Pacific Journal of Mathematics

HOW MANY SPRAYS COVER SPACE?

ALESSANDRO ANDRETTA AND IVAN IZMESTIEV

Volume 341 No. 1 March 2026

PACIFIC JOURNAL OF MATHEMATICS
Vol. 341, No. 1, 2026
https://doi.org/10.2140/pjm.2026.341.1

HOW MANY SPRAYS COVER SPACE?

ALESSANDRO ANDRETTA AND IVAN IZMESTIEV

For all $d \geq 3$ we show that the cardinality of $\mathbb{R}$ is at most $\aleph_n$ if and only if $\mathbb{R}^d$ can be covered with $(n+1)(d-1)+1$ sprays whose centers are in general position in a hyperplane. This extends previous results by Schmerl when $d = 2$.

1. Introduction

The general theme of this paper is the possibility of covering $\mathbb{R}^d$ with few small sets — smallness means having finite or countable intersection with prescribed geometric objects. For example $\mathbb{R}^2$ can be covered with countably many curves with distinct axes [Davies 1974], but cannot be covered with finitely many curves [Mazurkiewicz 1933]. (A curve C is the rotated graph of a function $f : \mathbb{R} \to \mathbb{R}$, and its axis is the image of the y-axis via the rotation; the smallness condition is that C intersected any line parallel to its axis has cardinality 1.)

A spray in the plane is a subset of $\mathbb{R}^2$ together with a distinguished point (called center) such that all circles centered in that point have finite intersection with the spray. These objects were introduced in [Schmerl 2003], where it is observed that CH, the continuum hypothesis, implies that the plane can be covered with three sprays. In [de la Vega 2009] it is shown that CH is equivalent to the plane being covered with three sprays with collinear centers — in fact the statement "the cardinality of $\mathbb{R}$ is at most $\aleph_n$" (in symbols: $2^{\aleph_0} \leq \aleph_n$) is equivalent to $\mathbb{R}^2$ being covered with $n+2$ sprays with collinear centers [Schmerl 2010]. Collinearity is essential, since ZFC proves that the plane can be covered with three sprays centered in arbitrary noncollinear points [de la Vega 2009; Schmerl 2010]. It is easy to check that if a spray is measurable, then it must be null, so if the plane is covered with at most countably many sprays, they cannot be all measurable. Any construction of a covering of the plane with countably many sprays (or curves) requires the axiom of choice.

The notion of spray can be extended to $\mathbb{R}^d$ for any $d \geq 3$, with $(d-1)$-dimensional spheres in place of 1-dimensional spheres, i.e. circles, and the natural question is the relation (if any) between the size of the continuum and the number of sprays

MSC2020: primary 03E50; secondary 51N20.
Keywords: continuum hypothesis, spray.

needed to cover $\mathbb{R}^d$. By [Erdős et al. 1994, Theorem 2], $2^{\aleph_0} \leq \aleph_n$ implies that $\mathbb{R}^3$ can be covered with $2n+3$ sprays such that their centers are coplanar, and no three of them are collinear. In particular, CH implies that given five coplanar points such that no three of them are collinear, then there are sprays centered around them that cover $\mathbb{R}^3$. By [Schmerl 2012] $\mathbb{R}^3$ cannot be covered with three sprays, but the problem whether the space can be covered with four sprays remains open.

We prove that $2^{\aleph_0} \leq \aleph_n$ is equivalent to $\mathbb{R}^3$ being covered with $2n+3$ sprays whose centers are coplanar, and no three of them are collinear. Moreover $2n+3$ is optimal — in particular $\mathbb{R}^3$ cannot be covered with four sprays with coplanar centers. In fact we prove a similar result for $\mathbb{R}^d$ with $d > 3$, namely: $2^{\aleph_0} \leq \aleph_n$ is equivalent to $\mathbb{R}^d$ being covered with $(n+1)(d-1)+1$-many sprays whose centers lie on a hyperplane H, and the affine span of any d of them is H (Theorem 5.4). Again the number $(n+1)(d-1)+1$ of sprays is optimal. Finally we show that, irrespective of the size of the continuum, $\mathbb{R}^d$ can be covered with countably many sprays whose centers lie on a hyperplane H, and the affine span of any d of them is H (Theorem 5.8).

We do not know if $\mathbb{R}^d$ can be covered with $d+1$-many sprays such that the affine span of their centers is $\mathbb{R}^d$, even in the case $d=3$. In other words: can $\mathbb{R}^3$ be covered with four sprays whose centers are not coplanar, i.e. form a tetrahedron? We suspect that the answer is affirmative and that it can be proved in ZFC. If true, this would be analogous to the fact that $\mathbb{R}^2$ can be covered with three sprays with noncollinear centers.

The results in this paper lie on the interface between set theory and the geometry of euclidean spaces. The reader may find them odd as *a priori* there is no obvious relation between geometric notions (like being in general position, or lying on a hyperplane, or having a specific dimension) and the value of $2^{\aleph_0} = |\mathbb{R}| = |\mathbb{R}^d|$ — note that there is a Borel bijection between $\mathbb{R}$ and $\mathbb{R}^d$. On the other hand there are several results in this area that connect set-theoretic issues with geometric properties. Examples are the classical theorems of Sierpiński — see Theorem 2.6 below and the beginning of Section 4 for some context and references. Most of these results deal with linear objects like lines, or hyperplanes in $\mathbb{R}^d$, while sprays are essentially nonlinear objects. We extend a construction in [Schmerl 2010] from $\mathbb{R}^2$ to $\mathbb{R}^d$, transforming the quadratic problem of covering the space with sprays to the linear problem of covering the space with sets having finite intersections with certain families of hyperplanes. This latter problem has been studied before [Bagemihl 1959/60; Erdős et al. 1994; Simms 1997]. Extending these earlier results we are able to prove our results on sprays.

The paper is organized as follows. After recalling the notations and the basic notions that will be used throughout the paper, we show in Section 3 how to transform a covering of $\mathbb{R}^d$ with sprays with centers on a given hyperplane into a covering of (an open subset of) $\mathbb{R}^d$ with small intersection with certain families of hyperplanes. Section 4 is devoted to studying the following problem: given distinct, nonzero vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ in $\mathbb{R}^d$, are there $A_1, \ldots, A_k$ covering $\mathbb{R}^d$ such that every plane orthogonal to $\boldsymbol{u}_i$ has finite intersection with A_i? It turns out that this problem is closely related to the size of the continuum, and by the results in Section 3 it is equivalent to the existence of sprays $X_1, \ldots, X_k$ with centers on a given hyperplane H, and covering $\mathbb{R}^d$, as the (directions of the) $\boldsymbol{u}_i$s correspond to the position of the centers of the X_is on H. Finally in Section 5 we prove the results about the existence of sprays covering $\mathbb{R}^d$ and the size of the continuum.

This research was conducted while the first author was visiting TU Wien in 2024. He would like to thank the Mathematics Department and the Institute of Discrete Mathematics and Geometry for gracious hospitality and a stimulating environment.

2. Notation and preliminary results

2.1. *Notation.*

2.1.1. *Set theory.* We work in ZFC — this is the standard framework that most mathematicians (implicitly or explicitly) adopt to prove theorems. Our notation is standard: $\mathscr{P}(X)$ is the power set of X, the pointwise image of $Z \subseteq X$ via some $f : X \to Y$ is $f[Z] := \{f(z) \mid z \in Z\}$, and the pointwise preimage of $W \subseteq Y$ is $f^{-1}[W] := \{x \in X \mid f(x) \in W\}$.

The *cardinality* of a set X is denoted by $|X|$. A set X is finite if it is in bijection with a natural number i.e. $|X| < \aleph_0$; otherwise it is infinite i.e. $\aleph_0 \leq |X|$. We say X is *countable* if either $|X| < \aleph_0$, or else $|X| = \aleph_0$ — this can be written in a compact way as $|X| \leq \aleph_0$ or as $|X| < \aleph_1$. A set X is *uncountable* if it is not countable, that is $\aleph_1 \leq |X|$.

The set $\mathbb{R}$ is in bijection with the set of all infinite sequences of 0s and 1s, and for this reason the cardinality of $\mathbb{R}$ is denoted with $2^{\aleph_0}$. By Cantor's theorem $\aleph_1 \leq |\mathbb{R}|$, and Cantor's continuum hypothesis CH asserts that the inequality can be replaced with an equality. Since $\aleph_1 \leq 2^{\aleph_0}$, CH can be stated as $2^{\aleph_0} \leq \aleph_1$. By Cohen's results, it is consistent that the cardinality of $\mathbb{R}$ be any $\aleph_{n+1}$, or even larger cardinals, like $\aleph_{\omega+n+1}$. (But $2^{\aleph_0}$ cannot be $\aleph_\omega$ by König's theorem.)

2.1.2. *Geometry.* The standard basis of $\mathbb{R}^d$ is denoted by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$. A hyperplane of a vector space is a linear subspace of codimension 1. The following notation will be used throughout the paper.

Notation 2.1. Given a vector $\boldsymbol{u} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}$ and $\boldsymbol{p} \in \mathbb{R}^d$ let

$$H_{\boldsymbol{u}}(\boldsymbol{p}) = \boldsymbol{p} + \{\boldsymbol{v} \mid \boldsymbol{v} \cdot \boldsymbol{u} = 0\}$$

be the hyperplane of $\mathbb{R}^d$ orthogonal to $\boldsymbol{u}$ passing through $\boldsymbol{p}$, where $\cdot$ is the (standard) inner product. We also let $H_i(x) = \{(p_1, \ldots, p_d) \in \mathbb{R}^d \mid p_i = x\}$.

An *affine subspace* E of $\mathbb{R}^d$ is a translate of some vector subspace of $\mathbb{R}^d$, that is a set of the form $\boldsymbol{p} + V$ where V is a vector subspace of $\mathbb{R}^d$; the vector subspace V is unique, and it is the vector space associated to E, while $\boldsymbol{p} + V = \boldsymbol{p}' + V$ if and only if $\boldsymbol{p} - \boldsymbol{p}' \in V$. In this paper, a (finite dimensional) *affine space* is an affine subspace of some $\mathbb{R}^d$. Elements of an affine space are called points, and since every vector subspace V of $\mathbb{R}^d$ is also an affine space, we can refer to elements of it as points or vectors, depending if we privilege the affine or vector space structure. If E is an affine subspace of $\mathbb{R}^d$ and $\boldsymbol{p}, \boldsymbol{q} \in E$ then $\boldsymbol{q} - \boldsymbol{p}$ belongs to V, the underlying vector space of E, and if $\boldsymbol{v} \in V$ then $\boldsymbol{p} + \boldsymbol{v}$ belongs to E. The dimension of an affine space is, by definition, the dimension of the associated vector space. An *affine hyperplane* of $\mathbb{R}^d$ is a translate of linear subspace of dimension $d - 1$. More generally, a hyperplane of an affine space E of dimension d is an affine subspace of dimension $d - 1$. Given an affine space E with associated vector space V, the *affine envelope* or *affine span* of a non-empty set $S \subseteq E$ is $\boldsymbol{p} + \operatorname{span}\{\boldsymbol{q} - \boldsymbol{p} \mid \boldsymbol{q} \in S\}$ where $\boldsymbol{p} \in S$ and span X is the smallest vector subspace of V containing $X \subseteq V$. This definition does not depend on the point $\boldsymbol{p}$, and the affine span of S is the intersection of all affine subspaces of E containing S. Two affine subspaces E and F of $\mathbb{R}^d$ are *complementary* if their associated vector spaces V and W are complementary, that is $\mathbb{R}^d = V \oplus W$. It is easy to check that the intersection of two complementary spaces E, F is a single point. If $V \oplus W = \mathbb{R}^d$, then V and W are orthogonal if $\forall \boldsymbol{v} \in V \, \forall \boldsymbol{w} \in W \, (\boldsymbol{v} \cdot \boldsymbol{w} = 0)$. In this case we call one of the two subspaces the *orthogonal complement* of the other, and write $V^\perp$ to denote W, the orthogonal complement of V. Two complementary affine subspaces E and F of $\mathbb{R}^d$ are orthogonal if their underlying vector spaces are orthogonal.

Let E be an affine subspace of $\mathbb{R}^d$. The *sphere in E with center $\boldsymbol{c} \in E$ and radius* $r \in \mathbb{R}$ is the set

$$\mathbb{S}(E; \boldsymbol{c}, r) := \{\boldsymbol{x} \in E \mid \|\boldsymbol{x} - \boldsymbol{c}\| = r\}.$$

We convene that $\mathbb{S}(E; \boldsymbol{c}, r)$ is empty if $r < 0$, and it is the singleton $\{\boldsymbol{c}\}$ when $r = 0$. Observe that if $r > 0$ then $\mathbb{S}(E; \boldsymbol{c}, r)$ has cardinality 2 when $\dim(E) = 1$; if $\dim(E) \geq 2$, then $\mathbb{S}(E; \boldsymbol{c}, r)$ has cardinality $2^{\aleph_0}$. Whenever the ambient space (i.e. $\mathbb{R}^d$, E, ...) is clear we will simply write $\mathbb{S}(\boldsymbol{c}, r)$. A $(k-1)$*-dimensional sphere* is a sphere in an affine subspace E of some $\mathbb{R}^d$ with $\dim E = k$.

2.2. ***Families with finite mesh.*** Let $N, d \geq 2$ be natural numbers and suppose that $\mathcal{H}_i \subseteq \mathscr{P}(\mathbb{R}^d)$ with $1 \leq i \leq N$ are pairwise disjoint. Following [Erdős et al. 1994], the sequence $(\mathcal{H}_i)_{i=1}^N$ is (r, s) finitely determined if $r \geq 2$ and $s \geq 1$ are natural numbers such that

- for any distinct $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s \in \mathbb{R}^d$, the set of all $H \in \bigcup_{i=1}^N \mathcal{H}_i$ such that $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_s \in H$ is finite;

- the intersection of r sets belonging to distinct $\mathcal{H}_i$s is a finite set.

We say that $(\mathcal{H}_i)_{i=1}^N$ is of *mesh r* if r is least such that the sequence is (r, s) finitely determined, for some s.

Examples 2.2. (i) Let $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ be noncollinear points of $\mathbb{R}^3$, and let $\mathcal{H}_i$ be the collection of all spheres of with center $\boldsymbol{c}_i$. Then $(\mathcal{H}_i)_{i=1}^3$ is $(3, 1)$ finitely determined, and of mesh 3.

(ii) Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ be a basis for $\mathbb{R}^3$, and let $\mathcal{H}_i$ be the set of all planes orthogonal to $\boldsymbol{u}_i$. Then $(\mathcal{H}_i)_{i=1}^3$ is $(3, 1)$ finitely determined, and of mesh 3.

The reason for the notion of mesh is the following very general result by Erdős, Jackson, and Mauldin [Erdős et al. 1994, Theorem 2]:

Theorem 2.3. *For all* $d \geq 2$, $n \geq 0$, *and* $\delta = 0, 1$ *the following are equivalent*:

(a) $2^{\aleph_0} \leq \aleph_{\delta+n}$;

(b) *for any* $r \geq 2$, *letting* $N = (n+1)(r-1)+1$, *and for any sequence of pairwise disjoint* $\mathcal{H}_i \subseteq \mathscr{P}(\mathbb{R}^d)$ *with* $1 \leq i \leq N$ *of mesh* r, *there are* $A_1, \ldots, A_N$ *covering* $\mathbb{R}^d$ *such that* $\forall 1 \leq i \leq N \, \forall H \in \mathcal{H}_i \, (|H \cap A_i| < \aleph_\delta)$.

Actually the result in [Erdős et al. 1994] is more general than Theorem 2.3, as $\mathbb{R}^d$ can be replaced by an arbitrary infinite set X, and $2^{\aleph_0}$ can be replaced by $|X|$. Also δ can be any ordinal, not just 0 or 1.

The main use of Theorem 2.3 in this paper is the direction (a)⇒(b). Given $(\mathcal{H}_i)_{i=1}^N$ of mesh r in $\mathbb{R}^d$, then $|\mathbb{R}| \leq \aleph_{\delta+n}$ implies that there are $A_1, \ldots, A_N \subseteq \mathbb{R}^d$ covering $\mathbb{R}^d$ such that for all $H \in \mathcal{H}_i$, the set $H \cap A_i$ is *finite* if $\delta = 0$ or *countable* if $\delta = 1$. Actually, the "forward implication" (i.e. from a bound on the size of the continuum, to the existence of specific subsets of the space) in [Bagemihl 1959/60; 1968; Davies 1962; 1963a; 1963b; Komjáth 2001] follows from Theorem 2.3.

2.3. *Points and vectors in general position.*

Definition 2.4. A set of vectors of $\mathbb{R}^d$ is in *general position* if any subset of size $\leq d$ is linearly independent — in other words: the vectors are as linearly independent as possible.

The following is a straightforward generalization of Examples 2.2(ii).

Example 2.5. If $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ are vectors in general position in $\mathbb{R}^d$, and $\mathcal{H}_i := \{H_{\boldsymbol{u}_i}(\boldsymbol{p}) \mid \boldsymbol{p} \in \mathbb{R}^d\}$ is the family of all affine hyperplanes orthogonal to $\boldsymbol{u}_i$, then $(\mathcal{H}_i)_{i=1}^N$ is of mesh d.

By Example 2.5 and Theorem 2.3, if $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{(n+1)(d-1)+1}$ are vectors in general position in $\mathbb{R}^d$, then

- $2^{\aleph_0} \leq \aleph_n$ implies that there are $A_1, \ldots, A_{(n+1)(d-1)+1}$ covering $\mathbb{R}^d$ such that $H_{\boldsymbol{u}_i}(\boldsymbol{p}) \cap A_i$ is finite for every $\boldsymbol{p} \in \mathbb{R}^d$ and $1 \leq i \leq (n+1)(d-1)+1$;

- $2^{\aleph_0} \leq \aleph_{n+1}$ implies that there are $A_1, \ldots, A_{(n+1)(d-1)+1}$ covering $\mathbb{R}^d$ such that $H_{\boldsymbol{u}_i}(\boldsymbol{p}) \cap A_i$ is countable, for every $\boldsymbol{p} \in \mathbb{R}^d$ and $1 \leq i \leq (n+1)(d-1)+1$.

Therefore CH implies that if $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d$ are a basis of $\mathbb{R}^d$, there are $A_1, \ldots, A_d$ covering $\mathbb{R}^d$ such that every affine hyperplane orthogonal to $\boldsymbol{u}_i$ has countable intersection with A_i.

In fact, the implications above are actually equivalences [Komjáth and Totik 2006, pp. 71, 327–328].

Theorem 2.6 (Sierpiński). CH *is equivalent to either one of the following*:

(a) *There are A_1, A_2 covering $\mathbb{R}^2$ such that every vertical line has countable intersection with A_1 and every horizontal line has countable intersection with A_2.*

(b) *There are A_1, A_2, A_3 covering $\mathbb{R}^3$ such that every plane orthogonal to $\boldsymbol{e}_i$ has countable intersection with A_i.*

Definition 2.4 can be extended to points in affine spaces.

Definition 2.7. Let S be a set of points of an affine space E of dimension d. We say that S is

- in *general position in E* if the affine span of any of its subset of size $k+1$ has dimension k, for all $k \leq d$,
- *well-placed in E* if $S \subseteq H$ for some hyperplane H of E, and S is in general position in H.

It is common practice in geometry to use "general position" for *vectors* (a linear algebra notion, Definition 2.4) and for *points* (an affine notion, Definition 2.7), and this could give rise to some ambiguity when the affine space is $\mathbb{R}^d$. For example, the *vectors* $(-1, 1)$, $(0, 1)$, $(1, 1)$ are in general position, but the set of *points* $\{(-1, 1), (0, 1), (1, 1)\}$ is not. For this reason we will use the terms "vectors" and "points" to help the reader sort out which of the two notions is being used.

Remarks 2.8. Let E be an affine space of dimension $d \geq 2$ and let $S \subseteq E$.

(a) If S is in general position (well-placed), and $S' \subseteq S$ then S' is in general position (well-placed). In other words, the notions of being in general position/well-placed are downward persistent with respect to inclusion.

(b) Suppose S is well-placed:
   - if $|S| \geq d$ then the hyperplane in the definition is unique, being the affine span of any subset of size d;
   - if $|S| \leq d$ then S is in general position in E.

(c) Suppose H is a hyperplane of E, and $S \subseteq H$. Then S is well-placed in E if and only if S is in general position in H.

(d) The set of points S is in general position in E if and only if the vectors in $\{\boldsymbol{q} - \boldsymbol{p} \mid \boldsymbol{q} \in S \setminus \{\boldsymbol{p}\}\}$ are in general position in $\mathbb{R}^d$, for any $\boldsymbol{p} \in S$.

Every set of points in $\mathbb{R}$ is in general position, a set of points in $\mathbb{R}^2$ is in general position if no three of them are collinear, a set of points in $\mathbb{R}^3$ is in general position if no four of them are coplanar, and so on. A set of points in $\mathbb{R}^2$ is well-placed if they are collinear, a set of points in $\mathbb{R}^3$ is well-placed if they are coplanar, and no three of them are collinear, a set of points in $\mathbb{R}^4$ is well-placed if they belong to the same 3-dimensional affine subspace, and no four of them are coplanar, and so on.

Finally observe that for any affine space E of dimension $d \geq 2$

(1) if $S \subseteq E$ is a set of points in general position or well-placed in E, then the affine span of any subset of S of size $k+1$ has dimension k, for any $k < d$.

Lemma 2.9. *Suppose $\boldsymbol{c}_1, \dots, \boldsymbol{c}_{k+1}$ are distinct points in general position in $\mathbb{R}^d$, with $d \geq k$. Let H be an affine hyperplane orthogonal to $\boldsymbol{c}_k - \boldsymbol{c}_{k+1}$, let $\pi : \mathbb{R}^d \to H$ be the orthogonal projection. Let $\bar{\boldsymbol{c}}_k = \pi(\boldsymbol{c}_k) = \pi(\boldsymbol{c}_{k+1})$ and let $\bar{\boldsymbol{c}}_i = \pi(\boldsymbol{c}_i)$ for $i < k$. Then the points $\bar{\boldsymbol{c}}_1, \dots, \bar{\boldsymbol{c}}_k$ are distinct and in general position in H.*

Proof. If $\bar{\boldsymbol{c}}_i = \bar{\boldsymbol{c}}_k$ for some $i < k$, then $\boldsymbol{c}_i, \boldsymbol{c}_k, \boldsymbol{c}_{k+1}$ must be collinear, and if $\bar{\boldsymbol{c}}_i = \bar{\boldsymbol{c}}_j$ for some $i < j < k$, then $\boldsymbol{c}_i - \boldsymbol{c}_j$ and $\boldsymbol{c}_k - \boldsymbol{c}_{k+1}$ must be parallel, so the four points $\boldsymbol{c}_i, \boldsymbol{c}_j, \boldsymbol{c}_k, \boldsymbol{c}_{k+1}$ are coplanar. In either case this contradicts the assumption that $\boldsymbol{c}_1, \dots, \boldsymbol{c}_{k+1}$ are in general position. Therefore the points $\bar{\boldsymbol{c}}_1, \dots, \bar{\boldsymbol{c}}_k$ are distinct.

In order to conclude the proof it is enough to show that the vectors $\bar{\boldsymbol{c}}_1 - \bar{\boldsymbol{c}}_k$, $\bar{\boldsymbol{c}}_2 - \bar{\boldsymbol{c}}_k, \dots, \bar{\boldsymbol{c}}_{k-1} - \bar{\boldsymbol{c}}_k$ of $\mathbb{R}^d$ are linearly independent. Suppose

$$\boldsymbol{0} = r_1(\bar{\boldsymbol{c}}_1 - \bar{\boldsymbol{c}}_k) + r_2(\bar{\boldsymbol{c}}_2 - \bar{\boldsymbol{c}}_k) + \cdots + r_{k-1}(\bar{\boldsymbol{c}}_{k-1} - \bar{\boldsymbol{c}}_k).$$

As $\bar{\boldsymbol{c}}_i$ is the projection of $\boldsymbol{c}_i$ along the vector $\boldsymbol{c}_{k+1} - \boldsymbol{c}_k$, there are $s_1, \dots, s_k \in \mathbb{R}$ such that $\bar{\boldsymbol{c}}_i = \boldsymbol{c}_i + s_i(\boldsymbol{c}_{k+1} - \boldsymbol{c}_k)$. Substituting in the previous formula we obtain

$$\boldsymbol{0} = r_1(\boldsymbol{c}_1 - \boldsymbol{c}_k) + \cdots + r_{k-1}(\boldsymbol{c}_{k-1} - \boldsymbol{c}_k) + [r_1(s_1 - s_k) + \cdots + r_{k-1}(s_{k-1} - s_k)](\boldsymbol{c}_{k+1} - \boldsymbol{c}_k),$$

and since $\boldsymbol{c}_1, \dots, \boldsymbol{c}_{k+1}$ are in general position in $\mathbb{R}^d$, then $[r_1(s_1 - s_k) + \cdots + r_{k-1}(s_{k-1} - s_k)] = 0$ and $r_1 = \cdots = r_{k-1} = 0$, which is what we had to prove. □

Lemma 2.10. *Suppose H is a hyperplane of an affine space E of dimension $d+1$. Let $\pi : E \to H$ be the orthogonal projection, and let h be the distance between a point $\boldsymbol{c} \in E$ and H. Given R let $r = \sqrt{R^2 - h^2}$ if $R \geq h$ and $r = -1$ if $R \leq h$. Then $\mathbb{S}(E; \boldsymbol{c}, R) \cap H = \mathbb{S}(H; \pi(\boldsymbol{c}), r)$.*

Conversely, if $r > 0$ then $\mathbb{S}(E; \boldsymbol{c}, \sqrt{r^2 + h^2}) \cap H = \mathbb{S}(H, \pi(\boldsymbol{c}), r)$.

Proof. Without loss of generality we may assume that $E = \mathbb{R}^{d+1}$, that $H = \mathbb{R}^d \times \{0\}$, $\boldsymbol{c} = (0, \dots, 0, h)$, and $\pi(\boldsymbol{c}) = \boldsymbol{0}$, so that $h = \|\boldsymbol{c} - \boldsymbol{0}\|$. It is clear that $R < h$ if and only if $\mathbb{S}(\boldsymbol{c}, R) \cap H = \varnothing$, and that $R = h$ if and only if $\mathbb{S}(\boldsymbol{c}, R) \cap H = \{\pi(\boldsymbol{c})\} = \{\mathbf{0}\}$. If $R > h$ then $r^2 + h^2 = R^2$ so

$$\begin{aligned} \boldsymbol{x} \in \mathbb{S}(E; \boldsymbol{c}, R) \cap H &\iff \boldsymbol{x} = (x_1, \dots, x_d, 0) \wedge (\textstyle\sum_{i=1}^d x_i^2) + h^2 = R^2 \\ &\iff x_1^2 + \cdots + x_d^2 = r^2 \\ &\iff \|\boldsymbol{x}\| = r \\ &\iff \boldsymbol{x} \in \mathbb{S}(H; \pi(\boldsymbol{c}), r). \end{aligned}$$

The equivalences above prove the second part of the statement as well. □

Lemma 2.11. *Suppose $\boldsymbol{c}_1, \boldsymbol{c}_2$ are distinct points of an affine space E of dimension $d \geq 2$, and let $r_1, r_2 > 0$ be such that $|r_1 - r_2| \leq \|\boldsymbol{c}_1 - \boldsymbol{c}_2\| \leq r_1 + r_2$. Then*

$$\mathbb{S}(E; \boldsymbol{c}_1, r_1) \cap \mathbb{S}(E; \boldsymbol{c}_2, r_2) = \mathbb{S}(H; \boldsymbol{c}, r),$$

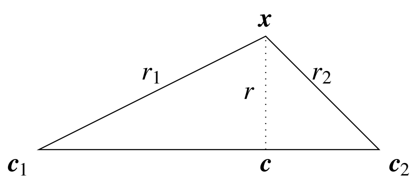
where $\boldsymbol{c}$ is a point of the open segment $(\boldsymbol{c}_1; \boldsymbol{c}_2)$, H is the hyperplane passing through $\boldsymbol{c}$ and orthogonal to $\boldsymbol{c}_1 - \boldsymbol{c}_2$, and $r \geq 0$.

Proof. We may assume that $E = \mathbb{R}^d$. The hypothesis $|r_1 - r_2| \leq \|\boldsymbol{c}_1 - \boldsymbol{c}_2\|$ guarantees that neither sphere properly contains inside the other sphere, while the hypothesis $\|\boldsymbol{c}_1 - \boldsymbol{c}_2\| \leq r_1 + r_2$ ensures that if neither sphere contains the other, then they are close enough to intersect.

Let $\boldsymbol{x}$ be a point of $\mathbb{S}(\boldsymbol{c}_1, r_1) \cap \mathbb{S}(\boldsymbol{c}_2, r_2)$, and let P be the plane passing through $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{x}$. (If $d = 2$ then P coincides with $\mathbb{R}^2$.) Consider the triangle with vertices $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{x}$.

If $r_1 + r_2 = \|\boldsymbol{c}_1 - \boldsymbol{c}_2\|$ then the two spheres are tangent in the point $\boldsymbol{c} := \boldsymbol{x}$ which belongs to the open segment $(\boldsymbol{c}_1; \boldsymbol{c}_2)$, so the triangle is degenerate and letting $r = 0$ we have $\mathbb{S}(H; \boldsymbol{c}, r) = \{\boldsymbol{c}\}$.

So we may assume that $r_1 + r_2 > \|\boldsymbol{c}_1 - \boldsymbol{c}_2\|$ and that the triangle



is nondegenerate. Let $\boldsymbol{c}$ be the projection of $\boldsymbol{x}$ on the open segment $(\boldsymbol{c}_1; \boldsymbol{c}_2)$, let $r = \|\boldsymbol{x} - \boldsymbol{c}\| > 0$, and let H be the hyperplane orthogonal to $\boldsymbol{c}_1 - \boldsymbol{c}_2$ passing through $\boldsymbol{c}$. Observe that r, the point $\boldsymbol{c}$, and the hyperplane H, depend only on the points $\boldsymbol{c}_1, \boldsymbol{c}_2$ and on r_1, r_2, and not on the point $\boldsymbol{x}$ or the plane P. Therefore, for all $\boldsymbol{x} \in \mathbb{R}^d$,

$$\begin{aligned} \boldsymbol{x} \in \mathbb{S}(\boldsymbol{c}_1, r_1) \cap \mathbb{S}(\boldsymbol{c}_2, r_2) &\iff \|\boldsymbol{x} - \boldsymbol{c}_1\| = r_1 \wedge \|\boldsymbol{x} - \boldsymbol{c}_2\| = r_2 \\ &\iff \boldsymbol{x} \in H \wedge \|\boldsymbol{c} - \boldsymbol{x}\| = r. \end{aligned}$$

□

Proposition 2.12. *Let E be an affine space of dimension $d \geq 2$, let $\boldsymbol{c}_1, \dots, \boldsymbol{c}_k$ be points in general position in E with $k \leq d$, and let K be the affine span of the $\boldsymbol{c}_i$s. Let S_i be a sphere centered in $\boldsymbol{c}_i$ for $i \leq k$.*

(a) *There are $r \in \mathbb{R}$ and an affine subspace H of E of dimension $d - (k-1)$, such that $H \cap K$ is a singleton $\{\boldsymbol{c}\}$ and H and K are orthogonal, so that $S_1 \cap \dots \cap S_k = \mathbb{S}(H; \boldsymbol{c}, r)$.*

(b) *If $\boldsymbol{c}_{k+1} \in E \setminus \{\boldsymbol{c}_1, \dots, \boldsymbol{c}_k\}$ and $\{\boldsymbol{c}_1, \dots, \boldsymbol{c}_k, \boldsymbol{c}_{k+1}\}$ is not in general position in E, then there is $R \in \mathbb{R}$ such that*

$$S_1 \cap \dots \cap S_k \subseteq \mathbb{S}(\boldsymbol{c}_{k+1}, R).$$

Proof. We may assume that $E = \mathbb{R}^d$.

If $d = 2$ then the result is clear. Part (a) amounts to say that: given two circles C_1, C_2 in the plane with distinct centers $\boldsymbol{c}_1, \boldsymbol{c}_2$, either $I := C_1 \cap C_2$ is empty, or else there is a point $\boldsymbol{c}$ on the open segment $(\boldsymbol{c}_1; \boldsymbol{c}_2)$ such that $I = \{\boldsymbol{c}\}$, or else I is the set of two points on line orthogonal to $(\boldsymbol{c}_1; \boldsymbol{c}_2)$ passing through $\boldsymbol{c}$, and symmetric with respect to $\boldsymbol{c}$. Part (b) says that if three points are collinear, then given any two circles centered in the first two points there is a (possibly degenerate) circle centered in the third point that passes through the intersection of the first two circles.

Therefore we may assume that $d \geq 3$.

(a) We proceed by induction on k. If $k = 1$ the result is trivial, so we may assume the result holds for some k towards proving it for $k + 1 \leq d$. Suppose $S_1, \dots, S_{k+1}$ are spheres centered in distinct points $\boldsymbol{c}_1, \dots, \boldsymbol{c}_{k+1}$. By an isometry we may assume that K, the affine span of the $\boldsymbol{c}_i$s, is contained in $\mathbb{R}^{d-1} \times \{0\}$. Let $\bar{S}_k := S_k \cap S_{k+1}$. We have three cases:

(1) $\bar{S}_k = \varnothing$.

(2) $\bar{S}_k$ is a singleton.

(3) $\bar{S}_k$ is a nondegenerate sphere in some affine hyperplane H of $\mathbb{R}^d$ (Lemma 2.11).

If either (1) or (2) holds, then $S_1 \cap \dots \cap S_{k+1}$ is empty or a singleton, and the result follows trivially. So we may assume case (3). The hyperplane H is orthogonal to the open segment $(\boldsymbol{c}_k; \boldsymbol{c}_{k+1})$, and $\bar{\boldsymbol{c}}_k$, the center of $\bar{S}_k$, belongs to this segment. Let $\pi : \mathbb{R}^d \to H$ be the orthogonal projection. Then $\pi(\boldsymbol{c}_k) = \pi(\boldsymbol{c}_{k+1}) = \bar{\boldsymbol{c}}_k$ and for $i < k$ let $\bar{\boldsymbol{c}}_i := \pi(\boldsymbol{c}_i)$. By Lemma 2.10 for every $i < k$ the set $\bar{S}_i := S_i \cap H$ is a sphere of H with center $\bar{\boldsymbol{c}}_i$. As $S_k \cap S_{k+1} \subseteq H$

$$S_1 \cap \dots \cap S_k \cap S_{k+1} = \bar{S}_1 \cap \dots \cap \bar{S}_k.$$

By Lemma 2.9 $\bar{\boldsymbol{c}}_1, \dots, \bar{\boldsymbol{c}}_k$ are in general position in $H \cong \mathbb{R}^{d-1}$, and since $d - 1 \geq 2$, by inductive assumption we can conclude that $\bar{S}_1 \cap \dots \cap \bar{S}_k = \mathbb{S}(H'; \boldsymbol{c}, r)$ where

H' a subspace of H (and hence of $\mathbb{R}^d$) of dimension $(d-1)-(k-1)=d-k$, such that $\{\boldsymbol{c}\}=H'\cap K$, and H' is orthogonal to K.

(b) By assumption $\boldsymbol{c}_{k+1}$ belongs to K, and $k-1=\dim(K)$. By part (a) there is a subspace H orthogonal to K, of dimension $d-(k-1)$, and $r\in\mathbb{R}$ such that $S_1\cap\dots\cap S_k=\mathbb{S}(H;\boldsymbol{c},r)$ where $\boldsymbol{c}$ is the unique element of $H\cap K$. It is enough to find R such that $\mathbb{S}(H;\boldsymbol{c},r)$ is contained in $\mathbb{S}(E;\boldsymbol{c}_{k+1},R)$. This is clear if $r<0$; otherwise take $R=\sqrt{\|\boldsymbol{c}_{k+1}-\boldsymbol{c}\|^2+r^2}$, this value works by Pythagoras theorem as the vectors $\boldsymbol{x}-\boldsymbol{c}$ and $\boldsymbol{c}_{k+1}-\boldsymbol{c}$ are orthogonal whenever $\boldsymbol{x}\in\mathbb{S}(H;\boldsymbol{c},r)$. □

Theorem 2.13. *Let E be an affine space of dimension $d\geq 2$ and let $\boldsymbol{c}_1,\dots,\boldsymbol{c}_d$ be distinct points in general position in E.*

(a) *For all spheres S_i in E centered in $\boldsymbol{c}_i$ with $1\leq i\leq d$, the set $S_1\cap\dots\cap S_d$ is finite.*
(b) *For all $k<d$ and $1\leq i\leq k$ there are spheres S_i in E centered in $\boldsymbol{c}_i$ such that $S_1\cap\dots\cap S_k$ is a nondegenerate sphere in a subspace of dimension $d-(k-1)\geq 2$.*

Proof. Recall that a nondegenerate sphere in space of dimension n has cardinality $2^{\aleph_0}$ or 2 depending whether $n>1$ or $n=1$. Part (a) then follows at once form Proposition 2.12.

(b) We prove by induction on k a stronger statement: for any $r_k>0$ and for any affine space E of dimension $d>k$, there exist $r_1,\dots,r_{k-1}>0$ such that

$$\mathbb{S}(E;\boldsymbol{c}_1,r_1)\cap\dots\cap\mathbb{S}(E;\boldsymbol{c}_k,r_k)=\mathbb{S}(H;\boldsymbol{c},r),$$

for some $r>0$, $\boldsymbol{c}\in H$ and H a subspace of E of dimension $d-(k-1)$.

The base case $k=1$ is immediate, so we may assume that the result holds for some k towards proving it for $k+1<d$. Fix $r_{k+1}>0$ and pick $r_k>0$ such that $|r_k-r_{k+1}|<\|\boldsymbol{c}_k-\boldsymbol{c}_{k+1}\|<r_k+r_{k+1}$. By Lemma 2.11 $\mathbb{S}(E;\boldsymbol{c}_{k+1},r_{k+1})\cap\mathbb{S}(E;\boldsymbol{c}_k,r_k)$ is a nondegenerate sphere $\mathbb{S}(\bar{E};\bar{\boldsymbol{c}}_k,\bar{r}_k)$ in some hyperplane $\bar{E}$ of E. Let $\pi:E\to\bar{E}$ be the orthogonal projection. By Lemma 2.9 the points

$$\bar{\boldsymbol{c}}_1=\pi(\boldsymbol{c}_1),\quad \bar{\boldsymbol{c}}_2=\pi(\boldsymbol{c}_2),\quad \dots\quad \bar{\boldsymbol{c}}_{k-1}=\pi(\boldsymbol{c}_{k-1}),\quad \bar{\boldsymbol{c}}_k=\pi(\boldsymbol{c}_k)=\pi(\boldsymbol{c}_{k+1})$$

are distinct and in general position in the affine space $\bar{E}$ of dimension $d-1>k$. Since $\bar{r}_k>0$, by inductive assumption there exist $\bar{r}_1,\dots,\bar{r}_{k-1}>0$ such that

$$S:=\mathbb{S}(\bar{E},\bar{\boldsymbol{c}}_1,\bar{r}_1)\cap\dots\cap\mathbb{S}(\bar{E},\bar{\boldsymbol{c}}_k,\bar{r}_k)$$

is a nondegenerate sphere in a subspace H of dimension $(d-1)-(k-1)=d-k$. Letting $r_i=\sqrt{\bar{r}_i^2+h_i^2}$ where h_i is the distance between $\boldsymbol{c}_i$ and the hyperplane $\bar{E}$, we have

$$S=\mathbb{S}(E,\boldsymbol{c}_1,r_1)\cap\dots\cap\mathbb{S}(E,\boldsymbol{c}_k,r_k)\cap\mathbb{S}(E,\boldsymbol{c}_{k+1},r_{k+1})$$

as required. □

Corollary 2.14. *Suppose E is an affine space of dimension $d \geq 2$ and $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{d+1}$ are distinct points that are not in general position in E. Then there are spheres S_i centered in $\boldsymbol{c}_i$ such that $S_1 \cap \cdots \cap S_{d+1}$ is infinite.*

Proof. Let k be the dimension of the affine span K of $\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{d+1}\}$. By assumption $k < d$, and without loss of generality we may assume that $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{k+1}$ are in general position in K. By part (b) of Theorem 2.13 there are spheres $S_1, \ldots, S_{k+1}$ centered in $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{k+1}$ whose intersection is a nondegenerate sphere in a space of dimension ≥ 2. By repeated applications of part (b) of Proposition 2.12, there are spheres $S_{k+2}, \ldots, S_{d+1}$ centered in $\boldsymbol{c}_{k+2}, \ldots, \boldsymbol{c}_{d+1}$ such that

$$S_1 \cap \cdots \cap S_{k+1} \subseteq S_i \quad \text{for } k+2 \leq i \leq d+1$$

and hence $S_1 \cap \cdots \cap S_{k+1} = S_1 \cap \cdots \cap S_{d+1}$ is infinite. □

Observe that by (1), both "S is in general position" and "S is well-placed" imply the assumption of the next corollary.

Corollary 2.15. *Suppose S is a set of at least d points of an affine space E of dimension $d \geq 2$. Assume that any d-many points of S span a subspace of dimension $d-1$. Then the intersection of d-many spheres centered in these points is a finite set.*

Lemma 2.16. *Suppose $N \geq d$ and $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_N$ are distinct points of $\mathbb{R}^d$ such that any d of them span an affine subspace of dimension $d-1$. If $\mathcal{H}_i$ is the collection of all spheres of $\mathbb{R}^d$ with center $\boldsymbol{c}_i$, then $(\mathcal{H}_i)_{i=1}^N$ is of mesh d.*

Proof. It is enough to consider the case $N = d$. If $d = 1$ this is trivial since a sphere in $\mathbb{R}$ is just a pair of points that are symmetric with respect to the center, and if $d = 2$ it follows immediately from the fact that the intersection of two circles with distinct centers has size at most 2. If $d = 3$ this follows from the fact that the intersection of three spheres in $\mathbb{R}^3$ with noncollinear centers has size at most 2. The case for $d > 3$ follows from Theorem 2.13. □

2.4. *Sprays.* J. Schmerl [2003] defined a spray to be a set $X \subseteq \mathbb{R}^2$ such that $C \cap X$ is finite, for all circles C centered in some given point. Thus a spray X is a very sparse subset of the plane, and one may investigate what happens if one relaxes the finiteness condition on $C \cap X$ with, say, being countable.

Definition 2.17. Let E be an affine subspace of $\mathbb{R}^d$, with $d \geq 2$ be a natural number. An *$\aleph_\delta$-spray in E with center $\boldsymbol{c}$* is a set $X \subseteq E$ such that $|\mathbb{S}(E; \boldsymbol{c}, r) \cap X| < \aleph_\delta$ for every $r > 0$. When $\delta = 0$ it is called a *spray*, when $\delta = 1$ we speak of *σ-spray*.

Note that a center need not to belong to the $\aleph_\delta$-spray, and that a $\aleph_\delta$-spray may have more than one center.

By Lemma 2.16, if $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{(n+1)(d-1)+1}$ are either in general position, or well-placed in $\mathbb{R}^d$, then

- $2^{\aleph_0} \leq \aleph_n$ implies that there are $X_1, \ldots, X_{(n+1)(d-1)+1}$ covering $\mathbb{R}^d$ such that each X_i is a spray centered in $\boldsymbol{c}_i$;
- $2^{\aleph_0} \leq \aleph_{n+1}$ implies that there are $X_1, \ldots, X_{(n+1)(d-1)+1}$ covering $\mathbb{R}^d$ such that each X_i is a σ-spray.

In particular if the points are well-placed, that is if we assume that the $\boldsymbol{c}_i$s lie on a hyperplane, then:

Theorem 2.18. *Let $d \geq 2$, $n \geq 0$, and $\delta = 0, 1$. Assume that $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{(n+1)(d-1)+1}$ are well-placed points in $\mathbb{R}^d$. If $2^{\aleph_0} \leq \aleph_{\delta+n}$ then there are $X_1, \ldots, X_{(n+1)(d-1)+1}$ covering $\mathbb{R}^d$ such that each X_i is an $\aleph_\delta$-spray centered in $\boldsymbol{c}_i$.*

By (1) and Corollary 2.15, the thesis of Theorem 2.18 holds if $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{(n+1)(d-1)+1}$ are in general position in $\mathbb{R}^d$. The reason for focusing on well-placed points is that under this assumption the implication in Theorem 2.18 can be reversed, and this is the main goal of this paper. In Section 3 we argue that any covering of $\mathbb{R}^d$ with sprays (σ-sprays) with well-placed centers $\boldsymbol{c}_i$s can be transformed into a covering (of an open subset) of $\mathbb{R}^d$ with sets whose intersection with the hyperplanes orthogonal to $\boldsymbol{u}_i$ is finite (countable) for all i, where the vector $\boldsymbol{u}_i$ is obtained from the point $\boldsymbol{c}_i$. Appealing to the results from Section 4 the equivalence will be established.

The case of points in general position, but not well-placed, i.e. not belonging to a hyperplane, is more problematic. If $d = 2$ and three points are not collinear (i.e. are in general position, but not well-placed in $\mathbb{R}^2$) then by [Schmerl 2010, Theorem 1], the plane is the union of three sprays centered in these points, irrespective of the size of the continuum. The analogous statement for $d > 2$ is open, and if true it would show that the assumption that the points are well-placed cannot be dropped.

Conjecture 2.19. *For all $d \geq 3$, for all $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{d+1}$ in general position in $\mathbb{R}^d$, there are $X_1, \ldots, X_{d+1}$ covering $\mathbb{R}^d$, with X_i a spray centered in $\boldsymbol{c}_i$ for $1 \leq i \leq d+1$.*

2.5. *Elementary results about sprays.* The following basic facts are nonetheless useful.

Gluing: If X and Y are $\aleph_\delta$-sprays in an affine space E with the same center $\boldsymbol{c}$, then $X \cup Y$ is a $\aleph_\delta$-spray in E with center $\boldsymbol{c}$.

Projection: Suppose that $X_1, \ldots, X_n$ are $\aleph_\delta$-sprays in an affine space E of dimension $d+1$, and $X_1 \cup \cdots \cup X_n = E$ with centers $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$. Suppose H is a hyperplane of E, and let $\pi : E \to H$ be the orthogonal projection. Then $X_1 \cap H, \ldots, X_n \cap H$ are $\aleph_\delta$-sprays in H with centers $\pi(\boldsymbol{c}_1), \ldots, \pi(\boldsymbol{c}_n) \in H$, that cover H.

Projection follows from Lemma 2.10. Observe that the points $\pi(\boldsymbol{c}_1), \dots, \pi(\boldsymbol{c}_n)$ need not be distinct, even if the $\boldsymbol{c}_1, \dots, \boldsymbol{c}_n$ are distinct. This, together with gluing, allows to transform n-many $\aleph_\delta$-sprays covering $\mathbb{R}^{d+1}$ into n'-many $\aleph_\delta$-sprays covering $\mathbb{R}^d$, with $n' < n$.

Proposition 2.20. *Let $\delta \in \mathrm{Ord}$ and suppose $\mathbb{R}^2 = X_1 \cup X_2$ where X_1 is an $\aleph_\delta$-spray and X_2 is an $\aleph_{\delta+1}$-spray. Then $2^{\aleph_0} \leq \aleph_\delta$.*

Proof. For $i = 1, 2$, let $\boldsymbol{c}_i$ be the center of X_i, and for $r > 0$ let

$$C_i(r) := \{\boldsymbol{p} \in \mathbb{R}^2 \mid \|\boldsymbol{p} - \boldsymbol{c}_i\| = r\}$$

be the circle centered in $\boldsymbol{c}_i$ of radius r. If $\boldsymbol{c}_1 = \boldsymbol{c}_2$ then by gluing $X_1 \cup X_2$ would be an $\aleph_{\delta+1}$-spray in $\mathbb{R}^2$, and every circle centered in $\boldsymbol{c}_1$ is contained in $X_1 \cup X_2 = \mathbb{R}^2$ is of cardinality $\leq \aleph_\delta$. As any circle is in bijection with $\mathbb{R}$, the result follows. Therefore we may assume that $\boldsymbol{c}_1 \neq \boldsymbol{c}_2$. Applying an isometry if needed, we may assume that $\boldsymbol{c}_1 = (0, 0)$ and $\boldsymbol{c}_2 = (a, 0)$ for some $a > 0$.

Towards a contradiction, suppose $2^{\aleph_0} > \aleph_\delta$. Fix distinct reals r_α in the interval $(a/2; a)$, for $\alpha < \aleph_\delta$. By assumption $X_2(\alpha) := X_2 \cap C_2(r_\alpha)$ has size $\leq \aleph_\delta$, so the set

$$\{r \in (a/2; a) \mid C_1(r) \cap (\textstyle\bigcup_{\alpha \in \aleph_\delta} X_2(\alpha)) \neq \emptyset\}$$

has size $\leq \aleph_\delta$. As $|\mathbb{R}| > \aleph_\delta$ we may pick $r \in (a/2; a)$ outside of this set. For each $\alpha \in \aleph_\delta$ the set $C_1(r) \cap C_2(r_\alpha)$ has size 2, and its points belong to X_1. As the $C_2(r_\alpha)$s are disjoint, it follows that $C_1(r) \cap X_1$ has size $\geq \aleph_\delta$, contradicting that X_1 is an $\aleph_\delta$-spray. □

When $\delta = 0$ we obtain at once:

Corollary 2.21. *$\mathbb{R}^2$ is not the union of a spray and a σ-spray. In particular, $\mathbb{R}^2$ is not the union of two sprays.*

Corollary 2.22. *The following are equivalent:*

(a) CH *holds.*

(b) *$\mathbb{R}^2$ is the union of two σ-sprays with prescribed, distinct centers.*

Proposition 2.23. *Let $n \geq d \geq 3$, and suppose that the points $\boldsymbol{c}_1, \dots, \boldsymbol{c}_n$ belong to an affine hyperplane H of $\mathbb{R}^d$. Suppose there is $L \subseteq H$ an affine subspace of dimension $d - 2$ such that $\{\pi(\boldsymbol{c}_1), \dots, \pi(\boldsymbol{c}_n)\}$ has size $\leq d - 1$, where $\pi : H \to L$ is the orthogonal projection. Then there are no sprays $X_1, \dots, X_n$ that cover $\mathbb{R}^d$ with X_i centered in $\boldsymbol{c}_i$, for $i \leq n$.*

Proof. Towards a contradiction, let $d \geq 3$ be least such that the statement fails, and suppose X_i is a spray centered in $\boldsymbol{c}_i$ such that $\mathbb{R}^d = X_1 \cup \dots \cup X_n$. Then $H \cong \mathbb{R}^{d-1}$ is covered by the sprays $X_1 \cap H, \dots, X_n \cap H$ centered in $\{\pi(\boldsymbol{c}_1), \dots, \pi(\boldsymbol{c}_n)\}$. This means that $\mathbb{R}^{d-1}$ can be covered by $d - 1$-many sprays whose centers lie on the hyperplane L. This contradicts the minimality of d, if $d-1 \geq 3$. If $d=3$, then we would have that $\mathbb{R}^2$ can be covered by two sprays, against Corollary 2.21. □

In [Schmerl 2012, p. 1169] it is observed that the next result follows from results of Sikorski.

Theorem 2.24. *For $d \geq 2$ the space $\mathbb{R}^d$ is not the union of d-many sprays.*

Proof. The case $d=2$ is Corollary 2.21. Suppose $d \geq 3$ and that $\boldsymbol{c}_1, \dots, \boldsymbol{c}_d$ are centers of sprays $X_1, \dots, X_d$ that cover $\mathbb{R}^d$. Let H be a hyperplane containing these points, and apply Proposition 2.23 with $n = d-1$: if L is an affine subspace of H of dimension $d-2 > 0$ that is orthogonal to the vector $\boldsymbol{c}_d - \boldsymbol{c}_{d-1}$, then $\{\pi(\boldsymbol{c}_1), \dots, \pi(\boldsymbol{c}_d)\}$ has size $\leq d-1$ since $\pi(\boldsymbol{c}_d) = \pi(\boldsymbol{c}_{d-1})$. But Proposition 2.23 implies that the X_is cannot cover $\mathbb{R}^d$. □

A similar argument shows:

Theorem 2.25. *For $d \geq 2$ the space $\mathbb{R}^d$ is not the union of $(d-1)$-many σ-sprays.*

Theorem 2.26. *For $d \geq 2$ the following are equivalent*:

(a) CH;

(b) *for all well-placed $\boldsymbol{c}_1, \dots, \boldsymbol{c}_d \in \mathbb{R}^d$ there are $X_1, \dots, X_d$ covering $\mathbb{R}^d$ such that each X_i is a σ-spray with center $\boldsymbol{c}_i$*;

(c) *there are well-placed $\boldsymbol{c}_1, \dots, \boldsymbol{c}_d \in \mathbb{R}^d$ and $X_1, \dots, X_d$ covering $\mathbb{R}^d$ such that each X_i is a σ-spray with center $\boldsymbol{c}_i$.*

Proof. (a)⇒(b) follows from Theorem 2.18; (b)⇒(c) is trivial, while (c)⇒(a) is established by induction on $d \geq 2$.

When $d=2$ the result follows at once from Proposition 2.20. Suppose the result holds for some d towards proving it for $d+1$. Fix σ-sprays $X_1, \dots, X_{d+1}$ covering $\mathbb{R}^{d+1}$ and centered in well-placed $\boldsymbol{c}_1, \dots, \boldsymbol{c}_{d+1}$. Let $H \subseteq \mathbb{R}^d$ be the hyperplane determined by the $\boldsymbol{c}_i$s. Let H' be a hyperplane orthogonal to $\boldsymbol{c}_{d+1} - \boldsymbol{c}_d$, and let $\pi : \mathbb{R}^{d+1} \to H'$ be the orthogonal projection. By projecting and gluing $X_1 \cap H$, $X_2 \cap H$, ..., $(X_d \cup X_{d+1}) \cap H$ are σ-sprays centered in the points $\pi(\boldsymbol{c}_1)$, $\pi(\boldsymbol{c}_2)$, ..., $\pi(\boldsymbol{c}_d) = \pi(\boldsymbol{c}_{d+1})$ which are well-placed, and belong to $H' \cong \mathbb{R}^d$. By inductive assumption, CH holds. □

3. Transforming sprays into linear objects

In this section we construct, for every $d \geq 2$, a continuous map Φ that transforms any spray of $\mathbb{R}^d$ with center $\boldsymbol{c}$ into a set $A \subseteq \mathbb{R}^d$ such that $A \cap H$ is finite, for every hyperplane H orthogonal to some vector $\boldsymbol{u}$, and conversely. (The vector $\boldsymbol{u}$ depends only on the point $\boldsymbol{c}$.) This is an extension and an elaboration of the construction used by Schmerl when $d=2$ to prove that if $\mathbb{R}^2$ is the union of $(n+2)$-many sprays with collinear centers, then $2^{\aleph_0} \leq \aleph_n$ [Schmerl 2010, Theorem 7].

Let $\boldsymbol{p}_1, \dots, \boldsymbol{p}_d$ be distinct points of an affine space E of dimension $d-1 \geq 1$. For each $\boldsymbol{q} \in E$ the vectors $\boldsymbol{p}_1 - \boldsymbol{q}, \dots, \boldsymbol{p}_d - \boldsymbol{q}$ are linearly dependent, and hence

$$\mathcal{U}_{\boldsymbol{p}_1,\dots,\boldsymbol{p}_d}(\boldsymbol{q}) := \{(u_1, \dots, u_d) \in \mathbb{R}^d \mid u_1(\boldsymbol{p}_1 - \boldsymbol{q}) + \dots + u_d(\boldsymbol{p}_d - \boldsymbol{q}) = \boldsymbol{0}\} \tag{2}$$

is a vector subspace of $\mathbb{R}^d$ of dimension ≥ 1. When the points $\boldsymbol{p}_1, \dots, \boldsymbol{p}_d$ are clear from the context we write $\mathcal{U}(\boldsymbol{q})$. Note that if $\boldsymbol{e}_i \in \mathcal{U}(\boldsymbol{q})$ then $\boldsymbol{q} = \boldsymbol{p}_i$, and hence

$$\boldsymbol{q} \notin \{\boldsymbol{p}_1, \dots, \boldsymbol{p}_d\} \Rightarrow \boldsymbol{e}_1, \dots, \boldsymbol{e}_d \notin \mathcal{U}(\boldsymbol{q}). \tag{3}$$

Theorem 3.1. *Suppose $\boldsymbol{p}_1, \dots, \boldsymbol{p}_d, \boldsymbol{q} \in E$, an affine space of dimension $d-1$, and that $\boldsymbol{p}_1, \dots, \boldsymbol{p}_d$ are distinct. For every $(u_1, \dots, u_d) \in \mathcal{U}(\boldsymbol{q}) \setminus \{\boldsymbol{0}\}$ letting $b := -\sum_{i=1}^d u_i$ and $c := -\sum_{i=1}^d u_i(\|\boldsymbol{p}_i\|^2 - \|\boldsymbol{q}\|^2)$ we have*

$$\forall \boldsymbol{x} \in E \left(u_1\|\boldsymbol{x} - \boldsymbol{p}_1\|^2 + \dots + u_d\|\boldsymbol{x} - \boldsymbol{p}_d\|^2 + b\|\boldsymbol{x} - \boldsymbol{q}\|^2 + c = 0\right).$$

Proof. For notational ease let $F_i(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{p}_i\|^2$ and $F(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{q}\|^2$. For $\boldsymbol{x} \in E$ and $1 \leq i \leq d$

$$\begin{aligned} F_i(\boldsymbol{x}) - F(\boldsymbol{x}) &= \|\boldsymbol{x} - \boldsymbol{p}_i\|^2 - \|\boldsymbol{x} - \boldsymbol{q}\|^2 \\ &= \|\boldsymbol{x}\|^2 + \|\boldsymbol{p}_i\|^2 - 2\boldsymbol{x} \cdot \boldsymbol{p}_i - \|\boldsymbol{x}\|^2 - \|\boldsymbol{q}\|^2 + 2\boldsymbol{x} \cdot \boldsymbol{q} \\ &= \|\boldsymbol{p}_i\|^2 - \|\boldsymbol{q}\|^2 - 2\boldsymbol{x} \cdot (\boldsymbol{p}_i - \boldsymbol{q}). \end{aligned}$$

Since $(u_1, \dots, u_d) \in \mathcal{U}(\boldsymbol{q})$, this implies

$$\sum_{i=1}^d u_i(F_i(\boldsymbol{x}) - F(\boldsymbol{x})) = \sum_{i=1}^d u_i(\|\boldsymbol{p}_i\|^2 - \|\boldsymbol{q}\|^2) - 2\boldsymbol{x} \cdot \sum_{i=1}^d u_i(\boldsymbol{p}_i - \boldsymbol{q}) = -c.$$

By the definition of b one obtains $\left(\sum_{i=1}^d u_i F_i(\boldsymbol{x})\right) + bF(\boldsymbol{x}) + c = 0$ for all $\boldsymbol{x} \in E$, as required. □

The definition of $\aleph_\delta$-spray was given for $\mathbb{R}^d$, but it can be adapted to the space

$$\mathbb{H}^d := \mathbb{R}^{d-1} \times (0; +\infty) \subseteq \mathbb{R}^d$$

as follows: an $\aleph_\delta$-spray in $\mathbb{H}^d$ is a set $X \subseteq \mathbb{H}^d$ together with a point $\boldsymbol{c}$, the center of X, belonging to the hyperplane $\mathbb{R}^{d-1} \times \{0\}$ of $\mathbb{R}^d$, such that $|S \cap X| < \aleph_\delta$ for any sphere S of $\mathbb{R}^d$ centered in $\boldsymbol{c}$. (Observe that $\boldsymbol{c}$ does not belong to $\mathbb{H}$.)

If $X_1, X_2, \dots$ are $\aleph_\delta$-sprays in $\mathbb{R}^d$ with centers $\boldsymbol{c}_1, \boldsymbol{c}_2, \dots \in \mathbb{R}^{d-1} \times \{0\}$, then $X_1 \cap \mathbb{H}^d, X_2 \cap \mathbb{H}^d, \dots$ are $\aleph_\delta$-sprays in $\mathbb{H}^d$ with the same centers; moreover, if $X_1, X_2, \dots$ cover $\mathbb{R}^d$, then $X_1 \cap \mathbb{H}^d, X_2 \cap \mathbb{H}^d, \dots$ cover $\mathbb{H}^d$. Therefore we may focus on covering $\mathbb{H}^d$ with $\aleph_\delta$-sprays in $\mathbb{H}^d$ with centers on $\mathbb{R}^{d-1} \times \{0\}$.

Let $\boldsymbol{c}_1, \dots, \boldsymbol{c}_d$ be points in general position in $\mathbb{R}^d$. These points belong to a hyperplane, and without loss of generality we may assume that $\boldsymbol{c}_1, \dots, \boldsymbol{c}_d \in$

$\mathbb{R}^{d-1} \times \{0\}$. Let $\mathbb{R}^d_+ = \{\boldsymbol{x} \in \mathbb{R}^d \mid \forall i \leq d\, (x_i \geq 0)\}$, let

$$\Phi : \mathbb{H}^d \to \mathbb{R}^d_+, \qquad \boldsymbol{x} \mapsto (\|\boldsymbol{x} - \boldsymbol{c}_1\|^2, \dots, \|\boldsymbol{x} - \boldsymbol{c}_d\|^2), \tag{4}$$

and let $E^d = \operatorname{ran} \Phi$. Then Φ is a homeomorphism between $\mathbb{H}^d$ and the open set E^d.

The map Φ transforms any sphere centered around $\boldsymbol{c}_i$ into a subset of a hyperplane of $\mathbb{R}^d$ orthogonal to $\boldsymbol{e}_i$, and conversely. To be specific — and recalling Notation 2.1 — if S is the sphere of $\mathbb{R}^d$ centered in $\boldsymbol{c}_i$ of radius r, then $\Phi[S \cap \mathbb{H}^d]$ is the intersection between E and the hyperplane $H_i(r^2)$ of $\mathbb{R}^d$; conversely, $\Phi^{-1}[H_i(r^2) \cap E] = S \cap \mathbb{H}^d$. Therefore if $X \subseteq \mathbb{H}^d$ is an $\aleph_\delta$-spray centered in $\boldsymbol{c}_i$, then $\Phi[X] \subseteq E$ is such that $|\Phi[X] \cap H_i(r)| < \aleph_\delta$ for every $r > 0$; conversely if $Y \subseteq E$ intersects all hyperplanes orthogonal to $\boldsymbol{e}_i$ in a set of size $< \aleph_\delta$ then $\Phi^{-1}[Y] \subseteq \mathbb{H}^d$ is an $\aleph_\delta$-spray centered in $\boldsymbol{c}_i$.

Suppose $\boldsymbol{c}_{d+1} \in \mathbb{R}^{d-1} \times \{0\}$ is distinct from $\boldsymbol{c}_1, \dots, \boldsymbol{c}_d$. Letting $\boldsymbol{p}_i = \boldsymbol{c}_i$ and $\boldsymbol{q} = \boldsymbol{c}_{d+1}$ in (2), fix $\boldsymbol{u} = (u_1, \dots, u_d) \in \mathcal{U}(\boldsymbol{c}_{d+1}) \setminus \{\boldsymbol{0}\}$. By Theorem 3.1 there are b, c such that $u_1\|\boldsymbol{x} - \boldsymbol{c}_1\|^2 + \dots + u_d\|\boldsymbol{x} - \boldsymbol{c}_d\|^2 + b\|\boldsymbol{x} - \boldsymbol{c}_{d+1}\|^2 + c = 0$, for all $\boldsymbol{x} \in \mathbb{R}^d$. The set

$$P = \{(w_1, \dots, w_{d+1}) \mid u_1 w_1 + \dots + u_d w_d + b w_{d+1} + c = 0\}$$

is an affine hyperplane of $\mathbb{R}^{d+1}$, and since $\boldsymbol{u}$ is not a multiple of any $\boldsymbol{e}_i$ with $1 \leq i \leq d$ by (3), it follows that P is not orthogonal to any such vector. Therefore the projection $\pi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is a bijection between P and $\mathbb{R}^d$. The map

$$\check{\Phi} : \mathbb{H}^d \to \mathbb{R}^{d+1}_+, \quad \boldsymbol{x} \mapsto (\|\boldsymbol{x} - \boldsymbol{c}_1\|^2, \dots, \|\boldsymbol{x} - \boldsymbol{c}_{d+1}\|^2),$$

is a homeomorphism on its image $\check{E}$, and clearly $\check{E} \subseteq P$. Then

$$\check{E} = \{(r_1, \dots, r_{d+1}) \in \mathbb{R}^{d+1}_+ \mid (r_1, \dots, r_d) \in E \wedge (r_1, \dots, r_{d+1}) \in P\},$$

that is: $\check{E}$ is the subset of P that projects onto E, and π is a homeomorphism from $\check{E}$ to E.

For any $k \in \mathbb{R}$ the set

$$\begin{aligned} L(\boldsymbol{u}, k) &:= \{(r_1, \dots, r_d) \mid u_1 r_1 + \dots + u_d r_d + bk + c = 0\} \\ &= \{(r_1, \dots, r_d) \mid (r_1, \dots, r_d, k) \in P\} \end{aligned}$$

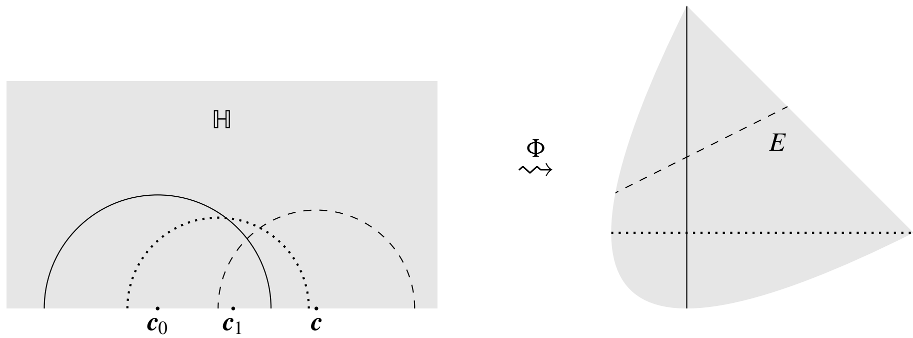
is an affine hyperplane of $\mathbb{R}^d$ orthogonal to $\boldsymbol{u}$, and all affine hyperplanes of $\mathbb{R}^d$ orthogonal to $\boldsymbol{u}$ are of this form. Arguing as before, if $X \subseteq \mathbb{H}^d$ is an $\aleph_\delta$-spray centered in $\boldsymbol{c}_{d+1}$ then $\Phi[X]$ is a subset of E that intersects every $L(\boldsymbol{u}, k)$ in a set of size $< \aleph_\delta$; conversely if $Y \subseteq E$ is such that every $L(\boldsymbol{u}, k)$ intersects Y in $< \aleph_\delta$-many points, then $\Phi^{-1}[Y]$ is an $\aleph_\delta$-spray centered in $\boldsymbol{c}_{d+1}$.

Let us summarize what we proved so far.

Theorem 3.2. *Suppose $\boldsymbol{c}_1, \dots, \boldsymbol{c}_d$ are in general position in $\mathbb{R}^d$, and without loss of generality we may assume that they belong to $\mathbb{R}^{d-1} \times \{0\}$. There is a nonempty open* *set* $E^d \subseteq \mathbb{R}^d$ *and a homeomorphism* $\Phi : \mathbb{H}^d \to E^d$ *that transforms any* $\aleph_\delta$*-spray of* $\mathbb{H}^d$ *centered in* $\boldsymbol{c}_i$ *into a subset of* E^d *that intersects any hyperplane orthogonal to* $\boldsymbol{e}_i$ *in a set of size* $< \aleph_\delta$*, and conversely.*

Let $\boldsymbol{c} \in \mathbb{R}^{d-1} \times \{0\}$ *be distinct from the* $\boldsymbol{c}_i$*s, and let* $\boldsymbol{u} \in \mathcal{U}_{\boldsymbol{c}_1,\dots,\boldsymbol{c}_d}(\boldsymbol{c}) \setminus \{\boldsymbol{0}\}$*. Then* Φ *maps any* $\aleph_\delta$*-spray centered in* $\boldsymbol{c}$ *into a subset of* E *that intersects any hyperplane orthogonal to* $\boldsymbol{u}$ *in a set of size* $< \aleph_\delta$*, and conversely.*

The following picture when $d = 2$ may help to visualize the previous construction.



We can now sketch Schmerl's argument that if the plane is covered with three sprays with collinear centers, then CH holds. First of all we may assume that the centers lie on the x-axis so that the three sprays cover $\mathbb{H}^2$. Then E^2 can be covered with three sets A_1, A_2, A_3 such that the intersection of A_i with any line orthogonal to some vector $\boldsymbol{u}_i$ is finite ($i = 1, 2, 3$), and by a theorem of Bagemihl [1961] this implies CH. Our strategy is to replace lines in $\mathbb{R}^2$ with hyperplanes in $\mathbb{R}^d$, and this is the topic of the next section.

4. Hyperplane sections

Over the last century several results were obtained, establishing connections between the size of the continuum and elementary properties of the euclidean spaces. The first such result is Sierpiński's theorem form → from from 1919, asserting that CH is equivalent to a covering of the plane with two sets such that the intersection of the first set with any vertical line is countable and the intersection of every horizontal line with the second set is countable (see Theorem 2.6(a)). Sierpiński [1951] sharpened his previous result by replacing "countable" with "finite", but at the cost of increasing the dimension: CH is equivalent to a decomposition A_1, A_2, A_3 of $\mathbb{R}^3$ such that the intersection of any line with direction $\boldsymbol{e}_i$ with A_i is finite (Theorem 2.6(b)), and this was quickly generalized by Kuratowski to higher dimensions [Kuratowski 1951]: $2^{\aleph_0} \leq \aleph_n$ if and only if there is a decomposition $A_1, \dots, A_{n+2}$ of $\mathbb{R}^{n+2}$ such that every line parallel to $\boldsymbol{e}_i$ has finite intersection with A_i. In the 1960s, Bagemihl and Davies showed that Kuratowski's result could be proved for $\mathbb{R}^2$ by taking intersections with lines of prescribed directions [Bagemihl 1960; 1961; 1968; Davies 1962; 1963b].

A line is a hyperplane in $\mathbb{R}^2$, so Sierpiński's result from 1919 could be stated as: if $d = 2$ then CH is equivalent to a decomposition of $\mathbb{R}^d$ into $A_1, \dots, A_d$ such that every hyperplane orthogonal to $\boldsymbol{e}_i$ has countable intersection with A_i. By Corollary 4.7 this result holds for all $d \geq 2$. Sierpiński [1951] states and proves it for $d = 3$, and he observes that the analogous result with "finite" replacing "countable" is false: there is no decomposition A_1, A_2, A_3 of $\mathbb{R}^3$ such that every plane orthogonal to $\boldsymbol{e}_i$ has finite intersection with A_i. Erdős, Jackson, and Mauldin prove that CH is equivalent to $\mathbb{R}^3$ being decomposable in five pieces $A_1, \dots, A_5$ so that every plane orthogonal to $\boldsymbol{u}_i$ has finite intersection with A_i, where $\boldsymbol{u}_1, \dots, \boldsymbol{u}_5$ are vectors in general position in $\mathbb{R}^3$ [Erdős et al. 1994, Corollary 6], but no decomposition exists if we just allow four vectors and four sets [Erdős et al. 1994, Lemma 1]. In that Lemma it is stated (but not proved) that an analogous result holds when CH is weakened to $2^{\aleph_0} \leq \aleph_n$, but the case for $\mathbb{R}^d$ with $d > 3$ is not mentioned. Since we need a detailed analysis of the positions of the various (hyper)planes we will state and prove these results in full generality below. Much of what follows is an elaboration of ideas from [Erdős et al. 1994], and in doing so we fill a gap in the proof of Lemma 1 of that paper.

This section is devoted to the following problem:

Problem 4.1. Given $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n$ distinct, nonzero vectors of $\mathbb{R}^d$, what conditions on the cardinality of $\mathbb{R}$ are equivalent to the existence of $A_1, \dots, A_n$ covering $\mathbb{R}^d$ such that each $A_i \cap H_{\boldsymbol{u}_i}(\boldsymbol{p})$ is finite (or countable)?

Lemma 4.2. *Suppose $\boldsymbol{u}_1, \dots, \boldsymbol{u}_d \in \mathbb{R}^d$ are linearly independent, and $d \geq 2$. There is a linear isomorphism $T : \mathbb{R}^d \to \mathbb{R}^d$ that maps every hyperplane orthogonal to $\boldsymbol{u}_i$ to a hyperplane orthogonal to $\boldsymbol{e}_i$, and conversely, for any $i \leq d$.*

Proof. The inner product $\boldsymbol{u} \cdot \boldsymbol{v}$ of two vectors in $\mathbb{R}^d$ is the matrix-product $\boldsymbol{u}^{\mathrm{t}}\boldsymbol{v}$, where superscript "t" stand for "transpose". Let M be the matrix of the linear transformation that maps $\boldsymbol{e}_i$ to $\boldsymbol{u}_i$ for $i = 1, \dots, d$, and let T be the linear transformation given by the matrix M^{t}. Then T is an isomorphism, since $\det(M^{\mathrm{t}}) = \det(M) \neq 0$. For all $\boldsymbol{x} \in \mathbb{R}^d$

$$\boldsymbol{u}_i^{\mathrm{t}}\boldsymbol{x} = (M\boldsymbol{e}_i)^{\mathrm{t}}\boldsymbol{x} = \boldsymbol{e}_i^{\mathrm{t}}(M^{\mathrm{t}}\boldsymbol{x})$$

so $\boldsymbol{x}$ is orthogonal to $\boldsymbol{u}_i$ if and only if $\boldsymbol{e}_i$ is orthogonal to $T(\boldsymbol{x})$. □

Lemma 4.3. *Let $m \geq 1$, let $\boldsymbol{v}$ be a unit vector of $\mathbb{R}^m$, let $\varepsilon > 0$ and let κ be an infinite cardinal. For all $X \subseteq \mathbb{R}^m$ with $\aleph_0 \leq |X| < \kappa \leq 2^{\aleph_0}$, there is $S \subseteq \{r\boldsymbol{v} \mid |r| < \varepsilon\} \subseteq (-\varepsilon; \varepsilon)^m$ such that $|S| = \kappa$ and $(X - X) \cap (S - S) = \{\boldsymbol{0}\}$.*

Proof. Let $V = \{r\boldsymbol{v} \mid r \in \mathbb{R}\}$, let G be the subgroup of $(\mathbb{R}^m, +)$ generated by X, and let $H = G \cap V$. Then $|H| \leq |G| = |X| < 2^{\aleph_0}$ and $|V/H| = 2^{\aleph_0}$.

Suppose first H is dense in V. We can construct $T \subseteq \{r\boldsymbol{v} \mid |r| < \varepsilon\}$ a transversal for the quotient, i.e. a set picking exactly one element from each coset of V/H. Then $2^{\aleph_0} = |T| = |V/H|$, and let $S \subseteq T$ be of size κ. If $s_1 - s_2 = x_1 - x_2$ for some $s_1, s_2 \in S$ and $x_1, x_2 \in X$, since $s_1 - s_2 \in H \subseteq G$ this means that the cosets $s_1 + H$ and $s_2 + H$ are the same, so $s_1 = s_2$ by definition of S. Therefore $(S - S) \cap (X - X) = \{\mathbf{0}\}$.

Now suppose that H is not dense in V. Then there is $0 < \delta < \varepsilon$ such that the segment $(\mathbf{0}; \delta\boldsymbol{v})$ is disjoint from H. Let $S \subseteq (\mathbf{0}; \delta\boldsymbol{v})$ be of size κ. As H is a group, then $(-\delta\boldsymbol{v}; \delta\boldsymbol{v}) \cap H = \{\mathbf{0}\}$, so the intersection of $S - S \subseteq (-\delta\boldsymbol{v}; \delta\boldsymbol{v})$ with $G - G \supseteq X - X$ is $\{\mathbf{0}\}$. □

The next result asserts that if the sets $A_i \subseteq \mathbb{R}^d$ have small intersection with the hyperplanes orthogonal to $\boldsymbol{u}_i$, and if $2^{\aleph_0}$ is large enough, then there is a set $Z \subseteq \mathbb{R}^d$ such that no translate of it can be covered by the A_is — in particular the A_is do not cover $\mathbb{R}^d$.

Theorem 4.4. *Let* $\delta \in \mathrm{Ord}$, $d \geq 3$ *and let* $N = 2(d-1)$. *Suppose* $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ *are distinct, nonzero vectors of* $\mathbb{R}^d$. *Suppose* $A_1, \ldots, A_N$ *are subsets of* $\mathbb{R}^d$ *such that for all* $\boldsymbol{p} \in \mathbb{R}^d$, *and for all* $1 \leq i \leq N$

$$|H_{\boldsymbol{u}_i}(\boldsymbol{p}) \cap A_i| < \aleph_\delta.$$

If $2^{\aleph_0} > \aleph_\delta$ *then for every* $\varepsilon > 0$ *there is* $Z \subseteq (-\varepsilon; \varepsilon)^d$ *of size* $\aleph_{\delta+1}$ *such that*

$$\forall \boldsymbol{p} \in \mathbb{R}^d \left(\boldsymbol{p} + Z \nsubseteq \textstyle\bigcup_{i=1}^N A_i\right).$$

Before proving this let us draw a few corollaries. With the same notation as before let us state a contrapositive of the preceding result:

Theorem 4.5. *Let* $\delta \in \mathrm{Ord}$, $d \geq 3$ *and* $N = 2(d-1)$. *Suppose* $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ *are nonzero vectors of* $\mathbb{R}^d$. *Suppose* $D \subseteq \mathbb{R}^d$ *is such that* $\mathrm{Int}(D) \neq \emptyset$. *If* $A_1, \ldots, A_N$ *cover* D *and for all* $\boldsymbol{p} \in \mathbb{R}^d$ *and for all* $1 \leq i \leq N$

$$|H_{\boldsymbol{u}_i}(\boldsymbol{p}) \cap A_i| < \aleph_\delta.$$

Then $2^{\aleph_0} \leq \aleph_\delta$.

Proof. Towards a contradiction, suppose $2^{\aleph_0} \geq \aleph_{\delta+1}$. Let $\varepsilon > 0$ be small enough so that $\boldsymbol{p} + (-\varepsilon; \varepsilon)^d \subseteq D$ for some $\boldsymbol{p} \in \mathbb{R}^d$, and, towards a contradiction, suppose $A_1, \ldots, A_N$ are as in the statement. Let $Z \subseteq (-\varepsilon; \varepsilon)^d$ be as in Theorem 4.4: then $\boldsymbol{p} + Z$ is contained in D but on other hand it is not contained in $A_1 \cup \cdots \cup A_N$, a contradiction. □

The presence of the set D in the statement of Theorem 4.5 may seem unwarranted right now, but it will be crucial for the results in Section 5. For the time being the reader can safely replace D with $\mathbb{R}^d$ without losing much.

Theorem 4.6. *For all $d \geq 3$ and all $D \subseteq \mathbb{R}^d$ such that $\varnothing \neq \mathrm{Int}(D)$, the following are equivalent*:

(a) CH *holds.*

(b) *For all nonzero vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_d$ spanning $\mathbb{R}^d$, there are $A_1, \dots, A_d$ covering D such that $H_{\boldsymbol{u}_i}(\boldsymbol{p}) \cap A_i$ is countable for all $\boldsymbol{p} \in \mathbb{R}^d$ and $1 \leq i \leq d$.*

(c) *There are nonzero vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{2d-2}$ and there are $A_1, \dots, A_{2d-2}$ covering D such that $H_{\boldsymbol{u}_i}(\boldsymbol{p}) \cap A_i$ is countable for all $\boldsymbol{p} \in \mathbb{R}^d$ and $1 \leq i \leq 2d-2$.*

Proof. The implication (a)⇒(b) follows from Theorem 2.3 and Example 2.5, while (b)⇒(c) is trivial — take $A_{d+1} = \cdots = A_{2d-2} = \varnothing$. The implication (c)⇒(a) follows from Theorem 4.5 when $\delta = 1$. □

Corollary 4.7. *For all $d \geq 3$,* CH *is equivalent to the existence of $A_1, \dots, A_d$ covering $\mathbb{R}^d$ such that $H_i(x) \cap A_i$ is countable, for all $x \in \mathbb{R}$ and all $1 \leq i \leq d$.*

Cantor's theorem says that $2^{\aleph_0} > \aleph_\delta$ if $\delta = 0$, so when $d = 3$ Theorem 4.4 implies the following result, which is Lemma 1 of [Erdős et al. 1994]:

Corollary 4.8. *Suppose $\boldsymbol{u}_1, \dots, \boldsymbol{u}_4$ are nonzero vectors of $\mathbb{R}^3$. There are no $A_1, \dots, A_4$ covering $\mathbb{R}^3$ such that $H_{\boldsymbol{u}_i}(\boldsymbol{p}) \cap A_i$ is finite, for all $\boldsymbol{p} \in \mathbb{R}^3$ and all $1 \leq i \leq 4$.*

Remark 4.9. Corollary 4.8 is a negative result, asserting that $\mathbb{R}^3$ is not the union of four sets such that each intersects any plane orthogonal to a given vector in a finite set. But looking at the proof of Theorem 4.4, it could be recast in a positive way:

> *Suppose $\Bbbk$ is an infinite field. If there are nonzero vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_4$ of $\Bbbk^3$ and sets $A_1, \dots, A_4$ covering $\Bbbk^3$ such that every hyperplane orthogonal to $\boldsymbol{u}_i$ has finite intersection with A_i, then $|\Bbbk| = \aleph_0$.*

Proof of Theorem 4.4. Let $A_1, \dots, A_N$ and $\varepsilon > 0$ be as in the statement. We consider two cases, depending whether the $\boldsymbol{u}_i$s span $\mathbb{R}^d$.

Case 1: the vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_N$ do not span $\mathbb{R}^d$.

Let $\boldsymbol{v}$ be a nonzero vector such that $\boldsymbol{v} \cdot \boldsymbol{u}_i = 0$ for all $1 \leq i \leq N$, and let $V = \{r\boldsymbol{v} \mid r \in \mathbb{R}\}$. Since $|V| = 2^{\aleph_0} \geq \aleph_{\delta+1}$ we can take $Z \subseteq V \cap (-\varepsilon; \varepsilon)^d$ of size $\aleph_{\delta+1}$. Since $\boldsymbol{p} + V \subseteq H_{\boldsymbol{u}_i}(\boldsymbol{p})$ we have $|(\boldsymbol{p} + V) \cap \bigcup_{i=1}^N A_i| < \aleph_\delta$, and hence $\boldsymbol{p} + Z \nsubseteq \bigcup_{i=1}^N A_i$.

Case 2: the vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_N$ span $\mathbb{R}^d$.

Without loss of generality we may assume that $(\boldsymbol{u}_1, \dots, \boldsymbol{u}_d)$ is a basis of $\mathbb{R}^d$. We first prove the result under the additional assumption that

$$(\boldsymbol{u}_1, \dots, \boldsymbol{u}_d) \text{ is the standard basis } (\boldsymbol{e}_1, \dots, \boldsymbol{e}_d).$$

Let

$$U = \operatorname{span}\{\boldsymbol{u}_{d+1}, \dots, \boldsymbol{u}_N\}, \qquad U_j = \operatorname{span}\{\boldsymbol{e}_j, \boldsymbol{u}_{d+1}, \dots, \boldsymbol{u}_N\} \quad \text{for } 1 \leq j \leq d.$$

From $\dim U \leq d-2$ it follows that $\dim U_j \leq d-1$, so there is a nonzero vector orthogonal to U_j.

Claim 4.4.1. *There is* $1 \leq j \leq d$ *such that there is* $\boldsymbol{v} \in U_j^{\perp} \setminus \{\boldsymbol{0}\}$ *that is not multiple of any* $\boldsymbol{e}_i$.

Proof of the claim. Suppose $\dim U_j \leq d-2$ for some j, so that $\dim(U_j^{\perp}) \geq 2$. Then not every vector in $U_j^{\perp}$ can be multiple of some $\boldsymbol{e}_i$.

Therefore we may assume that $\dim U_j = d-1$ for all j, and hence $\dim U = d-2$. Towards a contradiction, suppose that for every $j \in \{1, \dots, d\}$ there is $j^* \neq j$ such that $\boldsymbol{e}_{j^*} \in U_j^{\perp}$ — such j^* is unique since $\dim(U_j^{\perp}) = 1$. Fix j and let $i = j^*$ so that $i^* \neq j^*$. As $d \geq 3$ pick k distinct from i^*, j^*. Both vectors $\boldsymbol{e}_{i^*}$ and $\boldsymbol{e}_{j^*}$ are orthogonal to U and to $\boldsymbol{e}_k$, so both belong to $U_k^{\perp}$, which is impossible as $\dim(U_k^{\perp}) = 1$. □

Fix j and $\boldsymbol{v} = (a_1, \dots, a_d)$ as in the Claim. By reindexing, if needed, we may assume that $j = d$, that is $a_d = 0$, and that a_1, a_2 are nonzero. By rescaling we may assume that

$$\boldsymbol{v} = (1, a_2, \dots, a_{d-1}, 0), \quad a_2 \neq 0,$$

and that

$$d \leq k \leq N \Rightarrow \boldsymbol{v} \cdot \boldsymbol{u}_k = 0. \tag{5}$$

Let $\nu := \max\{1, |a_2|, \dots, |a_{d-1}|\}$. For $1 \leq i \leq d$ let $X_i \subseteq (-\varepsilon/2; \varepsilon/2)$ such that $|X_1| = \aleph_{\delta+1}$ and $|X_i| = \aleph_\delta$ for all $2 \leq i \leq d$. By Lemma 4.3 with $m = 1$ and $\kappa = \aleph_{\delta+1}$, let $S \subseteq (-\varepsilon/(2\nu); \varepsilon/(2\nu))$ be of size $\aleph_{\delta+1}$ such that $(S - S) \cap (a_2^{-1} X_2 - a_2^{-1} X_2) = \{0\}$, and let

$$V = \{s\boldsymbol{v} \mid s \in S\}.$$

By (5), if $\boldsymbol{q} \in \mathbb{R}^d$ and $s \in S$ then

$$d \leq k \leq N \Rightarrow \boldsymbol{q} + V \subseteq H_{\boldsymbol{u}_k}(\boldsymbol{q} + s\boldsymbol{v}). \tag{6}$$

As $V \subseteq (-\varepsilon/2; \varepsilon/2)^d$ it follows that

$$Z := V + \prod_{i=1}^{d} X_i$$

is a subset of $(-\varepsilon; \varepsilon)^d$, and it is of cardinality $\aleph_{\delta+1}$. Fix $\boldsymbol{p} = (p_1, \dots, p_d) \in \mathbb{R}^d$ and, towards a contradiction, suppose that $\boldsymbol{p} + Z \subseteq \bigcup_{i=1}^{N} A_i$. Observe that $Z = \bigcup_{x \in X_1} Z_x$, where

$$Z_x := V + \big(\{x\} \times \prod_{i=2}^{d} X_i\big).$$

Claim 4.4.2. *The sets Z_x are pairwise disjoint.*

Proof. If $Z_{x'} \cap Z_{x''} \neq \varnothing$, then there are $s', s'' \in S$, $x'_i, x''_i \in X_i$ for $2 \leq i \leq d$ such that

$$s'\boldsymbol{v} + (x', x'_2, \ldots, x'_d) = s''\boldsymbol{v} + (x'', x''_2, \ldots, x''_d).$$

The second component yields that $s'a_2 + x'_2 = s''a_2 + x''_2$, and hence

$$s' - s'' = a_2^{-1}(x''_2 - x'_2) \in (S - S) \cap (a_2^{-1}X_2 - a_2^{-1}X_2) = \{0\}\,.$$

It follows that $s' = s'' = s$, and hence

$$s\boldsymbol{v} + (x', x'_2, \ldots, x'_d) = s\boldsymbol{v} + (x'', x''_2, \ldots, x''_d).$$

Looking at the first component we obtain $s + x' = s + x''$ and hence $x' = x''$. □

Summarizing,

$$\boldsymbol{p} + Z = \bigcup_{x \in X_1} \boldsymbol{p} + Z_x \quad \text{and} \quad x' \neq x'' \Rightarrow \boldsymbol{p} + Z_{x'} \cap \boldsymbol{p} + Z_{x''} = \varnothing. \tag{7}$$

Claim 4.4.3. $$|A_d \cap (\boldsymbol{p} + Z)| < \aleph_{\delta+1}.$$

Proof. As the last component of $\boldsymbol{v}$ is 0, the set V does not contribute to Z on the d-th coordinate, so for all $w \in \mathbb{R}$

$$\begin{aligned}(\boldsymbol{p} + Z) \cap H_d(p_d + w) &= (\boldsymbol{p} + \prod_{i=1}^{d} X_i) \cap H_d(p_d + w) \\ &= \boldsymbol{p} + \big((\prod_{i=1}^{d} X_i) \cap H_d(w)\big)\end{aligned}$$

and this set is $\boldsymbol{p} + \big(\prod_{i=1}^{d-1} X_i\big) \times \{w\}$ if $w \in X_d$, and it is empty if $w \notin X_d$. Therefore $\boldsymbol{p} + Z \subseteq \bigcup_{w \in X_d} H_d(p_d + w)$ and

$$A_d \cap (\boldsymbol{p} + Z) \subseteq \bigcup_{w \in X_d} A_d \cap H_d(p_d + w).$$

By assumption $|A_d \cap H_d(p_d + w)| < \aleph_\delta$ for any $w \in \mathbb{R}$, and $|X_d| = \aleph_\delta$; therefore $|A_d \cap (\boldsymbol{p} + Z)| \leq \aleph_\delta < \aleph_{\delta+1}$. □

As $|X_1| = \aleph_{\delta+1}$ and by (7), there is $\bar{x}_1 \in X_1$ such that $\boldsymbol{p} + Z_{\bar{x}_1}$ is disjoint from A_d. This implies that

$$\boldsymbol{p} + Z_{\bar{x}_1} \subseteq A_1 \cup \cdots \cup A_{d-1} \cup \bigcup_{k=d+1}^{N} A_k.$$

Claim 4.4.4. *There is an $\bar{s} \in S$ such that $\boldsymbol{p} + \bar{s}\boldsymbol{v} + (\{\bar{x}_1\} \times \prod_{i=2}^{d} X_i)$ is disjoint from $\bigcup_{k=d+1}^{N} A_k$.*

Proof. Towards a contradiction, suppose that for all $s \in S$ and all $2 \leq i \leq d$ there are $x_i(s) \in X_i$ such that

$$\boldsymbol{p} + s\boldsymbol{v} + (\bar{x}_1, x_2(s), \dots, x_d(s)) \in \bigcup_{k=d+1}^{N} A_k.$$

As $|X_i| = \aleph_\delta$ for $2 \leq i \leq d$, there are $x'_i \in X_i$ and $S' \subseteq S$ of size $\aleph_{\delta+1}$ such that $x_i(s) = x'_i$ for all $s \in S'$. Therefore

$$\forall s \in S' \left(\boldsymbol{p} + s\boldsymbol{v} + (\bar{x}_1, x'_2, \dots, x'_d) \in \bigcup_{k=d+1}^{N} A_k\right).$$

As $|S'| = \aleph_{\delta+1}$ there is $S^* \subseteq S'$ of size $\aleph_{\delta+1}$, and there is k with $d < k \leq N$ such that

$$\forall s \in S^* \left(\boldsymbol{p} + s\boldsymbol{v} + (\bar{x}_1, x'_2, \dots, x'_d) \in A_k\right). \tag{8}$$

Fix $s^* \in S^*$ and let

$$Q_k := H_{\boldsymbol{u}_k}\left(\boldsymbol{p} + s^*\boldsymbol{v} + (\bar{x}_1, x'_2, \dots, x'_d)\right).$$

By (6) with $\boldsymbol{q} = \boldsymbol{p} + (\bar{x}_1, x'_2, \dots, x'_d)$ we have

$$\forall s \in S^* \left(\boldsymbol{p} + s\boldsymbol{v} + (\bar{x}_1, x'_2, \dots, x'_d) \in Q_k\right).$$

Therefore by (8) $\boldsymbol{p} + s\boldsymbol{v} + (\bar{x}_1, x'_2, \dots, x'_d) \in A_k \cap Q_k$ for all $s \in S^*$, and this is a contradiction, since $|A_k \cap Q_k| < \aleph_\delta$, and yet it must contain $\aleph_{\delta+1}$ points. □

Let $\bar{s}$ be as in Claim 4.4.4. Then $|\boldsymbol{p} + \bar{s}\boldsymbol{v} + (\{\bar{x}_1\} \times \prod_{i=2}^{d} X_i)| = \aleph_\delta$ and

$$W_1 := \boldsymbol{p} + \bar{s}\boldsymbol{v} + (\{\bar{x}_1\} \times \textstyle\prod_{i=2}^{d} X_i) \subseteq A_1 \cup \dots \cup A_{d-1}. \tag{9}$$

As W_1 is included in $H_1(p_1 + \bar{s} + \bar{x}_1)$, and $|H_1(p_1 + \bar{s} + \bar{x}_1) \cap A_1| < \aleph_\delta$, then $|W_1 \cap A_1| < \aleph_\delta$. As $|X_2| = \aleph_\delta$, there is some $\bar{x}_2 \in X_2$ such that the set $W_2 := \boldsymbol{p} + \bar{s}\boldsymbol{v} + (\{\bar{x}_1\} \times \{\bar{x}_2\} \times \prod_{i=3}^{d} X_i)$ is disjoint from A_1, and hence it is contained in $A_2 \cup \dots \cup A_{d-1}$ and in the hyperplane $H_2(p_2 + \bar{s}a_2 + \bar{x}_2)$. As before $|W_2 \cap A_2| < \aleph_\delta$. Repeating this argument we construct $\bar{x}_3 \in X_3, \dots, \bar{x}_d \in X_d$ such that

$$\boldsymbol{p} + \bar{s}\boldsymbol{v} + (\bar{x}_1, \dots, \bar{x}_d) \notin A_1 \cup \dots \cup A_{d-1}$$

against (9). This concludes the proof, assuming that $\boldsymbol{u}_1, \dots, \boldsymbol{u}_d$ is the standard basis.

If $\boldsymbol{u}_1, \dots, \boldsymbol{u}_d$ is an arbitrary basis of $\mathbb{R}^d$, by Lemma 4.2 there is a linear injective transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ that maps every hyperplane orthogonal to $\boldsymbol{u}_i$ to a hyperplane orthogonal to $\boldsymbol{e}_i$, for $1 \leq i \leq d$. The transformation T maps parallel hyperplanes to parallel hyperplanes, so for $d \leq k < N$ let $\bar{\boldsymbol{u}}_k$ be a vector such that T maps every hyperplane orthogonal to $\boldsymbol{u}_k$ to a hyperplane orthogonal $\bar{\boldsymbol{u}}_k$. For

$1 \leq k \leq d$ set $\bar{\boldsymbol{u}}_k$ to be $\boldsymbol{e}_k$. As T^{-1} is continuous, pick $\bar{\varepsilon} > 0$ small enough so that $T^{-1}[(-\bar{\varepsilon}; \bar{\varepsilon})^d] \subseteq (-\varepsilon; \varepsilon)^d$. Arguing as above there is a $\bar{Z} \subseteq (-\bar{\varepsilon}; \bar{\varepsilon})^d$ such that $\boldsymbol{p} + \bar{Z} \nsubseteq \bigcup_{i=1}^N T[A_i]$ for all $\boldsymbol{p} \in \mathbb{R}^d$. Then $Z = T^{-1}[\bar{Z}] \subseteq (-\varepsilon; \varepsilon)^d$ is such that $\boldsymbol{p} + Z \nsubseteq \bigcup_{i=1}^N A_i$ for all $\boldsymbol{p} \in \mathbb{R}^d$. □

The next result extends Theorem 4.4 by relaxing the size of the continuum.

Theorem 4.9. *Let* $\delta \in \mathrm{Ord}$, $d \geq 3$, $n \geq 0$, *and let* $N = (n+1)(d-1)+1$. *Suppose* $\boldsymbol{u}_1, \dots, \boldsymbol{u}_N$ *are distinct, nonzero vectors of* $\mathbb{R}^d$, *and that* $A_1, \dots, A_N \subseteq \mathbb{R}^d$ *are such that for all* $\boldsymbol{p} \in \mathbb{R}^d$

$$\forall 1 \leq k \leq N \ \left(|H_{\boldsymbol{u}_k}(\boldsymbol{p}) \cap A_k| < \aleph_\delta\right).$$

If $2^{\aleph_0} > \aleph_{\delta+n}$ *then for every* $\varepsilon > 0$ *there is* $Z_{n,\varepsilon} \subseteq (-\varepsilon; \varepsilon)^d$ *of size* $\aleph_{\delta+n+1}$ *such that*

$$\forall \boldsymbol{p} \in \mathbb{R}^d \left(\boldsymbol{p} + Z_{n,\varepsilon} \nsubseteq \bigcup_{k=1}^N A_k\right).$$

Let us draw some consequences from Theorem 4.9.

Theorem 4.10. *Let* $\delta \in \mathrm{Ord}$, *let* $n \geq 1$, *and let* $d \geq 3$. *Let also* $(n+1)(d-1) < N \leq (n+2)(d-1)$. *For all* $D \subseteq \mathbb{R}^d$ *such that* $\mathrm{Int}(D) \neq \varnothing$, *the following are equivalent*:

(a) $2^{\aleph_0} \leq \aleph_{\delta+n}$.

(b) *For all* $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{(n+1)(d-1)+1}$ *in general position there are* $A_1, \dots, A_{(n+1)(d-1)+1}$ *covering* D *such that* $|A_k \cap H_{\boldsymbol{u}_k}(\boldsymbol{p})| < \aleph_\delta$ *for all* $\boldsymbol{p} \in \mathbb{R}^d$ *and all* $1 \leq k \leq (n+1)(d-1)+1$.

(c) *For all distinct nonzero vectors* $\boldsymbol{u}_1, \dots, \boldsymbol{u}_N$ *such that* $(n+1)(d-1)+1$ *of them are in general position, there are* $A_1, \dots, A_N$ *covering* D *such that* $|A_k \cap H_{\boldsymbol{u}_k}(\boldsymbol{p})| < \aleph_\delta$ *for all* $\boldsymbol{p} \in \mathbb{R}^d$ *and all* $1 \leq k \leq N$.

(d) *There are distinct nonzero vectors* $\boldsymbol{u}_1, \dots, \boldsymbol{u}_N$ *and there are* $A_1, \dots, A_N$ *covering* D *such that* $|A_k \cap H_{\boldsymbol{u}_k}(\boldsymbol{p})| < \aleph_\delta$ *for all* $\boldsymbol{p} \in \mathbb{R}^d$ *and all* $1 \leq k \leq N$.

Proof. The implication (a)⇒(b) follows from the Erdős–Jackson–Mauldin result we stated as Theorem 2.3, together with Example 2.5.

The implication (b)⇒(c) is easy. Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_N$ be as in (c). Without loss of generality we may assume that $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{(n+1)(d-1)+1}$ are in general position in $\mathbb{R}^d$, and let $A_1, \dots, A_{(n+1)(d-1)+1}$ be as in (b). Letting $A_{(n+1)(d-1)+2} = \cdots = A_N = \varnothing$ we have sets as in (c).

The implication (c)⇒(d) is trivial.

Assume (d) and towards a contradiction suppose that $2^{\aleph_0} > \aleph_{\delta+n}$. By a translation, we may assume that $(-\varepsilon; \varepsilon)^d \subseteq D$. The hypotheses of Theorem 4.9 are satisfied so there is a set $Z \subseteq D$ that is not contained in $\bigcup_{k=1}^N A_k$, so the sequence $A_1, \dots, A_N$ does not cover D. □

When $d = 3$ Theorem 4.10 yields the following results — recall that vectors in $\mathbb{R}^3$ are in general position if any three of them are linearly independent. First we consider the case when the intersection of a plane with a set is finite.

Corollary 4.11. *Let $D \subseteq \mathbb{R}^3$ be such that $\varnothing \neq \operatorname{Int}(D)$, and let $n \geq 1$. The following are equivalent.*

(a) $2^{\aleph_0} \leq \aleph_n$.

(b) *For all $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{2n+3}$ vectors in general position in $\mathbb{R}^3$, there are $A_1, \dots, A_{2n+3}$ covering D such that every plane orthogonal to $\boldsymbol{u}_i$ intersects A_i in a finite set, for all $1 \leq i \leq 2n+3$.*

(c) *There are nonzero vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{2n+4}$ in $\mathbb{R}^3$, and sets $A_1, \dots, A_{2n+4}$ covering D such that every plane orthogonal to $\boldsymbol{u}_i$ intersects A_i in a finite set, for all $1 \leq i \leq 2n+4$.*

With $n = 1$, this recovers the result of [Erdős et al. 1994; Simms 1997] that CH is equivalent to each of the following:

- For all $\boldsymbol{u}_1, \dots, \boldsymbol{u}_5$ in general position there are $A_1, \dots, A_5$ covering $\mathbb{R}^3$ such that any plane orthogonal to $\boldsymbol{u}_i$ has finite intersection with A_i.
- There are nonzero vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_6$ and $A_1, \dots, A_6$ covering $\mathbb{R}^3$ such that any plane orthogonal to $\boldsymbol{u}_i$ has finite intersection with A_i.

If the intersections between planes and sets are taken to be countable we have a result analogous to Corollary 4.11, with a shift of 2 in the number of vectors/pieces.

Corollary 4.12. *Let $D \subseteq \mathbb{R}^3$ be such that $\varnothing \neq \operatorname{Int}(D)$, let $n \geq 1$ and let N be such that $2n+2 \leq N$. The following are equivalent:*

(a) $2^{\aleph_0} \leq \aleph_n$.

(b) *For all $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{2n+1}$ vectors in general position there are $A_1, \dots, A_{2n+1}$ covering D so that every plane orthogonal to $\boldsymbol{u}_k$ intersects A_k in a countable set, for all $1 \leq k \leq 2n+1$.*

(c) *For all vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{2n+2}$ nonzero vectors such that at least $2n+1$ of them are in general position, there are $A_1, \dots, A_{2n+2}$ covering D such that every plane orthogonal to $\boldsymbol{u}_k$ intersects A_k in a countable set, for all $1 \leq k \leq 2n+2$;*

(d) *There are vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{2n+2}$ and sets $A_1, \dots, A_{2n+2}$ covering D such that every plane orthogonal to $\boldsymbol{u}_k$ intersects A_k in a countable set, for all $1 \leq k \leq 2n+2$.*

Table 1 summarize the results proved so far. On the left, for any dimension d and any n such that $2^{\aleph_0} \leq \aleph_n$, we give $(n+1)(d-1)+1$, the minimum number of vectors/pieces of a covering of $\mathbb{R}^d$ so that each piece has *finite* intersection with any hyperplane orthogonal to the given vector. The maximum number for such a

$d \downarrow$ \ n	1	2	3	4	…
2	3	4	5	6	…
3	5	7	9	11	…
4	7	10	13	16	…
5	9	13	16	20	…
⋮					⋱

$d \downarrow$ \ n	1	2	3	4	…
2	2	3	4	5	…
3	3	5	7	9	…
4	4	7	10	13	…
5	5	9	13	17	…
⋮					⋱

Table 1. Minimum number of pieces of a decomposition of $\mathbb{R}^d$ equivalent to $2^{\aleph_0} \leq \aleph_n$, for *finite* intersections (left) and for *countable* intersections (right).

decomposition is $(n+2)(d-1)$, the integer in the square with coordinates $(d, n+1)$ decreased by 1. The table on the right is similar, but the intersections with the hyperplanes are *countable*.

Proof of Theorem 4.9. Fix $\delta \in \mathrm{Ord}$ and $d \geq 3$, and proceed by induction on n.

If $n = 0$ then $N = d < 2(d-1)$, and pick $\boldsymbol{u}_{d+1}, \ldots, \boldsymbol{u}_{2(d-1)}$ so that $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{2(d-1)}$ are distinct nonzero vectors of $\mathbb{R}^d$, and let $A_{d+1} = \cdots = A_{2(d-1)} = \varnothing$. Given $\varepsilon > 0$, if $2^{\aleph_0} > \aleph_{\delta+0}$ then the assumptions of Theorem 4.4 are fulfilled, so we can find the set Z as required. Therefore we may assume the result holds for some $\bar{n}$ towards proving it for $\bar{n}+1$.

Suppose $2^{\aleph_0} > \aleph_{\delta+\bar{n}+1}$. Let $N = (\bar{n}+2)(d-1)+1$, and let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ and $A_1, \ldots, A_N$ be as in the statement. Given $\varepsilon > 0$ we must construct $Z = Z_{\bar{n}+1,\varepsilon}$ such that

$$Z \subseteq (-\varepsilon; \varepsilon)^d, \quad |Z| = \aleph_{\delta+\bar{n}+2}, \quad \forall \boldsymbol{p} \in \mathbb{R}^d \left(\boldsymbol{p} + Z \nsubseteq \bigcup_{k=1}^N A_k\right).$$

Let $\bar{N} = (\bar{n}+1)(d-1)+1$. Then $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{\bar{N}}$ and $A_1, \ldots, A_{\bar{N}}$ satisfy the hypotheses of the statement for $\bar{n}$. By the inductive assumption there is $\bar{Z} = Z_{\bar{n},\varepsilon/2} \subseteq (-\varepsilon/2; \varepsilon/2)^d$ of size $\aleph_{\delta+\bar{n}+1}$ such that

$$\forall \boldsymbol{p} \in \mathbb{R}^d \left(\boldsymbol{p} + \bar{Z} \nsubseteq \bigcup_{k=1}^{\bar{N}} A_k\right). \tag{10}$$

As $N - \bar{N} = d - 1$, there is a unit vector $\boldsymbol{v}$ such that

$$\bar{N} < k \leq N \Rightarrow \boldsymbol{v} \cdot \boldsymbol{u}_k = 0. \tag{11}$$

The subspace $V = \{r\boldsymbol{v} \mid r \in \mathbb{R}\}$ is of cardinality $2^{\aleph_0} \geq \aleph_{\delta+\bar{n}+2} > \aleph_{\delta+\bar{n}+1} = |\bar{Z}|$. By Lemma 4.3 with $m = d$ and $\kappa = \aleph_{\delta+\bar{n}+1}$ we obtain $Y \subseteq \{r\boldsymbol{v} \mid |r| < \varepsilon/2\} \subseteq V$ of size $\aleph_{\delta+\bar{n}+2}$ such that

$$(Y - Y) \cap (\bar{Z} - \bar{Z}) = \{\boldsymbol{0}\}.$$

Let

$$Z := Y + \bar{Z} = \bigcup_{\boldsymbol{y} \in Y} \boldsymbol{y} + \bar{Z} = \bigcup_{\boldsymbol{z} \in \bar{Z}} \boldsymbol{z} + Y.$$

Observe that $Z \subseteq (-\varepsilon; \varepsilon)^d$ and $|Z| = \aleph_{\delta+\bar{n}+2}$. We must argue that $\boldsymbol{p} + Z \nsubseteq \bigcup_{k=1}^{N} A_k$ for all $\boldsymbol{p} \in \mathbb{R}^d$.

Towards a contradiction, let $\hat{\boldsymbol{p}} \in \mathbb{R}^d$ be such that $\hat{\boldsymbol{p}} + Z \subseteq \bigcup_{k=1}^{N} A_k$.

Claim 4.9.1. *If* $\boldsymbol{y}, \boldsymbol{y}' \in Y$ *and* $(\boldsymbol{y} + \bar{Z}) \cap (\boldsymbol{y}' + \bar{Z}) \neq \emptyset$ *then* $\boldsymbol{y} = \boldsymbol{y}'$.

Proof. Suppose $\boldsymbol{y} + \boldsymbol{z} = \boldsymbol{y}' + \boldsymbol{z}'$ with $\boldsymbol{y}, \boldsymbol{y}' \in Y$ and $\boldsymbol{z}, \boldsymbol{z}' \in \bar{Z}$. Then $\boldsymbol{y} - \boldsymbol{y}' = \boldsymbol{z}' - \boldsymbol{z} \in (Y - Y) \cap (\bar{Z} - \bar{Z}) = \{\boldsymbol{0}\}$, so $\boldsymbol{y} = \boldsymbol{y}'$. □

By (11) V is contained in $H_{\boldsymbol{u}_k}(\boldsymbol{0})$, for $\bar{N} < k \leq N$. By assumption $|A_k \cap H_{\boldsymbol{u}_k}(\boldsymbol{q})| < \aleph_\delta$, and since $Y \subseteq V$, then $|A_k \cap (\boldsymbol{q} + Y)| < \aleph_\delta$, for each $\boldsymbol{q} \in \mathbb{R}^d$. Therefore for any $\bar{N} < k \leq N$,

$$|A_k \cap (\hat{\boldsymbol{p}} + Z)| = |A_k \cap (\hat{\boldsymbol{p}} + Y + \bar{Z})| = \Big|\bigcup_{\boldsymbol{z} \in \bar{Z}} A_k \cap (\hat{\boldsymbol{p}} + \boldsymbol{z} + Y)\Big| \leq \aleph_{\delta+\bar{n}+1}. \tag{12}$$

Claim 4.9.2. *There is* $\hat{\boldsymbol{y}} \in Y$ *such that* $(\hat{\boldsymbol{p}} + \hat{\boldsymbol{y}} + \bar{Z}) \cap \bigcup_{\bar{N} < k \leq N} A_k = \emptyset$.

Proof. Towards a contradiction, suppose that for all $\boldsymbol{y} \in Y$ there are $\boldsymbol{z}(\boldsymbol{y}) \in \bar{Z}$ and $\bar{N} < k(\boldsymbol{y}) \leq N$ such that $\hat{\boldsymbol{p}} + \boldsymbol{y} + \boldsymbol{z}(\boldsymbol{y}) \in A_{k(\boldsymbol{y})}$. As

$$\aleph_{\delta+\bar{n}+2} = |Y| > \aleph_{\delta+\bar{n}+1} = \bar{Z} > \Big|\{k \mid \bar{N} < k \leq N\}\Big|,$$

there is $\hat{Y} \subseteq Y$ of size $\aleph_{\delta+\bar{n}+2}$, and $\hat{\boldsymbol{z}} \in \bar{Z}$ and $\bar{N} < \hat{k} \leq N$ such that $\boldsymbol{z}(\boldsymbol{y}) = \hat{\boldsymbol{z}}$ and $k(\boldsymbol{y}) = \hat{k}$ for all $\boldsymbol{y} \in \hat{Y}$. Therefore $\forall \boldsymbol{y} \in \hat{Y}\, (\hat{\boldsymbol{p}} + \boldsymbol{y} + \hat{\boldsymbol{z}} \in A_{\hat{k}})$. By Claim 4.9.1 the map $\hat{Y} \to A_{\hat{k}}$, $\boldsymbol{y} \mapsto \hat{\boldsymbol{p}} + \boldsymbol{y} + \hat{\boldsymbol{z}}$ is injective, so $|A_{\hat{k}} \cap (\hat{\boldsymbol{p}} + Y + \bar{Z})| \geq |\hat{Y}| = \aleph_{\delta+\bar{n}+2}$, against (12). □

Fix $\hat{\boldsymbol{y}}$ as in Claim 4.9.2. Then $(\hat{\boldsymbol{p}} + \hat{\boldsymbol{y}}) + \bar{Z} \subseteq A_1 \cup \cdots \cup A_{\bar{N}}$ against (10). Having reached a contradiction, we conclude that $\boldsymbol{p} + Z \nsubseteq \bigcup_{k=1}^{N} A_k$ for all $\boldsymbol{p} \in \mathbb{R}^d$. □

Remark 4.13. Theorems 4.4 and 4.9 (together with Theorem 2.3) provide a complete solution to Problem 4.1 when the vectors are in general position, yet some further generalizations are possible. For the sake of readability we have opted for less generality, and here we would like to mention some of these extensions. (The proof of these results will appear elsewhere.)

Focusing on $d = 3$ and $\delta = 0$, the requirement on the size of the intersections in Theorem 4.4 could be relaxed to

$$|A_i \cap H_{\boldsymbol{u}_i}(\boldsymbol{p})| < \aleph_0 \quad \text{for } i = 1, 2, \qquad |A_i \cap H_{\boldsymbol{u}_i}(\boldsymbol{p})| \leq \aleph_0 \quad \text{for } i = 3, 4,$$

strengthening a theorem of Bagemihl [1959/60] that there are no A_1, A_2, A_3 covering $\mathbb{R}^3$ so that all planes orthogonal to $\boldsymbol{e}_2, \boldsymbol{e}_3$ have countable intersections with A_2, A_3 and all planes orthogonal to $\boldsymbol{e}_1$ have finite intersections with A_1. (By Theorem 2.6 and Lemma 4.2 CH is equivalent to the fact that the sets $A_i \cap H_{\boldsymbol{u}_i}(\boldsymbol{p})$ are countable for three distinct is, if $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_4$ are in general position.)

Another possible generalization of Theorem 4.4 is that for any $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n \in \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_2) \setminus \{\boldsymbol{0}\}$ and $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_3) \setminus \{\boldsymbol{0}\}$, there are no $A_1, A_2, A_3, B_1, \dots, B_n, C_1, \dots, C_m$ covering $\mathbb{R}^3$ such that each $A_i \cap H_{\boldsymbol{e}_i}(\boldsymbol{p})$, $B_j \cap H_{\boldsymbol{u}_j}(\boldsymbol{p})$, $C_k \cap H_{\boldsymbol{v}_k}(\boldsymbol{p})$ is finite. (Clearly $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ can be replaced with any other basis.) This shows that Corollary 4.11 can fail badly, if the vectors are not in general position. On the other hand, not being in general position does not preclude a positive result. For example CH is equivalent to the existence of $A_1, \dots, A_6$ covering $\mathbb{R}^3$ such that each $A_i \cap H_{\boldsymbol{u}_i}(\boldsymbol{p})$ is finite, where $\boldsymbol{u}_1 = \boldsymbol{e}_1$, $\boldsymbol{u}_2 = \boldsymbol{e}_2$, $\boldsymbol{u}_3 = \boldsymbol{e}_3$, $\boldsymbol{u}_4 \in \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_2) \setminus \{\boldsymbol{0}\}$, $\boldsymbol{u}_5 \in \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_4) \setminus \{\boldsymbol{0}\}$, $\boldsymbol{u}_6 \in \operatorname{span}(\boldsymbol{e}_2, \boldsymbol{e}_3) \setminus \{\boldsymbol{0}\}$.

5. The main results

Combining Theorem 3.2 with the results from Section 4 we are ready to prove the results about sprays in $\mathbb{R}^d$.

Theorem 5.1. *Let $\boldsymbol{c}_1, \dots, \boldsymbol{c}_4$ be four coplanar points in $\mathbb{R}^3$. There are no $X_1, \dots, X_4$ covering $\mathbb{R}^3$ such that X_i is a spray with centers $\boldsymbol{c}_i$.*

Proof. Let $X_1, \dots, X_4$ be sprays in $\mathbb{R}^3$ with $\boldsymbol{c}_i$ the center of X_i.

If $\boldsymbol{c}_1, \dots, \boldsymbol{c}_4$ belong to the same line ℓ, let P be a plane orthogonal to ℓ, and let $\pi : \mathbb{R}^3 \to P$ be the orthogonal projection. Then $Y := \bigcup_{1 \le i \le 4} X_i \cap P$ is a spray of P centered in $\pi(\boldsymbol{c}_1) = \cdots = \pi(\boldsymbol{c}_4) \in P$. Since Y cannot cover P then $X_1, \dots, X_4$ cannot cover $\mathbb{R}^3$.

Suppose $\boldsymbol{c}_1, \dots, \boldsymbol{c}_4$ are not collinear. By relabeling and applying an isometry, we may assume that $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ are not collinear and belong to $\mathbb{R}^2 \times \{0\}$. By Theorem 3.2 there is an open $E^3 \subseteq \mathbb{R}^3$ and there are $A_1, \dots, A_4$ covering E^3 such that $A_i \cap H_i(r)$ are finite for $i = 1, 2, 3$, and $A_4 \cap H_{\boldsymbol{u}}(\boldsymbol{p})$ is finite, where $\boldsymbol{u} = (u_1, u_2, u_3) \neq \boldsymbol{0}$ is such that $u_1(\boldsymbol{c}_1 - \boldsymbol{c}_4) + u_2(\boldsymbol{c}_2 - \boldsymbol{c}_4) + u_3(\boldsymbol{c}_3 - \boldsymbol{c}_4) = \boldsymbol{0}$ and r and $\boldsymbol{p}$ range in $\mathbb{R}$ and $\mathbb{R}^3$, respectively. The result follows from Theorem 4.4. □

The next result uses the vector space $\mathcal{U}_{\boldsymbol{c}_1, \dots, \boldsymbol{c}_d}(\boldsymbol{q})$ defined in (2).

Proposition 5.2. *Suppose H is an affine hyperplane of $\mathbb{R}^d$, $\boldsymbol{c}_1, \dots, \boldsymbol{c}_n \in H$ are distinct, well-placed points in $\mathbb{R}^d$, and $n \ge d$. For $k \le n$ let $\boldsymbol{u}_k \in \mathcal{U}_{\boldsymbol{c}_1, \dots, \boldsymbol{c}_d}(\boldsymbol{c}_k) \setminus \{\boldsymbol{0}\}$. Then the vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n$ are in general position in $\mathbb{R}^d$.*

Proof. As the definition of $\mathcal{U}(\boldsymbol{c}_k)$ is not affected by a translation of H, we may assume that $\boldsymbol{0} \notin H$, so that the vectors $(\boldsymbol{c}_1, \dots, \boldsymbol{c}_d)$ form a basis of $\mathbb{R}^d$. Let $\boldsymbol{u}_k = (u_k^1, \dots, u_k^d)$. We claim that $\sum_{i=1}^d u_k^i \neq 0$, for any $k \le n$. Otherwise

$$\begin{aligned}\boldsymbol{0} = \sum_{i=1}^d u_k^i(\boldsymbol{c}_i - \boldsymbol{c}_k) &= \sum_{i=1}^d u_k^i \boldsymbol{c}_i = u_k^1 \boldsymbol{c}_1 + \sum_{i=2}^d u_k^i \boldsymbol{c}_i \\ &= -\Big(\sum_{i=2}^d u_k^i\Big)\boldsymbol{c}_1 + \sum_{i=2}^d u_k^i \boldsymbol{c}_i = \sum_{i=2}^d u_k^i(\boldsymbol{c}_i - \boldsymbol{c}_1)\end{aligned}$$

shows that $\boldsymbol{c}_1, \dots, \boldsymbol{c}_d$ are not in general position in H, against our assumption. Scaling of vectors does not affect their general position, so we may assume that $\sum_{i=1}^d u_k^i = 1$ for all $k \leq n$. Then $\sum_{i=1}^d u_k^i(\boldsymbol{c}_i - \boldsymbol{c}_k) = \boldsymbol{0}$ implies that

$$\boldsymbol{c}_k = \sum_{i=1}^d u_k^i \boldsymbol{c}_i$$

that is to say: $(u_k^1, \dots, u_k^d)$ are the components of the vector $\boldsymbol{c}_k$ with respect to the basis $(\boldsymbol{c}_1, \dots, \boldsymbol{c}_d)$. Therefore the general position of the vectors $\boldsymbol{u}_k$s follows from the general position of the points $\boldsymbol{c}_k$s in H. □

The very same proof yields:

Proposition 5.3. *Let H be a hyperplane of $\mathbb{R}^d$, and suppose $\{\boldsymbol{c}_k \mid 1 \leq k < \omega\} \subseteq H$ are distinct points in general position in H. For all $k \geq 1$ let $\boldsymbol{u}_k \in \mathcal{U}_{\boldsymbol{c}_1,\dots,\boldsymbol{c}_d}(\boldsymbol{c}_k) \setminus \{\boldsymbol{0}\}$. Then the vectors $\{\boldsymbol{u}_k \mid 1 \leq k < \omega\}$ are in general position in $\mathbb{R}^d$.*

Theorem 5.4. *Fix $d \geq 2$ and $n \geq 1$, and let $N = (n+1)(d-1)+1$ and $M = (n+2)(d-1)$. The following are equivalent*:

(a) $2^{\aleph_0} \leq \aleph_n$.

(b) *For all distinct, well-placed points $\boldsymbol{c}_1, \dots, \boldsymbol{c}_N \in \mathbb{R}^d$ there are sprays $X_1, \dots, X_N$ covering $\mathbb{R}^d$ with X_i centered in $\boldsymbol{c}_i$.*

(c) *There are sprays $X_1, \dots, X_M$ covering $\mathbb{R}^d$ with X_i centered in $\boldsymbol{c}_i$ such that $\boldsymbol{c}_1, \dots, \boldsymbol{c}_N$ are distinct and well-placed.*

Proof. (a)⇒(b) follows from Theorem 2.18 with $\delta = 0$, and (b)⇒(c) is trivial — set $X_{N+1} = \cdots = X_M = \varnothing$ and $\boldsymbol{c}_{N+1}, \dots, \boldsymbol{c}_M$ any points belonging to the same hyperplane passing through $\boldsymbol{c}_1, \dots, \boldsymbol{c}_N$. So it is enough to prove (c)⇒(a).

If $d = 2$ then $N = M = n+2$ and (c) says that $\mathbb{R}^2$ is the union of $n+2$ sprays with aligned centers, so $2^{\aleph_0} \leq \aleph_n$ follows from [Schmerl 2010, Theorem 7]. Therefore we may assume that $d \geq 3$. Towards a contradiction, assume $2^{\aleph_0} \geq \aleph_{n+1}$ and suppose that $\boldsymbol{c}_1, \dots, \boldsymbol{c}_M \in \mathbb{R}^d$ are well-placed, and that $X_1, \dots, X_M$ are sprays as above. For ease of notation, let $\boldsymbol{u}_i = \boldsymbol{e}_i$ for $1 \leq i \leq d$. By Proposition 5.2 the vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_N$ are in general position in $\mathbb{R}^d$, and the map $\Phi : \mathbb{H}^d \to E^d$ transforms the X_i into a covering $A_1, \dots, A_M$ of E^d such that every hyperplane orthogonal to $\boldsymbol{u}_i$ intersects A_i in a finite set. But this contradicts Theorem 4.10 with $\delta = 0$. □

In particular:

Theorem 5.5. *The following are equivalent.*

(a) CH *holds.*

(b) *For all well-placed $\boldsymbol{c}_1, \dots, \boldsymbol{c}_5 \in \mathbb{R}^3$ there are sprays $X_1, \dots, X_5$ covering $\mathbb{R}^3$ with X_i centered in $\boldsymbol{c}_i$.*

(c) *There are sprays* $X_1, \ldots, X_6$ *covering* $\mathbb{R}^3$ *with* X_i *centered in* $\boldsymbol{c}_i$ *such that* $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_6$ *are well-placed.*

More generally:

Theorem 5.6. *Fix* $d \geq 2$ *and* $n \geq 1$, *and let* $M = (n+2)(d-1)$, $N = (n+1)(d-1)+1$, $\overline{M} = (n+1)(d-1) = N-1$, *and* $\overline{N} = n(d-1)+1$. *The following are equivalent*:

(a) $2^{\aleph_0} \leq \aleph_n$.

(b) *For all well-placed points* $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_N \in \mathbb{R}^d$ *there are sprays* $X_1, \ldots, X_N$ *covering* $\mathbb{R}^d$ *with* X_i *centered in* $\boldsymbol{c}_i$.

(c) *There are sprays* $X_1, \ldots, X_M$ *covering* $\mathbb{R}^d$ *with* X_i *centered in* $\boldsymbol{c}_i$ *such that* $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_M$ *are well-placed.*

(d) *For all well-placed points* $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{\overline{N}} \in \mathbb{R}^d$ *there are* σ*-sprays* $X_1, \ldots, X_N$ *covering* $\mathbb{R}^d$ *with* X_i *centered in* $\boldsymbol{c}_i$.

(e) *There are* σ*-sprays* $X_1, \ldots, X_{\overline{M}}$ *covering* $\mathbb{R}^d$ *with* X_i *centered in* $\boldsymbol{c}_i$ *such that* $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{\overline{M}}$ *are well-placed.*

Proposition 5.3 is the bridge connecting Problem 4.1 with the next problem.

Problem 5.7. Given $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$ distinct points of H, a hyperplane of $\mathbb{R}^d$, what conditions on the cardinality of $\mathbb{R}$ are equivalent to the existence of $X_1, \ldots, X_n$ covering $\mathbb{R}^d$, each X_i a spray (or σ-spray) centered in $\boldsymbol{c}_i$?

Theorem 5.6 yields a complete solution to Problem 5.7 when the $\boldsymbol{c}_i$s are well-placed, i.e. in general position in H. Using the results mentioned in Remark 4.13 it is possible to distill a few more results on sprays in $\mathbb{R}^3$. Let us mention just two of them. The first is that given four coplanar points $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \boldsymbol{c}_4$ in $\mathbb{R}^3$, there exist no X_1, X_2, X_3, X_4 covering $\mathbb{R}^3$ such that X_1, X_2 are sprays centered in $\boldsymbol{c}_1, \boldsymbol{c}_2$, and X_3, X_4 are σ-sprays centered in $\boldsymbol{c}_3, \boldsymbol{c}_4$. The second result is that the six points in $\mathbb{R}^2 \times \{0\}$ of Figure 1 are not in general position in the plane, and

- no five sprays centered in these points can cover $\mathbb{R}^3$, but
- CH is equivalent to the existence of six sprays, centered in these points, covering $\mathbb{R}^3$.

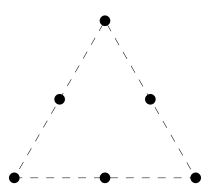


Figure 1. CH is equivalent to the existence of six sprays centered in these points and covering $\mathbb{R}^3$. Unconditionally, no five sprays centered in these points can achieve this.

5.1. ***Covering the space with infinitely many sprays.*** We have seen how covering the space with sprays with well-placed centers is equivalent to giving an upper bound for the size of the continuum, the larger the number of sprays, the weaker the bound. Next we show that, irrespective of the size of the continuum, the space can be covered with $\aleph_0$-many sprays with well-placed centers.

A *drizzle* in $\mathbb{R}^d$ with center $\boldsymbol{c}$ is a set $X \subseteq \mathbb{R}^d$ such that any sphere centered in $\boldsymbol{c}$ intersects X in at most one point — thus a drizzle is a very sparse spray.

Theorem 5.8. *If* $\{\boldsymbol{c}_n \mid n \geq 1\}$ *are distinct, well-placed points in* $\mathbb{R}^d$, *with* $d \geq 2$, *then there are* X_n *covering* $\mathbb{R}^d$ *such that* X_n *is a drizzle centered in* $\boldsymbol{c}_n$.

Proof. Without loss of generality we may assume that the $\boldsymbol{c}_n$s belong to $\mathbb{R}^{d-1} \times \{0\}$. Pick $\boldsymbol{u}_n \in \mathcal{U}(\boldsymbol{c}_n) \setminus \{\boldsymbol{0}\}$, where $\mathcal{U}(\boldsymbol{q}) = \mathcal{U}_{\boldsymbol{c}_1,\ldots,\boldsymbol{c}_d}(\boldsymbol{q})$. By Proposition 5.3 the vectors $\boldsymbol{u}_n$ $(n \geq 1)$ are in general position in $\mathbb{R}^d$. By [Davies 1974] there are sets A_k $(k \geq 1)$ such that for all $\boldsymbol{p} \in \mathbb{R}^d$, $H_{\boldsymbol{u}_k}(\boldsymbol{p}) \cap A_k$ has at most one point, and such that $\mathbb{R}^d = \bigcup_{n \geq 1} A_{2n} = \bigcup_{n \geq 0} A_{2n+1}$. The map $\Phi : \mathbb{H}^d \to E^d$ of (4) can be extended to the closures of $\mathbb{H}^d$ and E^d, so we can assume that $X_{2n} := \Phi^{-1}[A_{2n}]$ is a subset of $\mathrm{Cl}(\mathbb{H}^d) = \mathbb{R}^{d-1} \times [0; +\infty)$, and that $\bigcup_{n \geq 1} X_{2n} = \mathrm{Cl}(\mathbb{H}^d)$. Letting $\tau : \mathbb{R}^d \to \mathbb{R}^d$ be the reflection with respect to the hyperplane $\mathbb{R}^{d-1} \times \{0\}$, let $X_{2n+1} = \tau[\Phi^{-1}[A_{2n+1}]]$. Then $\bigcup_{n \geq 0} X_{2n+1} = \tau[\mathbb{H}^d] = \mathbb{R}^{d-1} \times (-\infty; 0)$. Therefore $\mathbb{R}^d = \bigcup_n X_n$, and by construction each X_n is a drizzle centered in $\boldsymbol{c}_n$. □

References

[Bagemihl 1959/60] F. Bagemihl, "Planar sections of spatial sets", *Math. Z.* **72** (1959/60), 362–366. MR

[Bagemihl 1960] F. Bagemihl, "Decompositions of the plane into three sets", *J. Reine Angew. Math.* **203** (1960), 209–215. MR

[Bagemihl 1961] F. Bagemihl, "A proposition of elementary plane geometry that implies the continuum hypothesis", *Z. Math. Logik Grundlagen Math.* **7** (1961), 77–79. MR

[Bagemihl 1968] F. Bagemihl, "Decompositions of the plane into sets, and coverings of the plane with curves", *Czechoslovak Math. J.* **18(93)** (1968), 616–621. MR

[Davies 1962] R. O. Davies, "Equivalence to the continuum hypothesis of a certain proposition of elementary plane geometry", *Z. Math. Logik Grundlagen Math.* **8** (1962), 109–111. MR

[Davies 1963a] R. O. Davies, "On a problem of Erdős concerning decompositions of the plane", *Proc. Cambridge Philos. Soc.* **59** (1963), 33–36. MR

[Davies 1963b] R. O. Davies, "The power of the continuum and some propositions of plane geometry", *Fund. Math.* **52** (1963), 277–281. MR

[Davies 1974] R. O. Davies, "Covering space with denumerably many curves", *Bull. London Math. Soc.* **6** (1974), 189–190. MR

[Erdős et al. 1994] P. Erdős, S. Jackson, and R. D. Mauldin, "On partitions of lines and space", *Fund. Math.* **145**:2 (1994), 101–119. MR

[Komjáth 2001] P. Komjáth, "Three clouds may cover the plane", *Ann. Pure Appl. Logic* **109**:1-2 (2001), 71–75. MR

[Komjáth and Totik 2006] P. Komjáth and V. Totik, *Problems and theorems in classical set theory*, Springer, 2006. MR

[Kuratowski 1951] C. Kuratowski, "Sur une caractérisation des alephs", *Fund. Math.* **38** (1951), 14–17. MR

[Mazurkiewicz 1933] S. Mazurkiewicz, "Sur la décomposition du plan en courbes", *Fund. Math.* **21** (1933), 43–45.

[Schmerl 2003] J. H. Schmerl, "How many clouds cover the plane?", *Fund. Math.* **177**:3 (2003), 209–211. MR

[Schmerl 2010] J. H. Schmerl, "Covering the plane with sprays", *Fund. Math.* **208**:3 (2010), 263–272. MR

[Schmerl 2012] J. H. Schmerl, "A generalization of Sierpiński's paradoxical decompositions: coloring semialgebraic grids", *J. Symbolic Logic* **77**:4 (2012), 1165–1183. MR

[Sierpiński 1951] W. Sierpiński, "Sur quelques propositions concernant la puissance du continu", *Fund. Math.* **38** (1951), 1–13. MR

[Simms 1997] J. C. Simms, "Another characterization of alephs: decompositions of hyperspace", *Notre Dame J. Formal Logic* **38**:1 (1997), 19–36. MR

[de la Vega 2009] R. de la Vega, "Decompositions of the plane and the size of the continuum", *Fund. Math.* **203**:1 (2009), 65–74. MR

Received October 11, 2024. Revised August 26, 2025.

ALESSANDRO ANDRETTA
DIPARTIMENTO DI MATEMATICA "GIUSEPPE PEANO"
UNIVERSITÀ DEGLI STUDI DI TORINO
TORINO
ITALY
alessandro.andretta@unito.it

IVAN IZMESTIEV
INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE
TECHNISCHE UNIVERSITÄT WIEN
VIENNA
AUSTRIA
izmestiev@dmg.tuwien.ac.at

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2026 is US $710/year for the electronic version, and $965/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 341 No. 1 March 2026