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**EVALUATION 2-FUNCTORS FOR KAC-MOODY  
2-CATEGORIES OF TYPE  $A_2$**

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## EVALUATION 2-FUNCTORS FOR KAC–MOODY 2-CATEGORIES OF TYPE $A_2$

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**We construct a 2-functor from the Kac–Moody 2-category for the extended quantum affine  $\mathfrak{sl}_3$  to the homotopy 2-category of bounded chain complexes with values in the Kac–Moody 2-category for quantum  $\mathfrak{gl}_3$ , categorifying the evaluation map between the corresponding quantum Kac–Moody algebras. Our approach establishes and exploits a categorical analogue of the well-known relation between the evaluation map and Lusztig’s internal braid group action for quantum  $\mathfrak{gl}_3$ .**

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### 1. Introduction

In the 90s, Chari and Pressley launched a systematic study of finite-dimensional representations of quantum affine algebras, starting with affine  $\mathfrak{sl}_2$  in [4]. Since then, these representations have been studied intensively and continue to be an active research topic with important open questions and interesting links to other research areas, e.g., mathematical physics and cluster algebras; see [8], for example, for more information.

In affine type  $A$ , there is a special class of irreducible finite-dimensional representations, the so-called *evaluation representations*. These are obtained by pulling back irreducible representations of finite type  $A$  through a so-called *evaluation map*, which is an algebra homomorphism  $\text{ev}_{a,n} : \mathbf{U}_\Delta(n) \rightarrow \mathbf{U}(n)$ , where  $a \in \mathbb{C}^\times$  is

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a scalar,  $\mathbf{U}_\Delta(n)$  is the so-called *extended quantum affine*  $\mathfrak{sl}_n$  and  $\mathbf{U}(n) = \mathbf{U}_q(\mathfrak{gl}_n)$ , see Section 2. As we will recall in more detail in that section, the level zero weight lattice of the former quantum affine Kac–Moody algebra can be identified with the  $\mathfrak{gl}_n$ -weight lattice. The fact that we have to pass from  $\mathfrak{sl}_n$  to  $\mathfrak{gl}_n$  is important and has a categorical counterpart, as we will explain below. For more information on evaluation maps and evaluation representations in general, see, e.g., [6; 5; 7]. Since we wish to categorify this construction, we must pass to the idempotent forms of these algebras, which can be considered as categories, and the evaluation map can therefore be considered as a functor.

Quantum Kac–Moody algebras were *categorified* by Khovanov and Lauda [10], and independently by Rouquier [18]. We call these 2-categories *Kac–Moody 2-categories* after [3]. The ones of interest to us in this paper are  $\tilde{\mathbf{U}}_\Delta(n)$  and  $\tilde{\mathbf{U}}(n)$ , which categorify  $\mathbf{U}_\Delta(n)$  and  $\mathbf{U}(n)$ , respectively. The tilde indicates that our choice of signs in their definition differs from Khovanov and Lauda’s original choices; see below for more comments on this. In finite Dynkin types, all irreducible finite-dimensional representations can be categorified by certain quotients of the Kac–Moody 2-categories, which nowadays go under the name of *cyclotomic KLR algebras*. In other Dynkin types, such as affine Dynkin types, this is not true. In particular, evaluation representations in affine type  $A$  cannot be categorified by cyclotomic KLR-algebras, because the latter categorify highest weight representations and evaluation representations do not have a highest weight. However, we conjecture that the evaluation map (considered as an evaluation functor)  $\text{ev}_n^t := \text{ev}_{q^t, n}$ , for any  $t \in \mathbb{Z}$  and  $n \in \mathbb{N}_{>2}$ , can be categorified by an evaluation 2-functor  $\mathcal{E}_{v_n^t}: \tilde{\mathbf{U}}_\Delta(n) \rightarrow K^b(\tilde{\mathbf{U}}(n))$ , which can be used to define evaluation 2-representations (i.e., categorified evaluation representations) of  $\tilde{\mathbf{U}}_\Delta(n)$  by pulling back “irreducible” 2-representations (i.e., cyclotomic KLR algebras) of  $\tilde{\mathbf{U}}(n)$ . Here  $K^b(\tilde{\mathbf{U}}(n))$  denotes the homotopy 2-category of bounded complexes in  $\tilde{\mathbf{U}}(n)$ , so the 1-morphisms of  $\tilde{\mathbf{U}}_\Delta(n)$  act by composing with bounded complexes in  $\tilde{\mathbf{U}}(n)$ . As a matter of fact, we not only conjecture  $\mathcal{E}_{v_n^t}$  to exist but also an extension of it to  $K^b(\tilde{\mathbf{U}}_\Delta(n))$ .

In this paper, we prove the first conjecture for  $\tilde{\mathbf{U}}_\Delta(3)$  and hope that it serves as the base case for an inductive proof for  $\tilde{\mathbf{U}}_\Delta(n)$ , when  $n > 3$ , in a forthcoming paper. Proving that there is no obstruction to extending  $\mathcal{E}_{v_n^t}$  to  $K^b(\tilde{\mathbf{U}}_\Delta(n))$  is not easy and certainly beyond the scope of this paper and its sequel.

There are two good reasons for publishing the case  $n = 3$  separately. Firstly, in this case there is a close relation with the categorification of the internal braid group action on  $\mathbf{U}_q(\mathfrak{gl}_3)$  in [1] (strictly speaking, in that paper they consider  $\mathbf{U}_q(\mathfrak{sl}_3)$ , so part of our work consists in adapting their results to our setting - see the following paragraph for more details). This is the categorical analogue of a relation between the evaluation map and the braid group action on the decategorified level, which is

certainly known to experts, although we couldn't find a reference in the literature. We therefore spell it out in Section 2.2.1, because it is not completely straightforward. Its categorification is conceptually clear, but requires solving multiple nontrivial sign problems, which we do by using certain 2-isomorphisms. This is also why we define two versions of the evaluation 2-functor, denoted  $\mathcal{E}_\nu$  and  $\mathcal{E}_{\nu'}$ , respectively. The former uses relatively nice sign conventions, whereas the signs in the definition of  $\mathcal{E}_{\nu'}$  are much more complicated. However, the latter are easier to match with the signs in the categorified internal braid group action (for our choice of signs in  $\tilde{U}_\Delta(3)$  and  $\tilde{U}(3)$ ), which is necessary to prove that  $\mathcal{E}_{\nu'}$  is well-defined in our approach; see Theorem 4.3 and its proof in Section 6.4. The relation between  $\mathcal{E}_\nu$  and  $\mathcal{E}_{\nu'}$ , given in Lemma 4.4, guarantees that well-definedness of the latter implies well-definedness of the former. In principle, all of this should also work for  $n > 3$ , but only if the categorified braid group action extends to  $K^b(\tilde{U}(n))$  (to include the action of longer braids), which has been conjectured to be the case but not yet proved (see [1, Conjecture 1.2]). This is why our approach for  $n > 3$  will be completely different. We hope that presenting the base case  $n = 3$  here will prepare the ground for the general case and also keep the size of the forthcoming paper within reasonable bounds.

The second reason for publishing this case separately, is that it reveals the need to pass from  $\mathfrak{sl}_n$  to  $\mathfrak{gl}_n$  once more, but now on the categorical level. Recall that the definition of a Kac–Moody 2-category depends on a choice of invertible scalars and compatible bubble parameters; see, e.g., [13]. In finite type  $A$  all choices yield essentially the same 2-category, i.e., up to 2-isomorphism, but in affine type  $A$  they don't. In particular, Khovanov and Lauda's original affine type  $A$  *unsigned* Kac–Moody 2-category in [10], with all scalars and bubble parameters equal to one, and the Kac–Moody 2-category defined in [15], with nontrivial bubble parameters depending on level zero  $\widehat{\mathfrak{gl}}_n$ -weights (instead of level zero  $\widehat{\mathfrak{sl}}_n$ -weights), are not 2-isomorphic when  $n$  is odd. This was mentioned in [11] without proof and, therefore, we prove it in Theorem A.4. Although this does not by itself imply that there is no evaluation 2-functor for trivial scalars and bubble parameters when  $n = 3$ , we failed to find one. More generally, it seems that one is forced to use the scalars and level zero  $\widehat{\mathfrak{gl}}_n$ -bubble parameters from [15] when  $n$  is odd. When  $n$  is even, everything is simpler because in that case both choices of scalars and bubble parameters yield essentially the same Kac–Moody 2-category, see Section 6.4.

There is an analogous story for the affine Hecke algebra and its finite-dimensional representations. The categorification of the corresponding evaluation map was carried out in [17] and was technically less challenging than the categorification of the evaluation map for the affine type  $A$  Kac–Moody algebra. In both cases, the target (2-)category of the evaluation (2-)functor is a homotopy category of bounded complexes and, as was argued in [17], one motivation for defining and studying

evaluation 2-representations is that they might provide some important clues for the development of triangulated 2-representation theory, which at the moment is very poorly understood, even at the most basic level. For example, it was shown in [17] that every evaluation 2-representation of extended affine Soergel bimodules has a *finitary cover*, somehow relating finitary and triangulated 2-representations. The same might hold for the evaluation 2-representations of  $\tilde{U}_\Delta(n)$ , but that question is outside the scope of this paper. Also in both cases, one would like to categorify tensor products of evaluation representations, which play a fundamental role in the finite-dimensional representation theory of the affine quantum algebras in question; see, e.g., [5, Chapter 12, Section 2C] for the case of  $\tilde{U}_\Delta(n)$ . However, it is far from clear how to do that at this point. Perhaps it is possible to somehow adapt Webster's tensor algebras of Stendhal diagrams [19] in that case. We hope to address these and some other interesting questions about evaluation 2-representations in the future.

The structure of the paper is as follows. Section 2 reviews the evaluation map/functor  $\text{ev}_3^t$  and Section 3 presents the definitions of the affine and finite type A Kac–Moody 2-categories  $\tilde{U}_\Delta(3)$  and  $\tilde{U}(3)$  that we will be working with. In Section 4 we define the two evaluation 2-functors  $\mathcal{E}_v$  and  $\mathcal{E}_{v'}$  and prove their relationship to each other. We translate the categorified braid group actions to our choice of scalars in Section 5, and then in Section 6 we prove Theorem 4.3, saying that  $\mathcal{E}_{v'}$  is a well-defined 2-functor that decategorifies to  $\text{ev}_3^t$ , from which Theorem 4.1 follows. We finish the paper with Section 6.4, where we justify our choice of the scalars and bubble parameters in the definition of  $\tilde{U}_\Delta(3)$  over a choice in [10] by proving in Theorem A.4 that the two choices are not related by a 2-isomorphism that fixes objects and 1-morphisms.

## 2. The decategorified setting

Our main reference for this section is [7], though the evaluation map was first considered in [9]. Note that we are interested in the idempotent version of some of the quantum algebras in that paper, so we have to adapt Du and Fu's definitions. We use the idempotent versions because these are the ones that are categorified by Kac–Moody 2-categories.

**2.1. Finite type  $\dot{U}(n)$  and affine type  $\dot{U}_\Delta(n)$  of level zero.** Throughout this paper we identify both the (integral)  $\mathfrak{gl}_n$ -weight lattice and the level-zero (integral)  $\widehat{\mathfrak{gl}}_n$ -weight lattice with  $\mathbb{Z}^n$ , denoting either sort of (integral) weight by, e.g.,  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ . The simple  $\widehat{\mathfrak{gl}}_n$ -roots  $\alpha_1, \dots, \alpha_n$  are then given by

$$\alpha_i = \begin{cases} (0, \dots, 0, 1, -1, 0, \dots, 0) & \text{if } 1 \leq i \leq n-1, \\ (-1, 0, \dots, 0, 1) & \text{if } i = n, \end{cases}$$

where the 1 is always the  $i$ -th entry. Note that  $\alpha_1, \dots, \alpha_{n-1}$  are the simple  $\mathfrak{gl}_n$ -roots.

Under this identification, the bilinear form on these weight lattices corresponds to the Euclidean inner product on  $\mathbb{Z}^n$ . Its restriction to the root lattices then reads

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i \equiv j \pm 1 \pmod{n}, \\ 0 & \text{else,} \end{cases}$$

for  $1 \leq i, j \leq n$ . Note that, in the affine case, the indices  $1, \dots, n$  are interpreted as representatives of the residue classes modulo  $n$ . From now on, we will always tacitly use this interpretation of the indices of affine weights and roots. We also recall the standard notation  $i \cdot j := (\alpha_i, \alpha_j)$ , which we will often use below.

Finally, given  $\lambda \in \mathbb{Z}^n$ , define  $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_n) \in \mathbb{Z}^n$ , where  $\bar{\lambda}_i = \lambda_i - \lambda_{i+1}$  for all  $i = 1, \dots, n$ . By the above convention for affine weights, we have  $\bar{\lambda}_n = \lambda_n - \lambda_1$ , so  $\bar{\lambda}_1 + \dots + \bar{\lambda}_n = 0$ . In other words,  $\bar{\lambda}$  belongs to a rank  $n - 1$  sublattice of  $\mathbb{Z}^n$ , which can be identified with the level-zero integral  $\widehat{\mathfrak{sl}}_n$ -weight lattice. The element  $(\bar{\lambda}_1, \dots, \bar{\lambda}_{n-1}) \in \mathbb{Z}^{n-1}$  can then be identified with an (integral)  $\mathfrak{sl}_n$ -weight.

Recall that the quantum integer  $[m]$ , for  $m \in \mathbb{Z}$ , is defined as

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

**Definition 2.1.** The *idempotent extended quantum affine  $\mathfrak{sl}_n$* , denoted by  $\dot{\mathbf{U}}_{\Delta}(n)$ , is the associative idempotent  $\mathbb{Q}(q)$ -algebra generated by  $1_{\lambda}$ ,  $E_i 1_{\lambda}$  and  $F_i 1_{\lambda}$ , for  $\lambda \in \mathbb{Z}^n$  and  $i = 1, \dots, n$ , subject to the relations

$$\begin{aligned} 1_{\lambda} 1_{\mu} &= \delta_{\lambda, \mu} 1_{\lambda}, \\ E_i 1_{\lambda} 1_{\lambda'} &= \delta_{\lambda, \lambda'} E_i 1_{\lambda}, \\ F_i 1_{\lambda} 1_{\lambda'} &= \delta_{\lambda, \lambda'} F_i 1_{\lambda}, \\ 1_{\mu} E_i 1_{\lambda} &= \delta_{\mu, \lambda + \alpha_i} E_i 1_{\lambda}, \\ 1_{\mu} F_i 1_{\lambda} &= \delta_{\mu, \lambda - \alpha_i} F_i 1_{\lambda}, \\ E_i F_j 1_{\lambda} - F_j E_i 1_{\lambda} &= \delta_{i, j} [\bar{\lambda}_i] 1_{\lambda}, \\ E_i E_j 1_{\lambda} &= E_j E_i 1_{\lambda} && \text{if } i \cdot j = 0, \\ F_i F_j 1_{\lambda} &= F_j F_i 1_{\lambda} && \text{if } i \cdot j = 0, \\ E_i^2 E_j 1_{\lambda} + E_j E_i^2 1_{\lambda} &= [2] E_i E_j E_i 1_{\lambda} && \text{if } i \cdot j = -1, \\ F_i^2 F_j 1_{\lambda} + F_j F_i^2 1_{\lambda} &= [2] F_i F_j F_i 1_{\lambda} && \text{if } i \cdot j = -1. \end{aligned}$$

Note that  $E_i 1_{\lambda} = 1_{\lambda + \alpha_i} E_i 1_{\lambda}$ , so we can use the notation  $E_i E_j 1_{\lambda} := E_i 1_{\lambda + \alpha_j} \cdot E_j 1_{\lambda}$  without ambiguity. Similarly, we will use the notation  $1_{\mu} E_i = 1_{\mu} E_i 1_{\mu - \alpha_i}$  and  $1_{\mu} F_i = 1_{\mu} F_i 1_{\mu + \alpha_i}$ , so that  $E_i 1_{\lambda} = 1_{\lambda + \alpha_i} E_i$  and  $F_i 1_{\lambda} = 1_{\lambda - \alpha_i} F_i$ .

**Definition 2.2.** The *idempotent quantum*  $\mathfrak{gl}_n$ , denoted by  $\dot{\mathbf{U}}(n)$ , is the idempotent subalgebra of  $\dot{\mathbf{U}}_\Delta(n)$  generated by  $1_\lambda$ ,  $E_i 1_\lambda$  and  $F_i 1_\lambda$ , for  $i = 1, \dots, n-1$  and  $\lambda \in \mathbb{Z}^n$ .

Note that  $\dot{\mathbf{U}}_\Delta(n)$  and  $\dot{\mathbf{U}}(n)$  share the same idempotents, but, whereas  $\dot{\mathbf{U}}(n) = \dot{\mathbf{U}}(\mathfrak{gl}_n)$ , the idempotent algebra  $\dot{\mathbf{U}}_\Delta(n)$  is only an idempotent subalgebra of  $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$ , which is why it is called the idempotent *extended* quantum affine  $\mathfrak{sl}_n$  and not the idempotent quantum affine  $\mathfrak{gl}_n$ . See [7, Section 2] for more details.

**Remark 2.3.** Recall that these idempotent algebras can be seen as linear categories whose object sets are given by the sets of weights and whose hom-spaces are given by, e.g.,

$$\mathrm{Hom}_{\dot{\mathbf{U}}(n)}(\lambda, \mu) = 1_\mu \dot{\mathbf{U}}(n) 1_\lambda$$

with composition corresponding to multiplication. This is why these idempotent algebras are categorified by 2-categories rather than categories.

**2.2. Evaluation maps.** Fix  $t \in \mathbb{Z}$  and let  $[X, Y]_{q^{\pm 1}} = XY - q^{\pm 1}YX$  be the  $q^{\pm 1}$ -commutator. From now on we will always assume that  $n > 2$ .

**Definition 2.4.** The *evaluation map*  $\mathrm{ev}_n^t: \dot{\mathbf{U}}_\Delta(n) \rightarrow \dot{\mathbf{U}}(n)$  is the homomorphism of idempotent algebras defined by

- (1)  $\mathrm{ev}_n^t(1_\lambda) = 1_\lambda$ ,
- (2)  $\mathrm{ev}_n^t(E_i 1_\lambda) = E_i 1_\lambda$  for  $i \neq n$ ,
- (3)  $\mathrm{ev}_n^t(F_i 1_\lambda) = F_i 1_\lambda$  for  $i \neq n$ ,
- (4)  $\mathrm{ev}_n^t(E_n 1_\lambda) = q^{\lambda_1 + \lambda_n + t - 1} [\dots [[F_1, F_2]_q, F_3]_q \dots]_q, F_{n-1} 1_\lambda$ ,
- (5)  $\mathrm{ev}_n^t(F_n 1_\lambda) = q^{-\lambda_1 - \lambda_n - t + 1} [E_{n-1}, [E_{n-2}, [\dots [E_2, E_1]_{q^{-1}}]_{q^{-1}} \dots]_{q^{-1}}]_{q^{-1}} 1_\lambda$ .

**Remark 2.5.** (a) Note that

$$[\dots [[F_1, F_2]_q, F_3]_q \dots]_q, F_{n-1} 1_\lambda = 1_{\lambda - \alpha_1 - \dots - \alpha_{n-1}} [\dots [[F_1, F_2]_q, F_3]_q \dots]_q, F_{n-1} 1_\lambda,$$

so  $\mathrm{ev}_n^t(E_n 1_\lambda) = \mathrm{ev}_n^t(1_{\lambda + \alpha_n} E_n 1_\lambda)$  is well defined, because  $\alpha_1 + \dots + \alpha_{n-1} + \alpha_n = 0$ . The same is true for  $\mathrm{ev}_n^t(F_n 1_\lambda) = \mathrm{ev}_n^t(1_{\lambda - \alpha_n} F_n 1_\lambda)$ .

(b) When we consider the idempotent algebras as categories as in Remark 2.3,  $\mathrm{ev}_n^t$  becomes a linear functor. This is why it is categorified by a 2-functor rather than a functor.

The expressions for  $\mathrm{ev}_n^t(E_n 1_\lambda)$  and  $\mathrm{ev}_n^t(F_n 1_\lambda)$  in (4) and (5) can be written as alternating sums, which will be important later on. For  $\xi = (\xi_1, \dots, \xi_{n-2}) \in$

$\{0, 1\}^{n-2}$  set

$$(6) \quad E_\xi 1_\lambda := E_{n-1}^{1-\xi_{n-2}} E_{n-2}^{1-\xi_{n-3}} \cdots E_2^{1-\xi_1} E_1 E_2^{\xi_1} \cdots E_{n-2}^{\xi_{n-3}} E_{n-1}^{\xi_{n-2}} 1_\lambda,$$

$$(7) \quad F_\xi 1_\lambda := F_{n-1}^{\xi_{n-2}} F_{n-2}^{\xi_{n-3}} \cdots F_2^{\xi_1} F_1 F_2^{1-\xi_1} \cdots F_{n-2}^{1-\xi_{n-3}} F_{n-1}^{1-\xi_{n-2}} 1_\lambda.$$

and let  $|\xi| = \xi_1 + \cdots + \xi_{n-2}$ . The following can be obtained by direct computation.

**Lemma 2.6.** *We have*

$$(8) \quad \text{ev}_n^t(E_n 1_\lambda) = q^{\lambda_1 + \lambda_n + t - 1} \sum_{\xi \in \{0,1\}^{n-2}} (-q)^{|\xi|} F_\xi 1_\lambda,$$

$$(9) \quad \text{ev}_n^t(F_n 1_\lambda) = q^{-\lambda_1 - \lambda_n - t + 1} \sum_{\xi \in \{0,1\}^{n-2}} (-q)^{-|\xi|} E_\xi 1_\lambda.$$

For more details on the evaluation map, see [7, Section 5].

**2.2.1. Connection with the braid group action for  $n = 3$ .** For each  $i = 1, \dots, n-1$  and  $e = \pm 1$ , Lusztig defined algebra automorphisms  $T'_{i,e}$  and  $T''_{i,e}$  of  $\dot{\mathbf{U}}(\mathfrak{sl}_n)$ ; see, e.g., [14, Section 37.1] for their definition, which we can adapt to  $\dot{\mathbf{U}}(n)$  without issue. The two automorphisms are related by the equation  $(T'_{i,e})^{-1} = T''_{i,-e}$  (see [14, Proposition 37.1.2]) and, for a fixed choice of  $e$ , the  $T'_{1,e}, \dots, T'_{n-1,e}$ , resp. the  $T''_{1,e}, \dots, T''_{n-1,e}$ , satisfy the braid relations (see [14, Theorem 39.4.3]) and, therefore, define two actions of the braid group  $B_n$  on  $\dot{\mathbf{U}}(n)$ , called the *internal braid group actions*.

Let  $n = 3$  and  $t \in \mathbb{Z}$ , and set  $\text{ev} = \text{ev}_3^t$ . Comparison of the expressions in [14, Subsection 37.1.3] with the ones in Definition 2.4 shows that  $\text{ev}$  can be partially expressed in terms of the above algebra automorphisms. For  $i = 1, 3$  and  $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$ , we have

$$\begin{aligned} \text{ev}(E_3 1_\lambda) &= q^{\lambda_1 + \lambda_3 + t - 1} T'_{1,-1}(F_2 1_{s_1(\lambda)}), \\ \text{ev}(F_3 1_\lambda) &= q^{-\lambda_1 - \lambda_3 - t + 1} T'_{1,-1}(E_2 1_{s_1(\lambda)}), \\ \text{ev}(E_1 1_\lambda) &= -q^{\lambda_1 - \lambda_2} T'_{1,-1}(F_1 1_{s_1(\lambda)}), \\ \text{ev}(F_1 1_\lambda) &= -q^{-\lambda_1 + \lambda_2 + 2} T'_{1,-1}(E_1 1_{s_1(\lambda)}), \end{aligned}$$

where  $s_1(\lambda) = (\lambda_2, \lambda_1, \lambda_3)$ . For  $i = 2, 3$  and  $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$ , we have

$$\begin{aligned} \text{ev}(E_3 1_\lambda) &= q^{\lambda_1 + \lambda_3 + t - 1} T''_{2,1}(F_1 1_{s_2(\lambda)}), \\ \text{ev}(F_3 1_\lambda) &= q^{-\lambda_1 - \lambda_3 - t + 1} T''_{2,1}(E_1 1_{s_2(\lambda)}), \\ \text{ev}(E_2 1_\lambda) &= -q^{-\lambda_2 + \lambda_3 + 2} T''_{2,1}(F_2 1_{s_2(\lambda)}), \\ \text{ev}(F_2 1_\lambda) &= -q^{\lambda_2 - \lambda_3} T''_{2,1}(E_2 1_{s_2(\lambda)}), \end{aligned}$$

where  $s_2(\lambda) = (\lambda_1, \lambda_3, \lambda_2)$ . Using the fact that  $T'_{1,-1}$  and  $T'_{2,1}$  are well-defined algebra automorphisms of  $\dot{\mathbf{U}}(3)$ , it is easy to prove that  $\text{ev}: \dot{\mathbf{U}}_{\Delta}(3) \rightarrow \dot{\mathbf{U}}(3)$  is a well-defined algebra homomorphism. Specifically, the fact that  $T'_{1,-1}$  is an algebra automorphism implies that  $\text{ev}$  preserves the relations in Definition 2.1 for  $i = 1, 3$ , the fact that  $T''_{2,1}$  is an algebra automorphism implies that  $\text{ev}$  preserves the relations in Definition 2.1 for  $i = 2, 3$ , and  $\text{ev}$  preserves the relations in Definition 2.1 for  $i = 1, 2$  by definition, of course. Since all relations in Definition 2.1 involve either one colour  $i$  or two colours  $i, j$ , and it's very easy to check that  $\text{ev}$  preserves the one-colour relations directly, we see that  $\text{ev}$  preserves all relations in  $\dot{\mathbf{U}}_{\Delta}(3)$  and is therefore a well-defined algebra homomorphism.

Of course, one can also prove that  $\text{ev}$  preserves the relations in  $\dot{\mathbf{U}}_{\Delta}(3)$  directly, but that is besides the point. To show that the evaluation 2-functor  $\mathcal{E}_v$  preserves the relations in  $\tilde{\mathbf{U}}_{\Delta}(3)$ , we will follow the same reasoning as above for all one- and two-colour KLR relations, taking advantage of the categorification of  $T'_{i,1}$  in [1]. For the three-colour KLR relations, the results in that paper cannot be used and we will give a direct proof.

### 3. Kac–Moody 2-categories

We will move on to recalling in detail the 2-categories  $\tilde{\mathbf{U}}(n)$  and  $\tilde{\mathbf{U}}_{\Delta}(n)$  as defined in [16, Definition 3.1] and [15, Definition 3.19], respectively. These decategorify to  $\dot{\mathbf{U}}(n)$  and  $\dot{\mathbf{U}}_{\Delta}(n)$ .

**3.1. Definition.** We define  $\tilde{\mathbf{U}}_{\Delta}(n)$  and  $\tilde{\mathbf{U}}(n)$  simultaneously, because only the range of the indices of the 1-morphisms and of the colours of the 2-morphisms differ. For concreteness, we will work over  $\mathbb{Q}$ , but any field of characteristic zero would serve.

**Definition 3.1.** The 2-category  $\tilde{\mathbf{U}}_{\Delta}(n)$  (resp.  $\tilde{\mathbf{U}}(n)$ ) is the graded  $\mathbb{Q}$ -linear 2-category with:

- Objects:  $\lambda \in \mathbb{Z}^n$ .
- 1-morphisms: formal direct sums of shifts of

$$\mathbf{1}_{\lambda}, \quad \mathcal{E}_i \mathbf{1}_{\lambda} = \mathbf{1}_{\lambda + \alpha_i} \mathcal{E}_i \mathbf{1}_{\lambda} = \mathbf{1}_{\lambda + \alpha_i} \mathcal{E}_i, \quad \mathcal{F}_i \mathbf{1}_{\lambda} = \mathbf{1}_{\lambda - \alpha_i} \mathcal{F}_i \mathbf{1}_{\lambda} = \mathbf{1}_{\lambda - \alpha_i} \mathcal{F}_i,$$

for  $\lambda \in \mathbb{Z}^n$  and for  $i \in \{1, \dots, n\}$  (resp.  $i \in \{1, \dots, n-1\}$ ).

- 2-morphisms: equivalence classes of  $\mathbb{Q}$ -linear combinations of diagrams obtained by horizontally concatenating and vertically gluing the generators below. By convention, a 2-morphism  $\alpha: X \langle r \rangle \rightarrow Y \langle s \rangle$ , for  $r, s \in \mathbb{Z}$ , is given by a linear combination of homogeneous diagrams of degree  $s - r$ , as defined in [10].

$$\begin{array}{c} \lambda + \alpha_i \\ \uparrow \\ \bullet \\ \downarrow \\ i \end{array} \lambda : \mathcal{E}_i \mathbf{1}_{\lambda} \rightarrow \mathcal{E}_i \mathbf{1}_{\lambda} \langle 2 \rangle, \quad \begin{array}{c} \lambda - \alpha_i \\ \downarrow \\ \bullet \\ \downarrow \\ i \end{array} \lambda : \mathcal{F}_i \mathbf{1}_{\lambda} \rightarrow \mathcal{F}_i \mathbf{1}_{\lambda} \langle 2 \rangle,$$

$$\begin{array}{ll}
 \begin{array}{c} \color{red}{\nearrow} \\ \color{blue}{\searrow} \\ i \quad j \end{array} \lambda : \mathcal{E}_i \mathcal{E}_j \mathbf{1}_\lambda \rightarrow \mathcal{E}_j \mathcal{E}_i \mathbf{1}_\lambda \langle -i \cdot j \rangle, & \begin{array}{c} \color{blue}{\nearrow} \\ \color{red}{\searrow} \\ i \quad j \end{array} \lambda : \mathcal{F}_i \mathcal{F}_j \mathbf{1}_\lambda \rightarrow \mathcal{F}_j \mathcal{F}_i \mathbf{1}_\lambda \langle -i \cdot j \rangle, \\
 \begin{array}{c} \color{red}{\nearrow} \\ \color{blue}{\searrow} \\ i \quad j \end{array} \lambda : \mathcal{E}_i \mathcal{F}_j \mathbf{1}_\lambda \rightarrow \mathcal{F}_j \mathcal{E}_i \mathbf{1}_\lambda, & \begin{array}{c} \color{blue}{\nearrow} \\ \color{red}{\searrow} \\ i \quad j \end{array} \lambda : \mathcal{F}_i \mathcal{E}_j \mathbf{1}_\lambda \rightarrow \mathcal{E}_j \mathcal{F}_i \mathbf{1}_\lambda, \\
 \begin{array}{c} \color{red}{\curvearrowright} \\ i \end{array} \lambda : \mathbf{1}_\lambda \rightarrow \mathcal{F}_i \mathcal{E}_i \mathbf{1}_\lambda \langle 1 + \bar{\lambda}_i \rangle, & \begin{array}{c} \color{red}{\curvearrowleft} \\ i \end{array} \lambda : \mathbf{1}_\lambda \rightarrow \mathcal{E}_i \mathcal{F}_i \mathbf{1}_\lambda \langle 1 - \bar{\lambda}_i \rangle, \\
 \begin{array}{c} \color{blue}{\curvearrowright} \\ i \end{array} \lambda : \mathcal{F}_i \mathcal{E}_i \mathbf{1}_\lambda \rightarrow \mathbf{1}_\lambda \langle 1 + \bar{\lambda}_i \rangle, & \begin{array}{c} \color{blue}{\curvearrowleft} \\ i \end{array} \lambda : \mathcal{E}_i \mathcal{F}_i \mathbf{1}_\lambda \rightarrow \mathbf{1}_\lambda \langle 1 - \bar{\lambda}_i \rangle.
 \end{array}$$

The equivalence relation is defined by the equations below.

(KM1) Right and left adjunction:

$$(10) \quad \begin{array}{c} \color{red}{\uparrow} \\ \color{red}{\curvearrowright} \\ i \end{array} \lambda = \begin{array}{c} \color{red}{\uparrow} \\ \color{red}{\downarrow} \\ i \end{array} \lambda = \begin{array}{c} \color{red}{\uparrow} \\ \color{red}{\curvearrowleft} \\ i \end{array} \lambda \quad \begin{array}{c} \color{red}{\downarrow} \\ \color{red}{\curvearrowleft} \\ i \end{array} \lambda = \begin{array}{c} \color{red}{\downarrow} \\ \color{red}{\downarrow} \\ i \end{array} \lambda = \begin{array}{c} \color{red}{\downarrow} \\ \color{red}{\curvearrowright} \\ i \end{array} \lambda$$

(KM2) Dot cyclicity:

$$(11) \quad \begin{array}{c} \color{red}{\curvearrowright} \\ \bullet \\ \color{red}{\downarrow} \\ i \end{array} \lambda = \begin{array}{c} \color{red}{\downarrow} \\ \bullet \\ \color{red}{\downarrow} \\ i \end{array} \lambda = \begin{array}{c} \color{red}{\downarrow} \\ \bullet \\ \color{red}{\curvearrowright} \\ i \end{array} \lambda$$

(KM3) Crossing cyclicity:

$$(12) \quad \begin{array}{c} \color{red}{\curvearrowright} \\ \color{blue}{\curvearrowright} \\ \color{red}{\downarrow} \\ i \end{array} \lambda = \begin{array}{c} \color{red}{\searrow} \\ \color{blue}{\nearrow} \\ i \quad j \end{array} \lambda = \begin{array}{c} \color{red}{\curvearrowleft} \\ \color{blue}{\curvearrowleft} \\ \color{red}{\downarrow} \\ i \end{array} \lambda$$

$$(13) \quad \begin{array}{c} \color{red}{\curvearrowleft} \\ \color{blue}{\curvearrowleft} \\ \color{red}{\downarrow} \\ i \end{array} \lambda = \begin{array}{c} \color{red}{\nearrow} \\ \color{blue}{\searrow} \\ i \quad j \end{array} \lambda = \begin{array}{c} \color{red}{\curvearrowright} \\ \color{blue}{\curvearrowright} \\ \color{red}{\downarrow} \\ i \end{array} \lambda$$

$$(14) \quad \begin{array}{c} \color{red}{\curvearrowright} \\ \color{blue}{\curvearrowright} \\ \color{red}{\downarrow} \\ i \end{array} \lambda = \begin{array}{c} \color{red}{\searrow} \\ \color{blue}{\nearrow} \\ i \quad j \end{array} \lambda = \begin{array}{c} \color{red}{\curvearrowleft} \\ \color{blue}{\curvearrowleft} \\ \color{red}{\downarrow} \\ i \end{array} \lambda$$

(KM4) Quadratic KLR:

$$(15) \quad \begin{array}{c} \text{Diagram: two strands } i \text{ and } j \text{ crossing twice.} \\ \lambda \end{array} = \begin{cases} \begin{array}{c} \text{Diagram: two parallel strands } i \text{ and } j. \\ \lambda \end{array} & \text{if } i \cdot j = 0, \\ \varepsilon(i, j) \left( \begin{array}{c} \text{Diagram: strand } i \text{ with a red dot, then strand } j. \\ \lambda \end{array} - \begin{array}{c} \text{Diagram: strand } j \text{ with a blue dot, then strand } i. \\ \lambda \end{array} \right) & \text{if } i \cdot j = -1, \end{cases}$$

$$\text{where } \varepsilon(i, j) = \begin{cases} 1 & \text{if } i = j + 1 \pmod{n}, \\ -1 & \text{if } i = j - 1 \pmod{n}, \\ 0 & \text{else.} \end{cases}$$

(KM5) Dot slide:

$$(16) \quad \begin{array}{c} \text{Diagram: strand } i \text{ with a red dot, then strand } j. \\ \lambda \end{array} - \begin{array}{c} \text{Diagram: strand } j \text{ with a blue dot, then strand } i. \\ \lambda \end{array} = \begin{array}{c} \text{Diagram: strand } i \text{ with a red dot, then strand } j. \\ \lambda \end{array} - \begin{array}{c} \text{Diagram: strand } j \text{ with a blue dot, then strand } i. \\ \lambda \end{array} = \begin{cases} \begin{array}{c} \text{Diagram: two parallel strands } i \text{ and } j. \\ \lambda \end{array} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(KM6) Cubic KLR:

$$(17) \quad \begin{array}{c} \text{Diagram: three strands } i, j, k \text{ crossing in a cubic pattern.} \\ \lambda \end{array} - \begin{array}{c} \text{Diagram: three strands } i, j, k \text{ crossing in a different cubic pattern.} \\ \lambda \end{array} = \begin{cases} \begin{array}{c} \text{Diagram: three parallel strands } i, j, k. \\ \lambda \end{array} & \text{if } i = k \text{ and } i \cdot j = -1, \\ 0 & \text{if } i \neq k \text{ or } i \cdot j \neq -1. \end{cases}$$

Before we list more relations, we introduce a shorthand using  $\clubsuit$  in lieu of  $\bullet$ :

$$(18) \quad \begin{array}{c} \text{Diagram: a circle with a dot at the bottom, labeled } i \text{ at the top and } +m \text{ at the bottom.} \\ \lambda \end{array} := \begin{array}{c} \text{Diagram: a circle with a dot at the bottom, labeled } i \text{ at the top and } \bar{\lambda}_i - 1 + m \text{ at the bottom.} \\ \lambda \end{array} \quad \begin{array}{c} \text{Diagram: a circle with a dot at the bottom, labeled } i \text{ at the top and } +m \text{ at the bottom.} \\ \lambda \end{array} := \begin{array}{c} \text{Diagram: a circle with a dot at the bottom, labeled } i \text{ at the top and } -\bar{\lambda}_i - 1 + m \text{ at the bottom.} \\ \lambda \end{array}$$

Using this notation, the other relations on diagrams are:

(KM7) Mixed EF:

$$(19) \quad \begin{array}{c} \text{Diagram: two strands } i \text{ and } j \text{ crossing twice.} \\ \lambda \end{array} = \begin{cases} \begin{array}{c} \text{Diagram: two parallel strands } i \text{ and } j. \\ \lambda \end{array} & \text{if } i \neq j, \\ \begin{array}{c} \text{Diagram: two parallel strands } i \text{ and } j. \\ \lambda \end{array} - \sum_{a+b+c=-\bar{\lambda}_i-1} \begin{array}{c} \text{Diagram: a circle with a dot at the bottom, labeled } i \text{ at the top and } +c \text{ at the bottom.} \\ \lambda \end{array} & \text{if } i = j, \end{cases}$$

$$(20) \quad \text{Diagram} = \begin{cases} \begin{matrix} \uparrow \lambda \\ i \\ \downarrow \lambda \\ j \end{matrix} & \text{if } i \neq j, \\ \begin{matrix} \uparrow \lambda \\ i \\ \downarrow \lambda \\ i \end{matrix} - \sum_{a+b+c=\bar{\lambda}_i-1} \text{Diagram} & \text{if } i = j. \end{cases}$$

(KM8) Bubble relations:

$$(21) \quad \begin{matrix} \text{Bubble} \\ \uparrow \lambda \\ \downarrow +m \end{matrix} = \begin{cases} (-1)^{\lambda_i+1} & \text{if } m = 0, \\ 0 & \text{if } m < 0, \end{cases}$$

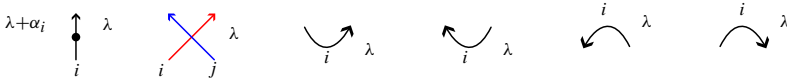
$$\begin{matrix} \text{Bubble} \\ \uparrow \lambda \\ \downarrow +m \end{matrix} = \begin{cases} (-1)^{\lambda_i+1-1} & \text{if } m = 0, \\ 0 & \text{if } m < 0. \end{cases}$$

(KM9) Infinite Grassmannian relation:

$$(22) \quad \left( \begin{matrix} \text{Bubble} \\ \uparrow \lambda \\ \downarrow +0 \end{matrix} + \begin{matrix} \text{Bubble} \\ \uparrow \lambda \\ \downarrow +1 \end{matrix} t + \dots + \begin{matrix} \text{Bubble} \\ \uparrow \lambda \\ \downarrow +m \end{matrix} t^m + \dots \right) \left( \begin{matrix} \text{Bubble} \\ \uparrow \lambda \\ \downarrow +0 \end{matrix} + \begin{matrix} \text{Bubble} \\ \uparrow \lambda \\ \downarrow +1 \end{matrix} t + \dots + \begin{matrix} \text{Bubble} \\ \uparrow \lambda \\ \downarrow +m \end{matrix} t^m + \dots \right) = -1.$$

This ends the definition of the 2-category  $\tilde{\mathcal{U}}_{\Delta}(n)$  (resp.  $\tilde{\mathcal{U}}(n)$ ).

**Remark 3.2.** Thanks to adjunction and cyclicity (equations (10) through (14)), the 2-morphisms of  $\tilde{\mathcal{U}}_{\Delta}(n)$  are already generated by



for  $\lambda \in \mathbb{Z}^n$  and  $i, j = 1, \dots, n$  (and similarly for  $\tilde{\mathcal{U}}(n)$ ). It therefore suffices to define the evaluation functor on this smaller set of generators.

**Remark 3.3.** The choice of signs in Definition 3.1 is not covered by [1]. This choice of signs is referred to as a *choice of scalars and bubble parameters*; see [13] for an in-depth explanation. Since we will be adapting various proofs from [1] for the proof of our main result, we will therefore need to take care when translating them across the different sign conventions. We will discuss difference choices of scalars and bubble parameters further in Section 6.4, since they might have implications for the existence of an evaluation 2-functor.

**3.2. Some additional relations.** Some well-known consequences of the above relations are listed below. For the proofs, see [2], for instance.

$$(23) \quad \text{Diagram} = - \sum_{a+b=m-\bar{\lambda}_i} \begin{matrix} \uparrow \lambda \\ a \\ \text{Bubble} \\ \downarrow +b \\ i \end{matrix}, \quad \text{Diagram} = \sum_{a+b=m+\bar{\lambda}_i} \begin{matrix} \text{Bubble} \\ \uparrow \lambda \\ \downarrow +b \\ a \\ i \end{matrix}$$

$$(24) \quad \left| \begin{array}{c} i \\ \circlearrowleft \\ +m \\ \uparrow \lambda \\ j \end{array} \right. = \begin{cases} - \sum_{a+b=m} (a+1) \left| \begin{array}{c} i \\ \circlearrowleft \\ +b \\ \uparrow \lambda \\ i \end{array} \right. & \text{if } i = j, \\ -\varepsilon(i, j) \left( \left| \begin{array}{c} i \\ \circlearrowleft \\ +m-1 \\ \uparrow \lambda \\ j \end{array} \right. - \left| \begin{array}{c} i \\ \circlearrowleft \\ +m \\ \uparrow \lambda \\ j \end{array} \right. \right) & \text{if } i \cdot j = -1, \\ \left| \begin{array}{c} i \\ \circlearrowleft \\ +m \\ \uparrow \lambda \\ j \end{array} \right. & \text{if } i \cdot j = 0, \end{cases}$$

$$(25) \quad \left| \begin{array}{c} i \\ \circlearrowleft \\ +m \\ \uparrow \lambda \\ j \end{array} \right. = \begin{cases} - \left| \begin{array}{c} i \\ \circlearrowleft \\ +m-2 \\ \uparrow \lambda \\ i \end{array} \right. + 2 \left| \begin{array}{c} i \\ \circlearrowleft \\ +m-1 \\ \uparrow \lambda \\ i \end{array} \right. - \left| \begin{array}{c} i \\ \circlearrowleft \\ +m \\ \uparrow \lambda \\ i \end{array} \right. & \text{if } i = j, \\ \varepsilon(i, j) \sum_{a+b=m} \left| \begin{array}{c} i \\ \circlearrowleft \\ +b \\ \uparrow \lambda \\ j \end{array} \right. & \text{if } i \cdot j = -1, \end{cases}$$

$$(26) \quad \left| \begin{array}{c} i \\ \circlearrowleft \\ +m \\ \uparrow \lambda \\ j \end{array} \right. = \begin{cases} - \sum_{a+b=m} (a+1) \left| \begin{array}{c} i \\ \circlearrowleft \\ +b \\ \uparrow \lambda \\ i \end{array} \right. & \text{if } i = j, \\ -\varepsilon(i, j) \left( \left| \begin{array}{c} i \\ \circlearrowleft \\ +m-1 \\ \uparrow \lambda \\ j \end{array} \right. - \left| \begin{array}{c} i \\ \circlearrowleft \\ +m \\ \uparrow \lambda \\ j \end{array} \right. \right) & \text{if } i \cdot j = -1, \\ \left| \begin{array}{c} i \\ \circlearrowleft \\ +m \\ \uparrow \lambda \\ j \end{array} \right. & \text{if } i \cdot j = 0, \end{cases}$$

$$(27) \quad \left| \begin{array}{c} i \\ \circlearrowleft \\ +m \\ \uparrow \lambda \\ j \end{array} \right. = \begin{cases} - \left| \begin{array}{c} i \\ \circlearrowleft \\ +m-2 \\ \uparrow \lambda \\ i \end{array} \right. + 2 \left| \begin{array}{c} i \\ \circlearrowleft \\ +m-1 \\ \uparrow \lambda \\ i \end{array} \right. - \left| \begin{array}{c} i \\ \circlearrowleft \\ +m \\ \uparrow \lambda \\ i \end{array} \right. & \text{if } i = j, \\ \varepsilon(i, j) \sum_{a+b=m} \left| \begin{array}{c} i \\ \circlearrowleft \\ +b \\ \uparrow \lambda \\ j \end{array} \right. & \text{if } i \cdot j = -1, \end{cases}$$

$$(28) \quad \left| \begin{array}{c} i \\ \circlearrowleft \\ +m \\ \uparrow \lambda \\ j \end{array} \right. - \left| \begin{array}{c} i \\ \circlearrowleft \\ +m \\ \uparrow \lambda \\ j \end{array} \right. = \begin{cases} \sum_{\substack{a+b+c+d \\ = \lambda_i}} \left| \begin{array}{c} i \\ \circlearrowleft \\ +b \\ \uparrow \lambda \\ i \end{array} \right. + \sum_{\substack{a+b+c+d \\ = -\lambda_i - 2}} \left| \begin{array}{c} i \\ \circlearrowleft \\ +b \\ \uparrow \lambda \\ i \end{array} \right. & \text{if } i = j = k, \\ 0 & \text{else.} \end{cases}$$

**4. Two versions of the evaluation 2-functor for  $n = 3$**

In this section, we will define the 2-functors  $\varepsilon_v$  and  $\varepsilon_{v'}$  discussed in the introduction, which are  $\mathbb{Q}$ -linear monoidal functors

$$\varepsilon_v, \varepsilon_{v'}: \tilde{U}_\Delta(3) \rightarrow K^b(\tilde{U}(3)),$$

defined in the next pages. Note that in this case, Definition 2.4 is particularly simple, because (4) and (5) only involve one  $q$ -commutator each:

$$\begin{aligned} \text{ev}(E_3 1_\lambda) &= q^{\lambda_1 + \lambda_3 + t - 1} (F_1 F_2 1_\lambda - q F_2 F_1 1_\lambda), \\ \text{ev}(F_3 1_\lambda) &= q^{-\lambda_1 - \lambda_3 - t + 1} (E_2 E_1 1_\lambda - q^{-1} E_1 E_2 1_\lambda), \end{aligned}$$

for  $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$  (recall that  $t \in \mathbb{Z}$  is arbitrary but fixed). For the remainder of this paper, we set  $S(\lambda) = \lambda_1 + \lambda_3 + t - 1$ , and we suppress the  $\lambda$  when there is no confusion. We also use the notation  $\mathcal{E}_{i_1 i_2 \dots i_k} \mathbf{1}_\lambda = \mathcal{E}_{i_1} \mathcal{E}_{i_2} \dots \mathcal{E}_{i_k} \mathbf{1}_\lambda$  and  $\mathcal{F}_{i_1 i_2 \dots i_k} \mathbf{1}_\lambda = \mathcal{F}_{i_1} \mathcal{F}_{i_2} \dots \mathcal{F}_{i_k} \mathbf{1}_\lambda$ .

**4.1. The definition of  $\varepsilon_v$ .** In the definition below, we underline the 1-morphism in homological degree zero in each complex.

- For objects, we define  $\varepsilon_v(\lambda) = \lambda$ , where  $\lambda \in \mathbb{Z}^3$ .
- For 1-morphisms, we define the action of  $\varepsilon_v$  on generating 1-morphisms and extend it to all 1-morphisms via composition and direct sums, using the standard composition of complexes:

$$(1) \quad \varepsilon_v(\mathcal{E}_1 \mathbf{1}_\lambda) = \underline{\mathcal{E}_1 \mathbf{1}_\lambda}, \quad \varepsilon_v(\mathcal{E}_2 \mathbf{1}_\lambda) = \underline{\mathcal{E}_2 \mathbf{1}_\lambda},$$

$$(2) \quad \varepsilon_v(\mathcal{F}_1 \mathbf{1}_\lambda) = \underline{\mathcal{F}_1 \mathbf{1}_\lambda}, \quad \varepsilon_v(\mathcal{F}_2 \mathbf{1}_\lambda) = \underline{\mathcal{F}_2 \mathbf{1}_\lambda},$$

$$(3) \quad \varepsilon_v(\mathcal{E}_3 \mathbf{1}_\lambda) = \underline{\mathcal{F}_{12} \mathbf{1}_\lambda} \langle S \rangle \xrightarrow{\begin{array}{c} \text{red } \swarrow \lambda \\ \text{blue } \searrow \lambda \\ \text{red } \downarrow 2 \quad \text{blue } \downarrow 1 \end{array}} \underline{\mathcal{F}_{21} \mathbf{1}_\lambda} \langle S + 1 \rangle ,$$

$$(4) \quad \varepsilon_v(\mathcal{F}_3 \mathbf{1}_\lambda) = \underline{\mathcal{E}_{12} \mathbf{1}_\lambda} \langle -S - 1 \rangle \xrightarrow{\begin{array}{c} \text{blue } \swarrow \lambda \\ \text{red } \searrow \lambda \\ \text{red } \downarrow 2 \quad \text{blue } \downarrow 1 \end{array}} \underline{\mathcal{E}_{21} \mathbf{1}_\lambda} \langle -S \rangle .$$

- We set  $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3$  and call  $|\lambda|$  the *Schur level* of  $\lambda$ .

For compatibility with later proofs, we give the definition of  $\varepsilon_v$  on downwards-pointing generating 2-morphisms (that is, the horizontally mirrored versions of the 2-morphisms in Remark 3.2). For generating 2-morphisms consisting only of strands between  $\mathcal{E}_1 \mathbf{1}_\lambda, \mathcal{E}_2 \mathbf{1}_\lambda, \mathcal{F}_1 \mathbf{1}_\lambda$  and  $\mathcal{F}_2 \mathbf{1}_\lambda$ , the 2-functor  $\varepsilon_v$  acts as the identity, with the following exceptions:

$$(29) \quad \varepsilon_v \left( \begin{array}{c} \text{red} \diagdown \text{blue} \\ \text{blue} \diagup \text{red} \\ \hline 2 \quad 1 \end{array} \lambda \right) = (-1)^{\bar{\lambda}_1(\bar{\lambda}_2+1)} \begin{array}{c} \text{red} \diagdown \text{blue} \\ \text{red} \diagup \text{blue} \\ \hline 2 \quad 1 \end{array} \lambda, \quad \varepsilon_v \left( \begin{array}{c} \text{blue} \diagdown \text{red} \\ \text{red} \diagup \text{blue} \\ \hline 1 \quad 2 \end{array} \lambda \right) = (-1)^{\bar{\lambda}_1(\bar{\lambda}_2+1)} \begin{array}{c} \text{blue} \diagdown \text{red} \\ \text{blue} \diagup \text{red} \\ \hline 1 \quad 2 \end{array} \lambda.$$

For the remaining generating 2-morphisms, we define the images as follows, emphasising these are commutative diagrams:

$$(30) \quad \varepsilon_v \left( \begin{array}{c} \downarrow \lambda \\ \hline 3 \end{array} \right) = \begin{array}{ccc} \mathcal{E}_{12} \mathbf{1}_\lambda \langle -S+1 \rangle & \xrightarrow{\begin{array}{c} \text{red} \diagdown \text{blue} \\ \text{blue} \diagup \text{red} \\ \hline 1 \quad 2 \end{array} \lambda} & \mathcal{E}_{21} \mathbf{1}_\lambda \langle -S+2 \rangle \\ \uparrow \begin{array}{c} \text{blue} \uparrow \text{red} \\ \hline 1 \quad 2 \end{array} \lambda & & \uparrow \begin{array}{c} \text{red} \uparrow \text{blue} \\ \hline 2 \quad 1 \end{array} \lambda \\ \mathcal{E}_{12} \mathbf{1}_\lambda \langle -S-1 \rangle & \xrightarrow{\begin{array}{c} \text{red} \diagdown \text{blue} \\ \text{red} \diagup \text{blue} \\ \hline 1 \quad 2 \end{array} \lambda} & \mathcal{E}_{21} \mathbf{1}_\lambda \langle -S \rangle \end{array}$$

$$(31) \quad \varepsilon_v \left( \begin{array}{c} \diagdown \diagup \lambda \\ \hline 3 \quad 3 \end{array} \right) = \begin{array}{ccccc} & & \begin{array}{c} \text{red} \uparrow \text{blue} \\ \text{red} \uparrow \text{blue} \\ \hline 2 \quad 2 \end{array} \lambda & & \begin{array}{c} \text{red} \uparrow \text{blue} \\ \text{red} \uparrow \text{blue} \\ \hline 2 \quad 2 \end{array} \lambda \\ & & \mathcal{E}_{2112} \mathbf{1}_\lambda \langle -2S-3 \rangle & & \mathcal{E}_{2121} \mathbf{1}_\lambda \langle -2S-2 \rangle \\ \begin{array}{c} \text{red} \uparrow \text{blue} \\ \text{red} \uparrow \text{blue} \\ \hline 2 \quad 2 \end{array} \lambda & \xrightarrow{- \begin{array}{c} \text{red} \uparrow \text{blue} \\ \text{red} \uparrow \text{blue} \\ \hline 2 \quad 2 \end{array} \lambda} & & \xrightarrow{- \begin{array}{c} \text{red} \uparrow \text{blue} \\ \text{red} \uparrow \text{blue} \\ \hline 2 \quad 2 \end{array} \lambda} & \\ \mathcal{E}_{1212} \mathbf{1}_\lambda \langle -2S-4 \rangle & & \mathcal{E}_{1221} \mathbf{1}_\lambda \langle -2S-3 \rangle & & \\ \begin{array}{c} \text{red} \uparrow \text{blue} \\ \text{red} \uparrow \text{blue} \\ \hline 2 \quad 2 \end{array} \lambda & \xrightarrow{- \begin{array}{c} \text{red} \uparrow \text{blue} \\ \text{red} \uparrow \text{blue} \\ \hline 2 \quad 2 \end{array} \lambda} & & \xrightarrow{- \begin{array}{c} \text{red} \uparrow \text{blue} \\ \text{red} \uparrow \text{blue} \\ \hline 2 \quad 2 \end{array} \lambda} & \\ \mathcal{E}_{1212} \mathbf{1}_\lambda \langle -2S-2 \rangle & & \mathcal{E}_{2112} \mathbf{1}_\lambda \langle -2S-1 \rangle & & \mathcal{E}_{2121} \mathbf{1}_\lambda \langle -2S \rangle \\ \begin{array}{c} \text{red} \uparrow \text{blue} \\ \text{red} \uparrow \text{blue} \\ \hline 2 \quad 2 \end{array} \lambda & \xrightarrow{- \begin{array}{c} \text{red} \uparrow \text{blue} \\ \text{red} \uparrow \text{blue} \\ \hline 2 \quad 2 \end{array} \lambda} & & \xrightarrow{- \begin{array}{c} \text{red} \uparrow \text{blue} \\ \text{red} \uparrow \text{blue} \\ \hline 2 \quad 2 \end{array} \lambda} & \\ & & \mathcal{E}_{1221} \mathbf{1}_\lambda \langle -2S-1 \rangle & & \end{array}$$

$$(32) \quad \varepsilon_v \left( \begin{array}{c} \diagdown \diagup \lambda \\ \hline 3 \quad 1 \end{array} \right) = \begin{array}{ccc} \mathcal{F}_1 \mathcal{E}_{12} \mathbf{1}_\lambda \langle -S \rangle & \xrightarrow{\begin{array}{c} \text{red} \uparrow \text{blue} \\ \text{red} \uparrow \text{blue} \\ \hline 1 \quad 2 \end{array} \lambda} & \mathcal{F}_1 \mathcal{E}_{21} \mathbf{1}_\lambda \langle -S+1 \rangle \\ \uparrow \begin{array}{c} \text{red} \uparrow \text{blue} \\ \text{red} \uparrow \text{blue} \\ \hline 1 \quad 2 \end{array} \lambda & & \uparrow \begin{array}{c} \text{red} \uparrow \text{blue} \\ \text{red} \uparrow \text{blue} \\ \hline 2 \quad 1 \end{array} \lambda \\ \mathcal{E}_{12} \mathcal{F}_1 \mathbf{1}_\lambda \langle -S \rangle & \xrightarrow{- \begin{array}{c} \text{red} \uparrow \text{blue} \\ \text{red} \uparrow \text{blue} \\ \hline 1 \quad 2 \end{array} \lambda} & \mathcal{E}_{21} \mathcal{F}_1 \mathbf{1}_\lambda \langle -S+1 \rangle \end{array}$$

(33)  $\varepsilon_v \left( \begin{array}{c} \text{---} \\ \diagup \lambda \\ \diagdown \\ \text{---} \end{array} \right) = - \begin{array}{c} \text{---} \\ \diagup \lambda \\ \diagdown \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \diagup \lambda \\ \diagdown \\ \text{---} \end{array}$

$$\begin{array}{ccc} \mathcal{E}_{12} \mathcal{F}_1 \mathbf{1}_\lambda \langle -S+1 \rangle & \xrightarrow{\begin{array}{c} \text{---} \\ \diagup \lambda \\ \diagdown \\ \text{---} \end{array}} & \mathcal{E}_{21} \mathcal{F}_1 \mathbf{1}_\lambda \langle -S+2 \rangle \\ \uparrow & & \uparrow \\ \mathcal{F}_1 \mathcal{E}_{12} \mathbf{1}_\lambda \langle -S-1 \rangle & \xrightarrow{\begin{array}{c} \text{---} \\ \diagup \lambda \\ \diagdown \\ \text{---} \end{array}} & \mathcal{F}_1 \mathcal{E}_{21} \mathbf{1}_\lambda \langle -S \rangle \end{array}$$

(34)  $\varepsilon_v \left( \begin{array}{c} \text{---} \\ \diagup \lambda \\ \diagdown \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \\ \diagup \lambda \\ \diagdown \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \diagup \lambda \\ \diagdown \\ \text{---} \end{array}$

$$\begin{array}{ccc} \mathcal{E}_{12} \mathcal{F}_2 \mathbf{1}_\lambda \langle -S-1 \rangle & \xrightarrow{\begin{array}{c} \text{---} \\ \diagup \lambda \\ \diagdown \\ \text{---} \end{array}} & \mathcal{E}_{21} \mathcal{F}_2 \mathbf{1}_\lambda \langle -S \rangle \\ \uparrow & & \uparrow \\ \mathcal{F}_2 \mathcal{E}_{12} \mathbf{1}_\lambda \langle -S-1 \rangle & \xrightarrow{\begin{array}{c} \text{---} \\ \diagup \lambda \\ \diagdown \\ \text{---} \end{array}} & \mathcal{F}_2 \mathcal{E}_{21} \mathbf{1}_\lambda \langle -S \rangle \end{array}$$

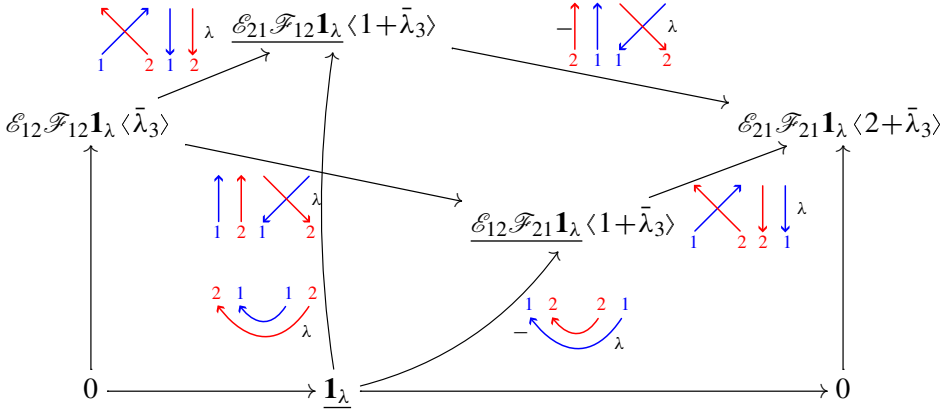
(35)  $\varepsilon_v \left( \begin{array}{c} \text{---} \\ \diagup \lambda \\ \diagdown \\ \text{---} \end{array} \right) = \begin{array}{c} \text{---} \\ \diagup \lambda \\ \diagdown \\ \text{---} \end{array} - \begin{array}{c} \text{---} \\ \diagup \lambda \\ \diagdown \\ \text{---} \end{array}$

$$\begin{array}{ccc} \mathcal{F}_2 \mathcal{E}_{12} \mathbf{1}_\lambda \langle -S \rangle & \xrightarrow{\begin{array}{c} \text{---} \\ \diagup \lambda \\ \diagdown \\ \text{---} \end{array}} & \mathcal{F}_2 \mathcal{E}_{21} \mathbf{1}_\lambda \langle -S+1 \rangle \\ \uparrow & & \uparrow \\ \mathcal{E}_{12} \mathcal{F}_2 \mathbf{1}_\lambda \langle -S-2 \rangle & \xrightarrow{\begin{array}{c} \text{---} \\ \diagup \lambda \\ \diagdown \\ \text{---} \end{array}} & \mathcal{F}_{12} \mathcal{E}_{21} \mathbf{1}_\lambda \langle -S-1 \rangle \end{array}$$

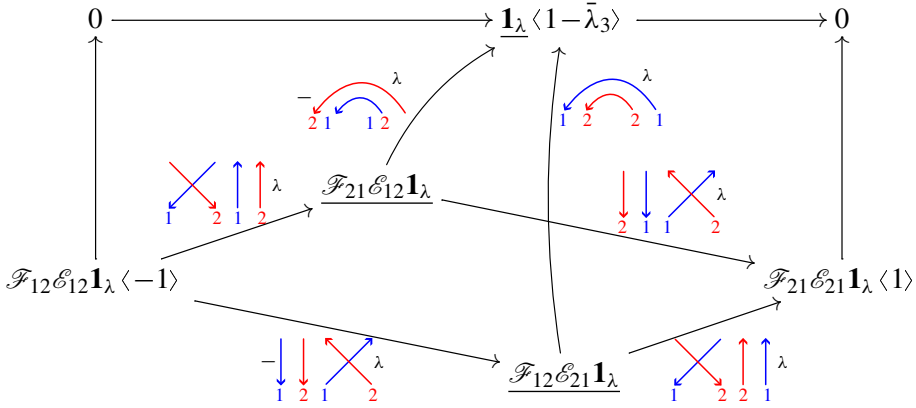
(36)  $\varepsilon_v \left( \begin{array}{c} \text{---} \\ \diagup \lambda \\ \diagdown \\ \text{---} \end{array} \right) =$

$$\begin{array}{ccccc} & & \mathcal{F}_{21} \mathcal{E}_{12} \mathbf{1}_\lambda \langle 1-\bar{\lambda}_3 \rangle & & \mathcal{F}_{21} \mathcal{E}_{21} \mathbf{1}_\lambda \langle 2-\bar{\lambda}_3 \rangle \\ & \nearrow & & \searrow & \\ \mathcal{F}_{12} \mathcal{E}_{12} \mathbf{1}_\lambda \langle -\bar{\lambda}_3 \rangle & & & & \\ & \searrow & & \nearrow & \\ & & \mathcal{F}_{12} \mathcal{E}_{21} \mathbf{1}_\lambda \langle 1-\bar{\lambda}_3 \rangle & & \\ & \nearrow & & \searrow & \\ 0 & \xrightarrow{\mathbf{1}_\lambda} & & \xrightarrow{\mathbf{1}_\lambda} & 0 \end{array}$$

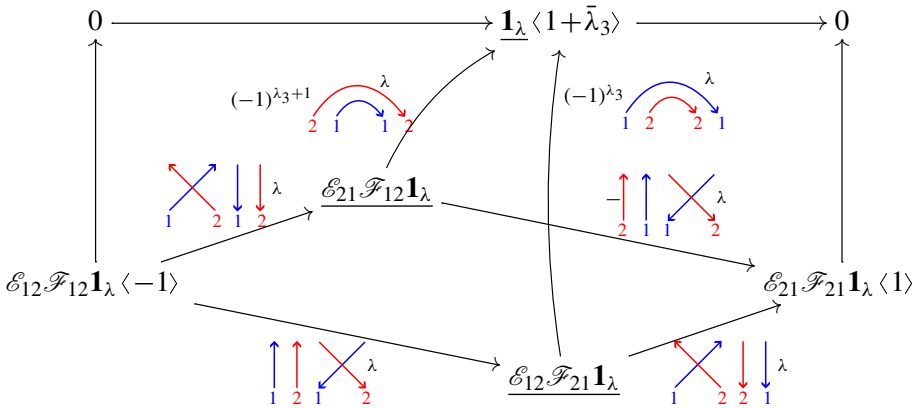
(37)  $\varepsilon v(\overset{3}{\curvearrowright}_\lambda) =$



(38)  $\varepsilon v(\overset{\lambda}{\curvearrowright}_3) =$



(39)  $\varepsilon v(\underset{3}{\curvearrowright}_\lambda) =$



The following theorem is the main result of the paper and we will prove in Section 4.3 that it is a consequence of Theorem 4.3 and Lemma 4.4.

**Theorem 4.1.**  $\varepsilon_v$  is a well-defined 2-functor that decategorifies to  $ev$ .

**4.2. The definition of  $\varepsilon_{v'}$ .** To define  $\varepsilon_{v'}$ , we need some notation. For the rest of the paper, we let  $\equiv$  denote  $\equiv_4$ , that is, congruence modulo 4. We define

$$(40) \quad k_i^{a_1, \dots, a_n}(\lambda) := \begin{cases} 1 & \text{if } \bar{\lambda}_i \equiv a_1, \dots, a_n, \\ -1 & \text{if otherwise.} \end{cases}$$

We will often omit the argument when it is  $\lambda$ . For example,  $k_1^{0,1} = 1$  (resp.  $-1$ ) if  $\bar{\lambda}_1 \equiv 0, 1$  (resp.  $\equiv 2, 3$ ).

**Remark 4.2.** We will use the following relations involving the  $k_i^{a_1, \dots, a_n}$  at various points:  $k_i^0 k_i^2 = (-1)^{\bar{\lambda}_i + 1}$ ,  $k_i^1 k_i^3 = (-1)^{\bar{\lambda}_i}$ ,  $k_1^a(\lambda + \alpha_2) = k_1^{a+1}(\lambda)$ ,  $k_1^a(s_1(\lambda)) = k_1^{-a}(\lambda)$ , and  $k_1^a(s_2(\lambda)) = k_3^{-a}(\lambda)$ .

We will also be using the more compact notation for 2-morphisms between complexes found in [1], where they are presented as ordered tuples (most commonly ordered pairs).

We now define  $\varepsilon_{v'}$  to be identical to  $\varepsilon_v$  on objects, 1-morphisms and all generating 2-morphisms except for the following:

- $\varepsilon_{v'} \left( \begin{array}{c} \diagup \diagdown \\ \color{red}{2} \quad \color{blue}{3} \end{array} \lambda \right) = \left( -k_3^{0,3} k_1^{0,3} \begin{array}{c} \diagup \diagdown \\ \color{red}{2} \quad \color{blue}{1} \quad \color{red}{2} \end{array} \lambda, -k_3^{0,3} k_1^{0,1} \begin{array}{c} \diagup \diagdown \\ \color{red}{2} \quad \color{blue}{2} \quad \color{red}{1} \end{array} \lambda \right).$
- $\varepsilon_{v'} \left( \begin{array}{c} \diagup \diagdown \\ \color{red}{3} \quad \color{blue}{2} \end{array} \lambda \right) = \left( k_3^{0,3} k_1^{0,3} \left( \begin{array}{c} \diagup \diagdown \\ \color{blue}{1} \quad \color{red}{2} \quad \color{red}{2} \end{array} \lambda - \begin{array}{c} \diagup \diagdown \\ \color{blue}{1} \quad \color{red}{2} \quad \color{red}{2} \end{array} \lambda \right), k_3^{0,3} k_1^{0,1} \left( \begin{array}{c} \diagup \diagdown \\ \color{red}{2} \quad \color{blue}{1} \quad \color{red}{2} \end{array} \lambda - \begin{array}{c} \diagup \diagdown \\ \color{red}{2} \quad \color{blue}{1} \quad \color{red}{2} \end{array} \lambda \right) \right).$
- $\varepsilon_{v'} \left( \begin{array}{c} \color{red}{3} \\ \curvearrowright \\ \color{blue}{\lambda} \end{array} \right) = (-1)^{\lambda_3 + 1} \left( k_1^0 \begin{array}{c} \color{red}{1} \quad \color{red}{2} \quad \color{red}{2} \quad \color{red}{1} \\ \curvearrowright \\ \color{blue}{\lambda} \end{array} - k_1^1 \begin{array}{c} \color{red}{2} \quad \color{red}{1} \quad \color{red}{1} \quad \color{red}{2} \\ \curvearrowright \\ \color{blue}{\lambda} \end{array} \right).$
- $\varepsilon_{v'} \left( \begin{array}{c} \color{red}{3} \\ \curvearrowleft \\ \color{blue}{\lambda} \end{array} \right) = k_1^0 \begin{array}{c} \color{red}{1} \quad \color{red}{2} \quad \color{red}{2} \quad \color{red}{1} \\ \curvearrowleft \\ \color{blue}{\lambda} \end{array} - k_1^3 \begin{array}{c} \color{red}{2} \quad \color{red}{1} \quad \color{red}{1} \quad \color{red}{2} \\ \curvearrowleft \\ \color{blue}{\lambda} \end{array}.$
- $\varepsilon_{v'} \left( \begin{array}{c} \color{red}{\lambda} \\ \curvearrowright \\ \color{blue}{3} \end{array} \right) = k_1^3 \begin{array}{c} \color{red}{\lambda} \\ \curvearrowright \\ \color{red}{2} \quad \color{blue}{1} \quad \color{red}{2} \end{array} - k_1^2 \begin{array}{c} \color{red}{\lambda} \\ \curvearrowright \\ \color{blue}{1} \quad \color{red}{2} \quad \color{red}{2} \quad \color{blue}{1} \end{array}.$
- $\varepsilon_{v'} \left( \begin{array}{c} \color{red}{\lambda} \\ \curvearrowleft \\ \color{blue}{3} \end{array} \right) = (-1)^{\lambda_3} \left( k_1^1 \begin{array}{c} \color{red}{\lambda} \\ \curvearrowleft \\ \color{red}{2} \quad \color{blue}{1} \quad \color{red}{2} \end{array} - k_1^2 \begin{array}{c} \color{red}{\lambda} \\ \curvearrowleft \\ \color{blue}{1} \quad \color{red}{2} \quad \color{red}{2} \quad \color{blue}{1} \end{array} \right).$

We will prove the following result in Section 6.4.

**Theorem 4.3.**  $\varepsilon_{v'}$  is a well-defined 2-functor that decategorifies to  $ev$ .

**4.3. Relating  $\varepsilon_v$  and  $\varepsilon_{v'}$ .** We now show that  $\varepsilon_v$  can be given by composing  $\varepsilon_{v'}$  with 2-isomorphisms. The first such 2-isomorphism is  $\gamma$ , which acts as the identity

on objects, 1-morphisms, and generating 2-morphisms with the exception of:

$$\begin{array}{ccc} \begin{array}{c} \curvearrowright \xrightarrow{\lambda} \\ \curvearrowright \end{array} \xrightarrow{\gamma} -k_1^0 \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} & & \begin{array}{c} \curvearrowright \xrightarrow{\lambda} \\ \curvearrowright \end{array} \xrightarrow{\gamma} -k_1^0 \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \\ \begin{array}{c} \curvearrowright \xrightarrow{\lambda} \\ \curvearrowright \end{array} \xrightarrow{\gamma} -k_1^2 \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} & & \begin{array}{c} \curvearrowright \xrightarrow{\lambda} \\ \curvearrowright \end{array} \xrightarrow{\gamma} -k_1^2 \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \end{array}$$

We abuse notation by also using  $\gamma$  to refer to the 2-isomorphism of  $K^b(\tilde{u}(3))$  that acts in the same fashion. Similarly to  $\beta$ , it is straightforward to see that  $\gamma$  preserves (KM1) through (KM9), and is therefore a (pair of) well-defined 2-isomorphism(s).

The second 2-isomorphism is  $\delta$ , which is again the identity on all objects, 1-morphisms and generating 2-morphisms except

$$\begin{array}{c} \begin{array}{c} \curvearrowright \xrightarrow{\lambda} \\ \curvearrowright \end{array} \xrightarrow{\delta} -k_3^{0,3} k_1^{0,3} \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array}, \quad \begin{array}{c} \curvearrowright \xrightarrow{\lambda} \\ \curvearrowright \end{array} \xrightarrow{\delta} -k_3^{0,3} k_1^{0,3} \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array}. \end{array}$$

It is again an easy calculation that  $\delta$  preserves (KM1) through (KM9) and is therefore a well-defined 2-isomorphism.

**Lemma 4.4.**  $\varepsilon v = \gamma \varepsilon v' \delta \gamma.$

*Proof.* Outside of the generating 2-morphisms that  $\varepsilon v'$  sends to a 2-morphism of complexes with a dependency on  $\bar{\lambda}_i$  modulo 4,  $\varepsilon v$  and  $\gamma \varepsilon v' \delta \gamma$  agree with  $\varepsilon v'$ . In particular, we have, e.g.,

$$\gamma \varepsilon v' \delta \gamma \left( \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right) = \gamma \varepsilon v' \left( -k_1^2 \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right) = (k_1^2)^2 \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} = \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} = \varepsilon v' \left( \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right).$$

For the 2-morphisms that differ we have

$$\begin{aligned} (41) \quad \gamma \varepsilon v' \delta \gamma \left( \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right) &= \gamma \left( k_1^3(\lambda) \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} - k_1^2(\lambda) \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right) \\ &= -k_1^3(\lambda) k_1^2(\lambda + \alpha_2) \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} + k_1^2(\lambda)^2 \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \\ &= -k_1^3(\lambda)^2 \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} + k_1^2(\lambda)^2 \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \\ &= - \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} + \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} = \varepsilon v \left( \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right), \\ (42) \quad \gamma \varepsilon v' \delta \gamma \left( \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right) &= \gamma \left( (-1)^{\lambda_3} \left( k_1^1(\lambda) \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} - k_1^2(\lambda) \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right) \right) \\ &= (-1)^{\lambda_3} \gamma \left( k_1^2(\lambda - \alpha_2) \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} - k_1^2(\lambda) \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right) \\ &= (-1)^{\lambda_3+1} \left( k_1^2(\lambda - \alpha_2)^2 \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} - k_1^2(\lambda)^2 \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right) \\ &= (-1)^{\lambda_3+1} \left( \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} - \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right) = \varepsilon v \left( \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array} \right), \end{aligned}$$

$$\begin{aligned}
 (43) \quad \gamma \varepsilon v' \delta \gamma \left( \begin{array}{c} \text{3} \\ \curvearrowright_\lambda \end{array} \right) &= \gamma \left( k_1^0(\lambda) \begin{array}{c} \text{1} \text{ 2} \text{ 2} \text{ 1} \\ \curvearrowright_\lambda \end{array} - k_1^0(\lambda - \alpha_2) \begin{array}{c} \text{2} \text{ 1} \text{ 1} \text{ 2} \\ \curvearrowright_\lambda \end{array} \right) \\
 &= -k_1^0(\lambda)^2 \begin{array}{c} \text{1} \text{ 2} \text{ 2} \text{ 1} \\ \curvearrowright_\lambda \end{array} + k_1^0(\lambda - \alpha_2)^2 \begin{array}{c} \text{2} \text{ 1} \text{ 1} \text{ 2} \\ \curvearrowright_\lambda \end{array} \\
 &= - \begin{array}{c} \text{1} \text{ 2} \text{ 2} \text{ 1} \\ \curvearrowright_\lambda \end{array} + \begin{array}{c} \text{2} \text{ 1} \text{ 1} \text{ 2} \\ \curvearrowright_\lambda \end{array} = \varepsilon v \left( \begin{array}{c} \text{3} \\ \curvearrowright_\lambda \end{array} \right), \\
 (44) \quad \gamma \varepsilon v' \delta \gamma \left( \begin{array}{c} \text{3} \\ \curvearrowleft_\lambda \end{array} \right) &= \gamma \left( (-1)^{\lambda_3+1} k_1^0(\lambda) \begin{array}{c} \text{1} \text{ 2} \text{ 2} \text{ 1} \\ \curvearrowleft_\lambda \end{array} + (-1)^{\lambda_3} k_1^0(\lambda) \begin{array}{c} \text{2} \text{ 1} \text{ 1} \text{ 2} \\ \curvearrowleft_\lambda \end{array} \right) \\
 &= (-1)^{\lambda_3+1} \left( (-1)^{\lambda_3+1} k_1^0(\lambda) \begin{array}{c} \text{1} \text{ 2} \text{ 2} \text{ 1} \\ \curvearrowleft_\lambda \end{array} - k_1^0(\lambda + \alpha_2) \begin{array}{c} \text{2} \text{ 1} \text{ 1} \text{ 2} \\ \curvearrowleft_\lambda \end{array} \right) \\
 &= (-1)^{\lambda_3+1} \left( -k_1^0(\lambda)^2 \begin{array}{c} \text{1} \text{ 2} \text{ 2} \text{ 1} \\ \curvearrowleft_\lambda \end{array} + k_1^0(\lambda + \alpha_2)^2 \begin{array}{c} \text{2} \text{ 1} \text{ 1} \text{ 2} \\ \curvearrowleft_\lambda \end{array} \right) \\
 &= (-1)^{\lambda_3+1} \left( - \begin{array}{c} \text{1} \text{ 2} \text{ 2} \text{ 1} \\ \curvearrowleft_\lambda \end{array} + \begin{array}{c} \text{2} \text{ 1} \text{ 1} \text{ 2} \\ \curvearrowleft_\lambda \end{array} \right) = \varepsilon v \left( \begin{array}{c} \text{3} \\ \curvearrowleft_\lambda \end{array} \right).
 \end{aligned}$$

Recalling that

$$\begin{aligned}
 \varepsilon v' \left( \begin{array}{c} \text{3} \\ \times_\lambda \end{array} \right) &= \left( -k_3^{0,3} k_1^{0,3} \begin{array}{c} \text{3} \\ \times_\lambda \end{array}, -k_3^{0,3} k_1^{0,1} \begin{array}{c} \text{3} \\ \times_\lambda \end{array} \right) \quad \text{and} \\
 \varepsilon v' \left( \begin{array}{c} \text{3} \\ \times_\lambda \end{array} \right) &= \left( k_3^{0,3} k_1^{0,3} \left( \begin{array}{c} \text{3} \\ \times_\lambda \end{array} - \begin{array}{c} \text{3} \\ \times_\lambda \end{array} \right), k_3^{0,3} k_1^{0,1} \left( \begin{array}{c} \text{3} \\ \times_\lambda \end{array} - \begin{array}{c} \text{3} \\ \times_\lambda \end{array} \right) \right),
 \end{aligned}$$

it is straightforward to see that

$$\gamma \varepsilon v' \delta \gamma \left( \begin{array}{c} \text{3} \\ \times_\lambda \end{array} \right) = \varepsilon v \left( \begin{array}{c} \text{3} \\ \times_\lambda \end{array} \right) \quad \text{and} \quad \gamma \varepsilon v' \delta \gamma \left( \begin{array}{c} \text{3} \\ \times_\lambda \end{array} \right) = \varepsilon v \left( \begin{array}{c} \text{3} \\ \times_\lambda \end{array} \right). \quad \square$$

This shows that Theorem 4.3 implies Theorem 4.1, and so it suffices to prove the former.

**4.4. Essential uniqueness of the image of the dotted 3-strand.** Finally, let us show that the above choice for the image of a dotted 3-strand under  $\varepsilon v$  is the only one possible, up to multiplication by a scalar. We will be using this on occasion in the proof of Theorem 4.3 and elsewhere.

**Lemma 4.5.**  $\text{End}_{K^b(\tilde{u}(3))}^*(\varepsilon v(\mathcal{E}_3 \mathbf{1}_\lambda))$  and  $\text{End}_{K^b(\tilde{u}(3))}^*(\varepsilon v(\mathcal{F}_3 \mathbf{1}_\lambda))$  are isomorphic to  $\mathbb{Q}[x]$ , where  $\deg x = 2$ .

*Proof.* We tackle the case  $\text{End}_{K^b(\tilde{u}(3))}^*(\varepsilon v(\mathcal{E}_3 \mathbf{1}_\lambda))$ , the other one being similar. We first work in the 2-category of bounded complexes  $\mathcal{C}^b(\tilde{u}(3))$ . We claim that

$$\text{End}_{\mathcal{C}^b(\tilde{u}(3))}^*(\varepsilon v(\mathcal{E}_3 \mathbf{1}_\lambda)) \cong \mathbb{Q}[x_1, x_2],$$

where  $\deg x_1 = \deg x_2 = 2$ . An element of  $\text{End}_{\mathcal{C}^b(\tilde{u}(3))}^*(\varepsilon v(\mathcal{E}_3 \mathbf{1}_\lambda))$  is a commutative

square of the form

$$\begin{array}{ccc}
 \mathcal{F}_{12}\mathbf{1}_\lambda \langle S+r \rangle & \xrightarrow{\quad \downarrow \begin{array}{c} \lambda \\ 2 \end{array} \quad} & \mathcal{F}_{21}\mathbf{1}_\lambda \langle S+r+1 \rangle \\
 \uparrow f_0 & & \uparrow f_1 \\
 \mathcal{F}_{12}\mathbf{1}_\lambda \langle S \rangle & \xrightarrow{\quad \downarrow \begin{array}{c} \lambda \\ 2 \end{array} \quad} & \mathcal{F}_{21}\mathbf{1}_\lambda \langle S+1 \rangle .
 \end{array}$$

By [10, Theorem 2.7], the shift  $r$  has to be even and  $f_0$  is a linear combination of 2-morphisms of the form  $g_0 = a \downarrow \begin{array}{c} \lambda \\ 2 \end{array} b^\lambda$ , where  $a + b = r/2$ .

The equality

$$\boxed{f_1} \begin{array}{c} \lambda \\ \downarrow \\ 2 \end{array} = \boxed{f_0} \begin{array}{c} \lambda \\ \downarrow \\ 2 \end{array}$$

implies that, for each summand  $g_0$  of  $f_0$ , there is a corresponding summand  $g_1$  of  $f_1$  that is determined by the choice of  $g_0$ , i.e.,

$$(g_0, g_1) = \left( a \downarrow \begin{array}{c} \lambda \\ 2 \end{array} b^\lambda, b \downarrow \begin{array}{c} \lambda \\ 2 \end{array} a^\lambda \right),$$

where we use the presentation used in [1] of only giving the vertical 2-morphisms as an ordered pair, for clarity of reading.

This proves the claim, with

$$x_1 = \left( \downarrow \begin{array}{c} \lambda \\ 2 \end{array}, \downarrow \begin{array}{c} \lambda \\ 2 \end{array} \right), \quad x_2 = \left( \downarrow \begin{array}{c} \lambda \\ 2 \end{array}, \downarrow \begin{array}{c} \lambda \\ 2 \end{array} \right).$$

We further claim that  $x_1$  and  $x_2$  are homotopic. Indeed, consider the diagram

$$(45) \quad \begin{array}{ccc}
 \mathcal{F}_{12}\mathbf{1}_\lambda \langle S+2 \rangle & \xrightarrow{\quad \downarrow \begin{array}{c} \lambda \\ 2 \end{array} \quad} & \mathcal{F}_{21}\mathbf{1}_\lambda \langle S+3 \rangle \\
 \uparrow \downarrow \begin{array}{c} \lambda \\ 2 \end{array} - \downarrow \begin{array}{c} \lambda \\ 2 \end{array} & \swarrow \downarrow \begin{array}{c} \lambda \\ 2 \end{array} & \uparrow \downarrow \begin{array}{c} \lambda \\ 2 \end{array} - \downarrow \begin{array}{c} \lambda \\ 2 \end{array} \\
 \mathcal{F}_{12}\mathbf{1}_\lambda \langle S \rangle & \xrightarrow{\quad \downarrow \begin{array}{c} \lambda \\ 2 \end{array} \quad} & \mathcal{F}_{21}\mathbf{1}_\lambda \langle S+1 \rangle .
 \end{array}$$

One sees that this diagram is commutative, by the downward version of relation (15), and hence  $x_1 - x_2 \simeq_h 0$ , which proves the lemma.  $\square$

A directly analogous result and proof hold for  $\mathcal{E}_v'$ .

### 5. Categorized braid group action

To categorify the connection between our desired evaluation functor and Lusztig’s algebra automorphisms  $T'_{1,-1}$  and  $T''_{2,1}$ , discussed in Section 2.2.1, we need to introduce various 2-functors to deal with some complications. While the automorphisms have already been categorized in [1], that paper works over  $\mathfrak{sl}_3$  and does not cover our choice of scalars and bubble parameters. We therefore adapt their constructions to our setup through composition with 2-isomorphisms.

**5.1. The braid group actions.** Denote by  $\mathcal{U}(3)$  the  $\mathfrak{gl}_3$  version of the (unsigned version of the) 2-category  $\mathcal{U}_Q(\mathfrak{sl}_3)$  defined in [1, Definition 3.3] with the trivial choice of scalars and bubble parameters. For this section, we will be utilising the 2-functors  $\mathcal{F}'_{1,1}$  and  $\mathcal{F}''_{2,1}$  as defined in [1, Section 4], and the 2-isomorphisms  $\omega, \psi$  defined in [11] as follows. The 2-isomorphism  $\omega : \mathcal{U}(3) \rightarrow \mathcal{U}(3)$  is 1- and 2-covariant and degree-preserving, and sends a weight  $\lambda$  to  $-\lambda$ , reverses the orientation of 2-morphisms, and scales the 1, 1- and 2, 2-crossings by a factor of  $-1$ . Similarly,  $\psi : \mathcal{U}(3) \rightarrow \mathcal{U}(3)^{co}$  is a 1-covariant, 2-contravariant 2-isomorphism that is the identity on objects, scales weights of 1-morphisms by a factor of  $-1$ , and reflects diagrams of 2-morphisms in the horizontal axis and then reverses their orientation. We recall that  $k_i^{a_1, \dots, a_n}(\mu)$ , defined in (40), omits the argument  $\mu$  when it is equal to  $\lambda$ , but retains it otherwise (generally when it is  $s_1(\lambda)$  or  $s_2(\lambda)$ ).

We also use the 2-isomorphism  $\zeta : \mathcal{U}(3) \rightarrow \tilde{\mathcal{U}}(3)$ , first defined as  $\Sigma$  in [10, Section 4.2] and [12], which is the identity on objects and 1-morphisms, and the identity on 2-morphisms except for the following generating 2-morphisms (and hence the 2-morphisms derived from them):

$$\begin{aligned}
 (46) \quad & \begin{array}{c} \uparrow \\ \lambda \\ \uparrow \end{array} \xrightarrow{\zeta} \begin{array}{c} \uparrow \\ -\lambda \\ \uparrow \end{array} & \begin{array}{c} \nearrow \\ \lambda \\ \searrow \end{array} \xrightarrow{\zeta} \begin{array}{c} \nwarrow \\ \lambda \\ \nearrow \end{array} \\
 (47) \quad & \begin{array}{c} \curvearrowright \\ \lambda \\ \downarrow \end{array} \xrightarrow{\zeta} (-1)^{\lambda_1+1} k_1^2 \begin{array}{c} \curvearrowleft \\ \lambda \\ \downarrow \end{array} & \begin{array}{c} \downarrow \\ \lambda \\ \curvearrowright \end{array} \xrightarrow{\zeta} (-1)^{\lambda_1} k_1^0 \begin{array}{c} \downarrow \\ \lambda \\ \curvearrowleft \end{array} \\
 (48) \quad & \begin{array}{c} \curvearrowright \\ \lambda \\ \downarrow \end{array} \xrightarrow{\zeta} -k_1^0 \begin{array}{c} \curvearrowleft \\ \lambda \\ \downarrow \end{array} & \begin{array}{c} \downarrow \\ \lambda \\ \curvearrowright \end{array} \xrightarrow{\zeta} -k_1^2 \begin{array}{c} \downarrow \\ \lambda \\ \curvearrowleft \end{array} \\
 (49) \quad & \begin{array}{c} \curvearrowright \\ \lambda \\ \downarrow \\ 2 \end{array} \xrightarrow{\zeta} (-1)^{\lambda_3-1} \begin{array}{c} \curvearrowleft \\ \lambda \\ \downarrow \\ 2 \end{array} & \begin{array}{c} \downarrow \\ \lambda \\ \curvearrowright \\ 2 \end{array} \xrightarrow{\zeta} (-1)^{\lambda_3} \begin{array}{c} \downarrow \\ \lambda \\ \curvearrowleft \\ 2 \end{array}
 \end{aligned}$$

We now define two 2-functors  $\tilde{\mathcal{F}}'_{1,-1}, \tilde{\mathcal{F}}''_{2,1} : \tilde{\mathcal{U}}(3) \rightarrow K^b(\tilde{\mathcal{U}}(3))$  using composites of the above 2-functors:

$$\tilde{\mathcal{F}}'_{1,-1} := \zeta \psi \mathcal{F}'_{1,1} \psi \zeta^{-1} = \zeta \mathcal{F}'_{1,-1} \zeta^{-1}, \quad \tilde{\mathcal{F}}''_{2,1} := \zeta \omega \mathcal{F}''_{2,1} \omega \zeta^{-1} = \zeta \mathcal{F}''_{2,1} \zeta^{-1}.$$

We let  $X[y]\langle z \rangle$  denote the 1-term complex with the 1-morphism at homological degree  $-y$  with internal degree shift of  $z$ . In detail,  $\tilde{\mathcal{F}}'_{1,-1}$  acts as follows:



In detail,  $\tilde{\mathcal{F}}''_{2,1}$  acts as follows:

- On objects,  $\lambda \xrightarrow{\tilde{\mathcal{F}}''_{2,1}} s_2(\lambda)$ .
- On 1-morphisms,

$$\begin{aligned} \mathcal{E}_1 \mathbf{1}_\lambda &\xrightarrow{\tilde{\mathcal{F}}''_{2,1}} \left( \mathcal{E}_{12} \mathbf{1}_{s_2(\lambda)} \langle -1 \rangle \xrightarrow{\begin{array}{c} \text{red } \lambda \\ \text{blue } 2 \end{array}} \mathcal{E}_{21} \mathbf{1}_{s_2(\lambda)} \right), \\ \mathcal{F}_1 \mathbf{1}_\lambda &\xrightarrow{\tilde{\mathcal{F}}''_{2,1}} \left( \mathcal{F}_{12} \mathbf{1}_{s_2(\lambda)} \xrightarrow{\begin{array}{c} k_3^{2,3} \\ \text{red } \lambda \\ \text{blue } 2 \end{array}} \mathcal{F}_{21} \mathbf{1}_{s_2(\lambda)} \langle 1 \rangle \right), \\ \mathcal{E}_2 \mathbf{1}_\lambda &\xrightarrow{\tilde{\mathcal{F}}''_{2,1}} \mathcal{F}_2 \mathbf{1}_{s_2(\lambda)} [-1] \langle \bar{\lambda}_2 \rangle, \quad \mathcal{F}_2 \mathbf{1}_\lambda \xrightarrow{\tilde{\mathcal{F}}''_{2,1}} \mathcal{E}_2 \mathbf{1}_{s_2(\lambda)} [1] \langle -2 - \bar{\lambda}_2 \rangle. \end{aligned}$$

- On nonidentity generating 2-morphisms,

$$(61) \quad \begin{array}{c} \uparrow \\ \text{blue } 1 \end{array} \xrightarrow{\tilde{\mathcal{F}}''_{2,1}} \left( \begin{array}{c} \uparrow \\ \text{blue } 1 \\ \text{red } 2 \end{array} s_2(\lambda), \begin{array}{c} \uparrow \\ \text{red } 2 \\ \text{blue } 1 \end{array} s_2(\lambda) \right), \quad \begin{array}{c} \uparrow \\ \text{red } 2 \end{array} \xrightarrow{\tilde{\mathcal{F}}''_{2,1}} \begin{array}{c} \downarrow \\ \text{red } 2 \end{array} s_2(\lambda), \quad \begin{array}{c} \text{red } \lambda \\ \text{blue } 2 \\ \text{red } 2 \end{array} \xrightarrow{\tilde{\mathcal{F}}''_{2,1}} - \begin{array}{c} \text{red } \lambda \\ \text{blue } 2 \\ \text{red } 2 \end{array} s_2(\lambda),$$

$$(62) \quad \begin{array}{c} \text{blue } \lambda \\ \text{red } 1 \\ \text{blue } 1 \end{array} \xrightarrow{\tilde{\mathcal{F}}''_{2,1}} \left( \begin{array}{c} \text{blue } \lambda \\ \text{red } 2 \\ \text{blue } 1 \\ \text{red } 2 \end{array} s_2(\lambda), \right. \\ \left. k_3^{1,3} \begin{array}{c} \text{red } \lambda \\ \text{blue } 2 \\ \text{red } 1 \\ \text{blue } 2 \end{array} s_2(\lambda) + k_3^{1,3} \begin{array}{c} \text{red } \lambda \\ \text{blue } 2 \\ \text{red } 2 \\ \text{blue } 1 \end{array} s_2(\lambda) + \begin{array}{c} \uparrow \\ \text{red } 2 \\ \text{blue } 1 \\ \text{red } 2 \end{array} s_2(\lambda) + \begin{array}{c} \uparrow \\ \text{red } 2 \\ \text{blue } 2 \\ \text{red } 1 \end{array} s_2(\lambda), - \begin{array}{c} \text{red } \lambda \\ \text{blue } 1 \\ \text{red } 2 \\ \text{blue } 1 \end{array} s_2(\lambda) \right),$$

$$(63) \quad \begin{array}{c} \text{red } \lambda \\ \text{blue } 1 \\ \text{red } 2 \end{array} \xrightarrow{\tilde{\mathcal{F}}''_{2,1}} \left( \begin{array}{c} \text{red } \lambda \\ \text{blue } 2 \\ \text{red } 1 \\ \text{red } 2 \end{array} s_2(\lambda), - \begin{array}{c} \text{red } \lambda \\ \text{blue } 2 \\ \text{red } 2 \\ \text{blue } 1 \end{array} s_2(\lambda) \right),$$

$$(64) \quad \begin{array}{c} \text{red } \lambda \\ \text{blue } 1 \\ \text{red } 2 \end{array} \xrightarrow{\tilde{\mathcal{F}}''_{2,1}} \left( \begin{array}{c} \text{red } \lambda \\ \text{blue } 2 \\ \text{red } 2 \\ \text{red } 2 \end{array} s_2(\lambda) - \begin{array}{c} \text{red } \lambda \\ \text{blue } 2 \\ \text{red } 2 \\ \text{red } 2 \end{array} s_2(\lambda), \begin{array}{c} \text{red } \lambda \\ \text{blue } 2 \\ \text{red } 1 \\ \text{red } 2 \end{array} s_2(\lambda) - \begin{array}{c} \text{red } \lambda \\ \text{blue } 2 \\ \text{red } 1 \\ \text{red } 2 \end{array} s_2(\lambda) \right),$$

$$(65) \quad \begin{array}{c} \text{red } \lambda \\ \text{red } 2 \end{array} \xrightarrow{\tilde{\mathcal{F}}''_{2,1}} (-1)^{\lambda_2+1} \begin{array}{c} \text{red } \lambda \\ \text{red } 2 \end{array} s_2(\lambda), \quad \begin{array}{c} \text{red } \lambda \\ \text{red } 2 \end{array} \xrightarrow{\tilde{\mathcal{F}}''_{2,1}} (-1)^{\lambda_3+1} \begin{array}{c} \text{red } \lambda \\ \text{red } 2 \end{array} s_2(\lambda),$$

$$(66) \quad \begin{array}{c} \text{red } 2 \\ \text{red } \lambda \end{array} \xrightarrow{\tilde{\mathcal{F}}''_{2,1}} (-1)^{\lambda_2} \begin{array}{c} \text{red } 2 \\ \text{red } \lambda \end{array} s_2(\lambda), \quad \begin{array}{c} \text{red } 2 \\ \text{red } \lambda \end{array} \xrightarrow{\tilde{\mathcal{F}}''_{2,1}} (-1)^{\lambda_3} \begin{array}{c} \text{red } 2 \\ \text{red } \lambda \end{array} s_2(\lambda),$$

$$(67) \quad \begin{array}{c} \text{red } \lambda \\ \text{blue } 1 \end{array} \xrightarrow{\tilde{\mathcal{F}}''_{2,1}} -k_1^0 \left( k_3^3 \begin{array}{c} \text{red } \lambda \\ \text{red } 2 \\ \text{blue } 1 \end{array} s_2(\lambda) - k_3^2 \begin{array}{c} \text{red } \lambda \\ \text{red } 2 \\ \text{blue } 1 \end{array} s_2(\lambda) \right),$$

$$(68) \quad \begin{array}{c} \text{red } \lambda \\ \text{blue } 1 \end{array} \xrightarrow{\tilde{\mathcal{F}}''_{2,1}} (-1)^{\lambda_1} k_1^2 \left( k_3^1 \begin{array}{c} \text{red } \lambda \\ \text{red } 2 \\ \text{blue } 1 \end{array} s_2(\lambda) - k_3^2 \begin{array}{c} \text{red } \lambda \\ \text{red } 2 \\ \text{blue } 1 \end{array} s_2(\lambda) \right),$$

$$(69) \quad \begin{array}{c} \text{red } \lambda \\ \text{blue } 1 \end{array} \xrightarrow{\tilde{\mathcal{F}}''_{2,1}} (-1)^{\lambda_1} k_1^0 \left( k_3^1 \begin{array}{c} \text{red } 2 \\ \text{blue } 1 \\ \text{red } \lambda \end{array} s_2(\lambda) - k_3^0 \begin{array}{c} \text{red } 2 \\ \text{blue } 1 \\ \text{red } \lambda \end{array} s_2(\lambda) \right),$$

$$(70) \quad \begin{array}{c} \text{red } \lambda \\ \text{blue } 1 \end{array} \xrightarrow{\tilde{\mathcal{F}}''_{2,1}} k_1^2 \left( k_3^3 \begin{array}{c} \text{red } 2 \\ \text{blue } 1 \\ \text{red } \lambda \end{array} s_2(\lambda) - k_3^0 \begin{array}{c} \text{red } 2 \\ \text{blue } 1 \\ \text{red } \lambda \end{array} s_2(\lambda) \right).$$

**5.2. Relating the two actions.** We define a 2-automorphism  $\beta : \tilde{u}(3) \rightarrow \tilde{u}(3)$ , which is the identity on objects, 1-morphisms and all generating 2-morphisms with the exception of

$$\begin{array}{cc} \begin{array}{c} \lambda \\ \curvearrowright \\ 1 \end{array} \xrightarrow{\beta} -k_1^2 \begin{array}{c} \lambda \\ \curvearrowleft \\ 1 \end{array}, & \begin{array}{c} 1 \\ \curvearrowright \\ \lambda \end{array} \xrightarrow{\beta} -k_1^2 \begin{array}{c} 1 \\ \curvearrowleft \\ \lambda \end{array}, \\ \begin{array}{c} \lambda \\ \curvearrowleft \\ 1 \end{array} \xrightarrow{\beta} -k_1^0 \begin{array}{c} \lambda \\ \curvearrowright \\ 1 \end{array}, & \begin{array}{c} 1 \\ \curvearrowleft \\ \lambda \end{array} \xrightarrow{\beta} -k_1^0 \begin{array}{c} 1 \\ \curvearrowright \\ \lambda \end{array}. \end{array}$$

It is a straightforward calculation to confirm that this preserves the axioms (KM1)–(KM3) and (KM7)–(KM9) and is therefore a well-defined 2-automorphism.

We also define a 2-automorphism  $\alpha : \tilde{u}(3) \rightarrow \tilde{u}(3)$  which is the identity on all objects and 1-morphisms and on all generating 2-morphisms except

$$\begin{array}{cc} \begin{array}{c} \lambda \\ \times \\ 2 \quad 1 \end{array} \xrightarrow{\alpha} k_1^{0,1} \begin{array}{c} \lambda \\ \times \\ 2 \quad 1 \end{array}, & \begin{array}{c} \lambda \\ \times \\ 1 \quad 2 \end{array} \xrightarrow{\alpha} k_1^{0,1} \begin{array}{c} \lambda \\ \times \\ 1 \quad 2 \end{array}. \end{array}$$

It is easy to see that  $\alpha$  preserves axioms (KM3)–(KM7), and is therefore a 2-automorphism. We also use  $\alpha$  to denote its extension to a 2-automorphism of  $K^b(\tilde{u}(3))$ .

We now define some notation for ease of stating the following lemma. Let  $D_i(\lambda)$  denote a 2-morphism diagram with strands mono-coloured in colour  $i$  such that the weight to the right of the diagram is  $\lambda$ .

**Lemma 5.1.** *For any diagram  $D_i(\lambda)$ ,*

$$\tilde{\mathcal{F}}'_{1,-1}(D_2(s_1(\lambda))) \sim_h \alpha \tilde{\mathcal{F}}''_{2,1}(\beta(D_1(s_2(\lambda)))).$$

*Proof.* The proof follows from calculations that, while not complicated, are liable to confuse. We therefore present them below.

$$\begin{aligned} & \alpha \tilde{\mathcal{F}}''_{2,1}(\beta(\begin{array}{c} \lambda \\ \times \\ 1 \quad 1 \end{array}_{s_2(\lambda)})) \\ &= \alpha \tilde{\mathcal{F}}''_{2,1}(\begin{array}{c} \lambda \\ \times \\ 1 \quad 1 \end{array}_{s_2(\lambda)}) \\ &= \alpha(\begin{array}{c} \lambda \\ \times \\ 1 \quad 2 \end{array}, k_3^{1,3}(s_2(\lambda)) \begin{array}{c} \lambda \\ \times \\ 2 \quad 1 \end{array} + k_3^{1,3}(s_2(\lambda)) \begin{array}{c} \lambda \\ \times \\ 1 \quad 2 \end{array} + \begin{array}{c} \lambda \\ \uparrow \\ 2 \quad 1 \end{array} \begin{array}{c} \lambda \\ \uparrow \\ 2 \quad 1 \end{array} + \begin{array}{c} \lambda \\ \uparrow \\ 1 \quad 2 \end{array} \begin{array}{c} \lambda \\ \uparrow \\ 1 \quad 2 \end{array} - \begin{array}{c} \lambda \\ \times \\ 2 \quad 1 \end{array}) \\ &= (\begin{array}{c} \lambda \\ \times \\ 1 \quad 2 \end{array}, -\begin{array}{c} \lambda \\ \times \\ 2 \quad 1 \end{array} - \begin{array}{c} \lambda \\ \times \\ 1 \quad 2 \end{array} + \begin{array}{c} \lambda \\ \uparrow \\ 2 \quad 1 \end{array} \begin{array}{c} \lambda \\ \uparrow \\ 2 \quad 1 \end{array} + \begin{array}{c} \lambda \\ \uparrow \\ 1 \quad 2 \end{array} \begin{array}{c} \lambda \\ \uparrow \\ 1 \quad 2 \end{array}, -\begin{array}{c} \lambda \\ \times \\ 2 \quad 1 \end{array}) \\ &= \tilde{\mathcal{F}}'_{1,-1}(\begin{array}{c} \lambda \\ \times \\ 2 \quad 2 \end{array}_{s_1(\lambda)}). \end{aligned}$$

$$\begin{aligned}
 \alpha \tilde{\mathcal{T}}'_{2,1}''(\beta(\downarrow_1 \curvearrowright^{s_2(\lambda)})) &= \alpha \tilde{\mathcal{T}}'_{2,1}''(-k_1^2(s_2(\lambda)) \downarrow_1 \curvearrowright^{s_2(\lambda)}) \\
 &= \alpha(-k_1^2(s_2(\lambda)) k_1^0(s_2(\lambda)) (-k_3^3(s_2(\lambda)) \downarrow_2 \curvearrowright^{\lambda} + k_3^2(s_2(\lambda)) \downarrow_1 \curvearrowright^{\lambda})) \\
 &= (-1)^{\bar{\lambda}_3+1} (k_1^3(s_1(\lambda)) \downarrow_2 \curvearrowright^{\lambda} - k_1^2(s_1(\lambda)) \downarrow_1 \curvearrowright^{\lambda}) \\
 &= \tilde{\mathcal{T}}'_{1,-1}(\downarrow_2 \curvearrowright^{s_1(\lambda)}).
 \end{aligned}$$

$$\begin{aligned}
 \alpha \tilde{\mathcal{T}}'_{2,1}''(\beta(\uparrow_1 \curvearrowleft^{s_2(\lambda)})) &= \alpha \tilde{\mathcal{T}}'_{2,1}''(-k_1^2(s_2(\lambda)) \uparrow_1 \curvearrowleft^{s_2(\lambda)}) \\
 &= \alpha((-1)^{s_2(\lambda)_1+1} k_1^2(s_2(\lambda)) k_1^0(s_2(\lambda)) (k_3^1(s_2(\lambda)) \uparrow_2 \curvearrowleft^{\lambda} - k_3^0(s_2(\lambda)) \uparrow_1 \curvearrowleft^{\lambda})) \\
 &= (-1)^{\lambda_3} (k_1^1(s_1(\lambda)) \uparrow_2 \curvearrowleft^{\lambda} - k_1^0(s_1(\lambda)) \uparrow_1 \curvearrowleft^{\lambda}) \\
 &= \tilde{\mathcal{T}}'_{1,-1}(\uparrow_2 \curvearrowleft^{s_1(\lambda)}).
 \end{aligned}$$

$$\begin{aligned}
 \alpha \tilde{\mathcal{T}}'_{2,1}''(\beta(\downarrow_1 \curvearrowleft^{s_2(\lambda)})) &= \alpha \tilde{\mathcal{T}}'_{2,1}''(-k_1^0(s_2(\lambda)) \downarrow_1 \curvearrowleft^{s_2(\lambda)}) \\
 &= \alpha((-1)^{s_2(\lambda)_1+1} k_1^0(s_2(\lambda)) k_1^2(s_2(\lambda)) (k_3^1(s_2(\lambda)) \downarrow_2 \curvearrowleft^{\lambda} - k_3^2(s_2(\lambda)) \downarrow_1 \curvearrowleft^{\lambda})) \\
 &= (-1)^{\lambda_3} (k_1^1(s_1(\lambda)) \downarrow_2 \curvearrowleft^{\lambda} - k_1^2(s_1(\lambda)) \downarrow_1 \curvearrowleft^{\lambda}) \\
 &= \tilde{\mathcal{T}}'_{1,-1}(\downarrow_2 \curvearrowleft^{s_1(\lambda)}).
 \end{aligned}$$

$$\begin{aligned}
 \alpha \tilde{\mathcal{T}}'_{2,1}''(\beta(\uparrow_1 \curvearrowright^{s_2(\lambda)})) &= \alpha \tilde{\mathcal{T}}'_{2,1}''(-k_1^0(s_2(\lambda)) \uparrow_1 \curvearrowright^{s_2(\lambda)}) \\
 &= \alpha(-k_1^0(s_2(\lambda)) k_1^2(s_2(\lambda)) (k_3^3(s_2(\lambda)) \uparrow_2 \curvearrowright^{\lambda} - k_3^0(s_2(\lambda)) \uparrow_1 \curvearrowright^{\lambda})) \\
 &= (-1)^{\bar{\lambda}_3} (k_1^3(s_1(\lambda)) \uparrow_2 \curvearrowright^{\lambda} - k_1^0(s_1(\lambda)) \uparrow_1 \curvearrowright^{\lambda}) \\
 &= \tilde{\mathcal{T}}'_{1,-1}(\uparrow_2 \curvearrowright^{s_1(\lambda)}).
 \end{aligned}$$

The remaining element of the proof is comparing

$$\tilde{\mathcal{T}}'_{1,-1}(\uparrow_2 \curvearrowright^{s_1(\lambda)}) = \left( \uparrow_1 \uparrow_2 \uparrow_2 \uparrow_1 \right) \curvearrowright^{\lambda} \quad \text{and} \quad \alpha \tilde{\mathcal{T}}'_{2,1}''(\beta(\uparrow_1 \curvearrowright^{s_2(\lambda)})) = \left( \uparrow_1 \uparrow_2 \uparrow_2 \uparrow_1 \right) \curvearrowright^{\lambda},$$

which are equal up to homotopy, by (45).  $\square$

The following result will often be used in proving the main theorem:

**Lemma 5.2.**  $\tilde{\mathcal{T}}'_{1,-1}$  and  $\alpha \tilde{\mathcal{T}}'_{2,1}'' \beta$  preserve all the KM identities.

*Proof.* All component 2-functors of these 2-functors are either 2-isomorphisms (which clearly preserve defining axioms) or preserve identities (KM1)–(KM9) by [1, Section 4].  $\square$

### 6. Proof of Theorem 4.3

As a last bit of preparation before proving Theorem 4.3, we define some extra 2-functors to account for  $\varepsilon_{\nu'}$  having a different domain 2-category to  $\tilde{\mathcal{T}}'_{1,-1}$  and  $\tilde{\mathcal{T}}''_{2,1}$ , and for categorifying the powers of  $q$  and signs found in Section 2.2.1. We remind the reader that  $k_i^{a_1, \dots, a_n}(\mu)$  omits the argument  $\mu$  when it is equal to  $\lambda$ , but retains it otherwise (generally when it is  $s_1(\lambda)$  or  $s_2(\lambda)$ ).

**6.1. Embeddings.** We define 2-embeddings  $\iota, \iota' : \tilde{\mathcal{U}}(3) \rightarrow \tilde{\mathcal{U}}_{\Delta}(3)$  using other component 2-functors. First, we define  $\tilde{\omega} = \zeta \omega \zeta^{-1} : \tilde{\mathcal{U}}(3) \rightarrow \tilde{\mathcal{U}}(3)$ . Second, we define 2-functors  $\eta, \eta' : \tilde{\mathcal{U}}(3) \rightarrow \tilde{\mathcal{U}}_{\Delta}(3)$  as follows.

For  $\eta$ :

- On objects,  $\lambda \xrightarrow{\eta} -s_1(\lambda)$ .
- On 1-morphisms,

$$\begin{aligned} \mathcal{E}_1 \mathbf{1}_{\lambda} &\xrightarrow{\eta} \mathcal{E}_1 \mathbf{1}_{-s_1(\lambda)}, & \mathcal{F}_1 \mathbf{1}_{\lambda} &\xrightarrow{\eta} \mathcal{F}_1 \mathbf{1}_{-s_1(\lambda)}, \\ \mathcal{E}_2 \mathbf{1}_{\lambda} &\xrightarrow{\eta} \mathcal{E}_3 \mathbf{1}_{-s_1(\lambda)}, & \mathcal{F}_2 \mathbf{1}_{\lambda} &\xrightarrow{\eta} \mathcal{F}_3 \mathbf{1}_{-s_1(\lambda)}. \end{aligned}$$

- On 2-morphisms,

$$\begin{aligned} (71) \quad & \begin{array}{ccc} \uparrow \lambda & \xrightarrow{\eta} & \uparrow -s_1(\lambda) \\ | & & | \\ \downarrow & & \downarrow \end{array} \\ (72) \quad & \begin{array}{ccc} \begin{array}{ccc} \uparrow & \xrightarrow{\eta} & \uparrow \\ \diagdown & & \diagup \\ \downarrow & & \downarrow \end{array} & \xrightarrow{\eta} & \begin{array}{ccc} \uparrow & \xrightarrow{\eta} & \uparrow \\ \diagdown & & \diagup \\ \downarrow & & \downarrow \end{array} \\ \downarrow & & \downarrow \end{array} \\ (73) \quad & \begin{array}{ccc} \begin{array}{ccc} \uparrow & \xrightarrow{\eta} & \uparrow \\ \diagdown & & \diagup \\ \downarrow & & \downarrow \end{array} & \xrightarrow{\eta} & \begin{array}{ccc} \uparrow & \xrightarrow{\eta} & \uparrow \\ \diagdown & & \diagup \\ \downarrow & & \downarrow \end{array} \\ \downarrow & & \downarrow \end{array} \\ (74) \quad & \begin{array}{ccc} \begin{array}{ccc} \uparrow & \xrightarrow{\eta} & \uparrow \\ \downarrow & & \downarrow \end{array} & \xrightarrow{\eta} & \begin{array}{ccc} \uparrow & \xrightarrow{\eta} & \uparrow \\ \downarrow & & \downarrow \end{array} \\ \downarrow & & \downarrow \end{array} \\ (75) \quad & \begin{array}{ccc} \begin{array}{ccc} \uparrow & \xrightarrow{\eta} & \uparrow \\ \downarrow & & \downarrow \end{array} & \xrightarrow{\eta} & \begin{array}{ccc} \uparrow & \xrightarrow{\eta} & \uparrow \\ \downarrow & & \downarrow \end{array} \\ \downarrow & & \downarrow \end{array} \\ (76) \quad & \begin{array}{ccc} \begin{array}{ccc} \uparrow & \xrightarrow{\eta} & \uparrow \\ \downarrow & & \downarrow \end{array} & \xrightarrow{\eta} & \begin{array}{ccc} \uparrow & \xrightarrow{\eta} & \uparrow \\ \downarrow & & \downarrow \end{array} \\ \downarrow & & \downarrow \end{array} \\ (77) \quad & \begin{array}{ccc} \begin{array}{ccc} \uparrow & \xrightarrow{\eta} & \uparrow \\ \downarrow & & \downarrow \end{array} & \xrightarrow{\eta} & \begin{array}{ccc} \uparrow & \xrightarrow{\eta} & \uparrow \\ \downarrow & & \downarrow \end{array} \\ \downarrow & & \downarrow \end{array} \end{aligned}$$

For  $\eta'$ :

- On objects,  $\lambda \xrightarrow{\eta'} -s_2(\lambda)$ .
- On 1-morphisms,

$$\begin{aligned} \mathcal{E}_2 \mathbf{1}_{\lambda} &\xrightarrow{\eta'} \mathcal{E}_2 \mathbf{1}_{-s_2(\lambda)}, & \mathcal{F}_2 \mathbf{1}_{\lambda} &\xrightarrow{\eta'} \mathcal{F}_2 \mathbf{1}_{-s_2(\lambda)}, \\ \mathcal{E}_1 \mathbf{1}_{\lambda} &\xrightarrow{\eta'} \mathcal{E}_3 \mathbf{1}_{-s_2(\lambda)}, & \mathcal{F}_1 \mathbf{1}_{\lambda} &\xrightarrow{\eta'} \mathcal{F}_3 \mathbf{1}_{-s_2(\lambda)}. \end{aligned}$$

- On 2-morphisms,

$$\begin{array}{ll}
 (78) & \begin{array}{ccc} \uparrow \lambda & \xrightarrow{\eta'} & \uparrow -s_2(\lambda) \\ 2 & & 2 \end{array} & \begin{array}{ccc} \uparrow \lambda & \xrightarrow{\eta'} & \uparrow -s_2(\lambda) \\ 1 & & 3 \end{array} \\
 (79) & \begin{array}{ccc} \begin{array}{c} \diagup \lambda \\ 2 \quad 2 \end{array} & \xrightarrow{\eta'} & \begin{array}{c} \diagup -s_2(\lambda) \\ 2 \quad 2 \end{array} & \begin{array}{ccc} \begin{array}{c} \diagup \lambda \\ 1 \quad 1 \end{array} & \xrightarrow{\eta'} & \begin{array}{c} \diagup -s_2(\lambda) \\ 3 \quad 3 \end{array} \\
 (80) & \begin{array}{ccc} \begin{array}{c} \diagup \lambda \\ 2 \quad 1 \end{array} & \xrightarrow{\eta'} & \begin{array}{c} \diagup -s_2(\lambda) \\ 2 \quad 3 \end{array} & \begin{array}{ccc} \begin{array}{c} \diagup \lambda \\ 1 \quad 2 \end{array} & \xrightarrow{\eta'} & \begin{array}{c} \diagup -s_2(\lambda) \\ 3 \quad 2 \end{array} \\
 (81) & \begin{array}{ccc} \begin{array}{c} \curvearrowright \lambda \\ 2 \end{array} & \xrightarrow{\eta'} & \begin{array}{c} \curvearrowright -s_2(\lambda) \\ 2 \end{array} & \begin{array}{ccc} \begin{array}{c} \curvearrowright \lambda \\ 2 \end{array} & \xrightarrow{\eta'} & \begin{array}{c} \curvearrowright -s_2(\lambda) \\ (-1)^{\bar{\lambda}_2} 1 \end{array} \\
 (82) & \begin{array}{ccc} \begin{array}{c} \curvearrowright \lambda \\ 2 \end{array} & \xrightarrow{\eta'} & \begin{array}{c} \curvearrowright -s_2(\lambda) \\ 2 \end{array} & \begin{array}{ccc} \begin{array}{c} \curvearrowright \lambda \\ 2 \end{array} & \xrightarrow{\eta'} & \begin{array}{c} \curvearrowright -s_2(\lambda) \\ (-1)^{\bar{\lambda}_2} 2 \end{array} \\
 (83) & \begin{array}{ccc} \begin{array}{c} \curvearrowleft \lambda \\ 1 \end{array} & \xrightarrow{\eta'} & \begin{array}{c} \curvearrowleft -s_2(\lambda) \\ 3 \end{array} & \begin{array}{ccc} \begin{array}{c} \curvearrowleft \lambda \\ 1 \end{array} & \xrightarrow{\eta'} & \begin{array}{c} \curvearrowleft -s_2(\lambda) \\ (-1)^{\bar{\lambda}_1} 3 \end{array} \\
 (84) & \begin{array}{ccc} \begin{array}{c} \curvearrowleft \lambda \\ 1 \end{array} & \xrightarrow{\eta'} & \begin{array}{c} \curvearrowleft -s_2(\lambda) \\ 3 \end{array} & \begin{array}{ccc} \begin{array}{c} \curvearrowleft \lambda \\ 1 \end{array} & \xrightarrow{\eta'} & \begin{array}{c} \curvearrowleft -s_2(\lambda) \\ (-1)^{\bar{\lambda}_1} 3 \end{array}
 \end{array}$$

It is straightforward to check that  $\eta$  and  $\eta'$  preserve (KM1)–(KM9) and are therefore well-defined.

We now define  $\iota = \eta\tilde{\omega}$  and  $\iota' = \eta'\tilde{\omega}$ . Explicitly,  $\iota$  is given by:

- On objects,  $\lambda \xrightarrow{\iota} s_1(\lambda)$ .
- On 1-morphisms,

$$\begin{array}{ll}
 \mathcal{E}_1 \mathbf{1}_\lambda & \xrightarrow{\iota} \mathcal{F}_1 \mathbf{1}_{s_1(\lambda)}, & \mathcal{F}_1 \mathbf{1}_\lambda & \xrightarrow{\iota} \mathcal{E}_1 \mathbf{1}_{s_1(\lambda)}, \\
 \mathcal{E}_2 \mathbf{1}_\lambda & \xrightarrow{\iota} \mathcal{F}_3 \mathbf{1}_{s_1(\lambda)}, & \mathcal{F}_2 \mathbf{1}_\lambda & \xrightarrow{\iota} \mathcal{E}_3 \mathbf{1}_{s_1(\lambda)}.
 \end{array}$$

- On 2-morphisms,

$$\begin{array}{ll}
 (85) & \begin{array}{ccc} \uparrow \lambda & \xrightarrow{\iota} & \downarrow s_1(\lambda) \\ 1 & & 1 \end{array} & \begin{array}{ccc} \uparrow \lambda & \xrightarrow{\iota} & \downarrow s_1(\lambda) \\ 2 & & 3 \end{array} \\
 (86) & \begin{array}{ccc} \begin{array}{c} \diagup \lambda \\ 1 \quad 1 \end{array} & \xrightarrow{\iota} & \begin{array}{c} \diagup s_1(\lambda) \\ 1 \quad 1 \end{array} & \begin{array}{ccc} \begin{array}{c} \diagup \lambda \\ 2 \quad 2 \end{array} & \xrightarrow{\iota} & \begin{array}{c} \diagup s_1(\lambda) \\ 3 \quad 3 \end{array} \\
 (87) & \begin{array}{ccc} \begin{array}{c} \diagup \lambda \\ 1 \quad 2 \end{array} & \xrightarrow{\iota} & \begin{array}{c} \diagup s_1(\lambda) \\ 1 \quad 3 \end{array} & \begin{array}{ccc} \begin{array}{c} \diagup \lambda \\ 2 \quad 1 \end{array} & \xrightarrow{\iota} & \begin{array}{c} \diagup s_1(\lambda) \\ 3 \quad 1 \end{array} \\
 (88) & \begin{array}{ccc} \begin{array}{c} \curvearrowleft \lambda \\ 1 \end{array} & \xrightarrow{\iota} & \begin{array}{c} \curvearrowleft s_1(\lambda) \\ 1 \end{array} & \begin{array}{ccc} \begin{array}{c} \curvearrowleft \lambda \\ 1 \end{array} & \xrightarrow{\iota} & \begin{array}{c} \curvearrowleft s_1(\lambda) \\ 1 \end{array} \\
 (89) & \begin{array}{ccc} \begin{array}{c} \curvearrowleft \lambda \\ 1 \end{array} & \xrightarrow{\iota} & \begin{array}{c} \curvearrowleft s_1(\lambda) \\ 1 \end{array} & \begin{array}{ccc} \begin{array}{c} \curvearrowleft \lambda \\ 1 \end{array} & \xrightarrow{\iota} & \begin{array}{c} \curvearrowleft s_1(\lambda) \\ 1 \end{array} \\
 (90) & \begin{array}{ccc} \begin{array}{c} \curvearrowright \lambda \\ 2 \end{array} & \xrightarrow{\iota} & \begin{array}{c} \curvearrowright s_1(\lambda) \\ 3 \end{array} & \begin{array}{ccc} \begin{array}{c} \curvearrowright \lambda \\ 2 \end{array} & \xrightarrow{\iota} & \begin{array}{c} \curvearrowright s_1(\lambda) \\ 3 \end{array} \\
 (91) & \begin{array}{ccc} \begin{array}{c} \curvearrowright \lambda \\ 2 \end{array} & \xrightarrow{\iota} & \begin{array}{c} \curvearrowright s_1(\lambda) \\ 3 \end{array} & \begin{array}{ccc} \begin{array}{c} \curvearrowright \lambda \\ 2 \end{array} & \xrightarrow{\iota} & \begin{array}{c} \curvearrowright s_1(\lambda) \\ 3 \end{array}
 \end{array}$$

Explicitly,  $\iota'$  is given by:

- On objects,  $\lambda \xrightarrow{\iota'} s_2(\lambda)$ .
- On 1-morphisms,

$$\begin{aligned} \mathcal{E}_2 \mathbf{1}_\lambda &\xrightarrow{\iota'} \mathcal{F}_2 \mathbf{1}_{s_2(\lambda)}, & \mathcal{F}_2 \mathbf{1}_\lambda &\xrightarrow{\iota'} \mathcal{E}_2 \mathbf{1}_{s_2(\lambda)}, \\ \mathcal{E}_1 \mathbf{1}_\lambda &\xrightarrow{\iota'} \mathcal{F}_3 \mathbf{1}_{s_2(\lambda)}, & \mathcal{F}_1 \mathbf{1}_\lambda &\xrightarrow{\iota'} \mathcal{E}_3 \mathbf{1}_{s_2(\lambda)}. \end{aligned}$$

- On 2-morphisms,

$$(92) \quad \begin{array}{c} \uparrow \lambda \\ \downarrow 2 \\ \downarrow 2 \\ \downarrow s_2(\lambda) \end{array} \xrightarrow{\iota'} \begin{array}{c} \downarrow \lambda \\ \downarrow 2 \\ \downarrow 2 \\ \downarrow s_2(\lambda) \end{array}$$

$$(93) \quad \begin{array}{c} \nearrow \lambda \\ \searrow 2 \\ \searrow 2 \\ \searrow s_2(\lambda) \end{array} \xrightarrow{\iota'} \begin{array}{c} \nwarrow \lambda \\ \swarrow 2 \\ \swarrow 2 \\ \swarrow s_2(\lambda) \end{array}$$

$$(94) \quad \begin{array}{c} \nearrow \lambda \\ \searrow 2 \\ \swarrow 1 \\ \searrow s_2(\lambda) \end{array} \xrightarrow{\iota'} \begin{array}{c} \nwarrow \lambda \\ \swarrow 2 \\ \swarrow 3 \\ \swarrow s_2(\lambda) \end{array}$$

$$(95) \quad \begin{array}{c} \curvearrowright \lambda \\ \downarrow 2 \\ \downarrow 2 \\ \downarrow s_2(\lambda) \end{array} \xrightarrow{\iota'} (-1)^{\lambda_2+1} \begin{array}{c} \curvearrowright \lambda \\ \downarrow 2 \\ \downarrow 2 \\ \downarrow s_2(\lambda) \end{array}$$

$$(96) \quad \begin{array}{c} \curvearrowright \lambda \\ \downarrow 2 \\ \downarrow 2 \\ \downarrow s_3(\lambda) \end{array} \xrightarrow{\iota'} (-1)^{\lambda_2} \begin{array}{c} \curvearrowright \lambda \\ \downarrow 2 \\ \downarrow 2 \\ \downarrow s_3(\lambda) \end{array}$$

$$(97) \quad \begin{array}{c} \curvearrowright \lambda \\ \downarrow 1 \\ \downarrow 1 \\ \downarrow s_2(\lambda) \end{array} \xrightarrow{\iota'} (-1)^{\lambda_1+1} \begin{array}{c} \curvearrowright \lambda \\ \downarrow 1 \\ \downarrow 1 \\ \downarrow s_2(\lambda) \end{array}$$

$$(98) \quad \begin{array}{c} \curvearrowright \lambda \\ \downarrow 1 \\ \downarrow 1 \\ \downarrow s_2(\lambda) \end{array} \xrightarrow{\iota'} (-1)^{\lambda_1} \begin{array}{c} \curvearrowright \lambda \\ \downarrow 1 \\ \downarrow 1 \\ \downarrow s_2(\lambda) \end{array}$$

$$\begin{array}{c} \uparrow \lambda \\ \downarrow 1 \\ \downarrow 3 \\ \downarrow s_2(\lambda) \end{array} \xrightarrow{\iota'} \begin{array}{c} \downarrow \lambda \\ \downarrow 1 \\ \downarrow 3 \\ \downarrow s_2(\lambda) \end{array}$$

$$\begin{array}{c} \nwarrow \lambda \\ \swarrow 1 \\ \swarrow 1 \\ \swarrow s_2(\lambda) \end{array} \xrightarrow{\iota'} \begin{array}{c} \nearrow \lambda \\ \searrow 1 \\ \searrow 1 \\ \searrow s_2(\lambda) \end{array}$$

$$\begin{array}{c} \nwarrow \lambda \\ \swarrow 1 \\ \swarrow 2 \\ \swarrow s_2(\lambda) \end{array} \xrightarrow{\iota'} \begin{array}{c} \nearrow \lambda \\ \searrow 1 \\ \searrow 3 \\ \searrow s_2(\lambda) \end{array}$$

$$\begin{array}{c} \curvearrowright \lambda \\ \downarrow 2 \\ \downarrow 2 \\ \downarrow s_2(\lambda) \end{array} \xrightarrow{\iota'} (-1)^{\lambda_3+1} \begin{array}{c} \curvearrowright \lambda \\ \downarrow 2 \\ \downarrow 2 \\ \downarrow s_2(\lambda) \end{array}$$

$$\begin{array}{c} \curvearrowright \lambda \\ \downarrow 2 \\ \downarrow 2 \\ \downarrow s_2(\lambda) \end{array} \xrightarrow{\iota'} (-1)^{\lambda_3} \begin{array}{c} \curvearrowright \lambda \\ \downarrow 2 \\ \downarrow 2 \\ \downarrow s_2(\lambda) \end{array}$$

$$\begin{array}{c} \curvearrowright \lambda \\ \downarrow 1 \\ \downarrow 1 \\ \downarrow s_2(\lambda) \end{array} \xrightarrow{\iota'} (-1)^{\lambda_2+1} \begin{array}{c} \curvearrowright \lambda \\ \downarrow 1 \\ \downarrow 1 \\ \downarrow s_2(\lambda) \end{array}$$

$$\begin{array}{c} \curvearrowright \lambda \\ \downarrow 1 \\ \downarrow 1 \\ \downarrow s_2(\lambda) \end{array} \xrightarrow{\iota'} (-1)^{\lambda_2} \begin{array}{c} \curvearrowright \lambda \\ \downarrow 1 \\ \downarrow 1 \\ \downarrow s_2(\lambda) \end{array}$$

**6.2. Degree shifts.** Finally, we introduce two “shift” 2-isomorphisms

$$\sigma_1, \sigma_2 : K^b(\tilde{U}(3)) \rightarrow K^b(\tilde{U}(3)).$$

They shift the homological degree and internal degree of complexes, but otherwise act as the identity. We recall that  $X[y]\langle z \rangle$  is the 1-term complex with the 1-morphism at homological degree  $-y$  with grade shift  $z$ . Specifically, we define  $\sigma_1$  on the generating 1-morphisms by

- $\mathcal{E}_1 \mathbf{1}_\lambda[y]\langle z \rangle \xrightarrow{\sigma_1} \mathcal{E}_1 \mathbf{1}_\lambda[y-1]\langle z - \bar{\lambda}_1 \rangle$ ,
- $\mathcal{F}_1 \mathbf{1}_\lambda[y]\langle z \rangle \xrightarrow{\sigma_1} \mathcal{F}_1 \mathbf{1}_\lambda[y+1]\langle z - 2 + \bar{\lambda}_1 \rangle$ ,
- $\mathcal{E}_2 \mathbf{1}_\lambda[y]\langle z \rangle \xrightarrow{\sigma_1} \mathcal{E}_2 \mathbf{1}_\lambda[y+1]\langle z - \lambda_2 - \lambda_3 - t \rangle$ ,
- $\mathcal{F}_2 \mathbf{1}_\lambda[y]\langle z \rangle \xrightarrow{\sigma_1} \mathcal{F}_2 \mathbf{1}_\lambda[y-1]\langle z + \lambda_2 + \lambda_3 + t \rangle$ ,

and  $\sigma_2$  on the generating 1-morphisms by

- $\mathcal{E}_1 \mathbf{1}_\lambda[y]\langle z \rangle \xrightarrow{\sigma_2} \mathcal{E}_1 \mathbf{1}_\lambda[y+1]\langle z - 2\lambda_3 - \bar{\lambda}_1 - t + 4 \rangle$ ,
- $\mathcal{F}_1 \mathbf{1}_\lambda[y]\langle z \rangle \xrightarrow{\sigma_2} \mathcal{F}_1 \mathbf{1}_\lambda[y-1]\langle z + 2\lambda_3 + \bar{\lambda}_1 + t - 4 \rangle$ ,





For  $\varepsilon_{\nu'} \circ \iota'$ :

$$\begin{aligned}
 (117) \quad \varepsilon_{\nu'} \iota'(\mathcal{E}_1 \mathbf{1}_\lambda) &= \varepsilon_{\nu'}(\mathcal{F}_3 \mathbf{1}_{s_2(\lambda)}) \\
 &= \mathcal{E}_{12} \mathbf{1}_{s_2(\lambda)} \langle -S(s_2(\lambda)) - 1 \rangle \xrightarrow{\text{diagram}} \underline{\mathcal{E}_{21} \mathbf{1}_{s_2(\lambda)}} \langle -S(s_2(\lambda)) \rangle \\
 &= \sigma_2 \alpha \tilde{\mathcal{T}}_{2,1}'' \beta(\mathcal{E}_1 \mathbf{1}_\lambda),
 \end{aligned}$$

$$\begin{aligned}
 (118) \quad \varepsilon_{\nu'} \iota'(\mathcal{F}_1 \mathbf{1}_\lambda) &= \varepsilon_{\nu'}(\mathcal{E}_3 \mathbf{1}_{s_2(\lambda)}) \\
 &= \underline{\mathcal{F}_{12} \mathbf{1}_{s_2(\lambda)}} \langle S(s_2(\lambda)) \rangle \xrightarrow{\text{diagram}} \mathcal{F}_{21} \mathbf{1}_{s_2(\lambda)} \langle S(s_2(\lambda)) + 1 \rangle \\
 &= \sigma_2 \alpha \tilde{\mathcal{T}}_{2,1}'' \beta(\mathcal{F}_1 \mathbf{1}_\lambda),
 \end{aligned}$$

$$(119) \quad \varepsilon_{\nu'} \iota'(\mathcal{E}_2 \mathbf{1}_\lambda) = \varepsilon_{\nu'}(\mathcal{F}_2 \mathbf{1}_\lambda) = \underline{\mathcal{F}_2 \mathbf{1}_{s_2(\lambda)}} = \sigma_2 \alpha \tilde{\mathcal{T}}_{2,1}'' \beta(\mathcal{E}_2 \mathbf{1}_\lambda),$$

$$(120) \quad \varepsilon_{\nu'} \iota'(\mathcal{F}_2 \mathbf{1}_\lambda) = \varepsilon_{\nu'}(\mathcal{E}_2 \mathbf{1}_\lambda) = \underline{\mathcal{E}_2 \mathbf{1}_{s_2(\lambda)}} = \sigma_2 \alpha \tilde{\mathcal{T}}_{2,1}'' \beta(\mathcal{F}_2 \mathbf{1}_\lambda),$$

$$(121) \quad \varepsilon_{\nu'} \iota' \left( \begin{array}{c} \uparrow \lambda \\ \downarrow 2 \end{array} \right) = \varepsilon_{\nu'} \left( \begin{array}{c} \downarrow s_2(\lambda) \\ \downarrow 2 \end{array} \right) = \begin{array}{c} \downarrow s_2(\lambda) \\ \downarrow 2 \end{array} = \tilde{\mathcal{T}}_{2,1}' \left( \begin{array}{c} \uparrow \lambda \\ \downarrow 2 \end{array} \right) = \sigma_2 \alpha \tilde{\mathcal{T}}_{2,1}'' \beta \left( \begin{array}{c} \uparrow \lambda \\ \downarrow 2 \end{array} \right),$$

$$(122) \quad \varepsilon_{\nu'} \iota' \left( \begin{array}{c} \uparrow \lambda \\ \uparrow 1 \end{array} \right) = \varepsilon_{\nu'} \left( \begin{array}{c} \downarrow s_2(\lambda) \\ \downarrow 3 \end{array} \right) = \left( \begin{array}{c} \uparrow s_2(\lambda) \\ \uparrow 2 \end{array}, \begin{array}{c} \uparrow s_2(\lambda) \\ \uparrow 2 \end{array} \right) \sim_h \sigma_2 \alpha \tilde{\mathcal{T}}_{2,1}'' \beta \left( \begin{array}{c} \uparrow \lambda \\ \uparrow 1 \end{array} \right),$$

$$(123) \quad \varepsilon_{\nu'} \iota' \left( \begin{array}{c} \times \lambda \\ \downarrow 2 \end{array} \right) = \varepsilon_{\nu'} \left( - \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array} \right) = - \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array} = \sigma_2 \alpha \tilde{\mathcal{T}}_{2,1}'' \beta \left( \begin{array}{c} \times \lambda \\ \downarrow 2 \end{array} \right),$$

$$\begin{aligned}
 (124) \quad \varepsilon_{\nu'} \iota' \left( \begin{array}{c} \times \lambda \\ \downarrow 2 \end{array} \right) &= \varepsilon_{\nu'} \left( k_1^{2,3}(\lambda) \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array} \right) \\
 &= k_1^{0,3}(s_2(\lambda)) \left( \left( \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array} - \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array} \right), - \left( \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array} - \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array} \right) \right) \\
 &= k_1^{0,3}(s_2(\lambda)) \tilde{\mathcal{T}}_{2,1}'' \left( \begin{array}{c} \times \lambda \\ \downarrow 2 \end{array} \right) = \sigma_2 \alpha \tilde{\mathcal{T}}_{2,1}'' \beta \left( \begin{array}{c} \times \lambda \\ \downarrow 2 \end{array} \right),
 \end{aligned}$$

$$\begin{aligned}
 (125) \quad \varepsilon_{\nu'} \iota' \left( \begin{array}{c} \times \lambda \\ \downarrow 1 \end{array} \right) &= \varepsilon_{\nu'} \left( k_1^{0,1} \begin{array}{c} \times s_2(\lambda) \\ \downarrow 3 \end{array} \right) = k_1^{0,3}(s_2(\lambda)) \left( \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array}, - \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array} \right) \\
 &= k_1^{0,3}(s_2(\lambda)) \tilde{\mathcal{T}}_{2,1}'' \left( \begin{array}{c} \times \lambda \\ \downarrow 2 \end{array} \right) = \sigma_2 \alpha \tilde{\mathcal{T}}_{2,1}'' \beta \left( \begin{array}{c} \times \lambda \\ \downarrow 2 \end{array} \right),
 \end{aligned}$$

$$\begin{aligned}
 (126) \quad \varepsilon_{\nu'} \iota' \left( \begin{array}{c} \times \lambda \\ \downarrow 1 \end{array} \right) &= \varepsilon_{\nu'} \left( \begin{array}{c} \times s_2(\lambda) \\ \downarrow 3 \end{array} \right) \\
 &= \left( \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array}, \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array} \right) \\
 &\quad - \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array} + \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array} + \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array} + \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array} + \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array} \\
 &\quad - \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array} \\
 &= \sigma_2 \alpha \tilde{\mathcal{T}}_{2,1}'' \beta \left( \begin{array}{c} \times \lambda \\ \downarrow 2 \end{array} \right),
 \end{aligned}$$

$$(127) \quad \varepsilon_{\nu'} \iota' \left( \begin{array}{c} \times \lambda \\ \downarrow 2 \end{array} \right) = (-1)^{\lambda_2+1} \varepsilon_{\nu'} \left( \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array} \right) = (-1)^{\lambda_2+1} \begin{array}{c} \times s_2(\lambda) \\ \downarrow 2 \end{array} = \sigma_2 \alpha \tilde{\mathcal{T}}_{2,1}'' \beta \left( \begin{array}{c} \times \lambda \\ \downarrow 2 \end{array} \right),$$

$$(128) \quad \varepsilon_{v'} l' \left( \begin{array}{c} \lambda \\ \curvearrowright_2 \end{array} \right) = (-1)^{\lambda_3+1} \varepsilon_{v'} \left( \begin{array}{c} s_2(\lambda) \\ \curvearrowright_2 \end{array} \right) = (-1)^{\lambda_3+1} \begin{array}{c} s_2(\lambda) \\ \curvearrowright_2 \end{array} = \sigma_2 \alpha \tilde{\mathcal{J}}''_{2,1} \beta \left( \begin{array}{c} \lambda \\ \curvearrowright_2 \end{array} \right),$$

$$(129) \quad \varepsilon_{v'} l' \left( \begin{array}{c} \lambda \\ \curvearrowright_1 \end{array} \right) = (-1)^{\lambda_2} \varepsilon_{v'} \left( \begin{array}{c} s_2(\lambda) \\ \curvearrowright_1 \end{array} \right) = (-1)^{\lambda_2} \begin{array}{c} s_2(\lambda) \\ \curvearrowright_1 \end{array} = \sigma_2 \alpha \tilde{\mathcal{J}}''_{2,1} \beta \left( \begin{array}{c} \lambda \\ \curvearrowright_1 \end{array} \right),$$

$$(130) \quad \varepsilon_{v'} l' \left( \begin{array}{c} \lambda \\ \curvearrowright_3 \end{array} \right) = (-1)^{\lambda_3} \varepsilon_{v'} \left( \begin{array}{c} s_2(\lambda) \\ \curvearrowright_3 \end{array} \right) = (-1)^{\lambda_3} \begin{array}{c} s_2(\lambda) \\ \curvearrowright_3 \end{array} = \sigma_2 \alpha \tilde{\mathcal{J}}''_{2,1} \beta \left( \begin{array}{c} \lambda \\ \curvearrowright_3 \end{array} \right),$$

$$(131) \quad \begin{aligned} \varepsilon_{v'} l' \left( \begin{array}{c} \lambda \\ \curvearrowright_3 \end{array} \right) &= (-1)^{\lambda_1+1} \varepsilon_{v'} \left( \begin{array}{c} s_2(\lambda) \\ \curvearrowright_3 \end{array} \right) \\ &= (-1)^{\lambda_1+1} (-1)^{s_2(\lambda)} \left( k_1^1(s_2(\lambda)) \begin{array}{c} s_2(\lambda) \\ \curvearrowright_2 \end{array} - k_1^2(s_2(\lambda)) \begin{array}{c} s_2(\lambda) \\ \curvearrowright_1 \end{array} \right) \\ &= (-1)^{\bar{\lambda}_1+1} \left( k_3^3(\lambda) \begin{array}{c} s_2(\lambda) \\ \curvearrowright_2 \end{array} - k_3^2(\lambda) \begin{array}{c} s_2(\lambda) \\ \curvearrowright_1 \end{array} \right) \\ &= \left( k_1^2 k_1^0 k_3^3 \begin{array}{c} s_2(\lambda) \\ \curvearrowright_2 \end{array} - k_1^2 k_1^0 k_3^2 \begin{array}{c} s_2(\lambda) \\ \curvearrowright_1 \end{array} \right) = \sigma_2 \alpha \tilde{\mathcal{J}}''_{2,1} \beta \left( \begin{array}{c} \lambda \\ \curvearrowright_3 \end{array} \right), \end{aligned}$$

$$(132) \quad \begin{aligned} \varepsilon_{v'} l' \left( \begin{array}{c} \lambda \\ \curvearrowright_1 \end{array} \right) &= (-1)^{\lambda_2+1} \varepsilon_{v'} \left( \begin{array}{c} s_2(\lambda) \\ \curvearrowright_1 \end{array} \right) \\ &= (-1)^{\lambda_1} k_1^0 k_1^2 \left( k_3^1 \begin{array}{c} s_2(\lambda) \\ \curvearrowright_2 \end{array} - k_3^2 \begin{array}{c} s_2(\lambda) \\ \curvearrowright_1 \end{array} \right) \\ &= \sigma_2 \alpha \tilde{\mathcal{J}}''_{2,1} \beta \left( \begin{array}{c} \lambda \\ \curvearrowright_1 \end{array} \right), \end{aligned}$$

$$(133) \quad \begin{aligned} \varepsilon_{v'} l' \left( \begin{array}{c} \lambda \\ \curvearrowright_3 \end{array} \right) &= (-1)^{\lambda_1} \varepsilon_{v'} \left( \begin{array}{c} s_2(\lambda) \\ \curvearrowright_3 \end{array} \right) = -k_1^0 k_1^2 \left( k_3^3 \begin{array}{c} s_2(\lambda) \\ \curvearrowright_1 \end{array} - k_3^2 \begin{array}{c} s_2(\lambda) \\ \curvearrowright_2 \end{array} \right) \\ &= \sigma_2 \alpha \tilde{\mathcal{J}}''_{2,1} \beta \left( \begin{array}{c} \lambda \\ \curvearrowright_3 \end{array} \right), \end{aligned}$$

$$(134) \quad \begin{aligned} \varepsilon_{v'} l' \left( \begin{array}{c} \lambda \\ \curvearrowright_1 \end{array} \right) &= (-1)^{\lambda_2} \varepsilon_{v'} \left( \begin{array}{c} s_2(\lambda) \\ \curvearrowright_1 \end{array} \right) \\ &= (-1)^{\lambda_1+1} k_1^0 k_1^2 \left( k_3^2 \begin{array}{c} s_2(\lambda) \\ \curvearrowright_3 \end{array} - k_3^1 \begin{array}{c} s_2(\lambda) \\ \curvearrowright_2 \end{array} \right) \\ &= \sigma_2 \alpha \tilde{\mathcal{J}}''_{2,1} \beta \left( \begin{array}{c} \lambda \\ \curvearrowright_1 \end{array} \right). \end{aligned}$$

This finishes the proof.  $\square$

**6.4. Proof of Theorem 4.3.** Because  $\iota$  reverses the orientation of the diagrams, we felt that this proof would be clearer to the reader if we proved that  $\varepsilon_{v'}$  preserved the 180 degree rotated versions of relations (10)-(22). By the cyclicity relations (KM2) and (KM3), this is equivalent to proving the original relations are preserved.

For any KM relation that only involves strands labelled 1 and 2 and does not involve a crossing of a 1-strand and a 2-strand,  $\varepsilon_{v'}$  acts as the identity and therefore trivially preserves the relation. For relations that do involve these crossings, the calculations are generally straightforward. For example,

$$\begin{aligned} \varepsilon_{v'} \left( \begin{array}{c} \lambda \\ \curvearrowright_{2,1} \end{array} \right) &= (-1)^{\bar{\lambda}_1(\bar{\lambda}_2+1)} \begin{array}{c} \lambda \\ \curvearrowright_{2,1} \end{array} \\ &= (-1)^{\bar{\lambda}_1(\bar{\lambda}_2+1)} \begin{array}{c} \lambda \\ \curvearrowright_{1,2} \end{array} = \varepsilon_{v'} \left( \begin{array}{c} \lambda \\ \curvearrowright_{1,2} \end{array} \right), \end{aligned}$$

with the other cyclicity relations following similarly, as does the relevant (KM5)

relation (since  $(-1)^{\bar{\lambda}_1(\bar{\lambda}_2+1)} \cdot 0 = 0$ ). For (KM4), we have

$$\mathcal{E}v' \left( \begin{array}{c} \text{Diagram with two crossings and strands 1, 2} \\ \lambda \end{array} \right) = ((-1)^{\bar{\lambda}_1(\bar{\lambda}_2+1)})^2 \left( \begin{array}{c} \text{Diagram with two crossings and strands 1, 2} \\ \lambda \end{array} \right) - \left( \begin{array}{c} \text{Diagram with two crossings and strands 1, 2} \\ \lambda \end{array} \right) = \mathcal{E}v' \left( \begin{array}{c} \text{Diagram with two crossings and strands 1, 2} \\ \lambda \end{array} \right) - \left( \begin{array}{c} \text{Diagram with two crossings and strands 1, 2} \\ \lambda \end{array} \right).$$

Identity (KM7) is similar, as is (KM6); any (multicolour) cubic KLR diagram consisting only of strands labelled 1 and 2 will have precisely two multicoloured crossings, leading to a similar squaring of the sign. It therefore remains to consider only those diagrams with at least one strand labelled 3.

For most of the KM identities discussed below, we are able to use that  $\iota$  and  $\iota'$  are locally faithful 2-functors, and therefore we are able to consider the unique preimage of any 2-morphism in their images. The results will then follow from liberal use of Proposition 6.1 (we give an example in first equation below of where it is used). We also implicitly make use of Lemma 5.1 when there is a diagram in the image of both  $\iota$  and  $\iota'$ . We will present a representative sampling of the identities of each KM axiom. The exception is the six instances of (KM6) where, using the notation of (17),  $\{i, j, k\} = \{1, 2, 3\}$ , since such 2-morphisms are not in the image of either  $\iota$  or  $\iota'$ . In these cases, we will be proving directly that  $\mathcal{E}v'$  preserves (KM6).

(KM1) and (KM2):

$$\begin{aligned} \mathcal{E}v' \left( \begin{array}{c} \text{Diagram with a loop and strand 3} \\ \lambda \end{array} \right) &= (-1)^{s_1(\lambda)_2+1} (-1)^{(s_1(\lambda)+\alpha_2)_2} \mathcal{E}v' \iota \left( \begin{array}{c} \text{Diagram with a loop and strand 2} \\ s_1(\lambda) \end{array} \right) \\ &\stackrel{6.1}{=} \sigma_1 \tilde{\mathcal{F}}'_{1,-1} \left( \begin{array}{c} \text{Diagram with a loop and strand 2} \\ s_1(\lambda) \end{array} \right) \sim_h \sigma_1 \tilde{\mathcal{F}}'_{1,-1} \left( \begin{array}{c} \text{Diagram with a loop and strand 2} \\ s_1(\lambda) \end{array} \right) \stackrel{6.1}{=} \mathcal{E}v' \iota \left( \begin{array}{c} \text{Diagram with a loop and strand 2} \\ s_1(\lambda) \end{array} \right) = \mathcal{E}v' \left( \begin{array}{c} \text{Diagram with a loop and strand 3} \\ \lambda \end{array} \right), \end{aligned}$$

where the homotopy (and all future homotopies in this proof) follows from Lemma 5.2 and from the  $\sigma_i$  being 2-isomorphisms. The other adjunction relation and dot cyclicity work similarly.

(KM3):

$$\begin{aligned} &\mathcal{E}v' \left( \begin{array}{c} \text{Diagram with a loop and strand 3} \\ \lambda \end{array} \right) \\ &= k_1^{2,3}(s_1(\lambda)) (-1)^{s_1(\lambda)_2+1+(s_1(\lambda)-\alpha_2)_1+1+(s_1(\lambda)+\alpha_3)_2+(s_1(\lambda)-\alpha_1)_1} \mathcal{E}v' \iota \left( \begin{array}{c} \text{Diagram with a loop and strand 2} \\ s_1(\lambda) \end{array} \right) \\ &= k_1^{2,3}(s_1(\lambda)) \sigma_1 \tilde{\mathcal{F}}'_{1,-1} \left( \begin{array}{c} \text{Diagram with a loop and strand 2} \\ s_1(\lambda) \end{array} \right) \sim_h k_1^{2,3}(s_1(\lambda)) \sigma_1 \tilde{\mathcal{F}}'_{1,-1} \left( \begin{array}{c} \text{Diagram with a loop and strand 2} \\ s_1(\lambda) \end{array} \right) \\ &= k_1^{2,3}(s_1(\lambda)) \mathcal{E}v' \iota \left( \begin{array}{c} \text{Diagram with a loop and strand 2} \\ s_1(\lambda) \end{array} \right) = \mathcal{E}v' \left( \begin{array}{c} \text{Diagram with a loop and strand 3} \\ \lambda \end{array} \right), \end{aligned}$$

$$\begin{aligned}
& \mathfrak{E}v' \left( \text{Diagram with red strands } \lambda \right) \\
&= k_1^{0,1}(s_2(\lambda))(-1)^{s_2(\lambda)_2+1+(s_2(\lambda)-\alpha_2)_1+1+(s_2(\lambda)+\alpha_3)_2+(s_2(\lambda)-\alpha_1)_1} \mathfrak{E}v'l' \left( \text{Diagram with blue and red strands } s_2(\lambda) \right) \\
&= k_1^{0,1}(s_2(\lambda))\sigma_2\alpha\tilde{\mathcal{J}}''_{2,1}\beta \left( \text{Diagram with blue and red strands } s_1(\lambda) \right) \sim_h k_1^{0,1}(s_2(\lambda))\sigma_1\tilde{\mathcal{J}}'_{1,-1} \left( \text{Diagram with blue and red strands } s_1(\lambda) \right) \\
&= k_1^{0,1}(s_2(\lambda))\mathfrak{E}v'l' \left( \text{Diagram with blue and red strands } s_1(\lambda) \right) = \mathfrak{E}v' \left( \text{Diagram with blue strands } \lambda \right), \\
& \mathfrak{E}v' \left( \text{Diagram with red strands } \lambda \right) = (-1)^{s_1(\lambda)_2+1+2(s_1(\lambda)-\alpha_2)_2+1+(s_1(\lambda)-2\alpha_2)_2} \mathfrak{E}v'l' \left( \text{Diagram with red strands } s_1(\lambda) \right) \\
&= \sigma_1\tilde{\mathcal{J}}'_{1,-1} \left( \text{Diagram with red strands } s_1(\lambda) \right) \sim_h \sigma_1\tilde{\mathcal{J}}'_{1,-1} \left( \text{Diagram with red strands } s_1(\lambda) \right) \\
&= \mathfrak{E}v'l' \left( \text{Diagram with red strands } s_1(\lambda) \right) = \mathfrak{E}v' \left( \text{Diagram with red strands } \lambda \right).
\end{aligned}$$

The other crossing cyclicity identities are similar.

(KM4):

$$\begin{aligned}
& \mathfrak{E}v' \left( \text{Diagram with blue and red strands } \lambda \right) = k_1^{0,1}(s_1(\lambda))k_1^{2,3}(s_1(\lambda))\mathfrak{E}v'l' \left( \text{Diagram with blue and red strands } s_1(\lambda) \right) = -\sigma_1\tilde{\mathcal{J}}'_{1,-1} \left( \text{Diagram with blue and red strands } s_1(\lambda) \right) \\
&\sim_h -\sigma_1\tilde{\mathcal{J}}'_{1,-1} \left( \text{Diagram with blue and red strands } s_1(\lambda) \right) \\
&= \mathfrak{E}v'l' \left( \text{Diagram with blue and red strands } s_1(\lambda) \right) = \mathfrak{E}v' \left( \text{Diagram with blue and red strands } \lambda \right), \\
& \mathfrak{E}v' \left( \text{Diagram with blue and red strands } \lambda \right) = k_1^{0,1}(s_2(\lambda))k_1^{2,3}(s_2(\lambda))\mathfrak{E}v'l' \left( \text{Diagram with blue and red strands } s_2(\lambda) \right) = -\sigma_2\alpha\tilde{\mathcal{J}}''_{2,1}\beta \left( \text{Diagram with blue and red strands } s_2(\lambda) \right) \\
&\sim_h -\sigma_2\alpha\tilde{\mathcal{J}}''_{2,1}\beta \left( \text{Diagram with blue and red strands } s_2(\lambda) \right) \\
&= \mathfrak{E}v'l' \left( \text{Diagram with blue and red strands } s_2(\lambda) \right) = \mathfrak{E}v' \left( \text{Diagram with blue and red strands } \lambda \right),
\end{aligned}$$

$$\mathcal{E}_{\mathcal{V}'} \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \hline \diagup \diagdown \\ \diagdown \diagup \end{array} \lambda \right) = \mathcal{E}_{\mathcal{V}'} \iota \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \hline \diagup \diagdown \\ \diagdown \diagup \end{array} s_1(\lambda) \right) = \mathcal{E}_{\mathcal{V}'} \iota(0) = 0 = \mathcal{E}_{\mathcal{V}'}(0).$$

The other two identities are similar.

(KM5):

$$\begin{aligned} \mathcal{E}_{\mathcal{V}'} \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \hline \diagup \diagdown \\ \diagdown \diagup \end{array} \lambda - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \hline \diagdown \diagup \\ \diagup \diagdown \end{array} \lambda \right) &= k_1^{0,1}(s_1(\lambda)) \mathcal{E}_{\mathcal{V}'} \iota \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \hline \diagup \diagdown \\ \diagdown \diagup \end{array} s_1(\lambda) - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \hline \diagdown \diagup \\ \diagup \diagdown \end{array} s_1(\lambda) \right) = \mathcal{E}_{\mathcal{V}'} \iota(0) = 0 = \mathcal{E}_{\mathcal{V}'}(0), \\ \mathcal{E}_{\mathcal{V}'} \left( \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \hline \diagdown \diagup \\ \diagup \diagdown \end{array} \lambda - \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \hline \diagup \diagdown \\ \diagdown \diagup \end{array} \lambda \right) &= k_1^{0,1}(s_2(\lambda)) \mathcal{E}_{\mathcal{V}'} \iota' \left( \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \hline \diagdown \diagup \\ \diagup \diagdown \end{array} s_2(\lambda) - \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \hline \diagup \diagdown \\ \diagdown \diagup \end{array} s_2(\lambda) \right) = \mathcal{E}_{\mathcal{V}'} \iota(0) = 0 = \mathcal{E}_{\mathcal{V}'}(0), \\ \mathcal{E}_{\mathcal{V}'} \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \hline \diagup \diagdown \\ \diagdown \diagup \end{array} \lambda - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \hline \diagdown \diagup \\ \diagup \diagdown \end{array} \lambda \right) &= \mathcal{E}_{\mathcal{V}'} \iota \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \hline \diagup \diagdown \\ \diagdown \diagup \end{array} s_1(\lambda) - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \hline \diagdown \diagup \\ \diagup \diagdown \end{array} s_1(\lambda) \right) = \sigma_1 \tilde{\mathcal{F}}'_{1,-1} \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \hline \diagup \diagdown \\ \diagdown \diagup \end{array} s_1(\lambda) - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \hline \diagdown \diagup \\ \diagup \diagdown \end{array} s_1(\lambda) \right) \\ &\sim_h \sigma_1 \tilde{\mathcal{F}}'_{1,-1} \left( \begin{array}{c} \uparrow \uparrow \\ \hline \uparrow \uparrow \end{array} s_1(\lambda) \right) = \mathcal{E}_{\mathcal{V}'} \iota \left( \begin{array}{c} \uparrow \uparrow \\ \hline \uparrow \uparrow \end{array} s_1(\lambda) \right) = \mathcal{E}_{\mathcal{V}'} \left( \begin{array}{c} \downarrow \downarrow \\ \hline \downarrow \downarrow \end{array} \lambda \right). \end{aligned}$$

The other two identities are similar.

(KM6):

$$\begin{aligned} \mathcal{E}_{\mathcal{V}'} \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \hline \diagup \diagdown \\ \diagdown \diagup \end{array} \lambda - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \hline \diagdown \diagup \\ \diagup \diagdown \end{array} \lambda \right) &= \mathcal{E}_{\mathcal{V}'} \iota \left( -k_1^{0,1}(s_1(\lambda) + \alpha_1) k_1^{2,3}(s_1(\lambda) + \alpha_1) \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \hline \diagup \diagdown \\ \diagdown \diagup \end{array} s_1(\lambda) + k_1^{0,1}(s_1(\lambda)) k_1^{2,3}(s_1(\lambda)) \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \hline \diagdown \diagup \\ \diagup \diagdown \end{array} s_1(\lambda) \right) \\ &= \sigma_1 \tilde{\mathcal{F}}'_{1,-1} \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \hline \diagup \diagdown \\ \diagdown \diagup \end{array} s_1(\lambda) - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \hline \diagdown \diagup \\ \diagup \diagdown \end{array} s_1(\lambda) \right) = -\sigma_1 \tilde{\mathcal{F}}'_{1,-1} \left( \begin{array}{c} \uparrow \uparrow \\ \hline \uparrow \uparrow \end{array} s_1(\lambda) \right) \\ &= -\mathcal{E}_{\mathcal{V}'} \iota \left( \begin{array}{c} \uparrow \uparrow \\ \hline \uparrow \uparrow \end{array} s_1(\lambda) \right) = -\mathcal{E}_{\mathcal{V}'} \left( \begin{array}{c} \downarrow \downarrow \\ \hline \downarrow \downarrow \end{array} s_1(\lambda) \right), \\ \mathcal{E}_{\mathcal{V}'} \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \hline \diagup \diagdown \\ \diagdown \diagup \end{array} \lambda - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \hline \diagdown \diagup \\ \diagup \diagdown \end{array} \lambda \right) &= -k_1^{2,3}(s_2(\lambda) + \alpha_2) k_1^{2,3}(s_2(\lambda)) \mathcal{E}_{\mathcal{V}'} \iota' \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \hline \diagup \diagdown \\ \diagdown \diagup \end{array} s_2(\lambda) - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \hline \diagdown \diagup \\ \diagup \diagdown \end{array} s_2(\lambda) \right) = \mathcal{E}_{\mathcal{V}'} \iota'(0) = 0 = \mathcal{E}_{\mathcal{V}'}(0), \\ \mathcal{E}_{\mathcal{V}'} \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \hline \diagup \diagdown \\ \diagdown \diagup \end{array} \lambda - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \hline \diagdown \diagup \\ \diagup \diagdown \end{array} \lambda \right) &= \mathcal{E}_{\mathcal{V}'} \iota \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \hline \diagup \diagdown \\ \diagdown \diagup \end{array} s_1(\lambda) - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \hline \diagdown \diagup \\ \diagup \diagdown \end{array} s_1(\lambda) \right) = \mathcal{E}_{\mathcal{V}'} \iota(0) = 0 = \mathcal{E}_{\mathcal{V}'}(0). \end{aligned}$$

With the exception of the three-coloured identities discussed below, the other cubic KLR relations are similar. We prove three of these identities directly; the other three are similar. We have

$$\mathcal{E}_{\mathcal{V}'} \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \hline \diagup \diagdown \\ \diagdown \diagup \end{array} \lambda \right) = k_3^{0,3}(\lambda - \alpha_1) k_1^{0,3}(\lambda - \alpha_1) (-1)^{\bar{\lambda}_1(\bar{\lambda}_2+1)} \left( \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \hline \diagup \diagdown \\ \diagdown \diagup \end{array} \lambda, \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \\ \hline \diagdown \diagup \\ \diagup \diagdown \end{array} \lambda \right),$$



$$\begin{aligned}
 &\sim_h \sigma_1 \tilde{\mathcal{F}}'_{1,-1} \left( \begin{array}{c} \uparrow \\ 2 \\ \downarrow \\ 2 \end{array} \middle| s_1(\lambda) - \sum_{\substack{a+b+c= \\ s_1(\lambda)_2-1}} \begin{array}{c} \text{bubble} \\ \text{with } a, b, c \text{ labels} \end{array} \right) \\
 &= \varepsilon_{v'} l \left( \begin{array}{c} \uparrow \\ 2 \\ \downarrow \\ 2 \end{array} \middle| s_1(\lambda) - \sum_{\substack{a+b+c= \\ s_1(\lambda)_2-1}} \begin{array}{c} \text{bubble} \\ \text{with } a, b, c \text{ labels} \end{array} \right) \\
 &= \varepsilon_{v'} \left( \begin{array}{c} \uparrow \\ 3 \\ \downarrow \\ 3 \end{array} \middle| \lambda - \sum_{\substack{a+b+c= \\ -\tilde{\lambda}_3-1}} \begin{array}{c} \text{bubble} \\ \text{with } a, b, c \text{ labels} \end{array} \right).
 \end{aligned}$$

The other two identities are similar.

$$\begin{aligned}
 \text{(KM8): } \varepsilon_{v'} \left( \begin{array}{c} \text{clockwise bubble} \\ \text{with } m \text{ label} \end{array} \right) &= (-1)^{\tilde{\lambda}_3+1} \varepsilon_{v'} l \left( \begin{array}{c} \text{bubble} \\ \text{with } m \text{ label} \end{array} \right) = (-1)^{\tilde{\lambda}_3+1} \sigma_1 \tilde{\mathcal{F}}'_{1,-1} \left( \begin{array}{c} \text{bubble} \\ \text{with } m \text{ label} \end{array} \right) \\
 &= \sigma_1 \tilde{\mathcal{F}}'_{1,-1} \left( -\max(0, m+1) \right)^{s_1(\lambda)_2} = \varepsilon_{v'} \left( -\max(0, m+1) \right)^{\lambda_1}.
 \end{aligned}$$

The anticlockwise bubble is similar.

(KM9):

$$\begin{aligned}
 &\varepsilon_{v'} \left( \left( \begin{array}{c} \text{bubble} \\ \text{with } 0 \text{ label} \end{array} + \begin{array}{c} \text{bubble} \\ \text{with } 1 \text{ label} \end{array} t + \dots \right) \left( \begin{array}{c} \text{bubble} \\ \text{with } 0 \text{ label} \end{array} + \begin{array}{c} \text{bubble} \\ \text{with } 1 \text{ label} \end{array} t + \dots \right) \right) \\
 &= (-1)^{2\tilde{\lambda}_3+2} \varepsilon_{v'} l \left( \left( \begin{array}{c} \text{bubble} \\ \text{with } 0 \text{ label} \end{array} + \begin{array}{c} \text{bubble} \\ \text{with } 1 \text{ label} \end{array} t + \dots \right) \left( \begin{array}{c} \text{bubble} \\ \text{with } 0 \text{ label} \end{array} + \begin{array}{c} \text{bubble} \\ \text{with } 1 \text{ label} \end{array} t + \dots \right) \right) \\
 &= \sigma_1 \tilde{\mathcal{F}}'_{1,-1} \left( \left( \begin{array}{c} \text{bubble} \\ \text{with } 0 \text{ label} \end{array} + \begin{array}{c} \text{bubble} \\ \text{with } 1 \text{ label} \end{array} t + \dots \right) \left( \begin{array}{c} \text{bubble} \\ \text{with } 0 \text{ label} \end{array} + \begin{array}{c} \text{bubble} \\ \text{with } 1 \text{ label} \end{array} t + \dots \right) \right) \\
 &= \sigma_1 \tilde{\mathcal{F}}'_{1,-1} (-1) = -1 = \varepsilon_{v'} (-1).
 \end{aligned}$$

### Appendix: A remark on 2-isomorphism classes

In Definition 3.1 of [10], Khovanov and Lauda chose a different set of scalars and bubble parameters for the Kac–Moody 2-categories than those we use in this paper. We were unable to define an evaluation 2-functor for their choice and we conjecture that no such evaluation 2-functor exists. Although we have yet to prove this conjecture, this would be consistent with the first paragraph of [11, p. 2699], which mentions the existence of a one-parameter family of mutually non-2-isomorphic sub-2-categories of the affine type A Kac–Moody 2-categories categorifying the Borel subalgebra in affine type A. This contrasts with the finite type A case, where [13, Theorem 3.5] proves that any two choices of scalars and bubble parameters yield 2-isomorphic Kac–Moody 2-categories. Since Khovanov and Lauda did not include a proof in the above paper, we prove here that the two different choices discussed do lead to non-2-isomorphic Kac–Moody 2-categories in affine type A.

First, a small comment on the choice of weights for the 2-categories in question. The objects of the cyclic 2-categories  $\tilde{u}(n)$  and  $\tilde{u}_\Delta(n)$  from Definition 3.1 are  $\mathfrak{gl}_n$ -weights and level-zero  $\widehat{\mathfrak{gl}}_n$ -weights, respectively, which coincide (as explained in 2.1). Moreover, the relations satisfied by the generating 2-morphisms of those two 2-categories really depend on those weights, and not on the induced  $\mathfrak{sl}_n$ -weights and level-zero  $\widehat{\mathfrak{sl}}_n$ -weights, respectively. Specifically, the degree-zero  $i$ -coloured bubbles, in a region labelled by  $\lambda \in \mathbb{Z}^n$ , are equal to  $(-1)^{\lambda_{i+1}}$  or  $(-1)^{\lambda_{i+1}-1}$  (depending on orientation), so they cannot be expressed in terms of  $\bar{\lambda}$  (unless we choose and fix a certain Schur level).

On the other hand, the cyclic 2-categories  $u_Q(\mathfrak{sl}_n)$  and  $u_Q(\widehat{\mathfrak{sl}}_n)$ , defined in [2, Definition 1.3] (generalizing [10, Definition 3.1]), trivially induce cyclic 2-categories  $u_Q(\mathfrak{gl}_n)$  and  $u_Q(\widehat{\mathfrak{gl}}_n)$  whose objects are  $\mathfrak{gl}_n$ -weights and level-zero  $\widehat{\mathfrak{gl}}_n$ -weights, respectively: Simply label the regions of the string diagrams by  $\lambda \in \mathbb{Z}^n$  and let the relations be those for  $\bar{\lambda}$ , see Section 2.1 for the notation. We write  $u_Q(\widehat{\mathfrak{gl}}'_n)$  to indicate that it is actually an extended version of  $u_Q(\widehat{\mathfrak{sl}}_n)$  rather than the full  $u_Q(\widehat{\mathfrak{gl}}_n)$  (whatever that would be), see remarks below Definition 2.2.

Recall that, following [13],  $u_Q(\mathfrak{sl}_n)$  and  $u_Q(\widehat{\mathfrak{sl}}_n)$  depend on a choice of scalars  $t_{ij} \in \mathbb{Q}^\times$  satisfying  $t_{ii} = 1$  and  $t_{ij} = t_{ji}$  when  $j \neq i \pm 1 \pmod n$ , and bubble parameters  $\beta_i = \beta_{i,\lambda}$ ,  $c_{i,\lambda}^+$ ,  $c_{i,\lambda}^- \in \mathbb{Q}^\times$  satisfying

$$c_{i,\lambda}^+ c_{i,\lambda}^- = -1/\beta_i = 1/t_{ii} \quad \text{and} \quad c_{i,\lambda+\alpha_j}^\pm = t_{ij} c_{i,\lambda}^\pm.$$

Here  $i, j \in 1, \dots, n-1$  and  $\lambda \in \mathbb{Z}^{n-1}$ , for  $\mathfrak{sl}_n$ , and  $i, j \in 1, \dots, n$  and  $\lambda \in \mathbb{Z}^n$ , for  $\widehat{\mathfrak{sl}}_n$ . For Khovanov and Lauda's original choice in [10, Definition 3.1], with all scalars and bubble parameters equal to one, we will follow their notation and denote the corresponding 2-categories by  $u(\mathfrak{sl}_n)$  and  $u(\widehat{\mathfrak{sl}}_n)$ , and the trivially induced  $\mathfrak{gl}_n$  versions of these by  $u(\mathfrak{gl}_n)$  and  $u(\widehat{\mathfrak{gl}}'_n)$ , respectively. The 2-categories  $\tilde{u}(n)$  and  $\tilde{u}_\Delta(n)$  correspond to the choice  $t_{ii} = -1 = t_{i,i+1} = -1$  and  $t_{ij} = 1$  for all  $i$  and  $j \neq i, i+1$  in the respective ranges, and  $c_{i,\lambda}^+ = (-1)^{\lambda_{i+1}} = -c_{i,\lambda}^-$  for all  $i$  in the respective ranges.

For any  $n \in \mathbb{N}_{\geq 2}$ , the 2-categories  $\tilde{u}(n)$  and  $u(\mathfrak{gl}_n)$  are 2-isomorphic, with the 2-isomorphism being obtained by composing the 2-isomorphism from [16, (6)] and the 2-isomorphism  $\Sigma$  from [10, Section 4.2.1] (see also [12]). When  $n \in \mathbb{N}_{\geq 2}$  is even, that composite 2-isomorphism extends to a 2-isomorphism between  $\tilde{u}_\Delta(n)$  and  $u_Q(\widehat{\mathfrak{gl}}'_n)$ . When  $n$  is odd, it does not extend to the affine 2-categories, because Khovanov and Lauda's 2-isomorphism  $\Sigma$  is no longer well-defined in that case. The reason is that in the definition of  $\Sigma$  occur factors like  $(-1)^i$ , for  $i = 1, \dots, n-1$ , which are not well-defined for  $i \in \mathbb{Z}/n\mathbb{Z}$  when  $n$  is odd.

We show that there is no 2-isomorphism between  $\tilde{u}_\Delta(n)$  and  $u_Q(\widehat{\mathfrak{gl}}'_n)$  for odd  $n$ , for any choice of scalars and bubble parameters satisfying the above conditions.

**Lemma A.3.** *Let  $Q$  be a choice of scalars and bubble parameters for  $\widehat{\mathfrak{sl}}_n$  and let  $\Xi : \mathcal{U}_Q(\widehat{\mathfrak{gl}}'_n) \rightarrow \tilde{\mathcal{U}}_\Delta(n)$  be a 2-isomorphism which is the identity on objects and 1-morphisms. Then*

$$(135) \quad \Xi \left( \begin{array}{c} \uparrow \\ \downarrow \\ \uparrow \\ \downarrow \end{array} \lambda \right) = o_i(\lambda) \begin{array}{c} \uparrow \\ \downarrow \\ \uparrow \\ \downarrow \end{array} \lambda \quad \text{and} \quad \Xi \left( \begin{array}{c} \nearrow \\ \searrow \\ \nwarrow \\ \nearrow \end{array} \lambda \right) = f_{ij}(\lambda) \begin{array}{c} \nearrow \\ \searrow \\ \nwarrow \\ \nearrow \end{array} \lambda$$

for some  $o_i(\lambda), f_{ij}(\lambda) \in \mathbb{Q}^\times$  and for all  $i, j \in \{1, \dots, n\}$  and all  $\lambda \in \mathbb{Z}^n$ . Moreover, these scalars satisfy  $o_i(\lambda) f_{ii}(\lambda) = 1$  for all  $i \in \hat{I}$  and all  $\lambda \in \mathbb{Z}^n$ .

*Proof.* For degree reasons, the second equality in (135) is immediate, but the first one requires an argument. A priori, we have

$$\Xi \left( \begin{array}{c} \uparrow \\ \downarrow \\ \uparrow \\ \downarrow \end{array} \lambda \right) = o_i(\lambda) \begin{array}{c} \uparrow \\ \downarrow \\ \uparrow \\ \downarrow \end{array} \lambda + \sum_{j=1}^n b_{ij}(\lambda) \begin{array}{c} \circlearrowleft \\ \downarrow \\ \uparrow \\ \downarrow \end{array} \lambda.$$

Now consider the image of the nil-Hecke relation:

$$\begin{aligned} & \Xi \left( \begin{array}{c} \nearrow \\ \searrow \\ \nwarrow \\ \nearrow \end{array} \lambda - \begin{array}{c} \nwarrow \\ \nearrow \\ \searrow \\ \nearrow \end{array} \lambda \right) \\ &= o_i(\lambda) f_{ii}(\lambda) \left( \begin{array}{c} \nearrow \\ \searrow \\ \nwarrow \\ \nearrow \end{array} \lambda - \begin{array}{c} \nwarrow \\ \nearrow \\ \searrow \\ \nearrow \end{array} \lambda \right) + f_{ii}(\lambda) \sum_{j=1}^n b_{ij}(\lambda) \left( \begin{array}{c} \circlearrowleft \\ \searrow \\ \nwarrow \\ \nearrow \end{array} \lambda - \begin{array}{c} \circlearrowright \\ \nwarrow \\ \nearrow \\ \searrow \end{array} \lambda \right) \\ &= o_i(\lambda) f_{ii}(\lambda) \begin{array}{c} \uparrow \\ \downarrow \\ \uparrow \\ \downarrow \end{array} \lambda + 2f_{ii}(\lambda) \left( b_{ii}(\lambda) \begin{array}{c} \circlearrowleft \\ \downarrow \\ \uparrow \\ \downarrow \end{array} \lambda + b_{i,i+1}(\lambda) \begin{array}{c} \circlearrowright \\ \downarrow \\ \uparrow \\ \downarrow \end{array} \lambda \right) \\ & \quad + f_{ii}(\lambda) (2b_{ii}(\lambda) + b_{i,i-1}(\lambda) - b_{i,i+1}(\lambda)) \begin{array}{c} \nearrow \\ \searrow \\ \nwarrow \\ \nearrow \end{array} \lambda. \end{aligned}$$

The fact that  $\Xi$  has to preserve the nil-Hecke relation implies that  $o_i(\lambda) f_{ii}(\lambda) = 1$  and  $b_{ii}(\lambda) = b_{i,i-1}(\lambda) = b_{i,i+1}(\lambda) = 0$  for all  $i \in \hat{I}$  and all  $\lambda \in \mathbb{Z}^n$ .

To see this, first note that

$$\begin{array}{c} \uparrow \\ \downarrow \\ \uparrow \\ \downarrow \end{array} \lambda, \quad \begin{array}{c} \circlearrowleft \\ \searrow \\ \nwarrow \\ \nearrow \end{array} \lambda, \quad \begin{array}{c} \circlearrowright \\ \nwarrow \\ \nearrow \\ \searrow \end{array} \lambda, \quad \begin{array}{c} \nearrow \\ \searrow \\ \nwarrow \\ \nearrow \end{array} \lambda$$

are linearly independent in  $\text{Hom}_{\tilde{\mathcal{U}}_\Delta(n)}(\mathcal{E}_{ii} \mathbf{1}_\lambda, \mathcal{E}_{ii} \mathbf{1}_\lambda)$ . Just as in the proof of [10, Lemma 6.16], this follows from looking at their images under the 2-representation  $\mathcal{F}_{\text{Bim}}$  from [16, Section 4.2] (in particular, see (45) in that paper), and its extension for the affine case in [15, Definition 5.6].

The condition  $o_i(\lambda) f_{ii}(\lambda) = 1$  is therefore immediate. Further, for each  $i \in \hat{I}$ , linear independence of the two degree-two bubbles above, coloured  $i - 1$  and  $i$  implies that  $b_{i,i+1} = b_{ii} = 0$ . Using this and

$$2b_{ii}(\lambda) + b_{i,i-1}(\lambda) - b_{i,i+1}(\lambda) = 0,$$

we see that  $b_{i,i-1} = 0$  as well.



$\{o_i, f_{ij} | i, j = 1, \dots, n\}$  as above. Recall that  $o_i(\lambda), g_{ij}(\lambda), t_{ij} \in \mathbb{Q}^\times$  and that  $g_{ij} = g_{ji}$  for all  $i, j \in \hat{I}$ . Thus, suppressing  $\lambda$  for readability, we get

$$\begin{aligned} o_1 &= -g_{12}t_{12}^{-1} = -o_2t_{21}t_{12}^{-1} = (-1)^2g_{23}t_{23}^{-1}t_{21}t_{12}^{-1} = \dots = (-1)^n g_{n,1}t_{n,1}^{-1} \dots t_{21}t_{12}^{-1} \\ &= (-1)^n o_1 \prod_{i \in \hat{I}} t_{i+1,i}t_{i,i+1}^{-1}. \end{aligned}$$

This implies that  $\prod_{i \in \hat{I}} t_{i+1,i}t_{i,i+1}^{-1} = (-1)^n$  has to hold. But by the definition of  $Q$  and the fact that  $\sum_{k=1}^n \alpha_k = 0$  in the (level zero)  $\widehat{\mathfrak{sl}}_n$ -root lattice, we have the following:

- $t_{ii} = 1$  for all  $i = 1, \dots, n$ , so in particular  $\prod_{i=1}^n t_{ii} = 1$ .
- $t_{ij} = t_{ji}$  whenever  $|i - j| > 1 \pmod n$ , so in particular  $\prod_{\substack{i,j=1,\dots,n \\ |i-j|>1}} t_{ij} = x^2$  for some  $x \in \mathbb{Q}^\times$ .
- $1 = \frac{c_{i,\bar{\lambda}}}{c_{i,\bar{\lambda}}} = \frac{c_{i,\bar{\lambda}+\sum_{k=1}^n \alpha_k}}{c_{i,\bar{\lambda}}} = \prod_{j=1}^n \frac{c_{i,\bar{\lambda}+\sum_{k=1}^j \alpha_k}}{c_{i,\bar{\lambda}+\sum_{k=1}^{j-1} \alpha_k}} = \prod_{j=1}^n t_{ij}$  for any  $\lambda \in \mathbb{Z}^n$  and any  $i = 1, \dots, n$ .

Therefore, for any  $\lambda \in \mathbb{Z}^n$

$$\prod_{i,j=1,\dots,n} t_{ij} = \prod_{i=1}^n \frac{c_{i,\bar{\lambda}}}{c_{i,\bar{\lambda}}} = 1.$$

But

$$\prod_{i,j=1,\dots,n} t_{ij} = \left(\prod_{i=1}^n t_{ii}\right) \left(\prod_{\substack{i,j=1,\dots,n \\ |i-j|>1}} t_{ij}\right) \left(\prod_{\substack{i=1,\dots,n \\ |i-j|=1}} t_{ij}\right) = x^2 \prod_{\substack{i=1,\dots,n \\ |i-j|=1}} t_{ij},$$

for some  $x \in \mathbb{Q}^\times$ , by the above remarks. Multiplying this by  $\prod_{i \in \hat{I}} t_{i+1,i}t_{i,i+1}^{-1}$  yields

$$\prod_{i=1}^n t_{i+1,i}^2 = (-1)^n x^{-2},$$

which implies  $n$  has to be even, completing our proof. □

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