes the space of intertwiners between A-modules $L^2(\varphi)$ and $L^2(\psi)$.

In view of $L^2(\varphi) = L^2(A)[\varphi]$ with $L^2(A)$ the standard space of the second dual W*-algebra A^{**} and $[\varphi]$ the support projection of φ in A^{**}, we have

$$\mathrm{Hom}(L^2(\varphi), L^2(\psi)) \cong A^{**}[\varphi][\psi].$$

Thus $\mathrm{Hom}(L^2(\varphi), L^2(\psi))$ is isomorphic to the L^∞-space on $[-1, 1]$ with respect to the common measure class of φ and ψ in the Lebesgue decomposition.

When $\varphi = \delta_t$ with $\varepsilon_t = \varphi \circ E$ a spherical state on $C^*(G)$, $\mathcal{R}_\varphi$ is simply denoted by $\mathcal{R}_t$. As observed in [22], $(\mathcal{R}_t)_{-1\le t\le 1}$ is a continuous family of $C^*(G)$-A-bimodules and therefore it provides a Borel field structure so that the following holds.

Theorem 1.5 (Plancherel Formula). *Under the identification of φ with the associated Radon measure $\varphi(dt)$ on the spectrum $[-1, 1]$ of A, we have a natural isometric isomorphism between $\mathcal{R}_\varphi$ and the direct integral $\oint_{[-1,1]} \mathcal{R}_t \sqrt{\varphi(dt)}$ specified by the correspondence*

$$\mathcal{R}_\varphi \ni (\varphi \circ E)^{1/2} \mapsto \oint_{[-1,1]} \varepsilon_t^{1/2} \sqrt{\varphi(dt)} \in \oint_{[-1,1]} \mathcal{R}_t \sqrt{\varphi(dt)}.$$

For the positive definite function $\phi_\upsilon = \varphi_\upsilon \circ E$ $(-1 \le \upsilon \le 1)$ of Haagerup, where $\varphi_\upsilon(h_n) = \upsilon^n$ $(n \ge 0)$, the accompanied Radon measure $\varphi_\upsilon(dt)$ takes the following form ([22]):

(i) If $|\upsilon| \le \sqrt{r/(1-r)}$, $\varphi_\upsilon(dt)$ is supported by the interval σ_r and of the form

$$\varphi_\upsilon(dt) = \frac{\upsilon^{-1} - \upsilon}{2\pi} \frac{\sqrt{4r(1-r) - t^2}}{(1-t^2)(c_r(\upsilon) - t)}\, dt.$$

(ii) If $\sqrt{r/(1-r)} < |\upsilon| \le 1$, adding to the continuous measure in (i), there appears an atomic measure of the form

$$\frac{1 - c_r(\upsilon^2)}{1 - c_r(\upsilon)^2} \delta(t - c_r(\upsilon)).$$

In the special case $\upsilon = 0$ for which $\phi_0 = \tau$, $\varphi_0(dt)$ is reduced to the Kesten measure

$$\varphi_0(dt) = \lim_{\upsilon \to 0} \varphi_\upsilon(dt) = \frac{1}{2\pi r} \frac{\sqrt{4r(1-r)-t^2}}{1-t^2}\, dt$$

on $\sigma_r = [-2\sqrt{r(1-r)}, 2\sqrt{r(1-r)}]$ as stated in [3] (see also [18; 14]).

Let π be the GNS-representation of the spherical state ε_s for a spectral parameter $s = 2\sqrt{r(1-r)}$ and $\mathcal{H} = \overline{C^*(G)\varsigma}$ be the representation space of π with the GNS-vector denoted by $\varsigma = \varepsilon_s^{1/2}$. Notice that the spectral parameter $s = 2\sqrt{r(1-r)}$ is critical in the sense that it is located at the boundary of principal and complementary series. It is also critical from the viewpoint of C*-completion relative to $\ell^p(G)$ (see [2; 11]).

As a deformation of π, an analytic family (π_c) of bounded representations on $\mathcal{H}$ is constructed in [15] for $c \in \mathbb{C}$ satisfying $c^2 \notin (1, \infty)$ in such a way that $\pi_s = \pi$ and the following properties hold.

(i) For each $g \in G$, there is a finite dimensional subspace $\mathcal{H}_g$ of $\mathcal{H}$ so that $\pi_c(g) = \pi(g)$ on $\mathcal{H}_g^\perp$ for any c, i.e., $\pi_c(g) - \pi(g)$ is a finite-rank operator.

(ii) For any $g \in G$, $\pi_c(g)$ is continuous in c and holomorphic on

$$\{c \in \mathbb{C} \setminus ((-\infty, -1] \sqcup [1, \infty))\}.$$

(iii) We have $\pi_c(g)^* = \pi_{\bar{c}}(g^{-1})$ for $g \in G$ and any parameter c. Consequently π_c is unitary if $c \in [-1, 1]$.

(iv) If $c \neq \pm 1$, ς is a cyclic vector of π_c and satisfies $(\varsigma | \pi_c(g)\varsigma) = \varepsilon_c(g) = \delta_c(h_{|g|})$ $(g \in G)$.

(v) For $c = \pm 1$, we have $\pi_{\pm 1}(g)\varsigma = \varepsilon_{\pm 1}(g)\varsigma$ $(g \in G)$ but ς is not cyclic.

From property (iv), we see $(\varsigma | (\pi_c(h_1) - c)^2 \varsigma) = 0$, i.e., $\pi_c(h_1)\varsigma = c\varsigma$ for $c \in (-1, 1)$, and then, by analytic continuation on c, $\pi_c(h_1)\varsigma = c\varsigma$ for any c. Thus ς is an eigenvector of $\pi_c(h_1)$ of eigenvalue c.

More is known on the spectrum of $\pi_c(h_1)$:

Theorem 1.6 [19, Theorem 5; 15, Lemma 3]. *Any spectral parameter $c \in \mathbb{C}$ in the closed elliptic disk of ℓ^1-boundedness is an eigenvalue of $\pi_c(h_1)$ with ς its eigenvector so that $\pi_c(h_1)\varsigma^\perp \subset \varsigma^\perp$ and the spectrum of $\pi_c(h_1)$ as a bounded operator on the reducing subspace $\varsigma^\perp \subset \mathcal{H}$ is contained in the regular spectrum $\sigma_r = [-2\sqrt{r(1-r)}, 2\sqrt{r(1-r)}]$.*

Remark 2. A common-space-realization Π_z of the complementary component of λ_z in [19] turns out to be extended to the region $z^2 \notin [1, \infty)$ by an analytic continuation, which seems to be globally similar to (π_z) in [15].

2. The primitive ideal space

We begin with a universal construction of radial bimodules. For two positive functionals φ and ψ of A satisfying $\varphi \leq \psi$, the spherical decomposition in the Plancherel formula enables us to define an isometric embedding by

$$\oint_{[-1,1]} x\sqrt{\varphi(dt)} \mapsto \oint_{[-1,1]} x\sqrt{\frac{\varphi(dt)}{\psi(dt)}}\sqrt{\psi(dt)}$$

($\varphi(dt)/\psi(dt)$ being the Radon–Nikodym derivative), which in turn is converted to an isometric embedding of $\mathcal{R}_\varphi$ into $\mathcal{R}_\psi$ thanks to the identity $(\varphi \circ E)^{1/2} = \sqrt{d\varphi/d\psi}(\psi \circ E)^{1/2}$ in the standard representation space $L^2(C^*(G))$ of the second dual $C^*(G)^{**}$ (see [21]).

By the way of this construction, embeddings $\mathcal{R}_\varphi \to \mathcal{R}_\psi$ give an inductive system of $C^*(G)$-A-bimodules and we obtain the universal $C^*(G)$-A-bimodule $\mathcal{R}$ as an inductive limit. The image of the left action of $C^*(G)$ on $\mathcal{R}$ is then called the radial C*-algebra of G and denoted by $C^*_{\mathrm{rad}}(G)$, which is minimally universal with respect to radial representations in the sense that it includes all $\mathcal{R}_\varphi$ and satisfies the property stated in Proposition 2.1 below. Our main concern is in describing the primitive ideal space of $C^*_{\mathrm{rad}}(G)$.

Note that $C^*_{\mathrm{rad}}(G)$ itself is realized without minimality by a direct sum representation on $\bigoplus_\varphi \mathcal{R}_\varphi$. Remark also that we have a series of natural *-homomorphisms

$$\mathbb{C}G \to C^*(G) \to C^*_{\mathrm{rad}}(G) \to C^*_r(G)$$

with the total composition $\mathbb{C}G \to C^*_r(G)$ being injective because so is the regular representation of G. As a result, the obvious *-homomorphism $\mathbb{C}G \to C^*_{\mathrm{rad}}(G)$ is injective as well.

In terms of this universal radial bimodule, irreducibility of spherical representations is now rephrased in the following manner:

Proposition 2.1. *The von Neumann algebra* $\mathrm{End}({}_{C^*(G)}\mathcal{R}_A)$ *of self-intertwiners is isomorphic to* A^{**} *through the right multiplication of* A^{**}, *i.e.*, $\mathrm{End}({}_{C^*(G)}\mathcal{R}_A)$ *is generated by the right action of* A *and isomorphic to the second dual W*-algebra* A^{**} *associated with* A.

Here ${}_{C^*(G)}\mathcal{R}_A$ *indicates that* $\mathcal{R}$ *is a* $C^*(G)$-A-*bimodule and a bounded linear operator* T *on* $\mathcal{R}$ *is called a self-intertwiner if* $T(x\xi a) = xT(\xi)a$ *for* $x \in C^*(G)$, $a \in A$ *and* $\xi \in \mathcal{R}$.

Proof. By [22], the space $\mathrm{End}(\mathcal{R}_\varphi)$ of self-intertwiners is generated by the right action of A and $\mathcal{R}$ contains $L^2(A)$ as an A-A-subbimodule, whence the problem is reduced to showing that $\mathrm{End}(\mathcal{R})$ is generated by the right action of A as a von Neumann algebra.

To see this, observe first that the projection e_φ to the subspace $\mathcal{R}_\varphi$ belongs to $\mathrm{End}({}_{C^*(G)}\mathcal{R}_A)$ and satisfies $\lim_{\varphi\uparrow\infty} e_\varphi = 1_{\mathcal{R}}$ in the σ-strong operator topology. Let $T \in \mathrm{End}({}_{C^*(G)}\mathcal{R}_A)$. If its reduction $e_\varphi T e_\varphi \in \mathrm{End}({}_{C^*(G)}\mathcal{H}_A)$ commutes with $e_\varphi \,\mathrm{End}(\mathcal{R}_A) e_\varphi$ for any $\varphi \in A_+^*$, $a' \in \mathrm{End}(\mathcal{R}_A)$ and $\xi, \eta \in \mathcal{R}_\psi$ with $\psi \in A_+^*$ satisfy

$$(\xi | T e_\varphi a' \eta) = (\xi | e_\varphi T e_\varphi a' e_\varphi \eta) = (\xi | e_\varphi a' e_\varphi T e_\varphi \eta) = (\xi | a' e_\varphi T \eta) \quad (\varphi \geq \psi)$$

and then we have $(\xi | T a' \eta) = (\xi | a' T \eta)$ by taking limit $\varphi \uparrow \infty$. Since the inductive (algebraic) limit of $C^*(G)(\psi \circ E)^{1/2}$ for $\psi \uparrow \infty$ is dense in $\mathcal{R}$, this implies $Ta' = a'T$, i.e., $T \in \mathrm{End}(\mathcal{R}_A)'$. As a final step, apply the double commutant theorem of von Neumann. □

Corollary 2.2. *For a positive functional φ of A, $\mathrm{End}(\mathcal{R}_\varphi) \cong A^{**}[\varphi]$, where $[\varphi]$ denotes the support projection of φ in A^{**}.*

We use the notation $\mathcal{C}(\mathcal{H})$ to stand for the compact operator algebra on a Hilbert space $\mathcal{H}$.

Proposition 2.3. *For a continuous function $f \in C([-1, 1])$ vanishing on $\sigma_r \subset [-1, 1]$, we have $\pi_t(f(h_1)) = f(t)|\varsigma)(\varsigma|$ $(-1 \leq t \leq 1)$ and the representation π_t for $t \in (-1, 1) \setminus \sigma_r$ is well-behaved in the sense that $\mathcal{C}(\mathcal{H}) \subset \pi_t(C^*(G))$.*

Proof. By the spectral property in Theorem 1.6, $\pi_t(f(h_1)) = f(t)|\varsigma)(\varsigma| = 0$ if $t \in \sigma_r$.

For $t \in [-1, 1] \setminus \sigma_r$, the spectrum of $\pi_t(f(h_1))$ is supported by the point t with $|\varsigma)$ a unique eigenvector and hence $\pi_t(f(h_1)) = f(t)|\varsigma)(\varsigma|$ by functional calculus.

Thus $\pi_t(C^*(G))$ contains a rank-one projection $|\varsigma)(\varsigma|$ and the irreducibility of π_t for $-1 < t < 1$ implies $\mathcal{C}(\mathcal{H}) \subset \pi_t(C^*(G))$. □

Remark 3. Properties of complementary series representations concerning compact operator algebras are also pointed out in [20].

Proposition 2.4. *Primitive ideals of $C^*(G)$ associated to pure states ε_t are different for $t \in [-1, 1] \setminus \sigma_r$ and coincide with the kernel of the regular representation λ for $t \in \sigma_r$.*

Proof. Clearly characters $\varepsilon_{\pm 1}$ give rise to ideals different from those for ε_t $(-1 < t < 1)$ and we focus on irreducible representations π_t $(-1 < t < 1)$.

If two primitive ideals coincide for spectral parameters in $(-1, 1)$, the associated C*-algebras $\pi_t(C^*(G))$ for various $t \in (-1, 1)$ are canonically isomorphic and give rise to the same spectrum of $\pi_t(h_1)$. In the case when one of t is outside of σ_r, this necessitates the coincidence of t by Theorem 1.6.

Let $t \in \sigma_r$. If $x \in C^*(G)$ is in the kernel of λ, then the Plancherel formula ensures $\pi_{t'}(x^* x) = 0$ for almost any $t' \in \sigma_r$ with respect to the Kesten measure (being equivalent to the Lebesgue measure on σ_r) and we can find a sequence $t_n \in \sigma_r$ converging to t so that $\pi_{t_n}(x^*x) = 0$. Since $\varepsilon_{t'}$ is weak*-continuous in t', we have

$$\varepsilon_t(y^*x^*xy) = \lim_{n\to\infty} \varepsilon_{t_n}(y^*x^*xy) = 0$$

for any $y \in C^*(G)$. Thus, by cyclicity of ς, we have $\pi_t(x^*x) = 0$, i.e., $\ker\lambda \subset \ker\pi_t$ with $\lambda(\ker\pi_t)$ a closed ideal of a simple C*-algebra $C_r^*(G)$, which proves that $\ker\pi_t = \ker\lambda$. □

Corollary 2.5. (i) *We have* $\pi_t(C^*(G)) \cap \mathcal{C}(\mathcal{H}) = \{0\}$ *for* $t \in \sigma_r$.

(ii) *For* $f \in C([-1, 1])$, *the condition* $f(h_1) \in \ker\pi$ *is equivalent to* $f|_{\sigma_r} = 0$.

Proof. (i) If $\pi_t(C^*(G)) \cap \mathcal{C}(\mathcal{H}) \neq 0$, $\mathcal{C}(\mathcal{H}) \subset \pi_t(C^*(G))$ by irreducibility of π_t. In view of $\ker\pi_{t'} = \ker\pi_t$ for $t' \in \sigma_r$, this implies $\pi_{t'} \cong \pi_t$ ([5] Corollary 4.1.10), which contradicts with the disjointness of π_t and $\pi_{t'}$ for $t \neq t'$ in σ_r.

(ii) From the kernel coincidence, the spectrum of $\pi(h_1)$ is

$$\sigma_{C^*(G)/\ker\pi}(h_1 + \ker\pi) = \sigma_{C^*(G)/\ker\lambda}(h_1 + \ker\lambda) = \sigma(\lambda(h_1)) = \sigma_r,$$

whence $\pi(f(h_1)) = f(\pi(h_1)) = 0$ if and only if $f|_{\sigma_r} = 0$. □

Proposition 2.6. *Let* $-1 \leq t \leq 1$. *In the orthogonal decomposition* $\pi_t(h_1) = t|\varsigma)(\varsigma| \oplus \pi_t(h_1)|_{\varsigma^\perp}$, *the spectrum of* $\pi_t(h_1)|_{\varsigma^\perp}$ *is* σ_r *and therefore the spectrum of* $\pi_t(h_1)$ *is* $\{t\} \cup \sigma_r$.

Proof. Let σ (σ') be the (essential) spectrum of $\pi_t(h_1)$ on $\mathcal{H}$, i.e., σ' consists of accumulation points of σ or eigenvalues of infinite multiplicity of $\pi_t(h_1)$. Since $\pi_t(h_1)$ for various t coincide up to finite-rank operators, σ' does not depend on $t \in [-1, 1]$ (Weyl's stability theorem, see [17] XIII.4 for example).

By Proposition 2.4, $\ker\pi_t = \ker\lambda$ for $t \in \sigma_r$ and the spectrum of $\pi_t(h_1)$ in $\pi_t(C^*(G))$ is equal to that of $\lambda(h_1)$ in $C_r^*(G)$. Consequently, if $t \in \sigma_r$, $\sigma = \sigma_r$ and hence $\sigma' = \sigma_r$.

Now $\sigma(\pi_t(h_1)|_{\varsigma^\perp}) \subset \sigma_r$ (Theorem 1.6) is combined with $\sigma_r = \sigma' \subset \sigma(\pi_t(h_1)|_{\varsigma^\perp})$ to conclude that $\sigma(\pi_t(h_1)|_{\varsigma^\perp}) = \sigma_r$. □

Proposition 2.7. *The continuous family* π_t $(-1 \leq t \leq 1)$ *of representations gives rise to the same *-homomorphism* $C^*(G) \to \mathcal{B}(\mathcal{H})/\mathcal{C}(\mathcal{H})$ *by taking the quotient to the Calkin algebra* $\mathcal{B}(\mathcal{H})/\mathcal{C}(\mathcal{H})$.

In particular, we have $\pi_t(\ker\pi) \subset \mathcal{C}(\mathcal{H})$ *for* $-1 \leq t \leq 1$. *Recall that* $\pi = \pi_s$ *with* $s = 2\sqrt{r(1-r)}$ *and* $\ker\pi_t = \ker\pi$ *for* $t \in \sigma_r$.

Proof. Since $\pi_t(g) - \pi(g)$ is a finite-rank operator for each $g \in G$, the same holds for $\pi_t(x) - \pi(x)$ if $x \in \mathbb{C}G$. Since $x \in C^*(G)$ is norm-approximated by a sequence x_n in $\mathbb{C}G$ and $\pi_t : C^*(G) \to \mathcal{B}(\mathcal{H})$ is contractive in norm, $\lim_{n\to\infty} \|\pi_t(x_n) - \pi_t(x)\| = 0$ shows that $\pi_t(x) - \pi(x)$ is norm-approximated by a sequence $(\pi_t(x_n) - \pi(x_n))_{n\geq 1}$ of finite-rank operators. Thus $\pi_t(x) - \pi(x)$ is a compact operator on $\mathcal{H}$. □

Corollary 2.8. *For* $-1 \le t \le 1$, *we have* $\ker \pi = \pi_t^{-1}(\mathcal{C}(\mathcal{H})) \supset \ker \pi_t$ *and hence* $\pi_t(\ker \pi) = \pi_t(C^*(G)) \cap \mathcal{C}(\mathcal{H})$.

Proof. Since $\pi(C^*(G)) \cap \mathcal{C}(\mathcal{H}) = 0$ (Corollary 2.5) and

$$\pi_t(C^*(G)) + \mathcal{C}(\mathcal{H}) = \pi(C^*(G)) + \mathcal{C}(\mathcal{H}),$$

we see

$$\begin{aligned} C^*(G)/\pi_t^{-1}(\mathcal{C}(\mathcal{H})) &\cong \pi_t(C^*(G))/\big(\pi_t(C^*(G)) \cap \mathcal{C}(\mathcal{H})\big) \\ &\cong \big(\pi_t(C^*(G)) + \mathcal{C}(\mathcal{H})\big)/\mathcal{C}(\mathcal{H}) \\ &\cong \pi(C^*(G)) \cong C^*(G)/\ker \pi, \end{aligned}$$

which means that $\ker \pi = \pi_t^{-1}(\mathcal{C}(\mathcal{H})) \supset \ker \pi_t$. □

By the continuity of $\pi_t(g)$ in $t \in [-1, 1]$ for each $g \in G$, a norm-continuous family $(\pi_t(x))_{-1 \le t \le 1}$ of operators in $\mathcal{B}(\mathcal{H})$ is associated to any $x \in C^*(G)$ as a uniform limit of continuous functions and the correspondence $x \mapsto (\pi_t(x))$ defines a *-homomorphism Π of $C^*(G)$ into $C([-1, 1]) \otimes \mathcal{B}(\mathcal{H})$ so that

$$\ker \Pi = \bigcap_{-1 \le t \le 1} \ker \pi_t = \bigcap_{t \in (-1,1) \setminus \sigma_r} \ker \pi_t$$

and $\Pi(\ker \pi) \subset C([-1, 1]) \otimes \mathcal{C}(\mathcal{H})$ by Corollary 2.8.

In view of $\ker \pi_t = \ker \pi$ $(t \in \sigma_r)$, $\Pi(\ker \pi)$ is in fact included in the ideal $C_{\sigma_r}([-1, 1]) \otimes \mathcal{C}(\mathcal{H})$ of $C([-1, 1]) \otimes \mathcal{C}(\mathcal{H})$, where

$$C_{\sigma_r}([-1, 1]) = \{f \in C([-1, 1]);\ f|_{\sigma_r} = 0\} \cong C_0([-1, 1] \setminus \sigma_r)$$

is the ideal of $C([-1, 1])$ vanishing on $\sigma_r \subset [-1, 1]$. Recall also that π_t is irreducible for $-1 < t < 1$, whereas $\pi_{\pm 1}$ is decomposed as $\varepsilon_{\pm 1} \oplus \pi'_{\pm}$ according to the orthogonal decomposition $\mathcal{H} = \mathbb{C}\varsigma \oplus \varsigma^{\perp}$.

Choose $f \in C_{\sigma_r}([-1, 1])$ so that $f(\pm 1) = 1$. By Theorem 1.6 or Proposition 2.6, $f(h_1) \in A$ satisfies $\pi_{\pm 1}(f(h_1)) = |\varsigma)(\varsigma|$ and then

$$\pi_{\pm 1}\big(zf(h_1) + x(1 - f(h_1))\big) = z|\varsigma)(\varsigma| + \pi_{\pm 1}(x)(1 - |\varsigma)(\varsigma|) = z|\varsigma)(\varsigma| + \pi'_{\pm}(x)$$

$(z \in \mathbb{C}, x \in C^*(G))$ reveals that $\pi_{\pm 1}(C^*(G)) = \mathbb{C}|\varsigma)(\varsigma| \oplus \pi'_{\pm}(C^*(G))$.

Consequently, Corollary 2.8 is used again to see

$$\begin{aligned} \ker \pi &= \pi_{\pm 1}^{-1}(\mathcal{C}(\mathcal{H})) = \pi_{\pm 1}^{-1}\big(\mathbb{C}|\varsigma)(\varsigma| \oplus \pi'_{\pm}(C^*(G)) \cap \mathcal{C}(\varsigma^{\perp})\big) \\ &= (\pi'_{\pm})^{-1}(\mathcal{C}(\varsigma^{\perp})) \supset \ker \pi'_{\pm} \end{aligned}$$

and hence $\ker \pi / \ker \pi'_{\pm} \cong \pi'_{\pm}(\ker \pi) = \pi'_{\pm}(C^*(G)) \cap \mathcal{C}(\varsigma^{\perp})$.

Define the residual representation of $C^*(G)$ on $\varsigma^{\perp} \oplus \varsigma^{\perp}$ to be a direct sum $\pi' = \pi'_+ \oplus \pi'_-$, which satisfies $\ker \pi' = \ker \pi'_+ \cap \ker \pi'_- \subset \ker \pi$, $\pi'(\ker \pi) \subset$

$\pi'(C^*(G)) \cap \big(\mathcal{C}(\varsigma^\perp) \oplus \mathcal{C}(\varsigma^\perp)\big)$ and $\Pi(\ker \pi') \subset C$ with

$$C = \{f \in C_{\sigma_r}([-1,1]) \otimes \mathcal{C}(\mathcal{H});\ f(\pm 1) \in \mathbb{C}|\varsigma)(\varsigma|\}$$

a C*-subalgebra of $C_{\sigma_r}([-1,1]) \otimes \mathcal{C}(\mathcal{H})$.

Proposition 2.9. *We have the equality* $\Pi(\ker\pi') = C$ *and the associated isomorphism* $C \cong \ker\pi' / \ker\Pi$ *induces a bijection between* $\mathrm{Prim}(C)$ *and*

$$\mathrm{Prim}(\ker\pi') \setminus \mathrm{Prim}(\ker\Pi) \subset \mathrm{Prim}(C^*(G))$$

so that $\mathrm{Prim}(C)$ *is the set of primitive ideals of spherical representations in the complementary series*, *i.e.*,

$$\{\ker\pi_t;\ t \in (-1,1) \setminus \sigma_r\} \sqcup \{\ker\varepsilon_{\pm 1}\}.$$

Proof. We first describe pure states of C. Let $C_0 = \{f \in C;\ f(\pm 1) = 0\}$. Since a closed ideal of C_0 is isomorphic to $C_0((-1,1) \setminus \sigma_r) \otimes \mathcal{C}(\mathcal{H})$ and satisfies $C/C_0 \cong \mathbb{C} \oplus \mathbb{C}$, irreducible representations of C are given by evaluation at $t \in [-1,1] \setminus \sigma_r$ and hence pure states of C are evaluations followed by applying pure states of fibers ($\mathcal{C}(\mathcal{H})$ or $\mathbb{C}|\varsigma)(\varsigma|$). In particular, C is a CCR (completely continuous representation) algebra.

The equality then follows from the Stone–Weierstrass theorem on CCR C*-algebras due to Kaplansky (see [1] for a survey) if these pure states are separated by elements in $\Pi(\ker\pi')$.

To see this, we recall Proposition 2.3 that $\pi_t(f(h_1)) = f(t)|\varsigma)(\varsigma|$ holds for $f \in C_{\sigma_r}([-1,1])$ and $t \in [-1,1]$, whence $f(h_1) \in \ker\pi'$.

Let φ and ψ be pure states of C given by $\varphi(x) = (\xi|x(s)\xi)$ and $\psi(x) = (\eta|x(t)\eta)$, where $s, t \in [-1,1] \setminus \sigma_r$ and ξ, η are unit vectors in $\mathcal{H}$.

If $s \neq t$ with $t \in (-1,1)$, π_t is irreducible and we can find $y \in C^*(G)$ such that $\pi_t(y) = |\eta)(\varsigma|$ by Kadison's transitivity[1]. Then $yf(h_1)y^* \in \ker\pi'$ is represented by

$$\pi_s(yf(h_1)y^*) = f(s)|\pi_s(y)\varsigma)(\pi_s(y)\varsigma|, \quad \pi_t(yf(h_1)y^*) = f(t)|\eta)(\eta|.$$

Thus, for a choice $f \in C_{\sigma_r}([-1,1])$ satisfying $f(s) = 0$ and $f(t) = 1$, φ and ψ are separated by $\Pi(yf(h_1)y^*)$:

$$\varphi(\Pi(yf(h_1)y^*)) = (\xi|\pi_s(yf(h_1)y^*)\xi) = f(s)(\xi|\pi_s(y)\varsigma)(\pi_s(y)\varsigma|\xi) = 0,$$
$$\psi(\Pi(yf(h_1)y^*)) = (\eta|\pi_t(yf(h_1)y^*)\eta) = f(t)(\eta|\pi_t(y)\varsigma)(\pi_t(y)\varsigma|\eta) = 1.$$

When both $s \neq t$ come from $\{\pm 1\}$, we may assume that $s = -1$ and $t = 1$, with $\varphi(x) = (\varsigma|x(-1)\varsigma)$ and $\psi(x) = (\varsigma|x(1)\varsigma)$ for $x = (x(t))_{-1 \le t \le 1} \in C$. Then

[1] We can dispense with Kadison's transitivity if η is approximated by $\pi_t(y)\varsigma$.

$\varphi(\Pi(f(h_1))) = f(-1)$ and $\psi(\Pi(f(h_1))) = f(1)$ and we see that φ and ψ are separated by $f(h_1) \in \ker\pi'$ if $f \in C_{\sigma_r}([-1,1])$ satisfies $f(-1) \neq f(1)$.

Finally consider the case $s = t$. Since states are unique for $t = \pm 1$, we assume $-1 < t < 1$ with $\varphi(x) = (\xi|x(t)\xi)$ and $\psi(x) = (\eta|x(t)\eta)$. Then the condition $\varphi \neq \psi$ is equivalent to $|(\xi|\eta)| < 1$ and $yf(h_1)y^* \in \ker\pi'$ described above is evaluated by

$$\varphi(\Pi(yf(h_1)y^*)) = f(t)(\xi|\eta)(\eta|\xi), \quad \psi(\Pi(yf(h_1)y^*)) = f(t).$$

Thus φ and ψ are separated by $yf(h_1)y^* \in \ker\pi'$ if $f(t) \neq 0$. □

Lemma 2.10. *We have* $\ker\pi = \ker\pi'$.

Proof. To see this, we use the continuous deformation $(\lambda_\upsilon)_{-1\leq\upsilon\leq 1}$ of the regular representation $\lambda = \lambda_0$ in [16]. Recall that these are unitary representations of G with the following properties (Theorem 1.4):

(i) λ_υ is unitarily equivalent to λ if $\upsilon \in \sigma_r$, to $\pi_\upsilon \oplus \lambda$ if $\upsilon \in (-1,1)\setminus\sigma_r$ and to $\epsilon_{\pm 1} \oplus \overbrace{\lambda \oplus \cdots \oplus \lambda}^{l \text{ times}}$ if $\upsilon = \pm 1$.

(ii) $\ker\lambda_\upsilon = \ker\pi_{c_r(\upsilon)} \cap \ker\lambda = \ker\pi_{c_r(\upsilon)} \cap \ker\pi = \ker\pi_{c_r(\upsilon)}$ if $|\upsilon| > \sqrt{r/(1-r)}$, $\ker\lambda_\upsilon = \ker\lambda$ if $|\upsilon| \leq \sqrt{r/(1-r)}$, and finally $\ker\lambda_{\pm 1} = \ker\epsilon_{\pm 1} \cap \ker\lambda = \ker\epsilon_{\pm 1} \cap \ker\pi$. Notice that $c_r(\upsilon) \notin \sigma_r$ for $\upsilon \neq \pm\sqrt{r/(1-r)}$.

($\pi'_\pm$ would be unitarily equivalent to $\lambda \otimes 1_{\mathbb{C}^{l-1}}$ but its validity is irrelevant here.)

As observed for (π_t) before, the family $(\lambda_\upsilon)_{-1\leq\upsilon\leq 1}$ of *-representations of $C^*(G)$ satisfies $\lambda_\upsilon(x) - \lambda(x) \in \mathcal{C}(\ell^2(G))$ for $x \in C^*(G)$ and gives rise to a *-homomorphism $\Lambda : C^*(G) \to C([-1,1]) \otimes \mathcal{B}(\ell^2(G))$ in such a way that $\Lambda(\ker\lambda) = \Lambda(\ker\pi) \subset C([-1,1]) \otimes \mathcal{C}(\ell^2(G))$. In view of norm-continuity of $\lambda_\upsilon(x)$ in $\upsilon \in [-1,1]$ and Theorem 1.4,

$$\ker\Lambda = \bigcap_{\sqrt{r/(1-r)}<|\upsilon|<1} \ker\lambda_\upsilon = \bigcap_{4r(1-r)<t^2<1} \ker\pi_t = \ker\Pi$$

and $\Lambda(\ker\lambda) = \Lambda(\ker\pi) \subset C([-1,1]) \otimes \mathcal{C}(\ell^2(G))$. Note here that

$$\{c_r(\upsilon);\ \sqrt{r/(1-r)} < |\upsilon| < 1\} = \{t \in \mathbb{R};\ 4r(1-r) < t^2 < 1\}.$$

Thus pure states of $\ker\pi/\ker\Pi = \ker\lambda/\ker\Lambda \cong \Lambda(\ker\lambda)$ are given by pure states of $\lambda_\upsilon(\ker\lambda)$ after the evaluation at υ satisfying $|\upsilon| > \sqrt{r/(1-r)}$. Here recall that

$$\lambda_\upsilon(\ker\lambda) \cong \begin{cases} 0 & (|\upsilon| \leq \sqrt{r/(1-r)}), \\ \pi_{c_r(\upsilon)}(\ker\pi) & (\sqrt{r/(1-r)} < |\upsilon| < 1), \\ \mathbb{C} & (\upsilon = \pm 1). \end{cases}$$

Hence the set of associated primitive ideals of $\ker\lambda/\ker\Lambda = \ker\pi/\ker\Pi$ is

$$\{\ker\pi_t;\ t \in (-1,1)\setminus\sigma_r\} \cup \{\ker\epsilon_{\pm 1}\}.$$

At this point, there might be overlapping in the union but the comparison of this with Proposition 2.9 enables us to conclude that these are in fact distinct and $\ker\pi/\ker\pi' = 0$. □

Remark 4. We have an inclusion $\overline{C^*(G)f(h_1)C^*(G)} \subset \ker\pi$ for $f \in C_{\sigma_r}([-1,1])$ and the equality $\ker\pi' = \ker\pi$ is rephrased by

$$\ker\pi = \overline{\bigcup_{f\uparrow[-1,1]\setminus\sigma_r} C^*(G)f(h_1)C^*(G)}.$$

Here $f \uparrow [-1,1]\setminus\sigma_r$ means an inductive limit on $f \in C_{\sigma_r}([-1,1])$ satisfying $0 \le f \le 1$.

We summarize our considerations so far:

Theorem 2.11. *Primitive ideals of the radial C^*-algebra $C^*_{\mathrm{rad}}(G)$ are exactly kernels of spherical representations of G:*

$$\mathrm{Prim}(C^*_{\mathrm{rad}}(G)) = \{\ker\pi\} \sqcup \{\ker\pi_t;\, t \in (-1,1)\setminus\sigma_r\} \sqcup \{\ker\varepsilon_{\pm1}\}$$

with the primitive ideal space of $\ker\lambda/\ker\Lambda = \ker\pi/\ker\Pi$ *identified with*

$$\{\ker\pi_t;\, t \in (-1,1)\setminus\sigma_r\} \sqcup \{\ker\varepsilon_{\pm1}\}.$$

We now look into the topology of the primitive ideal space $\Delta \equiv \mathrm{Prim}(C^*_{\mathrm{rad}}(G))$, which is a closed subset of $\mathrm{Prim}(C^*(G))$. Since primitive ideals in Δ are of the form $[t] = \ker\pi_t$ $(-1 < t < 1)$ or $[\pm1] = \ker\varepsilon_{\pm1}$, Δ is identified with a quotient of $[-1,1]$ in such a way that σ_r is shrunken to one point in Δ.

We first check the continuity of $[t] \in \Delta$ in $t \in [-1,1]$: If a sequence (t_n) in $[-1,1]$ converges to $t \in [-1,1]$, then $\bigcap_{n\ge1}\ker\pi_{t_n} \subset \ker\pi_t$ because $x \in \bigcap_{n\ge1}\ker\pi_{t_n}$ and $y \in C^*(G)$ satisfy

$$\|\pi_t(xy)\varsigma\| = \|xy\epsilon_t^{1/2}\| = \lim_{n\to\infty}\|xy\epsilon_{t_n}^{1/2}\| = \lim_{n\to\infty}\|\pi_{t_n}(x)\pi_{t_n}(y)\varsigma\| = 0.$$

Theorem 2.12. *For a nonempty subset T of $(-1,1)\setminus\sigma_r$, we have*

$$\overline{\{[t];\, t \in T\}} = \{[t];\, t \in \overline{T}\} \cup \{\ker\pi\}$$

in $\Delta = \mathrm{Prim}(C^*_{\mathrm{rad}}(G))$. Here the left hand side is the closure in Δ and $\overline{T}$ denotes the closure of T in $[-1,1]$.

Since $\ker\varepsilon_{\pm1}$ and $\ker\pi$ are maximal ideals, they are closed as one-point sets. Thus the closure operation in Δ is completely described by this.

Proof. Let us begin with showing that Δ contains relatively open intervals in $[-1,1]\setminus\sigma_r$ as open subsets of Δ: For a continuous function $f \in C([-1,1])$, Proposition 2.6 gives

$$\|\pi_t(f(h_1))\| = \begin{cases} \max\{|f(t')|;\, t' \in \{t\} \cup \sigma_r\} & (-1 < t < 1), \\ |\varepsilon_t(f(h_1))| = |f(\pm 1)| & (t = \pm 1). \end{cases}$$

Since relatively open subsets of $[-1, 1] \setminus \sigma_r$ are then realized in the form

$$\{t \in [-1, 1] \setminus \sigma_r;\, \|\pi_t(f(h_1))\| > 0\}$$

with $f = 0$ on σ_r (relatively open subsets being disjoint unions of countably many relatively open intervals), they are open in Δ as well because $\Delta \ni [t] \mapsto \|\pi_t(f(h_1)\|$ is a lower semicontinuous function ([5] Proposition 3.3.2).

Thus a relatively open subset $[-1, 1] \setminus (\sigma_r \cup \overline{T})$ of $[-1, 1] \setminus \sigma_r$ is open in Δ and hence its complement $[\overline{T}] \cup \{\ker \pi\}$ in Δ is closed. Consequently

$$[\overline{T}] \subset \overline{[T]} \subset [\overline{T}] \cup \{\ker \pi\}.$$

Here the first inclusion is due to the continuity of $[t]$ in $t \in [-1, 1]$. Since T is nonempty, its closure $\overline{[T]}$ in Δ contains $\ker \pi$ (Corollary 2.8) and the assertion is proved. □

Corollary 2.13. *Open sets of Δ are exactly of the following form*:

(i) $[U]$ *with* $U \subset [-1, 1] \setminus \sigma_r$ *an open subset in the relative topology of* $[-1, 1]$.

(ii) $\Delta \setminus F$ *with* F *a subset* (*including the empty set*) *of* $\{[1], [-1]\}$.

Corollary 2.14. *The radial C^*-algebra $C^*_{\text{rad}}(G)$ of G is isomorphic to $C^*(G)/\ker \Pi$ and the center of $C^*_{\text{rad}}(G)$ is trivial.*

Proof. Let $f : \Delta \to \mathbb{C}$ be a continuous function. Since $\ker \pi \in \overline{\{\ker \pi_t\}}$ $(-1 < t < 1)$, $f([t]) = f(\ker \pi)$ for $-1 < t < 1$ and then for $-1 \le t \le 1$ by continuity of $[t]$ in t. Thus f is constant on Δ and the assertion follows from the Dauns–Hofmann theorem ([12], §4.4). □

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