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We investigate groups that act amenably on their Higson corona (also known as bi-exact groups) and provide reformulations of this in relation to the stable Higson corona, nuclearity of crossed products and to positive type kernels.

We further investigate implications of this in relation to the Baum–Connes conjecture, and prove that Gromov hyperbolic groups have isomorphic equivariant *K*-theories of their Gromov boundary and their stable Higson corona.

1. Introduction

This paper arose from pursuing a side idea of [EWZ21], namely to investigate groups that act amenably on their own (reduced) stable Higson corona $\mathfrak{c}^{\mathrm{red}}G$. The

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motivation comes from the commutative diagram

$$\begin{array}{ccc} K_*^{\mathrm{top}}(G;\mathfrak{c}^{\mathrm{red}}G) & \xrightarrow{\mu^*_{\mathrm{EM}}} & K_G^{1-*}(\underline{E}G) \\ & \searrow^{\mu_*^{\mathrm{BC}}} \quad \nearrow^{\mu_G^*} & \\ & K_*(\mathfrak{c}^{\mathrm{red}}G \rtimes_\mu G) & \end{array} \tag{1-1}$$

where μ^*_{EM} is the co-assembly map of Emerson and Meyer, μ_*^{BC} is the Baum–Connes assembly map, μ_G^* is the equivariant coarse co-assembly map, and where we have by definition $K_G^{1-*}(\underline{E}G) := K_{*-1}(C_0(\underline{E}G) \rtimes_{\mathrm{red}} G)$ [EWZ21, Section 5.1].

In the diagram we may use any crossed product functor $- \rtimes_\mu G$ which is exact, and one of our questions was: In which cases does the choice not matter, i.e., when do $\mathfrak{c}^{\mathrm{red}}G \rtimes_{\max} G$ and $\mathfrak{c}^{\mathrm{red}}G \rtimes_{\mathrm{red}} G$ coincide (assuming for simplicity that G is exact)? As is now known, this question is intimately connected to amenability of the G-C^*-algebra $\mathfrak{c}^{\mathrm{red}}G$ [BEW20]. We tried to answer this question in [EWZ21, Propositions 5.8 and 5.12], but unfortunately, there is an error in the proofs and one of the goals of this paper is to provide an erratum to it.[1] We will discuss the error in detail in Section 3.3.

As it turns out, the corrected version of [EWZ21, Proposition 5.8] uses the *unreduced* stable Higson compactification $\bar{\mathfrak{c}}G$ and stable Higson corona. Concretely, we prove the following (we only state the version for the compactification):

Theorem 1.1 (part of Proposition 3.6). *Let G be a countable and discrete group. The following statements are equivalent to each other*:

(a) *The group G acts amenably on its Higson compactification hG.*

(b) *$\bar{\mathfrak{c}}G$ is an amenable G-C*-algebra.*

(c) *We have $\bar{\mathfrak{c}}G \rtimes_{\max} G \cong \bar{\mathfrak{c}}G \rtimes_{\mathrm{red}} G$ and G is exact.*

Groups acting amenably on their Higson compactification hG have been already studied before, but with another focus: One can prove that this condition is equivalent to the group G being bi-exact (see Definition 2.2) and from this we get a plethora of examples. This will be quickly summarized in Section 2.

In Section 3.4 we will find more reformulations for a group to act amenably on its Higson compactification. Concretely, we prove the following:

Theorem 1.2 (Proposition 3.17). *Let G be a countable and discrete group. The following statements are equivalent to each other*:

(a) *The group G acts amenably on its Higson compactification.*

(b) *The C^*-algebra $C(hG) \rtimes_{\mathrm{red}} G$ is nuclear.*

[1] Fortunately, since this question was just a side hustle in [EWZ21], none of the main results of it are affected by this.

(c) *The embedding* $\mathbb{C} \rtimes_{\text{red}} G \to C(hG) \rtimes_{\text{red}} G$ *is nuclear.*

(d) *There is a sequence* $(k_n)_{n\in\mathbb{N}}$ *in* $C_c(G \times G, \Delta)$ *of normalized positive type kernels having vanishing variation on diagonals and converging to* 1 *uniformly on all finite width neighbourhoods of the diagonal* Δ *in* $G \times G$.

As we will discuss in Remarks 3.18, the above result shows that the condition of acting amenably on its Higson compactification sits naturally between amenability and exactness of the group. Because both amenability and exactness have profound implications for the Baum–Connes conjecture for the group (amenability implies bijectivity whereas exactness implies injectivity of the assembly map), the question arises whether one can prove something in this direction for groups acting amenably on their Higson compactifications. We will pursue this line of thought in Section 4, and our main result in this direction is the following:

Theorem 1.3 (Proposition 4.3). *Let G be a bi-exact group and assume that it admits a G-finite classifying space for proper G-actions* $\underline{E}G$. *Then we have the split short exact sequence*

$$0 \to K_{*+1}(\bar{\mathfrak{c}}^{\text{red}}\underline{E}G \rtimes_{\text{red}} G) \to K_*(C^*_{\text{red}}(G)) \to K_*(\bar{\mathfrak{c}}(\underline{E}G) \rtimes_{\text{red}} G) \to 0. \tag{1-2}$$

Further, the Baum–Connes conjecture for trivial coefficients $\mathbb{C}$ *and coefficients* $\bar{\mathfrak{c}}^{\text{red}}\underline{E}G$ *are equivalent to each other for G and imply the isomorphism*

$$K_*(C^*_{\text{red}}(G)) \xrightarrow{\cong} K_*(\bar{\mathfrak{c}}(\underline{E}G) \rtimes_{\text{red}} G), \tag{1-3}$$

which is induced from the inclusion of $\mathcal{K}$ *as the constant functions in* $\bar{\mathfrak{c}}(\underline{E}G)$.

Our final result is a computation of $K_*(\mathfrak{c}(\underline{E}G) \rtimes_{\text{red}} G)$ for Gromov hyperbolic groups (note that this result now refers to the stable Higson corona, contrary to the compactification as above in Theorem 1.3):

Theorem 1.4 (Example 4.5). *Let G be a finitely generated, Gromov hyperbolic group. Then we have an isomorphism*

$$K_*(C(\partial G) \rtimes_{\text{red}} G) \cong K_*(\mathfrak{c}G \rtimes_{\text{red}} G), \tag{1-4}$$

where ∂G *denotes the Gromov boundary of G.*

2. Bi-exact groups

Let G be a countable and discrete group. If needed, we will equip it without further mentioning with any proper, left-invariant metric.

Definition 2.1 (Amenable actions, [HR00, Definitions 2.1 and 2.2] and [ADR00, Section 2.2]).

(a) For any countable set Z we denote by $\operatorname{prob}(Z)$ the set of Borel probability measures on Z, i.e., the set of functions $b\colon Z \to [0,1]$ such that $\sum_{z\in Z} b(z) = 1$.

We view $\operatorname{prob}(Z)$ as a subset of $\ell^1(Z)$ and equip it with the weak-*-topology (recall that $\ell^1(Z)$ is the Banach space dual of $c_0(Z)$). By $\|-\|_1$ we denote the usual norm on $\ell^1(Z)$.

If Z is equipped with an action of the group G, then the induced action of G on $\operatorname{prob}(Z)$ is defined by $g.b(z) := b(g^{-1}.z)$.

(b) Let X be a locally compact Hausdorff space on which G acts by homeomorphisms. The action is called amenable if there is a sequence of weak-*-continuous maps $b^n : X \to \operatorname{prob}(G)$ such that for every $g \in G$ we have

$$\lim_{n\to\infty} \|g.b^n_x - b^n_{g.x}\|_1 = 0$$

uniformly on compact subsets of X.

Definition 2.2 (bi-exact groups [BO08, Chapter 15]). We consider the $G\times G$-action on the group G given by left and right translations.

We call G *bi-exact* if the induced action of $G\times G$ on the Stone–Čech boundary $\partial_\beta G$ is amenable.

Remarks 2.3. It is known that G is bi-exact if and only if it acts amenably on its Higson corona (which will be defined in Section 3.1) [BO08, Proposition 15.2.3]. Because the latter is the class of groups that we consider in this paper, everything we prove here holds for bi-exact groups.

Since the Higson corona is a compact Hausdorff space, being bi-exact implies being exact (see [HR00] or [BO08, Theorem 5.1.7]).

Examples 2.4. The following groups are bi-exact:

(a) amenable groups, since being amenable implies that any action on any space is amenable and so especially the action on its Higson corona [AD07, Example 1.4(1)];

(b) groups hyperbolic relative to a family of amenable subgroups [Oza06b, Proposition 12] or, more generally, hyperbolic relative to a family of bi-exact subgroups [Oya23b];

(c) small cancellation groups, such as finitely generated $C'\left(\frac{1}{33}\right)$-groups [Oya23a];

(d) discrete subgroups of connected, simple Lie groups of rank one [Ska88, Section 4];

(e) $\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ [Oza09].

The class of bi-exact groups also enjoys the following closure properties:

(f) It is closed under passage to subgroups.

The argument is as follows: Assume that H is a subgroup of G and that G is a bi-exact group, i.e., the action of $G \times G$ on $\partial_\beta G$ is amenable. Then also $H \times H$ acts amenably on $\partial_\beta G$, cf. [BEW24, Proposition 5.19] or [OS21, Corollary 3.4] (keep in mind for these references that here we are only considering discrete groups). By the universal property of the Stone–Čech compactification we have an $(H \times H)$-map $\partial_\beta H \to \partial_\beta G$ and hence $H \times H$ will also act amenably on $\partial_\beta H$.

(g) It is closed under free products (with finite amalgamation) [Oza06a, Theorem 3.3]

(h) Wreath products $\Upsilon \wr G$, where Υ is amenable and G bi-exact, are again bi-exact [BO08, Corollary 15.3.9].

(i) It is closed under measure equivalence [Sak09].

(j) It is closed under W^*-equivalence [DP, Corollary 6.3].

Remarks 2.5. The class of bi-exact groups is in general not closed under products: The group $\mathbb{Z} \times \mathrm{SL}(2, \mathbb{Z})$ is not bi-exact.[2]

It follows that $\mathrm{SL}(3, \mathbb{Z})$ is not bi-exact either: The group $\mathbb{Z} \times \mathrm{SL}(2, \mathbb{Z})$ ME-embeds into it[3] and hence by Sako's result[4] [Sak09] bi-exactness of $\mathrm{SL}(3, \mathbb{Z})$ would propagate to $\mathbb{Z} \times \mathrm{SL}(2, \mathbb{Z})$.

Weakly amenable groups are in general not bi-exact, since weak amenability is, contrary to bi-exactness, closed under products: A product of free groups (which are weakly amenable by [Oza08]) is weakly amenable but not bi-exact (by note 2).

Remarks 2.6. The importance of bi-exact groups stems from the fact that the group von Neumann algebra of a bi-exact group is solid [Oza04].

3. Amenable actions on the Higson compactification

3.1. ***The commutative case.*** We will discuss the relevant notions in the commutative case, i.e., for the corresponding spaces of complex-valued functions.

[2] In general, any product of an infinite group G and a non-amenable group H will not be bi-exact. The argument is as follows: In advance of Section 3.1 we use the characterization of bi-exactness of a group by acting amenably on its Higson corona. Let x be a point in the Higson corona of $G \times H$ with x lying in the closure of $G \times \{e_H\}$ therein. (This needs G to be infinite, otherwise there is no such x.) Acting with $\{e_G\} \times H$ from the left on $G \times \{e_H\}$ is the same as acting on it from the right and hence $\{e_G\} \times H$ acts trivially on the point x (to see this one has to use the concrete definition of the Higson corona). If $G \times H$ would act amenably on its Higson corona, then all point stabilizers would be amenable subgroups and therefore H would be amenable.

[3] The ME-embedding comes from a discrete embedding of $\mathbb{Z} \times \mathrm{SL}(2, \mathbb{Z})$ into $\mathrm{SL}(3, \mathbb{R})$.

[4] In the version that if G ME-embeds into a bi-exact group, then G is bi-exact.

Definition 3.1 (Higson compactification and corona). Let Y be any metric space. If $\vartheta : Y \to M$ is a map to another metric space M, then for each $r > 0$ the r-variation of ϑ is defined as the function

$$\mathrm{Var}_r\vartheta : Y \to [0,\infty)\,, \qquad y \mapsto \sup\{d(\vartheta(y),\vartheta(x)) : x \in Y \text{ with } d(y,x) \le r\}\,.$$

The function ϑ is said to have vanishing variation if for all $r > 0$ the r-variation $\mathrm{Var}_r\vartheta$ converges to zero at infinity.[5]

The *Higson compactification* hY of Y is the Gelfand dual of the C^*-algebra of all complex-valued, bounded, continuous functions of vanishing variation on Y. The *Higson corona* $\partial_h Y$ is defined as the complement $hY \setminus Y$.

If $\bar{Y}$ is any compactification of Y, it is called *Higson dominated* if $C(\bar{Y}) \subset C(hY)$, i.e., if we have surjective map $hY \to \bar{Y}$ extending the identity on Y.

Because the Higson compactification hG is a compact Hausdorff space, acting amenably on it implies that G is exact in the usual sense (see [Oza00; HR00] and [BEW20, Theorem 5.3]).

In the case of the Stone–Čech compactification and corona the following lemma is well-known, see e.g. [BCL17].

Lemma 3.2. *Let G be a countable, discrete group. The following statements are equivalent:*

- *G acts amenably on its Higson compactification hG.*
- *G acts amenably on its Higson corona $\partial_h G$.*
- *$C(hG) \rtimes_{\max} G = C(hG) \rtimes_{\mathrm{red}} G$ and G is exact.*
- *$C(\partial_h G) \rtimes_{\max} G = C(\partial_h G) \rtimes_{\mathrm{red}} G$ and G is exact.*

Proof. In general, the group G acts amenably on a G-C^*-algebra A if and only if it acts amenably on both a G-ideal I in A and the quotient A/I [BEW24, Proposition 3.23]. The equivalence of the first two points in the lemma follows from this since G always acts amenably on $C_0(G)$ by [AD02, Example 2.7(3)] (and using [BEW24, Proposition 3.9] that an action on X is amenable if and only if the action on $C_0(X)$ is strongly amenable, and using [BEW24, Remark 3.8] that acting amenably and strongly amenably are equivalent in this case since these are commutative C^*-algebras).

For equivalence with the other points see [Mat14] and [BEW20, Theorem 5.2]. □

[5] If this is defined to mean that for every $\varepsilon > 0$ exists a compact subset $K \subset Y$ with $|\mathrm{Var}_r\vartheta(y)| < \varepsilon$ for all $y \in Y$, then one should assume Y to be locally compact. If Y is not locally compact, then one should instead demand that K be just bounded.

3.2. ***Groups acting amenably on the stable Higson compactification and stable Higson corona.*** For a discrete group G we will first recall the different notions of amenability of G-C*-algebras from [BEW20, Definitions 2.1 and 4.13] and then apply them to the (unreduced) stable Higson compactification and corona (see Definition 3.4). (Variants of some of these notions occur in e.g. [AD02; BO08]. How these variants relate to each other is explained in [BEW20, Remark 2.2].)

Definition 3.3. Let G be a discrete group.

(a) The G-C*-algebra A is called strongly amenable if there is a net

$$(\theta_i : G \to Z\mathcal{M}(A))_{i\in I} \, ,$$

where $Z\mathcal{M}(A)$ is the center of the multiplier algebra, of positive type functions[6] such that

- each θ_i is finitely supported,
- for each i we have $\theta_i(e) \leq 1$, and
- for each $g \in G$ we have $\theta_i(g) \to 1$ strictly as $i \to \infty$.

(b) The G-C*-algebra A is called amenable if there is a net $(\theta_i : G \to Z(A^{**}))_{i\in I}$ of positive type functions such that

- each θ_i is finitely supported,
- for each i we have $\theta_i(e) \leq 1$, and
- for each $g \in G$ we have $\theta_i(g) \to 1$ ultra-weakly as $i \to \infty$.[7]

(c) The G-C*-algebra A is called commutant amenable if for every covariant pair $(\pi, u) : (A, G) \to \mathcal{B}(H)$ there exists a net $(\theta_i : G \to \pi(A)')_{i\in I}$ of positive type functions such that

- each θ_i is finitely supported,
- for each i we have $\theta_i(e) \leq 1$, and
- for each $g \in G$ we have $\theta_i(g) \to 1$ ultra-weakly as $i \to \infty$.

Note that strong amenability implies amenability [BEW20, Remark 2.2], and amenability implies commutant amenability [BEW20, Remark 4.14].

The main players of this section are the (unreduced[8]) stable Higson compactification $\bar{\mathfrak{c}}G$ and corona $\mathfrak{c}G$ of G, defined thus:

[6] In general, a function $\vartheta : G \to B$ is of positive type if for any finite subset $\{g_1, \ldots, g_n\}$ of G the matrix $(\alpha_{g_i}(\vartheta(g_i^{-1} g_j)))_{i,j} \in M_n(B)$ is positive, where α is the action of G on B [AD87, Definition 2.1].

[7] Recall that a net $(T_\lambda)_{\lambda\in\Lambda}$ in A^{**} converges ultra-weakly to T if and only if $(T_\lambda(\varphi))_{\lambda\in\Lambda}$ converges to $T(\varphi)$ for every $\varphi \in A^*$.

[8] There are also reduced versions of these C^*-algebras, but the main results of the present section (Proposition 3.6, Proposition 3.8) do not hold for them; see the next Section 3.3 for a discussion.

Definition 3.4 ([EM06a, Definition 3.2]). Let G be a countable and discrete group, and equip it with any proper, left-invariant metric.[9] Fix any separable Hilbert space H.

The (*unreduced*) *stable Higson compactification* $\bar{\mathfrak{c}}G$ is the C^*-algebra of all bounded (continuous) functions of vanishing variation $f : G \to \mathcal{K}(H)$.

The (*unreduced*) *stable Higson corona* $\mathfrak{c}G$ is defined as the quotient

$$\mathfrak{c}G := \bar{\mathfrak{c}}G / C_0(G, \mathcal{K}(H))\,.$$

Proposition 3.6 below answers the variant of [EWZ21, Question 5.10] for the unreduced stable Higson compactification in the case of the metric space acted on by G being the group G itself. But before we can prove it, we first need to identify the multiplier algebra of $\bar{\mathfrak{c}}G$:

Lemma 3.5. *Let G be a countable discrete group. Then $\mathcal{M}(\bar{\mathfrak{c}}G)$ is the G-C*-algebra of bounded* (*strictly continuous*)[10] *functions of vanishing variation $G \to \mathcal{B}(H)$.*

Proof. Let us first discuss a general fact about C^*-algebras: If B is a C^*-algebra between a C^*-algebra A and its multiplier algebra, i.e., we have a chain of inclusions $A \subset B \subset \mathcal{M}(A)$, then we also have an inclusion $\mathcal{M}(B) \to \mathcal{M}(A)$. To see this, first note that because A is an ideal in $\mathcal{M}(A)$, it is also an ideal in B; and since B is an ideal in $\mathcal{M}(B)$, we get that A is an ideal in $\mathcal{M}(B)$.[11] This provides us with the map $\mathcal{M}(B) \to \mathcal{M}(A)$. Now since A is essential in $\mathcal{M}(A)$ by the universal property of the multiplier algebra, it is also essential in B; and B is essential in $\mathcal{M}(B)$, hence A is also essential in $\mathcal{M}(B)$. This shows that the map $\mathcal{M}(B) \to \mathcal{M}(A)$ is injective.

We apply this general fact to our situation: $\bar{\mathfrak{c}}G$ contains the ideal $C_0(G, \mathcal{K}(H))$. By [APT73, Corollary 3.4] the multiplier algebra of $C_0(G, \mathcal{K}(H))$ are the bounded (strictly continuous) $\mathcal{B}(H)$-valued functions on G; and $\bar{\mathfrak{c}}G$ is a subalgebra of them. Therefore we get by the above an inclusion

$$\mathcal{M}(\bar{\mathfrak{c}}G) \subset \mathcal{M}(C_0(G, \mathcal{K}(H))) = C_b(G, \mathcal{B}(H)_\beta),$$

where the notation $-_\beta$ is taken from [APT73, Section 3] and denotes that we are meaning strictly continuous functions (which, as we already mentioned, is in our discrete case a vacuous condition).

Now $\bar{\mathfrak{c}}G$ is essential in the C^*-algebra $C_{\mathfrak{w}}(G, \mathcal{B}(H)_\beta)$ of the bounded (and strictly continuous) functions of vanishing variation $G \to \mathcal{B}(H)$ and hence we have an inclusion $C_{\mathfrak{w}}(G, \mathcal{B}(H)_\beta) \subset \mathcal{M}(\bar{\mathfrak{c}}G)$. To show the other inclusion (and by this

[9] Since any two such metrics are coarsely equivalent to each other, the defined C^*-algebras are independent of this choice.

[10] Since G is discrete by assumption, every function is automatically strictly continuous (and even continuous). We mention this here in case one wants to treat the case of a nondiscrete group or space (as we have to do in the proof of Lemma 4.2).

[11] Note that this needs us to work in C^*-algebras. Such a fact is not generally true in all rings.

finish the proof), let $f \in C_b(G, \mathcal{B}(H)_\beta)$ multiply $\bar{\mathfrak{c}}G$ and we have to show that f has vanishing variation. To show this, we assume the contrary, i.e., that f does not have vanishing variation. Then there exist $\delta, r > 0$ and a sequence (x_n, y_n) in $G \times G$ with $(x_n, y_n) \to \infty$ such that $d(x_n, y_n) < r$ and $\|f(x_n) - f(y_n)\| \geq \delta$ for every $n \in \mathbb{N}$. Choose now for each $n \in \mathbb{N}$ a compact operator K_n satisfying $\|(f(x_n) - f(y_n))K_n\| \geq \delta/2$ and $\sup_{n\in\mathbb{N}} \|K_n\| < \infty$; for example, K_n could be a rank-1 projection onto a (normed) vector witnessing the norm of $f(x_n) - f(y_n)$. Passing to a subsequence of (x_n, y_n) we may assume that the n-neighbourhoods $B_n(\{x_n, y_n\})$ are disjoint from each other; and then we set

$$g(x) := \Big(1 - \frac{\operatorname{dist}(x, \{x_n, y_n\})}{n}\Big) \cdot K_n$$

for $x \in B_n(\{x_n, y_n\})$, and 0 otherwise. We have $g \in \bar{\mathfrak{c}}G$ and

$$\|(fg)(x_n) - (fg)(y_n)\| = \|(f(x_n) - f(y_n))K_n\| \geq \delta/2$$

for all $n \in \mathbb{N}$, which shows that fg does not have vanishing variation. But this is a contradiction to our assumption that f multiplies $\bar{\mathfrak{c}}G$.[12] □

Proposition 3.6. *Let G be a countable and discrete group. The following statements are equivalent to each other*:

(a) *The group G acts amenably on its Higson compactification hG.*

(b) $\bar{\mathfrak{c}}G$ *is an amenable* $(hG \rtimes^{\mathcal{G}} G)$-C*-*algebra.*[13]

(c) $\bar{\mathfrak{c}}G$ *is a strongly amenable* G-C*-*algebra.*

(d) $\bar{\mathfrak{c}}G$ *is an amenable* G-C*-*algebra.*

(e) $\bar{\mathfrak{c}}G$ *is a commutant amenable* G-C*-*algebra.*

(f) *We have* $\bar{\mathfrak{c}}G \rtimes_{\max} G \cong \bar{\mathfrak{c}}G \rtimes_{\mathrm{red}} G$ *and G is exact.*

Proof. To see the equivalence (a) ⇔ (b) we will provide a G-equivariant, non-degenerate map $\phi : C(hG) \to Z\mathcal{M}(\bar{\mathfrak{c}}G)$. By Lemma 3.5 we know that $\mathcal{M}(\bar{\mathfrak{c}}G)$ is the G-C*-algebra of bounded (strictly continuous) functions of vanishing variation $G \to \mathcal{B}(H)$ and therefore $Z\mathcal{M}(\bar{\mathfrak{c}}G)$ are the bounded (strictly continuous) complex-valued functions on G of vanishing variation. The map ϕ is then just the identity.

[12] The author thanks Christopher Wulff for providing the idea for how to construct this function g.

[13] We denote by $hG \rtimes^{\mathcal{G}} G$ the crossed product groupoid and by $\bar{\mathfrak{c}}G$ being an amenable $(hG \rtimes^{\mathcal{G}} G)$-C*-algebra we mean that there exists a G-equivariant, nondegenerate *-homomorphism $\phi : C(hG) \to Z\mathcal{M}(\bar{\mathfrak{c}}G)$ and that G acts amenably on hG (the notion of an $X \rtimes^{\mathcal{G}} G$-C*-algebra is mentioned in the introduction of [CEOO03] whose Corollary 0.4 we will use below in the proof of Proposition 4.3).

This is the same notion as $\bar{\mathfrak{c}}G$ being a strongly amenable G-$C(hG)$-algebra in the sense of [AD02, Definition 6.1] which can be seen by directly comparing the definitions.

The implication (b) ⇒ (c) is a general fact [BEW20, Lemma 2.5], and the same is true for (c) ⇒ (d) by [BEW20, Remark 2.2] and for (d) ⇒ (e) by [BEW20, Remark 4.14].

Let us show (d) ⇒ (a). Let $p \in \bar{\mathfrak{c}}G$ be the constant function to a fixed rank-one projection, so that $p\bar{\mathfrak{c}}Gp$ G-equivariantly identifies with $C(hG)$. The claim follows since amenability passes to G-invariant hereditary C^*-subalgebras [BEW24, Corollary 3.24].

Let us show the implication (e) ⇒ (a). We first note that $\bar{\mathfrak{c}}G$ is G-equivariantly C^*-isomorphic to its opposite $(\bar{\mathfrak{c}}G)^{\mathrm{op}}$ by applying a C^*-isomorphism $\mathcal{B}(H) \to \mathcal{B}(H)^{\mathrm{op}}$ point-wise to functions in $\bar{\mathfrak{c}}G$. Next we employ the standard form representations for von Neumann algebras as developed by Haagerup [Haa75] in the form presented in [BEW20, Theorem 5.1]: There exists a normal, unital and faithful representation $\pi^{\mathrm{op}} : ((\bar{\mathfrak{c}}G)^{\mathrm{op}})^{**} \to \mathcal{B}(V)$ on a Hilbert space V and a unitary representation u of G on V such that (π^{op}, u) is a covariant pair and we have $\pi^{\mathrm{op}}((\bar{\mathfrak{c}}G)^{\mathrm{op}})' \cong (\bar{\mathfrak{c}}G)^{**}$. Composing π^{op} with an equivariant C^*-isomorphism $\bar{\mathfrak{c}}G \cong (\bar{\mathfrak{c}}G)^{\mathrm{op}}$ we get a covariant pair (ρ, u) for $\bar{\mathfrak{c}}G$ with the property that $\rho(\bar{\mathfrak{c}}G)' \cong (\bar{\mathfrak{c}}G)^{**}$. To this covariant pair we can now apply the assumed (e) to get a net

$$(\theta_i : G \to \rho(\bar{\mathfrak{c}}G)')_{i \in I} \tag{3-1}$$

of positive type functions having the properties listed in Definition 3.3.(c). Let $p \in \bar{\mathfrak{c}}G$ be the constant function to a fixed rank-one projection, so that $p\bar{\mathfrak{c}}Gp$ G-equivariantly identifies with $C(hG)$. The corresponding map $\bar{\mathfrak{c}}G \to C(hG)$ given by compression by p extends to a unital[14] normal[15] conditional expectation

$$\psi : (\bar{\mathfrak{c}}G)^{**} \to C(hG)^{**} \tag{3-2}$$

by [Bla06, Section III.5.2.10]. Using the isomorphism $\rho(\bar{\mathfrak{c}}G)' \cong (\bar{\mathfrak{c}}G)^{**}$ and composing the net in (3-1) with the map ψ from (3-2) we conclude that $C(hG)$ is amenable. This implies that G acts amenably on hG [BEW20, Remark 2.2].

Finally, let us prove the equivalence (e) ⇔ (f). That (e) implies the weak containment property $\bar{\mathfrak{c}}G \rtimes_{\max} G \cong \bar{\mathfrak{c}}G \rtimes_{\mathrm{red}} G$ is a general fact: It is the implication (2) ⇒ (4) in [BEW24, Proposition 5.10]. Using the already proven implication (e) ⇒ (a) and since hG is a compact Hausdorff space, we conclude that (e) also implies that G is exact. The reverse implication (f) ⇒ (e) is true in general [BEW20, Theorem 4.17]. □

[14]As a general fact, the double dual $(\bar{\mathfrak{c}}G)^{**}$ contains the multiplier algebra $\mathcal{M}(\bar{\mathfrak{c}}G)$ as a subalgebra: It is the idealizer of $\bar{\mathfrak{c}}G$ in $(\bar{\mathfrak{c}}G)^{**}$; and hence the unit of $\mathcal{M}(\bar{\mathfrak{c}}G)$ is the one of $(\bar{\mathfrak{c}}G)^{**}$. From this description of the unit we see that ψ is unital.

[15]This means that for every bounded, increasing net (x_i) of positive elements we have $\phi(\sup x_i) = \sup \phi(x_i)$ [Bla06, Definition III.2.2.1].

Let us now quickly turn to the corresponding statements for the (unreduced) stable Higson corona (thus resolving the unreduced variant of [EWZ21, Conj. 1.25] in the case that the metric space is the group itself). Again we will first identify the corresponding multiplier algebra:

Lemma 3.7. *Let G be a countable, discrete group. Then we have an exact sequence*

$$0 \to C_0(G, \mathcal{B}(H)) \to \mathcal{M}(\bar{\mathfrak{c}}G) \to \mathcal{M}(\mathfrak{c}G),$$

i.e., $\mathcal{M}(\mathfrak{c}G)$ contains the quotient of the G-C-algebra $\mathcal{M}(\bar{\mathfrak{c}}G)$ of the bounded (strictly continuous) functions $G \to \mathcal{B}(H)$ of vanishing variation by its ideal $C_0(G, \mathcal{B}(H))$ as a sub-C^*-algebra.*

Further, we have a G-equivariant, unital (hence nondegenerate) *-homomorphism

$$\phi : C(\partial_h G) \to Z\mathcal{M}(\mathfrak{c}G).$$

Proof. The quotient map $q : \bar{\mathfrak{c}}G \to \mathfrak{c}G$ extends to a map $\mathcal{M}(q) : \mathcal{M}(\bar{\mathfrak{c}}G) \to \mathcal{M}(\mathfrak{c}G)$ in the canonical way. By [AS11, Proposition 1.1(i)] the kernel of $\mathcal{M}(q)$ is given by the strict closure of the kernel of q in $\mathcal{M}(\bar{\mathfrak{c}}G)$; which is $C_0(G, \mathcal{B}(H))$. We therefore have the exact sequence

$$0 \to C_0(G, \mathcal{B}(H)) \to \mathcal{M}(\bar{\mathfrak{c}}G) \xrightarrow{\mathcal{M}(q)} \mathcal{M}(\mathfrak{c}G)$$

as claimed.[16]

We have a G-equivariant *-homomorphism $C(hG) \to \mathcal{M}(\bar{\mathfrak{c}}G)$ given by $f \mapsto f \otimes \mathrm{id}_H$ and the kernel of the composition $C(hG) \to \mathcal{M}(\bar{\mathfrak{c}}G) \to \mathcal{M}(\mathfrak{c}G)$ is exactly $C_0(G)$, i.e., we get a map $C(\partial_h G) \to \mathcal{M}(\mathfrak{c}G)$. It is G-equivariant and unital, and is our sought map ϕ provided we can show that it takes values in the center of $\mathcal{M}(\mathfrak{c}G)$.

To show the above, we will use the following general fact about multiplier algebras: Assume that $f \in \mathcal{M}(A)$ centralizes A, i.e., we have $fa = af$ for all $a \in A$. Then f even centralizes all of $\mathcal{M}(A)$, i.e., lies in the center of $\mathcal{M}(A)$. We prove this in two steps:

- First we will prove that multiplication by f is strictly continuous on $\mathcal{M}(A)$, i.e., if $(x_\lambda)_{\lambda\in\Lambda} \subset \mathcal{M}(A)$ converges strictly to x, then $(fx_\lambda)_{\lambda\in\Lambda}$, $(x_\lambda f)_{\lambda\in\Lambda} \subset \mathcal{M}(A)$ converge strictly to fx, resp. to xf. We treat only the case of left multiplication by f since the other case is analogous. We have to show that for any $c \in A$ we have norm convergence $fx_\lambda c \to fxc$ and $cfx_\lambda \to cfx$. The first case follows since $x_\lambda \to x$ strictly implies $x_\lambda c \to xc$ in norm and hence $fx_\lambda c \to fxc$ since

[16] In general, if $A \to B$ is a surjective map, then the induced map $\mathcal{M}(A) \to \mathcal{M}(B)$ will be again surjective if A is separable [APT73, Theorem 4.2] or if A is σ-unital [Lan95, Proposition 6.8]. Unfortunately, $\mathcal{M}(\bar{\mathfrak{c}}G)$ is in general neither of these. But still the author believes that the map $\mathcal{M}(q)$ should be surjective. Since we don't need it, we have not tried to prove it by hand.

multiplication is norm continuous. For the second case we use that $cf \in A$ and hence $cfx_\lambda \to cfx$ in norm. Note that this does not use that f centralizes A.

- To finish the proof of this general subclaim, let $f \in \mathcal{M}(A)$ centralize A and choose any $x \in \mathcal{M}(A)$. We have to show that $fx = xf$. Now we use that A is strictly dense in $\mathcal{M}(A)$, i.e., there exists a net $(x_\lambda)_{\lambda\in\Lambda} \subset A$ with $x_\lambda \to x$ strictly. Then $fx = f(\lim x_\lambda) = \lim(fx_\lambda) = \lim(x_\lambda f) = (\lim x_\lambda)f = xf$, where for the second and the second-to-last equality sign we used that multiplication by f is strictly continuous and for the middle equality sign that f centralizes A.

Let us now use this general fact to conclude the proof. We have to show that every element in the image of the map $C(\partial_h G) \to \mathcal{M}(\mathfrak{c}G)$ centralizes $\mathfrak{c}G$. But this map is induced from the map $C(hG) \to \mathcal{M}(\bar{\mathfrak{c}}G)$, $f \mapsto f \otimes \mathrm{id}_H$ which clearly takes values in the center. □

Proposition 3.8. *Let G be a countable and discrete group. The following statements are equivalent to each other*:

(a) *The group G acts amenably on its Higson corona $\partial_h G$.*

(b) *$\mathfrak{c}G$ is an amenable $(\partial_h G \rtimes^{\mathcal{G}} G)$-C*-algebra.*

(c) *$\mathfrak{c}G$ is a strongly amenable G-C*-algebra.*

(d) *$\mathfrak{c}G$ is an amenable G-C*-algebra.*

(e) *$\mathfrak{c}G$ is a commutant amenable G-C*-algebra.*

(f) *We have $\mathfrak{c}G \rtimes_{\max} G \cong \mathfrak{c}G \rtimes_{\mathrm{red}} G$ and G is exact.*

Proof. The proof is analogous to the one of Proposition 3.6. For the equivalence of (a) with (b) we use the map ϕ from Lemma 3.7. □

By Lemma 3.2 the group G acts amenably on its Higson compactification hG if and only if it acts amenably on its Higson corona $\partial_h G$. We therefore conclude the following corollary:

Corollary 3.9. *The conditions in Proposition 3.6 are equivalent to the conditions in Proposition 3.8.*

3.3. *The case of the reduced algebras.*

Definition 3.10 [EM06a, Definition 5.4]. Let G be any countable and discrete group, and equip it with any proper, left-invariant metric. Fix a separable Hilbert space H.

The *reduced* stable Higson compactification $\bar{\mathfrak{c}}^{\mathrm{red}}G$ is the C^*-algebra of all bounded (continuous) functions of vanishing variation $f : G \to \mathcal{B}(H)$ with $f(x) - f(y) \in \mathcal{K}(H)$ for all $x, y \in G$.

The *reduced* stable Higson corona $\mathfrak{c}^{\mathrm{red}}G$ is defined as the quotient

$$\mathfrak{c}^{\mathrm{red}}G := \bar{\mathfrak{c}}^{\mathrm{red}}G / C_0(G, \mathcal{K}(H)) .$$

Contrary to what was claimed in [EWZ21, Proposition 5.8] the analogue of Proposition 3.6 for the reduced stable Higson compactification $\bar{\mathfrak{c}}^{\mathrm{red}}G$ is in general not true (and similarly for Proposition 5.12 in loc. cit.), as we will discuss below. This also answers in the negative Question 5.10 and Conjecture 1.25 in loc. cit.

Lemma 3.11. *Assume that $\bar{\mathfrak{c}}^{\mathrm{red}}G$ or $\mathfrak{c}^{\mathrm{red}}G$ is an amenable G-C*-algebra. Then G is amenable.*

Proof. Let us discuss the case of $\bar{\mathfrak{c}}^{\mathrm{red}}G$. The case of $\mathfrak{c}^{\mathrm{red}}G$ is completely analogously.

Note that $\bar{\mathfrak{c}}G$ is an ideal in $\bar{\mathfrak{c}}^{\mathrm{red}}G$ with quotient the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$ equipped with the trivial G-action. Since amenability descends to quotients [BEW24, Proposition 3.23], the claim follows because the trivial action can only be amenable for amenable groups.[17] □

Example 3.12. Consider a Gromov hyperbolic group G. It is known that G acts amenably on its Gromov boundary ([AD02, Example 2.7.4], originally proved in [Ada94]) and hence, because the Gromov boundary is Higson dominated, G acts also amenably on its Higson corona.

Because hyperbolic groups are in general not amenable, this provides concrete counter-examples to [EWZ21, Propositions 5.8 and 5.12] and invalidates Example 5.13 and Proposition 1.24 in loc. cit.

Remarks 3.13. The mistake in [EWZ21] occurs in Lemma 5.6 therein: The map constructed there is in general not unital, contrary to what is claimed there. But the unitality of this map was crucial for the proof of Proposition 5.8 therein.

3.4. ***Nuclearity of crossed products and positive type kernels.*** Let us first recall the necessary notions related to kernels and Schur multipliers:

Definition 3.14 [Roe03, Section 11.2]. A symmetric function $k : G \times G \to \mathbb{R}$ is called a *positive type kernel*, if for every $n \in \mathbb{N}$ and every $g_1, \dots, g_n \in G$ the matrix given by $[k(g_i, g_j)]_{i,j} \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is positive semidefinite.

- A positive type kernel k is called *normalized* if $k(g, g) = 1$ for every $g \in G$.
  Note that in this case the kernel will be uniformly bounded; concretely, we will have $|k(g, h)| \leq 1$ for all $g, h \in G$ [BO08, Theorem D.3].
- A positive type kernel k is called *equivariant* if $k(h_1 g, h_2 g) = k(h_1, h_2)$ for every $g, h_1, h_2 \in G$.

If k is a positive type kernel on $G \times G$ with $k(g, g) \leq 1$ for every $g \in G$, then the Schur multiplier

$$\theta_k : \mathcal{B}(\ell^2(G)) \to \mathcal{B}(\ell^2(G)), \quad [T_{g,h}]_{g,h\in G} \mapsto [k(g, h)T_{g,h}]_{g,h\in G} \tag{3-3}$$

[17] For actions on spaces the last statement can be found in e.g. [AD07, Example 1.4(1)]. Unfortunately, the author could not find in the literature the generalization of this to trivial actions on (unital) C^*-algebras, though it seems likely to be also true.

is a completely positive contraction;[18] if k is additionally normalized, the corresponding Schur multiplier θ_k is unital and completely positive ([Roe03, Lemma 11.17], [BO08, Theorem D.3]).

We will write $C_c(G \times G, \Delta)$ for the algebra of all functions f on $G \times G$ for which there is an $R > 0$ such that $f(g, h) = 0$ whenever $d(g, h) > R$; and we call a subset E of $G \times G$ a *finite width neighbourhood of the diagonal* Δ if there is an $R > 0$ such that $d(x, y) < R$ for all $(x, y) \in E$.

The following important properties that the group G might have were originally defined in different terms, but can be equivalently defined by the existence of positive type functions with certain properties:

Fact 3.15. (a) *The group G is **amenable** if there is a sequence $(k_n)_{n\in\mathbb{N}}$ in $C_c(G\times G, \Delta)$ of normalized, equivariant positive type kernels converging to* 1 *uniformly on all finite width neighbourhoods of the diagonal Δ in $G \times G$.*

(References are [BO08, Theorem 2.6.8], [AD02, Example 2.7(1)] and [AD07, Section 4.3 and Definition-Proposition 1.1].)

(b) *The group G is **exact** if there exists a sequence $(k_n)_{n\in\mathbb{N}}$ in $C_c(G \times G, \Delta)$ of normalized positive type kernels converging to* 1 *uniformly on all finite width neighbourhoods of the diagonal Δ in $G \times G$.*

*(*References are [Oza00, Theorem 3], [Roe03, Lemma 11.37], [Tu01, Proposition 3.2] and [AD02, Proposition 3.5].)

We next introduce a condition on kernels that is related to the Higson compactification:

Definition 3.16. Let k be a function on $G \times G$. We will say that k has *vanishing variation on diagonals* if for every $g \in G$ the function $h \mapsto k(h, gh)$ on G has vanishing variation.

Obviously, if k is equivariant, then it has vanishing variation on diagonals. This means that in the following Proposition 3.17 condition (d) sits naturally between the two conditions in Fact 3.15:

Proposition 3.17. *Let G be a countable and discrete group. The following statements are equivalent to each other*:

(a) *The group G acts amenably on its Higson compactification.*
(b) *The C^*-algebra $C(hG) \rtimes_{\mathrm{red}} G$ is nuclear.*
(c) *The embedding $\mathbb{C} \rtimes_{\mathrm{red}} G \to C(hG) \rtimes_{\mathrm{red}} G$ is nuclear.*

[18] To see this, we first convince ourselves that the following version of [BO08, Theorem D.3] holds: A kernel k on G is of positive type if and only if there exists a Hilbert space $\mathcal{H}$ and vectors $\xi_g \in \mathcal{H}$ such that $k(g, h) = \langle \xi_h, \xi_g \rangle$ for every $g, h \in G$ if and only if the multiplier θ_k is completely positive. That $k(g, g) \le 1$ for every $g \in G$ then implies that θ_k is completely contractive is [BO08, Theorem D.4]. This result is also stated, without proof, at the beginning of [Oza00].

(d) *There is a sequence* $(k_n)_{n\in\mathbb{N}}$ *in* $C_c(G\times G,\Delta)$ *of normalized positive type kernels having vanishing variation on diagonals and converging to* 1 *uniformly on all finite width neighbourhoods of the diagonal* Δ *in* $G\times G$.

Proof. The equivalence of (a) and (b) follows from [AD02, Theorem 5.8]. Note that since we assume G to be discrete, the Property (W) in the statement of [AD02, Theorem 5.8] is automatically satisfied by [AD02, Example 4.4].[19]

The equivalence of (a) with (d) follows from [AD02, Proposition 2.5] in combination with the reformulation of it discussed directly before Proposition 3.5 in loc. cit.

That (b) implies (c) is clear: Nuclearity of $C(hG)\rtimes_{\mathrm{red}}G$ means that its identity map is nuclear, whence the composition $\mathbb{C}\rtimes_{\mathrm{red}}G\to C(hG)\rtimes_{\mathrm{red}}G\xrightarrow{\mathrm{id}}C(hG)\rtimes_{\mathrm{red}}G$ will be also nuclear [BO08, Exercise 2.1.4].

For the proof that (c) implies (d) we follow the corresponding proof for exactness in [Oza00, Lemma 2 and Theorem 3]. By the nuclearity assumption,[20] for any finite subset $E\subset G$ (with $e\in E$), regarded as a subset $E\subset\mathbb{C}\rtimes_{\mathrm{red}}G$, and $\varepsilon>0$ there is an $n\in\mathbb{N}$ and unital completely positive maps $\phi\colon\mathbb{C}\rtimes_{\mathrm{red}}G\to\mathrm{Mat}_n(\mathbb{C})$ and $\psi\colon\mathrm{Mat}_n(\mathbb{C})\to C(hG)\rtimes_{\mathrm{red}}G$ such that

$$\|(\psi\circ\phi)(x)-x\|<\varepsilon/2 \text{ for all } x\in E\,.$$

The map ϕ can be extended to a unital completely positive map $\mathcal{B}(\ell^2(G))\to\mathrm{Mat}_n(\mathbb{C})$ along the inclusion $\mathbb{C}\rtimes_{\mathrm{red}}G=C^*_{\mathrm{red}}(G)\subset\mathcal{B}(\ell^2(G))$ [BO08, Corollary 1.5.16]; let us continue to denote the extended map by ϕ. We can now do the approximation argument from the proof of [Oza00, Lemma 2] with ϕ to finally obtain the unital completely positive map $\phi''\colon\mathbb{C}\rtimes_{\mathrm{red}}G\to\mathrm{Mat}_n(\mathbb{C})$ satisfying

$$\|\phi''(x)-\phi(x)\|<\varepsilon/2\quad\text{for all }\ x\in E$$

and the following property: Putting $\theta:=\psi\circ\phi''$ we get a map θ which is

(a) unital completely positive and of finite rank,

(b) satisfies $\|\theta(x)-x\|<\varepsilon$ for all $x\in E$, and

(c) has the form

$$\theta(x)=\sum_{k=1}^{d}\omega_{\delta_{p(k)},\delta_{q(k)}}(x)\otimes y_k$$

for elements $y_k\in C(hG)\rtimes_{\mathrm{red}}G$ and the linear functionals $\omega_{\delta_{p(k)},\delta_{q(k)}}$ on $\mathcal{B}(\ell^2(G))$ given by $\omega_{\delta_{p(k)},\delta_{q(k)}}(x)=\langle x(\delta_{p(k)}),\delta_{q(k)}\rangle$ for $p(k),q(k)\in G$.

Setting $k\colon G\times G\to\mathbb{C}$ as $k(s,t):=\langle\delta_s,\theta(st^{-1})\delta_t\rangle$, where we have secretly used

[19] The implication (a) ⇒ (b) is also shown in [Mat14, Theorem on last page] using Lemma 3.2.

[20] See [BO08, Exercise 2.1.1] for this version of nuclearity of maps.

the canonical inclusion $C(hG) \rtimes_{\mathrm{red}} G \subset \mathcal{B}(\ell^2(G))$, we get a normalized positive type kernel which is

- supported on the finite width neighbourhood defined by
$$F := \{q(k)p(k)^{-1} : k = 1, \dots, d\}$$
of the diagonal $\Delta \subset G \times G$,[21]
- has vanishing variation on diagonals, and
- which is ε-close to 1 on the finite width neighbourhood defined by E.

This finishes the proof that (c) implies (d).

The proof of the proposition is now complete. As a remark, let us note that the implication from (d) to (b) can be directly proven by doing the obvious modifications to the proof of [Oza00, Theorem 3(ii) ⇒ (iii)]. □

Remarks 3.18. (a) in Proposition 3.17 sits naturally between amenability and exactness, and the same is true for (d) by Fact 3.15.

It is known that nuclearity of the reduced group C^*-algebra $C^*_{\mathrm{red}}(G) = \mathbb{C} \rtimes_{\mathrm{red}} G$ is equivalent to amenability [Lan73, Theorem 4.2]. Moreover, nuclearity of the uniform Roe algebra (which is isomorphic to $\ell^\infty(G) \rtimes_{\mathrm{red}} G$) is equivalent to exactness of G [Oza00]. Therefore (b) in Proposition 3.17 sits naturally between amenability and exactness.

In [GK02, Remark 2 in Section 5] it was suggested to consider C^*-algebras A satisfying $C^*_{\mathrm{red}}(G) \subset A \subset \ell^\infty(G) \rtimes_{\mathrm{red}} G$ and impose the requirement that the inclusion of $C^*_{\mathrm{red}}(G)$ into A be a nuclear map in order to get conditions interpolating between amenability and exactness. Proposition 3.17 and the further results of this article show that $A = C(hG) \rtimes_{\mathrm{red}} G$ is a good choice.

4. Isomorphism results

Let G be a countable, discrete group and assume that it admits a G-finite classifying space for proper G-actions $\underline{EG}$. Then we have a short exact sequence

$$0 \to C_0(\underline{EG}) \otimes \mathcal{K} \to \bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG} \to \mathfrak{c}^{\mathrm{red}}G \to 0 \tag{4-1}$$

of G-C^*-algebras.

Proposition 4.1. *Let G be a countable, discrete group and assume that it admits a G-finite classifying space for proper G-actions $\underline{EG}$.*

If G is exact, the boundary morphism $\partial : K_*^{\mathrm{top}}(G; \mathfrak{c}^{\mathrm{red}}G) \to K_{*-1}^{\mathrm{top}}(G; C_0(\underline{EG}))$ resulting from (4-1) is an isomorphism and consequently

$$K_{*-1}^{\mathrm{top}}(G; \bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG}) \cong 0 \tag{4-2}$$

[21] This is the subset of all $(s, t) \in G \times G$ satisfying $st^{-1} \in F$.

for all $* \in \mathbb{Z}$.

Proof. We have the commutative diagram with bijective top horizontal map

$$\begin{array}{ccc} K_*^{\mathrm{top}}(G;\mathfrak{c}^{\mathrm{red}}G) & \xrightarrow{\mu_{\mathrm{EM}}^*,\,\cong} & K_G^{1-*}(\underline{E}G) \\ & {\scriptstyle \mu_*^{\mathrm{BC}}}\searrow \qquad \nearrow{\scriptstyle \mu_G^*} & \\ & K_*(\mathfrak{c}^{\mathrm{red}}G \rtimes_{\mathrm{red}} G) & \end{array} \tag{4-3}$$

where μ_{EM}^* is the co-assembly map of Emerson and Meyer, μ_*^{BC} is the Baum–Connes assembly map, μ_G^* is the equivariant coarse co-assembly map, and where we have by definition $K_G^{1-*}(\underline{E}G) := K_{*-1}(C_0(\underline{E}G) \rtimes_{\mathrm{red}} G)$ [EWZ21, Section 5.1].

The short exact sequence (4-1) induces the following commutative diagram whose rows are exact and the vertical maps are the respective assembly maps:

$$\begin{array}{ccccccc} \cdots \to K_*^{\mathrm{top}}(G;\mathfrak{c}^{\mathrm{red}}G) & \xrightarrow{\partial} & K_{*-1}^{\mathrm{top}}(G;C_0(\underline{E}G)) & \longrightarrow & K_{*-1}^{\mathrm{top}}(G;\bar{\mathfrak{c}}^{\mathrm{red}}\underline{E}G) \to \cdots \\ \downarrow & \overset{\mu_{\mathrm{EM}}^*,\,\cong}{\searrow} & \downarrow\cong & & \downarrow \\ \cdots \to K_*(\mathfrak{c}^{\mathrm{red}}G \rtimes_{\mathrm{red}} G) & \xrightarrow{\partial} & K_{*-1}(C_0(\underline{E}G) \rtimes_{\mathrm{red}} G) & \to & K_{*-1}(\bar{\mathfrak{c}}^{\mathrm{red}}\underline{E}G \rtimes_{\mathrm{red}} G) \to \cdots \end{array}$$

The middle vertical map is an isomorphism, since $C_0(\underline{E}G)$ is a proper G-C^*-algebra, and the diagonal dashed map is the one from Diagram (4-3). The claim follows. □

Strengthening the assumption on G from exactness to bi-exactness, we arrive at Proposition 4.3 below. But first we prove a lemma about amenability of $\bar{\mathfrak{c}}(\underline{E}G)$:

Lemma 4.2. *Let G be a countable, discrete and bi-exact group and we assume that it admits a G-finite classifying space for proper G-actions $\underline{E}G$. Then $\bar{\mathfrak{c}}(\underline{E}G)$ is an amenable $(h\underline{E}G \rtimes^{\mathcal{G}} G)$-$C^*$-algebra.*

Proof. We have the short exact sequence $0 \to C_0(\underline{E}G) \otimes \mathcal{K} \to \bar{\mathfrak{c}}(\underline{E}G) \to \mathfrak{c}(\underline{E}G) \to 0$. Because we assume $\underline{E}G$ to be G-finite, we have a coarse equivalence $G \to \underline{E}G$ which is G-equivariant by identifying G with an orbit of a point in $\underline{E}G$; this gives an isomorphism of G-C^*-algebras $\mathfrak{c}(G) \to \mathfrak{c}(\underline{E}G)$. Hence G being bi-exact is equivalent to $\mathfrak{c}(\underline{E}G)$ being an amenable G-C^*-algebra by Proposition 3.8.

Since the G-action on $C_0(\underline{E}G)$ is always amenable (by an adaption of the proof of [AD02, Example 2.7(3)]), we get with [BEW24, Proposition 3.23] that $\mathfrak{c}(\underline{E}G)$ being an amenable G-C^*-algebra is equivalent to $\bar{\mathfrak{c}}(\underline{E}G)$ being one.

The argument for the implication (d)⇒(a) of Proposition 3.6 also works for $\underline{E}G$ and shows that then G acts amenably on $h\underline{E}G$. From this we can conclude the claim by using the variant of Lemma 3.5 for $\underline{E}G$. (It is exactly the same proof, but now with the words "strictly continuous" necessarily included.) □

Proposition 4.3. *Let G be a countable, discrete and bi-exact group and we assume that it admits a G-finite classifying space for proper G-actions $\underline{EG}$. Then we have a split short exact sequence*

$$0 \to K_{*+1}(\bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG} \rtimes_{\mathrm{red}} G) \to K_*(C^*_{\mathrm{red}}(G)) \to K_*(\bar{\mathfrak{c}}(\underline{EG}) \rtimes_{\mathrm{red}} G) \to 0. \tag{4-4}$$

Further, the Baum–Connes conjecture for trivial coefficients $\mathbb{C}$ and coefficients $\bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG}$ are equivalent to each other for G and imply the isomorphism

$$K_*(C^*_{\mathrm{red}}(G)) \xrightarrow{\cong} K_*(\bar{\mathfrak{c}}(\underline{EG}) \rtimes_{\mathrm{red}} G), \tag{4-5}$$

which is induced from the inclusion of $\mathcal{K}$ as the constant functions in $\bar{\mathfrak{c}}(\underline{EG})$.

Proof. We have a short exact sequence of G-C^*-algebras

$$0 \to \bar{\mathfrak{c}}(\underline{EG}) \to \bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG} \to \mathcal{Q} \to 0, \tag{4-6}$$

where $\mathcal{Q}$ is the Calkin algebra of a separable, ∞-dimensional Hilbert space, and $\mathcal{Q}$ is equipped with the trivial G-action. We consider the resulting commutative diagram with exact rows and where the vertical maps are the respective assembly maps:

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\partial} & K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}(\underline{EG})) & \longrightarrow & K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG}) & \longrightarrow & K_*^{\mathrm{top}}(G; \mathcal{Q}) & \xrightarrow{\partial} \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \xrightarrow{\partial} & K_*(\bar{\mathfrak{c}}(\underline{EG}) \rtimes_{\mathrm{red}} G) & \longrightarrow & K_*(\bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG} \rtimes_{\mathrm{red}} G) & \longrightarrow & K_*(\mathcal{Q} \rtimes_{\mathrm{red}} G) & \xrightarrow{\partial} \cdots
\end{array}$$

By Proposition 4.1 we have $K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG}) \cong 0$ and therefore the boundary maps in the top row are isomorphisms. Since we assume that G is bi-exact, $\bar{\mathfrak{c}}(\underline{EG})$ is an amenable $(h\underline{EG} \rtimes^{\mathcal{G}} G)$-$C^*$-algebra by Lemma 4.2 and we conclude that the assembly map for it is an isomorphism [CEOO03, Corollary 0.4]. We therefore arrive at the diagram

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\partial,\cong} & K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}(\underline{EG})) & \longrightarrow & 0 & \longrightarrow & K_*^{\mathrm{top}}(G; \mathcal{Q}) & \xrightarrow{\partial,\cong} \cdots \\
& \downarrow \cong & & \downarrow & & \downarrow & \\
\cdots \xrightarrow{\partial} & K_*(\bar{\mathfrak{c}}(\underline{EG}) \rtimes_{\mathrm{red}} G) & \longrightarrow & K_*(\bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG} \rtimes_{\mathrm{red}} G) & \longrightarrow & K_*(\mathcal{Q} \rtimes_{\mathrm{red}} G) & \xrightarrow{\partial} \cdots
\end{array} \tag{4-7}$$

showing that $K_*(\bar{\mathfrak{c}}(\underline{EG}) \rtimes_{\mathrm{red}} G) \to K_*(\bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG} \rtimes_{\mathrm{red}} G)$ is the zero map. The isomorphisms in this diagram provide the split for the resulting short exact sequence

$$0 \to K_{*+1}(\bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG} \rtimes_{\mathrm{red}} G) \to K_{*+1}(\mathcal{Q} \rtimes_{\mathrm{red}} G) \xrightarrow{\partial} K_*(\bar{\mathfrak{c}}(\underline{EG}) \rtimes_{\mathrm{red}} G) \to 0\,.$$

The proof of (4-4) is finished with the isomorphism $K_{*+1}(\mathcal{Q} \rtimes_{\mathrm{red}} G) \cong K_*(C^*_{\mathrm{red}}(G))$ which is the boundary map in the long exact sequence induced from the short exact sequence of G-C^*-algebras $0 \to \mathcal{K} \to \mathcal{B} \to \mathcal{Q} \to 0$ with the trivial G-action. To see that (4-5) is induced from the inclusion of $\mathcal{K}$ as the constant functions in $\bar{\mathfrak{c}}(\underline{EG})$ we refer to the proof of Proposition 5.5 in [EM06a].

The final statement of the proposition about the Baum–Connes conjecture follows from Diagram (4-7). □

We can now prove an equivariant version of [Wil13, Proposition 4.5] and an alternative version of [Wil09, Proposition 6.2.1]. To be able to state it we need the canonical inclusion $i : C(\partial G) \otimes \mathcal{K} \to \mathfrak{c}G$ from [EM06a, Proposition 3.6] for boundaries at infinity ∂G of suitable compactifications of G. Each such boundary will arise as the boundary of a compactification $\bar{P}$ of a suitable model P for $\underline{EG}$.

Proposition 4.4. *Let G be a countable, discrete group admitting a G-finite model P for its classifying space for proper G-actions $\underline{EG}$, subject to the condition that P admits a metrizable compactification $\bar{P}$ such that*

(a) *$\bar{P}$ is Higson-dominated,*

(b) *$\bar{P}$ is H-equivariantly contractible for every finite subgroup $H < G$, and*

(c) *the G-action on P extends to an amenable action on $\bar{P}$.*

Then the inclusion $i : C(\partial G) \otimes \mathcal{K} \to \mathfrak{c}G$, where ∂G is the boundary of P inside $\bar{P}$, induces an isomorphism

$$K_*(C(\partial G) \rtimes_{\mathrm{red}} G) \cong K_*(\mathfrak{c}G \rtimes_{\mathrm{red}} G). \tag{4-8}$$

Proof. There is a G-equivariant quasi-isometry $G \to P$ (canonical up to closeness) inducing G-equivariant C^*-isomorphisms $\mathfrak{c}G \to \mathfrak{c}P$ and $\mathfrak{c}^{\mathrm{red}}G \to \mathfrak{c}^{\mathrm{red}}P$. We conclude

$$K_*(\mathfrak{c}G \rtimes_{\mathrm{red}} G) \cong K_*(\mathfrak{c}P \rtimes_{\mathrm{red}} G) \quad \text{and} \quad K_*(\mathfrak{c}^{\mathrm{red}}G \rtimes_{\mathrm{red}} G) \cong K_*(\mathfrak{c}^{\mathrm{red}}P \rtimes_{\mathrm{red}} G).$$

In the following we will use that since G acts amenably on a Higson-dominated compactification, it also acts amenably on its Higson compactification and hence G is bi-exact. This implies further that G is exact.

We consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (C_0(P)\otimes\mathcal{K})\rtimes_{\mathrm{red}}G & \longrightarrow & \bar{\mathfrak{c}}P\rtimes_{\mathrm{red}}G & \longrightarrow & \mathfrak{c}P\rtimes_{\mathrm{red}}G & \longrightarrow & 0\\
 & & \Vert & & \uparrow & & \uparrow{\scriptstyle i'\rtimes_{\mathrm{red}}G} & & \\
0 & \longrightarrow & (C_0(P)\otimes\mathcal{K})\rtimes_{\mathrm{red}}G & \longrightarrow & (C(\bar{P})\otimes\mathcal{K})\rtimes_{\mathrm{red}}G & \longrightarrow & (C(\partial G)\otimes\mathcal{K})\rtimes_{\mathrm{red}}G & \longrightarrow & 0
\end{array}$$

whose rows are exact since G is an exact group. The map i' is the map i composed with the G-equivariant C^*-isomorphism $\mathfrak{c}G \to \mathfrak{c}P$. From this diagram and the induced transformation of the corresponding long exact sequences in K-theory, together with the previously noted isomorphism $K_*(\mathfrak{c}G \rtimes_{\mathrm{red}} G) \cong K_*(\mathfrak{c}P \rtimes_{\mathrm{red}} G)$, we see that to prove (4-8) it suffices to prove that the middle vertical map in the above diagram induces isomorphisms in K-theory.

Consider the commutative diagram whose horizontal maps are the respective assembly maps and the vertical maps are induced by $\bar{i}' : C(\bar{P}) \otimes \mathcal{K} \to \bar{\mathfrak{c}}P$:

$$\begin{array}{ccc} K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}P) & \xrightarrow{\cong} & K_*(\bar{\mathfrak{c}}P \rtimes_{\mathrm{red}} G) \\ \uparrow & & \uparrow \\ K_*^{\mathrm{top}}(G; C(\bar{P}) \otimes \mathcal{K}) & \xrightarrow{\cong} & K_*((C(\bar{P}) \otimes \mathcal{K}) \rtimes_{\mathrm{red}} G) \end{array}$$

Similarly as in Lemma 4.2, because G is bi-exact we know that $\bar{\mathfrak{c}}P$ is an amenable $(hP \rtimes G)$-C^*-algebra and we conclude that the assembly map for it, which is the top horizontal map in the above diagram, is an isomorphism [CEOO03, Corollary 0.4]. By the same argument, because we assume that G acts amenably on $\bar{P}$, we conclude that the lower horizontal map is an isomorphism [Tu99].

We have a short exact sequence of G-C^*-algebras

$$0 \to \bar{\mathfrak{c}}P \to \bar{\mathfrak{c}}^{\mathrm{red}}P \to \mathcal{Q} \to 0, \tag{4-9}$$

where $\mathcal{Q}$ is the Calkin algebra of a separable, ∞-dimensional Hilbert space and $\mathcal{Q}$ is equipped with the trivial G-action. We consider the resulting long exact sequence:

$$\cdots \xrightarrow{\partial} K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}P) \to K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}^{\mathrm{red}}P) \to K_*^{\mathrm{top}}(G; \mathcal{Q}) \xrightarrow{\partial} \cdots$$

Proposition 4.1 implies $K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}^{\mathrm{red}}P) \cong 0$ and we conclude that the boundary map is an isomorphism

$$K_{*+1}^{\mathrm{top}}(G; \mathcal{Q}) \xrightarrow{\partial, \cong} K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}P). \tag{4-10}$$

On the other hand, since $\bar{P}$ is H-equivariantly contractible for every finite subgroup H of G we conclude by [Hig00, Proposition 3.7] that we have the isomorphism

$$K_*^{\mathrm{top}}(G; \mathcal{K}) \xrightarrow{\cong} K_*^{\mathrm{top}}(G; C(\bar{P}) \otimes \mathcal{K}) \tag{4-11}$$

induced from the inclusion of $\mathbb{C}$ into $C(\bar{P})$ as constant functions. We check (as in [EM06a, proof of Proposition 5.5]) that the diagram

$$\begin{array}{ccc} K_{*+1}^{\mathrm{top}}(G; \mathcal{Q}) & \xrightarrow{\partial, \cong} & K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}P) \\ {\scriptstyle \partial, \cong}\downarrow & & \uparrow \\ K_*^{\mathrm{top}}(G; \mathcal{K}) & \xrightarrow{\cong} & K_*^{\mathrm{top}}(G; C(\bar{P}) \otimes \mathcal{K}) \end{array}$$

with horizontal maps (4-10) and (4-11) commutes, where the left vertical map is the boundary map induced from the short exact sequence $0 \to \mathcal{K} \to \mathcal{B} \to \mathcal{Q} \to 0$ and the right vertical map is the one where we have to show that it is an isomorphism. But this follows from the diagram. □

Note that sometimes the K-theory groups of $C(\partial G) \rtimes_{\mathrm{red}} G$ are computable, for example with the techniques from [EM06b]; cf. [Wil09, Remark 6.2.4].

Example 4.5. Let G be a Gromov hyperbolic group. We collect in the following the references which verify the assumptions of Proposition 4.4 for these groups.

Let $P_d(G)$ be the Rips complex at scale $d \geq 1$ of the group G (equipped with any word metric) and P be its second barycentric subdivision. It is known that for large d this is a model for the classifying space $\underline{E}G$ for proper actions of G [MS02]. There is a G-equivariant quasi-isometry $G \to P$ (canonical up to closeness) and hence P is also hyperbolic and its Gromov boundary is canonically equivariantly homeomorphic to the boundary ∂G of G.

It is known that the Gromov compactification $\bar{P}$ is H-equivariantly contractible for every finite subgroup $H < G$,[22] and it is furthermore known that G acts amenably on ∂G ([AD02, Example 2.7.4], originally proved in [Ada94]) and therefore also on $\bar{P}$.[23] By [Roe91, Corollary 2.2] we know that the Gromov compactification is Higson dominated.

Finally, let us mention that hyperbolic groups satisfy the Baum–Connes conjecture for all coefficients [Laf12] and hence the isomorphism (4-5) holds true.

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[22] This is a general result and follows from the Whitehead theorem for families [Lüc05, Theorem 1.6]: Apply it to $Y = \bar{P}$, $Z = \mathrm{pt}$, H as the group, and all its subgroups as the family $\mathcal{F}$. In point (i) of its statement we choose $X = \bar{P}$ and get that the set $[X, Y]^H$ of H-homotopy classes of H-maps consists of a single element. But this set contains both the identity and the map to a single H-fixed point.

[23] G acts amenably on ∂G and therefore also on ∂P since they are equivariantly homeomorphic. Since P is a model for $\underline{E}G$, the G-action on it is amenable (adapt the proof of [AD02, Ex. 2.7(3)]); therefore we conclude that the G-action on $\bar{P} = P \sqcup \partial P$ is amenable [BEW24, Proposition 3.23].

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