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We show that the first order theory of the homeomorphism group of a compact manifold interprets the full second order theory of countable groups of homeomorphisms of the manifold. The interpretation is uniform across manifolds of bounded dimension. As a consequence, many classical problems in group theory and geometry (e.g., the linearity of mapping classes of compact 2-manifolds) are encoded as elementary properties of homeomorphism groups of manifolds. Furthermore, the homeomorphism group uniformly interprets the Borel and projective hierarchies of the homeomorphism group, which gives a characterization of definable subsets of the homeomorphism group. Finally, we prove analogues of Rice's theorem from computability theory for homeomorphism groups of manifolds. As a consequence, it follows that the collection of sentences that isolate the homeomorphism group of a particular manifold, or that isolate the homeomorphism groups of manifolds in general, is not definable in second order arithmetic, and that membership of particular sentences in these collections cannot be proved in ZFC.

1. Introduction

Let M be a compact, connected, topological manifold of positive dimension. In this paper, we investigate countable subgroups of the group $\text{Homeo}(M)$ from the point of view of the first order logic of groups, thus continuing a research program initiated together with Kim [28]. There, we proved that for each compact manifold M , there is a sentence in the language of groups which isolates the group $\text{Homeo}(M)$; that is, there exists a sentence in the language of group theory that is true in the group of homeomorphisms of an arbitrary compact manifold N if and only if N is homeomorphic to M .

Our overarching theme is that the first order theory of $\text{Homeo}(M)$ is expressive enough to interpret arbitrary sequences of elements of $\text{Homeo}(M)$. More concretely: on the one hand, the question of determining the isomorphism type of the subgroup

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of $\text{Homeo}(M)$ generated by a finite list of elements is difficult, and in general is intractable. On the other hand, it can be shown by general Baire category arguments (Proposition 4.5 in [17], cf. Chapter 3 in [27]) that generically, pairs of homeomorphisms will generate nonabelian free groups. Even in the case of one-dimensional manifolds, general finitely generated groups of homeomorphisms (and even diffeomorphisms) can be extremely complicated; cf. [2; 23; 26; 30; 24; 25; 6; 29].

Since $\text{Homeo}(M)$ can interpret arbitrary sequences of elements in the underlying group, the first order theory of $\text{Homeo}(M)$ is expressive enough to decide if a countable subgroup is isomorphic to a given finitely presented group; as another example, by identifying tuples of homeomorphisms which generate a particular isomorphism type of groups (e.g., a free group of rank two), we obtain an upper bound on the complexity of the set of tuples which generate that type of group. Thus, the elementary theory of the homeomorphism group $\text{Homeo}(M)$ encodes a substantial amount of the algebraic structure of this group.

1.1. Main results. All results stated in this section hold for arbitrary compact, connected manifolds; we assume connectedness mostly for convenience. There is a dependence of the formulae on the dimension of the underlying manifold, but otherwise all formulae are uniform across manifolds of fixed dimension. Throughout, we let

$$\text{Homeo}_0(M) \leq \mathcal{H} \leq \text{Homeo}(M),$$

where here $\text{Homeo}_0(M)$ denotes the identity component of $\text{Homeo}(M)$. Unless otherwise noted, formulae are uniform in \mathcal{H} , which is to say they do not depend on which subgroup between $\text{Homeo}_0(M)$ and $\text{Homeo}(M)$ we consider. We suppress M from the notation \mathcal{H} since it will not cause confusion.

To begin, \mathcal{H} is viewed as a structure in the language of group theory. The content of the paper [28] is that the language of group theory in \mathcal{H} admits a conservative expansion wherein many more things can be interpreted: specifically, the sorts of regular open sets $\text{RO}(M)$ in M , the natural numbers \mathbb{N} , the real numbers \mathbb{R} , and points in M can be parameter-free interpreted. Moreover, natural predicates, both internal to these sorts (e.g., arithmetic) and relating these sorts to each other, are uniformly definable; see [Theorem 2.3](#) below.

The main result of this paper is the conservative interpretation of a sequence of new sorts in \mathcal{H} , which are written $\text{HS}_i(M)$ for $i \geq 0$. The meanings of these sorts are as follows:

- The elements of $\text{HS}_0(M)$ are in canonical correspondence with homeomorphisms of M .

- For $i \geq 1$, the elements of $\text{HS}_i(M)$ are in canonical correspondence with sequences of elements in $\text{HS}_{i-1}(M)$.
- These sorts admit parameter-free definable predicates for manipulating them and for relating them to each other and to the home sort.

We call $\text{HS}(M)$, the union of the sorts $\{\text{HS}_i(M)\}_{i \in \mathbb{N}}$, *hereditarily sequential subsets of* $\text{Homeo}(M)$; this is by analogy to (and by generalization of) *hereditarily finite sets* (cf. Section 3 in [22], for instance).

Note that for $n \geq 2$, elements of $\text{HS}_n(M)$ are not really subsets of $\text{Homeo}(M)$. One would be justified in calling an interpretation of $\text{HS}_1(M)$ *countable second order logic*, since then one can quantify freely over countable subsets of $\text{Homeo}(M)$. Then, for $n \geq 2$ one would be justified in calling an interpretation of $\text{HS}_n(M)$ *countable $(n+1)^{\text{st}}$ order logic*. The distinction between countable second order logic and countable higher order logics collapses in our situation; this is because our interpretation of countable second order logic (i.e., $\text{HS}_1(M)$) encodes countable sequences via fixed length definable tuples, up to a definable equivalence relation. Thus for all $n \geq 2$, an interpretation of $\text{HS}_n(M)$ would consist of sequences of fixed length finite tuples, which themselves would be encoded by fixed length finite tuples in \mathcal{H} .

Hereditarily sequential sets subsume hereditarily finite sets via a straightforward padding construction.

Theorem 1.1. *Let $D \geq 1$ be a natural number, and let*

$$\text{Homeo}_0(M) \leq \mathcal{H} \leq \text{Homeo}(M).$$

Then there is a conservative expansion of the language of group theory and a uniform interpretation of the union of the sorts $\text{HS}(M)$ in \mathcal{H} that is valid for all manifolds M with $\dim M \leq D$. The elements in the sort $\text{HS}_0(M)$ canonically correspond to elements of $\text{Homeo}(M)$.

Moreover, the following predicates are definable without parameters:

- (1) *For each i and each $j \in \mathbb{N}$, the j -th element $s(j)$ of a sequence $s \in \text{HS}_i(M)$;*
- (2) *For each $i \geq 0$, a membership predicate*

$$\in_i \subseteq \text{HS}_i(M) \times \text{HS}_{i+1}(M)$$

defined recursively by:

- (a) $\Gamma \in_0 s$ *if and only if there is a j such that $\Gamma = s(j)$.*
 - (b) $s \in_i t$ *if and only if there is a j such that $s = t(j)$.*
- (3) *Memberwise group multiplication within $\text{HS}_1(M)$, i.e., a predicate $\text{mult}_{i,j,k}(\sigma)$ such that for all sequences $s \in \text{HS}_1(M)$, we have $\text{mult}_{i,j,k}(s)$ if and only if $s(i) \cdot s(j) = s(k)$.*

(4) *Membership of an element in $\text{HS}_0(M)$ in \mathcal{H} , i.e., a predicate*

$$R \subseteq \mathcal{H} \times \text{HS}_0(M)$$

such that $(g, \Gamma) \in R$ if and only if Γ canonically encodes g .

(5) *The extended support $\text{supp}^e f$ of an element $f \in \text{HS}_0(M)$, i.e., a predicate*

$$\text{supp}^e \subseteq \text{HS}_0(M) \times \text{RO}(M)$$

such that $(\Gamma, U) \in \text{supp}^e$ if and only if the homeomorphism encoded by Γ has extended support equal to U .

We will sometimes abuse notation and suppress the subscript in \in_i when no confusion can occur. We note that Item (4) is crucial and what makes [Theorem 1.1](#) not a consequence of [\[28\]](#). Moreover, Item (4) will allow us to characterize definable sets in \mathcal{H} below (see [Theorem 1.5](#)).

The key step in interpreting $\text{HS}(M)$ yields the following, which is of independent interest. See [Lemma 3.1](#).

Proposition 1.2. *For manifolds of fixed dimension, the group \mathcal{H} admits a uniform, parameter-free interpretation of the sort $\text{seq}(M)$ of countable sequences of points in M , which is uniform for all manifolds of dimension d . Moreover, the predicate $p \in \sigma$ expressing membership of a point p in a sequence σ , and the predicate $\sigma(i) = p$ expressing that p is the i -th term of σ , are both parameter-free definable.*

The interpretability of hereditarily sequential sets in \mathcal{H} has a large number of consequences with regard to definability in \mathcal{H} .

Proposition 1.3. *The class \mathcal{H} of subgroups of $\text{Homeo}(M)$ that contain $\text{Homeo}_0(M)$ is uniformly interpretable (with parameters) in \mathcal{H} , as definable subsets of the sort $\text{HS}_0(M)$. Among the elements of \mathcal{H} are three canonical parameter-free interpretable subgroups, namely*

$$\{\text{Homeo}_0(M), \text{Homeo}(M), \mathcal{H}\}.$$

Combining [Theorem 1.1](#) and [Proposition 1.3](#), we will be able to interpret hereditarily sequential sets in other groups lying between $\text{Homeo}_0(M)$ and $\text{Homeo}(M)$, and in various parameter-free interpretable quotients such as the topological mapping class group $\text{Mod}(M) := \text{Homeo}(M)/\text{Homeo}_0(M)$.

1.2. Group theoretic consequences of the main results. [Theorem 1.1](#) immediately implies that within the first order theory of \mathcal{H} , we have unfettered access to the full second order theory of countable subgroups of $\text{Homeo}(M)$; in particular, we may freely quantify over countable subgroups, as well as their subgroups, and homomorphisms between them. Since \mathcal{H} also interprets second order arithmetic, we may uniformly interpret combinatorial (and even analytic) group theory within

the first order theory of \mathcal{H} ; that is, we can encode arbitrary recursively presented groups within second order arithmetic, and we may also manipulate them (i.e., test for nontriviality of words, solve the conjugacy problem, test for isomorphism, determining if a subgroup has finite index, measure the index of a finite index subgroup, test for amenability, test Kazhdan's property (T), etc.; the reader is directed to [46] for an extensive discussion of mathematics that can be developed within second order arithmetic). Observe that an abstract countable group will generally have to be specified with parameters, in the form of a sequence of natural numbers.

For abstract finitely generated groups, the standard concepts from geometric group theory can also be interpreted, such as the Cayley graph with respect to a finite generating set, growth, hyperbolicity, and quasi-isometry.

Below, we give a (non-exhaustive) list some concepts that can be encoded within the elementary theory of \mathcal{H} .

Theorem 1.4. *The following group-theoretic sorts and predicates are parameter-free interpretable in \mathcal{H} , uniformly for all compact manifolds M of fixed dimension:*

- (1) *countable subgroups of $\text{Homeo}(M)$ and their full second order theory;*
- (2) *the topological mapping class group $\text{Mod}(M)$ of M , i.e., the group*

$$\pi_0(\text{Homeo}(M)) = \text{Homeo}(M) / \text{Homeo}_0(M),$$

and the full second order theory of $\text{Mod}(M)$;

- (3) *for a sequence \underline{g} of homeomorphisms or mapping classes, the membership predicate for the subgroup $\langle \underline{g} \rangle$;*
- (4) *finite generation and finite presentability of arbitrary countable subgroups of $\text{Homeo}(M)$ or $\text{Mod}(M)$;*
- (5) *residual finiteness of arbitrary countable subgroups of $\text{Homeo}(M)$ and $\text{Mod}(M)$;*
- (6) *linearity of arbitrary countable subgroups of $\text{Homeo}(M)$ and $\text{Mod}(M)$, i.e., a predicate which holds if and only if the corresponding group is linear over a field of characteristic zero;*
- (7) *a predicate expressing isomorphism with a particular group that is parameter-free definable in second order arithmetic (e.g., isomorphism with some finite index subgroup of $\text{SL}_n(\mathbb{Z})$);*
- (8) *for a finitely generated subgroup of $\text{Homeo}(M)$ or $\text{Mod}(M)$, a predicate expressing whether this group is amenable or has Kazhdan's Property (T).*

Thus, the first order theory of \mathcal{H} encodes many well-known conjectures as elementary properties of homeomorphism groups. These include the linearity of mapping class groups of compact 2-manifolds (see [13] for a general reference, and

Question 1.1 of [35]), property (T) for mapping class groups of compact 2-manifolds, finite presentability of the Torelli group of a compact 2-manifold (see [43; 36], and especially Section 5 of [35]) the existence of an infinite, discrete, property (T) group of homeomorphisms of the circle (see [11; 39; 1], and especially Question 2 of [40]), the amenability of Thompson’s group F [8; 7], and many cases of the Zimmer program (i.e., faithful continuous actions of finite index subgroups of lattices in semisimple Lie groups on compact manifolds [14; 15; 4; 5]).

1.3. Descriptive set theory. Much of the foregoing discussion treats $\text{Homeo}(M)$ as a discrete group. We wish to observe further that the first order theory of \mathcal{H} recovers the topology of $\text{Homeo}(M)$, and in fact the full projective hierarchy of subsets of $\text{Homeo}(M)$. More precisely:

Theorem 1.5. *The following sorts are uniformly interpretable in \mathcal{H} , viewed as a subset of $\text{HS}_0(M)$, uniformly in manifolds of fixed dimension:*

- (1) *open and closed sets in $\text{Homeo}(M)$;*
- (2) *Borel sets in $\text{Homeo}(M)$, and the full Borel hierarchy of $\text{Homeo}(M)$;*
- (3) *the projective hierarchy in $\text{Homeo}(M)$.*

The membership predicate \in is parameter-free interpretable for sets in these sorts.

Moreover, the topology of \mathcal{H} , as well as the Borel hierarchy and projective hierarchy of \mathcal{H} are all uniformly definable among manifolds of bounded dimension.

As a consequence, we will obtain the following general fact about definable subsets of $\text{Homeo}(M)$:

Theorem 1.6. *A set is definable (with parameters) in \mathcal{H} if and only if it lies in the projective hierarchy.*

1.4. Undefinability and independence. As is implicit from the uniform parameter-free interpretation of second order arithmetic in \mathcal{H} as produced in [28], not only is the first order theory of \mathcal{H} (and of $\text{Homeo}(M)$ in particular) undecidable, but in fact there are elementary properties of homeomorphism groups of manifolds whose validity is independent of ZFC. A question therefore is whether or not there are “natural” first order group theoretic statements in \mathcal{H} that are independent of ZFC, and this is unclear to the authors.

There are also many natural undefinable sets in arithmetic which are directly related to compact manifolds and their homeomorphism groups, which we record here. Manifolds and their homeomorphism groups can be formalized in second order arithmetic; however, there is some sense in which the manifold homeomorphism group recognition problem is at least as complicated as full true second order arithmetic, which we now make precise.

Choosing a numbering of the language of groups, we obtain a Gödel numbering of sentences in group theory. For a fixed compact manifold M , one can consider the set of sentences in group theory (viewed as a subset of \mathbb{N} via their Gödel numbers) which isolate $\text{Homeo}(M)$. Similarly, one may consider the set of sentences in group theory which isolate some isomorphism type of compact manifold homeomorphism group. It turns out that neither of these sets is definable in arithmetic. For a sentence ψ , we write $\#\psi$ for its Gödel number with respect to a fixed numbering of the language.

Theorem 1.7. *Let M be a fixed compact manifold and let N be an arbitrary compact manifold.*

(1) *The set*

$$\text{Sent}_M := \{\#\psi \mid (\text{Homeo}(N) \models \psi) \iff (M \cong N)\}$$

is not definable in second order arithmetic.

(2) *The set*

$$\text{Sent} := \{\#\psi \mid \psi \text{ isolates } \text{Homeo}(N) \text{ for some compact manifold } N\}$$

is not definable in second order arithmetic.

In particular, these sets are not decidable.

In [Theorem 1.7](#), the group Homeo can be replaced by any group lying between Homeo_0 and Homeo . We will show in [Section 6](#) that membership of Gödel numbers in Sent_M or Sent cannot be proved within ZFC.

More generally than [Theorem 1.7](#), we will prove that for any class \mathcal{M} of compact manifold homeomorphism groups which is isolated by a single sentence, the set of Gödel numbers of sentences isolating \mathcal{M} is undefinable in second order arithmetic; this gives an analogue of Rice's theorem (i.e., nontrivial classes of partially recursive functions are not computable) for homeomorphism groups of manifolds. In fact, we will prove that if \mathcal{F} consists of nonempty sets of homeomorphism groups of compact manifolds which are isolated by first order sentences, and if $A \subseteq F$ is proper, then the set of Gödel numbers of sentences isolating elements of A is not definable in second order arithmetic. See [Theorem 6.1](#) and [Theorem 6.2](#) for precise statements.

1.5. Organization of the paper. In [Section 2](#), we gather preliminary material about topological manifolds and the first order theory of homeomorphism groups of manifolds. [Section 3](#) proves [Theorem 1.1](#), the main result of the paper. [Section 4](#) interprets mapping class groups of manifolds as well as intermediate subgroups lying between Homeo_0 and Homeo of manifolds, and discusses [Theorem 1.4](#). [Section 5](#) discusses descriptive set theory and the projective hierarchy in $\text{Homeo}(M)$. [Section 6](#) proves [Theorem 1.7](#) and the analogues of Rice's theorem.

Throughout, we have tried to balance mathematical precision with clarity. To give completely precise and explicit formulae is possible, though extremely unwieldy and unlikely to yield deeper insight. Thus, we have often avoided giving explicit formulae, either explaining how to obtain them in English with enough precision that the formulae could be produced if desired, or we have avoided them entirely when certain predicates are obviously definable in second order arithmetic or in the countable second order theory of a group.

2. Background

We first gather some preliminary results. Throughout, we will always assume that all manifolds are compact, connected, and second countable.

2.1. Results from geometric topology of manifolds. We will appeal to the following fact about compact topological manifolds. We write $B(i) \subset \mathbb{R}^d$ for the closed ball of radius i about the origin. We write $H(i) \subset \mathbb{R}_{\geq 0}^d$ for the half-ball of radius i about the origin in the half-space $\mathbb{R}_{\geq 0}^d$. That is, $H(i) = B(i) \cap \mathbb{R}_{\geq 0}^d$. A *collared ball* in a d -dimensional manifold M is a map

$$B(1) \rightarrow M$$

which is a homeomorphism onto its image, and which extends to a homeomorphism of $B(2)$ onto its image, and a *collared half-ball* in a manifold with boundary is defined analogously in the usual sense, so that the image of the origin in \mathbb{R}^d lands in the boundary $\partial M \subseteq M$ and the intersection of the image of $H(i)$ with ∂M is a collared open ball in ∂M .

An open set in M is *regular* if it is equal to the interior of its closure. We will say that a regular open set U is a *regular open collared ball* if it is the interior of a collared open ball. A *regular open collared half-ball* is a regular open set that is the interior of a collared half-ball. A regular open collared half-ball meets the boundary of M in a regular open collared ball.

Proposition 2.1 (see Chapter 3 in [10], Theorem IV.2 in [19], Theorem 3 in [41], Section 6.1 in [28]). *Let M be a compact, connected manifold of dimension d . Then there exists a computable function $n(d)$ such that the following conclusions hold.*

- (1) *If M is a closed topological manifold then there exist $n(d)$ collections of disjoint collared balls $\{B_1, \dots, B_{n(d)}\}$ such that*

$$M = \bigcup_{i=1}^{n(d)} B_i.$$

- (2) *If $\partial M \neq \emptyset$ then the following conclusions hold.*

- (a) For every collar neighborhood $U \supseteq \partial M$, there exist collections of disjoint collared balls $\{B_1, \dots, B_{n(d)}\}$ and collections of disjoint collared half-balls $\{H_1, \dots, H_{n(d-1)}\}$ such that

$$M \setminus U \subseteq \bigcup_{i=1}^{n(d)} B_i \quad \text{and} \quad \text{cl } U \subseteq \bigcup_{j=1}^{n(d-1)} H_j.$$

In [Proposition 2.1](#), note that each B_i and each H_i is a (possibly disconnected) set, each component of which a collared ball or collared half-ball, respectively.

Proof of Proposition 2.1. We will assume that M is closed; the argument for manifolds with boundary is a minor variation on the proof given here.

This essentially follows from the fact that M can be embedded in \mathbb{R}^{2d+1} . Choose such an embedding, which by scaling we may assume lies in the unit cube I^{2d+1} . For any positive threshold $\epsilon > 0$, we may cover I^{2d+1} by $2d+2$ collections of regular open sets $\{B_1, \dots, B_{2d+2}\}$, each consisting of disjoint collared open Euclidean balls, with each ball having diameter at most ϵ . Moreover, we may assume that any two components of any B_i are separated by a distance that is uniformly bounded away from zero. These claims follow from standard constructions in Lebesgue covering dimension; see Chapter 3 in [\[10\]](#), Chapter 50 in [\[38\]](#).

Choose an atlas for M such that for an arbitrary component V of some B_i , we have that the intersection $V \cap M$ lies in a coordinate chart. This can be achieved by setting ϵ small enough with respect to a fixed atlas for M , as follows from the Lebesgue covering lemma.

Let $U \cong \mathbb{R}^d$ be such a coordinate chart of M and let $B = B_i$ for some i . Then, $U \cap B$ is a collection of open sets which are separated by a definite distance $\delta > 0$ which is independent of U . For any component $V \in \pi_0(B)$ such that $V \cap M$ is entirely contained in U , we may cover $V \cap M$ with collared open balls (in M) which are contained in a $\delta/3$ neighborhood of the closure of V in M . This covering may be further refined to be a covering by regular collared balls having order at most $d+2$; in particular, the closure of V is covered by at most $d+2$ collections of regular open sets, whose components consist of disjoint collared open balls. Repeating this construction for each component $V \in \pi_0(B)$, we obtain a collection of $d+1$ regular open sets whose components are collared open balls that cover $B \cap M$. Allowing B to range over $\{B_1, \dots, B_{2d+2}\}$, we obtain $(d+1)(2d+2)$ regular open sets covering M , all of whose components are collared open balls, as desired. \square

The importance of [Proposition 2.1](#) is that many of the formulae we build in this paper will be uniform in the underlying manifold, provided that the dimension is bounded. This is reflected in the dependence of $n(d)$ on d . The proof of the following corollary is straightforward, and we omit it.

Corollary 2.2. *Let M be a compact, connected manifold of dimension d , and let $n(d)$ be as in Proposition 2.1.*

- (1) *If M is closed then M can be covered by $n(d)$ regular open collared balls.*
- (2) *If $\partial M \neq \emptyset$ and if N is a component of ∂M , then there is a tubular neighborhood of N whose closure can be covered by $n(d-1)$ regular open collared half-balls. Moreover, for all tubular neighborhoods $U \supseteq \partial M$, we have $M \setminus U$ can be covered by $n(d)$ regular open collared balls.*

2.2. Results about the first order theory of homeomorphism groups of manifolds.

The present paper builds on the results of the authors' joint paper with Kim [28]. In that paper, we investigated the first order theory of $\text{Homeo}(M)$ for a compact manifold M , and in particular proved that each group $\text{Homeo}(M)$ is quasi-finitely axiomatizable within the class of homeomorphism groups of manifolds.

The central result of this paper is the interpretation of $\text{HS}(M)$, which does not follow from the paper [28]. However, we shall require tools which were developed in that paper in order to prove the results in this paper. We will briefly list the relevant results that we use here. In the following theorem, if $U \subseteq M$ is an open set and $G \leq \text{Homeo}(M)$, then we write $G[U]$ for the *rigid stabilizer* of U , consisting of all elements of G which are the identity outside of U .

The following result follows from the fact that \mathcal{H} conservatively interprets, without parameters, a structure called AGAPE; see Section 3 of [28]. We have given more precise citations for most enumerated statements that refer to [28]. The statements below differ slightly from the way they are stated in [28] in order to better serve our purposes, though there is no difference in content.

Theorem 2.3 (see [28]). *Let M be a compact, connected, topological manifold of dimension at least one, and let*

$$\text{Homeo}_0(M) \leq \mathcal{H} \leq \text{Homeo}(M).$$

Then there exists a sentence ψ_M in the language of group theory such that for all compact manifolds N and all subgroups

$$\text{Homeo}_0(N) \leq \mathcal{H}' \leq \text{Homeo}(N),$$

we have $\mathcal{H}' \models \psi_M$ if and only if $N \cong M$. Moreover, the following sorts and predicates are interpretable without parameters in \mathcal{H} , uniformly in M .

- (1) *The Boolean algebra $\text{RO}(M)$ of regular open sets of M , equipped with an action of \mathcal{H} ; that is, a predicate*

$$\text{Act} \subseteq \mathcal{H} \times \text{RO}(M) \times \text{RO}(M)$$

such that $(g, U, V) \in \text{Act}$ if and only if $g(U) = V$ in M ; the interpretation of

$\text{RO}(M)$ is uniform for all manifolds, including noncompact ones. (See Section 2.2 and Theorem 3.4.)

(2) Predicates expressing connectedness of regular open sets, as well as that a regular open set U is a connected component of a regular open set V . (See Lemma 3.6 and Corollary 3.7.)

(3) A predicate $\text{RCB} \subseteq \text{RO}(M)$ such that $U \in \text{RCB}$ if and only if the closure of U lies in a collared open ball in M . (See Lemma 3.10.)

(4) A predicate $\text{RCB}^\partial \subseteq \text{RO}(M)$ such that $U \in \text{RCB}^\partial$ if and only if the closure of U lies in a collared open half-ball in M .

(5) Second order arithmetic $(\mathbb{N}, 0, +, \times, <, \subset)$, and a definable predicate

$$\# \subseteq \mathbb{N} \times \text{RO}(M)$$

such that $(n, U) \in \#$ if and only if U has exactly n components; moreover, if $\emptyset \neq U \in \text{RO}(M)$, then second order arithmetic can be interpreted using only U and $\mathcal{H}[U]$. (See Section 4.)

(6) Points $\mathcal{P}(M)$ of M , and more generally finite tuples $\mathcal{P}^{<\infty}(M)$ of points in M ; moreover, a predicate $\in_{\mathcal{P}} \subseteq \mathcal{P}(M) \times \text{RO}(M)$ such that (p, U) lies in $\in_{\mathcal{P}}$ if and only if the statement $p \in U$ is true in M . (See Section 5.)

(7) Predicates expressing that a point of M belongs to a union of two regular open sets, and that a point belongs to the closure of a regular open set. (See Section 5.)

(8) For each n , predicate expressing that a collection of n regular open sets covers the closure of a regular open set U .

(9) Exponentiation, i.e., a definable function

$$\exp: \mathcal{H} \times \mathbb{Z} \times M \rightarrow M$$

with the property that

$$\exp(g, n, p) = g^n(p) \quad \text{in } M.$$

(See Section 5.3.)

(10) A predicate which holds for a regular open set U if and only if U contains a tubular neighborhood of ∂M in M . (See Theorem 7.1.)

In view of [Theorem 2.3](#), we will assume that \mathcal{H} is implicitly equipped with the sorts of regular open sets of M , second order arithmetic, and points, as well as the relevant predicates listed in the theorem.

Some items in [Theorem 2.3](#) require special comment. Item (3) was only formally proved for manifolds of dimension 2 or higher, though for manifolds of dimension

one, the proof is even easier. By the characterization of connected sets in one-manifolds, it suffices to express that U is contained in a connected regular open set V , and that there is a homeomorphism h of M such that $V \cap h(V) = \emptyset$.

Item (4) was not formally stated in [28], though it is not difficult to find such a formula. One expresses that a regular open set U accumulates on a single component N of ∂M , as is easily deduced from 3.4.3. One then requires the existence of a homeomorphism h fixing each component of the boundary of M , which moves U into an arbitrary half-ball in N ; half-balls are interpreted explicitly in Section 7 of [28].

In item (6), a point $p \in M$ is encoded by an equivalence class of regular open sets, up to definable equivalence. If $U \subseteq M$ is a regular open set and $p \in U$ then there is a regular open set $V \subseteq U$ which encodes or *isolates* p ; this is implicit in Section 5 of [28]. In particular, if U is a regular open set with infinitely many components $\{U_i\}_{i \in \mathbb{N}}$ and if $p_i \in U_i$ is a point for each i , then the set of points $\bigcup_i p_i$ is encoded by a single regular open set V , which has the property that $V \subseteq \bigcup_i U_i$ and such that $V \cap U_i$ encodes the point p_i . We will abbreviate the predicate $\in_{\mathcal{P}} \text{ by } \in$.

Observe that the exponentiation function, together with the membership predicate relating $\mathcal{P}(M)$ to $\text{RO}(M)$ allows one to express that $g^n(U) = V$ for group elements in \mathcal{H} , integer exponents, and pairs of regular open sets, since we may express that

$$\exp(g, n, p) \in V \leftrightarrow p \in U.$$

The sentence ψ_M in [Theorem 2.3](#) is said to *isolate* M (or its homeomorphism group). We note that in [28], the proof of the content of [Theorem 2.3](#) was given for manifolds of dimension at least two. This was done purely to simplify some of the arguments and shorten the exposition; the proofs themselves can easily be generalized to manifolds of dimension one.

We note that even though we will refer to collared balls and half-balls in the sequel, these are concepts in the metalanguage; we will never appeal to these objects directly in the formal language.

To make one further observation about the relationship between $\text{Homeo}(M)$, its countable subgroups, and arithmetic, we remark the following: $\text{Homeo}(M)$ clearly contains many countable subgroups that are definable in arithmetic, including cyclic groups and free groups. Some subgroups of $\text{Homeo}(M)$ are in fact bi-interpretable with first order arithmetic, such as Thompson's groups F and T by [32]; it is not difficult to show that F in fact arises as a subgroup of $\text{Homeo}(M)$ for all positive dimensional manifolds. Most countable subgroups of $\text{Homeo}(M)$ are not definable in first order arithmetic, simply because $\text{Homeo}(M)$ interprets second order arithmetic. Indeed, then any countable elementary subgroup of $\text{Homeo}(M)$ (which exists by the Löwenheim–Skolem theorem) has too complicated a theory to be interpretable in arithmetic. A more detailed discussion can be found in [31].

3. Hereditarily sequential sets of homeomorphisms of a manifold

Let M and \mathcal{H} be as above and fixed, and fix the notation $d \geq 1$ for the dimension of M . In this section, we prove [Theorem 1.1](#); the uniformity of the interpretation among manifolds of a fixed dimension will be clear, and by taking disjunctions we obtain an interpretation that is valid for all manifolds of dimension bounded by a prescribed constant D . We prove the result in several steps.

3.1. Interpreting $\text{HS}_0(M)$. We begin by interpreting the sort $\text{HS}_0(M)$ in \mathcal{H} , and show that its members canonically correspond to elements of $\text{Homeo}(M)$. This itself is done in several steps. The reader should remember for the duration of the proof that we are encoding a homeomorphism of M by a proxy for its graph; the reader may pretend M is closed on a first reading, for simplicity.

The scheme for finding parameter-free interpretations of new sorts in \mathcal{H} will follow the basic scheme:

- (1) Encode data describing the new sort within various sorts of topological data to which we have access in view of [Theorem 2.3](#); oftentimes this data requires making choices, which amounts to an interpretation with parameters.
- (2) Observe that the set of suitable parameters is itself parameter-free definable within the relevant sort.
- (3) Eliminate parameters by quantifying over the relevant space of parameters.

The basic idea to interpret $\text{HS}_0(M)$ is to fix a finite cover of M , move the charts in the cover to a single chart in M (forming a finite set of “pages”), and then taking countably many disjoint copies of these pages. In each copy, we choose a point, which gives us the intermediate result of being able to interpret the sort of countable sequences of points in M ; since points in M are encoded by equivalence classes of regular open sets in M wherein only the local structure of the open set near the point being encoded matters, we may encode the countable sequence of points by a single suitable equivalence class of regular open sets. By considering a sequence σ of points in M , we may consider the odd and even index points in σ , thus obtaining a countable collection of points in $M \times M$. We then place definable conditions on such pairs to make sure the points occurring in each coordinate are dense in M , and so that these pairs actually arise from the graph of a homeomorphism of M . We have included some figures to aid the reader.

Lemma 3.1. *The group \mathcal{H} admits a parameter-free interpretation of the sort $\text{seq}(M)$ of countable sequences of points in M , which is uniform for all manifolds of dimension d . Moreover, the predicate $p \in \sigma$ expressing membership of a point p in a sequence σ , and the predicate $\sigma(i) = p$ expressing that p is the i -th term of σ , are both parameter-free definable.*

For technical reasons, we first prove the lemma in the case where M is not the interval, and give an adapted proof for the interval later.

Proof of Lemma 3.1 for $M \neq I$. We retain the notation $n(d)$ from Proposition 2.1. Choosing a cover M . We first fix a collection of regular open sets in M of bounded cardinality (depending on d) which cover M , and which can be used as charts in an atlas for M . Fix a collar neighborhood K of ∂M in M . Since M has dimension d and ∂M has dimension $d - 1$, Proposition 2.1 shows that $M \setminus K$ can be covered by $n(d)$ regular open sets, each component of which is a collared open ball, and each component of K can be cover by $n(d - 1)$ such sets consisting of collared open half-balls. By Items (3), (4), and (8) of Theorem 2.3, we may express the existence of collections

$$\mathfrak{B} = \{U_1, \dots, U_{n(d)}\} \quad \text{and} \quad \mathfrak{H} = \{V_1, \dots, V_{n(d-1)}\}$$

such that:

- (1) The sets $\mathfrak{B} \cup \mathfrak{H}$ cover M ; this is expressible since we simply require every point of M to lie in an element of $\mathfrak{B} \cup \mathfrak{H}$.
- (2) For $W \in \mathfrak{B} \cup \mathfrak{H}$ and W_0 a component of W , the closure of W_0 is contained inside of a collared open ball or open half-ball depending on whether $W \in \mathfrak{B}$ or $W \in \mathfrak{H}$, respectively.

Observe that the components of the sets U_i and V_i need not themselves be balls or half-balls, only have their closures be contained inside of balls or half-balls. Since the collections \mathfrak{B} and \mathfrak{H} have bounded cardinality depending only on d , the parameter space of choices for $(\mathfrak{B}, \mathfrak{H})$ is itself parameter-free definable.

The number of charts required in the atlas is the only part of the proof which depends on the dimension of M . All other dependencies on dimension fundamentally arise from the number of charts in the atlas.

Initializing a scratchpad. A schematic illustration of the initialized scratchpad is given in Figure 1. Fix regular open sets W and W^∂ with the following properties:

- (1) The closure of W is contained in a collared open ball in M .
- (2) For all components W_0 of W^∂ , the closure of W_0 is contained in a collared open half-ball in M .
- (3) If \hat{W} is an arbitrary regular open set whose closure is contained in a collared open half-ball in M , then there exists an element $g \in \mathcal{H}$ such that $g(\hat{W}) \subseteq W^\partial$.
- (4) Each component of ∂M meets at most one component of W^∂ .

It is straightforward to see that, in view of Theorem 2.3, the conditions defining W and W^∂ are expressible, and that such W and W^∂ always exist. Next, choose elements $\{g_U \mid U \in \mathfrak{B}\}$ and $\{g_V \mid V \in \mathfrak{H}\}$ such that:

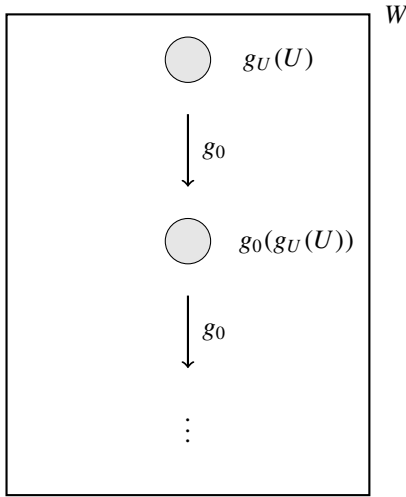


Figure 1. A schematic of the scratchpad; here we draw the image of one chart U in the atlas (which need not actually be a disk) in W , and the image under g_0 . The iterates under g_0 continue to infinity.

- (1) For all $U \in \mathfrak{B}$ and $V \in \mathfrak{H}$, we have $g_U(U)$ has compact closure inside of W and $g_V(V)$ has compact closure inside of W^∂ .
- (2) For distinct $U_1, U_2 \in \mathfrak{B}$, the images $g_{U_1}(U_1)$ and $g_{U_2}(U_2)$ are disjoint; we place the same requirement on distinct elements of \mathfrak{H} . Let

$$U_0 = \bigcup_{U \in \mathfrak{B}} g_U(U) \quad \text{and} \quad V_0 = \bigcup_{V \in \mathfrak{H}} g_V(V).$$

- (3) Choose elements $g_0 \in \mathcal{H}[W]$ and $g_0^\partial \in \mathcal{H}[W^\partial]$ such that for all distinct $i, j \geq 0$, we have

$$g_0^i(U_0) \cap g_0^j(U_0) = \emptyset,$$

and similarly

$$(g_0^\partial)^i(V_0) \cap (g_0^\partial)^j(V_0) = \emptyset.$$

Here, we are implicitly using the fact that we may quantify over the arguments of the (definable) exponentiation function.

We write $U^i = g_0^i(U_0)$ and $V^i = (g_0^\partial)^i(V_0)$, respectively. The reader may observe that this is the point where the argument fails for $M = I$, since in the case of the interval the homeomorphism g_0^∂ may not exist.

Encoding countable sequences of points in M . For a schematic of this part, see [Figure 2](#). We now choose a regular open set P , which together with the scratchpad will encode a countable sequence of points in M . Here, we require P to satisfy the following conditions:

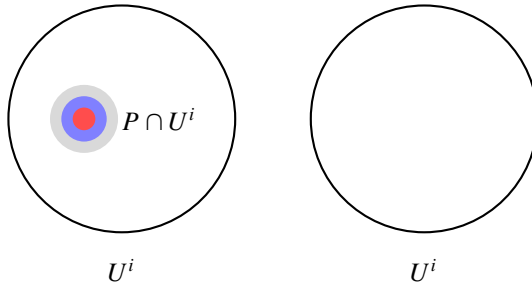


Figure 2. A schematic of two components in U^i . The sets P meets U^i and isolates a unique point in it.

- (1) The set P is contained in $\bigcup_i U^i \cup \bigcup_i V^i$. This can be expressed by requiring for each component of P , there is an i so that the $-i$ -th power of the relevant g_0 or g_0^∂ is contained in U_0 or V_0 , respectively.
- (2) For each i , exactly one of the intersections $P \cap U^i$ and $P \cap V^i$ is nonempty and isolates a unique point p_i in U^i or V^i . From here on, write q_i for the backwards image of p_i under the i -th power of g_0 or g_0^∂ respectively, followed by the relevant g_U^{-1} or g_V^{-1} .

Via the set P , we have thus encoded (with parameters), in an unambiguous way, a countably infinite sequence of points $\{q_i\}_{i \in \mathbb{N}} \subseteq M$. This is the sort $\text{seq}(M)$.

Since we can quantify over the arguments in the exponentiation function, it is straightforward to see from the construction that the membership predicate $p \in \sigma$ and $\sigma(i) = p$ are both definable, *a priori* with parameters.

Eliminating parameters. It is clear from the descriptions of the regular open sets chosen in the covers and the relevant homeomorphisms of \mathcal{H} that are chosen, that the choices are made over definable sets of parameters. Given two choices of parameters, we simply declare two interpretations of two sequences of points to be equivalent if for each $i \in \mathbb{N}$, the i -th terms of the sequences represent the same point of M ; this is possible in view of Item (6) of [Theorem 2.3](#). This completes the proof of the lemma. \square

We can now give a modified proof of [Lemma 3.1](#) for the interval. Technically we will only interpret sequences of points in the interior $(0, 1)$ of I , which is all that will be needed. It is not difficult to add “dummy entries” of two varieties to sequences which stand for possible choices of endpoints of I .

Proof of Lemma 3.1 for $M = I$. We begin by defining the set of homeomorphisms of I which attract to a point in the interior $(0, 1)$ of I . Fixing a point $p_0 \in (0, 1)$, we may define the set of elements $f \in \mathcal{H}$ such that for all U containing p_0 and with closure contained in $(0, 1)$, and for all $q \in (0, 1)$, there exists an $n \in \mathbb{N}$ such

that $f^n(q) \in U_0$. Call these elements of \mathcal{H} the p_0 -attracting homeomorphisms. In light of [Theorem 2.3](#), the p -attracting homeomorphisms of I are definable, with the point p_0 as the sole parameter.

Now, let $f \in \mathcal{H}$ be a p_0 -attracting homeomorphism for some $p_0 \in (0, 1)$, let $U \subseteq (0, 1)$ be a regular open set whose closure is contained in $(0, 1)$, let $U_0 \subseteq U$ be a regular open set containing p whose closure is contained in U , and let $g \in \mathcal{H}[U]$ have the property that for all distinct $i, j \in \mathbb{N}$, we have $g^i(U_0) \cap g^j(U_0) = \emptyset$. Write $U_i = g^i(U_0)$ for $i \in \mathbb{N}$. Up to now, we have carried out the interval analogue of initializing the scratchpad.

We now interpret countable sequences of points in $(0, 1)$. We do this by choosing a regular open set P which isolates a unique point p_i in each U_i . Defining $q_i = f^{-i}g^{-i}(p_i)$, we have unambiguously interpreted the sequence $\{q_i\}_{i \in \mathbb{N}}$ inside of \mathcal{H} . Moreover, every sequence of points in $(0, 1)$ arises as some such $\{q_i\}_{i \in \mathbb{N}}$, for various choices of P and f . This defines sequences of points in $(0, 1)$ with parameters. We declare two sequences σ_1 and σ_2 , with different choices of parameters, to be equivalent if for all $i \in \mathbb{N}$ the encoded points $\sigma_1(i)$ and $\sigma_2(i)$ represent the same point of $(0, 1)$. □

Interpreting pregraphs. Armed with the interpretation of sort $\text{seq}(M)$, we can interpret the sort of *pregraphs*; we define pregraphs to be countable subsets $\Gamma \subseteq M \times M$ such that the projection of Γ to each factor is dense in M .

Lemma 3.2. *The sort of pregraphs is uniformly interpretable for manifolds in dimension d , from the sort $\text{seq}(M)$. Moreover, the predicate $(x, y) \in \Gamma$ expressing that a pair $(x, y) \in M \times M$ is an element of Γ is parameter-free interpretable.*

Proof. We may quantify over terms of a sequence $\sigma \in \text{seq}(M)$ and thus encode a countable subset Γ of $M \times M$ from σ by declaring $(x, y) \in \Gamma$ if and only if there exists an $n \in \mathbb{N}$ such that $\sigma(2n) = x$ and $\sigma(2n + 1) = y$. Density of the projections is expressed by saying that for each nonempty regular $U \in \text{RO}(M)$, there is an odd index i and an even index j such that $\sigma(i), \sigma(j) \in U$. The set of Γ encoded by this definable set of sequences clearly coincides with pregraphs. We finally put an equivalence relation on elements of $\text{seq}(M)$ encoding pregraphs, which expresses that σ_1 and σ_2 are equivalent if and only if they encode pregraphs that are equal as subsets of $M \times M$; this is evidently a definable equivalence relation. This completes the parameter-free interpretation. □

From pregraphs to graphs. We now pass to graphs of homeomorphisms of M .

Lemma 3.3. *Pregraphs in dimension d admit a parameter-free interpretation of $\text{HS}_0(M)$.*

Proof. We put definable conditions on pregraphs to guarantee that they define graphs of homeomorphisms of M . Since M is compact, it suffices to require that a

pregraph Γ extend continuously to the graph of a continuous self-map of M which is injective and surjective.

Continuity: We need only require for all $(x_0, y_0) \in \Gamma$ that for all open V containing y_0 , there is a U containing x_0 such that for all $(x, y) \in \Gamma$ with $x \in U$, we have $y \in V$. This is clearly expressible. Any Γ satisfying this continuity requirement automatically encodes a continuous map

$$f_\Gamma : M \rightarrow M.$$

Injectivity: We need only require that for all disjoint open U_1 and U_2 there exist disjoint open V_1 and V_2 such that if $(x_i, y_i) \in \Gamma$ for $i \in \{1, 2\}$ with $x_i \in U_i$, then $y_i \in V_i$.

Surjectivity: We need only require that the image of f_Γ be dense in M . This can be achieved by requiring for all nonempty V that there be an $(x, y) \in \Gamma$ with $y \in V$.

Any pregraph Γ satisfying the foregoing conditions will automatically encode the graph of a homeomorphism of M . Moreover, every homeomorphism of M is encoded by some pregraph, simply by taking a dense subset of the graph of the homeomorphism. To complete the interpretation of $\text{HS}_0(M)$, we put an equivalence relation on pregraphs which expresses that two pregraphs Γ_1 and Γ_2 are equivalent if they encode the same homeomorphism of M . For this, it suffices to require that if $(x_1, y_1) \in \Gamma_1$ with $x_1 \in U$ and $y_1 \in V$ then there exists a pair $(x_2, y_2) \in \Gamma_2$ with $x_2 \in U$ and $y_2 \in V$. \square

3.2. Interpreting \mathcal{H} within $\text{HS}_0(M)$. Recall that the initial given data is \mathcal{H} , whereas here we have interpreted elements of $\text{Homeo}(M)$ via their graphs; *a priori*, $\text{Homeo}(M)$ may be substantially larger than \mathcal{H} . We note that it is straightforward to interpret \mathcal{H} as a set within $\text{HS}_0(M)$: indeed, consider the association $g \mapsto \Gamma_g$, which sends an element $g \in \text{Homeo}(M)$ to the graph of g as a homeomorphism of M . We have Γ_g corresponds to a graph of an element of \mathcal{H} if and only if

$$(\exists \gamma) [\forall x \forall y ((x, y) \in \Gamma_g \leftrightarrow \gamma(x) = y)].$$

Thus, we are justified in saying that \mathcal{H} can interpret its own elements via graphs, and we are justified in saying we have interpreted elements of \mathcal{H} inside of $\text{HS}_0(M)$. We will interpret the group operation below. We summarize with the following corollary:

Corollary 3.4. *There is a definable predicate $R \subseteq \mathcal{H} \times \text{HS}_0(M)$ defining the pairs (g, Γ) such that $\Gamma = \Gamma_g$ encodes the graph of g .*

3.3. Interpreting the sorts $\text{HS}_n(M)$ for $n \geq 1$. The interpretation of the sorts $\text{HS}_n(M)$ for $n \geq 1$ is now straightforward, because of the existence of a computable bijection $\phi : \mathbb{N}^2 \rightarrow \mathbb{N}$.

Lemma 3.5. *For all $n \geq 1$, the sort $\text{HS}_n(M)$ is parameter-free interpretable in $\text{seq}(M)$, uniformly interpretable for manifolds of dimension d .*

Proof. We proceed by induction, $\text{HS}_0(M)$ having been interpreted already. To interpret $\text{HS}_{n+1}(M)$ once $\text{HS}_n(M)$ has been parameter-free interpreted, we use the bijection ϕ as above to definably pass from \mathbb{N} -indexed sequences of points to \mathbb{N}^2 -indexed points $\{q_{(i,j)}\}_{(i,j) \in \mathbb{N}^2}$. For i fixed, we simply require that the (obviously parameter-free definable) subsequence $\{q_{(i,j)}\}_{j \in \mathbb{N}}$ encode an element of $\text{HS}_n(M)$. It is clear that this furnishes a parameter-free interpretation of $\text{HS}_{n+1}(M)$.

It is clear that the predicate $\in_n \subseteq \text{HS}_n(M) \times \text{HS}_{n+1}(M)$ is parameter-free definable, as is the predicate defining the i -th term in a sequence in $\text{HS}_n(M)$. \square

3.4. Predicates for manipulating $\text{HS}(M)$. Most predicates for manipulating sequences in $\text{HS}_n(M)$ are easily seen to be interpretable, as follows from the fact that one can freely quantify over the arguments in the exponentiation function; we have argued concerning membership \in_n and the predicate $s(i) = t$ for $s \in \text{HS}_{n+1}(M)$ and $t \in \text{HS}_n(M)$ already.

Let $f_1, f_2, f_3 \in \text{HS}_0(M)$ be terms in a sequence $\sigma \in \text{HS}_1(M)$. It is easy to see that there is a predicate expressing that $f_1 * f_2 = f_3$ in $\text{Homeo}(M)$. Indeed, let Γ_i be graphs of f_i for $i \in \{1, 2, 3\}$. To express that $f_1 * f_2 = f_3$, it suffices to express that for all $(x, z) \in \Gamma_3$ and all open sets U and V such that $x \in U$ and $z \in V$, whenever $(x', y) \in \Gamma_1$ with $x' \in U$ and all open W such that $y \in W$, there exists a $(y', z') \in \Gamma_2$ such that $y' \in W$ and $z' \in V$.

For homeomorphisms of M , extended supports are regular open sets which are interpretable via Rubin's interpretability theorem, and which is given by a purely first order group theoretic formula; see [45], and specifically Theorem 3.6.3 of [25] and Section 3.2 of [28]. It is clear then that we may interpret a new sort which represents the extended support of an element $f \in \text{HS}_0(M)$, and which is canonically identified with the extended support of the homeomorphism f . This completes the proof of [Theorem 1.1](#).

We have the following consequences of interpreting the sort $\text{HS}_1(M)$ and the preceding predicates.

Corollary 3.6. (1) *The set of sequences $s \in \text{HS}_1(M)$ which, via the identification of $\text{HS}_0(M)$ with $\text{Homeo}(M)$, form subgroups of $\text{Homeo}(M)$ is parameter-free definable.*

(2) *If $X \subseteq \text{HS}_0(M) = \text{Homeo}(M)$ is arbitrary, then there is a predicate*

$$\text{member}_X \subseteq \text{Homeo}(M),$$

using X as a parameter, which expresses whether an arbitrary $f \in \text{Homeo}(M)$ is a finite product of elements of X . In particular, if X is parameter-free definable then member_X is parameter-free definable.

Proof. The first part reduces to requiring for all $f, g \in s$, we have $f^{-1} \in s$ and $f \cdot g \in s$. The second part reduces to the existence of a sequence $s \in \text{HS}_1(M)$ with $s(0) = 1$, with $s(n) = f$ for some $n \in \mathbb{N}$, and such that for all $0 < m \leq n$ we have $s(n-1)^{-1}s(n) \in X$. \square

4. Intermediate subgroups, mapping class groups, and Theorem 1.4

We now use the interpretation of the sorts $\text{HS}(M)$ to extract group-theoretic consequences. Observe first that \mathcal{H} interprets $\text{Homeo}(M)$. Indeed, this is part of the content of Theorem 1.1. Next, we can interpret $\text{Homeo}_0(M)$. The key to interpreting $\text{Homeo}_0(M)$ is the following result, which appears as Corollary 1.3 in [12].

Theorem 4.1 (Edwards and Kirby). *Let \mathcal{U} be an open cover of a compact manifold M . An arbitrary element $g \in \text{Homeo}_0(M)$ admits a **fragmentation** subordinate to \mathcal{U} . That is, g can be written as a composition of homeomorphisms that are supported in elements of \mathcal{U} .*

Proposition 4.2. *The group \mathcal{H} interprets $\text{Homeo}_0(M) \subseteq \text{HS}_0(M)$.*

As always, the interpretation of $\text{Homeo}_0(M)$ in \mathcal{H} is uniform in manifolds of bounded dimension.

Proof of Proposition 4.2. It suffices to construct a formula $\text{isotopy}_0(\gamma)$ that is satisfied by a homeomorphism g if and only if g is isotopic to the identity. We will carry out the construction for closed manifolds, with the general case being similar.

Consider Γ_g , the graph of a homeomorphism as obtained from interpreting the sort $\text{HS}_1(M)$, and let $\mathfrak{B} = \{U_1, \dots, U_{n(d)}\}$ be a cover of M , with each component of each U_i having compact closure inside of a collared open ball.

By imposing suitable definable conditions on the data defining Γ_g , we may insist that there exists an i and a component \hat{U}_i of U_i such for all $(p, q) \in \Gamma_g$, we have $p = q$ unless $p \in \hat{U}_i$. Specifically, we may write

$$\text{small-sup}(\Gamma) := (\forall(x, y) \in \Gamma)(\exists i \leq n(d))(\exists \hat{u} \in \pi_0(u_i))[x \notin \hat{u} \rightarrow x = y];$$

in this formula we are implicitly treating elements of \mathfrak{B} as parameters.

This condition implies that the homeomorphism g encoded by Γ is the identity outside of \hat{U}_i . Since \hat{U}_i is compactly contained in the interior of a collared ball in M we have that g is isotopic to the identity, as follows from the Alexander trick.

By quantifying over all such covers \mathfrak{B} of M , we thus obtain a parameter-free definable set $X \subseteq \text{HS}_0(M)$ consisting of graphs of elements of $\text{Homeo}(M)$ which satisfy small-sup for some such cover.

By Theorem 4.1, we have that $g \in \text{Homeo}(M)$ is isotopic to the identity if and only if g is a product of a finite tuple of homeomorphisms lying in X . By

Corollary 3.6, it follows that $\text{Homeo}_0(M)$ is parameter-free definable as a subset of the sort $\text{HS}_0(M)$. \square

An arbitrary subgroup $\text{Homeo}_0(M) \leq \mathcal{H}' \leq \text{Homeo}(M)$ is automatically of countable index in $\text{Homeo}(M)$, as follows from the fact that for a compact manifold, $\text{Homeo}(M)$ is separable and therefore has countably many connected components.

Proof of Proposition 1.3. A subgroup

$$\text{Homeo}_0(M) \leq \mathcal{H}' \leq \text{Homeo}(M)$$

can be encoded by a definable equivalence class of countable subsets of $\text{Homeo}(M)$; indeed, if \underline{g} is a sequence then we obtain a subgroup $\mathcal{H}_{\underline{g}}$ (viewed as a subset of $\text{HS}_0(M)$) via

$$\mathcal{H}_{\underline{g}} = \{h \mid (\exists g \in \underline{g})[h \in g \cdot \text{Homeo}_0(M)]\},$$

after adding the further condition that $\mathcal{H}_{\underline{g}}$ be a group (which can be guaranteed by imposing the first order condition that \underline{g} be a group, for instance). Two sequences of homeomorphisms \underline{g} and \underline{h} are equivalent if $\mathcal{H}_{\underline{g}} = \mathcal{H}_{\underline{h}}$. Since the mapping class group of M is countable, any such subgroup \mathcal{H}' occurs as $\mathcal{H}_{\underline{g}}$ for some sequence \underline{g} . We thus obtain a canonical bijection between subgroups \mathcal{H}' as above and suitable equivalence classes of sequences of homeomorphisms, as desired.

We have already shown that $\text{Homeo}(M)$ and $\text{Homeo}_0(M)$ are interpretable without parameters. The group \mathcal{H} itself is also definable without parameters in the interpretation of $\text{Homeo}(M) = \text{HS}_0(M)$ in \mathcal{H} , as is part of the content of **Theorem 1.1**. \square

It is not difficult to argue the conclusions of **Theorem 1.4**, and so we only sketch the arguments. Because $\text{Homeo}(M)$ and $\text{Homeo}_0(M)$ are parameter-free interpretable in \mathcal{H} , so is $\text{Mod}(M)$. The sorts of countable subgroups of $\text{Homeo}(M)$ and $\text{Mod}(M)$ are parameter-free interpretable, by **Corollary 3.6**; it is immediate that one can quantify over arbitrary subsets of countable subgroups, since these subsets will always be countable. All of the countable algebra of groups can be formalized within ACA or slightly stronger systems, which is substantially weaker than full second order theory of countable groups to which we have access: see [46], page 14, and also Chapter III. Here and for the rest of the section, “subgroup” will refer to a subgroup of $\text{Homeo}(M)$ or of $\text{Mod}(M)$.

Membership in a fixed countable subgroup follows from **Corollary 3.6**. Finite generation asks whether for a countable subgroup, there exists a sequence σ wherein all but finitely many terms are the identity, so that every element in the subgroup can be written as a finite product of entries in σ ; this is clearly expressible: indeed, we think of the set X in **Corollary 3.6** as a sequence σ where there exists an $n \in \mathbb{N}$

such that for all $i > n$ the term $\sigma(i)$ is the identity. Finite presentability is slightly more complicated but still straightforward.

Finite index subgroups of a given countable subgroup are easily defined, using the subgroup itself as a parameter; thus, residual finiteness is expressible. For finitely generated groups, linearity can be expressed via the Lubotzky Linearity Criterion [33], and in general by quantifying suitably over countable subgroups of $\mathrm{GL}_n(\mathbb{C})$; we omit the tedious details.

Isomorphism of a countable subgroup with a particular definable group is straightforward. Amenability can be encoded with the Følner Criterion; see Chapter 2 of [9]. Kazhdan's Property (T) for finitely generated subgroups can be encoded using Ozawa's Criterion, which is the main result of [42].

5. Descriptive set theory in $\mathrm{Homeo}(M)$

In this section, we show how to interpret the projective hierarchy in $\mathrm{Homeo}(M)$ and characterize definability via [Theorem 1.6](#). The interpretation is modeled on the fact that the projective hierarchy in Euclidean space is definable in second order arithmetic.

In this section, we emphasize that all interpretations are *uniform*; that is, there is a single formula, depending only on the dimension of the manifold M , which defines all open sets of $\mathrm{Homeo}(M)$ (as subsets of $\mathrm{HS}_0(M)$) for various choices of parameters. The same holds for all sets at the various levels of the Borel hierarchy, analytic sets, and sets in the projective hierarchy.

5.1. Generalities on descriptive set theory. The reader is directed to [21; 37] for a more thorough background. Suppose that we are given a Polish (i.e., completely metrizable and separable) space X . We recall the definition of the Borel hierarchy. For every nonempty countable ordinal α one can define the families $\Sigma_\alpha^0(X)$ and $\Pi_\alpha^0(X)$ of subsets of X as follows.

- The class $\Sigma_1^0(X)$ consists of all open subsets of X .
- For all $\alpha < \aleph_1$ the class Π_α^0 is the collection of complements of subsets in $\Sigma_\alpha^0(X)$.
- For any limit ordinal $\alpha < \aleph_1$ we have $\Sigma_\alpha^0(X) = \bigcup_{\beta < \alpha} \Sigma_\beta^0(X)$.
- For all $\alpha < \aleph_1$ the family $\Sigma_{\alpha+1}^0(X)$ consists of all countable unions of sets in $\Pi_\alpha^0(X)$.

A subset of X is called *Borel* if it belongs to Σ_α^0 for some $\alpha < \aleph_1$.

A subset of X is *analytic* if it is a continuous image of Baire space $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$; equivalently, a subset $A \subseteq X$ is analytic if and only if there is a closed subset of $C \subseteq X \times \mathcal{N}$ such that A is the projection of C to X . Observe that Baire space and its topology are parameter-free interpretable in second order arithmetic.

In the projective hierarchy, analytic sets are called Σ_1^1 . One can extend the notation of the Borel hierarchy in order to define classes $\Sigma_\alpha^1(X)$ and $\Pi_\alpha^1(X)$ for all $\alpha < \aleph_1$; here we will need only concern ourselves with integer values of α , for which the definition can be given by usual induction as follows:

- For all n for which $\Sigma_n^1(X)$ has been defined, let $\Pi_n^1(X)$ be the class of complements of sets in Σ_n^1 . In particular, for $n = 1$, we obtain the family $\Pi_1^1(X)$ of projective sets in X .
- A set $Z \subseteq X$ is in Σ_{n+1}^1 if there is a Π_n^1 subset $Y \subseteq X \times \mathcal{N}$ such that Z is the projection of Y to X .

The sets $\{\Sigma_n^1(X)\}_{n \geq 1}$ form the *projective hierarchy*; when X is a Euclidean space, the projective hierarchy is easily seen to be definable in second order arithmetic. We note that in the definition of the projective hierarchy, the factor \mathcal{N} can be replaced by an arbitrary uncountable Polish space (e.g., $\text{Homeo}(M)$ itself).

It is a standard fact that a subset $X \subseteq \mathbb{R}$ is definable (with parameters) in second order arithmetic if and only if it lies in the projective hierarchy. [Theorem 1.6](#) establishes the corresponding characterization of definable subsets of $\text{Homeo}(M)$. We prove one direction first.

Proposition 5.1. *Let $X \subseteq \text{Homeo}(M)$ be definable with parameters, as a subset of the sort $\text{HS}_0(M)$. Then X lies in the projective hierarchy. If $X \subseteq \mathcal{H}$ is definable with parameters in the language of group theory, then X lies in the projective hierarchy.*

Proof. This follows by induction on the quantifier complexity ϕ of a formula defining X . Equalities and inequalities (with parameters) in $\text{Homeo}(M)$ define closed and open sets respectively, and so if X is defined by quantifier-free formula then it is certainly Borel.

Suppose that X is defined by

$$\phi(x) = (\exists y)\psi(x, y, a),$$

where ψ defines a set

$$Y \subseteq (\text{Homeo}(M))^k$$

in the projective hierarchy for some $k \in \mathbb{N}$, and where a is a tuple of parameters. Then X is given by a projection of Y to a smaller Cartesian power of copies of $\text{Homeo}(M)$, and so X lies at most one level higher than Y in the projective hierarchy. If $X \subseteq \mathcal{H}$ then a canonical definable identification between X and a subset of the sort $\text{HS}_0(M)$ is given by [Corollary 3.4](#). The proposition now follows easily. \square

In the remainder of this section, we will interpret sorts for levels of the projective hierarchy together with the predicate \in , and show that every subset of $\text{Homeo}(M)$ in the projective hierarchy is definable with parameters (as a subset of the sort

$\text{HS}_0(M)$), and every subset of the home sort \mathcal{H} in the projective hierarchy is definable with parameters.

5.2. Open sets. We interpret the compact-open topology on $\text{Homeo}(M)$ directly. First, cover M by regular open sets. Since regular open sets themselves are encoded by definable equivalence classes of homeomorphisms by associating their extended supports (cf. [Theorem 2.3](#)), finite covers of M can be encoded by equivalence classes of finite tuples of homeomorphisms. To define finite tuples τ of homeomorphisms whose supports cover M since one need only express that for all $p \in M$, there exists an $f \in \tau$ such that $p \in \text{supp}^e(f)$.

From a finite cover \mathcal{V} of M , one can define an open set $U_{\mathcal{V}} \subseteq \text{Homeo}(M)$ by considering homeomorphisms f such that for all $p \in M$, there exists a $V \in \mathcal{V}$ such that both p and $f(p)$ lie in V . It is not so difficult to see that $U_{\mathcal{V}}$ is indeed open.

Now, if $f \in \text{Homeo}(M)$ and \mathcal{V} is a finite covering of M then we set $U_{\mathcal{V}}(f)$ to be the set of homeomorphisms g such that $g^{-1}f$ lies in $U_{\mathcal{V}}$ as defined above. Observe that for a given cover \mathcal{V} and $f \in \text{Homeo}(M)$, the subset of $\text{HS}_0(M)$ contained $U_{\mathcal{V}}(f)$ is definable (with \mathcal{V} as a parameter).

As \mathcal{V} varies over finite covers of M and f varies over $\text{Homeo}(M)$, we have that the sets $U_{\mathcal{V}}(f)$ form a basis for the compact-open topology of $\text{Homeo}(M)$. Thus, a basis for the topology on $\text{Homeo}(M)$ is encoded by certain equivalence classes of finite tuples of elements of $\text{Homeo}(M)$, which is to say certain definable subsets of $\text{HS}_1(M)$, up to definable equivalence.

More explicitly, for a tuple (f_1, \dots, f_n) , we definably associate extended supports via $f_i \mapsto V_i = \text{supp}^e f_i$ for $i \geq 2$, while requiring that

$$M \subseteq \bigcup_{i=2}^n \text{supp}^e(f_i).$$

Setting $\mathcal{V} = \{V_2, \dots, V_n\}$, such a tuple of homeomorphisms encodes the set $U_{\mathcal{V}}(f_1)$. This interpretation is clearly uniform in the sense described at the beginning of the section. The predicate \in is trivial to interpret.

We see now that basic open sets are interpretable as a definable subset of $\text{HS}_1(M)$, up to definable equivalence by setting two tuples to be equivalent if and only if the basic open sets they encode contain the same homeomorphisms (viewed as elements in the sort $\text{HS}_0(M)$).

An arbitrary open set is then interpreted as a countable sequence of basic open sets, with a homeomorphism f being a member of the open set if and only if it is a member of one of the elements in the sequence. Since basic open sets are parameter-free interpretable in $\text{HS}_1(M)$, we see that open sets are parameter-free interpretable in $\text{HS}_2(M)$. Closed sets are then simply complements of open sets. It is trivial to interpret the membership relation of homeomorphisms in an open or closed set.

Corollary 5.2. *The sorts of open and closed sets of $\text{Homeo}(M)$, viewed as subsets of $\text{HS}_0(M)$, are uniformly interpretable with parameters. Open and closed subsets of \mathcal{H} are uniformly definable with parameters.*

Proof. We have argued that open sets in $\text{Homeo}(M)$ are encoded by elements in $\text{HS}_2(M)$. Thus, a particular open set U is identified with a parameter-free definable equivalence class of elements $\tau \in \text{HS}_2(M)$, and $f \in \text{HS}_0(M)$ if and only if there exists a $\sigma \in \tau$ such that $f \in \sigma$. The case of closed sets is identical. The definability of open and closed sets in the home sort follows now from [Corollary 3.4](#). \square

With a minor variation on the preceding arguments, we can recover the topology on $\text{Homeo}(M)^\ell \times \mathcal{N}^k$ for all $\ell, k \geq 0$. Note that for $k \geq 1$, we have $\text{Homeo}(M)^\ell \times \mathcal{N}^k \cong \text{Homeo}(M) \times \mathcal{N}$. We record the following corollary.

Corollary 5.3. *The sorts of open and closed subsets of $\text{Homeo}(M)^\ell \times \mathcal{N}^k$ are parameter-free interpretable in \mathcal{H} , and open and closed sets in these spaces are uniformly interpretable with parameters.*

5.3. The Borel hierarchy. The Borel hierarchy of $\text{Homeo}(M)$ and \mathcal{H} is now straightforward to interpret. We first indicate an interpretation of finite levels of the Borel hierarchy, followed by the case of arbitrary countable ordinals using Borel codes; see Section 1.4 of [\[16\]](#).

We have already interpreted open and closed sets in $\text{HS}_2(M)$, which corresponds to $\Sigma_1^0(\text{Homeo}(M))$ and $\Pi_1^0(\text{Homeo}(M))$, respectively; we will suppress the notation of $\text{Homeo}(M)$ since it will not cause confusion.

By induction, Σ_k^0 and Π_k^0 are uniformly interpreted in $\text{HS}_{k+1}(M)$. By definition, elements of Σ_{k+1}^0 are countable unions of elements of Π_k^0 , which are then encoded by definable equivalence classes of elements in $\text{HS}_{k+2}(M)$. Elements of Π_{k+1}^0 are just given by complementation. The proof of the following is nearly identical to that of [Corollary 5.2](#).

Corollary 5.4. *Let $X \subseteq \text{Homeo}(M)$ lie in a finite level of the Borel hierarchy. Then X is uniformly interpretable with parameters, viewed as a subset of $\text{HS}_0(M)$. If $X \subseteq \mathcal{H}$ lies in a finite level of the Borel hierarchy then X is uniformly definable with parameters in the home sort.*

For the general Borel hierarchy, it is helpful to use *Borel codes*, which are a standard tool in descriptive set theory. For a Polish space X , one chooses a countable basis $\{U_\tau\}_{\tau \in \mathbb{N}^{<\mathbb{N}}}$ for the topology of X . A Borel set $Y \subseteq X$ is encoded by a *labeled, well-founded tree*, the definition of which we briefly recall here; cf. Section 1.4 of [\[16\]](#). A tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a prefix-closed subset, where elements of $\mathbb{N}^{<\mathbb{N}}$ (also called nodes) are viewed as finite sequences; there is an obvious notion of length for a node. A tree T is *well-founded* if there is no infinite sequence $\{\tau_i\}_{i \in \mathbb{N}}$ where τ_{i-1} is a prefix of τ_i . An element $\tau \in T$ is *terminal* if it admits no proper extension in T .

If $\tau \in T$ then one writes T_τ for the set of suffixes of elements of T which have τ as a prefix, so that T_τ is itself a tree. A well-founded tree $T \neq \emptyset$ together with a label function $\lambda : T \rightarrow \mathbb{N}$ forms a Borel code provided that:

- (1) If $\tau \in T$ is non-terminal then $\lambda(\tau) \in \{0, 1\}$.
- (2) If $\tau \in T$ is non-terminal and $\lambda(\tau) = 0$ then there exists a unique $\sigma \in T$ extending τ by exactly one entry, i.e., of length exactly one more than τ .

If $\tau \in T$ and λ is a labeling of T then there is an obvious labeling of T_τ which we also call λ .

The *rank* of $\tau \in T$ is defined recursively:

- (1) If τ is terminal then the rank of τ is zero.
- (2) If $\tau \in T$ is not terminal, then the rank of τ is one more than the supremum of the ranks of the one-entry extensions of τ in T , i.e., of length exactly one more than τ .
- (3) The rank of T is the rank of the empty sequence $\emptyset \in T$.

Choose a bijection $\mathbb{N}^{<\mathbb{N}}$ with \mathbb{N} , which we write $\tau \mapsto \langle \tau \rangle$. A Borel set $B_{(T,\lambda)}$ in X is encoded by the pair (T, λ) as follows.

- (1) If \emptyset is the only node of T then $B_{(T,\lambda)} = U_\tau$, where $\langle \tau \rangle = \lambda(\emptyset)$.
- (2) If \emptyset is non-terminal and $\lambda(\emptyset) = 0$ then there is a unique node σ of length one extending \emptyset . We write $B_{(T,\lambda)} = X \setminus B_{(T_\sigma,\lambda)}$.
- (3) If \emptyset is non-terminal and $\lambda(\emptyset) = 1$, then write $\{\sigma_i\}_{i \in \mathbb{N}}$ for the nodes of length one in T and define

$$B_{(T,\lambda)} = \bigcup_i B_{(T_{\sigma_i},\lambda)}.$$

This encoding makes sense because of the well-foundedness of T . A set in X is Borel if and only if it admits a Borel code. Moreover, for a countable ordinal α , a Borel set lies in Σ_α^0 if and only if it is encoded by a Borel code encoded by (T, λ) of rank at most α with $\lambda(\emptyset) \neq 0$. Similarly, a Borel set lies in Π_α^0 if and only if it is encoded by a Borel code encoded by (T, λ) of rank at most α with $\lambda(\emptyset) = 0$.

Corollary 5.5. *The following are uniformly parameter-free interpretable in \mathcal{H} :*

- (1) the Borel sets \mathcal{B} of $\text{Homeo}(M)$, viewed as subsets of $\text{HS}_0(M)$;
- (2) the membership predicate for Borel subsets;
- (3) a rank predicate $\text{rk} \subseteq \mathcal{B} \times \aleph_1$, consisting of pairs (A, α) with $A \in \Sigma_\alpha^0$.

Proof. This is nearly immediate. First, countable ordinals are parameter-free definable in second order arithmetic. Moreover, there is a definable bijection between $\mathbb{N}^{<\mathbb{N}}$ and \mathbb{N} , so that in second order arithmetic we may define (without parameters) well-founded trees and hence Borel codes.

It is straightforward to see that, in light of [Section 5.2](#), we may have direct access to countable bases for the topology on $\text{Homeo}(M)$. It is similarly straightforward to see that via Borel codes, we may encode:

- (1) Borel sets;
- (2) a parameter-free predicate that expresses when two Borel codes encode the same Borel set;
- (3) the rank function rk ;
- (4) the membership predicate in members of the class of Borel sets.

Moreover, individual Borel sets are interpretable with parameters. We omit the remaining details. □

5.4. The projective hierarchy. We will now complete the proof of [Theorem 1.6](#); precisely, we will show that the levels Σ_n^1 and Π_n^1 of the projective hierarchy of $\text{Homeo}(M)$ are uniformly interpretable sorts, and that a set in the projective hierarchy is definable with parameters, uniformly within a level of the hierarchy.

By definition, an analytic set in a Polish space X is a continuous image of \mathcal{N} . Equivalently, an analytic set in X is the projection of a closed subset of $X \times \mathcal{N}$ to X . By [Corollary 5.3](#), we have interpreted closed subsets of $\text{Homeo}(M) \times \mathcal{N}$. More precisely, an open set in $\text{Homeo}(M) \times \mathcal{N}$ is a countable union of basic open sets in the product, which can be taken to be pairs of basic open sets in each factor. It is not difficult to see then that open sets in $\text{Homeo}(M) \times \mathcal{N}$ can be encoded in $\text{HS}_3(M)$, and closed sets by complementation. If

$$C \subseteq \text{Homeo}(M) \times \mathcal{N}$$

is a closed subset then the set

$$Y_C = \{f \mid (\exists x)[(f, x) \in C]\}$$

is analytic, and every analytic set arises this way. Thus, membership of a homeomorphism $f \in \text{Homeo}(M)$ in an analytic (or co-analytic) set is expressible.

Corollary 5.6. *Let $X \subseteq \text{Homeo}(M)$ be analytic or co-analytic. Then X is uniformly interpretable with parameters, viewed as a subset of $\text{HS}_0(M)$. If $X \subseteq \mathcal{H}$ is analytic or co-analytic then X is uniformly definable with parameters in the home sort.*

It is trivial to extend this discussion to analytic and co-analytic subsets of finite Cartesian powers of $\text{Homeo}(M)$.

To interpret the higher levels of the projective hierarchy, suppose by induction that $X \subseteq \text{Homeo}(M)^\ell$ is a Π_n^1 set that is definable with parameters. Then the set $Y \subseteq \text{Homeo}(M)^{\ell-1}$ given by projecting X to the first $\ell - 1$ factors is Σ_{n+1}^1 , and every Σ_{n+1}^1 occurs this way. Thus, we have:

Corollary 5.7. *Let $X \subseteq \text{Homeo}(M)$ lie in a fixed level the projective hierarchy. Then X is uniformly interpretable with parameters, viewed as a subset of $\text{HS}_0(M)$. If $X \subseteq \mathcal{H}$ lies in a fixed level of the projective hierarchy then X is uniformly definable with parameters in the home sort.*

This completes the proof of [Theorem 1.6](#).

6. Undefinability of sentences isolating manifolds

Throughout this section, we will limit ourselves to full homeomorphism groups of manifolds; it is easy to see that the entire discussion could be carried out for any subgroup between Homeo_0 and Homeo , and we make this choice for the sake of concision.

In this section, we prove [Theorems 1.7](#), [6.1](#), and [6.2](#); together, these results show that many natural sets of natural numbers associated to homeomorphism groups of manifolds are not definable in second order arithmetic.

We fix an arbitrary numbering of the symbols in the language of group theory, and thus obtain a computable Gödel numbering of strings of symbols in this language. As is standard, well-formed formulae and sentences are definable in arithmetic, which is to say the set of Gödel numberings of formulae and sentences are definable in first order arithmetic. For formulae and sentences ψ in the language of group theory (and occasionally, by abuse of notation, in arithmetic), we will write $\#\psi$ for the corresponding Gödel numbers. For a class of sentences in group theory, the definability of the set of Gödel numbers of sentences in that class is independent of the Gödel numbering used. The proof of [Theorem 1.7](#) will follow ultimately from Tarski's well-known undefinability of truth [[3](#); [18](#); [34](#)]. That is, there is no predicate True that is definable in arithmetic (first or second order) such that for all sentences ϕ in second order arithmetic, we have

$$\phi \longleftrightarrow \text{True}(\#\phi).$$

See [Theorem 12.7](#) of [[20](#)] for a general discussion.

For the remainder of this section, we will fix a uniform interpretation of second order arithmetic in homeomorphism groups of compact manifolds. If ψ is an arithmetic sentence, we will write $\tilde{\psi}$ for the corresponding interpreted group theoretic statement. Thus, we have $\text{Arith}_2 \models \psi$ if and only if $\text{Homeo}(M) \models \tilde{\psi}$ for all compact manifolds M ; here we use Arith_2 to denote second order arithmetic, as opposed to \mathbb{N} which usually denotes first order arithmetic. For a fixed Gödel numbering in arithmetic, the association $\#\psi \mapsto \#\tilde{\psi}$ is computable.

Let M be a fixed compact manifold and let ψ be a sentence in group theory. Recall that ψ *isolates* M if for all compact manifolds N , we have

$$\text{Homeo}(N) \models \psi \longleftrightarrow M \cong N.$$

Notice that if ψ isolates M then $\text{Homeo}(M) \models \psi$. Similarly, we will say ψ *isolates a manifold* if there is a unique compact manifold M such that $\text{Homeo}(M) \models \psi$.

Recall that Rice's theorem from computability theory asserts that if \mathcal{C} is a class of partial recursive functions then the set $\{n \mid \phi_n \in \mathcal{C}\}$ is computable if and only if \mathcal{C} is empty or equal to the whole class of partial recursive functions. Here, we have adopted the standard notation ϕ_n for the n -th function computed by the universal Turing machine. See Corollary 1.6.14 in [47].

Here we will prove two analogues of Rice's theorem for homeomorphism groups of manifolds. Let \mathcal{M} be a class of (homeomorphism classes) of compact manifolds. We will say that \mathcal{M} is *finitely axiomatized* if there is a first order sentence $\phi_{\mathcal{M}}$ in the language of group theory such that for all compact manifolds M , we have

$$M \in \mathcal{M} \Leftrightarrow \text{Homeo}(M) \models \phi_{\mathcal{M}};$$

in particular, $\phi_{\mathcal{M}}$ isolates precisely those manifolds M which lie in \mathcal{M} .

Theorem 6.1. *Let \mathcal{M} be a class of compact manifolds that is finitely axiomatized, and let*

$$\text{axiom}(\mathcal{M}) := \{\#\phi \mid \phi \text{ finitely axiomatizes } \mathcal{M}\}.$$

Then $\text{axiom}(\mathcal{M})$ is not definable in second order arithmetic.

The reader may note that [Theorem 6.1](#) implies that even the set of sentences which are *false* for all compact manifold homeomorphism groups (i.e., $\text{axiom}(\emptyset)$) is so complicated as to be undefinable in second order arithmetic.

Even more generally, let \mathcal{A} denote the set of all homeomorphism classes of compact manifolds, and let \mathcal{F} denote the set of nonempty subsets of \mathcal{A} that are finitely axiomatized by first order sentences in the language of group theory.

Theorem 6.2. *Let $A \subseteq \mathcal{F}$ be nonempty and proper. Then the set*

$$\chi(A) = \{\#\psi \mid \psi \text{ finitely axiomatizes some } a \in A\}$$

is not definable in second order arithmetic.

Before giving the proof of [Theorem 6.2](#), we note that it implies [Theorem 6.1](#), as well as [Theorem 1.7](#) from the introduction.

Proof of [Theorem 6.1](#). Suppose first that $A = \mathcal{M}$ is nonempty and finitely axiomatized. We have $A \neq \mathcal{F}$ because A is a subset of \mathcal{A} and because each of the countably infinitely many singletons of \mathcal{A} is finitely axiomatized; this is part of the content of [Theorem 2.3](#). By [Theorem 6.2](#), we have that $\chi(A) = \text{axiom}(\mathcal{M})$ is not definable in second order arithmetic.

To see that $\text{axiom}(\emptyset)$ is not definable in second order arithmetic, we simply note that for all arithmetic sentences ψ , we have $\#\tilde{\psi} \in \text{axiom}(\emptyset)$ if and only if ψ is false in Arith_2 . This violates the undefinability of truth. \square

Proof of Theorem 1.7. Let M be a fixed compact manifold. The undefinability of the set Sent_M is precisely the conclusion of [Theorem 6.1](#) when $\mathcal{M} = \{M\}$.

For the undefinability of Sent , we note that if ϕ isolates some compact manifold M then for all arithmetic sentences ψ , we have $\phi \wedge \tilde{\psi}$ isolates some compact manifold M if and only if $\text{Arith}_2 \models \psi$; this is simply because $\neg\tilde{\psi}$ is always false in compact manifolds homeomorphism groups, and $\neg\phi$ isolates no compact manifold because there are at least three pairwise non-homeomorphic compact manifolds. Thus, if Sent_M were definable then we would be able to define truth in Arith_2 , a contradiction. \square

To add to the complexity of the sets Sent_M and Sent , note that it is well-known that there is a Diophantine equation which does not admit a solution if and only if ZFC is consistent (or, if and only if PA is consistent); cf. Chapter 6 of [\[44\]](#). For such an equation, we may express the nonexistence of a solution to a particular Diophantine equation as a sentence ϕ in first order arithmetic. Interpreting this sentence in $\text{Homeo}(M)$ to get a group theoretic sentence $\tilde{\phi}$, we see that if ψ isolates M then $\psi \wedge \tilde{\phi}$ isolates M if and only if ZFC is consistent (or, if and only if PA is consistent). A similar argument works for sentences isolating some manifold. Thus, for a particular Gödel numbering, there are numbers whose membership in Sent_M and Sent cannot be proved in ZFC.

We finally establish [Theorem 6.2](#).

Proof of Theorem 6.2. Let $\phi \in \chi(A)$ finitely axiomatize some $a \in A$ and let θ finitely axiomatize some $\emptyset \neq b \notin A$; the sentence θ exists since A is assumed to be proper. For each arithmetic sentence ψ , we let

$$\psi^* := (\tilde{\psi} \wedge \phi) \vee (\neg\tilde{\psi} \wedge \theta).$$

Notice that $\#\psi^* \in \chi(A)$ if and only if $\text{Arith}_2 \models \psi$. Indeed, if ψ is true in arithmetic then $\tilde{\psi}$ is true for all compact manifolds and $\neg\tilde{\psi} \wedge \theta$ is false for all compact manifolds. In this case, ψ^* is true in $\text{Homeo}(M)$ if and only if ϕ holds in $\text{Homeo}(M)$, in which case $\#\psi^* \in \chi(A)$.

Conversely, suppose that ψ is false in arithmetic. Then $\tilde{\psi} \wedge \phi$ is false for all compact manifolds, and so ψ^* is true for $\text{Homeo}(M)$ if and only if $\text{Homeo}(M) \models \theta$, in which case $M \in b \notin A$. It follows that $\#\psi^* \notin \chi(A)$.

Thus, if $\chi(A)$ were definable in second order arithmetic then we could define truth, a contradiction. \square

[Theorem 6.2](#) has many other consequences regarding undefinability. As a single example, a finite list of compact manifolds is finitely axiomatized, in view of [Theorem 2.3](#); the set of sentences axiomatizing finite collections of manifolds is itself undefinable.

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