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# HOW MANY SPRAYS COVER SPACE?

ALESSANDRO ANDRETTA AND IVAN IZMESTIEV

**For all $d \geq 3$ we show that the cardinality of $\mathbb{R}$ is at most $\aleph_n$ if and only if $\mathbb{R}^d$ can be covered with $(n+1)(d-1)+1$ sprays whose centers are in general position in a hyperplane. This extends previous results by Schmerl when $d = 2$.**

## 1. Introduction

The general theme of this paper is the possibility of covering $\mathbb{R}^d$ with few small sets — smallness means having finite or countable intersection with prescribed geometric objects. For example $\mathbb{R}^2$ can be covered with countably many curves with distinct axes [Davies 1974], but cannot be covered with finitely many curves [Mazurkiewicz 1933]. (A curve $C$ is the rotated graph of a function $f : \mathbb{R} \to \mathbb{R}$, and its axis is the image of the $y$-axis via the rotation; the smallness condition is that $C$ intersected any line parallel to its axis has cardinality 1.)

A spray in the plane is a subset of $\mathbb{R}^2$ together with a distinguished point (called center) such that all circles centered in that point have finite intersection with the spray. These objects were introduced in [Schmerl 2003], where it is observed that CH, the continuum hypothesis, implies that the plane can be covered with three sprays. In [de la Vega 2009] it is shown that CH is equivalent to the plane being covered with three sprays with collinear centers — in fact the statement "the cardinality of $\mathbb{R}$ is at most $\aleph_n$" (in symbols: $2^{\aleph_0} \leq \aleph_n$) is equivalent to $\mathbb{R}^2$ being covered with $n+2$ sprays with collinear centers [Schmerl 2010]. Collinearity is essential, since ZFC proves that the plane can be covered with three sprays centered in arbitrary noncollinear points [de la Vega 2009; Schmerl 2010]. It is easy to check that if a spray is measurable, then it must be null, so if the plane is covered with at most countably many sprays, they cannot be all measurable. Any construction of a covering of the plane with countably many sprays (or curves) requires the axiom of choice.

The notion of spray can be extended to $\mathbb{R}^d$ for any $d \geq 3$, with $(d-1)$-dimensional spheres in place of 1-dimensional spheres, i.e. circles, and the natural question is the relation (if any) between the size of the continuum and the number of sprays

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needed to cover $\mathbb{R}^d$. By [Erdős et al. 1994, Theorem 2], $2^{\aleph_0} \leq \aleph_n$ implies that $\mathbb{R}^3$ can be covered with $2n+3$ sprays such that their centers are coplanar, and no three of them are collinear. In particular, CH implies that given five coplanar points such that no three of them are collinear, then there are sprays centered around them that cover $\mathbb{R}^3$. By [Schmerl 2012] $\mathbb{R}^3$ cannot be covered with three sprays, but the problem whether the space can be covered with four sprays remains open.

We prove that $2^{\aleph_0} \leq \aleph_n$ is equivalent to $\mathbb{R}^3$ being covered with $2n+3$ sprays whose centers are coplanar, and no three of them are collinear. Moreover $2n+3$ is optimal — in particular $\mathbb{R}^3$ cannot be covered with four sprays with coplanar centers. In fact we prove a similar result for $\mathbb{R}^d$ with $d > 3$, namely: $2^{\aleph_0} \leq \aleph_n$ is equivalent to $\mathbb{R}^d$ being covered with $(n+1)(d-1)+1$-many sprays whose centers lie on a hyperplane $H$, and the affine span of any $d$ of them is $H$ (Theorem 5.4). Again the number $(n+1)(d-1)+1$ of sprays is optimal. Finally we show that, irrespective of the size of the continuum, $\mathbb{R}^d$ can be covered with countably many sprays whose centers lie on a hyperplane $H$, and the affine span of any $d$ of them is $H$ (Theorem 5.8).

We do not know if $\mathbb{R}^d$ can be covered with $d+1$-many sprays such that the affine span of their centers is $\mathbb{R}^d$, even in the case $d=3$. In other words: can $\mathbb{R}^3$ be covered with four sprays whose centers are not coplanar, i.e. form a tetrahedron? We suspect that the answer is affirmative and that it can be proved in ZFC. If true, this would be analogous to the fact that $\mathbb{R}^2$ can be covered with three sprays with noncollinear centers.

The results in this paper lie on the interface between set theory and the geometry of euclidean spaces. The reader may find them odd as *a priori* there is no obvious relation between geometric notions (like being in general position, or lying on a hyperplane, or having a specific dimension) and the value of $2^{\aleph_0} = |\mathbb{R}| = |\mathbb{R}^d|$ — note that there is a Borel bijection between $\mathbb{R}$ and $\mathbb{R}^d$. On the other hand there are several results in this area that connect set-theoretic issues with geometric properties. Examples are the classical theorems of Sierpiński — see Theorem 2.6 below and the beginning of Section 4 for some context and references. Most of these results deal with linear objects like lines, or hyperplanes in $\mathbb{R}^d$, while sprays are essentially nonlinear objects. We extend a construction in [Schmerl 2010] from $\mathbb{R}^2$ to $\mathbb{R}^d$, transforming the quadratic problem of covering the space with sprays to the linear problem of covering the space with sets having finite intersections with certain families of hyperplanes. This latter problem has been studied before [Bagemihl 1959/60; Erdős et al. 1994; Simms 1997]. Extending these earlier results we are able to prove our results on sprays.

The paper is organized as follows. After recalling the notations and the basic notions that will be used throughout the paper, we show in Section 3 how to transform a covering of $\mathbb{R}^d$ with sprays with centers on a given hyperplane into a covering of (an open subset of) $\mathbb{R}^d$ with small intersection with certain families of hyperplanes. Section 4 is devoted to studying the following problem: given distinct, nonzero vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k$ in $\mathbb{R}^d$, are there $A_1, \ldots, A_k$ covering $\mathbb{R}^d$ such that every plane orthogonal to $\boldsymbol{u}_i$ has finite intersection with $A_i$? It turns out that this problem is closely related to the size of the continuum, and by the results in Section 3 it is equivalent to the existence of sprays $X_1, \ldots, X_k$ with centers on a given hyperplane $H$, and covering $\mathbb{R}^d$, as the (directions of the) $\boldsymbol{u}_i$s correspond to the position of the centers of the $X_i$s on $H$. Finally in Section 5 we prove the results about the existence of sprays covering $\mathbb{R}^d$ and the size of the continuum.

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## 2. Notation and preliminary results

### 2.1. *Notation.*

**2.1.1.** *Set theory.* We work in ZFC — this is the standard framework that most mathematicians (implicitly or explicitly) adopt to prove theorems. Our notation is standard: $\mathscr{P}(X)$ is the power set of $X$, the pointwise image of $Z \subseteq X$ via some $f : X \to Y$ is $f[Z] := \{f(z) \mid z \in Z\}$, and the pointwise preimage of $W \subseteq Y$ is $f^{-1}[W] := \{x \in X \mid f(x) \in W\}$.

The *cardinality* of a set $X$ is denoted by $|X|$. A set $X$ is finite if it is in bijection with a natural number i.e. $|X| < \aleph_0$; otherwise it is infinite i.e. $\aleph_0 \leq |X|$. We say $X$ is *countable* if either $|X| < \aleph_0$, or else $|X| = \aleph_0$ — this can be written in a compact way as $|X| \leq \aleph_0$ or as $|X| < \aleph_1$. A set $X$ is *uncountable* if it is not countable, that is $\aleph_1 \leq |X|$.

The set $\mathbb{R}$ is in bijection with the set of all infinite sequences of 0s and 1s, and for this reason the cardinality of $\mathbb{R}$ is denoted with $2^{\aleph_0}$. By Cantor's theorem $\aleph_1 \leq |\mathbb{R}|$, and Cantor's continuum hypothesis CH asserts that the inequality can be replaced with an equality. Since $\aleph_1 \leq 2^{\aleph_0}$, CH can be stated as $2^{\aleph_0} \leq \aleph_1$. By Cohen's results, it is consistent that the cardinality of $\mathbb{R}$ be any $\aleph_{n+1}$, or even larger cardinals, like $\aleph_{\omega+n+1}$. (But $2^{\aleph_0}$ cannot be $\aleph_\omega$ by König's theorem.)

**2.1.2.** *Geometry.* The standard basis of $\mathbb{R}^d$ is denoted by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$. A hyperplane of a vector space is a linear subspace of codimension 1. The following notation will be used throughout the paper.

**Notation 2.1.** Given a vector $\boldsymbol{u} \in \mathbb{R}^d \setminus \{\boldsymbol{0}\}$ and $\boldsymbol{p} \in \mathbb{R}^d$ let

$$H_{\boldsymbol{u}}(\boldsymbol{p}) = \boldsymbol{p} + \{\boldsymbol{v} \mid \boldsymbol{v} \cdot \boldsymbol{u} = 0\}$$

be the hyperplane of $\mathbb{R}^d$ orthogonal to $\boldsymbol{u}$ passing through $\boldsymbol{p}$, where $\cdot$ is the (standard) inner product. We also let $H_i(x) = \{(p_1, \ldots, p_d) \in \mathbb{R}^d \mid p_i = x\}$.

An *affine subspace* $E$ of $\mathbb{R}^d$ is a translate of some vector subspace of $\mathbb{R}^d$, that is a set of the form $\boldsymbol{p}+V$ where $V$ is a vector subspace of $\mathbb{R}^d$; the vector subspace $V$ is unique, and it is the vector space associated to $E$, while $\boldsymbol{p}+V=\boldsymbol{p}'+V$ if and only if $\boldsymbol{p}-\boldsymbol{p}'\in V$. In this paper, a (finite dimensional) *affine space* is an affine subspace of some $\mathbb{R}^d$. Elements of an affine space are called points, and since every vector subspace $V$ of $\mathbb{R}^d$ is also an affine space, we can refer to elements of it as points or vectors, depending if we privilege the affine or vector space structure. If $E$ is an affine subspace of $\mathbb{R}^d$ and $\boldsymbol{p},\boldsymbol{q}\in E$ then $\boldsymbol{q}-\boldsymbol{p}$ belongs to $V$, the underlying vector space of $E$, and if $\boldsymbol{v}\in V$ then $\boldsymbol{p}+\boldsymbol{v}$ belongs to $E$. The dimension of an affine space is, by definition, the dimension of the associated vector space. An *affine hyperplane* of $\mathbb{R}^d$ is a translate of linear subspace of dimension $d-1$. More generally, a hyperplane of an affine space $E$ of dimension $d$ is an affine subspace of dimension $d-1$. Given an affine space $E$ with associated vector space $V$, the *affine envelope* or *affine span* of a non-empty set $S\subseteq E$ is $\boldsymbol{p}+\operatorname{span}\{\boldsymbol{q}-\boldsymbol{p}\mid \boldsymbol{q}\in S\}$ where $\boldsymbol{p}\in S$ and $\operatorname{span} X$ is the smallest vector subspace of $V$ containing $X\subseteq V$. This definition does not depend on the point $\boldsymbol{p}$, and the affine span of $S$ is the intersection of all affine subspaces of $E$ containing $S$. Two affine subspaces $E$ and $F$ of $\mathbb{R}^d$ are *complementary* if their associated vector spaces $V$ and $W$ are complementary, that is $\mathbb{R}^d=V\oplus W$. It is easy to check that the intersection of two complementary spaces $E$, $F$ is a single point. If $V\oplus W=\mathbb{R}^d$, then $V$ and $W$ are orthogonal if $\forall \boldsymbol{v}\in V\,\forall \boldsymbol{w}\in W\,(\boldsymbol{v}\cdot\boldsymbol{w}=0)$. In this case we call one of the two subspaces the *orthogonal complement* of the other, and write $V^\perp$ to denote $W$, the orthogonal complement of $V$. Two complementary affine subspaces $E$ and $F$ of $\mathbb{R}^d$ are orthogonal if their underlying vector spaces are orthogonal.

Let $E$ be an affine subspace of $\mathbb{R}^d$. The *sphere in $E$ with center $\boldsymbol{c}\in E$ and radius* $r\in\mathbb{R}$ is the set

$$\mathbb{S}(E;\boldsymbol{c},r):=\{\boldsymbol{x}\in E\mid \|\boldsymbol{x}-\boldsymbol{c}\|=r\}.$$

We convene that $\mathbb{S}(E;\boldsymbol{c},r)$ is empty if $r<0$, and it is the singleton $\{\boldsymbol{c}\}$ when $r=0$. Observe that if $r>0$ then $\mathbb{S}(E;\boldsymbol{c},r)$ has cardinality 2 when $\dim(E)=1$; if $\dim(E)\geq 2$, then $\mathbb{S}(E;\boldsymbol{c},r)$ has cardinality $2^{\aleph_0}$. Whenever the ambient space (i.e. $\mathbb{R}^d$, $E$, …) is clear we will simply write $\mathbb{S}(\boldsymbol{c},r)$. A $(k-1)$*-dimensional sphere* is a sphere in an affine subspace $E$ of some $\mathbb{R}^d$ with $\dim E=k$.

**2.2. *Families with finite mesh.*** Let $N,d\geq 2$ be natural numbers and suppose that $\mathcal{H}_i\subseteq\mathscr{P}(\mathbb{R}^d)$ with $1\leq i\leq N$ are pairwise disjoint. Following [Erdős et al. 1994], the sequence $(\mathcal{H}_i)_{i=1}^N$ is $(r,s)$ finitely determined if $r\geq 2$ and $s\geq 1$ are natural numbers such that

- for any distinct $\boldsymbol{p}_1,\dots,\boldsymbol{p}_s\in\mathbb{R}^d$, the set of all $H\in\bigcup_{i=1}^N\mathcal{H}_i$ such that $\boldsymbol{p}_1,\dots,\boldsymbol{p}_s\in H$ is finite;

- the intersection of $r$ sets belonging to distinct $\mathcal{H}_i$s is a finite set.

We say that $(\mathcal{H}_i)_{i=1}^N$ is of *mesh* $r$ if $r$ is least such that the sequence is $(r, s)$ finitely determined, for some $s$.

**Examples 2.2.** (i) Let $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ be noncollinear points of $\mathbb{R}^3$, and let $\mathcal{H}_i$ be the collection of all spheres of with center $\boldsymbol{c}_i$. Then $(\mathcal{H}_i)_{i=1}^3$ is $(3, 1)$ finitely determined, and of mesh 3.

(ii) Let $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3$ be a basis for $\mathbb{R}^3$, and let $\mathcal{H}_i$ be the set of all planes orthogonal to $\boldsymbol{u}_i$. Then $(\mathcal{H}_i)_{i=1}^3$ is $(3, 1)$ finitely determined, and of mesh 3.

The reason for the notion of mesh is the following very general result by Erdős, Jackson, and Mauldin [Erdős et al. 1994, Theorem 2]:

**Theorem 2.3.** *For all $d \geq 2$, $n \geq 0$, and $\delta = 0, 1$ the following are equivalent*:

(a) $2^{\aleph_0} \leq \aleph_{\delta+n}$;

(b) *for any $r \geq 2$, letting $N = (n+1)(r-1)+1$, and for any sequence of pairwise disjoint $\mathcal{H}_i \subseteq \mathscr{P}(\mathbb{R}^d)$ with $1 \leq i \leq N$ of mesh $r$, there are $A_1, \dots, A_N$ covering $\mathbb{R}^d$ such that $\forall 1 \leq i \leq N \, \forall H \in \mathcal{H}_i \, (|H \cap A_i| < \aleph_\delta)$.*

Actually the result in [Erdős et al. 1994] is more general than Theorem 2.3, as $\mathbb{R}^d$ can be replaced by an arbitrary infinite set $X$, and $2^{\aleph_0}$ can be replaced by $|X|$. Also $\delta$ can be any ordinal, not just 0 or 1.

The main use of Theorem 2.3 in this paper is the direction (a)⇒(b). Given $(\mathcal{H}_i)_{i=1}^N$ of mesh $r$ in $\mathbb{R}^d$, then $|\mathbb{R}| \leq \aleph_{\delta+n}$ implies that there are $A_1, \dots, A_N \subseteq \mathbb{R}^d$ covering $\mathbb{R}^d$ such that for all $H \in \mathcal{H}_i$, the set $H \cap A_i$ is *finite* if $\delta = 0$ or *countable* if $\delta = 1$. Actually, the "forward implication" (i.e. from a bound on the size of the continuum, to the existence of specific subsets of the space) in [Bagemihl 1959/60; 1968; Davies 1962; 1963a; 1963b; Komjáth 2001] follows from Theorem 2.3.

### 2.3. *Points and vectors in general position.*

**Definition 2.4.** A set of vectors of $\mathbb{R}^d$ is in *general position* if any subset of size $\leq d$ is linearly independent — in other words: the vectors are as linearly independent as possible.

The following is a straightforward generalization of Examples 2.2(ii).

**Example 2.5.** If $\boldsymbol{u}_1, \dots, \boldsymbol{u}_N$ are vectors in general position in $\mathbb{R}^d$, and $\mathcal{H}_i := \{H_{\boldsymbol{u}_i}(\boldsymbol{p}) \mid \boldsymbol{p} \in \mathbb{R}^d\}$ is the family of all affine hyperplanes orthogonal to $\boldsymbol{u}_i$, then $(\mathcal{H}_i)_{i=1}^N$ is of mesh $d$.

By Example 2.5 and Theorem 2.3, if $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{(n+1)(d-1)+1}$ are vectors in general position in $\mathbb{R}^d$, then

- $2^{\aleph_0} \leq \aleph_n$ implies that there are $A_1, \dots, A_{(n+1)(d-1)+1}$ covering $\mathbb{R}^d$ such that $H_{\boldsymbol{u}_i}(\boldsymbol{p}) \cap A_i$ is finite for every $\boldsymbol{p} \in \mathbb{R}^d$ and $1 \leq i \leq (n+1)(d-1)+1$;

- $2^{\aleph_0} \leq \aleph_{n+1}$ implies that there are $A_1, \ldots, A_{(n+1)(d-1)+1}$ covering $\mathbb{R}^d$ such that $H_{\boldsymbol{u}_i}(\boldsymbol{p}) \cap A_i$ is countable, for every $\boldsymbol{p} \in \mathbb{R}^d$ and $1 \leq i \leq (n+1)(d-1)+1$.

Therefore CH implies that if $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d$ are a basis of $\mathbb{R}^d$, there are $A_1, \ldots, A_d$ covering $\mathbb{R}^d$ such that every affine hyperplane orthogonal to $\boldsymbol{u}_i$ has countable intersection with $A_i$.

In fact, the implications above are actually equivalences [Komjáth and Totik 2006, pp. 71, 327–328].

**Theorem 2.6** (Sierpiński). CH *is equivalent to either one of the following*:

(a) *There are $A_1, A_2$ covering $\mathbb{R}^2$ such that every vertical line has countable intersection with $A_1$ and every horizontal line has countable intersection with $A_2$.*

(b) *There are $A_1, A_2, A_3$ covering $\mathbb{R}^3$ such that every plane orthogonal to $\boldsymbol{e}_i$ has countable intersection with $A_i$.*

Definition 2.4 can be extended to points in affine spaces.

**Definition 2.7.** Let $S$ be a set of points of an affine space $E$ of dimension $d$. We say that $S$ is

- in *general position in $E$* if the affine span of any of its subset of size $k+1$ has dimension $k$, for all $k \leq d$,
- *well-placed in $E$* if $S \subseteq H$ for some hyperplane $H$ of $E$, and $S$ is in general position in $H$.

It is common practice in geometry to use "general position" for *vectors* (a linear algebra notion, Definition 2.4) and for *points* (an affine notion, Definition 2.7), and this could give rise to some ambiguity when the affine space is $\mathbb{R}^d$. For example, the *vectors* $(-1, 1)$, $(0, 1)$, $(1, 1)$ are in general position, but the set of *points* $\{(-1, 1), (0, 1), (1, 1)\}$ is not. For this reason we will use the terms "vectors" and "points" to help the reader sort out which of the two notions is being used.

**Remarks 2.8.** Let $E$ be an affine space of dimension $d \geq 2$ and let $S \subseteq E$.

(a) If $S$ is in general position (well-placed), and $S' \subseteq S$ then $S'$ is in general position (well-placed). In other words, the notions of being in general position/well-placed are downward persistent with respect to inclusion.

(b) Suppose $S$ is well-placed:
  - if $|S| \geq d$ then the hyperplane in the definition is unique, being the affine span of any subset of size $d$;
  - if $|S| \leq d$ then $S$ is in general position in $E$.

(c) Suppose $H$ is a hyperplane of $E$, and $S \subseteq H$. Then $S$ is well-placed in $E$ if and only if $S$ is in general position in $H$.

(d) The set of points $S$ is in general position in $E$ if and only if the vectors in $\{\boldsymbol{q}-\boldsymbol{p} \mid \boldsymbol{q} \in S \setminus \{\boldsymbol{p}\}\}$ are in general position in $\mathbb{R}^d$, for any $\boldsymbol{p} \in S$.

Every set of points in $\mathbb{R}$ is in general position, a set of points in $\mathbb{R}^2$ is in general position if no three of them are collinear, a set of points in $\mathbb{R}^3$ is in general position if no four of them are coplanar, and so on. A set of points in $\mathbb{R}^2$ is well-placed if they are collinear, a set of points in $\mathbb{R}^3$ is well-placed if they are coplanar, and no three of them are collinear, a set of points in $\mathbb{R}^4$ is well-placed if they belong to the same 3-dimensional affine subspace, and no four of them are coplanar, and so on.

Finally observe that for any affine space $E$ of dimension $d \geq 2$

(1) if $S \subseteq E$ is a set of points in general position or well-placed in $E$, then the affine span of any subset of $S$ of size $k+1$ has dimension $k$, for any $k < d$.

**Lemma 2.9.** *Suppose $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{k+1}$ are distinct points in general position in $\mathbb{R}^d$, with $d \geq k$. Let $H$ be an affine hyperplane orthogonal to $\boldsymbol{c}_k - \boldsymbol{c}_{k+1}$, let $\pi : \mathbb{R}^d \to H$ be the orthogonal projection. Let $\bar{\boldsymbol{c}}_k = \pi(\boldsymbol{c}_k) = \pi(\boldsymbol{c}_{k+1})$ and let $\bar{\boldsymbol{c}}_i = \pi(\boldsymbol{c}_i)$ for $i < k$. Then the points $\bar{\boldsymbol{c}}_1, \ldots, \bar{\boldsymbol{c}}_k$ are distinct and in general position in $H$.*

*Proof.* If $\bar{\boldsymbol{c}}_i = \bar{\boldsymbol{c}}_k$ for some $i < k$, then $\boldsymbol{c}_i, \boldsymbol{c}_k, \boldsymbol{c}_{k+1}$ must be collinear, and if $\bar{\boldsymbol{c}}_i = \bar{\boldsymbol{c}}_j$ for some $i < j < k$, then $\boldsymbol{c}_i - \boldsymbol{c}_j$ and $\boldsymbol{c}_k - \boldsymbol{c}_{k+1}$ must be parallel, so the four points $\boldsymbol{c}_i, \boldsymbol{c}_j, \boldsymbol{c}_k, \boldsymbol{c}_{k+1}$ are coplanar. In either case this contradicts the assumption that $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{k+1}$ are in general position. Therefore the points $\bar{\boldsymbol{c}}_1, \ldots, \bar{\boldsymbol{c}}_k$ are distinct.

In order to conclude the proof it is enough to show that the vectors $\bar{\boldsymbol{c}}_1 - \bar{\boldsymbol{c}}_k$, $\bar{\boldsymbol{c}}_2 - \bar{\boldsymbol{c}}_k, \ldots, \bar{\boldsymbol{c}}_{k-1} - \bar{\boldsymbol{c}}_k$ of $\mathbb{R}^d$ are linearly independent. Suppose

$$\mathbf{0} = r_1(\bar{\boldsymbol{c}}_1 - \bar{\boldsymbol{c}}_k) + r_2(\bar{\boldsymbol{c}}_2 - \bar{\boldsymbol{c}}_k) + \cdots + r_{k-1}(\bar{\boldsymbol{c}}_{k-1} - \bar{\boldsymbol{c}}_k).$$

As $\bar{\boldsymbol{c}}_i$ is the projection of $\boldsymbol{c}_i$ along the vector $\boldsymbol{c}_{k+1} - \boldsymbol{c}_k$, there are $s_1, \ldots, s_k \in \mathbb{R}$ such that $\bar{\boldsymbol{c}}_i = \boldsymbol{c}_i + s_i(\boldsymbol{c}_{k+1} - \boldsymbol{c}_k)$. Substituting in the previous formula we obtain

$$\mathbf{0} = r_1(\boldsymbol{c}_1 - \boldsymbol{c}_k) + \cdots + r_{k-1}(\boldsymbol{c}_{k-1} - \boldsymbol{c}_k) + [r_1(s_1 - s_k) + \cdots + r_{k-1}(s_{k-1} - s_k)](\boldsymbol{c}_{k+1} - \boldsymbol{c}_k),$$

and since $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{k+1}$ are in general position in $\mathbb{R}^d$, then $[r_1(s_1 - s_k) + \cdots + r_{k-1}(s_{k-1} - s_k)] = 0$ and $r_1 = \cdots = r_{k-1} = 0$, which is what we had to prove. □

**Lemma 2.10.** *Suppose $H$ is a hyperplane of an affine space $E$ of dimension $d+1$. Let $\pi : E \to H$ be the orthogonal projection, and let $h$ be the distance between a point $\boldsymbol{c} \in E$ and $H$. Given $R$ let $r = \sqrt{R^2 - h^2}$ if $R \geq h$ and $r = -1$ if $R \leq h$. Then $\mathbb{S}(E; \boldsymbol{c}, R) \cap H = \mathbb{S}(H; \pi(\boldsymbol{c}), r)$.*

*Conversely, if $r > 0$ then $\mathbb{S}(E; \boldsymbol{c}, \sqrt{r^2 + h^2}) \cap H = \mathbb{S}(H, \pi(\boldsymbol{c}), r)$.*

*Proof.* Without loss of generality we may assume that $E = \mathbb{R}^{d+1}$, that $H = \mathbb{R}^d \times \{0\}$, $\boldsymbol{c} = (0, \ldots, 0, h)$, and $\pi(\boldsymbol{c}) = \mathbf{0}$, so that $h = \|\boldsymbol{c} - \mathbf{0}\|$. It is clear that $R < h$ if and only if $\mathbb{S}(\boldsymbol{c}, R) \cap H = \emptyset$, and that $R = h$ if and only if $\mathbb{S}(\boldsymbol{c}, R) \cap H = \{\pi(\boldsymbol{c})\} = \{\boldsymbol{0}\}$. If $R > h$ then $r^2 + h^2 = R^2$ so

$$\begin{aligned} \boldsymbol{x} \in \mathbb{S}(E; \boldsymbol{c}, R) \cap H &\iff \boldsymbol{x} = (x_1, \ldots, x_d, 0) \wedge (\textstyle\sum_{i=1}^d x_i^2) + h^2 = R^2 \\ &\iff x_1^2 + \cdots + x_d^2 = r^2 \\ &\iff \|\boldsymbol{x}\| = r \\ &\iff \boldsymbol{x} \in \mathbb{S}(H; \pi(\boldsymbol{c}), r). \end{aligned}$$

The equivalences above prove the second part of the statement as well. □

**Lemma 2.11.** *Suppose $\boldsymbol{c}_1, \boldsymbol{c}_2$ are distinct points of an affine space $E$ of dimension $d \geq 2$, and let $r_1, r_2 > 0$ be such that $|r_1 - r_2| \leq \|\boldsymbol{c}_1 - \boldsymbol{c}_2\| \leq r_1 + r_2$. Then*

$$\mathbb{S}(E; \boldsymbol{c}_1, r_1) \cap \mathbb{S}(E; \boldsymbol{c}_2, r_2) = \mathbb{S}(H; \boldsymbol{c}, r),$$

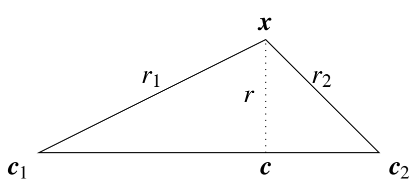
*where $\boldsymbol{c}$ is a point of the open segment $(\boldsymbol{c}_1; \boldsymbol{c}_2)$, $H$ is the hyperplane passing through $\boldsymbol{c}$ and orthogonal to $\boldsymbol{c}_1 - \boldsymbol{c}_2$, and $r \geq 0$.*

*Proof.* We may assume that $E = \mathbb{R}^d$. The hypothesis $|r_1 - r_2| \leq \|\boldsymbol{c}_1 - \boldsymbol{c}_2\|$ guarantees that neither sphere properly contains inside the other sphere, while the hypothesis $\|\boldsymbol{c}_1 - \boldsymbol{c}_2\| \leq r_1 + r_2$ ensures that if neither sphere contains the other, then they are close enough to intersect.

Let $\boldsymbol{x}$ be a point of $\mathbb{S}(\boldsymbol{c}_1, r_1) \cap \mathbb{S}(\boldsymbol{c}_2, r_2)$, and let $P$ be the plane passing through $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{x}$. (If $d = 2$ then $P$ coincides with $\mathbb{R}^2$.) Consider the triangle with vertices $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{x}$.

If $r_1 + r_2 = \|\boldsymbol{c}_1 - \boldsymbol{c}_2\|$ then the two spheres are tangent in the point $\boldsymbol{c} := \boldsymbol{x}$ which belongs to the open segment $(\boldsymbol{c}_1; \boldsymbol{c}_2)$, so the triangle is degenerate and letting $r = 0$ we have $\mathbb{S}(H; \boldsymbol{c}, r) = \{\boldsymbol{c}\}$.

So we may assume that $r_1 + r_2 > \|\boldsymbol{c}_1 - \boldsymbol{c}_2\|$ and that the triangle




is nondegenerate. Let $\boldsymbol{c}$ be the projection of $\boldsymbol{x}$ on the open segment $(\boldsymbol{c}_1; \boldsymbol{c}_2)$, let $r = \|\boldsymbol{x} - \boldsymbol{c}\| > 0$, and let $H$ be the hyperplane orthogonal to $\boldsymbol{c}_1 - \boldsymbol{c}_2$ passing through $\boldsymbol{c}$. Observe that $r$, the point $\boldsymbol{c}$, and the hyperplane $H$, depend only on the points $\boldsymbol{c}_1, \boldsymbol{c}_2$ and on $r_1, r_2$, and not on the point $\boldsymbol{x}$ or the plane $P$. Therefore, for all $\boldsymbol{x} \in \mathbb{R}^d$,

$$\begin{aligned} \boldsymbol{x} \in \mathbb{S}(\boldsymbol{c}_1, r_1) \cap \mathbb{S}(\boldsymbol{c}_2, r_2) &\iff \|\boldsymbol{x} - \boldsymbol{c}_1\| = r_1 \wedge \|\boldsymbol{x} - \boldsymbol{c}_2\| = r_2 \\ &\iff \boldsymbol{x} \in H \wedge \|\boldsymbol{c} - \boldsymbol{x}\| = r. \end{aligned}$$ □

**Proposition 2.12.** *Let $E$ be an affine space of dimension $d \geq 2$, let $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_k$ be points in general position in $E$ with $k \leq d$, and let $K$ be the affine span of the $\boldsymbol{c}_i$s. Let $S_i$ be a sphere centered in $\boldsymbol{c}_i$ for $i \leq k$.*

(a) *There are $r \in \mathbb{R}$ and an affine subspace $H$ of $E$ of dimension $d-(k-1)$, such that $H \cap K$ is a singleton $\{\boldsymbol{c}\}$ and $H$ and $K$ are orthogonal, so that $S_1 \cap \cdots \cap S_k = \mathbb{S}(H; \boldsymbol{c}, r)$.*

(b) *If $\boldsymbol{c}_{k+1} \in E \setminus \{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_k\}$ and $\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_k, \boldsymbol{c}_{k+1}\}$ is not in general position in $E$, then there is $R \in \mathbb{R}$ such that*

$$S_1 \cap \cdots \cap S_k \subseteq \mathbb{S}(\boldsymbol{c}_{k+1}, R).$$

*Proof.* We may assume that $E = \mathbb{R}^d$.

If $d = 2$ then the result is clear. Part (a) amounts to say that: given two circles $C_1, C_2$ in the plane with distinct centers $\boldsymbol{c}_1, \boldsymbol{c}_2$, either $I := C_1 \cap C_2$ is empty, or else there is a point $\boldsymbol{c}$ on the open segment $(\boldsymbol{c}_1; \boldsymbol{c}_2)$ such that $I = \{\boldsymbol{c}\}$, or else $I$ is the set of two points on line orthogonal to $(\boldsymbol{c}_1; \boldsymbol{c}_2)$ passing through $\boldsymbol{c}$, and symmetric with respect to $\boldsymbol{c}$. Part (b) says that if three points are collinear, then given any two circles centered in the first two points there is a (possibly degenerate) circle centered in the third point that passes through the intersection of the first two circles.

Therefore we may assume that $d \geq 3$.

(a) We proceed by induction on $k$. If $k = 1$ the result is trivial, so we may assume the result holds for some $k$ towards proving it for $k+1 \leq d$. Suppose $S_1, \ldots, S_{k+1}$ are spheres centered in distinct points $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{k+1}$. By an isometry we may assume that $K$, the affine span of the $\boldsymbol{c}_i$s, is contained in $\mathbb{R}^{d-1} \times \{0\}$. Let $\bar{S}_k := S_k \cap S_{k+1}$. We have three cases:

(1) $\bar{S}_k = \varnothing$.

(2) $\bar{S}_k$ is a singleton.

(3) $\bar{S}_k$ is a nondegenerate sphere in some affine hyperplane $H$ of $\mathbb{R}^d$ (Lemma 2.11).

If either (1) or (2) holds, then $S_1 \cap \cdots \cap S_{k+1}$ is empty or a singleton, and the result follows trivially. So we may assume case (3). The hyperplane $H$ is orthogonal to the open segment $(\boldsymbol{c}_k; \boldsymbol{c}_{k+1})$, and $\bar{\boldsymbol{c}}_k$, the center of $\bar{S}_k$, belongs to this segment. Let $\pi : \mathbb{R}^d \to H$ be the orthogonal projection. Then $\pi(\boldsymbol{c}_k) = \pi(\boldsymbol{c}_{k+1}) = \bar{\boldsymbol{c}}_k$ and for $i < k$ let $\bar{\boldsymbol{c}}_i := \pi(\boldsymbol{c}_i)$. By Lemma 2.10 for every $i < k$ the set $\bar{S}_i := S_i \cap H$ is a sphere of $H$ with center $\bar{\boldsymbol{c}}_i$. As $S_k \cap S_{k+1} \subseteq H$

$$S_1 \cap \cdots \cap S_k \cap S_{k+1} = \bar{S}_1 \cap \cdots \cap \bar{S}_k.$$

By Lemma 2.9 $\bar{\boldsymbol{c}}_1, \ldots, \bar{\boldsymbol{c}}_k$ are in general position in $H \cong \mathbb{R}^{d-1}$, and since $d-1 \geq 2$, by inductive assumption we can conclude that $\bar{S}_1 \cap \cdots \cap \bar{S}_k = \mathbb{S}(H'; \boldsymbol{c}, r)$ where

$H'$ a subspace of $H$ (and hence of $\mathbb{R}^d$) of dimension $(d-1)-(k-1)=d-k$, such that $\{\boldsymbol{c}\} = H' \cap K$, and $H'$ is orthogonal to $K$.

(b) By assumption $\boldsymbol{c}_{k+1}$ belongs to $K$, and $k-1=\dim(K)$. By part (a) there is a subspace $H$ orthogonal to $K$, of dimension $d-(k-1)$, and $r \in \mathbb{R}$ such that $S_1 \cap \cdots \cap S_k = \mathbb{S}(H; \boldsymbol{c}, r)$ where $\boldsymbol{c}$ is the unique element of $H \cap K$. It is enough to find $R$ such that $\mathbb{S}(H; \boldsymbol{c}, r)$ is contained in $\mathbb{S}(E; \boldsymbol{c}_{k+1}, R)$. This is clear if $r<0$; otherwise take $R = \sqrt{\|\boldsymbol{c}_{k+1} - \boldsymbol{c}\|^2 + r^2}$, this value works by Pythagoras theorem as the vectors $\boldsymbol{x} - \boldsymbol{c}$ and $\boldsymbol{c}_{k+1} - \boldsymbol{c}$ are orthogonal whenever $\boldsymbol{x} \in \mathbb{S}(H; \boldsymbol{c}, r)$. □

**Theorem 2.13.** *Let $E$ be an affine space of dimension $d \geq 2$ and let $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_d$ be distinct points in general position in $E$.*

(a) *For all spheres $S_i$ in $E$ centered in $\boldsymbol{c}_i$ with $1 \leq i \leq d$, the set $S_1 \cap \cdots \cap S_d$ is finite.*
(b) *For all $k<d$ and $1 \leq i \leq k$ there are spheres $S_i$ in $E$ centered in $\boldsymbol{c}_i$ such that $S_1 \cap \cdots \cap S_k$ is a nondegenerate sphere in a subspace of dimension $d-(k-1) \geq 2$.*

*Proof.* Recall that a nondegenerate sphere in space of dimension $n$ has cardinality $2^{\aleph_0}$ or 2 depending whether $n>1$ or $n=1$. Part (a) then follows at once form Proposition 2.12.

(b) We prove by induction on $k$ a stronger statement: for any $r_k>0$ and for any affine space $E$ of dimension $d>k$, there exist $r_1, \ldots, r_{k-1} > 0$ such that

$$\mathbb{S}(E; \boldsymbol{c}_1, r_1) \cap \cdots \cap \mathbb{S}(E; \boldsymbol{c}_k, r_k) = \mathbb{S}(H; \boldsymbol{c}, r),$$

for some $r>0$, $\boldsymbol{c} \in H$ and $H$ a subspace of $E$ of dimension $d-(k-1)$.

The base case $k=1$ is immediate, so we may assume that the result holds for some $k$ towards proving it for $k+1<d$. Fix $r_{k+1}>0$ and pick $r_k>0$ such that $|r_k - r_{k+1}| < \|\boldsymbol{c}_k - \boldsymbol{c}_{k+1}\| < r_k + r_{k+1}$. By Lemma 2.11 $\mathbb{S}(E; \boldsymbol{c}_{k+1}, r_{k+1}) \cap \mathbb{S}(E; \boldsymbol{c}_k, r_k)$ is a nondegenerate sphere $\mathbb{S}(\bar{E}; \bar{\boldsymbol{c}}_k, \bar{r}_k)$ in some hyperplane $\bar{E}$ of $E$. Let $\pi : E \to \bar{E}$ be the orthogonal projection. By Lemma 2.9 the points

$$\bar{\boldsymbol{c}}_1 = \pi(\boldsymbol{c}_1), \quad \bar{\boldsymbol{c}}_2 = \pi(\boldsymbol{c}_2), \quad \ldots \quad \bar{\boldsymbol{c}}_{k-1} = \pi(\boldsymbol{c}_{k-1}), \quad \bar{\boldsymbol{c}}_k = \pi(\boldsymbol{c}_k) = \pi(\boldsymbol{c}_{k+1})$$

are distinct and in general position in the affine space $\bar{E}$ of dimension $d-1>k$. Since $\bar{r}_k>0$, by inductive assumption there exist $\bar{r}_1, \ldots, \bar{r}_{k-1} > 0$ such that

$$S := \mathbb{S}(\bar{E}, \bar{\boldsymbol{c}}_1, \bar{r}_1) \cap \cdots \cap \mathbb{S}(\bar{E}, \bar{\boldsymbol{c}}_k, \bar{r}_k)$$

is a nondegenerate sphere in a subspace $H$ of dimension $(d-1)-(k-1)=d-k$. Letting $r_i = \sqrt{\bar{r}_i^2 + h_i^2}$ where $h_i$ is the distance between $\boldsymbol{c}_i$ and the hyperplane $\bar{E}$, we have

$$S = \mathbb{S}(E, \boldsymbol{c}_1, r_1) \cap \cdots \cap \mathbb{S}(E, \boldsymbol{c}_k, r_k) \cap \mathbb{S}(E, \boldsymbol{c}_{k+1}, r_{k+1})$$

as required. □

**Corollary 2.14.** *Suppose $E$ is an affine space of dimension $d \geq 2$ and $\boldsymbol{c}_1, \dots, \boldsymbol{c}_{d+1}$ are distinct points that are not in general position in $E$. Then there are spheres $S_i$ centered in $\boldsymbol{c}_i$ such that $S_1 \cap \cdots \cap S_{d+1}$ is infinite.*

*Proof.* Let $k$ be the dimension of the affine span $K$ of $\{\boldsymbol{c}_1, \dots, \boldsymbol{c}_{d+1}\}$. By assumption $k < d$, and without loss of generality we may assume that $\boldsymbol{c}_1, \dots, \boldsymbol{c}_{k+1}$ are in general position in $K$. By part (b) of Theorem 2.13 there are spheres $S_1, \dots, S_{k+1}$ centered in $\boldsymbol{c}_1, \dots, \boldsymbol{c}_{k+1}$ whose intersection is a nondegenerate sphere in a space of dimension $\geq 2$. By repeated applications of part (b) of Proposition 2.12, there are spheres $S_{k+2}, \dots, S_{d+1}$ centered in $\boldsymbol{c}_{k+2}, \dots, \boldsymbol{c}_{d+1}$ such that

$$S_1 \cap \cdots \cap S_{k+1} \subseteq S_i \quad \text{for } k+2 \leq i \leq d+1$$

and hence $S_1 \cap \cdots \cap S_{k+1} = S_1 \cap \cdots \cap S_{d+1}$ is infinite. □

Observe that by (1), both "$S$ is in general position" and "$S$ is well-placed" imply the assumption of the next corollary.

**Corollary 2.15.** *Suppose $S$ is a set of at least $d$ points of an affine space $E$ of dimension $d \geq 2$. Assume that any $d$-many points of $S$ span a subspace of dimension $d-1$. Then the intersection of $d$-many spheres centered in these points is a finite set.*

**Lemma 2.16.** *Suppose $N \geq d$ and $\boldsymbol{c}_1, \dots, \boldsymbol{c}_N$ are distinct points of $\mathbb{R}^d$ such that any $d$ of them span an affine subspace of dimension $d-1$. If $\mathcal{H}_i$ is the collection of all spheres of $\mathbb{R}^d$ with center $\boldsymbol{c}_i$, then $(\mathcal{H}_i)_{i=1}^N$ is of mesh $d$.*

*Proof.* It is enough to consider the case $N = d$. If $d = 1$ this is trivial since a sphere in $\mathbb{R}$ is just a pair of points that are symmetric with respect to the center, and if $d = 2$ it follows immediately from the fact that the intersection of two circles with distinct centers has size at most 2. If $d = 3$ this follows from the fact that the intersection of three spheres in $\mathbb{R}^3$ with noncollinear centers has size at most 2. The case for $d > 3$ follows from Theorem 2.13. □

**2.4.** ***Sprays.*** J. Schmerl [2003] defined a spray to be a set $X \subseteq \mathbb{R}^2$ such that $C \cap X$ is finite, for all circles $C$ centered in some given point. Thus a spray $X$ is a very sparse subset of the plane, and one may investigate what happens if one relaxes the finiteness condition on $C \cap X$ with, say, being countable.

**Definition 2.17.** Let $E$ be an affine subspace of $\mathbb{R}^d$, with $d \geq 2$ be a natural number. An *$\aleph_\delta$-spray in $E$ with center $\boldsymbol{c}$* is a set $X \subseteq E$ such that $|\mathbb{S}(E; \boldsymbol{c}, r) \cap X| < \aleph_\delta$ for every $r > 0$. When $\delta = 0$ it is called a *spray*, when $\delta = 1$ we speak of *$\sigma$-spray*.

Note that a center need not to belong to the $\aleph_\delta$-spray, and that a $\aleph_\delta$-spray may have more than one center.

By Lemma 2.16, if $\boldsymbol{c}_1, \dots, \boldsymbol{c}_{(n+1)(d-1)+1}$ are either in general position, or well-placed in $\mathbb{R}^d$, then

- $2^{\aleph_0} \leq \aleph_n$ implies that there are $X_1, \dots, X_{(n+1)(d-1)+1}$ covering $\mathbb{R}^d$ such that each $X_i$ is a spray centered in $\boldsymbol{c}_i$;
- $2^{\aleph_0} \leq \aleph_{n+1}$ implies that there are $X_1, \dots, X_{(n+1)(d-1)+1}$ covering $\mathbb{R}^d$ such that each $X_i$ is a $\sigma$-spray.

In particular if the points are well-placed, that is if we assume that the $\boldsymbol{c}_i$s lie on a hyperplane, then:

**Theorem 2.18.** *Let $d \geq 2$, $n \geq 0$, and $\delta = 0, 1$. Assume that $\boldsymbol{c}_1, \dots, \boldsymbol{c}_{(n+1)(d-1)+1}$ are well-placed points in $\mathbb{R}^d$. If $2^{\aleph_0} \leq \aleph_{\delta+n}$ then there are $X_1, \dots, X_{(n+1)(d-1)+1}$ covering $\mathbb{R}^d$ such that each $X_i$ is an $\aleph_\delta$-spray centered in $\boldsymbol{c}_i$.*

By (1) and Corollary 2.15, the thesis of Theorem 2.18 holds if $\boldsymbol{c}_1, \dots, \boldsymbol{c}_{(n+1)(d-1)+1}$ are in general position in $\mathbb{R}^d$. The reason for focusing on well-placed points is that under this assumption the implication in Theorem 2.18 can be reversed, and this is the main goal of this paper. In Section 3 we argue that any covering of $\mathbb{R}^d$ with sprays ($\sigma$-sprays) with well-placed centers $\boldsymbol{c}_i$s can be transformed into a covering (of an open subset) of $\mathbb{R}^d$ with sets whose intersection with the hyperplanes orthogonal to $\boldsymbol{u}_i$ is finite (countable) for all $i$, where the vector $\boldsymbol{u}_i$ is obtained from the point $\boldsymbol{c}_i$. Appealing to the results from Section 4 the equivalence will be established.

The case of points in general position, but not well-placed, i.e. not belonging to a hyperplane, is more problematic. If $d = 2$ and three points are not collinear (i.e. are in general position, but not well-placed in $\mathbb{R}^2$) then by [Schmerl 2010, Theorem 1], the plane is the union of three sprays centered in these points, irrespective of the size of the continuum. The analogous statement for $d > 2$ is open, and if true it would show that the assumption that the points are well-placed cannot be dropped.

**Conjecture 2.19.** *For all $d \geq 3$, for all $\boldsymbol{c}_1, \dots, \boldsymbol{c}_{d+1}$ in general position in $\mathbb{R}^d$, there are $X_1, \dots, X_{d+1}$ covering $\mathbb{R}^d$, with $X_i$ a spray centered in $\boldsymbol{c}_i$ for $1 \leq i \leq d+1$.*

**2.5. *Elementary results about sprays.*** The following basic facts are nonetheless useful.

**Gluing:** If $X$ and $Y$ are $\aleph_\delta$-sprays in an affine space $E$ with the same center $\boldsymbol{c}$, then $X \cup Y$ is a $\aleph_\delta$-spray in $E$ with center $\boldsymbol{c}$.

**Projection:** Suppose that $X_1, \dots, X_n$ are $\aleph_\delta$-sprays in an affine space $E$ of dimension $d+1$, and $X_1 \cup \dots \cup X_n = E$ with centers $\boldsymbol{c}_1, \dots, \boldsymbol{c}_n$. Suppose $H$ is a hyperplane of $E$, and let $\pi : E \to H$ be the orthogonal projection. Then $X_1 \cap H, \dots, X_n \cap H$ are $\aleph_\delta$-sprays in $H$ with centers $\pi(\boldsymbol{c}_1), \dots, \pi(\boldsymbol{c}_n) \in H$, that cover $H$.

Projection follows from Lemma 2.10. Observe that the points $\pi(\boldsymbol{c}_1), \ldots, \pi(\boldsymbol{c}_n)$ need not be distinct, even if the $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$ are distinct. This, together with gluing, allows to transform $n$-many $\aleph_\delta$-sprays covering $\mathbb{R}^{d+1}$ into $n'$-many $\aleph_\delta$-sprays covering $\mathbb{R}^d$, with $n' < n$.

**Proposition 2.20.** *Let $\delta \in \mathrm{Ord}$ and suppose $\mathbb{R}^2 = X_1 \cup X_2$ where $X_1$ is an $\aleph_\delta$-spray and $X_2$ is an $\aleph_{\delta+1}$-spray. Then $2^{\aleph_0} \leq \aleph_\delta$.*

*Proof.* For $i = 1, 2$, let $\boldsymbol{c}_i$ be the center of $X_i$, and for $r > 0$ let

$$C_i(r) := \{\boldsymbol{p} \in \mathbb{R}^2 \mid \|\boldsymbol{p} - \boldsymbol{c}_i\| = r\}$$

be the circle centered in $\boldsymbol{c}_i$ of radius $r$. If $\boldsymbol{c}_1 = \boldsymbol{c}_2$ then by gluing $X_1 \cup X_2$ would be an $\aleph_{\delta+1}$-spray in $\mathbb{R}^2$, and every circle centered in $\boldsymbol{c}_1$ is contained in $X_1 \cup X_2 = \mathbb{R}^2$ is of cardinality $\leq \aleph_\delta$. As any circle is in bijection with $\mathbb{R}$, the result follows. Therefore we may assume that $\boldsymbol{c}_1 \neq \boldsymbol{c}_2$. Applying an isometry if needed, we may assume that $\boldsymbol{c}_1 = (0, 0)$ and $\boldsymbol{c}_2 = (a, 0)$ for some $a > 0$.

Towards a contradiction, suppose $2^{\aleph_0} > \aleph_\delta$. Fix distinct reals $r_\alpha$ in the interval $(a/2; a)$, for $\alpha < \aleph_\delta$. By assumption $X_2(\alpha) := X_2 \cap C_2(r_\alpha)$ has size $\leq \aleph_\delta$, so the set

$$\{r \in (a/2; a) \mid C_1(r) \cap (\textstyle\bigcup_{\alpha \in \aleph_\delta} X_2(\alpha)) \neq \emptyset\}$$

has size $\leq \aleph_\delta$. As $|\mathbb{R}| > \aleph_\delta$ we may pick $r \in (a/2; a)$ outside of this set. For each $\alpha \in \aleph_\delta$ the set $C_1(r) \cap C_2(r_\alpha)$ has size 2, and its points belong to $X_1$. As the $C_2(r_\alpha)$s are disjoint, it follows that $C_1(r) \cap X_1$ has size $\geq \aleph_\delta$, contradicting that $X_1$ is an $\aleph_\delta$-spray. □

When $\delta = 0$ we obtain at once:

**Corollary 2.21.** *$\mathbb{R}^2$ is not the union of a spray and a $\sigma$-spray. In particular, $\mathbb{R}^2$ is not the union of two sprays.*

**Corollary 2.22.** *The following are equivalent:*

(a) $\mathsf{CH}$ *holds.*

(b) *$\mathbb{R}^2$ is the union of two $\sigma$-sprays with prescribed, distinct centers.*

**Proposition 2.23.** *Let $n \geq d \geq 3$, and suppose that the points $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$ belong to an affine hyperplane $H$ of $\mathbb{R}^d$. Suppose there is $L \subseteq H$ an affine subspace of dimension $d-2$ such that $\{\pi(\boldsymbol{c}_1), \ldots, \pi(\boldsymbol{c}_n)\}$ has size $\leq d-1$, where $\pi : H \to L$ is the orthogonal projection. Then there are no sprays $X_1, \ldots, X_n$ that cover $\mathbb{R}^d$ with $X_i$ centered in $\boldsymbol{c}_i$, for $i \leq n$.*

*Proof.* Towards a contradiction, let $d \geq 3$ be least such that the statement fails, and suppose $X_i$ is a spray centered in $\boldsymbol{c}_i$ such that $\mathbb{R}^d = X_1 \cup \cdots \cup X_n$. Then $H \cong \mathbb{R}^{d-1}$ is covered by the sprays $X_1 \cap H, \ldots, X_n \cap H$ centered in $\{\pi(\boldsymbol{c}_1), \ldots, \pi(\boldsymbol{c}_n)\}$. This means that $\mathbb{R}^{d-1}$ can be covered by $d-1$-many sprays whose centers lie on the hyperplane $L$. This contradicts the minimality of $d$, if $d-1 \geq 3$. If $d=3$, then we would have that $\mathbb{R}^2$ can be covered by two sprays, against Corollary 2.21. □

In [Schmerl 2012, p. 1169] it is observed that the next result follows from results of Sikorski.

**Theorem 2.24.** *For $d \geq 2$ the space $\mathbb{R}^d$ is not the union of $d$-many sprays.*

*Proof.* The case $d=2$ is Corollary 2.21. Suppose $d \geq 3$ and that $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_d$ are centers of sprays $X_1, \ldots, X_d$ that cover $\mathbb{R}^d$. Let $H$ be a hyperplane containing these points, and apply Proposition 2.23 with $n=d-1$: if $L$ is an affine subspace of $H$ of dimension $d-2>0$ that is orthogonal to the vector $\boldsymbol{c}_d - \boldsymbol{c}_{d-1}$, then $\{\pi(\boldsymbol{c}_1), \ldots, \pi(\boldsymbol{c}_d)\}$ has size $\leq d-1$ since $\pi(\boldsymbol{c}_d) = \pi(\boldsymbol{c}_{d-1})$. But Proposition 2.23 implies that the $X_i$s cannot cover $\mathbb{R}^d$. □

A similar argument shows:

**Theorem 2.25.** *For $d \geq 2$ the space $\mathbb{R}^d$ is not the union of $(d-1)$-many $\sigma$-sprays.*

**Theorem 2.26.** *For $d \geq 2$ the following are equivalent*:

(a) CH;

(b) *for all well-placed $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_d \in \mathbb{R}^d$ there are $X_1, \ldots, X_d$ covering $\mathbb{R}^d$ such that each $X_i$ is a $\sigma$-spray with center $\boldsymbol{c}_i$*;

(c) *there are well-placed $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_d \in \mathbb{R}^d$ and $X_1, \ldots, X_d$ covering $\mathbb{R}^d$ such that each $X_i$ is a $\sigma$-spray with center $\boldsymbol{c}_i$.*

*Proof.* (a)⇒(b) follows from Theorem 2.18; (b)⇒(c) is trivial, while (c)⇒(a) is established by induction on $d \geq 2$.

When $d=2$ the result follows at once from Proposition 2.20. Suppose the result holds for some $d$ towards proving it for $d+1$. Fix $\sigma$-sprays $X_1, \ldots, X_{d+1}$ covering $\mathbb{R}^{d+1}$ and centered in well-placed $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_{d+1}$. Let $H \subseteq \mathbb{R}^d$ be the hyperplane determined by the $\boldsymbol{c}_i$s. Let $H'$ be a hyperplane orthogonal to $\boldsymbol{c}_{d+1} - \boldsymbol{c}_d$, and let $\pi : \mathbb{R}^{d+1} \to H'$ be the orthogonal projection. By projecting and gluing $X_1 \cap H$, $X_2 \cap H$, ..., $(X_d \cup X_{d+1}) \cap H$ are $\sigma$-sprays centered in the points $\pi(\boldsymbol{c}_1)$, $\pi(\boldsymbol{c}_2)$, ..., $\pi(\boldsymbol{c}_d) = \pi(\boldsymbol{c}_{d+1})$ which are well-placed, and belong to $H' \cong \mathbb{R}^d$. By inductive assumption, CH holds. □

## 3. Transforming sprays into linear objects

In this section we construct, for every $d \geq 2$, a continuous map $\Phi$ that transforms any spray of $\mathbb{R}^d$ with center $\boldsymbol{c}$ into a set $A \subseteq \mathbb{R}^d$ such that $A \cap H$ is finite, for every hyperplane $H$ orthogonal to some vector $\boldsymbol{u}$, and conversely. (The vector $\boldsymbol{u}$ depends only on the point $\boldsymbol{c}$.) This is an extension and an elaboration of the construction used by Schmerl when $d=2$ to prove that if $\mathbb{R}^2$ is the union of $(n+2)$-many sprays with collinear centers, then $2^{\aleph_0} \leq \aleph_n$ [Schmerl 2010, Theorem 7].

Let $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_d$ be distinct points of an affine space $E$ of dimension $d-1 \geq 1$. For each $\boldsymbol{q} \in E$ the vectors $\boldsymbol{p}_1 - \boldsymbol{q}, \ldots, \boldsymbol{p}_d - \boldsymbol{q}$ are linearly dependent, and hence

$$\mathcal{U}_{\boldsymbol{p}_1,\ldots,\boldsymbol{p}_d}(\boldsymbol{q}) := \{(u_1, \ldots, u_d) \in \mathbb{R}^d \mid u_1(\boldsymbol{p}_1 - \boldsymbol{q}) + \cdots + u_d(\boldsymbol{p}_d - \boldsymbol{q}) = \boldsymbol{0}\} \tag{2}$$

is a vector subspace of $\mathbb{R}^d$ of dimension $\geq 1$. When the points $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_d$ are clear from the context we write $\mathcal{U}(\boldsymbol{q})$. Note that if $\boldsymbol{e}_i \in \mathcal{U}(\boldsymbol{q})$ then $\boldsymbol{q} = \boldsymbol{p}_i$, and hence

$$\boldsymbol{q} \notin \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_d\} \Rightarrow \boldsymbol{e}_1, \ldots, \boldsymbol{e}_d \notin \mathcal{U}(\boldsymbol{q}). \tag{3}$$

**Theorem 3.1.** *Suppose $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_d, \boldsymbol{q} \in E$, an affine space of dimension $d-1$, and that $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_d$ are distinct. For every $(u_1, \ldots, u_d) \in \mathcal{U}(\boldsymbol{q}) \setminus \{\boldsymbol{0}\}$ letting $b := -\sum_{i=1}^{d} u_i$ and $c := -\sum_{i=1}^{d} u_i(\|\boldsymbol{p}_i\|^2 - \|\boldsymbol{q}\|^2)$ we have*

$$\forall \boldsymbol{x} \in E \left(u_1\|\boldsymbol{x} - \boldsymbol{p}_1\|^2 + \cdots + u_d\|\boldsymbol{x} - \boldsymbol{p}_d\|^2 + b\|\boldsymbol{x} - \boldsymbol{q}\|^2 + c = 0\right).$$

*Proof.* For notational ease let $F_i(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{p}_i\|^2$ and $F(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{q}\|^2$. For $\boldsymbol{x} \in E$ and $1 \leq i \leq d$

$$\begin{aligned} F_i(\boldsymbol{x}) - F(\boldsymbol{x}) &= \|\boldsymbol{x} - \boldsymbol{p}_i\|^2 - \|\boldsymbol{x} - \boldsymbol{q}\|^2 \\ &= \|\boldsymbol{x}\|^2 + \|\boldsymbol{p}_i\|^2 - 2\boldsymbol{x} \cdot \boldsymbol{p}_i - \|\boldsymbol{x}\|^2 - \|\boldsymbol{q}\|^2 + 2\boldsymbol{x} \cdot \boldsymbol{q} \\ &= \|\boldsymbol{p}_i\|^2 - \|\boldsymbol{q}\|^2 - 2\boldsymbol{x} \cdot (\boldsymbol{p}_i - \boldsymbol{q}). \end{aligned}$$

Since $(u_1, \ldots, u_d) \in \mathcal{U}(\boldsymbol{q})$, this implies

$$\sum_{i=1}^{d} u_i(F_i(\boldsymbol{x}) - F(\boldsymbol{x})) = \sum_{i=1}^{d} u_i(\|\boldsymbol{p}_i\|^2 - \|\boldsymbol{q}\|^2) - 2\boldsymbol{x} \cdot \sum_{i=1}^{d} u_i(\boldsymbol{p}_i - \boldsymbol{q}) = -c.$$

By the definition of $b$ one obtains $\left(\sum_{i=1}^{d} u_i F_i(\boldsymbol{x})\right) + bF(\boldsymbol{x}) + c = 0$ for all $\boldsymbol{x} \in E$, as required. □

The definition of $\aleph_\delta$-spray was given for $\mathbb{R}^d$, but it can be adapted to the space

$$\mathbb{H}^d := \mathbb{R}^{d-1} \times (0; +\infty) \subseteq \mathbb{R}^d$$

as follows: an $\aleph_\delta$-spray in $\mathbb{H}^d$ is a set $X \subseteq \mathbb{H}^d$ together with a point $\boldsymbol{c}$, the center of $X$, belonging to the hyperplane $\mathbb{R}^{d-1} \times \{0\}$ of $\mathbb{R}^d$, such that $|S \cap X| < \aleph_\delta$ for any sphere $S$ of $\mathbb{R}^d$ centered in $\boldsymbol{c}$. (Observe that $\boldsymbol{c}$ does not belong to $\mathbb{H}$.)

If $X_1, X_2, \ldots$ are $\aleph_\delta$-sprays in $\mathbb{R}^d$ with centers $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots \in \mathbb{R}^{d-1} \times \{0\}$, then $X_1 \cap \mathbb{H}^d, X_2 \cap \mathbb{H}^d, \ldots$ are $\aleph_\delta$-sprays in $\mathbb{H}^d$ with the same centers; moreover, if $X_1, X_2, \ldots$ cover $\mathbb{R}^d$, then $X_1 \cap \mathbb{H}^d, X_2 \cap \mathbb{H}^d, \ldots$ cover $\mathbb{H}^d$. Therefore we may focus on covering $\mathbb{H}^d$ with $\aleph_\delta$-sprays in $\mathbb{H}^d$ with centers on $\mathbb{R}^{d-1} \times \{0\}$.

Let $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_d$ be points in general position in $\mathbb{R}^d$. These points belong to a hyperplane, and without loss of generality we may assume that $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_d \in$

$\mathbb{R}^{d-1} \times \{0\}$. Let $\mathbb{R}^d_+ = \{\boldsymbol{x} \in \mathbb{R}^d \mid \forall i \leq d\, (x_i \geq 0)\}$, let

$$\Phi : \mathbb{H}^d \to \mathbb{R}^d_+, \qquad \boldsymbol{x} \mapsto (\|\boldsymbol{x} - \boldsymbol{c}_1\|^2, \ldots, \|\boldsymbol{x} - \boldsymbol{c}_d\|^2), \tag{4}$$

and let $E^d = \operatorname{ran} \Phi$. Then $\Phi$ is a homeomorphism between $\mathbb{H}^d$ and the open set $E^d$.

The map $\Phi$ transforms any sphere centered around $\boldsymbol{c}_i$ into a subset of a hyperplane of $\mathbb{R}^d$ orthogonal to $\boldsymbol{e}_i$, and conversely. To be specific — and recalling Notation 2.1 — if $S$ is the sphere of $\mathbb{R}^d$ centered in $\boldsymbol{c}_i$ of radius $r$, then $\Phi[S \cap \mathbb{H}^d]$ is the intersection between $E$ and the hyperplane $H_i(r^2)$ of $\mathbb{R}^d$; conversely, $\Phi^{-1}[H_i(r^2) \cap E] = S \cap \mathbb{H}^d$. Therefore if $X \subseteq \mathbb{H}^d$ is an $\aleph_\delta$-spray centered in $\boldsymbol{c}_i$, then $\Phi[X] \subseteq E$ is such that $|\Phi[X] \cap H_i(r)| < \aleph_\delta$ for every $r > 0$; conversely if $Y \subseteq E$ intersects all hyperplanes orthogonal to $\boldsymbol{e}_i$ in a set of size $< \aleph_\delta$ then $\Phi^{-1}[Y] \subseteq \mathbb{H}^d$ is an $\aleph_\delta$-spray centered in $\boldsymbol{c}_i$.

Suppose $\boldsymbol{c}_{d+1} \in \mathbb{R}^{d-1} \times \{0\}$ is distinct from $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_d$. Letting $\boldsymbol{p}_i = \boldsymbol{c}_i$ and $\boldsymbol{q} = \boldsymbol{c}_{d+1}$ in (2), fix $\boldsymbol{u} = (u_1, \ldots, u_d) \in \mathcal{U}(\boldsymbol{c}_{d+1}) \setminus \{\boldsymbol{0}\}$. By Theorem 3.1 there are $b, c$ such that $u_1\|\boldsymbol{x} - \boldsymbol{c}_1\|^2 + \cdots + u_d\|\boldsymbol{x} - \boldsymbol{c}_d\|^2 + b\|\boldsymbol{x} - \boldsymbol{c}_{d+1}\|^2 + c = 0$, for all $\boldsymbol{x} \in \mathbb{R}^d$. The set

$$P = \{(w_1, \ldots, w_{d+1}) \mid u_1 w_1 + \cdots + u_d w_d + b w_{d+1} + c = 0\}$$

is an affine hyperplane of $\mathbb{R}^{d+1}$, and since $\boldsymbol{u}$ is not a multiple of any $\boldsymbol{e}_i$ with $1 \leq i \leq d$ by (3), it follows that $P$ is not orthogonal to any such vector. Therefore the projection $\pi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is a bijection between $P$ and $\mathbb{R}^d$. The map

$$\check{\Phi} : \mathbb{H}^d \to \mathbb{R}^{d+1}_+, \quad \boldsymbol{x} \mapsto (\|\boldsymbol{x} - \boldsymbol{c}_1\|^2, \ldots, \|\boldsymbol{x} - \boldsymbol{c}_{d+1}\|^2),$$

is a homeomorphism on its image $\check{E}$, and clearly $\check{E} \subseteq P$. Then

$$\check{E} = \{(r_1, \ldots, r_{d+1}) \in \mathbb{R}^{d+1}_+ \mid (r_1, \ldots, r_d) \in E \wedge (r_1, \ldots, r_{d+1}) \in P\},$$

that is: $\check{E}$ is the subset of $P$ that projects onto $E$, and $\pi$ is a homeomorphism from $\check{E}$ to $E$.

For any $k \in \mathbb{R}$ the set

$$\begin{aligned} L(\boldsymbol{u}, k) &:= \{(r_1, \ldots, r_d) \mid u_1 r_1 + \cdots + u_d r_d + bk + c = 0\} \\ &= \{(r_1, \ldots, r_d) \mid (r_1, \ldots, r_d, k) \in P\} \end{aligned}$$

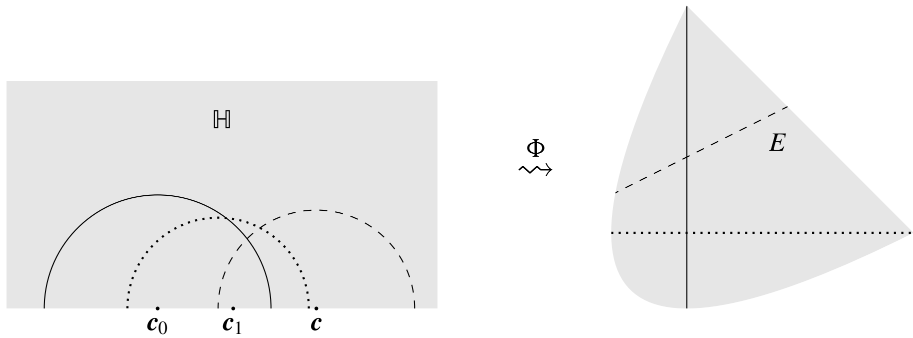
is an affine hyperplane of $\mathbb{R}^d$ orthogonal to $\boldsymbol{u}$, and all affine hyperplanes of $\mathbb{R}^d$ orthogonal to $\boldsymbol{u}$ are of this form. Arguing as before, if $X \subseteq \mathbb{H}^d$ is an $\aleph_\delta$-spray centered in $\boldsymbol{c}_{d+1}$ then $\Phi[X]$ is a subset of $E$ that intersects every $L(\boldsymbol{u}, k)$ in a set of size $< \aleph_\delta$; conversely if $Y \subseteq E$ is such that every $L(\boldsymbol{u}, k)$ intersects $Y$ in $< \aleph_\delta$-many points, then $\Phi^{-1}[Y]$ is an $\aleph_\delta$-spray centered in $\boldsymbol{c}_{d+1}$.

Let us summarize what we proved so far.

**Theorem 3.2.** *Suppose $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_d$ are in general position in $\mathbb{R}^d$, and without loss of generality we may assume that they belong to $\mathbb{R}^{d-1} \times \{0\}$. There is a nonempty open* *set* $E^d \subseteq \mathbb{R}^d$ *and a homeomorphism* $\Phi : \mathbb{H}^d \to E^d$ *that transforms any* $\aleph_\delta$*-spray of* $\mathbb{H}^d$ *centered in* $\boldsymbol{c}_i$ *into a subset of* $E^d$ *that intersects any hyperplane orthogonal to* $\boldsymbol{e}_i$ *in a set of size* $< \aleph_\delta$*, and conversely.*

*Let* $\boldsymbol{c} \in \mathbb{R}^{d-1} \times \{0\}$ *be distinct from the* $\boldsymbol{c}_i$*s, and let* $\boldsymbol{u} \in \mathcal{U}_{\boldsymbol{c}_1,\dots,\boldsymbol{c}_d}(\boldsymbol{c}) \setminus \{\boldsymbol{0}\}$*. Then* $\Phi$ *maps any* $\aleph_\delta$*-spray centered in* $\boldsymbol{c}$ *into a subset of* $E$ *that intersects any hyperplane orthogonal to* $\boldsymbol{u}$ *in a set of size* $< \aleph_\delta$*, and conversely.*

The following picture when $d = 2$ may help to visualize the previous construction.




We can now sketch Schmerl's argument that if the plane is covered with three sprays with collinear centers, then CH holds. First of all we may assume that the centers lie on the $x$-axis so that the three sprays cover $\mathbb{H}^2$. Then $E^2$ can be covered with three sets $A_1, A_2, A_3$ such that the intersection of $A_i$ with any line orthogonal to some vector $\boldsymbol{u}_i$ is finite ($i = 1, 2, 3$), and by a theorem of Bagemihl [1961] this implies CH. Our strategy is to replace lines in $\mathbb{R}^2$ with hyperplanes in $\mathbb{R}^d$, and this is the topic of the next section.

## 4. Hyperplane sections

Over the last century several results were obtained, establishing connections between the size of the continuum and elementary properties of the euclidean spaces. The first such result is Sierpiński's theorem form → from from 1919, asserting that CH is equivalent to a covering of the plane with two sets such that the intersection of the first set with any vertical line is countable and the intersection of every horizontal line with the second set is countable (see Theorem 2.6(a)). Sierpiński [1951] sharpened his previous result by replacing "countable" with "finite", but at the cost of increasing the dimension: CH is equivalent to a decomposition $A_1, A_2, A_3$ of $\mathbb{R}^3$ such that the intersection of any line with direction $\boldsymbol{e}_i$ with $A_i$ is finite (Theorem 2.6(b)), and this was quickly generalized by Kuratowski to higher dimensions [Kuratowski 1951]: $2^{\aleph_0} \le \aleph_n$ if and only if there is a decomposition $A_1, \dots, A_{n+2}$ of $\mathbb{R}^{n+2}$ such that every line parallel to $\boldsymbol{e}_i$ has finite intersection with $A_i$. In the 1960s, Bagemihl and Davies showed that Kuratowski's result could be proved for $\mathbb{R}^2$ by taking intersections with lines of prescribed directions [Bagemihl 1960; 1961; 1968; Davies 1962; 1963b].

A line is a hyperplane in $\mathbb{R}^2$, so Sierpiński's result from 1919 could be stated as: if $d = 2$ then CH is equivalent to a decomposition of $\mathbb{R}^d$ into $A_1, \dots, A_d$ such that every hyperplane orthogonal to $\boldsymbol{e}_i$ has countable intersection with $A_i$. By Corollary 4.7 this result holds for all $d \geq 2$. Sierpiński [1951] states and proves it for $d = 3$, and he observes that the analogous result with "finite" replacing "countable" is false: there is no decomposition $A_1$, $A_2$, $A_3$ of $\mathbb{R}^3$ such that every plane orthogonal to $\boldsymbol{e}_i$ has finite intersection with $A_i$. Erdős, Jackson, and Mauldin prove that CH is equivalent to $\mathbb{R}^3$ being decomposable in five pieces $A_1, \dots, A_5$ so that every plane orthogonal to $\boldsymbol{u}_i$ has finite intersection with $A_i$, where $\boldsymbol{u}_1, \dots, \boldsymbol{u}_5$ are vectors in general position in $\mathbb{R}^3$ [Erdős et al. 1994, Corollary 6], but no decomposition exists if we just allow four vectors and four sets [Erdős et al. 1994, Lemma 1]. In that Lemma it is stated (but not proved) that an analogous result holds when CH is weakened to $2^{\aleph_0} \leq \aleph_n$, but the case for $\mathbb{R}^d$ with $d > 3$ is not mentioned. Since we need a detailed analysis of the positions of the various (hyper)planes we will state and prove these results in full generality below. Much of what follows is an elaboration of ideas from [Erdős et al. 1994], and in doing so we fill a gap in the proof of Lemma 1 of that paper.

This section is devoted to the following problem:

**Problem 4.1.** Given $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n$ distinct, nonzero vectors of $\mathbb{R}^d$, what conditions on the cardinality of $\mathbb{R}$ are equivalent to the existence of $A_1, \dots, A_n$ covering $\mathbb{R}^d$ such that each $A_i \cap H_{\boldsymbol{u}_i}(\boldsymbol{p})$ is finite (or countable)?

**Lemma 4.2.** *Suppose $\boldsymbol{u}_1, \dots, \boldsymbol{u}_d \in \mathbb{R}^d$ are linearly independent, and $d \geq 2$. There is a linear isomorphism $T : \mathbb{R}^d \to \mathbb{R}^d$ that maps every hyperplane orthogonal to $\boldsymbol{u}_i$ to a hyperplane orthogonal to $\boldsymbol{e}_i$, and conversely, for any $i \leq d$.*

*Proof.* The inner product $\boldsymbol{u} \cdot \boldsymbol{v}$ of two vectors in $\mathbb{R}^d$ is the matrix-product $\boldsymbol{u}^{\mathrm{t}}\boldsymbol{v}$, where superscript "t" stand for "transpose". Let $M$ be the matrix of the linear transformation that maps $\boldsymbol{e}_i$ to $\boldsymbol{u}_i$ for $i = 1, \dots, d$, and let $T$ be the linear transformation given by the matrix $M^{\mathrm{t}}$. Then $T$ is an isomorphism, since $\det(M^{\mathrm{t}}) = \det(M) \neq 0$. For all $\boldsymbol{x} \in \mathbb{R}^d$

$$\boldsymbol{u}_i^{\mathrm{t}}\boldsymbol{x} = (M\boldsymbol{e}_i)^{\mathrm{t}}\boldsymbol{x} = \boldsymbol{e}_i^{\mathrm{t}}(M^{\mathrm{t}}\boldsymbol{x})$$

so $\boldsymbol{x}$ is orthogonal to $\boldsymbol{u}_i$ if and only if $\boldsymbol{e}_i$ is orthogonal to $T(\boldsymbol{x})$. □

**Lemma 4.3.** *Let $m \geq 1$, let $\boldsymbol{v}$ be a unit vector of $\mathbb{R}^m$, let $\varepsilon > 0$ and let $\kappa$ be an infinite cardinal. For all $X \subseteq \mathbb{R}^m$ with $\aleph_0 \leq |X| < \kappa \leq 2^{\aleph_0}$, there is $S \subseteq \{r\boldsymbol{v} \mid |r| < \varepsilon\} \subseteq (-\varepsilon; \varepsilon)^m$ such that $|S| = \kappa$ and $(X - X) \cap (S - S) = \{\boldsymbol{0}\}$.*

*Proof.* Let $V = \{r\boldsymbol{v} \mid r \in \mathbb{R}\}$, let $G$ be the subgroup of $(\mathbb{R}^m, +)$ generated by $X$, and let $H = G \cap V$. Then $|H| \leq |G| = |X| < 2^{\aleph_0}$ and $|V/H| = 2^{\aleph_0}$.

Suppose first $H$ is dense in $V$. We can construct $T \subseteq \{r\boldsymbol{v} \mid |r| < \varepsilon\}$ a transversal for the quotient, i.e. a set picking exactly one element from each coset of $V/H$. Then $2^{\aleph_0} = |T| = |V/H|$, and let $S \subseteq T$ be of size $\kappa$. If $s_1 - s_2 = x_1 - x_2$ for some $s_1, s_2 \in S$ and $x_1, x_2 \in X$, since $s_1 - s_2 \in H \subseteq G$ this means that the cosets $s_1 + H$ and $s_2 + H$ are the same, so $s_1 = s_2$ by definition of $S$. Therefore $(S - S) \cap (X - X) = \{\mathbf{0}\}$.

Now suppose that $H$ is not dense in $V$. Then there is $0 < \delta < \varepsilon$ such that the segment $(\mathbf{0}; \delta\boldsymbol{v})$ is disjoint from $H$. Let $S \subseteq (\mathbf{0}; \delta\boldsymbol{v})$ be of size $\kappa$. As $H$ is a group, then $(-\delta\boldsymbol{v}; \delta\boldsymbol{v}) \cap H = \{\mathbf{0}\}$, so the intersection of $S - S \subseteq (-\delta\boldsymbol{v}; \delta\boldsymbol{v})$ with $G - G \supseteq X - X$ is $\{\mathbf{0}\}$. □

The next result asserts that if the sets $A_i \subseteq \mathbb{R}^d$ have small intersection with the hyperplanes orthogonal to $\boldsymbol{u}_i$, and if $2^{\aleph_0}$ is large enough, then there is a set $Z \subseteq \mathbb{R}^d$ such that no translate of it can be covered by the $A_i$s — in particular the $A_i$s do not cover $\mathbb{R}^d$.

**Theorem 4.4.** *Let* $\delta \in \mathrm{Ord}$, $d \geq 3$ *and let* $N = 2(d-1)$. *Suppose* $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ *are distinct, nonzero vectors of* $\mathbb{R}^d$. *Suppose* $A_1, \ldots, A_N$ *are subsets of* $\mathbb{R}^d$ *such that for all* $\boldsymbol{p} \in \mathbb{R}^d$, *and for all* $1 \leq i \leq N$

$$|H_{\boldsymbol{u}_i}(\boldsymbol{p}) \cap A_i| < \aleph_\delta.$$

*If* $2^{\aleph_0} > \aleph_\delta$ *then for every* $\varepsilon > 0$ *there is* $Z \subseteq (-\varepsilon; \varepsilon)^d$ *of size* $\aleph_{\delta+1}$ *such that*

$$\forall \boldsymbol{p} \in \mathbb{R}^d \left(\boldsymbol{p} + Z \not\subseteq \textstyle\bigcup_{i=1}^N A_i\right).$$

Before proving this let us draw a few corollaries. With the same notation as before let us state a contrapositive of the preceding result:

**Theorem 4.5.** *Let* $\delta \in \mathrm{Ord}$, $d \geq 3$ *and* $N = 2(d-1)$. *Suppose* $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ *are nonzero vectors of* $\mathbb{R}^d$. *Suppose* $D \subseteq \mathbb{R}^d$ *is such that* $\mathrm{Int}(D) \neq \emptyset$. *If* $A_1, \ldots, A_N$ *cover* $D$ *and for all* $\boldsymbol{p} \in \mathbb{R}^d$ *and for all* $1 \leq i \leq N$

$$|H_{\boldsymbol{u}_i}(\boldsymbol{p}) \cap A_i| < \aleph_\delta.$$

*Then* $2^{\aleph_0} \leq \aleph_\delta$.

*Proof.* Towards a contradiction, suppose $2^{\aleph_0} \geq \aleph_{\delta+1}$. Let $\varepsilon > 0$ be small enough so that $\boldsymbol{p} + (-\varepsilon; \varepsilon)^d \subseteq D$ for some $\boldsymbol{p} \in \mathbb{R}^d$, and, towards a contradiction, suppose $A_1, \ldots, A_N$ are as in the statement. Let $Z \subseteq (-\varepsilon; \varepsilon)^d$ be as in Theorem 4.4: then $\boldsymbol{p} + Z$ is contained in $D$ but on other hand it is not contained in $A_1 \cup \cdots \cup A_N$, a contradiction. □

The presence of the set $D$ in the statement of Theorem 4.5 may seem unwarranted right now, but it will be crucial for the results in Section 5. For the time being the reader can safely replace $D$ with $\mathbb{R}^d$ without losing much.

**Theorem 4.6.** *For all $d \geq 3$ and all $D \subseteq \mathbb{R}^d$ such that $\varnothing \neq \operatorname{Int}(D)$, the following are equivalent*:

(a) CH *holds.*

(b) *For all nonzero vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d$ spanning $\mathbb{R}^d$, there are $A_1, \ldots, A_d$ covering $D$ such that $H_{\boldsymbol{u}_i}(\boldsymbol{p}) \cap A_i$ is countable for all $\boldsymbol{p} \in \mathbb{R}^d$ and $1 \leq i \leq d$.*

(c) *There are nonzero vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{2d-2}$ and there are $A_1, \ldots, A_{2d-2}$ covering $D$ such that $H_{\boldsymbol{u}_i}(\boldsymbol{p}) \cap A_i$ is countable for all $\boldsymbol{p} \in \mathbb{R}^d$ and $1 \leq i \leq 2d-2$.*

*Proof.* The implication (a)⇒(b) follows from Theorem 2.3 and Example 2.5, while (b)⇒(c) is trivial — take $A_{d+1} = \cdots = A_{2d-2} = \varnothing$. The implication (c)⇒(a) follows from Theorem 4.5 when $\delta = 1$. □

**Corollary 4.7.** *For all $d \geq 3$,* CH *is equivalent to the existence of $A_1, \ldots, A_d$ covering $\mathbb{R}^d$ such that $H_i(x) \cap A_i$ is countable, for all $x \in \mathbb{R}$ and all $1 \leq i \leq d$.*

Cantor's theorem says that $2^{\aleph_0} > \aleph_\delta$ if $\delta = 0$, so when $d = 3$ Theorem 4.4 implies the following result, which is Lemma 1 of [Erdős et al. 1994]:

**Corollary 4.8.** *Suppose $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_4$ are nonzero vectors of $\mathbb{R}^3$. There are no $A_1, \ldots, A_4$ covering $\mathbb{R}^3$ such that $H_{\boldsymbol{u}_i}(\boldsymbol{p}) \cap A_i$ is finite, for all $\boldsymbol{p} \in \mathbb{R}^3$ and all $1 \leq i \leq 4$.*

**Remark 4.9.** Corollary 4.8 is a negative result, asserting that $\mathbb{R}^3$ is not the union of four sets such that each intersects any plane orthogonal to a given vector in a finite set. But looking at the proof of Theorem 4.4, it could be recast in a positive way:

> *Suppose $\Bbbk$ is an infinite field. If there are nonzero vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_4$ of $\Bbbk^3$ and sets $A_1, \ldots, A_4$ covering $\Bbbk^3$ such that every hyperplane orthogonal to $\boldsymbol{u}_i$ has finite intersection with $A_i$, then $|\Bbbk| = \aleph_0$.*

*Proof of Theorem 4.4.* Let $A_1, \ldots, A_N$ and $\varepsilon > 0$ be as in the statement. We consider two cases, depending whether the $\boldsymbol{u}_i$s span $\mathbb{R}^d$.

Case 1: the vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ do not span $\mathbb{R}^d$.

Let $\boldsymbol{v}$ be a nonzero vector such that $\boldsymbol{v} \cdot \boldsymbol{u}_i = 0$ for all $1 \leq i \leq N$, and let $V = \{r\boldsymbol{v} \mid r \in \mathbb{R}\}$. Since $|V| = 2^{\aleph_0} \geq \aleph_{\delta+1}$ we can take $Z \subseteq V \cap (-\varepsilon; \varepsilon)^d$ of size $\aleph_{\delta+1}$. Since $\boldsymbol{p} + V \subseteq H_{\boldsymbol{u}_i}(\boldsymbol{p})$ we have $|(\boldsymbol{p} + V) \cap \bigcup_{i=1}^N A_i| < \aleph_\delta$, and hence $\boldsymbol{p} + Z \nsubseteq \bigcup_{i=1}^N A_i$.

Case 2: the vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ span $\mathbb{R}^d$.

Without loss of generality we may assume that $(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d)$ is a basis of $\mathbb{R}^d$. We first prove the result under the additional assumption that

$$(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_d) \text{ is the standard basis } (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d).$$

Let

$$U = \operatorname{span}\{\boldsymbol{u}_{d+1}, \dots, \boldsymbol{u}_N\}, \qquad U_j = \operatorname{span}\{\boldsymbol{e}_j, \boldsymbol{u}_{d+1}, \dots, \boldsymbol{u}_N\} \quad \text{for } 1 \le j \le d.$$

From $\dim U \le d-2$ it follows that $\dim U_j \le d-1$, so there is a nonzero vector orthogonal to $U_j$.

**Claim 4.4.1.** *There is $1 \le j \le d$ such that there is $\boldsymbol{v} \in U_j^{\perp} \setminus \{\boldsymbol{0}\}$ that is not multiple of any $\boldsymbol{e}_i$.*

*Proof of the claim.* Suppose $\dim U_j \le d-2$ for some $j$, so that $\dim(U_j^{\perp}) \ge 2$. Then not every vector in $U_j^{\perp}$ can be multiple of some $\boldsymbol{e}_i$.

Therefore we may assume that $\dim U_j = d-1$ for all $j$, and hence $\dim U = d-2$. Towards a contradiction, suppose that for every $j \in \{1, \dots, d\}$ there is $j^* \neq j$ such that $\boldsymbol{e}_{j^*} \in U_j^{\perp}$ — such $j^*$ is unique since $\dim(U_j^{\perp}) = 1$. Fix $j$ and let $i = j^*$ so that $i^* \neq j^*$. As $d \ge 3$ pick $k$ distinct from $i^*$, $j^*$. Both vectors $\boldsymbol{e}_{i^*}$ and $\boldsymbol{e}_{j^*}$ are orthogonal to $U$ and to $\boldsymbol{e}_k$, so both belong to $U_k^{\perp}$, which is impossible as $\dim(U_k^{\perp}) = 1$. □

Fix $j$ and $\boldsymbol{v} = (a_1, \dots, a_d)$ as in the Claim. By reindexing, if needed, we may assume that $j = d$, that is $a_d = 0$, and that $a_1$, $a_2$ are nonzero. By rescaling we may assume that

$$\boldsymbol{v} = (1, a_2, \dots, a_{d-1}, 0), \quad a_2 \neq 0,$$

and that

$$d \le k \le N \Rightarrow \boldsymbol{v} \cdot \boldsymbol{u}_k = 0. \tag{5}$$

Let $\nu := \max\{1, |a_2|, \dots, |a_{d-1}|\}$. For $1 \le i \le d$ let $X_i \subseteq (-\varepsilon/2; \varepsilon/2)$ such that $|X_1| = \aleph_{\delta+1}$ and $|X_i| = \aleph_\delta$ for all $2 \le i \le d$. By Lemma 4.3 with $m = 1$ and $\kappa = \aleph_{\delta+1}$, let $S \subseteq (-\varepsilon/(2\nu); \varepsilon/(2\nu))$ be of size $\aleph_{\delta+1}$ such that $(S-S) \cap (a_2^{-1}X_2 - a_2^{-1}X_2) = \{0\}$, and let

$$V = \{s\boldsymbol{v} \mid s \in S\}.$$

By (5), if $\boldsymbol{q} \in \mathbb{R}^d$ and $s \in S$ then

$$d \le k \le N \Rightarrow \boldsymbol{q} + V \subseteq H_{\boldsymbol{u}_k}(\boldsymbol{q} + s\boldsymbol{v}). \tag{6}$$

As $V \subseteq (-\varepsilon/2; \varepsilon/2)^d$ it follows that

$$Z := V + \prod_{i=1}^{d} X_i$$

is a subset of $(-\varepsilon; \varepsilon)^d$, and it is of cardinality $\aleph_{\delta+1}$. Fix $\boldsymbol{p} = (p_1, \dots, p_d) \in \mathbb{R}^d$ and, towards a contradiction, suppose that $\boldsymbol{p} + Z \subseteq \bigcup_{i=1}^{N} A_i$. Observe that $Z = \bigcup_{x \in X_1} Z_x$, where

$$Z_x := V + \Big(\{x\} \times \prod_{i=2}^{d} X_i\Big).$$

**Claim 4.4.2.** *The sets $Z_x$ are pairwise disjoint.*

*Proof.* If $Z_{x'} \cap Z_{x''} \neq \varnothing$, then there are $s', s'' \in S$, $x_i', x_i'' \in X_i$ for $2 \leq i \leq d$ such that

$$s'\boldsymbol{v} + (x', x_2', \ldots, x_d') = s''\boldsymbol{v} + (x'', x_2'', \ldots, x_d'').$$

The second component yields that $s'a_2 + x_2' = s''a_2 + x_2''$, and hence

$$s' - s'' = a_2^{-1}(x_2'' - x_2') \in (S - S) \cap (a_2^{-1}X_2 - a_2^{-1}X_2) = \{0\}\,.$$

It follows that $s' = s'' = s$, and hence

$$s\boldsymbol{v} + (x', x_2', \ldots, x_d') = s\boldsymbol{v} + (x'', x_2'', \ldots, x_d'').$$

Looking at the first component we obtain $s + x' = s + x''$ and hence $x' = x''$. ☐

Summarizing,

$$\boldsymbol{p} + Z = \bigcup_{x \in X_1} \boldsymbol{p} + Z_x \quad \text{and} \quad x' \neq x'' \Rightarrow \boldsymbol{p} + Z_{x'} \cap \boldsymbol{p} + Z_{x''} = \varnothing. \tag{7}$$

**Claim 4.4.3.** $$|A_d \cap (\boldsymbol{p} + Z)| < \aleph_{\delta+1}.$$

*Proof.* As the last component of $\boldsymbol{v}$ is 0, the set $V$ does not contribute to $Z$ on the $d$-th coordinate, so for all $w \in \mathbb{R}$

$$\begin{aligned}(\boldsymbol{p} + Z) \cap H_d(p_d + w) &= (\boldsymbol{p} + \prod_{i=1}^{d} X_i) \cap H_d(p_d + w) \\ &= \boldsymbol{p} + \big((\prod_{i=1}^{d} X_i) \cap H_d(w)\big)\end{aligned}$$

and this set is $\boldsymbol{p} + \big(\prod_{i=1}^{d-1} X_i\big) \times \{w\}$ if $w \in X_d$, and it is empty if $w \notin X_d$. Therefore $\boldsymbol{p} + Z \subseteq \bigcup_{w \in X_d} H_d(p_d + w)$ and

$$A_d \cap (\boldsymbol{p} + Z) \subseteq \bigcup_{w \in X_d} A_d \cap H_d(p_d + w).$$

By assumption $|A_d \cap H_d(p_d + w)| < \aleph_\delta$ for any $w \in \mathbb{R}$, and $|X_d| = \aleph_\delta$; therefore $|A_d \cap (\boldsymbol{p} + Z)| \leq \aleph_\delta < \aleph_{\delta+1}$. ☐

As $|X_1| = \aleph_{\delta+1}$ and by (7), there is $\bar{x}_1 \in X_1$ such that $\boldsymbol{p} + Z_{\bar{x}_1}$ is disjoint from $A_d$. This implies that

$$\boldsymbol{p} + Z_{\bar{x}_1} \subseteq A_1 \cup \cdots \cup A_{d-1} \cup \bigcup_{k=d+1}^{N} A_k.$$

**Claim 4.4.4.** *There is an $\bar{s} \in S$ such that $\boldsymbol{p} + \bar{s}\boldsymbol{v} + (\{\bar{x}_1\} \times \prod_{i=2}^{d} X_i)$ is disjoint from $\bigcup_{k=d+1}^{N} A_k$.*

*Proof.* Towards a contradiction, suppose that for all $s \in S$ and all $2 \leq i \leq d$ there are $x_i(s) \in X_i$ such that

$$\boldsymbol{p} + s\boldsymbol{v} + (\bar{x}_1, x_2(s), \dots, x_d(s)) \in \bigcup_{k=d+1}^{N} A_k.$$

As $|X_i| = \aleph_\delta$ for $2 \leq i \leq d$, there are $x'_i \in X_i$ and $S' \subseteq S$ of size $\aleph_{\delta+1}$ such that $x_i(s) = x'_i$ for all $s \in S'$. Therefore

$$\forall s \in S' \left(\boldsymbol{p} + s\boldsymbol{v} + (\bar{x}_1, x'_2, \dots, x'_d) \in \bigcup_{k=d+1}^{N} A_k\right).$$

As $|S'| = \aleph_{\delta+1}$ there is $S^* \subseteq S'$ of size $\aleph_{\delta+1}$, and there is $k$ with $d < k \leq N$ such that

$$\forall s \in S^* \left(\boldsymbol{p} + s\boldsymbol{v} + (\bar{x}_1, x'_2, \dots, x'_d) \in A_k\right). \tag{8}$$

Fix $s^* \in S^*$ and let

$$Q_k := H_{\boldsymbol{u}_k}\left(\boldsymbol{p} + s^*\boldsymbol{v} + (\bar{x}_1, x'_2, \dots, x'_d)\right).$$

By (6) with $\boldsymbol{q} = \boldsymbol{p} + (\bar{x}_1, x'_2, \dots, x'_d)$ we have

$$\forall s \in S^* \left(\boldsymbol{p} + s\boldsymbol{v} + (\bar{x}_1, x'_2, \dots, x'_d) \in Q_k\right).$$

Therefore by (8) $\boldsymbol{p} + s\boldsymbol{v} + (\bar{x}_1, x'_2, \dots, x'_d) \in A_k \cap Q_k$ for all $s \in S^*$, and this is a contradiction, since $|A_k \cap Q_k| < \aleph_\delta$, and yet it must contain $\aleph_{\delta+1}$ points. ☐

Let $\bar{s}$ be as in Claim 4.4.4. Then $|\boldsymbol{p} + \bar{s}\boldsymbol{v} + (\{\bar{x}_1\} \times \prod_{i=2}^{d} X_i)| = \aleph_\delta$ and

$$W_1 := \boldsymbol{p} + \bar{s}\boldsymbol{v} + (\{\bar{x}_1\} \times \textstyle\prod_{i=2}^{d} X_i) \subseteq A_1 \cup \dots \cup A_{d-1}. \tag{9}$$

As $W_1$ is included in $H_1(p_1 + \bar{s} + \bar{x}_1)$, and $|H_1(p_1 + \bar{s} + \bar{x}_1) \cap A_1| < \aleph_\delta$, then $|W_1 \cap A_1| < \aleph_\delta$. As $|X_2| = \aleph_\delta$, there is some $\bar{x}_2 \in X_2$ such that the set $W_2 := \boldsymbol{p} + \bar{s}\boldsymbol{v} + (\{\bar{x}_1\} \times \{\bar{x}_2\} \times \prod_{i=3}^{d} X_i)$ is disjoint from $A_1$, and hence it is contained in $A_2 \cup \dots \cup A_{d-1}$ and in the hyperplane $H_2(p_2 + \bar{s}a_2 + \bar{x}_2)$. As before $|W_2 \cap A_2| < \aleph_\delta$. Repeating this argument we construct $\bar{x}_3 \in X_3, \dots, \bar{x}_d \in X_d$ such that

$$\boldsymbol{p} + \bar{s}\boldsymbol{v} + (\bar{x}_1, \dots, \bar{x}_d) \notin A_1 \cup \dots \cup A_{d-1}$$

against (9). This concludes the proof, assuming that $\boldsymbol{u}_1, \dots, \boldsymbol{u}_d$ is the standard basis.

If $\boldsymbol{u}_1, \dots, \boldsymbol{u}_d$ is an arbitrary basis of $\mathbb{R}^d$, by Lemma 4.2 there is a linear injective transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ that maps every hyperplane orthogonal to $\boldsymbol{u}_i$ to a hyperplane orthogonal to $\boldsymbol{e}_i$, for $1 \leq i \leq d$. The transformation $T$ maps parallel hyperplanes to parallel hyperplanes, so for $d \leq k < N$ let $\bar{\boldsymbol{u}}_k$ be a vector such that $T$ maps every hyperplane orthogonal to $\boldsymbol{u}_k$ to a hyperplane orthogonal $\bar{\boldsymbol{u}}_k$. For

$1 \leq k \leq d$ set $\bar{\boldsymbol{u}}_k$ to be $\boldsymbol{e}_k$. As $T^{-1}$ is continuous, pick $\bar{\varepsilon} > 0$ small enough so that $T^{-1}[(-\bar{\varepsilon};\bar{\varepsilon})^d] \subseteq (-\varepsilon;\varepsilon)^d$. Arguing as above there is a $\bar{Z} \subseteq (-\bar{\varepsilon};\bar{\varepsilon})^d$ such that $\boldsymbol{p} + \bar{Z} \nsubseteq \bigcup_{i=1}^N T[A_i]$ for all $\boldsymbol{p} \in \mathbb{R}^d$. Then $Z = T^{-1}[\bar{Z}] \subseteq (-\varepsilon;\varepsilon)^d$ is such that $\boldsymbol{p} + Z \nsubseteq \bigcup_{i=1}^N A_i$ for all $\boldsymbol{p} \in \mathbb{R}^d$. □

The next result extends Theorem 4.4 by relaxing the size of the continuum.

**Theorem 4.9.** *Let* $\delta \in \mathrm{Ord}$, $d \geq 3$, $n \geq 0$, *and let* $N = (n+1)(d-1)+1$. *Suppose* $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ *are distinct, nonzero vectors of* $\mathbb{R}^d$, *and that* $A_1, \ldots, A_N \subseteq \mathbb{R}^d$ *are such that for all* $\boldsymbol{p} \in \mathbb{R}^d$

$$\forall 1 \leq k \leq N \ \left( |H_{\boldsymbol{u}_k}(\boldsymbol{p}) \cap A_k| < \aleph_\delta \right).$$

*If* $2^{\aleph_0} > \aleph_{\delta+n}$ *then for every* $\varepsilon > 0$ *there is* $Z_{n,\varepsilon} \subseteq (-\varepsilon;\varepsilon)^d$ *of size* $\aleph_{\delta+n+1}$ *such that*

$$\forall \boldsymbol{p} \in \mathbb{R}^d \left( \boldsymbol{p} + Z_{n,\varepsilon} \nsubseteq \bigcup_{k=1}^N A_k \right).$$

Let us draw some consequences from Theorem 4.9.

**Theorem 4.10.** *Let* $\delta \in \mathrm{Ord}$, *let* $n \geq 1$, *and let* $d \geq 3$. *Let also* $(n+1)(d-1) < N \leq (n+2)(d-1)$. *For all* $D \subseteq \mathbb{R}^d$ *such that* $\mathrm{Int}(D) \neq \varnothing$, *the following are equivalent*:

(a) $2^{\aleph_0} \leq \aleph_{\delta+n}$.

(b) *For all* $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{(n+1)(d-1)+1}$ *in general position there are* $A_1, \ldots, A_{(n+1)(d-1)+1}$ *covering* $D$ *such that* $|A_k \cap H_{\boldsymbol{u}_k}(\boldsymbol{p})| < \aleph_\delta$ *for all* $\boldsymbol{p} \in \mathbb{R}^d$ *and all* $1 \leq k \leq (n+1)(d-1)+1$.

(c) *For all distinct nonzero vectors* $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ *such that* $(n+1)(d-1)+1$ *of them are in general position, there are* $A_1, \ldots, A_N$ *covering* $D$ *such that* $|A_k \cap H_{\boldsymbol{u}_k}(\boldsymbol{p})| < \aleph_\delta$ *for all* $\boldsymbol{p} \in \mathbb{R}^d$ *and all* $1 \leq k \leq N$.

(d) *There are distinct nonzero vectors* $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ *and there are* $A_1, \ldots, A_N$ *covering* $D$ *such that* $|A_k \cap H_{\boldsymbol{u}_k}(\boldsymbol{p})| < \aleph_\delta$ *for all* $\boldsymbol{p} \in \mathbb{R}^d$ *and all* $1 \leq k \leq N$.

*Proof.* The implication (a)⇒(b) follows from the Erdős–Jackson–Mauldin result we stated as Theorem 2.3, together with Example 2.5.

The implication (b)⇒(c) is easy. Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ be as in (c). Without loss of generality we may assume that $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{(n+1)(d-1)+1}$ are in general position in $\mathbb{R}^d$, and let $A_1, \ldots, A_{(n+1)(d-1)+1}$ be as in (b). Letting $A_{(n+1)(d-1)+2} = \cdots = A_N = \varnothing$ we have sets as in (c).

The implication (c)⇒(d) is trivial.

Assume (d) and towards a contradiction suppose that $2^{\aleph_0} > \aleph_{\delta+n}$. By a translation, we may assume that $(-\varepsilon;\varepsilon)^d \subseteq D$. The hypotheses of Theorem 4.9 are satisfied so there is a set $Z \subseteq D$ that is not contained in $\bigcup_{k=1}^N A_k$, so the sequence $A_1, \ldots, A_N$ does not cover $D$. □

When $d = 3$ Theorem 4.10 yields the following results — recall that vectors in $\mathbb{R}^3$ are in general position if any three of them are linearly independent. First we consider the case when the intersection of a plane with a set is finite.

**Corollary 4.11.** *Let $D \subseteq \mathbb{R}^3$ be such that $\varnothing \neq \operatorname{Int}(D)$, and let $n \geq 1$. The following are equivalent.*

(a) $2^{\aleph_0} \leq \aleph_n$.

(b) *For all $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{2n+3}$ vectors in general position in $\mathbb{R}^3$, there are $A_1, \ldots, A_{2n+3}$ covering $D$ such that every plane orthogonal to $\boldsymbol{u}_i$ intersects $A_i$ in a finite set, for all $1 \leq i \leq 2n+3$.*

(c) *There are nonzero vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{2n+4}$ in $\mathbb{R}^3$, and sets $A_1, \ldots, A_{2n+4}$ covering $D$ such that every plane orthogonal to $\boldsymbol{u}_i$ intersects $A_i$ in a finite set, for all $1 \leq i \leq 2n+4$.*

With $n = 1$, this recovers the result of [Erdős et al. 1994; Simms 1997] that CH is equivalent to each of the following:

- For all $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_5$ in general position there are $A_1, \ldots, A_5$ covering $\mathbb{R}^3$ such that any plane orthogonal to $\boldsymbol{u}_i$ has finite intersection with $A_i$.
- There are nonzero vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_6$ and $A_1, \ldots, A_6$ covering $\mathbb{R}^3$ such that any plane orthogonal to $\boldsymbol{u}_i$ has finite intersection with $A_i$.

If the intersections between planes and sets are taken to be countable we have a result analogous to Corollary 4.11, with a shift of 2 in the number of vectors/pieces.

**Corollary 4.12.** *Let $D \subseteq \mathbb{R}^3$ be such that $\varnothing \neq \operatorname{Int}(D)$, let $n \geq 1$ and let $N$ be such that $2n+2 \leq N$. The following are equivalent*:

(a) $2^{\aleph_0} \leq \aleph_n$.

(b) *For all $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{2n+1}$ vectors in general position there are $A_1, \ldots, A_{2n+1}$ covering $D$ so that every plane orthogonal to $\boldsymbol{u}_k$ intersects $A_k$ in a countable set, for all $1 \leq k \leq 2n+1$.*

(c) *For all vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{2n+2}$ nonzero vectors such that at least $2n+1$ of them are in general position, there are $A_1, \ldots, A_{2n+2}$ covering $D$ such that every plane orthogonal to $\boldsymbol{u}_k$ intersects $A_k$ in a countable set, for all $1 \leq k \leq 2n+2$;*

(d) *There are vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_{2n+2}$ and sets $A_1, \ldots, A_{2n+2}$ covering $D$ such that every plane orthogonal to $\boldsymbol{u}_k$ intersects $A_k$ in a countable set, for all $1 \leq k \leq 2n+2$.*

Table 1 summarize the results proved so far. On the left, for any dimension $d$ and any $n$ such that $2^{\aleph_0} \leq \aleph_n$, we give $(n+1)(d-1)+1$, the minimum number of vectors/pieces of a covering of $\mathbb{R}^d$ so that each piece has *finite* intersection with any hyperplane orthogonal to the given vector. The maximum number for such a

| $d\downarrow$ \ $n$ | 1 | 2 | 3 | 4 | ... |
|---|---|---|---|---|---|
| 2 | 3 | 4 | 5 | 6 | ... |
| 3 | 5 | 7 | 9 | 11 | ... |
| 4 | 7 | 10 | 13 | 16 | ... |
| 5 | 9 | 13 | 16 | 20 | ... |
| $\vdots$ | | | | | $\ddots$ |

| $d\downarrow$ \ $n$ | 1 | 2 | 3 | 4 | ... |
|---|---|---|---|---|---|
| 2 | 2 | 3 | 4 | 5 | ... |
| 3 | 3 | 5 | 7 | 9 | ... |
| 4 | 4 | 7 | 10 | 13 | ... |
| 5 | 5 | 9 | 13 | 17 | ... |
| $\vdots$ | | | | | $\ddots$ |

**Table 1.** Minimum number of pieces of a decomposition of $\mathbb{R}^d$ equivalent to $2^{\aleph_0} \leq \aleph_n$, for *finite* intersections (left) and for *countable* intersections (right).

decomposition is $(n+2)(d-1)$, the integer in the square with coordinates $(d, n+1)$ decreased by 1. The table on the right is similar, but the intersections with the hyperplanes are *countable*.

*Proof of Theorem 4.9.* Fix $\delta \in \mathrm{Ord}$ and $d \geq 3$, and proceed by induction on $n$.

If $n=0$ then $N=d<2(d-1)$, and pick $\boldsymbol{u}_{d+1}, \dots, \boldsymbol{u}_{2(d-1)}$ so that $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{2(d-1)}$ are distinct nonzero vectors of $\mathbb{R}^d$, and let $A_{d+1} = \dots = A_{2(d-1)} = \emptyset$. Given $\varepsilon > 0$, if $2^{\aleph_0} > \aleph_{\delta+0}$ then the assumptions of Theorem 4.4 are fulfilled, so we can find the set $Z$ as required. Therefore we may assume the result holds for some $\bar{n}$ towards proving it for $\bar{n}+1$.

Suppose $2^{\aleph_0} > \aleph_{\delta+\bar{n}+1}$. Let $N = (\bar{n}+2)(d-1)+1$, and let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_N$ and $A_1, \dots, A_N$ be as in the statement. Given $\varepsilon > 0$ we must construct $Z = Z_{\bar{n}+1,\varepsilon}$ such that

$$Z \subseteq (-\varepsilon; \varepsilon)^d, \quad |Z| = \aleph_{\delta+\bar{n}+2}, \quad \forall \boldsymbol{p} \in \mathbb{R}^d \left(\boldsymbol{p} + Z \nsubseteq \bigcup_{k=1}^N A_k\right).$$

Let $\overline{N} = (\bar{n}+1)(d-1)+1$. Then $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{\overline{N}}$ and $A_1, \dots, A_{\overline{N}}$ satisfy the hypotheses of the statement for $\bar{n}$. By the inductive assumption there is $\bar{Z} = Z_{\bar{n},\varepsilon/2} \subseteq (-\varepsilon/2; \varepsilon/2)^d$ of size $\aleph_{\delta+\bar{n}+1}$ such that

$$\forall \boldsymbol{p} \in \mathbb{R}^d \left(\boldsymbol{p} + \bar{Z} \nsubseteq \bigcup_{k=1}^{\overline{N}} A_k\right). \tag{10}$$

As $N - \overline{N} = d-1$, there is a unit vector $\boldsymbol{v}$ such that

$$\overline{N} < k \leq N \Rightarrow \boldsymbol{v} \cdot \boldsymbol{u}_k = 0. \tag{11}$$

The subspace $V = \{r\boldsymbol{v} \mid r \in \mathbb{R}\}$ is of cardinality $2^{\aleph_0} \geq \aleph_{\delta+\bar{n}+2} > \aleph_{\delta+\bar{n}+1} = |\bar{Z}|$. By Lemma 4.3 with $m = d$ and $\kappa = \aleph_{\delta+\bar{n}+1}$ we obtain $Y \subseteq \{r\boldsymbol{v} \mid |r| < \varepsilon/2\} \subseteq V$ of size $\aleph_{\delta+\bar{n}+2}$ such that

$$(Y - Y) \cap (\bar{Z} - \bar{Z}) = \{\boldsymbol{0}\}.$$

Let

$$Z := Y + \bar{Z} = \bigcup_{\boldsymbol{y} \in Y} \boldsymbol{y} + \bar{Z} = \bigcup_{\boldsymbol{z} \in \bar{Z}} \boldsymbol{z} + Y.$$

Observe that $Z \subseteq (-\varepsilon; \varepsilon)^d$ and $|Z| = \aleph_{\delta+\bar{n}+2}$. We must argue that $\boldsymbol{p} + Z \nsubseteq \bigcup_{k=1}^{N} A_k$ for all $\boldsymbol{p} \in \mathbb{R}^d$.

Towards a contradiction, let $\hat{\boldsymbol{p}} \in \mathbb{R}^d$ be such that $\hat{\boldsymbol{p}} + Z \subseteq \bigcup_{k=1}^{N} A_k$.

**Claim 4.9.1.** *If $\boldsymbol{y}, \boldsymbol{y}' \in Y$ and $(\boldsymbol{y} + \bar{Z}) \cap (\boldsymbol{y}' + \bar{Z}) \neq \varnothing$ then $\boldsymbol{y} = \boldsymbol{y}'$.*

*Proof.* Suppose $\boldsymbol{y} + \boldsymbol{z} = \boldsymbol{y}' + \boldsymbol{z}'$ with $\boldsymbol{y}, \boldsymbol{y}' \in Y$ and $\boldsymbol{z}, \boldsymbol{z}' \in \bar{Z}$. Then $\boldsymbol{y} - \boldsymbol{y}' = \boldsymbol{z}' - \boldsymbol{z} \in (Y - Y) \cap (\bar{Z} - \bar{Z}) = \{\boldsymbol{0}\}$, so $\boldsymbol{y} = \boldsymbol{y}'$. □

By (11) $V$ is contained in $H_{\boldsymbol{u}_k}(\boldsymbol{0})$, for $\bar{N} < k \leq N$. By assumption $|A_k \cap H_{\boldsymbol{u}_k}(\boldsymbol{q})| < \aleph_\delta$, and since $Y \subseteq V$, then $|A_k \cap (\boldsymbol{q} + Y)| < \aleph_\delta$, for each $\boldsymbol{q} \in \mathbb{R}^d$. Therefore for any $\bar{N} < k \leq N$,

$$|A_k \cap (\hat{\boldsymbol{p}} + Z)| = |A_k \cap (\hat{\boldsymbol{p}} + Y + \bar{Z})| = \Big|\bigcup_{z \in \bar{Z}} A_k \cap (\hat{\boldsymbol{p}} + \boldsymbol{z} + Y)\Big| \leq \aleph_{\delta+\bar{n}+1}. \tag{12}$$

**Claim 4.9.2.** *There is $\hat{\boldsymbol{y}} \in Y$ such that $(\hat{\boldsymbol{p}} + \hat{\boldsymbol{y}} + \bar{Z}) \cap \bigcup_{\bar{N} < k \leq N} A_k = \varnothing$.*

*Proof.* Towards a contradiction, suppose that for all $\boldsymbol{y} \in Y$ there are $\boldsymbol{z}(\boldsymbol{y}) \in \bar{Z}$ and $\bar{N} < k(\boldsymbol{y}) \leq N$ such that $\hat{\boldsymbol{p}} + \boldsymbol{y} + \boldsymbol{z}(\boldsymbol{y}) \in A_{k(\boldsymbol{y})}$. As

$$\aleph_{\delta+\bar{n}+2} = |Y| > \aleph_{\delta+\bar{n}+1} = \bar{Z} > \Big|\{k \mid \bar{N} < k \leq N\}\Big|,$$

there is $\hat{Y} \subseteq Y$ of size $\aleph_{\delta+\bar{n}+2}$, and $\hat{\boldsymbol{z}} \in \bar{Z}$ and $\bar{N} < \hat{k} \leq N$ such that $\boldsymbol{z}(\boldsymbol{y}) = \hat{\boldsymbol{z}}$ and $k(\boldsymbol{y}) = \hat{k}$ for all $\boldsymbol{y} \in \hat{Y}$. Therefore $\forall \boldsymbol{y} \in \hat{Y}\ (\hat{\boldsymbol{p}} + \boldsymbol{y} + \hat{\boldsymbol{z}} \in A_{\hat{k}})$. By Claim 4.9.1 the map $\hat{Y} \to A_{\hat{k}}$, $\boldsymbol{y} \mapsto \hat{\boldsymbol{p}} + \boldsymbol{y} + \hat{\boldsymbol{z}}$ is injective, so $|A_{\hat{k}} \cap (\hat{\boldsymbol{p}} + Y + \bar{Z})| \geq |\hat{Y}| = \aleph_{\delta+\bar{n}+2}$, against (12). □

Fix $\hat{\boldsymbol{y}}$ as in Claim 4.9.2. Then $(\hat{\boldsymbol{p}} + \hat{\boldsymbol{y}}) + \bar{Z} \subseteq A_1 \cup \cdots \cup A_{\bar{N}}$ against (10). Having reached a contradiction, we conclude that $\boldsymbol{p} + Z \nsubseteq \bigcup_{k=1}^{N} A_k$ for all $\boldsymbol{p} \in \mathbb{R}^d$. □

**Remark 4.13.** Theorems 4.4 and 4.9 (together with Theorem 2.3) provide a complete solution to Problem 4.1 when the vectors are in general position, yet some further generalizations are possible. For the sake of readability we have opted for less generality, and here we would like to mention some of these extensions. (The proof of these results will appear elsewhere.)

Focusing on $d = 3$ and $\delta = 0$, the requirement on the size of the intersections in Theorem 4.4 could be relaxed to

$$|A_i \cap H_{\boldsymbol{u}_i}(\boldsymbol{p})| < \aleph_0 \quad \text{for } i = 1, 2, \qquad |A_i \cap H_{\boldsymbol{u}_i}(\boldsymbol{p})| \leq \aleph_0 \quad \text{for } i = 3, 4,$$

strengthening a theorem of Bagemihl [1959/60] that there are no $A_1$, $A_2$, $A_3$ covering $\mathbb{R}^3$ so that all planes orthogonal to $\boldsymbol{e}_2, \boldsymbol{e}_3$ have countable intersections with $A_2$, $A_3$ and all planes orthogonal to $\boldsymbol{e}_1$ have finite intersections with $A_1$. (By Theorem 2.6 and Lemma 4.2 CH is equivalent to the fact that the sets $A_i \cap H_{\boldsymbol{u}_i}(\boldsymbol{p})$ are countable for three distinct $i$s, if $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_4$ are in general position.)

Another possible generalization of Theorem 4.4 is that for any $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n \in \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_2) \setminus \{\boldsymbol{0}\}$ and $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_3) \setminus \{\boldsymbol{0}\}$, there are no $A_1, A_2, A_3$, $B_1, \dots, B_n$, $C_1, \dots, C_m$ covering $\mathbb{R}^3$ such that each $A_i \cap H_{\boldsymbol{e}_i}(\boldsymbol{p})$, $B_j \cap H_{\boldsymbol{u}_j}(\boldsymbol{p})$, $C_k \cap H_{\boldsymbol{v}_k}(\boldsymbol{p})$ is finite. (Clearly $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ can be replaced with any other basis.) This shows that Corollary 4.11 can fail badly, if the vectors are not in general position. On the other hand, not being in general position does not preclude a positive result. For example CH is equivalent to the existence of $A_1, \dots, A_6$ covering $\mathbb{R}^3$ such that each $A_i \cap H_{\boldsymbol{u}_i}(\boldsymbol{p})$ is finite, where $\boldsymbol{u}_1 = \boldsymbol{e}_1$, $\boldsymbol{u}_2 = \boldsymbol{e}_2$, $\boldsymbol{u}_3 = \boldsymbol{e}_3$, $\boldsymbol{u}_4 \in \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_2) \setminus \{\boldsymbol{0}\}$, $\boldsymbol{u}_5 \in \operatorname{span}(\boldsymbol{e}_1, \boldsymbol{e}_4) \setminus \{\boldsymbol{0}\}$, $\boldsymbol{u}_6 \in \operatorname{span}(\boldsymbol{e}_2, \boldsymbol{e}_3) \setminus \{\boldsymbol{0}\}$.

## 5. The main results

Combining Theorem 3.2 with the results from Section 4 we are ready to prove the results about sprays in $\mathbb{R}^d$.

**Theorem 5.1.** *Let $\boldsymbol{c}_1, \dots, \boldsymbol{c}_4$ be four coplanar points in $\mathbb{R}^3$. There are no $X_1, \dots, X_4$ covering $\mathbb{R}^3$ such that $X_i$ is a spray with centers $\boldsymbol{c}_i$.*

*Proof.* Let $X_1, \dots, X_4$ be sprays in $\mathbb{R}^3$ with $\boldsymbol{c}_i$ the center of $X_i$.

If $\boldsymbol{c}_1, \dots, \boldsymbol{c}_4$ belong to the same line $\ell$, let $P$ be a plane orthogonal to $\ell$, and let $\pi : \mathbb{R}^3 \to P$ be the orthogonal projection. Then $Y := \bigcup_{1 \le i \le 4} X_i \cap P$ is a spray of $P$ centered in $\pi(\boldsymbol{c}_1) = \dots = \pi(\boldsymbol{c}_4) \in P$. Since $Y$ cannot cover $P$ then $X_1, \dots, X_4$ cannot cover $\mathbb{R}^3$.

Suppose $\boldsymbol{c}_1, \dots, \boldsymbol{c}_4$ are not collinear. By relabeling and applying an isometry, we may assume that $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3$ are not collinear and belong to $\mathbb{R}^2 \times \{0\}$. By Theorem 3.2 there is an open $E^3 \subseteq \mathbb{R}^3$ and there are $A_1, \dots, A_4$ covering $E^3$ such that $A_i \cap H_i(r)$ are finite for $i = 1, 2, 3$, and $A_4 \cap H_{\boldsymbol{u}}(\boldsymbol{p})$ is finite, where $\boldsymbol{u} = (u_1, u_2, u_3) \neq \boldsymbol{0}$ is such that $u_1(\boldsymbol{c}_1 - \boldsymbol{c}_4) + u_2(\boldsymbol{c}_2 - \boldsymbol{c}_4) + u_3(\boldsymbol{c}_3 - \boldsymbol{c}_4) = \boldsymbol{0}$ and $r$ and $\boldsymbol{p}$ range in $\mathbb{R}$ and $\mathbb{R}^3$, respectively. The result follows from Theorem 4.4. □

The next result uses the vector space $\mathcal{U}_{\boldsymbol{c}_1,\dots,\boldsymbol{c}_d}(\boldsymbol{q})$ defined in (2).

**Proposition 5.2.** *Suppose $H$ is an affine hyperplane of $\mathbb{R}^d$, $\boldsymbol{c}_1, \dots, \boldsymbol{c}_n \in H$ are distinct, well-placed points in $\mathbb{R}^d$, and $n \ge d$. For $k \le n$ let $\boldsymbol{u}_k \in \mathcal{U}_{\boldsymbol{c}_1,\dots,\boldsymbol{c}_d}(\boldsymbol{c}_k) \setminus \{\boldsymbol{0}\}$. Then the vectors $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n$ are in general position in $\mathbb{R}^d$.*

*Proof.* As the definition of $\mathcal{U}(\boldsymbol{c}_k)$ is not affected by a translation of $H$, we may assume that $\boldsymbol{0} \notin H$, so that the vectors $(\boldsymbol{c}_1, \dots, \boldsymbol{c}_d)$ form a basis of $\mathbb{R}^d$. Let $\boldsymbol{u}_k = (u_k^1, \dots, u_k^d)$. We claim that $\sum_{i=1}^d u_k^i \neq 0$, for any $k \le n$. Otherwise

$$\begin{aligned}\boldsymbol{0} = \sum_{i=1}^d u_k^i(\boldsymbol{c}_i - \boldsymbol{c}_k) = \sum_{i=1}^d u_k^i \boldsymbol{c}_i = u_k^1 \boldsymbol{c}_1 + \sum_{i=2}^d u_k^i \boldsymbol{c}_i \\ = -\Big(\sum_{i=2}^d u_k^i\Big)\boldsymbol{c}_1 + \sum_{i=2}^d u_k^i \boldsymbol{c}_i = \sum_{i=2}^d u_k^i(\boldsymbol{c}_i - \boldsymbol{c}_1)\end{aligned}$$

shows that $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_d$ are not in general position in $H$, against our assumption. Scaling of vectors does not affect their general position, so we may assume that $\sum_{i=1}^d u_k^i = 1$ for all $k \leq n$. Then $\sum_{i=1}^d u_k^i (\boldsymbol{c}_i - \boldsymbol{c}_k) = \boldsymbol{0}$ implies that

$$\boldsymbol{c}_k = \sum_{i=1}^d u_k^i \boldsymbol{c}_i$$

that is to say: $(u_k^1, \ldots, u_k^d)$ are the components of the vector $\boldsymbol{c}_k$ with respect to the basis $(\boldsymbol{c}_1, \ldots, \boldsymbol{c}_d)$. Therefore the general position of the vectors $\boldsymbol{u}_k$s follows from the general position of the points $\boldsymbol{c}_k$s in $H$. □

The very same proof yields:

**Proposition 5.3.** *Let $H$ be a hyperplane of $\mathbb{R}^d$, and suppose $\{\boldsymbol{c}_k \mid 1 \leq k < \omega\} \subseteq H$ are distinct points in general position in $H$. For all $k \geq 1$ let $\boldsymbol{u}_k \in \mathcal{U}_{\boldsymbol{c}_1,\ldots,\boldsymbol{c}_d}(\boldsymbol{c}_k) \setminus \{\boldsymbol{0}\}$. Then the vectors $\{\boldsymbol{u}_k \mid 1 \leq k < \omega\}$ are in general position in $\mathbb{R}^d$.*

**Theorem 5.4.** *Fix $d \geq 2$ and $n \geq 1$, and let $N = (n+1)(d-1)+1$ and $M = (n+2)(d-1)$. The following are equivalent*:

(a) $2^{\aleph_0} \leq \aleph_n$.

(b) *For all distinct, well-placed points $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_N \in \mathbb{R}^d$ there are sprays $X_1, \ldots, X_N$ covering $\mathbb{R}^d$ with $X_i$ centered in $\boldsymbol{c}_i$.*

(c) *There are sprays $X_1, \ldots, X_M$ covering $\mathbb{R}^d$ with $X_i$ centered in $\boldsymbol{c}_i$ such that $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_N$ are distinct and well-placed.*

*Proof.* (a)⇒(b) follows from Theorem 2.18 with $\delta = 0$, and (b)⇒(c) is trivial—set $X_{N+1} = \cdots = X_M = \emptyset$ and $\boldsymbol{c}_{N+1}, \ldots, \boldsymbol{c}_M$ any points belonging to the same hyperplane passing through $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_N$. So it is enough to prove (c)⇒(a).

If $d = 2$ then $N = M = n+2$ and (c) says that $\mathbb{R}^2$ is the union of $n+2$ sprays with aligned centers, so $2^{\aleph_0} \leq \aleph_n$ follows from [Schmerl 2010, Theorem 7]. Therefore we may assume that $d \geq 3$. Towards a contradiction, assume $2^{\aleph_0} \geq \aleph_{n+1}$ and suppose that $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_M \in \mathbb{R}^d$ are well-placed, and that $X_1, \ldots, X_M$ are sprays as above. For ease of notation, let $\boldsymbol{u}_i = \boldsymbol{e}_i$ for $1 \leq i \leq d$. By Proposition 5.2 the vectors $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N$ are in general position in $\mathbb{R}^d$, and the map $\Phi : \mathbb{H}^d \to E^d$ transforms the $X_i$ into a covering $A_1, \ldots, A_M$ of $E^d$ such that every hyperplane orthogonal to $\boldsymbol{u}_i$ intersects $A_i$ in a finite set. But this contradicts Theorem 4.10 with $\delta = 0$. □

In particular:

**Theorem 5.5.** *The following are equivalent.*

(a) CH *holds.*

(b) *For all well-placed $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_5 \in \mathbb{R}^3$ there are sprays $X_1, \ldots, X_5$ covering $\mathbb{R}^3$ with $X_i$ centered in $\boldsymbol{c}_i$.*

(c) *There are sprays* $X_1, \dots, X_6$ *covering* $\mathbb{R}^3$ *with* $X_i$ *centered in* $\boldsymbol{c}_i$ *such that* $\boldsymbol{c}_1, \dots, \boldsymbol{c}_6$ *are well-placed.*

More generally:

**Theorem 5.6.** *Fix* $d \geq 2$ *and* $n \geq 1$, *and let* $M = (n+2)(d-1)$, $N = (n+1)(d-1)+1$, $\overline{M} = (n+1)(d-1) = N-1$, *and* $\overline{N} = n(d-1)+1$. *The following are equivalent*:

(a) $2^{\aleph_0} \leq \aleph_n$.

(b) *For all well-placed points* $\boldsymbol{c}_1, \dots, \boldsymbol{c}_N \in \mathbb{R}^d$ *there are sprays* $X_1, \dots, X_N$ *covering* $\mathbb{R}^d$ *with* $X_i$ *centered in* $\boldsymbol{c}_i$.

(c) *There are sprays* $X_1, \dots, X_M$ *covering* $\mathbb{R}^d$ *with* $X_i$ *centered in* $\boldsymbol{c}_i$ *such that* $\boldsymbol{c}_1, \dots, \boldsymbol{c}_M$ *are well-placed.*

(d) *For all well-placed points* $\boldsymbol{c}_1, \dots, \boldsymbol{c}_{\overline{N}} \in \mathbb{R}^d$ *there are* $\sigma$*-sprays* $X_1, \dots, X_N$ *covering* $\mathbb{R}^d$ *with* $X_i$ *centered in* $\boldsymbol{c}_i$.

(e) *There are* $\sigma$*-sprays* $X_1, \dots, X_{\overline{M}}$ *covering* $\mathbb{R}^d$ *with* $X_i$ *centered in* $\boldsymbol{c}_i$ *such that* $\boldsymbol{c}_1, \dots, \boldsymbol{c}_{\overline{M}}$ *are well-placed.*

Proposition 5.3 is the bridge connecting Problem 4.1 with the next problem.

**Problem 5.7.** Given $\boldsymbol{c}_1, \dots, \boldsymbol{c}_n$ distinct points of $H$, a hyperplane of $\mathbb{R}^d$, what conditions on the cardinality of $\mathbb{R}$ are equivalent to the existence of $X_1, \dots, X_n$ covering $\mathbb{R}^d$, each $X_i$ a spray (or $\sigma$-spray) centered in $\boldsymbol{c}_i$?

Theorem 5.6 yields a complete solution to Problem 5.7 when the $\boldsymbol{c}_i$s are well-placed, i.e. in general position in $H$. Using the results mentioned in Remark 4.13 it is possible to distill a few more results on sprays in $\mathbb{R}^3$. Let us mention just two of them. The first is that given four coplanar points $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \boldsymbol{c}_4$ in $\mathbb{R}^3$, there exist no $X_1, X_2, X_3, X_4$ covering $\mathbb{R}^3$ such that $X_1, X_2$ are sprays centered in $\boldsymbol{c}_1, \boldsymbol{c}_2$, and $X_3, X_4$ are $\sigma$-sprays centered in $\boldsymbol{c}_3, \boldsymbol{c}_4$. The second result is that the six points in $\mathbb{R}^2 \times \{0\}$ of Figure 1 are not in general position in the plane, and

- no five sprays centered in these points can cover $\mathbb{R}^3$, but
- CH is equivalent to the existence of six sprays, centered in these points, covering $\mathbb{R}^3$.

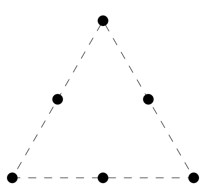


**Figure 1.** CH is equivalent to the existence of six sprays centered in these points and covering $\mathbb{R}^3$. Unconditionally, no five sprays centered in these points can achieve this.

**5.1.** ***Covering the space with infinitely many sprays.*** We have seen how covering the space with sprays with well-placed centers is equivalent to giving an upper bound for the size of the continuum, the larger the number of sprays, the weaker the bound. Next we show that, irrespective of the size of the continuum, the space can be covered with $\aleph_0$-many sprays with well-placed centers.

A *drizzle* in $\mathbb{R}^d$ with center $\boldsymbol{c}$ is a set $X \subseteq \mathbb{R}^d$ such that any sphere centered in $\boldsymbol{c}$ intersects $X$ in at most one point — thus a drizzle is a very sparse spray.

**Theorem 5.8.** *If* $\{\boldsymbol{c}_n \mid n \geq 1\}$ *are distinct, well-placed points in* $\mathbb{R}^d$, *with* $d \geq 2$, *then there are* $X_n$ *covering* $\mathbb{R}^d$ *such that* $X_n$ *is a drizzle centered in* $\boldsymbol{c}_n$.

*Proof.* Without loss of generality we may assume that the $\boldsymbol{c}_n$s belong to $\mathbb{R}^{d-1} \times \{0\}$. Pick $\boldsymbol{u}_n \in \mathcal{U}(\boldsymbol{c}_n) \setminus \{\boldsymbol{0}\}$, where $\mathcal{U}(\boldsymbol{q}) = \mathcal{U}_{\boldsymbol{c}_1,\ldots,\boldsymbol{c}_d}(\boldsymbol{q})$. By Proposition 5.3 the vectors $\boldsymbol{u}_n$ $(n \geq 1)$ are in general position in $\mathbb{R}^d$. By [Davies 1974] there are sets $A_k$ $(k \geq 1)$ such that for all $\boldsymbol{p} \in \mathbb{R}^d$, $H_{\boldsymbol{u}_k}(\boldsymbol{p}) \cap A_k$ has at most one point, and such that $\mathbb{R}^d = \bigcup_{n \geq 1} A_{2n} = \bigcup_{n \geq 0} A_{2n+1}$. The map $\Phi : \mathbb{H}^d \to E^d$ of (4) can be extended to the closures of $\mathbb{H}^d$ and $E^d$, so we can assume that $X_{2n} := \Phi^{-1}[A_{2n}]$ is a subset of $\mathrm{Cl}(\mathbb{H}^d) = \mathbb{R}^{d-1} \times [0; +\infty)$, and that $\bigcup_{n \geq 1} X_{2n} = \mathrm{Cl}(\mathbb{H}^d)$. Letting $\tau : \mathbb{R}^d \to \mathbb{R}^d$ be the reflection with respect to the hyperplane $\mathbb{R}^{d-1} \times \{0\}$, let $X_{2n+1} = \tau[\Phi^{-1}[A_{2n+1}]]$. Then $\bigcup_{n \geq 0} X_{2n+1} = \tau[\mathbb{H}^d] = \mathbb{R}^{d-1} \times (-\infty; 0)$. Therefore $\mathbb{R}^d = \bigcup_n X_n$, and by construction each $X_n$ is a drizzle centered in $\boldsymbol{c}_n$. □

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# DEFINABLE FUNCTORIALITY OF TENSOR-TRIANGULAR SPECTRA

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**We prove that the homological and Balmer spectra in tensor-triangular geometry are functorial in certain definable functors, thereby providing an alternative perspective on functoriality in tensor-triangular geometry from the viewpoint of purity, and generalising current results in the literature.**

## 1. Introduction

Tensor-triangular ("tt") geometry is the study of tensor-triangulated categories $\mathcal{K}$ from a geometric perspective via associated topological spaces, in analogy to the study of commutative rings through their Zariski spectra. There are two such associated spaces: the Balmer spectrum $\mathsf{Spc}(\mathcal{K})$ [1] and the homological spectrum $\mathsf{Spc}^{\mathsf{h}}(\mathcal{K})$ [4]. The former provides the universal solution to the problem of classifying (radical) thick $\otimes$-ideals of $\mathcal{K}$, whereas the latter parametrises abelian counterparts of residue fields of $\mathcal{K}$ and provides a nilpotence theorem. As such, understanding the Balmer and homological spectra of tt-categories is of interest across many fields: commutative algebra, representation theory, algebraic geometry, topology, and so on; see [2] for some highlights.

A standard yet powerful technique in tt-geometry is the use of descent [9; 10]. This relies upon the fact that the Balmer spectrum and the homological spectrum are functorial: suitable functors $\mathcal{K} \to \mathcal{L}$ induce continuous maps on the associated spectra. One can then use such maps to construct new points by comparison to well-understood examples, for instance by passage to residue fields.

Here we provide a new perspective on the functoriality of the homological and Balmer spectra by utilising techniques from model theory. We show that our result generalises and recovers Balmer's result that the homological spectrum is functorial in geometric functors. Using that the Balmer spectrum is the Kolmogorov quotient of the homological spectrum [8], we then deduce corresponding functoriality for

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the Balmer spectrum. Throughout the paper we work with essentially small rigid tt-categories rather than with the compact objects in some ambient "big" category. As such, we reformulate some key features of the homological spectrum in this small setting.

For ease of exposition in this introduction, we assume that we are in the world of rigidly-compactly generated tensor-triangulated categories. In [13], it was illustrated how one can recover the homological spectrum (and thus Balmer spectrum) of a rigidly-compactly generated tt-category from its pure structure. This pure structure consists of precisely the triangles (resp., objects) which become short exact sequences (resp., injective) when viewed as cohomological functors, and thus reflects the abelian structure of the functor category in the triangulated setting.

This connection between tt-geometry and model theory motivates the question of whether functors which preserve the pure structure of the categories, the *definable functors* [12], can also be used to provide functoriality of the homological spectrum. In this article we show that this indeed occurs.

**Theorem** (see Theorems 3.6 and 3.16). *Let $F : \mathsf{T} \to \mathsf{U}$ be a definable functor between rigidly-compactly generated tt-categories. If the induced adjunction*

$$\mathsf{Mod}(\mathsf{T}^{\mathsf{c}}) \underset{\bar{F}}{\overset{\Lambda}{\leftrightarrows}} \mathsf{Mod}(\mathsf{U}^{\mathsf{c}})$$

*satisfies the projection formula and $\Lambda$ preserves cohomological functors, then there is a continuous map $\mathsf{Spc}^{\mathsf{h}}(F) : \mathsf{Spc}^{\mathsf{h}}(\mathsf{T}^{\mathsf{c}}) \to \mathsf{Spc}^{\mathsf{h}}(\mathsf{U}^{\mathsf{c}})$ given by*

$$\mathcal{B} \mapsto \Lambda^{-1}(\mathcal{B}) \cap \mathsf{mod}(\mathsf{U}^{\mathsf{c}}).$$

*Thus, by taking Kolmogorov quotients, $F$ induces a continuous map*

$$\mathsf{Spc}(\mathsf{T}^{\mathsf{c}}) \to \mathsf{Spc}(\mathsf{U}^{\mathsf{c}}).$$

This theorem actually holds in more generality: one may replace the rigidly-compactly generated tt-categories by the categories of cohomological functors on essentially small rigid tt-categories; see Section 3.1 for details. We also remark that one can describe the induced map $\mathsf{Spc}^{\mathsf{h}}(F)$ in terms of the original functor $F$ rather than in terms of $\Lambda$: see Theorem 3.6 and Corollary 3.12.

The theorem just stated generalises the established functoriality of Balmer [3, Theorem 5.10], since the right adjoint to a geometric functor is a definable functor which satisfies the conditions of the above theorem, see Corollary 3.14. In particular, this motivates why the above theorem provides a *co*variant functoriality in definable functors, in contrast to the usual *contra*variant functoriality in geometric functors.

We envisage that this perspective to functoriality through purity may be useful in other settings such as that of non-rigid tt-categories.

## 2. Preliminaries

**2.1.** ***Rigid tt-categories and modules.*** Let $\mathcal{K}$ be an essentially small rigid tt-category: in short, this means that $\mathcal{K}$ is a triangulated category with a compatible closed symmetric monoidal structure $(\otimes, \underline{\mathsf{hom}}, \mathbb{1})$ for which every object is strongly dualisable (i.e., the natural map $\underline{\mathsf{hom}}(k, \mathbb{1}) \otimes k' \to \underline{\mathsf{hom}}(k, k')$ is an isomorphism for all $k, k' \in \mathcal{K}$). We let $\mathsf{Mod}(\mathcal{K})$ denote the category of additive functors $\mathcal{K}^{\mathrm{op}} \to \mathsf{Ab}$, which is a locally coherent Grothendieck abelian category. We write $\mathbb{y} : \mathcal{K} \to \mathsf{Mod}(\mathcal{K})$ for the Yoneda embedding given by $\mathbb{y}k := \mathrm{Hom}_{\mathcal{K}}(-, k)$. The full abelian subcategory of finitely presented objects in $\mathsf{Mod}(\mathcal{K})$ is denoted by $\mathsf{mod}(\mathcal{K})$, and consists of the functors $f$ which have a presentation

$$\mathbb{y}k \to \mathbb{y}k' \to f \to 0.$$

The Yoneda embedding gives an additive equivalence $\mathbb{y} : \mathcal{K} \xrightarrow{\sim} \mathsf{proj}(\mathcal{K})$, where $\mathsf{proj}(\mathcal{K})$ denotes the finitely presented projective objects in $\mathsf{Mod}(\mathcal{K})$. We write $\mathsf{Flat}(\mathcal{K})$ for the subcategory of $\mathsf{Mod}(\mathcal{K})$ consisting of the cohomological functors; the terminology is justified by the fact that $\mathsf{Flat}(\mathcal{K}) = \mathsf{Ind}(\mathcal{K}) = \varinjlim \mathsf{proj}(\mathcal{K})$ where $\varinjlim$ denotes the closure under direct limits. We note that any injective object in $\mathsf{Mod}(\mathcal{K})$ is flat by [11, Proposition 7.1] since $\mathcal{K}$ is a coherent category in the sense of [11, §4].

As $\mathcal{K}$ is closed symmetric monoidal, $\mathsf{Mod}(\mathcal{K})$ inherits a closed symmetric monoidal structure via Day convolution, see [6, Remark 2.4]. The monoidal product on $\mathsf{Mod}(\mathcal{K})$ is the unique functor $- \otimes - : \mathsf{Mod}(\mathcal{K}) \times \mathsf{Mod}(\mathcal{K}) \to \mathsf{Mod}(\mathcal{K})$ which commutes with colimits in both variables, and for which $\mathbb{y} : \mathcal{K} \to \mathsf{Mod}(\mathcal{K})$ is strong symmetric monoidal. We again write $\underline{\mathsf{hom}}$ for the internal hom on $\mathsf{Mod}(\mathcal{K})$ and also write $\mathbb{D} = \underline{\mathsf{hom}}(-, \mathbb{y}\mathbb{1})$ for the functional dual.

**2.2.** ***Purity.*** Let $\mathsf{C}$ be a finitely accessible category with products (for example, $\mathsf{Mod}(\mathcal{K})$ or $\mathsf{Flat}(\mathcal{K})$). A short exact sequence $0 \to L \to M \to N \to 0$ in $\mathsf{C}$ is called *pure* if the induced sequence

$$0 \to \mathrm{Hom}_{\mathsf{C}}(f, L) \to \mathrm{Hom}_{\mathsf{C}}(f, M) \to \mathrm{Hom}_{\mathsf{C}}(f, N) \to 0$$

is exact for all $f \in \mathsf{fp}(\mathsf{C})$, in which case $L \to M$ is called a *pure monomorphism*, and $L$ a *pure subobject* of $M$. The terms *pure epimorphism* and *pure quotient* are defined correspondingly for $M \to N$. An object $X \in \mathsf{C}$ is *pure injective* if any pure monomorphism $X \to Y$ splits.

A full subcategory $\mathcal{D} \subseteq \mathsf{C}$ is *definable* if it is closed under pure subobjects, direct limits, and products. The set $\mathsf{pinj}(\mathsf{C})$ of isomorphism classes of indecomposable pure injective objects in $\mathsf{C}$ underlies a topological space called the *Ziegler spectrum* of $\mathsf{C}$, denoted $\mathsf{Zg}(\mathsf{C})$. The closed sets are given by $\mathcal{D} \cap \mathsf{pinj}(\mathsf{C})$ where $\mathcal{D}$ is a definable subcategory of $\mathsf{C}$. If $\mathcal{X} \subseteq \mathsf{C}$, we let $\mathsf{Def}(\mathcal{X})$ denote the smallest definable subcategory of $\mathsf{C}$ containing $\mathcal{X}$.

The category $\mathsf{Flat}(\mathcal{K})$ is a definable subcategory of $\mathsf{Mod}(\mathcal{K})$, and for brevity we write $\mathsf{Zg}(\mathcal{K})$ for the Ziegler spectrum of $\mathsf{Flat}(\mathcal{K})$.

**2.3. *Definable functors.*** Given two finitely accessible categories $\mathsf{C}$, $\mathsf{D}$ with products, a functor $F : \mathsf{C} \to \mathsf{D}$ is *definable* if it preserves direct limits and products. Definable functors preserve pure exact sequences and pure injective objects, see [17, §13]. For the purpose of this paper, the key example is that for any $f \in \mathsf{mod}(\mathcal{K})$, the functor $f \otimes - : \mathsf{Mod}(\mathcal{K}) \to \mathsf{Mod}(\mathcal{K})$ is definable. This is easily seen by reduction to the case $f = \mathbb{y}k$ for $k \in \mathcal{K}$.

**2.4. *The fundamental correspondence.*** As $\mathsf{Flat}(\mathcal{K})$ is a finitely accessible category, we can consider the category $\mathsf{Mod}(\mathsf{proj}(\mathcal{K}))$ of additive functors $\mathsf{proj}(\mathcal{K})^{\mathrm{op}} \to \mathsf{Ab}$, and the restricted Yoneda functor $\mathsf{y} : \mathsf{Flat}(\mathcal{K}) \to \mathsf{Mod}(\mathsf{proj}(\mathcal{K}))$ given by $\mathsf{y} : X \mapsto \mathrm{Hom}_{\mathsf{Mod}(\mathcal{K})}(-, X)|_{\mathsf{proj}(\mathcal{K})}$.

There is, see [17, §8] or [16, Theorem 12.2.2], an order reversing bijection between definable subcategories $\mathcal{D} \subseteq \mathsf{Flat}(\mathcal{K})$ (equivalently, closed subsets of $\mathsf{Zg}(\mathcal{K})$) and Serre subcategories $\mathcal{S} \subseteq \mathsf{mod}(\mathsf{proj}(\mathcal{K}))$, which is given by the assignments

$$\mathcal{D} \mapsto \{f \in \mathsf{mod}(\mathsf{proj}(\mathcal{K})) : \mathrm{Hom}(f, \mathsf{y}X) = 0 \text{ for all } X \in \mathcal{D}\}$$

and

$$\mathcal{S} \mapsto \{X \in \mathsf{Flat}(\mathcal{K}) : \mathrm{Hom}(f, \mathsf{y}X) = 0 \text{ for all } f \in \mathcal{S}\}.$$

As $\mathbb{y} : \mathcal{K} \to \mathsf{proj}(\mathcal{K})$ is an equivalence, the restriction functor $\mathsf{mod}(\mathsf{proj}(\mathcal{K})) \to \mathsf{mod}(\mathcal{K})$, given by $F \mapsto F \circ \mathsf{y}$, is also an equivalence, and therefore, combining this with the above bijection, we obtain an order reversing bijection between definable subcategories $\mathcal{D}$ of $\mathsf{Flat}(\mathcal{K})$ and Serre subcategories $\mathcal{S}$ of $\mathsf{mod}(\mathcal{K})$. These are explicitly given by

$$\mathcal{D} \mapsto \mathbb{S}(\mathcal{D}) := \{f \in \mathsf{mod}(\mathcal{K}) : \mathrm{Hom}(f, X) = 0 \text{ for all } X \in \mathcal{D}\},$$
$$\mathcal{S} \mapsto \mathbb{D}(\mathcal{S}) := \{X \in \mathsf{Flat}(\mathcal{K}) : \mathrm{Hom}(f, X) = 0 \text{ for all } f \in \mathcal{S}\}.$$

**2.5. *The monoidal fundamental correspondence.*** A Serre subcategory $\mathcal{S} \subseteq \mathsf{mod}(\mathcal{K})$ is a *Serre $\otimes$-ideal* if for any $f \in \mathcal{S}$ and $g \in \mathsf{mod}(\mathcal{K})$ one has $f \otimes g \in \mathcal{S}$. A definable subcategory $\mathcal{D} \subseteq \mathsf{Flat}(\mathcal{K})$ is *$\otimes$-closed* if for any $X \in \mathcal{D}$ and $Y \in \mathsf{Flat}(\mathcal{K})$ one has $X \otimes Y \in \mathcal{D}$. Note that $\mathcal{D}$ is a $\otimes$-closed definable subcategory if and only if $X \otimes \mathbb{y}k \in \mathcal{D}$ for all $k \in \mathcal{K}$ since $\mathcal{D}$ is closed under direct limits. Given $\mathcal{X} \subseteq \mathsf{Flat}(\mathcal{K})$, we write $\mathsf{Def}^{\otimes}(\mathcal{X})$ for the smallest $\otimes$-closed definable subcategory of $\mathsf{Flat}(\mathcal{K})$ containing $\mathcal{X}$. One readily checks that

$$\mathsf{Def}^{\otimes}(\mathcal{X}) = \mathsf{Def}(\mathbb{y}k \otimes X : k \in \mathcal{K}, X \in \mathcal{X}).$$

Since $\mathcal{K}$ is rigid, the bijection of Section 2.4 given by $\mathcal{S}(-)$ and $\mathcal{D}(-)$ restricts to give a bijection between Serre $\otimes$-ideals of $\mathsf{mod}(\mathcal{K})$ and $\otimes$-closed definable subcategories of $\mathsf{Flat}(\mathcal{K})$. For example, if $\mathcal{S} \subseteq \mathsf{mod}(\mathcal{K})$ is a Serre $\otimes$-ideal, then $\mathcal{D}(\mathcal{S})$ is a $\otimes$-closed definable subcategory since for $X \in \mathcal{D}(\mathcal{S})$ and $f \in \mathcal{S}$ we have

$$\mathrm{Hom}_{\mathsf{Mod}(\mathcal{K})}(f, \mathrm{y}k \otimes X) \simeq \mathrm{Hom}_{\mathsf{Mod}(\mathcal{K})}(f \otimes \mathbb{D}(\mathrm{y}k), X) = 0$$

as $f \otimes \mathbb{D}(\mathrm{y}k) \in \mathcal{S}$. The other direction is similar.

**Remark 2.6.** Without an assumption of rigidity, one still obtains a bijection between Serre $\otimes$-ideals of $\mathsf{mod}(\mathcal{K})$ and definable coideals of $\mathsf{Flat}(\mathcal{K})$, that is, definable subcategories $\mathcal{D}$ such that if $X \in \mathcal{D}$ and $k \in \mathcal{K}$ one has $\underline{\mathsf{hom}}(\mathrm{y}k, X) \in \mathcal{D}$. However, it is no longer clear whether there is any assignment between $\otimes$-closed definable subcategories of $\mathsf{Flat}(\mathcal{K})$ and a type of Serre subcategory of $\mathsf{mod}(\mathcal{K})$ that respects a closure property inherited from the monoidal structure.

**2.7.** ***Injectives associated to Serre ⊗-ideals.*** As illustrated in [6, §3], associated to any Serre $\otimes$-ideal $\mathcal{S}$ is a localisation $Q_{\mathcal{S}} : \mathsf{Mod}(\mathcal{K}) \to \mathsf{Mod}(\mathcal{K})/\varinjlim \mathcal{S}$, which is a strong symmetric monoidal exact functor that admits a right adjoint $R_{\mathcal{S}} : \mathsf{Mod}(\mathcal{K})/\varinjlim \mathcal{S} \to \mathsf{Mod}(\mathcal{K})$. Letting $\alpha : Q_{\mathcal{S}}\mathrm{y}\mathbb{1} \hookrightarrow \mathbb{E}_{\mathcal{S}}$ denote the injective hull of $Q_{\mathcal{S}}\mathrm{y}\mathbb{1} \in \mathsf{Mod}(\mathcal{K})/\varinjlim \mathcal{S}$, applying $R_{\mathcal{S}}$ gives an injective object $J_{\mathcal{S}} := R_{\mathcal{S}}\mathbb{E}_{\mathcal{S}} \in \mathsf{Mod}(\mathcal{K})$. Since $R_{\mathcal{S}}$ is fully faithful, we have $Q_{\mathcal{S}}J_{\mathcal{S}} = \mathbb{E}_{\mathcal{S}}$. We note that $J_{\mathcal{S}} \in \mathcal{D}(\mathcal{S})$: if $f \in \mathcal{S}$ then

$$\mathrm{Hom}(f, J_{\mathcal{S}}) = \mathrm{Hom}(f, R_{\mathcal{S}}\mathbb{E}_{\mathcal{S}}) \simeq \mathrm{Hom}(Q_{\mathcal{S}}(f), \mathbb{E}_{\mathcal{S}}) = 0.$$

The Serre $\otimes$-ideal $\mathcal{S}$ can then be recovered from the injective object $J_{\mathcal{S}}$. This was proved in the rigidly-compactly generated setting in [6, Theorem 3.5].

**Lemma 2.8.** *Let $\mathcal{S}$ be a Serre $\otimes$-ideal of* $\mathsf{mod}(\mathcal{K})$. *Then* $\varinjlim \mathcal{S} = \mathsf{Ker}(- \otimes J_{\mathcal{S}})$, *and therefore* $\mathcal{S} = \mathsf{Ker}(- \otimes J_{\mathcal{S}}) \cap \mathsf{mod}(\mathcal{K})$.

*Proof.* The second claim follows from the first by intersecting with $\mathsf{mod}(\mathcal{K})$ so it suffices to prove the first. Since most of the proof of [6, Theorem 3.5] carries over to this setting, we only give a sketch. For the implication $\mathsf{Ker}(- \otimes J_{\mathcal{S}}) \subseteq \varinjlim \mathcal{S}$, we note that if $M \in \mathsf{Mod}(\mathcal{K})$, then there is a morphism $f : X \to Y$ in $\mathsf{Flat}(\mathcal{K})$ such that $M \simeq \mathsf{Im}(f)$. Indeed, we can consider the composition $Y \twoheadrightarrow M \hookrightarrow X$ where $Y \in \mathsf{Proj}(\mathcal{K})$ is projective, and $X$ is the injective envelope of $M$, which is flat by the discussion in Section 2.1. From this point the proof of the implication is the same as the corresponding at [6, Theorem 3.5].

For the implication $\varinjlim \mathcal{S} \subseteq \mathsf{Ker}(- \otimes J_{\mathcal{S}})$, it is sufficient, since $J_{\mathcal{S}}$ is flat, to restrict to finitely presented objects. At this point, one adapts [6, Proposition 3.3]. Define a map $\eta : \mathrm{y}\mathbb{1} \to J_{\mathcal{S}}$ via the composition

$$\mathrm{y}\mathbb{1} \xrightarrow{\text{unit}} R_{\mathcal{S}}Q_{\mathcal{S}}\mathrm{y}\mathbb{1} \xrightarrow{R_{\mathcal{S}}\alpha} J_{\mathcal{S}} = R_{\mathcal{S}}\mathbb{E}_{\mathcal{S}}$$

and consider the exact sequence

$$0 \to \mathsf{ker}(\eta) \to \mathsf{y}\mathbb{1} \xrightarrow{\eta} J_{\mathcal{S}} \to \mathsf{coker}(\eta) \to 0$$

in $\mathsf{Mod}(\mathcal{K})$, which is the replacement for the triangle $\Delta$ in [6, Proposition 3.3] and its proof. From this point, the adaptation of said proof is straightforward, and the conclusion follows. □

**2.9. *Weak rings.*** An object $R \in \mathsf{Mod}(\mathcal{K})$ is a *weak ring* if there is a map $\eta : \mathsf{y}\mathbb{1} \to R$ such that $R \otimes \eta : R \to R \otimes R$ is a split monomorphism. Note that if $R$ is a weak ring then $R \otimes R$ is nonzero. Given any Serre $\otimes$-ideal $\mathcal{S}$ of $\mathsf{mod}(\mathcal{K})$, the associated injective object $J_{\mathcal{S}}$ is a weak ring. Indeed, we have a map $\eta : \mathsf{y}\mathbb{1} \to J_{\mathcal{S}}$ defined as the composite

$$\mathsf{y}\mathbb{1} \xrightarrow{\text{unit}} R_{\mathcal{S}}Q_{\mathcal{S}}\mathsf{y}\mathbb{1} \xhookrightarrow{R_{\mathcal{S}}(\alpha)} J_{\mathcal{S}}.$$

Since $J_{\mathcal{S}}$ is injective and hence flat, the functor $Q_{\mathcal{S}}(J_{\mathcal{S}} \otimes -) \simeq \mathbb{E}_{\mathcal{S}} \otimes Q_{\mathcal{S}}(-)$ is exact, so we obtain a monomorphism

$$\mathbb{E}_{\mathcal{S}} \xrightarrow{\sim} \mathbb{E}_{\mathcal{S}} \otimes Q_{\mathcal{S}}\mathsf{y}\mathbb{1} \hookrightarrow \mathbb{E}_{\mathcal{S}} \otimes \mathbb{E}_{\mathcal{S}}.$$

This is a monomorphism out of an injective and hence splits. Now

$$\mathrm{Hom}(-, J_{\mathcal{S}}) = \mathrm{Hom}(-, R_{\mathcal{S}}(\mathbb{E}_{\mathcal{S}})) = \mathrm{Hom}(Q_{\mathcal{S}}(-), \mathbb{E}_{\mathcal{S}})$$

so this retraction lifts to a map $J_{\mathcal{S}} \otimes J_{\mathcal{S}} \to J_{\mathcal{S}}$ splitting $J_{\mathcal{S}} \otimes \eta$ as required.

**2.10. *The homological spectrum.*** Following [4], a *homological prime* of $\mathcal{K}$ is a maximal proper Serre $\otimes$-ideal of $\mathsf{mod}(\mathcal{K})$. The set of homological primes of $\mathcal{K}$ is denoted $\mathsf{Spc}^{\mathsf{h}}(\mathcal{K})$, and it is endowed with a topology with a basis of closed sets given by

$$\mathsf{supp}^{h}(k) := \{\mathcal{B} \in \mathsf{Spc}^{\mathsf{h}}(\mathcal{K}) : \mathsf{y}k \notin \mathcal{B}\}.$$

as $k$ ranges over $\mathcal{K}$.

**Remark 2.11.** One can conceptualise the relationship between the homological spectrum and definable subcategories as follows. Write $\mathsf{Zg}^{\otimes}(\mathcal{K})$ for the topological space whose points are the isomorphism classes of indecomposable pure injective objects of $\mathsf{Flat}(\mathcal{K})$, and whose closed sets are those of the form $\mathcal{D} \cap \mathsf{pinj}(\mathsf{Flat}(\mathcal{K}))$, where $\mathcal{D} \subseteq \mathsf{Flat}(\mathcal{K})$ is a $\otimes$-closed definable subcategory. There is a homeomorphism $\mathsf{Spc}^{\mathsf{h}}(\mathcal{K}) \simeq \mathsf{KZg}^{\otimes}_{\mathsf{Cl}}(\mathcal{K})^{\mathsf{GZ}}$ where the latter consists of the closed points of the Kolmogorov quotient of $\mathsf{Zg}^{\otimes}(\mathcal{K})$ equipped with the Gabriel–Zariski topology, see [13]. In loc. cit. this is proved when $\mathcal{K} = \mathsf{T}^{\mathsf{c}}$ for some rigidly-compactly generated tt-category $\mathsf{T}$. One can easily adapt the arguments to give the above claim since there is a homeomorphism $\mathsf{Zg}^{\otimes}(\mathsf{T}) \simeq \mathsf{Zg}^{\otimes}(\mathsf{Flat}(\mathsf{T}^{\mathsf{c}}))$.

## 3. Definable functoriality of the homological spectrum

**3.1.** ***Setup.*** Let $\mathcal{K}$ and $\mathcal{L}$ be essentially small rigid tt-categories, and let $F : \mathsf{Flat}(\mathcal{K}) \to \mathsf{Flat}(\mathcal{L})$ be a definable functor. We assume that $F$ has a left adjoint $\Lambda : \mathsf{Flat}(\mathcal{L}) \to \mathsf{Flat}(\mathcal{K})$, and that this adjunction satisfies the projection formula. Explicitly, this means that $F$ is lax monoidal and the canonical map

$$FX \otimes Y \xrightarrow{\sim} F(X \otimes \Lambda Y)$$

is an isomorphism for all $X \in \mathsf{Flat}(\mathcal{K})$ and $Y \in \mathsf{Flat}(\mathcal{L})$.

**Remark 3.2.** Given a definable functor $F : \mathsf{Flat}(\mathcal{K}) \to \mathsf{Flat}(\mathcal{L})$, by [15, Corollary 10.5], this functor extends to a definable functor $\bar{F} : \mathsf{Mod}(\mathcal{K}) \to \mathsf{Mod}(\mathcal{L})$ which fits into an adjoint pair

$$\mathsf{Mod}(\mathcal{K}) \underset{\bar{F}}{\overset{\bar{\Lambda}}{\leftrightarrows}} \mathsf{Mod}(\mathcal{L})$$

where $\bar{\Lambda}$ is exact and preserves finitely presented objects. As such, one can rephrase the requirement that $F$ has a left adjoint $\Lambda$ to the equivalent statement that $\bar{\Lambda}$ preserves flat objects.

**Remark 3.3.** The setup of starting with a definable functor $F : \mathsf{Flat}(\mathcal{K}) \to \mathsf{Flat}(\mathcal{L})$ may appear odd at first glance. However, this is done to allow one to work with essentially small tt-categories. If $\mathcal{K} = \mathsf{T}^{\mathrm{c}}$ and $\mathcal{L} = \mathsf{U}^{\mathrm{c}}$ are the full subcategories of compact objects in some rigidly-compactly generated tt-categories $\mathsf{T}$ and $\mathsf{U}$, then one can instead start with the data of a definable functor $\mathsf{T} \to \mathsf{U}$ of triangulated categories, that is, a (not necessarily triangulated) functor which preserves coproducts and filtered homology colimits, or equivalently, coproducts, products, and pure triangles. Indeed, such a definable functor of triangulated categories yields a definable functor $\mathsf{Flat}(\mathcal{K}) \to \mathsf{Flat}(\mathcal{L})$ by [12, Theorem B].

**Example 3.4.** The canonical example satisfying the above setup comes from a geometric functor $f^* : \mathsf{U} \to \mathsf{T}$, that is, a coproduct preserving strong monoidal triangulated functor between rigidly-compactly generated tt-categories. The functor $f^*$ has a right adjoint $f_* : \mathsf{T} \to \mathsf{U}$, which is a definable functor of triangulated categories, see [12, 5.3]. By [12, Theorems 4.16 and 4.21], there is a definable functor $\hat{f}_* : \mathsf{Flat}(\mathsf{T}^{\mathrm{c}}) \to \mathsf{Flat}(\mathsf{U}^{\mathrm{c}})$ which extends to a functor $\bar{f}_* : \mathsf{Mod}(\mathsf{T}^{\mathrm{c}}) \to \mathsf{Mod}(\mathsf{U}^{\mathrm{c}})$ which has a left adjoint $\bar{f}^*$. Since $\bar{f}^*$ commutes with the Yoneda embeddings, it preserves flats, and hence $\hat{f}_*$ has a left adjoint $\hat{f}^*$ given by the restriction of $\bar{f}^*$. The projection formula holds for the adjunction $(f^*, f_*)$ by [5, Proposition 2.15], and since both $\hat{f}^*$ and $\hat{f}_*$ preserve direct limits, it follows that the projection formula holds for $(\hat{f}^*, \hat{f}_*)$.

**Example 3.5.** Suppose that $f : \mathcal{L} \to \mathcal{K}$ is an exact functor between essentially small rigid tt-categories. By identifying $\mathcal{K}$ with $\mathsf{proj}(\mathcal{K})$, and likewise for $\mathcal{L}$, we extend

$f$ along direct limits to obtain a functor $\Lambda : \mathsf{Flat}(\mathcal{L}) \to \mathsf{Flat}(\mathcal{K})$ which preserves direct limits. We may then apply [15, Theorem 6.7] to obtain an adjoint pair

$$\mathsf{Flat}(\mathcal{K}) \underset{F}{\overset{\Lambda}{\rightleftarrows}} \mathsf{Flat}(\mathcal{L})$$

where $F$ is definable. Whenever $f$ is strong monoidal, so is $\Lambda$, and thus the adjunction satisfies the projection formula: the projection formula holds for the rigid objects by [14, Proposition 3.2], but since the rigid objects are the finitely presented projectives we may take colimits to deduce it on all objects. Therefore in this case, we are in the situation of Section 3.1. This is the 'small' version of Example 3.4; indeed, we could deduce the previous example from this by taking $f$ to be the restriction of $f^*$ to compact objects.

We now state our main result, and will assemble a proof throughout the rest of the section.

**Theorem 3.6.** *Let $F : \mathsf{Flat}(\mathcal{K}) \to \mathsf{Flat}(\mathcal{L})$ be a definable functor satisfying the hypotheses of Section 3.1. Then the assignment*

$$\mathcal{B} \mapsto \mathsf{ker}(- \otimes FJ_{\mathcal{B}}) \cap \mathsf{mod}(\mathcal{L})$$

*defines a continuous map $\mathsf{Spc}^{\mathsf{h}}(F) : \mathsf{Spc}^{\mathsf{h}}(\mathcal{K}) \to \mathsf{Spc}^{\mathsf{h}}(\mathcal{L})$.*

**3.7.** Given a definable subcategory $\mathcal{D} \subseteq \mathsf{Flat}(\mathcal{K})$ and a definable functor

$$F : \mathsf{Flat}(\mathcal{K}) \to \mathsf{Flat}(\mathcal{L}),$$

we write $\mathsf{pure}(F\mathcal{D})$ for the closure of $F\mathcal{D}$ under pure subobjects. Equivalently, this is the smallest definable subcategory of $\mathsf{Flat}(\mathcal{L})$ containing $F\mathcal{D}$, see for example after Corollary 13.4 in [17].

We say that a $\otimes$-closed definable subcategory is *simple* if it is nonzero, and contains no proper nonzero $\otimes$-closed definable subcategory.

**Lemma 3.8.** *Assume the setup of Section 3.1. Let $\mathcal{D} \subseteq \mathsf{Flat}(\mathcal{K})$ be a simple $\otimes$-closed definable subcategory. Then $\mathsf{pure}(F\mathcal{D})$ is also a simple $\otimes$-closed definable subcategory of $\mathsf{Flat}(\mathcal{L})$.*

*Proof.* Let us first prove that $\mathsf{pure}(F\mathcal{D})$ is a $\otimes$-closed definable subcategory of $\mathsf{Flat}(\mathcal{L})$. The fact that it is definable is standard as recalled in Section 3.7, so we must show that if $l \in \mathcal{L}$ and $X \in \mathsf{pure}(F\mathcal{D})$, then $\mathsf{y}l \otimes X \in \mathsf{pure}(F\mathcal{D})$. As $X \in \mathsf{pure}(F\mathcal{D})$ there is a pure monomorphism $X \to FD$ for some $D \in \mathcal{D}$. The functor $\mathsf{y}l \otimes -$ is definable so the induced map $\mathsf{y}l \otimes X \to \mathsf{y}l \otimes FD \simeq F(D \otimes \Lambda\mathsf{y}l)$ is also a pure monomorphism. Consequently, as $D \otimes \Lambda\mathsf{y}l \in \mathcal{D}$ we have $\mathsf{y}l \otimes X \in \mathsf{pure}(F\mathcal{D})$ as claimed.

Let us now suppose that there is a $\otimes$-closed definable subcategory $0 \neq \mathcal{E} \subseteq \mathsf{pure}(F\mathcal{D})$ in $\mathsf{Flat}(\mathcal{L})$; we must show that $\mathcal{E} = \mathsf{pure}(F\mathcal{D})$. Consider the Serre $\otimes$-ideal $\mathcal{S}(\mathcal{E}) \subseteq \mathsf{mod}(\mathcal{L})$ and the associated injective object $J_{\mathcal{S}(\mathcal{E})}$. As $J_{\mathcal{S}(\mathcal{E})} \in \mathcal{D}(\mathcal{S}(\mathcal{E})) = \mathcal{E}$ by Section 2.10, it follows that there is a pure (in fact split) monomorphism $\alpha : J_{\mathcal{S}(\mathcal{E})} \to FD$ for some $D \in \mathcal{D}$.

Consider the object $\Lambda J_{\mathcal{S}(\mathcal{E})} \otimes D \in \mathsf{Flat}(\mathcal{K})$: this makes sense since $J_{\mathcal{S}(\mathcal{E})}$ is injective hence flat. Observe that $\Lambda J_{\mathcal{S}(\mathcal{E})} \otimes D \neq 0$: if it were zero, we would have $J_{\mathcal{S}(\mathcal{E})} \otimes FD \simeq F(\Lambda J_{\mathcal{S}(\mathcal{E})} \otimes D) = 0$, and this cannot happen, since $J_{\mathcal{S}(\mathcal{E})} \otimes J_{\mathcal{S}(\mathcal{E})}$ is a summand of $J_{\mathcal{S}(\mathcal{E})} \otimes FD$ by the previous paragraph, and the former is nonzero as $J_{\mathcal{S}(\mathcal{E})}$ is a weak ring object, see Section 2.9.

As such, $\Lambda J_{\mathcal{S}(\mathcal{E})} \otimes D$ is a nonzero object in $\mathcal{D}$, and as $\mathcal{D}$ was assumed to be simple, we must have that $\mathsf{Def}^{\otimes}(\Lambda J_{\mathcal{S}(\mathcal{E})} \otimes D) = \mathcal{D}$. In other words, by Section 2.5 we have

$$\mathsf{Def}(\mathsf{y}k \otimes \Lambda J_{\mathcal{S}(\mathcal{E})} \otimes D : k \in \mathcal{K}) = \mathcal{D},$$

and as $F$ is a definable functor, we deduce that

$$\begin{aligned}\mathsf{pure}(F\mathcal{D}) &= \mathsf{Def}(F(\mathsf{y}k \otimes \Lambda J_{\mathcal{S}(\mathcal{E})} \otimes D) : k \in \mathcal{K}) \\ &= \mathsf{Def}(J_{\mathcal{S}(\mathcal{E})} \otimes F(\mathsf{y}k \otimes D) : k \in \mathcal{K}),\end{aligned}$$

where the latter equality follows from the projection formula. Since $\mathcal{E}$ is a $\otimes$-closed definable subcategory and $J_{\mathcal{S}(\mathcal{E})} \in \mathcal{E}$, we have that $J_{\mathcal{S}(\mathcal{E})} \otimes F(\mathsf{y}k \otimes D) \in \mathcal{E}$. Consequently, $\mathsf{pure}(F\mathcal{D}) \subseteq \mathcal{E}$, so that $\mathsf{pure}(F\mathcal{D}) = \mathcal{E}$, as required. Hence $\mathsf{pure}(F\mathcal{D})$ is simple as claimed. □

**Corollary 3.9.** *Assume the setup of Section 3.1 and let $\mathcal{B} \in \mathsf{Spc}^{\mathsf{h}}(\mathcal{K})$. Then $\mathsf{Def}^{\otimes}(FJ_{\mathcal{B}}) = \mathsf{pure}(F\mathsf{Def}^{\otimes}(J_{\mathcal{B}}))$.*

*Proof.* Firstly we note that the object $FJ_{\mathcal{B}}$ is nonzero: since $J_{\mathcal{B}}$ is a weak ring and $F$ is lax monoidal, $FJ_{\mathcal{B}}$ is also a weak ring and hence nonzero. The $\otimes$-closed definable subcategory $\mathcal{D}(\mathcal{B}) \subseteq \mathsf{Flat}(\mathcal{K})$ is simple since the fundamental correspondence is order-reversing, see Section 2.4 and Section 2.5. As $J_{\mathcal{B}} \in \mathcal{D}(\mathcal{B})$ by Section 2.10 and $\mathcal{D}(\mathcal{B})$ is simple, there is an equality

$$\mathcal{D}(\mathcal{B}) = \mathsf{Def}^{\otimes}(J_{\mathcal{B}}). \tag{3.10}$$

As $\mathsf{Def}^{\otimes}(J_{\mathcal{B}})$ is simple, by Lemma 3.8, so is $\mathsf{pure}(F\mathsf{Def}^{\otimes}(J_{\mathcal{B}}))$. It is clear that $FJ_{\mathcal{B}} \in \mathsf{pure}(F\mathsf{Def}^{\otimes}(J_{\mathcal{B}}))$, so there is an inclusion $\mathsf{Def}^{\otimes}(FJ_{\mathcal{B}}) \subseteq \mathsf{pure}(F\mathsf{Def}^{\otimes}(J_{\mathcal{B}}))$, but the simplicity of the latter, and the former being nonzero, means the two classes are equal. □

We are now in a position to prove the main theorem.

*Proof of Theorem 3.6.* Let $\mathcal{B} \in \mathsf{Spc}^{\mathsf{h}}(\mathcal{K})$ and consider the simple $\otimes$-closed definable subcategory $\mathcal{D}(\mathcal{B}) \subseteq \mathsf{Flat}(\mathcal{K})$. By Lemma 3.8, Corollary 3.9 and Equation (3.10), we have that $\mathsf{Def}^{\otimes}(FJ_{\mathcal{B}})$ is a simple $\otimes$-closed definable subcategory of $\mathsf{Flat}(\mathcal{L})$, and thus by the monoidal fundamental correspondence there is a homological prime $\mathcal{S} := \mathbb{S}(\mathsf{Def}^{\otimes}(FJ_{\mathcal{B}})) \in \mathsf{Spc}^{\mathsf{h}}(\mathcal{L})$. We define $\mathsf{Spc}^{\mathsf{h}}(F)(\mathcal{B}) := \mathcal{S}$, and it remains to show that this has the form as claimed in the statement.

Observe that $\mathsf{Def}^{\otimes}(FJ_{\mathcal{B}}) = \mathbb{D}(\mathcal{S}) = \mathsf{Def}^{\otimes}(J_{\mathcal{S}})$ where the latter equality holds as in Equation (3.10). For any $f \in \mathsf{mod}(\mathcal{L})$, the functor $f \otimes - : \mathsf{Mod}(\mathcal{L}) \to \mathsf{Mod}(\mathcal{L})$ is definable, so it follows that $f \otimes J_{\mathcal{S}} = 0$ if and only if $f \otimes FJ_{\mathcal{B}} = 0$. Thus

$$\mathcal{S} = \ker(- \otimes J_{\mathcal{S}}) \cap \mathsf{mod}(\mathcal{L}) = \ker(- \otimes FJ_{\mathcal{B}}) \cap \mathsf{mod}(\mathcal{L}) \tag{3.11}$$

by Lemma 2.8 as desired.

For continuity, we show that for any $l \in \mathcal{L}$, the preimage of the basic closed set $\mathsf{supp}^h(l)$ under $\mathsf{Spc}^{\mathsf{h}}(F)$ is closed. By the above, for any $\mathcal{B} \in \mathsf{Spc}^{\mathsf{h}}(\mathcal{K})$ there is an equality $\mathsf{Def}^{\otimes}(FJ_{\mathcal{B}}) = \mathsf{Def}^{\otimes}(J_{\mathsf{Spc}^{\mathsf{h}}(\mathsf{F})(\mathcal{B})})$. Therefore, since $\mathsf{y}l \otimes -$ is a definable functor, we have $\mathsf{y}l \otimes FJ_{\mathcal{B}} = 0$ if and only if $\mathsf{y}l \otimes J_{\mathsf{Spc}^{\mathsf{h}}(F)(\mathcal{B})} = 0$, and

$$\mathsf{Spc}^{\mathsf{h}}(F)^{-1}(\mathsf{supp}^h(l)) = \{\mathcal{B} \in \mathsf{Spc}^{\mathsf{h}}(\mathcal{K}) : \mathsf{y}l \otimes FJ_{\mathcal{B}} \neq 0\}.$$

By the projection formula, we have $\mathsf{y}l \otimes FJ_{\mathcal{B}} \simeq F(\Lambda \mathsf{y}l \otimes J_{\mathcal{B}})$. Since $J_{\mathcal{B}}$ is a weak ring object, the functor $F$ is conservative on objects of the form $\Lambda(f) \otimes J_{\mathcal{B}}$ for any $f \in \mathsf{Flat}(\mathcal{L})$ by [7, Remark 13.12], and thus $\mathsf{y}l \otimes FJ_{\mathcal{B}} \neq 0$ if and only if $\Lambda \mathsf{y}l \otimes J_{\mathcal{B}} \neq 0$. Being the left adjoint to a direct limit preserving functor, $\Lambda$ preserves finitely presented objects. Therefore $\Lambda \mathsf{y}l \simeq \mathsf{y}k$ for some $k \in \mathcal{K}$, and thus $\mathsf{Spc}^{\mathsf{h}}(F)^{-1}(\mathsf{supp}^h(l)) = \mathsf{supp}^h(k)$, which is a closed set, as required. □

One can also give an alternative description of the continuous map $\mathsf{Spc}^{\mathsf{h}}(F)$. Recall the functor $\overline{\Lambda} : \mathsf{Mod}(\mathcal{L}) \to \mathsf{Mod}(\mathcal{K})$ from Remark 3.2.

**Corollary 3.12.** *Let $F : \mathsf{Flat}(\mathcal{K}) \to \mathsf{Flat}(\mathcal{L})$ be a definable functor as in Section 3.1. Then the continuous map $\mathsf{Spc}^{\mathsf{h}}(F) : \mathsf{Spc}^{\mathsf{h}}(\mathcal{K}) \to \mathsf{Spc}^{\mathsf{h}}(\mathcal{L})$ of Theorem 3.6 takes the form*

$$\mathsf{Spc}^{\mathsf{h}}(F)(\mathcal{B}) = \overline{\Lambda}^{-1}(\mathcal{B}) \cap \mathsf{mod}(\mathcal{L})$$

*for all $\mathcal{B} \in \mathsf{Spc}^{\mathsf{h}}(\mathcal{K})$.*

*Proof.* We have

$$\begin{aligned}
\ker(-\otimes FJ_{\mathcal{B}})\cap\mathsf{mod}(\mathcal{L}) &= \mathbb{S}(\mathsf{Def}^{\otimes}(FJ_{\mathcal{B}})) && \text{by Equation (3.11)}\\
&= \mathbb{S}(\mathsf{pure}(F\mathsf{Def}^{\otimes}(J_{\mathcal{B}}))) && \text{by Corollary 3.9}\\
&= \overline{\Lambda}^{-1}(\mathbb{S}(\mathsf{Def}^{\otimes}(J_{\mathcal{B}})))\cap\mathsf{mod}(\mathcal{L}) && \text{by [12, Corollary 4.13]}\\
&= \overline{\Lambda}^{-1}(\mathcal{B})\cap\mathsf{mod}(\mathcal{L}) && \text{by Equation (3.10)}
\end{aligned}$$

as required. □

**3.13.** ***A comparison with Balmer's functoriality.*** There is already a notion of functoriality for the homological spectrum and we now show that the functoriality of Theorem 3.6 extends this result. Let $f^* : \mathsf{U} \to \mathsf{T}$ be a geometric functor between rigidly-compactly generated tt-categories. By Example 3.4 this fits into the setting of Section 3.1, so we may apply Theorem 3.6 together with Corollary 3.12 to recover the following result of Balmer [3, Theorem 5.10]:

**Corollary 3.14.** *Let $f^* : \mathsf{U} \to \mathsf{T}$ be a geometric functor between rigidly-compactly generated tt-categories. There is a continuous map $\mathsf{Spc}^{\mathsf{h}}(\mathsf{T}^{\mathsf{c}}) \to \mathsf{Spc}^{\mathsf{h}}(\mathsf{U}^{\mathsf{c}})$ given by the assignment $\mathcal{B} \mapsto (\bar{f}^*)^{-1}(\mathcal{B}) \cap \mathsf{mod}(\mathsf{U}^{\mathsf{c}})$.* □

**3.15.** ***Induced functoriality on the Balmer spectrum.*** As a corollary of Theorem 3.6, we also obtain functoriality of the Balmer spectrum under the same assumptions:

**Theorem 3.16.** *Let $F : \mathsf{Flat}(\mathcal{K}) \to \mathsf{Flat}(\mathcal{L})$ be a definable functor satisfying the hypotheses of Section 3.1. There is a continuous map on Balmer spectra*

$$\mathsf{Spc}(F) : \mathsf{Spc}(\mathcal{K}) \to \mathsf{Spc}(\mathcal{L})$$

*induced by taking the Kolmogorov quotient.*

*Proof.* By [8, Lemma 4.2], the map $\mathsf{y}^{-1} : \mathsf{Spc}^{\mathsf{h}}(\mathcal{K}) \to \mathsf{Spc}(\mathcal{K})$ exhibits $\mathsf{Spc}(\mathcal{K})$ as the Kolmogorov quotient of $\mathsf{Spc}^{\mathsf{h}}(\mathcal{K})$. Since the Kolmogorov quotient is functorial, applying it to the continuous map $\mathsf{Spc}^{\mathsf{h}}(\mathcal{K}) \to \mathsf{Spc}^{\mathsf{h}}(\mathcal{L})$ of Theorem 3.6 yields the claim. □

**Remark 3.17.** In the case when $F$ arises from an exact, strong monoidal functor $f : \mathcal{L} \to \mathcal{K}$ as in Example 3.5, it follows from Corollary 3.12 that the continuous map $\mathsf{Spc}(\mathcal{K}) \to \mathsf{Spc}(\mathcal{L})$ of the previous theorem is given by $\mathfrak{p} \mapsto f^{-1}(\mathfrak{p})$. In particular, we recover the usual functoriality of the Balmer spectrum.

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# GROUPS ACTING AMENABLY ON THEIR HIGSON CORONA

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**We investigate groups that act amenably on their Higson corona (also known as bi-exact groups) and provide reformulations of this in relation to the stable Higson corona, nuclearity of crossed products and to positive type kernels.**

**We further investigate implications of this in relation to the Baum–Connes conjecture, and prove that Gromov hyperbolic groups have isomorphic equivariant $K$-theories of their Gromov boundary and their stable Higson corona.**



## 1. Introduction

This paper arose from pursuing a side idea of [EWZ21], namely to investigate groups that act amenably on their own (reduced) stable Higson corona $\mathfrak{c}^{\mathrm{red}}G$. The

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motivation comes from the commutative diagram

$$\begin{array}{ccc} K_*^{\mathrm{top}}(G;\mathfrak{c}^{\mathrm{red}}G) & \xrightarrow{\mu^*_{\mathrm{EM}}} & K_G^{1-*}(\underline{EG}) \\ & \searrow^{\mu_*^{\mathrm{BC}}} \quad \nearrow^{\mu_G^*} & \\ & K_*(\mathfrak{c}^{\mathrm{red}}G \rtimes_\mu G) & \end{array} \tag{1-1}$$

where $\mu^*_{\mathrm{EM}}$ is the co-assembly map of Emerson and Meyer, $\mu_*^{\mathrm{BC}}$ is the Baum–Connes assembly map, $\mu_G^*$ is the equivariant coarse co-assembly map, and where we have by definition $K_G^{1-*}(\underline{EG}) := K_{*-1}(C_0(\underline{EG}) \rtimes_{\mathrm{red}} G)$ [EWZ21, Section 5.1].

In the diagram we may use any crossed product functor $- \rtimes_\mu G$ which is exact, and one of our questions was: In which cases does the choice not matter, i.e., when do $\mathfrak{c}^{\mathrm{red}}G \rtimes_{\max} G$ and $\mathfrak{c}^{\mathrm{red}}G \rtimes_{\mathrm{red}} G$ coincide (assuming for simplicity that $G$ is exact)? As is now known, this question is intimately connected to amenability of the $G$-$C^*$-algebra $\mathfrak{c}^{\mathrm{red}}G$ [BEW20]. We tried to answer this question in [EWZ21, Propositions 5.8 and 5.12], but unfortunately, there is an error in the proofs and one of the goals of this paper is to provide an erratum to it.[1] We will discuss the error in detail in Section 3.3.

As it turns out, the corrected version of [EWZ21, Proposition 5.8] uses the *unreduced* stable Higson compactification $\bar{\mathfrak{c}}G$ and stable Higson corona. Concretely, we prove the following (we only state the version for the compactification):

**Theorem 1.1** (part of Proposition 3.6). *Let $G$ be a countable and discrete group. The following statements are equivalent to each other*:

(a) *The group $G$ acts amenably on its Higson compactification $hG$.*
(b) *$\bar{\mathfrak{c}}G$ is an amenable $G$-C*-algebra.*
(c) *We have $\bar{\mathfrak{c}}G \rtimes_{\max} G \cong \bar{\mathfrak{c}}G \rtimes_{\mathrm{red}} G$ and $G$ is exact.*

Groups acting amenably on their Higson compactification $hG$ have been already studied before, but with another focus: One can prove that this condition is equivalent to the group $G$ being bi-exact (see Definition 2.2) and from this we get a plethora of examples. This will be quickly summarized in Section 2.

In Section 3.4 we will find more reformulations for a group to act amenably on its Higson compactification. Concretely, we prove the following:

**Theorem 1.2** (Proposition 3.17). *Let $G$ be a countable and discrete group. The following statements are equivalent to each other*:

(a) *The group $G$ acts amenably on its Higson compactification.*
(b) *The $C^*$-algebra $C(hG) \rtimes_{\mathrm{red}} G$ is nuclear.*

[1]Fortunately, since this question was just a side hustle in [EWZ21], none of the main results of it are affected by this.

(c) *The embedding* $\mathbb{C} \rtimes_{\text{red}} G \to C(hG) \rtimes_{\text{red}} G$ *is nuclear.*

(d) *There is a sequence* $(k_n)_{n\in\mathbb{N}}$ *in* $C_c(G \times G, \Delta)$ *of normalized positive type kernels having vanishing variation on diagonals and converging to* 1 *uniformly on all finite width neighbourhoods of the diagonal* $\Delta$ *in* $G \times G$.

As we will discuss in Remarks 3.18, the above result shows that the condition of acting amenably on its Higson compactification sits naturally between amenability and exactness of the group. Because both amenability and exactness have profound implications for the Baum–Connes conjecture for the group (amenability implies bijectivity whereas exactness implies injectivity of the assembly map), the question arises whether one can prove something in this direction for groups acting amenably on their Higson compactifications. We will pursue this line of thought in Section 4, and our main result in this direction is the following:

**Theorem 1.3** (Proposition 4.3). *Let G be a bi-exact group and assume that it admits a G-finite classifying space for proper G-actions* $\underline{EG}$. *Then we have the split short exact sequence*

$$0 \to K_{*+1}(\bar{\mathfrak{c}}^{\text{red}}\underline{EG} \rtimes_{\text{red}} G) \to K_*(C^*_{\text{red}}(G)) \to K_*(\bar{\mathfrak{c}}(\underline{EG}) \rtimes_{\text{red}} G) \to 0. \tag{1-2}$$

*Further, the Baum–Connes conjecture for trivial coefficients* $\mathbb{C}$ *and coefficients* $\bar{\mathfrak{c}}^{\text{red}}\underline{EG}$ *are equivalent to each other for G and imply the isomorphism*

$$K_*(C^*_{\text{red}}(G)) \xrightarrow{\cong} K_*(\bar{\mathfrak{c}}(\underline{EG}) \rtimes_{\text{red}} G), \tag{1-3}$$

*which is induced from the inclusion of* $\mathcal{K}$ *as the constant functions in* $\bar{\mathfrak{c}}(\underline{EG})$.

Our final result is a computation of $K_*(\mathfrak{c}(\underline{EG}) \rtimes_{\text{red}} G)$ for Gromov hyperbolic groups (note that this result now refers to the stable Higson corona, contrary to the compactification as above in Theorem 1.3):

**Theorem 1.4** (Example 4.5). *Let G be a finitely generated, Gromov hyperbolic group. Then we have an isomorphism*

$$K_*(C(\partial G) \rtimes_{\text{red}} G) \cong K_*(\mathfrak{c}G \rtimes_{\text{red}} G), \tag{1-4}$$

*where* $\partial G$ *denotes the Gromov boundary of* $G$.

## 2. Bi-exact groups

Let $G$ be a countable and discrete group. If needed, we will equip it without further mentioning with any proper, left-invariant metric.

**Definition 2.1** (Amenable actions, [HR00, Definitions 2.1 and 2.2] and [ADR00, Section 2.2]).

(a) For any countable set $Z$ we denote by $\operatorname{prob}(Z)$ the set of Borel probability measures on $Z$, i.e., the set of functions $b\colon Z \to [0,1]$ such that $\sum_{z\in Z} b(z) = 1$.

We view $\operatorname{prob}(Z)$ as a subset of $\ell^1(Z)$ and equip it with the weak-*-topology (recall that $\ell^1(Z)$ is the Banach space dual of $c_0(Z)$). By $\|-\|_1$ we denote the usual norm on $\ell^1(Z)$.

If $Z$ is equipped with an action of the group $G$, then the induced action of $G$ on $\operatorname{prob}(Z)$ is defined by $g.b(z) := b(g^{-1}.z)$.

(b) Let $X$ be a locally compact Hausdorff space on which $G$ acts by homeomorphisms. The action is called amenable if there is a sequence of weak-*-continuous maps $b^n : X \to \operatorname{prob}(G)$ such that for every $g \in G$ we have

$$\lim_{n\to\infty} \|g.b^n_x - b^n_{g.x}\|_1 = 0$$

uniformly on compact subsets of $X$.

**Definition 2.2** (bi-exact groups [BO08, Chapter 15]). We consider the $G\times G$-action on the group $G$ given by left and right translations.

We call $G$ *bi-exact* if the induced action of $G \times G$ on the Stone–Čech boundary $\partial_\beta G$ is amenable.

**Remarks 2.3.** It is known that $G$ is bi-exact if and only if it acts amenably on its Higson corona (which will be defined in Section 3.1) [BO08, Proposition 15.2.3]. Because the latter is the class of groups that we consider in this paper, everything we prove here holds for bi-exact groups.

Since the Higson corona is a compact Hausdorff space, being bi-exact implies being exact (see [HR00] or [BO08, Theorem 5.1.7]).

**Examples 2.4.** The following groups are bi-exact:

(a) amenable groups, since being amenable implies that any action on any space is amenable and so especially the action on its Higson corona [AD07, Example 1.4(1)];

(b) groups hyperbolic relative to a family of amenable subgroups [Oza06b, Proposition 12] or, more generally, hyperbolic relative to a family of bi-exact subgroups [Oya23b];

(c) small cancellation groups, such as finitely generated $C'\left(\frac{1}{33}\right)$-groups [Oya23a];

(d) discrete subgroups of connected, simple Lie groups of rank one [Ska88, Section 4];

(e) $\mathbb{Z}^2 \rtimes \operatorname{SL}(2,\mathbb{Z})$ [Oza09].

The class of bi-exact groups also enjoys the following closure properties:

(f) It is closed under passage to subgroups.

The argument is as follows: Assume that $H$ is a subgroup of $G$ and that $G$ is a bi-exact group, i.e., the action of $G \times G$ on $\partial_\beta G$ is amenable. Then also $H \times H$ acts amenably on $\partial_\beta G$, cf. [BEW24, Proposition 5.19] or [OS21, Corollary 3.4] (keep in mind for these references that here we are only considering discrete groups). By the universal property of the Stone–Čech compactification we have an $(H \times H)$-map $\partial_\beta H \to \partial_\beta G$ and hence $H \times H$ will also act amenably on $\partial_\beta H$.

(g) It is closed under free products (with finite amalgamation) [Oza06a, Theorem 3.3]

(h) Wreath products $\Upsilon \wr G$, where $\Upsilon$ is amenable and $G$ bi-exact, are again bi-exact [BO08, Corollary 15.3.9].

(i) It is closed under measure equivalence [Sak09].

(j) It is closed under $W^*$-equivalence [DP, Corollary 6.3].

**Remarks 2.5.** The class of bi-exact groups is in general not closed under products: The group $\mathbb{Z} \times \mathrm{SL}(2, \mathbb{Z})$ is not bi-exact.[2]

It follows that $\mathrm{SL}(3, \mathbb{Z})$ is not bi-exact either: The group $\mathbb{Z} \times \mathrm{SL}(2, \mathbb{Z})$ ME-embeds into it[3] and hence by Sako's result[4] [Sak09] bi-exactness of $\mathrm{SL}(3, \mathbb{Z})$ would propagate to $\mathbb{Z} \times \mathrm{SL}(2, \mathbb{Z})$.

Weakly amenable groups are in general not bi-exact, since weak amenability is, contrary to bi-exactness, closed under products: A product of free groups (which are weakly amenable by [Oza08]) is weakly amenable but not bi-exact (by note 2).

**Remarks 2.6.** The importance of bi-exact groups stems from the fact that the group von Neumann algebra of a bi-exact group is solid [Oza04].

## 3. Amenable actions on the Higson compactification

**3.1.** ***The commutative case.*** We will discuss the relevant notions in the commutative case, i.e., for the corresponding spaces of complex-valued functions.

[2]In general, any product of an infinite group $G$ and a non-amenable group $H$ will not be bi-exact. The argument is as follows: In advance of Section 3.1 we use the characterization of bi-exactness of a group by acting amenably on its Higson corona. Let $x$ be a point in the Higson corona of $G \times H$ with $x$ lying in the closure of $G \times \{e_H\}$ therein. (This needs $G$ to be infinite, otherwise there is no such $x$.) Acting with $\{e_G\} \times H$ from the left on $G \times \{e_H\}$ is the same as acting on it from the right and hence $\{e_G\} \times H$ acts trivially on the point $x$ (to see this one has to use the concrete definition of the Higson corona). If $G \times H$ would act amenably on its Higson corona, then all point stabilizers would be amenable subgroups and therefore $H$ would be amenable.

[3]The ME-embedding comes from a discrete embedding of $\mathbb{Z} \times \mathrm{SL}(2, \mathbb{Z})$ into $\mathrm{SL}(3, \mathbb{R})$.

[4]In the version that if $G$ ME-embeds into a bi-exact group, then $G$ is bi-exact.

**Definition 3.1** (Higson compactification and corona). Let $Y$ be any metric space. If $\vartheta : Y \to M$ is a map to another metric space $M$, then for each $r > 0$ the $r$-variation of $\vartheta$ is defined as the function

$$\mathrm{Var}_r\vartheta : Y \to [0,\infty)\,, \qquad y \mapsto \sup\{d(\vartheta(y),\vartheta(x)) : x \in Y \text{ with } d(y,x) \leq r\}\,.$$

The function $\vartheta$ is said to have vanishing variation if for all $r > 0$ the $r$-variation $\mathrm{Var}_r\vartheta$ converges to zero at infinity.[5]

The *Higson compactification* $hY$ of $Y$ is the Gelfand dual of the $C^*$-algebra of all complex-valued, bounded, continuous functions of vanishing variation on $Y$. The *Higson corona* $\partial_h Y$ is defined as the complement $hY \setminus Y$.

If $\bar{Y}$ is any compactification of $Y$, it is called *Higson dominated* if $C(\bar{Y}) \subset C(hY)$, i.e., if we have surjective map $hY \to \bar{Y}$ extending the identity on $Y$.

Because the Higson compactification $hG$ is a compact Hausdorff space, acting amenably on it implies that $G$ is exact in the usual sense (see [Oza00; HR00] and [BEW20, Theorem 5.3]).

In the case of the Stone–Čech compactification and corona the following lemma is well-known, see e.g. [BCL17].

**Lemma 3.2.** *Let $G$ be a countable, discrete group. The following statements are equivalent:*

- *$G$ acts amenably on its Higson compactification $hG$.*
- *$G$ acts amenably on its Higson corona $\partial_h G$.*
- *$C(hG) \rtimes_{\max} G = C(hG) \rtimes_{\mathrm{red}} G$ and $G$ is exact.*
- *$C(\partial_h G) \rtimes_{\max} G = C(\partial_h G) \rtimes_{\mathrm{red}} G$ and $G$ is exact.*

*Proof.* In general, the group $G$ acts amenably on a $G$-$C^*$-algebra $A$ if and only if it acts amenably on both a $G$-ideal $I$ in $A$ and the quotient $A/I$ [BEW24, Proposition 3.23]. The equivalence of the first two points in the lemma follows from this since $G$ always acts amenably on $C_0(G)$ by [AD02, Example 2.7(3)] (and using [BEW24, Proposition 3.9] that an action on $X$ is amenable if and only if the action on $C_0(X)$ is strongly amenable, and using [BEW24, Remark 3.8] that acting amenably and strongly amenably are equivalent in this case since these are commutative $C^*$-algebras).

For equivalence with the other points see [Mat14] and [BEW20, Theorem 5.2]. □

[5] If this is defined to mean that for every $\varepsilon > 0$ exists a compact subset $K \subset Y$ with $|\mathrm{Var}_r\vartheta(y)| < \varepsilon$ for all $y \in Y$, then one should assume $Y$ to be locally compact. If $Y$ is not locally compact, then one should instead demand that $K$ be just bounded.

**3.2. *Groups acting amenably on the stable Higson compactification and stable Higson corona.*** For a discrete group $G$ we will first recall the different notions of amenability of $G$-C*-algebras from [BEW20, Definitions 2.1 and 4.13] and then apply them to the (unreduced) stable Higson compactification and corona (see Definition 3.4). (Variants of some of these notions occur in e.g. [AD02; BO08]. How these variants relate to each other is explained in [BEW20, Remark 2.2].)

**Definition 3.3.** Let $G$ be a discrete group.

(a) The $G$-C*-algebra $A$ is called strongly amenable if there is a net

$$(\theta_i : G \to Z\mathcal{M}(A))_{i\in I} ,$$

where $Z\mathcal{M}(A)$ is the center of the multiplier algebra, of positive type functions[6] such that

- each $\theta_i$ is finitely supported,
- for each $i$ we have $\theta_i(e) \leq 1$, and
- for each $g \in G$ we have $\theta_i(g) \to 1$ strictly as $i \to \infty$.

(b) The $G$-C*-algebra $A$ is called amenable if there is a net $(\theta_i : G \to Z(A^{**}))_{i\in I}$ of positive type functions such that

- each $\theta_i$ is finitely supported,
- for each $i$ we have $\theta_i(e) \leq 1$, and
- for each $g \in G$ we have $\theta_i(g) \to 1$ ultra-weakly as $i \to \infty$.[7]

(c) The $G$-C*-algebra $A$ is called commutant amenable if for every covariant pair $(\pi, u) : (A, G) \to \mathcal{B}(H)$ there exists a net $(\theta_i : G \to \pi(A)')_{i\in I}$ of positive type functions such that

- each $\theta_i$ is finitely supported,
- for each $i$ we have $\theta_i(e) \leq 1$, and
- for each $g \in G$ we have $\theta_i(g) \to 1$ ultra-weakly as $i \to \infty$.

Note that strong amenability implies amenability [BEW20, Remark 2.2], and amenability implies commutant amenability [BEW20, Remark 4.14].

The main players of this section are the (unreduced[8]) stable Higson compactification $\bar{\mathfrak{c}}G$ and corona $\mathfrak{c}G$ of $G$, defined thus:

[6]In general, a function $\vartheta : G \to B$ is of positive type if for any finite subset $\{g_1, \ldots, g_n\}$ of $G$ the matrix $(\alpha_{g_i}(\vartheta(g_i^{-1} g_j)))_{i,j} \in M_n(B)$ is positive, where $\alpha$ is the action of $G$ on $B$ [AD87, Definition 2.1].

[7]Recall that a net $(T_\lambda)_{\lambda\in\Lambda}$ in $A^{**}$ converges ultra-weakly to $T$ if and only if $(T_\lambda(\varphi))_{\lambda\in\Lambda}$ converges to $T(\varphi)$ for every $\varphi \in A^*$.

[8]There are also reduced versions of these $C^*$-algebras, but the main results of the present section (Proposition 3.6, Proposition 3.8) do not hold for them; see the next Section 3.3 for a discussion.

**Definition 3.4** ([EM06a, Definition 3.2]). Let $G$ be a countable and discrete group, and equip it with any proper, left-invariant metric.[9] Fix any separable Hilbert space $H$.

The (*unreduced*) *stable Higson compactification* $\bar{\mathfrak{c}}G$ is the $C^*$-algebra of all bounded (continuous) functions of vanishing variation $f\colon G \to \mathcal{K}(H)$.

The (*unreduced*) *stable Higson corona* $\mathfrak{c}G$ is defined as the quotient

$$\mathfrak{c}G := \bar{\mathfrak{c}}G / C_0(G, \mathcal{K}(H)) .$$

Proposition 3.6 below answers the variant of [EWZ21, Question 5.10] for the unreduced stable Higson compactification in the case of the metric space acted on by $G$ being the group $G$ itself. But before we can prove it, we first need to identify the multiplier algebra of $\bar{\mathfrak{c}}G$:

**Lemma 3.5.** *Let $G$ be a countable discrete group. Then $\mathcal{M}(\bar{\mathfrak{c}}G)$ is the $G$-C*-algebra of bounded* (*strictly continuous*)[10] *functions of vanishing variation $G \to \mathcal{B}(H)$.*

*Proof.* Let us first discuss a general fact about $C^*$-algebras: If $B$ is a $C^*$-algebra between a $C^*$-algebra $A$ and its multiplier algebra, i.e., we have a chain of inclusions $A \subset B \subset \mathcal{M}(A)$, then we also have an inclusion $\mathcal{M}(B) \to \mathcal{M}(A)$. To see this, first note that because $A$ is an ideal in $\mathcal{M}(A)$, it is also an ideal in $B$; and since $B$ is an ideal in $\mathcal{M}(B)$, we get that $A$ is an ideal in $\mathcal{M}(B)$.[11] This provides us with the map $\mathcal{M}(B) \to \mathcal{M}(A)$. Now since $A$ is essential in $\mathcal{M}(A)$ by the universal property of the multiplier algebra, it is also essential in $B$; and $B$ is essential in $\mathcal{M}(B)$, hence $A$ is also essential in $\mathcal{M}(B)$. This shows that the map $\mathcal{M}(B) \to \mathcal{M}(A)$ is injective.

We apply this general fact to our situation: $\bar{\mathfrak{c}}G$ contains the ideal $C_0(G, \mathcal{K}(H))$. By [APT73, Corollary 3.4] the multiplier algebra of $C_0(G, \mathcal{K}(H))$ are the bounded (strictly continuous) $\mathcal{B}(H)$-valued functions on $G$; and $\bar{\mathfrak{c}}G$ is a subalgebra of them. Therefore we get by the above an inclusion

$$\mathcal{M}(\bar{\mathfrak{c}}G) \subset \mathcal{M}(C_0(G, \mathcal{K}(H))) = C_b(G, \mathcal{B}(H)_\beta),$$

where the notation $-_\beta$ is taken from [APT73, Section 3] and denotes that we are meaning strictly continuous functions (which, as we already mentioned, is in our discrete case a vacuous condition).

Now $\bar{\mathfrak{c}}G$ is essential in the $C^*$-algebra $C_{\mathfrak{w}}(G, \mathcal{B}(H)_\beta)$ of the bounded (and strictly continuous) functions of vanishing variation $G \to \mathcal{B}(H)$ and hence we have an inclusion $C_{\mathfrak{w}}(G, \mathcal{B}(H)_\beta) \subset \mathcal{M}(\bar{\mathfrak{c}}G)$. To show the other inclusion (and by this

[9] Since any two such metrics are coarsely equivalent to each other, the defined $C^*$-algebras are independent of this choice.

[10] Since $G$ is discrete by assumption, every function is automatically strictly continuous (and even continuous). We mention this here in case one wants to treat the case of a nondiscrete group or space (as we have to do in the proof of Lemma 4.2).

[11] Note that this needs us to work in $C^*$-algebras. Such a fact is not generally true in all rings.

finish the proof), let $f \in C_b(G, \mathcal{B}(H)_\beta)$ multiply $\bar{\mathfrak{c}}G$ and we have to show that $f$ has vanishing variation. To show this, we assume the contrary, i.e., that $f$ does not have vanishing variation. Then there exist $\delta, r > 0$ and a sequence $(x_n, y_n)$ in $G \times G$ with $(x_n, y_n) \to \infty$ such that $d(x_n, y_n) < r$ and $\|f(x_n) - f(y_n)\| \geq \delta$ for every $n \in \mathbb{N}$. Choose now for each $n \in \mathbb{N}$ a compact operator $K_n$ satisfying $\|(f(x_n) - f(y_n))K_n\| \geq \delta/2$ and $\sup_{n\in\mathbb{N}} \|K_n\| < \infty$; for example, $K_n$ could be a rank-1 projection onto a (normed) vector witnessing the norm of $f(x_n) - f(y_n)$. Passing to a subsequence of $(x_n, y_n)$ we may assume that the $n$-neighbourhoods $B_n(\{x_n, y_n\})$ are disjoint from each other; and then we set

$$g(x) := \Big(1 - \frac{\operatorname{dist}(x, \{x_n, y_n\})}{n}\Big) \cdot K_n$$

for $x \in B_n(\{x_n, y_n\})$, and 0 otherwise. We have $g \in \bar{\mathfrak{c}}G$ and

$$\|(fg)(x_n) - (fg)(y_n)\| = \|(f(x_n) - f(y_n))K_n\| \geq \delta/2$$

for all $n \in \mathbb{N}$, which shows that $fg$ does not have vanishing variation. But this is a contradiction to our assumption that $f$ multiplies $\bar{\mathfrak{c}}G$.[12] □

**Proposition 3.6.** *Let G be a countable and discrete group. The following statements are equivalent to each other*:

(a) *The group G acts amenably on its Higson compactification $hG$.*

(b) *$\bar{\mathfrak{c}}G$ is an amenable $(hG \rtimes^{\mathcal{G}} G)$-C*-algebra.*[13]

(c) *$\bar{\mathfrak{c}}G$ is a strongly amenable $G$-C*-algebra.*

(d) *$\bar{\mathfrak{c}}G$ is an amenable $G$-C*-algebra.*

(e) *$\bar{\mathfrak{c}}G$ is a commutant amenable $G$-C*-algebra.*

(f) *We have $\bar{\mathfrak{c}}G \rtimes_{\max} G \cong \bar{\mathfrak{c}}G \rtimes_{\mathrm{red}} G$ and $G$ is exact.*

*Proof.* To see the equivalence (a) ⇔ (b) we will provide a $G$-equivariant, non-degenerate map $\phi : C(hG) \to Z\mathcal{M}(\bar{\mathfrak{c}}G)$. By Lemma 3.5 we know that $\mathcal{M}(\bar{\mathfrak{c}}G)$ is the $G$-C*-algebra of bounded (strictly continuous) functions of vanishing variation $G \to \mathcal{B}(H)$ and therefore $Z\mathcal{M}(\bar{\mathfrak{c}}G)$ are the bounded (strictly continuous) complex-valued functions on $G$ of vanishing variation. The map $\phi$ is then just the identity.

[12]The author thanks Christopher Wulff for providing the idea for how to construct this function $g$.

[13]We denote by $hG \rtimes^{\mathcal{G}} G$ the crossed product groupoid and by $\bar{\mathfrak{c}}G$ being an amenable $(hG \rtimes^{\mathcal{G}} G)$-C*-algebra we mean that there exists a $G$-equivariant, nondegenerate *-homomorphism $\phi : C(hG) \to Z\mathcal{M}(\bar{\mathfrak{c}}G)$ and that $G$ acts amenably on $hG$ (the notion of an $X \rtimes^{\mathcal{G}} G$-C*-algebra is mentioned in the introduction of [CEOO03] whose Corollary 0.4 we will use below in the proof of Proposition 4.3).

This is the same notion as $\bar{\mathfrak{c}}G$ being a strongly amenable $G$-$C(hG)$-algebra in the sense of [AD02, Definition 6.1] which can be seen by directly comparing the definitions.

The implication (b) ⇒ (c) is a general fact [BEW20, Lemma 2.5], and the same is true for (c) ⇒ (d) by [BEW20, Remark 2.2] and for (d) ⇒ (e) by [BEW20, Remark 4.14].

Let us show (d) ⇒ (a). Let $p \in \bar{\mathfrak{c}}G$ be the constant function to a fixed rank-one projection, so that $p\bar{\mathfrak{c}}Gp$ $G$-equivariantly identifies with $C(hG)$. The claim follows since amenability passes to $G$-invariant hereditary $C^*$-subalgebras [BEW24, Corollary 3.24].

Let us show the implication (e) ⇒ (a). We first note that $\bar{\mathfrak{c}}G$ is $G$-equivariantly $C^*$-isomorphic to its opposite $(\bar{\mathfrak{c}}G)^{\mathrm{op}}$ by applying a $C^*$-isomorphism $\mathcal{B}(H) \to \mathcal{B}(H)^{\mathrm{op}}$ point-wise to functions in $\bar{\mathfrak{c}}G$. Next we employ the standard form representations for von Neumann algebras as developed by Haagerup [Haa75] in the form presented in [BEW20, Theorem 5.1]: There exists a normal, unital and faithful representation $\pi^{\mathrm{op}} : ((\bar{\mathfrak{c}}G)^{\mathrm{op}})^{**} \to \mathcal{B}(V)$ on a Hilbert space $V$ and a unitary representation $u$ of $G$ on $V$ such that $(\pi^{\mathrm{op}}, u)$ is a covariant pair and we have $\pi^{\mathrm{op}}((\bar{\mathfrak{c}}G)^{\mathrm{op}})' \cong (\bar{\mathfrak{c}}G)^{**}$. Composing $\pi^{\mathrm{op}}$ with an equivariant $C^*$-isomorphism $\bar{\mathfrak{c}}G \cong (\bar{\mathfrak{c}}G)^{\mathrm{op}}$ we get a covariant pair $(\rho, u)$ for $\bar{\mathfrak{c}}G$ with the property that $\rho(\bar{\mathfrak{c}}G)' \cong (\bar{\mathfrak{c}}G)^{**}$. To this covariant pair we can now apply the assumed (e) to get a net

$$(\theta_i : G \to \rho(\bar{\mathfrak{c}}G)')_{i \in I} \tag{3-1}$$

of positive type functions having the properties listed in Definition 3.3.(c). Let $p \in \bar{\mathfrak{c}}G$ be the constant function to a fixed rank-one projection, so that $p\bar{\mathfrak{c}}Gp$ $G$-equivariantly identifies with $C(hG)$. The corresponding map $\bar{\mathfrak{c}}G \to C(hG)$ given by compression by $p$ extends to a unital[14] normal[15] conditional expectation

$$\psi : (\bar{\mathfrak{c}}G)^{**} \to C(hG)^{**} \tag{3-2}$$

by [Bla06, Section III.5.2.10]. Using the isomorphism $\rho(\bar{\mathfrak{c}}G)' \cong (\bar{\mathfrak{c}}G)^{**}$ and composing the net in (3-1) with the map $\psi$ from (3-2) we conclude that $C(hG)$ is amenable. This implies that $G$ acts amenably on $hG$ [BEW20, Remark 2.2].

Finally, let us prove the equivalence (e) ⇔ (f). That (e) implies the weak containment property $\bar{\mathfrak{c}}G \rtimes_{\max} G \cong \bar{\mathfrak{c}}G \rtimes_{\mathrm{red}} G$ is a general fact: It is the implication (2) ⇒ (4) in [BEW24, Proposition 5.10]. Using the already proven implication (e) ⇒ (a) and since $hG$ is a compact Hausdorff space, we conclude that (e) also implies that $G$ is exact. The reverse implication (f) ⇒ (e) is true in general [BEW20, Theorem 4.17]. □

[14] As a general fact, the double dual $(\bar{\mathfrak{c}}G)^{**}$ contains the multiplier algebra $\mathcal{M}(\bar{\mathfrak{c}}G)$ as a subalgebra: It is the idealizer of $\bar{\mathfrak{c}}G$ in $(\bar{\mathfrak{c}}G)^{**}$; and hence the unit of $\mathcal{M}(\bar{\mathfrak{c}}G)$ is the one of $(\bar{\mathfrak{c}}G)^{**}$. From this description of the unit we see that $\psi$ is unital.

[15] This means that for every bounded, increasing net $(x_i)$ of positive elements we have $\phi(\sup x_i) = \sup \phi(x_i)$ [Bla06, Definition III.2.2.1].

Let us now quickly turn to the corresponding statements for the (unreduced) stable Higson corona (thus resolving the unreduced variant of [EWZ21, Conj. 1.25] in the case that the metric space is the group itself). Again we will first identify the corresponding multiplier algebra:

**Lemma 3.7.** *Let $G$ be a countable, discrete group. Then we have an exact sequence*

$$0 \to C_0(G, \mathcal{B}(H)) \to \mathcal{M}(\bar{\mathfrak{c}}G) \to \mathcal{M}(\mathfrak{c}G),$$

*i.e., $\mathcal{M}(\mathfrak{c}G)$ contains the quotient of the $G$-C*-algebra $\mathcal{M}(\bar{\mathfrak{c}}G)$ of the bounded (strictly continuous) functions $G \to \mathcal{B}(H)$ of vanishing variation by its ideal $C_0(G, \mathcal{B}(H))$ as a sub-$C^*$-algebra.*

*Further, we have a $G$-equivariant, unital (hence nondegenerate) $^*$-homomorphism*

$$\phi : C(\partial_h G) \to Z\mathcal{M}(\mathfrak{c}G).$$

*Proof.* The quotient map $q : \bar{\mathfrak{c}}G \to \mathfrak{c}G$ extends to a map $\mathcal{M}(q) : \mathcal{M}(\bar{\mathfrak{c}}G) \to \mathcal{M}(\mathfrak{c}G)$ in the canonical way. By [AS11, Proposition 1.1(i)] the kernel of $\mathcal{M}(q)$ is given by the strict closure of the kernel of $q$ in $\mathcal{M}(\bar{\mathfrak{c}}G)$; which is $C_0(G, \mathcal{B}(H))$. We therefore have the exact sequence

$$0 \to C_0(G, \mathcal{B}(H)) \to \mathcal{M}(\bar{\mathfrak{c}}G) \xrightarrow{\mathcal{M}(q)} \mathcal{M}(\mathfrak{c}G)$$

as claimed.[16]

We have a $G$-equivariant $^*$-homomorphism $C(hG) \to \mathcal{M}(\bar{\mathfrak{c}}G)$ given by $f \mapsto f \otimes \mathrm{id}_H$ and the kernel of the composition $C(hG) \to \mathcal{M}(\bar{\mathfrak{c}}G) \to \mathcal{M}(\mathfrak{c}G)$ is exactly $C_0(G)$, i.e., we get a map $C(\partial_h G) \to \mathcal{M}(\mathfrak{c}G)$. It is $G$-equivariant and unital, and is our sought map $\phi$ provided we can show that it takes values in the center of $\mathcal{M}(\mathfrak{c}G)$.

To show the above, we will use the following general fact about multiplier algebras: Assume that $f \in \mathcal{M}(A)$ centralizes $A$, i.e., we have $fa = af$ for all $a \in A$. Then $f$ even centralizes all of $\mathcal{M}(A)$, i.e., lies in the center of $\mathcal{M}(A)$. We prove this in two steps:

- First we will prove that multiplication by $f$ is strictly continuous on $\mathcal{M}(A)$, i.e., if $(x_\lambda)_{\lambda\in\Lambda} \subset \mathcal{M}(A)$ converges strictly to $x$, then $(fx_\lambda)_{\lambda\in\Lambda}$, $(x_\lambda f)_{\lambda\in\Lambda} \subset \mathcal{M}(A)$ converge strictly to $fx$, resp. to $xf$. We treat only the case of left multiplication by $f$ since the other case is analogous. We have to show that for any $c \in A$ we have norm convergence $fx_\lambda c \to fxc$ and $cfx_\lambda \to cfx$. The first case follows since $x_\lambda \to x$ strictly implies $x_\lambda c \to xc$ in norm and hence $fx_\lambda c \to fxc$ since

[16] In general, if $A \to B$ is a surjective map, then the induced map $\mathcal{M}(A) \to \mathcal{M}(B)$ will be again surjective if $A$ is separable [APT73, Theorem 4.2] or if $A$ is $\sigma$-unital [Lan95, Proposition 6.8]. Unfortunately, $\mathcal{M}(\bar{\mathfrak{c}}G)$ is in general neither of these. But still the author believes that the map $\mathcal{M}(q)$ should be surjective. Since we don't need it, we have not tried to prove it by hand.

multiplication is norm continuous. For the second case we use that $cf \in A$ and hence $cfx_\lambda \to cfx$ in norm. Note that this does not use that $f$ centralizes $A$.

- To finish the proof of this general subclaim, let $f \in \mathcal{M}(A)$ centralize $A$ and choose any $x \in \mathcal{M}(A)$. We have to show that $fx = xf$. Now we use that $A$ is strictly dense in $\mathcal{M}(A)$, i.e., there exists a net $(x_\lambda)_{\lambda\in\Lambda} \subset A$ with $x_\lambda \to x$ strictly. Then $fx = f(\lim x_\lambda) = \lim(fx_\lambda) = \lim(x_\lambda f) = (\lim x_\lambda)f = xf$, where for the second and the second-to-last equality sign we used that multiplication by $f$ is strictly continuous and for the middle equality sign that $f$ centralizes $A$.

Let us now use this general fact to conclude the proof. We have to show that every element in the image of the map $C(\partial_h G) \to \mathcal{M}(\mathfrak{c}G)$ centralizes $\mathfrak{c}G$. But this map is induced from the map $C(hG) \to \mathcal{M}(\bar{\mathfrak{c}}G)$, $f \mapsto f \otimes \mathrm{id}_H$ which clearly takes values in the center. □

**Proposition 3.8.** *Let $G$ be a countable and discrete group. The following statements are equivalent to each other*:

(a) *The group $G$ acts amenably on its Higson corona $\partial_h G$.*

(b) *$\mathfrak{c}G$ is an amenable $(\partial_h G \rtimes^{\mathcal{G}} G)$-C*-algebra.*

(c) *$\mathfrak{c}G$ is a strongly amenable $G$-C*-algebra.*

(d) *$\mathfrak{c}G$ is an amenable $G$-C*-algebra.*

(e) *$\mathfrak{c}G$ is a commutant amenable $G$-C*-algebra.*

(f) *We have $\mathfrak{c}G \rtimes_{\max} G \cong \mathfrak{c}G \rtimes_{\mathrm{red}} G$ and $G$ is exact.*

*Proof.* The proof is analogous to the one of Proposition 3.6. For the equivalence of (a) with (b) we use the map $\phi$ from Lemma 3.7. □

By Lemma 3.2 the group $G$ acts amenably on its Higson compactification $hG$ if and only if it acts amenably on its Higson corona $\partial_h G$. We therefore conclude the following corollary:

**Corollary 3.9.** *The conditions in Proposition 3.6 are equivalent to the conditions in Proposition 3.8.*

### 3.3. *The case of the reduced algebras.*

**Definition 3.10** [EM06a, Definition 5.4]. Let $G$ be any countable and discrete group, and equip it with any proper, left-invariant metric. Fix a separable Hilbert space $H$.

The *reduced* stable Higson compactification $\bar{\mathfrak{c}}^{\mathrm{red}}G$ is the $C^*$-algebra of all bounded (continuous) functions of vanishing variation $f : G \to \mathcal{B}(H)$ with $f(x) - f(y) \in \mathcal{K}(H)$ for all $x, y \in G$.

The *reduced* stable Higson corona $\mathfrak{c}^{\mathrm{red}}G$ is defined as the quotient

$$\mathfrak{c}^{\mathrm{red}}G := \bar{\mathfrak{c}}^{\mathrm{red}}G / C_0(G, \mathcal{K}(H)) .$$

Contrary to what was claimed in [EWZ21, Proposition 5.8] the analogue of Proposition 3.6 for the reduced stable Higson compactification $\bar{\mathfrak{c}}^{\mathrm{red}}G$ is in general not true (and similarly for Proposition 5.12 in loc. cit.), as we will discuss below. This also answers in the negative Question 5.10 and Conjecture 1.25 in loc. cit.

**Lemma 3.11.** *Assume that $\bar{\mathfrak{c}}^{\mathrm{red}}G$ or $\mathfrak{c}^{\mathrm{red}}G$ is an amenable $G$-C*-algebra. Then $G$ is amenable.*

*Proof.* Let us discuss the case of $\bar{\mathfrak{c}}^{\mathrm{red}}G$. The case of $\mathfrak{c}^{\mathrm{red}}G$ is completely analogously.

Note that $\bar{\mathfrak{c}}G$ is an ideal in $\bar{\mathfrak{c}}^{\mathrm{red}}G$ with quotient the Calkin algebra $\mathcal{B}(H)/\mathcal{K}(H)$ equipped with the trivial $G$-action. Since amenability descends to quotients [BEW24, Proposition 3.23], the claim follows because the trivial action can only be amenable for amenable groups.[17] □

**Example 3.12.** Consider a Gromov hyperbolic group $G$. It is known that $G$ acts amenably on its Gromov boundary ([AD02, Example 2.7.4], originally proved in [Ada94]) and hence, because the Gromov boundary is Higson dominated, $G$ acts also amenably on its Higson corona.

Because hyperbolic groups are in general not amenable, this provides concrete counter-examples to [EWZ21, Propositions 5.8 and 5.12] and invalidates Example 5.13 and Proposition 1.24 in loc. cit.

**Remarks 3.13.** The mistake in [EWZ21] occurs in Lemma 5.6 therein: The map constructed there is in general not unital, contrary to what is claimed there. But the unitality of this map was crucial for the proof of Proposition 5.8 therein.

**3.4.** ***Nuclearity of crossed products and positive type kernels.*** Let us first recall the necessary notions related to kernels and Schur multipliers:

**Definition 3.14** [Roe03, Section 11.2]. A symmetric function $k\colon G \times G \to \mathbb{R}$ is called a *positive type kernel*, if for every $n \in \mathbb{N}$ and every $g_1, \dots, g_n \in G$ the matrix given by $[k(g_i, g_j)]_{i,j} \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ is positive semidefinite.

- A positive type kernel $k$ is called *normalized* if $k(g, g) = 1$ for every $g \in G$.
  Note that in this case the kernel will be uniformly bounded; concretely, we will have $|k(g, h)| \leq 1$ for all $g, h \in G$ [BO08, Theorem D.3].
- A positive type kernel $k$ is called *equivariant* if $k(h_1 g, h_2 g) = k(h_1, h_2)$ for every $g, h_1, h_2 \in G$.

If $k$ is a positive type kernel on $G \times G$ with $k(g, g) \leq 1$ for every $g \in G$, then the Schur multiplier

$$\theta_k\colon \mathcal{B}(\ell^2(G)) \to \mathcal{B}(\ell^2(G))\,, \quad [T_{g,h}]_{g,h\in G} \mapsto [k(g,h)T_{g,h}]_{g,h\in G} \tag{3-3}$$

[17] For actions on spaces the last statement can be found in e.g. [AD07, Example 1.4(1)]. Unfortunately, the author could not find in the literature the generalization of this to trivial actions on (unital) $C^*$-algebras, though it seems likely to be also true.

is a completely positive contraction;[18] if $k$ is additionally normalized, the corresponding Schur multiplier $\theta_k$ is unital and completely positive ([Roe03, Lemma 11.17], [BO08, Theorem D.3]).

We will write $C_c(G \times G, \Delta)$ for the algebra of all functions $f$ on $G \times G$ for which there is an $R > 0$ such that $f(g, h) = 0$ whenever $d(g, h) > R$; and we call a subset $E$ of $G \times G$ a *finite width neighbourhood of the diagonal* $\Delta$ if there is an $R > 0$ such that $d(x, y) < R$ for all $(x, y) \in E$.

The following important properties that the group $G$ might have were originally defined in different terms, but can be equivalently defined by the existence of positive type functions with certain properties:

**Fact 3.15.** (a) *The group $G$ is **amenable** if there is a sequence $(k_n)_{n\in\mathbb{N}}$ in $C_c(G\times G, \Delta)$ of normalized, equivariant positive type kernels converging to* 1 *uniformly on all finite width neighbourhoods of the diagonal $\Delta$ in $G \times G$.*

(References are [BO08, Theorem 2.6.8], [AD02, Example 2.7(1)] and [AD07, Section 4.3 and Definition-Proposition 1.1].)

(b) *The group $G$ is **exact** if there exists a sequence $(k_n)_{n\in\mathbb{N}}$ in $C_c(G \times G, \Delta)$ of normalized positive type kernels converging to* 1 *uniformly on all finite width neighbourhoods of the diagonal $\Delta$ in $G \times G$.*

*(*References are [Oza00, Theorem 3], [Roe03, Lemma 11.37], [Tu01, Proposition 3.2] and [AD02, Proposition 3.5].)

We next introduce a condition on kernels that is related to the Higson compactification:

**Definition 3.16.** Let $k$ be a function on $G \times G$. We will say that $k$ has *vanishing variation on diagonals* if for every $g \in G$ the function $h \mapsto k(h, gh)$ on $G$ has vanishing variation.

Obviously, if $k$ is equivariant, then it has vanishing variation on diagonals. This means that in the following Proposition 3.17 condition (d) sits naturally between the two conditions in Fact 3.15:

**Proposition 3.17.** *Let $G$ be a countable and discrete group. The following statements are equivalent to each other*:

(a) *The group $G$ acts amenably on its Higson compactification.*
(b) *The $C^*$-algebra $C(hG) \rtimes_{\mathrm{red}} G$ is nuclear.*
(c) *The embedding $\mathbb{C} \rtimes_{\mathrm{red}} G \to C(hG) \rtimes_{\mathrm{red}} G$ is nuclear.*

[18] To see this, we first convince ourselves that the following version of [BO08, Theorem D.3] holds: A kernel $k$ on $G$ is of positive type if and only if there exists a Hilbert space $\mathcal{H}$ and vectors $\xi_g \in \mathcal{H}$ such that $k(g, h) = \langle \xi_h, \xi_g \rangle$ for every $g, h \in G$ if and only if the multiplier $\theta_k$ is completely positive. That $k(g, g) \leq 1$ for every $g \in G$ then implies that $\theta_k$ is completely contractive is [BO08, Theorem D.4]. This result is also stated, without proof, at the beginning of [Oza00].

(d) *There is a sequence* $(k_n)_{n\in\mathbb{N}}$ *in* $C_c(G \times G, \Delta)$ *of normalized positive type kernels having vanishing variation on diagonals and converging to* 1 *uniformly on all finite width neighbourhoods of the diagonal* $\Delta$ *in* $G \times G$.

*Proof.* The equivalence of (a) and (b) follows from [AD02, Theorem 5.8]. Note that since we assume $G$ to be discrete, the Property (W) in the statement of [AD02, Theorem 5.8] is automatically satisfied by [AD02, Example 4.4].[19]

The equivalence of (a) with (d) follows from [AD02, Proposition 2.5] in combination with the reformulation of it discussed directly before Proposition 3.5 in loc. cit.

That (b) implies (c) is clear: Nuclearity of $C(hG) \rtimes_{\mathrm{red}} G$ means that its identity map is nuclear, whence the composition $\mathbb{C} \rtimes_{\mathrm{red}} G \to C(hG) \rtimes_{\mathrm{red}} G \xrightarrow{\mathrm{id}} C(hG) \rtimes_{\mathrm{red}} G$ will be also nuclear [BO08, Exercise 2.1.4].

For the proof that (c) implies (d) we follow the corresponding proof for exactness in [Oza00, Lemma 2 and Theorem 3]. By the nuclearity assumption,[20] for any finite subset $E \subset G$ (with $e \in E$), regarded as a subset $E \subset \mathbb{C} \rtimes_{\mathrm{red}} G$, and $\varepsilon > 0$ there is an $n \in \mathbb{N}$ and unital completely positive maps $\phi : \mathbb{C} \rtimes_{\mathrm{red}} G \to \mathrm{Mat}_n(\mathbb{C})$ and $\psi : \mathrm{Mat}_n(\mathbb{C}) \to C(hG) \rtimes_{\mathrm{red}} G$ such that

$$\|(\psi \circ \phi)(x) - x\| < \varepsilon/2 \text{ for all } x \in E\,.$$

The map $\phi$ can be extended to a unital completely positive map $\mathcal{B}(\ell^2(G)) \to \mathrm{Mat}_n(\mathbb{C})$ along the inclusion $\mathbb{C} \rtimes_{\mathrm{red}} G = C^*_{\mathrm{red}}(G) \subset \mathcal{B}(\ell^2(G))$ [BO08, Corollary 1.5.16]; let us continue to denote the extended map by $\phi$. We can now do the approximation argument from the proof of [Oza00, Lemma 2] with $\phi$ to finally obtain the unital completely positive map $\phi'' : \mathbb{C} \rtimes_{\mathrm{red}} G \to \mathrm{Mat}_n(\mathbb{C})$ satisfying

$$\|\phi''(x) - \phi(x)\| < \varepsilon/2 \quad \text{for all } x \in E$$

and the following property: Putting $\theta := \psi \circ \phi''$ we get a map $\theta$ which is

(a) unital completely positive and of finite rank,

(b) satisfies $\|\theta(x) - x\| < \varepsilon$ for all $x \in E$, and

(c) has the form

$$\theta(x) = \sum_{k=1}^{d} \omega_{\delta_{p(k)},\delta_{q(k)}}(x) \otimes y_k$$

for elements $y_k \in C(hG) \rtimes_{\mathrm{red}} G$ and the linear functionals $\omega_{\delta_{p(k)},\delta_{q(k)}}$ on $\mathcal{B}(\ell^2(G))$ given by $\omega_{\delta_{p(k)},\delta_{q(k)}}(x) = \langle x(\delta_{p(k)}), \delta_{q(k)}\rangle$ for $p(k), q(k) \in G$.

Setting $k : G \times G \to \mathbb{C}$ as $k(s,t) := \langle \delta_s, \theta(st^{-1})\delta_t\rangle$, where we have secretly used

[19] The implication (a) ⇒ (b) is also shown in [Mat14, Theorem on last page] using Lemma 3.2.

[20] See [BO08, Exercise 2.1.1] for this version of nuclearity of maps.

the canonical inclusion $C(hG) \rtimes_{\mathrm{red}} G \subset \mathcal{B}(\ell^2(G))$, we get a normalized positive type kernel which is

- supported on the finite width neighbourhood defined by
$$F := \{q(k)p(k)^{-1} : k = 1, \dots, d\}$$
of the diagonal $\Delta \subset G \times G$,[21]
- has vanishing variation on diagonals, and
- which is $\varepsilon$-close to 1 on the finite width neighbourhood defined by $E$.

This finishes the proof that (c) implies (d).

The proof of the proposition is now complete. As a remark, let us note that the implication from (d) to (b) can be directly proven by doing the obvious modifications to the proof of [Oza00, Theorem 3(ii) ⇒ (iii)]. □

**Remarks 3.18.** (a) in Proposition 3.17 sits naturally between amenability and exactness, and the same is true for (d) by Fact 3.15.

It is known that nuclearity of the reduced group $C^*$-algebra $C^*_{\mathrm{red}}(G) = \mathbb{C} \rtimes_{\mathrm{red}} G$ is equivalent to amenability [Lan73, Theorem 4.2]. Moreover, nuclearity of the uniform Roe algebra (which is isomorphic to $\ell^\infty(G) \rtimes_{\mathrm{red}} G$) is equivalent to exactness of $G$ [Oza00]. Therefore (b) in Proposition 3.17 sits naturally between amenability and exactness.

In [GK02, Remark 2 in Section 5] it was suggested to consider $C^*$-algebras $A$ satisfying $C^*_{\mathrm{red}}(G) \subset A \subset \ell^\infty(G) \rtimes_{\mathrm{red}} G$ and impose the requirement that the inclusion of $C^*_{\mathrm{red}}(G)$ into $A$ be a nuclear map in order to get conditions interpolating between amenability and exactness. Proposition 3.17 and the further results of this article show that $A = C(hG) \rtimes_{\mathrm{red}} G$ is a good choice.

## 4. Isomorphism results

Let $G$ be a countable, discrete group and assume that it admits a $G$-finite classifying space for proper $G$-actions $\underline{EG}$. Then we have a short exact sequence

$$0 \to C_0(\underline{EG}) \otimes \mathcal{K} \to \bar{\mathfrak{c}}^{\mathrm{red}} \underline{EG} \to \mathfrak{c}^{\mathrm{red}} G \to 0 \tag{4-1}$$

of $G$-$C^*$-algebras.

**Proposition 4.1.** *Let $G$ be a countable, discrete group and assume that it admits a $G$-finite classifying space for proper $G$-actions $\underline{EG}$.*

*If $G$ is exact, the boundary morphism $\partial : K^{\mathrm{top}}_*(G; \mathfrak{c}^{\mathrm{red}} G) \to K^{\mathrm{top}}_{*-1}(G; C_0(\underline{EG}))$ resulting from (4-1) is an isomorphism and consequently*

$$K^{\mathrm{top}}_{*-1}(G; \bar{\mathfrak{c}}^{\mathrm{red}} \underline{EG}) \cong 0 \tag{4-2}$$

[21] This is the subset of all $(s, t) \in G \times G$ satisfying $st^{-1} \in F$.

*for all* $* \in \mathbb{Z}$.

*Proof.* We have the commutative diagram with bijective top horizontal map

$$\begin{array}{ccc} K_*^{\mathrm{top}}(G;\mathfrak{c}^{\mathrm{red}}G) & \xrightarrow{\mu^*_{\mathrm{EM}},\,\cong} & K_G^{1-*}(\underline{E}G) \\ & {\scriptstyle\mu_*^{\mathrm{BC}}}\searrow \qquad \nearrow{\scriptstyle\mu_G^*} & \\ & K_*(\mathfrak{c}^{\mathrm{red}}G \rtimes_{\mathrm{red}} G) & \end{array} \tag{4-3}$$

where $\mu^*_{\mathrm{EM}}$ is the co-assembly map of Emerson and Meyer, $\mu_*^{\mathrm{BC}}$ is the Baum–Connes assembly map, $\mu_G^*$ is the equivariant coarse co-assembly map, and where we have by definition $K_G^{1-*}(\underline{E}G) := K_{*-1}(C_0(\underline{E}G) \rtimes_{\mathrm{red}} G)$ [EWZ21, Section 5.1].

The short exact sequence (4-1) induces the following commutative diagram whose rows are exact and the vertical maps are the respective assembly maps:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & K_*^{\mathrm{top}}(G;\mathfrak{c}^{\mathrm{red}}G) & \xrightarrow{\partial} & K_{*-1}^{\mathrm{top}}(G;C_0(\underline{E}G)) & \longrightarrow & K_{*-1}^{\mathrm{top}}(G;\bar{\mathfrak{c}}^{\mathrm{red}}\underline{E}G) & \longrightarrow \cdots \\ & \downarrow & \searrow^{\mu^*_{\mathrm{EM}},\,\cong} & \downarrow\cong & & \downarrow & \\ \cdots \longrightarrow & K_*(\mathfrak{c}^{\mathrm{red}}G \rtimes_{\mathrm{red}} G) & \xrightarrow{\partial} & K_{*-1}(C_0(\underline{E}G)\rtimes_{\mathrm{red}} G) & \longrightarrow & K_{*-1}(\bar{\mathfrak{c}}^{\mathrm{red}}\underline{E}G \rtimes_{\mathrm{red}} G) & \longrightarrow \cdots \end{array}$$

The middle vertical map is an isomorphism, since $C_0(\underline{E}G)$ is a proper $G$-$C^*$-algebra, and the diagonal dashed map is the one from Diagram (4-3). The claim follows. □

Strengthening the assumption on $G$ from exactness to bi-exactness, we arrive at Proposition 4.3 below. But first we prove a lemma about amenability of $\bar{\mathfrak{c}}(\underline{E}G)$:

**Lemma 4.2.** *Let $G$ be a countable, discrete and bi-exact group and we assume that it admits a $G$-finite classifying space for proper $G$-actions $\underline{E}G$. Then $\bar{\mathfrak{c}}(\underline{E}G)$ is an amenable $(h\underline{E}G \rtimes^{\mathcal{G}} G)$-$C^*$-algebra.*

*Proof.* We have the short exact sequence $0 \to C_0(\underline{E}G) \otimes \mathcal{K} \to \bar{\mathfrak{c}}(\underline{E}G) \to \mathfrak{c}(\underline{E}G) \to 0$. Because we assume $\underline{E}G$ to be $G$-finite, we have a coarse equivalence $G \to \underline{E}G$ which is $G$-equivariant by identifying $G$ with an orbit of a point in $\underline{E}G$; this gives an isomorphism of $G$-$C^*$-algebras $\mathfrak{c}(G) \to \mathfrak{c}(\underline{E}G)$. Hence $G$ being bi-exact is equivalent to $\mathfrak{c}(\underline{E}G)$ being an amenable $G$-$C^*$-algebra by Proposition 3.8.

Since the $G$-action on $C_0(\underline{E}G)$ is always amenable (by an adaption of the proof of [AD02, Example 2.7(3)]), we get with [BEW24, Proposition 3.23] that $\mathfrak{c}(\underline{E}G)$ being an amenable $G$-$C^*$-algebra is equivalent to $\bar{\mathfrak{c}}(\underline{E}G)$ being one.

The argument for the implication (d)⇒(a) of Proposition 3.6 also works for $\underline{E}G$ and shows that then $G$ acts amenably on $h\underline{E}G$. From this we can conclude the claim by using the variant of Lemma 3.5 for $\underline{E}G$. (It is exactly the same proof, but now with the words "strictly continuous" necessarily included.) □

**Proposition 4.3.** *Let $G$ be a countable, discrete and bi-exact group and we assume that it admits a $G$-finite classifying space for proper $G$-actions $\underline{EG}$. Then we have a split short exact sequence*

$$0 \to K_{*+1}(\bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG} \rtimes_{\mathrm{red}} G) \to K_*(C^*_{\mathrm{red}}(G)) \to K_*(\bar{\mathfrak{c}}(\underline{EG}) \rtimes_{\mathrm{red}} G) \to 0. \tag{4-4}$$

*Further, the Baum–Connes conjecture for trivial coefficients $\mathbb{C}$ and coefficients $\bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG}$ are equivalent to each other for $G$ and imply the isomorphism*

$$K_*(C^*_{\mathrm{red}}(G)) \xrightarrow{\cong} K_*(\bar{\mathfrak{c}}(\underline{EG}) \rtimes_{\mathrm{red}} G), \tag{4-5}$$

*which is induced from the inclusion of $\mathcal{K}$ as the constant functions in $\bar{\mathfrak{c}}(\underline{EG})$.*

*Proof.* We have a short exact sequence of $G$-$C^*$-algebras

$$0 \to \bar{\mathfrak{c}}(\underline{EG}) \to \bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG} \to \mathcal{Q} \to 0, \tag{4-6}$$

where $\mathcal{Q}$ is the Calkin algebra of a separable, $\infty$-dimensional Hilbert space, and $\mathcal{Q}$ is equipped with the trivial $G$-action. We consider the resulting commutative diagram with exact rows and where the vertical maps are the respective assembly maps:

$$\begin{array}{ccccccc}
\cdots \xrightarrow{\partial} & K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}(\underline{EG})) & \longrightarrow & K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG}) & \longrightarrow & K_*^{\mathrm{top}}(G; \mathcal{Q}) & \xrightarrow{\partial} \cdots \\
& \downarrow & & \downarrow & & \downarrow & \\
\cdots \xrightarrow{\partial} & K_*(\bar{\mathfrak{c}}(\underline{EG}) \rtimes_{\mathrm{red}} G) & \longrightarrow & K_*(\bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG} \rtimes_{\mathrm{red}} G) & \longrightarrow & K_*(\mathcal{Q} \rtimes_{\mathrm{red}} G) & \xrightarrow{\partial} \cdots
\end{array}$$

By Proposition 4.1 we have $K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG}) \cong 0$ and therefore the boundary maps in the top row are isomorphisms. Since we assume that $G$ is bi-exact, $\bar{\mathfrak{c}}(\underline{EG})$ is an amenable $(h\underline{EG} \rtimes^{\mathcal{G}} G)$-$C^*$-algebra by Lemma 4.2 and we conclude that the assembly map for it is an isomorphism [CEOO03, Corollary 0.4]. We therefore arrive at the diagram

$$\begin{array}{ccccccc}
\cdots \xrightarrow{\partial,\cong} & K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}(\underline{EG})) & \longrightarrow & 0 & \longrightarrow & K_*^{\mathrm{top}}(G; \mathcal{Q}) & \xrightarrow{\partial,\cong} \cdots \\
& \downarrow \cong & & \downarrow & & \downarrow & \\
\cdots \xrightarrow{\partial} & K_*(\bar{\mathfrak{c}}(\underline{EG}) \rtimes_{\mathrm{red}} G) & \longrightarrow & K_*(\bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG} \rtimes_{\mathrm{red}} G) & \longrightarrow & K_*(\mathcal{Q} \rtimes_{\mathrm{red}} G) & \xrightarrow{\partial} \cdots
\end{array} \tag{4-7}$$

showing that $K_*(\bar{\mathfrak{c}}(\underline{EG}) \rtimes_{\mathrm{red}} G) \to K_*(\bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG} \rtimes_{\mathrm{red}} G)$ is the zero map. The isomorphisms in this diagram provide the split for the resulting short exact sequence

$$0 \to K_{*+1}(\bar{\mathfrak{c}}^{\mathrm{red}}\underline{EG} \rtimes_{\mathrm{red}} G) \to K_{*+1}(\mathcal{Q} \rtimes_{\mathrm{red}} G) \xrightarrow{\partial} K_*(\bar{\mathfrak{c}}(\underline{EG}) \rtimes_{\mathrm{red}} G) \to 0\,.$$

The proof of (4-4) is finished with the isomorphism $K_{*+1}(\mathcal{Q} \rtimes_{\mathrm{red}} G) \cong K_*(C^*_{\mathrm{red}}(G))$ which is the boundary map in the long exact sequence induced from the short exact sequence of $G$-$C^*$-algebras $0 \to \mathcal{K} \to \mathcal{B} \to \mathcal{Q} \to 0$ with the trivial $G$-action. To see that (4-5) is induced from the inclusion of $\mathcal{K}$ as the constant functions in $\bar{\mathfrak{c}}(\underline{EG})$ we refer to the proof of Proposition 5.5 in [EM06a].

The final statement of the proposition about the Baum–Connes conjecture follows from Diagram (4-7). □

We can now prove an equivariant version of [Wil13, Proposition 4.5] and an alternative version of [Wil09, Proposition 6.2.1]. To be able to state it we need the canonical inclusion $i : C(\partial G) \otimes \mathcal{K} \to \mathfrak{c}G$ from [EM06a, Proposition 3.6] for boundaries at infinity $\partial G$ of suitable compactifications of $G$. Each such boundary will arise as the boundary of a compactification $\bar{P}$ of a suitable model $P$ for $\underline{EG}$.

**Proposition 4.4.** *Let $G$ be a countable, discrete group admitting a $G$-finite model $P$ for its classifying space for proper $G$-actions $\underline{EG}$, subject to the condition that $P$ admits a metrizable compactification $\bar{P}$ such that*

(a) *$\bar{P}$ is Higson-dominated,*

(b) *$\bar{P}$ is $H$-equivariantly contractible for every finite subgroup $H < G$, and*

(c) *the $G$-action on $P$ extends to an amenable action on $\bar{P}$.*

*Then the inclusion $i : C(\partial G) \otimes \mathcal{K} \to \mathfrak{c}G$, where $\partial G$ is the boundary of $P$ inside $\bar{P}$, induces an isomorphism*

$$K_*(C(\partial G) \rtimes_{\text{red}} G) \cong K_*(\mathfrak{c}G \rtimes_{\text{red}} G). \tag{4-8}$$

*Proof.* There is a $G$-equivariant quasi-isometry $G \to P$ (canonical up to closeness) inducing $G$-equivariant $C^*$-isomorphisms $\mathfrak{c}G \to \mathfrak{c}P$ and $\mathfrak{c}^{\text{red}}G \to \mathfrak{c}^{\text{red}}P$. We conclude

$$K_*(\mathfrak{c}G \rtimes_{\text{red}} G) \cong K_*(\mathfrak{c}P \rtimes_{\text{red}} G) \quad \text{and} \quad K_*(\mathfrak{c}^{\text{red}}G \rtimes_{\text{red}} G) \cong K_*(\mathfrak{c}^{\text{red}}P \rtimes_{\text{red}} G).$$

In the following we will use that since $G$ acts amenably on a Higson-dominated compactification, it also acts amenably on its Higson compactification and hence $G$ is bi-exact. This implies further that $G$ is exact.

We consider the commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & (C_0(P)\otimes\mathcal{K}) \rtimes_{\text{red}} G & \longrightarrow & \bar{\mathfrak{c}}P \rtimes_{\text{red}} G & \longrightarrow & \mathfrak{c}P \rtimes_{\text{red}} G & \longrightarrow 0 \\
& \| & & \uparrow & & \uparrow{\scriptstyle i' \rtimes_{\text{red}} G} & \\
0 \longrightarrow & (C_0(P)\otimes\mathcal{K}) \rtimes_{\text{red}} G & \longrightarrow & (C(\bar{P})\otimes\mathcal{K}) \rtimes_{\text{red}} G & \longrightarrow & (C(\partial G)\otimes\mathcal{K}) \rtimes_{\text{red}} G & \longrightarrow 0
\end{array}$$

whose rows are exact since $G$ is an exact group. The map $i'$ is the map $i$ composed with the $G$-equivariant $C^*$-isomorphism $\mathfrak{c}G \to \mathfrak{c}P$. From this diagram and the induced transformation of the corresponding long exact sequences in $K$-theory, together with the previously noted isomorphism $K_*(\mathfrak{c}G \rtimes_{\text{red}} G) \cong K_*(\mathfrak{c}P \rtimes_{\text{red}} G)$, we see that to prove (4-8) it suffices to prove that the middle vertical map in the above diagram induces isomorphisms in $K$-theory.

Consider the commutative diagram whose horizontal maps are the respective assembly maps and the vertical maps are induced by $\bar{i}' : C(\bar{P}) \otimes \mathcal{K} \to \bar{\mathfrak{c}} P$:

$$\begin{array}{ccc} K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}P) & \xrightarrow{\cong} & K_*(\bar{\mathfrak{c}}P \rtimes_{\mathrm{red}} G) \\ \uparrow & & \uparrow \\ K_*^{\mathrm{top}}(G; C(\bar{P}) \otimes \mathcal{K}) & \xrightarrow{\cong} & K_*((C(\bar{P}) \otimes \mathcal{K}) \rtimes_{\mathrm{red}} G) \end{array}$$

Similarly as in Lemma 4.2, because $G$ is bi-exact we know that $\bar{\mathfrak{c}}P$ is an amenable $(hP \rtimes G)$-$C^*$-algebra and we conclude that the assembly map for it, which is the top horizontal map in the above diagram, is an isomorphism [CEOO03, Corollary 0.4]. By the same argument, because we assume that $G$ acts amenably on $\bar{P}$, we conclude that the lower horizontal map is an isomorphism [Tu99].

We have a short exact sequence of $G$-$C^*$-algebras

$$0 \to \bar{\mathfrak{c}}P \to \bar{\mathfrak{c}}^{\mathrm{red}} P \to \mathcal{Q} \to 0\,, \tag{4-9}$$

where $\mathcal{Q}$ is the Calkin algebra of a separable, $\infty$-dimensional Hilbert space and $\mathcal{Q}$ is equipped with the trivial $G$-action. We consider the resulting long exact sequence:

$$\cdots \xrightarrow{\partial} K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}P) \to K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}^{\mathrm{red}} P) \to K_*^{\mathrm{top}}(G; \mathcal{Q}) \xrightarrow{\partial} \cdots$$

Proposition 4.1 implies $K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}^{\mathrm{red}} P) \cong 0$ and we conclude that the boundary map is an isomorphism

$$K_{*+1}^{\mathrm{top}}(G; \mathcal{Q}) \xrightarrow{\partial, \cong} K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}P). \tag{4-10}$$

On the other hand, since $\bar{P}$ is $H$-equivariantly contractible for every finite subgroup $H$ of $G$ we conclude by [Hig00, Proposition 3.7] that we have the isomorphism

$$K_*^{\mathrm{top}}(G; \mathcal{K}) \xrightarrow{\cong} K_*^{\mathrm{top}}(G; C(\bar{P}) \otimes \mathcal{K}) \tag{4-11}$$

induced from the inclusion of $\mathbb{C}$ into $C(\bar{P})$ as constant functions. We check (as in [EM06a, proof of Proposition 5.5]) that the diagram

$$\begin{array}{ccc} K_{*+1}^{\mathrm{top}}(G; \mathcal{Q}) & \xrightarrow{\partial, \cong} & K_*^{\mathrm{top}}(G; \bar{\mathfrak{c}}P) \\ {\scriptstyle \partial, \cong}\downarrow & & \uparrow \\ K_*^{\mathrm{top}}(G; \mathcal{K}) & \xrightarrow{\cong} & K_*^{\mathrm{top}}(G; C(\bar{P}) \otimes \mathcal{K}) \end{array}$$

with horizontal maps (4-10) and (4-11) commutes, where the left vertical map is the boundary map induced from the short exact sequence $0 \to \mathcal{K} \to \mathcal{B} \to \mathcal{Q} \to 0$ and the right vertical map is the one where we have to show that it is an isomorphism. But this follows from the diagram. □

Note that sometimes the $K$-theory groups of $C(\partial G) \rtimes_{\text{red}} G$ are computable, for example with the techniques from [EM06b]; cf. [Wil09, Remark 6.2.4].

**Example 4.5.** Let $G$ be a Gromov hyperbolic group. We collect in the following the references which verify the assumptions of Proposition 4.4 for these groups.

Let $P_d(G)$ be the Rips complex at scale $d \geq 1$ of the group $G$ (equipped with any word metric) and $P$ be its second barycentric subdivision. It is known that for large $d$ this is a model for the classifying space $\underline{EG}$ for proper actions of $G$ [MS02]. There is a $G$-equivariant quasi-isometry $G \to P$ (canonical up to closeness) and hence $P$ is also hyperbolic and its Gromov boundary is canonically equivariantly homeomorphic to the boundary $\partial G$ of $G$.

It is known that the Gromov compactification $\bar{P}$ is $H$-equivariantly contractible for every finite subgroup $H < G$,[22] and it is furthermore known that $G$ acts amenably on $\partial G$ ([AD02, Example 2.7.4], originally proved in [Ada94]) and therefore also on $\bar{P}$.[23] By [Roe91, Corollary 2.2] we know that the Gromov compactification is Higson dominated.

Finally, let us mention that hyperbolic groups satisfy the Baum–Connes conjecture for all coefficients [Laf12] and hence the isomorphism (4-5) holds true.

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[22] This is a general result and follows from the Whitehead theorem for families [Lüc05, Theorem 1.6]: Apply it to $Y = \bar{P}$, $Z = \text{pt}$, $H$ as the group, and all its subgroups as the family $\mathcal{F}$. In point (i) of its statement we choose $X = \bar{P}$ and get that the set $[X, Y]^H$ of $H$-homotopy classes of $H$-maps consists of a single element. But this set contains both the identity and the map to a single $H$-fixed point.

[23] $G$ acts amenably on $\partial G$ and therefore also on $\partial P$ since they are equivariantly homeomorphic. Since $P$ is a model for $\underline{EG}$, the $G$-action on it is amenable (adapt the proof of [AD02, Ex. 2.7(3)]); therefore we conclude that the $G$-action on $\bar{P} = P \sqcup \partial P$ is amenable [BEW24, Proposition 3.23].

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# UNIFORM FIRST ORDER INTERPRETATION OF THE SECOND ORDER THEORY OF COUNTABLE GROUPS OF HOMEOMORPHISMS

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**We show that the first order theory of the homeomorphism group of a compact manifold interprets the full second order theory of countable groups of homeomorphisms of the manifold. The interpretation is uniform across manifolds of bounded dimension. As a consequence, many classical problems in group theory and geometry (e.g., the linearity of mapping classes of compact 2-manifolds) are encoded as elementary properties of homeomorphism groups of manifolds. Furthermore, the homeomorphism group uniformly interprets the Borel and projective hierarchies of the homeomorphism group, which gives a characterization of definable subsets of the homeomorphism group. Finally, we prove analogues of Rice's theorem from computability theory for homeomorphism groups of manifolds. As a consequence, it follows that the collection of sentences that isolate the homeomorphism group of a particular manifold, or that isolate the homeomorphism groups of manifolds in general, is not definable in second order arithmetic, and that membership of particular sentences in these collections cannot be proved in ZFC.**

## 1. Introduction

Let $M$ be a compact, connected, topological manifold of positive dimension. In this paper, we investigate countable subgroups of the group $\mathrm{Homeo}(M)$ from the point of view of the first order logic of groups, thus continuing a research program initiated together with Kim [28]. There, we proved that for each compact manifold $M$, there is a sentence in the language of groups which isolates the group $\mathrm{Homeo}(M)$; that is, there exists a sentence in the language of group theory that is true in the group of homeomorphisms of an arbitrary compact manifold $N$ if and only if $N$ is homeomorphic to $M$.

Our overarching theme is that the first order theory of $\mathrm{Homeo}(M)$ is expressive enough to interpret arbitrary sequences of elements of $\mathrm{Homeo}(M)$. More concretely: on the one hand, the question of determining the isomorphism type of the subgroup

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of Homeo($M$) generated by a finite list of elements is difficult, and in general is intractable. On the other hand, it can be shown by general Baire category arguments (Proposition 4.5 in [17], cf. Chapter 3 in [27]) that generically, pairs of homeomorphisms will generate nonabelian free groups. Even in the case of one-dimensional manifolds, general finitely generated groups of homeomorphisms (and even diffeomorphisms) can be extremely complicated; cf. [2; 23; 26; 30; 24; 25; 6; 29].

Since Homeo($M$) can interpret arbitrary sequences of elements in the underlying group, the first order theory of Homeo($M$) is expressive enough to decide if a countable subgroup if isomorphic to a given finitely presented group; as another example, by identifying tuples of homeomorphisms which generate a particular isomorphism type of groups (e.g., a free group of rank two), we obtain an upper bound on the complexity of the set of tuples which generate that type of group. Thus, the elementary theory of the homeomorphism group Homeo($M$) encodes a substantial amount of the algebraic structure of this group.

**1.1. *Main results.*** All results stated in this section hold for arbitrary compact, connected manifolds; we assume connectedness mostly for convenience. There is a dependence of the formulae on the dimension of the underlying manifold, but otherwise all formulae are uniform across manifolds of fixed dimension. Throughout, we let

$$\mathrm{Homeo}_0(M) \leq \mathscr{H} \leq \mathrm{Homeo}(M),$$

where here $\mathrm{Homeo}_0(M)$ denotes the identity component of Homeo($M$). Unless otherwise noted, formulae are uniform in $\mathscr{H}$, which is to say they do not depend on which subgroup between $\mathrm{Homeo}_0(M)$ and Homeo($M$) we consider. We suppress $M$ from the notation $\mathscr{H}$ since it will not cause confusion.

To begin, $\mathscr{H}$ is viewed as a structure in the language of group theory. The content of the paper [28] is that the language of group theory in $\mathscr{H}$ admits a conservative expansion wherein many more things can be interpreted: specifically, the sorts of regular open sets RO($M$) in $M$, the natural numbers $\mathbb{N}$, the real numbers $\mathbb{R}$, and points in $M$ can be parameter-free interpreted. Moreover, natural predicates, both internal to these sorts (e.g., arithmetic) and relating these sorts to each other, are uniformly definable; see Theorem 2.3 below.

The main result of this paper is the conservative interpretation of a sequence of new sorts in $\mathscr{H}$, which are written $\mathrm{HS}_i(M)$ for $i \geq 0$. The meanings of these sorts are as follows:

- The elements of $\mathrm{HS}_0(M)$ are in canonical correspondence with homeomorphisms of $M$.

- For $i \geq 1$, the elements of $\mathrm{HS}_i(M)$ are in canonical correspondence with sequences of elements in $\mathrm{HS}_{i-1}(M)$.
- These sorts admit parameter-free definable predicates for manipulating them and for relating them to each other and to the home sort.

We call $\mathrm{HS}(M)$, the union of the sorts $\{\mathrm{HS}_i(M)\}_{i\in\mathbb{N}}$, *hereditarily sequential subsets of* $\mathrm{Homeo}(M)$; this is by analogy to (and by generalization of) *hereditarily finite sets* (cf. Section 3 in [22], for instance).

Note that for $n \geq 2$, elements of $\mathrm{HS}_n(M)$ are not really subsets of $\mathrm{Homeo}(M)$. One would be justified in calling an interpretation of $\mathrm{HS}_1(M)$ *countable second order logic*, since then one can quantify freely over countable subsets of $\mathrm{Homeo}(M)$. Then, for $n \geq 2$ one would be justified in calling an interpretation of $\mathrm{HS}_n(M)$ *countable* $(n+1)^{st}$ *order logic*. The distinction between countable second order logic and countable higher order logics collapses in our situation; this is because our interpretation of countable second order logic (i.e., $\mathrm{HS}_1(M)$) encodes countable sequences via fixed length definable tuples, up to a definable equivalence relation. Thus for all $n \geq 2$, an interpretation of $\mathrm{HS}_n(M)$ would consist of sequences of fixed length finite tuples, which themselves would be encoded by fixed length finite tuples in $\mathscr{H}$.

Hereditarily sequential sets subsume hereditarily finite sets via a straightforward padding construction.

**Theorem 1.1.** *Let $D \geq 1$ be a natural number, and let*

$$\mathrm{Homeo}_0(M) \leq \mathscr{H} \leq \mathrm{Homeo}(M).$$

*Then there is a conservative expansion of the language of group theory and a uniform interpretation of the union of the sorts* $\mathrm{HS}(M)$ *in $\mathscr{H}$ that is valid for all manifolds $M$ with* $\dim M \leq D$. *The elements in the sort* $\mathrm{HS}_0(M)$ *canonically correspond to elements of* $\mathrm{Homeo}(M)$.

*Moreover, the following predicates are definable without parameters:*

(1) *For each $i$ and each $j \in \mathbb{N}$, the $j$-th element $s(j)$ of a sequence $s \in$* $\mathrm{HS}_i(M)$;

(2) *For each $i \geq 0$, a membership predicate*

$$\in_i \subseteq \mathrm{HS}_i(M) \times \mathrm{HS}_{i+1}(M)$$

*defined recursively by:*

(a) *$\Gamma \in_0 s$ if and only if there is a $j$ such that $\Gamma = s(j)$.*

(b) *$s \in_i t$ if and only if there is a $j$ such that $s = t(j)$.*

(3) *Memberwise group multiplication within* $\mathrm{HS}_1(M)$, *i.e., a predicate* $\mathrm{mult}_{i,j,k}(\sigma)$ *such that for all sequences $s \in$* $\mathrm{HS}_1(M)$, *we have* $\mathrm{mult}_{i,j,k}(s)$ *if and only if $s(i) \cdot s(j) = s(k)$.*

(4) *Membership of an element in* $\mathrm{HS}_0(M)$ *in* $\mathscr{H}$, *i.e., a predicate*

$$R \subseteq \mathscr{H} \times \mathrm{HS}_0(M)$$

*such that* $(g, \Gamma) \in R$ *if and only if* $\Gamma$ *canonically encodes* $g$.

(5) *The extended support* $\mathrm{supp}^e\, f$ *of an element* $f \in \mathrm{HS}_0(M)$, *i.e., a predicate*

$$\mathrm{supp}^e \subseteq \mathrm{HS}_0(M) \times \mathrm{RO}(M)$$

*such that* $(\Gamma, U) \in \mathrm{supp}^e$ *if and only if the homeomorphism encoded by* $\Gamma$ *has extended support equal to* $U$.

We will sometimes abuse notation and suppress the subscript in $\in_i$ when no confusion can occur. We note that Item (4) is crucial and what makes Theorem 1.1 not a consequence of [28]. Moreover, Item (4) will allow us to characterize definable sets in $\mathscr{H}$ below (see Theorem 1.5).

The key step in interpreting $\mathrm{HS}(M)$ yields the following, which is of independent interest. See Lemma 3.1.

**Proposition 1.2.** *For manifolds of fixed dimension, the group* $\mathscr{H}$ *admits a uniform, parameter-free interpretation of the sort* $\mathrm{seq}(M)$ *of countable sequences of points in* $M$, *which is uniform for all manifolds of dimension* $d$. *Moreover, the predicate* $p \in \sigma$ *expressing membership of a point* $p$ *in a sequence* $\sigma$, *and the predicate* $\sigma(i) = p$ *expressing that* $p$ *is the* $i$*-th term of* $\sigma$, *are both parameter-free definable.*

The interpretability of hereditarily sequential sets in $\mathscr{H}$ has a large number of consequences with regard to definability in $\mathscr{H}$.

**Proposition 1.3.** *The class* $\mathscr{K}$ *of subgroups of* $\mathrm{Homeo}(M)$ *that contain* $\mathrm{Homeo}_0(M)$ *is uniformly interpretable (with parameters) in* $\mathscr{H}$, *as definable subsets of the sort* $\mathrm{HS}_0(M)$. *Among the elements of* $\mathscr{K}$ *are three canonical parameter-free interpretable subgroups, namely*

$$\{\mathrm{Homeo}_0(M), \mathrm{Homeo}(M), \mathscr{H}\}.$$

Combining Theorem 1.1 and Proposition 1.3, we will be able to interpret hereditarily sequential sets in other groups lying between $\mathrm{Homeo}_0(M)$ and $\mathrm{Homeo}(M)$, and in various parameter-free interpretable quotients such as the topological mapping class group $\mathrm{Mod}(M) := \mathrm{Homeo}(M)/\mathrm{Homeo}_0(M)$.

**1.2.** ***Group theoretic consequences of the main results.*** Theorem 1.1 immediately implies that within the first order theory of $\mathscr{H}$, we have unfettered access to the full second order theory of countable subgroups of $\mathrm{Homeo}(M)$; in particular, we may freely quantify over countable subgroups, as well as their subgroups, and homomorphisms between them. Since $\mathscr{H}$ also interprets second order arithmetic, we may uniformly interpret combinatorial (and even analytic) group theory within the first order theory of $\mathscr{H}$; that is, we can encode arbitrary recursively presented groups within second order arithmetic, and we may also manipulate them (i.e., test for nontriviality of words, solve the conjugacy problem, test for isomorphism, determining if a subgroup has finite index, measure the index of a finite index subgroup, test for amenability, test Kazhdan's property (T), etc.; the reader is directed to [46] for an extensive discussion of mathematics that can be developed within second order arithmetic). Observe that an abstract countable group will generally have to be specified with parameters, in the form of a sequence of natural numbers.

For abstract finitely generated groups, the standard concepts from geometric group theory can also be interpreted, such as the Cayley graph with respect to a finite generating set, growth, hyperbolicity, and quasi-isometry.

Below, we give a (non-exhaustive) list some concepts that can be encoded within the elementary theory of $\mathscr{H}$.

**Theorem 1.4.** *The following group-theoretic sorts and predicates are parameter-free interpretable in $\mathscr{H}$, uniformly for all compact manifolds $M$ of fixed dimension*:

(1) *countable subgroups of* $\mathrm{Homeo}(M)$ *and their full second order theory*;

(2) *the topological mapping class group* $\mathrm{Mod}(M)$ *of $M$, i.e., the group*

$$\pi_0(\mathrm{Homeo}(M)) = \mathrm{Homeo}(M)/\,\mathrm{Homeo}_0(M),$$

*and the full second order theory of* $\mathrm{Mod}(M)$;

(3) *for a sequence $\underline{g}$ of homeomorphisms or mapping classes, the membership predicate for the subgroup $\langle\underline{g}\rangle$*;

(4) *finite generation and finite presentability of arbitrary countable subgroups of* $\mathrm{Homeo}(M)$ *or* $\mathrm{Mod}(M)$;

(5) *residual finiteness of arbitrary countable subgroups of* $\mathrm{Homeo}(M)$ *and* $\mathrm{Mod}(M)$;

(6) *linearity of arbitrary countable subgroups of* $\mathrm{Homeo}(M)$ *and* $\mathrm{Mod}(M)$, *i.e., a predicate which holds if and only if the corresponding group is linear over a field of characteristic zero*;

(7) *a predicate expressing isomorphism with a particular group that is parameter-free definable in second order arithmetic* (*e.g., isomorphism with some finite index subgroup of* $\mathrm{SL}_n(\mathbb{Z})$);

(8) *for a finitely generated subgroup of* $\mathrm{Homeo}(M)$ *or* $\mathrm{Mod}(M)$, *a predicate expressing whether this group is amenable of has Kazhdan's Property* (T).

Thus, the first order theory of $\mathscr{H}$ encodes many well-known conjectures as elementary properties of homeomorphism groups. These include the linearity of mapping class groups of compact 2-manifolds (see [13] for a general reference, and

Question 1.1 of [35]), property (T) for mapping class groups of compact 2-manifolds, finite presentability of the Torelli group of a compact 2-manifold (see [43; 36], and especially Section 5 of [35]) the existence of an infinite, discrete, property (T) group of homeomorphisms of the circle (see [11; 39; 1], and especially Question 2 of [40]), the amenability of Thompson's group $F$ [8; 7], and many cases of the Zimmer program (i.e., faithful continuous actions of finite index subgroups of lattices in semisimple Lie groups on compact manifolds [14; 15; 4; 5]).

**1.3.** ***Descriptive set theory.*** Much of the foregoing discussion treats $\mathrm{Homeo}(M)$ as a discrete group. We wish to observe further that the first order theory of $\mathscr{H}$ recovers the topology of $\mathrm{Homeo}(M)$, and in fact the full projective hierarchy of subsets of $\mathrm{Homeo}(M)$. More precisely:

**Theorem 1.5.** *The following sorts are uniformly interpretable in $\mathscr{H}$, viewed as a subset of* $\mathrm{HS}_0(M)$, *uniformly in manifolds of fixed dimension*:

(1) *open and closed sets in* $\mathrm{Homeo}(M)$;

(2) *Borel sets in* $\mathrm{Homeo}(M)$, *and the full Borel hierarchy of* $\mathrm{Homeo}(M)$;

(3) *the projective hierarchy in* $\mathrm{Homeo}(M)$.

*The membership predicate $\in$ is parameter-free interpretable for sets in these sorts.*

*Moreover, the topology of $\mathscr{H}$, as well as the Borel hierarchy and projective hierarchy of $\mathscr{H}$ are all uniformly definable among manifolds of bounded dimension.*

As a consequence, we will obtain the following general fact about definable subsets of $\mathrm{Homeo}(M)$:

**Theorem 1.6.** *A set is definable* (*with parameters*) *in $\mathscr{H}$ if and only if it lies in the projective hierarchy.*

**1.4.** ***Undefinability and independence.*** As is implicit from the uniform parameter-free interpretation of second order arithmetic in $\mathscr{H}$ as produced in [28], not only is the first order theory of $\mathscr{H}$ (and of $\mathrm{Homeo}(M)$ in particular) undecidable, but in fact there are elementary properties of homeomorphism groups of manifolds whose validity is independent of ZFC. A question therefore is whether or not there are "natural" first order group theoretic statements in $\mathscr{H}$ that are independent of ZFC, and this is unclear to the authors.

There are also many natural undefinable sets in arithmetic which are directly related to compact manifolds and their homeomorphism groups, which we record here. Manifolds and their homeomorphism groups can be formalized in second order arithmetic; however, there is some sense in which the manifold homeomorphism group recognition problem is at least as complicated as full true second order arithmetic, which we now make precise.

Choosing a numbering of the language of groups, we obtain a Gödel numbering of sentences in group theory. For a fixed compact manifold $M$, one can consider the set of sentences in group theory (viewed as a subset of $\mathbb{N}$ via their Gödel numbers) which isolate $\mathrm{Homeo}(M)$. Similarly, one may consider the set of sentences in group theory which isolate some isomorphism type of compact manifold homeomorphism group. It turns out that neither of these sets is definable in arithmetic. For a sentence $\psi$, we write $\#\psi$ for its Gödel number with respect to a fixed numbering of the language.

**Theorem 1.7.** *Let $M$ be a fixed compact manifold and let $N$ be an arbitrary compact manifold.*

(1) *The set*

$$\mathrm{Sent}_M := \{\#\psi \mid (\mathrm{Homeo}(N) \models \psi) \longleftrightarrow (M \cong N)\}$$

*is not definable in second order arithmetic.*

(2) *The set*

$$\mathrm{Sent} := \{\#\psi \mid \psi \textit{ isolates } \mathrm{Homeo}(N) \textit{ for some compact manifold } N\}$$

*is not definable in second order arithmetic.*

*In particular, these sets are not decidable.*

In Theorem 1.7, the group Homeo can be replaced by any group lying between $\mathrm{Homeo}_0$ and Homeo. We will show in Section 6 that membership of Gödel numbers in $\mathrm{Sent}_M$ or Sent cannot be proved within ZFC.

More generally than Theorem 1.7, we will prove that for any class $\mathscr{M}$ of compact manifold homeomorphism groups which is isolated by a single sentence, the set of Gödel numbers of sentences isolating $\mathscr{M}$ is undefinable in second order arithmetic; this gives an analogue of Rice's theorem (i.e., nontrivial classes of partially recursive functions are not computable) for homeomorphism groups of manifolds. In fact, we will prove that if $\mathscr{F}$ consists of nonempty sets of homeomorphism groups of compact manifolds which are isolated by first order sentences, and if $A \subseteq F$ is proper, then the set of Gödel numbers of sentences isolating elements of $A$ is not definable in second order arithmetic. See Theorem 6.1 and Theorem 6.2 for precise statements.

**1.5.** ***Organization of the paper.*** In Section 2, we gather preliminary material about topological manifolds and the first order theory of homeomorphism groups of manifolds. Section 3 proves Theorem 1.1, the main result of the paper. Section 4 interprets mapping class groups of manifolds as well as intermediate subgroups lying between $\mathrm{Homeo}_0$ and Homeo of manifolds, and discusses Theorem 1.4. Section 5 discusses descriptive set theory and the projective hierarchy in $\mathrm{Homeo}(M)$. Section 6 proves Theorem 1.7 and the analogues of Rice's theorem.

Throughout, we have tried to balance mathematical precision with clarity. To give completely precise and explicit formulae is possible, though extremely unwieldy and unlikely to yield deeper insight. Thus, we have often avoided giving explicit formulae, either explaining how to obtain them in English with enough precision that the formulae could be produced if desired, or we have avoided them entirely when certain predicates are obviously definable in second order arithmetic or in the countable second order theory of a group.

## 2. Background

We first gather some preliminary results. Throughout, we will always assume that all manifolds are compact, connected, and second countable.

**2.1.** ***Results from geometric topology of manifolds.*** We will appeal to the following fact about compact topological manifolds. We write $B(i) \subset \mathbb{R}^d$ for the closed ball of radius $i$ about the origin. We write $H(i) \subset \mathbb{R}^d_{\geq 0}$ for the half-ball of radius $i$ about the origin in the half-space $\mathbb{R}^d_{\geq 0}$. That is, $H(i) = B(i) \cap \mathbb{R}^d_{\geq 0}$. A *collared ball* in a $d$-dimensional manifold $M$ is a map

$$B(1) \to M$$

which is a homeomorphism onto its image, and which extends to a homeomorphism of $B(2)$ onto its image, and a *collared half-ball* in a manifold with boundary is defined analogously in the usual sense, so that the image of the origin in $\mathbb{R}^d$ lands in the boundary $\partial M \subseteq M$ and the intersection of the image of $H(i)$ with $\partial M$ is a collared open ball in $\partial M$.

An open set in $M$ is *regular* if it is equal to the interior of its closure. We will say that a regular open set $U$ is a *regular open collared ball* if it is the interior of a collared open ball. A *regular open collared half-ball* is a regular open set that is the interior of a collared half-ball. A regular open collared half-ball meets the boundary of $M$ in a regular open collared ball.

**Proposition 2.1** (see Chapter 3 in [10], Theorem IV.2 in [19], Theorem 3 in [41], Section 6.1 in [28]). *Let $M$ be a compact, connected manifold of dimension $d$. Then there exists a computable function $n(d)$ such that the following conclusions hold.*

(1) *If $M$ is a closed topological manifold then there exist $n(d)$ collections of disjoint collared balls $\{B_1, \ldots, B_{n(d)}\}$ such that*

$$M = \bigcup_{i=1}^{n(d)} B_i.$$

(2) *If $\partial M \neq \varnothing$ then the following conclusions hold.*

(a) *For every collar neighborhood* $U \supseteq \partial M$, *there exist collections of disjoint collared balls* $\{B_1, \ldots, B_{n(d)}\}$ *and collections of disjoint collared half-balls* $\{H_1, \ldots, H_{n(d-1)}\}$ *such that*

$$M \setminus U \subseteq \bigcup_{i=1}^{n(d)} B_i \quad \textit{and} \quad \operatorname{cl} U \subseteq \bigcup_{j=1}^{n(d-1)} H_j.$$

In Proposition 2.1, note that each $B_i$ and each $H_i$ is a (possibly disconnected) set, each component of which a collared ball or collared half-ball, respectively.

*Proof of Proposition 2.1.* We will assume that $M$ is closed; the argument for manifolds with boundary is a minor variation on the proof given here.

This essentially follows from the fact that $M$ can be embedded in $\mathbb{R}^{2d+1}$. Choose such an embedding, which by scaling we may assume lies in the unit cube $I^{2d+1}$. For any positive threshold $\epsilon > 0$, we may cover $I^{2d+1}$ by $2d+2$ collections of regular open sets $\{B_1, \ldots, B_{2d+2}\}$, each consisting of disjoint collared open Euclidean balls, with each ball having diameter at most $\epsilon$. Moreover, we may assume that any two components of any $B_i$ are separated by a distance that is uniformly bounded away from zero. These claims follow from standard constructions in Lebesgue covering dimension; see Chapter 3 in [10], Chapter 50 in [38].

Choose an atlas for $M$ such that for an arbitrary component $V$ of some $B_i$, we have that the intersection $V \cap M$ lies in a coordinate chart. This can be achieved by setting $\epsilon$ small enough with respect to a fixed atlas for $M$, as follows from the Lebesgue covering lemma.

Let $U \cong \mathbb{R}^d$ be such a coordinate chart of $M$ and let $B = B_i$ for some $i$. Then, $U \cap B$ is a collection of open sets which are separated by a definite distance $\delta > 0$ which is independent of $U$. For any component $V \in \pi_0(B)$ such that $V \cap M$ is entirely contained in $U$, we may cover $V \cap M$ with collared open balls (in $M$) which are contained in a $\delta/3$ neighborhood of the closure of $V$ in $M$. This covering may be further refined to be a covering by regular collared balls having order at most $d+2$; in particular, the closure of $V$ is covered by at most $d+2$ collections of regular open sets, whose components consist of disjoint collared open balls. Repeating this construction for each component $V \in \pi_0(B)$, we obtain a collection of $d+1$ regular open sets whose components are collared open balls that cover $B \cap M$. Allowing $B$ to range over $\{B_1, \ldots, B_{2d+2}\}$, we obtain $(d+1)(2d+2)$ regular open sets covering $M$, all of whose components are collared open balls, as desired. □

The importance of Proposition 2.1 is that many of the formulae we build in this paper will be uniform in the underlying manifold, provided that the dimension is bounded. This is reflected in the dependence of $n(d)$ on $d$. The proof of the following corollary is straightforward, and we omit it.

**Corollary 2.2.** *Let $M$ be a compact, connected manifold of dimension $d$, and let $n(d)$ be as in Proposition 2.1.*

(1) *If $M$ is closed then $M$ can be covered by $n(d)$ regular open collared balls.*

(2) *If $\partial M \neq \varnothing$ and if $N$ is a component of $\partial M$, then there is a tubular neighborhood of $N$ whose closure can be covered by $n(d-1)$ regular open collared half-balls. Moreover, for all tubular neighborhoods $U \supseteq \partial M$, we have $M \setminus U$ can be covered by $n(d)$ regular open collared balls.*

**2.2.** ***Results about the first order theory of homeomorphism groups of manifolds.*** The present paper builds on the results of the authors' joint paper with Kim [28]. In that paper, we investigated the first order theory of $\mathrm{Homeo}(M)$ for a compact manifold $M$, and in particular proved that each group $\mathrm{Homeo}(M)$ is quasi-finitely axiomatizable within the class of homeomorphism groups of manifolds.

The central result of this paper is the interpretation of $\mathrm{HS}(M)$, which does not follow from the paper [28]. However, we shall require tools which were developed in that paper in order to prove the results in this paper. We will briefly list the relevant results that we use here. In the following theorem, if $U \subseteq M$ is an open set and $G \leq \mathrm{Homeo}(M)$, then we write $G[U]$ for the *rigid stabilizer* of $U$, consisting of all elements of $G$ which are the identity outside of $U$.

The following result follows from the fact that $\mathscr{H}$ conservatively interprets, without parameters, a structure called AGAPE; see Section 3 of [28]. We have given more precise citations for most enumerated statements that refer to [28]. The statements below differ slightly from the way they are stated in [28] in order to better serve our purposes, though there is no difference in content.

**Theorem 2.3** (see [28]). *Let $M$ be a compact, connected, topological manifold of dimension at least one, and let*

$$\mathrm{Homeo}_0(M) \leq \mathscr{H} \leq \mathrm{Homeo}(M).$$

*Then there exists a sentence $\psi_M$ in the language of group theory such that for all compact manifolds $N$ and all subgroups*

$$\mathrm{Homeo}_0(N) \leq \mathscr{H}' \leq \mathrm{Homeo}(N),$$

*we have $\mathscr{H}' \models \psi_M$ if and only if $N \cong M$. Moreover, the following sorts and predicates are interpretable without parameters in $\mathscr{H}$, uniformly in $M$.*

(1) *The Boolean algebra* $\mathrm{RO}(M)$ *of regular open sets of $M$, equipped with an action of $\mathscr{H}$; that is, a predicate*

$$\mathrm{Act} \subseteq \mathscr{H} \times \mathrm{RO}(M) \times \mathrm{RO}(M)$$

*such that $(g, U, V) \in$* Act *if and only if $g(U) = V$ in $M$; the interpretation of*

$\mathrm{RO}(M)$ *is uniform for all manifolds, including noncompact ones.* (See Section 2.2 and Theorem 3.4.)

(2) *Predicates expressing connectedness of regular open sets, as well as that a regular open set $U$ is a connected component of a regular open set $V$.* (See Lemma 3.6 and Corollary 3.7.)

(3) *A predicate* $\mathrm{RCB} \subseteq \mathrm{RO}(M)$ *such that* $U \in \mathrm{RCB}$ *if and only if the closure of $U$ lies in a collared open ball in $M$.* (See Lemma 3.10.)

(4) *A predicate* $\mathrm{RCB}^\partial \subseteq \mathrm{RO}(M)$ *such that* $U \in \mathrm{RCB}^\partial$ *if and only if the closure of $U$ lies in a collared open half-ball in $M$.*

(5) *Second order arithmetic* $(\mathbb{N}, 0, +, \times, <, \subset)$, *and a definable predicate*

$$\# \subseteq \mathbb{N} \times \mathrm{RO}(M)$$

*such that $(n, U) \in \#$ if and only if $U$ has exactly $n$ components; moreover, if $\varnothing \neq U \in \mathrm{RO}(M)$, then second order arithmetic can be interpreted using only $U$ and $\mathscr{H}[U]$.* (See Section 4.)

(6) *Points $\mathscr{P}(M)$ of $M$, and more generally finite tuples $\mathscr{P}^{<\infty}(M)$ of points in $M$; moreover, a predicate $\in_{\mathscr{P}} \subseteq \mathscr{P}(M) \times \mathrm{RO}(M)$ such that $(p, U)$ lies in $\in_{\mathscr{P}}$ if and only if the statement $p \in U$ is true in $M$.* (See Section 5.)

(7) *Predicates expressing that a point of $M$ belongs to a union of two regular open sets, and that a point belongs to the closure of a regular open set.* (See Section 5.)

(8) *For each $n$, predicate expressing that a collection of $n$ regular open sets covers the closure of a regular open set $U$.*

(9) *Exponentiation, i.e., a definable function*

$$\exp : \mathscr{H} \times \mathbb{Z} \times M \to M$$

*with the property that*

$$\exp(g, n, p) = g^n(p) \quad \textit{in } M.$$

(See Section 5.3.)

(10) *A predicate which holds for a regular open set $U$ if and only if $U$ contains a tubular neighborhood of $\partial M$ in $M$.* (See Theorem 7.1.)

In view of Theorem 2.3, we will assume that $\mathscr{H}$ is implicitly equipped with the sorts of regular open sets of $M$, second order arithmetic, and points, as well as the relevant predicates listed in the theorem.

Some items in Theorem 2.3 require special comment. Item (3) was only formally proved for manifolds of dimension 2 or higher, though for manifolds of dimension one, the proof is even easier. By the characterization of connected sets in one-manifolds, it suffices to express that $U$ is contained in a connected regular open set $V$, and that there is a homeomorphism $h$ of $M$ such that $V \cap h(V) = \varnothing$.

Item (4) was not formally stated in [28], though it is not difficult to find such a formula. One expresses that a regular open set $U$ accumulates on a single component $N$ of $\partial M$, as is easily deduced from 3.4.3. One then requires the existence of a homeomorphism $h$ fixing each component of the boundary of $M$, which moves $U$ into an arbitrary half-ball in $N$; half-balls are interpreted explicitly in Section 7 of [28].

In item (6), a point $p \in M$ is encoded by an equivalence class of regular open sets, up to definable equivalence. If $U \subseteq M$ is a regular open set and $p \in U$ then there is a regular open set $V \subseteq U$ which encodes or *isolates* $p$; this is implicit in Section 5 of [28]. In particular, if $U$ is a regular open set with infinitely many components $\{U_i\}_{i\in\mathbb{N}}$ and if $p_i \in U_i$ is a point for each $i$, then the set of points $\bigcup_i p_i$ is encoded by a single regular open set $V$, which has the property that $V \subseteq \bigcup_i U_i$ and such that $V \cap U_i$ encodes the point $p_i$. We will abbreviate the predicate $\in_{\mathscr{P}}$ by $\in$.

Observe that the exponentiation function, together with the membership predicate relating $\mathscr{P}(M)$ to $\mathrm{RO}(M)$ allows one to express that $g^n(U) = V$ for group elements in $\mathscr{H}$, integer exponents, and pairs of regular open sets, since we may express that

$$\exp(g, n, p) \in V \leftrightarrow p \in U.$$

The sentence $\psi_M$ in Theorem 2.3 is said to *isolate* $M$ (or its homeomorphism group). We note that in [28], the proof of the content of Theorem 2.3 was given for manifolds of dimension at least two. This was done purely to simplify some of the arguments and shorten the exposition; the proofs themselves can easily be generalized to manifolds of dimension one.

We note that even though we will refer to collared balls and half-balls in the sequel, these are concepts in the metalanguage; we will never appeal to these objects directly in the formal language.

To make one further observation about the relationship between $\mathrm{Homeo}(M)$, its countable subgroups, and arithmetic, we remark the following: $\mathrm{Homeo}(M)$ clearly contains many countable subgroups that are definable in arithmetic, including cyclic groups and free groups. Some subgroups of $\mathrm{Homeo}(M)$ are in fact bi-interpretable with first order arithmetic, such as Thompson's groups $F$ and $T$ by [32]; it is not difficult to show that $F$ in fact arises as a subgroup of $\mathrm{Homeo}(M)$ for all positive dimensional manifolds. Most countable subgroups of $\mathrm{Homeo}(M)$ are not definable in first order arithmetic, simply because $\mathrm{Homeo}(M)$ interprets second order arithmetic. Indeed, then any countable elementary subgroup of $\mathrm{Homeo}(M)$ (which exists by the Löwenheim–Skolem theorem) has too complicated a theory to be interpretable in arithmetic. A more detailed discussion can be found in [31].

## 3. Hereditarily sequential sets of homeomorphisms of a manifold

Let $M$ and $\mathscr{H}$ be as above and fixed, and fix the notation $d \geq 1$ for the dimension of $M$. In this section, we prove Theorem 1.1; the uniformity of the interpretation among manifolds of a fixed dimension will be clear, and by taking disjunctions we obtain an interpretation that is valid for all manifolds of dimension bounded by a prescribed constant D. We prove the result in several steps.

**3.1.** ***Interpreting*** $\mathbf{HS_0}(M)$. We begin by interpreting the sort $\mathrm{HS}_0(M)$ in $\mathscr{H}$, and show that its members canonically correspond to elements of $\mathrm{Homeo}(M)$. This itself is done in several steps. The reader should remember for the duration of the proof that we are encoding a homeomorphism of $M$ by a proxy for its graph; the reader may pretend $M$ is closed on a first reading, for simplicity.

The scheme for finding parameter-free interpretations of new sorts in $\mathscr{H}$ will follow the basic scheme:

(1) Encode data describing the new sort within various sorts of topological data to which we have access in view of Theorem 2.3; oftentimes this data requires making choices, which amounts to an interpretation with parameters.

(2) Observe that the set of suitable parameters is itself parameter-free definable within the relevant sort.

(3) Eliminate parameters by quantifying over the relevant space of parameters.

The basic idea to interpret $\mathrm{HS}_0(M)$ is to fix a finite cover of $M$, move the charts in the cover to a single chart in $M$ (forming a finite set of “pages"), and then taking countably many disjoint copies of these pages. In each copy, we choose a point, which gives us the intermediate result of being able to interpret the sort of countable sequences of points in $M$; since points in $M$ are encoded by equivalence classes of regular open sets in $M$ wherein only the local structure of the open set near the point being encoded matters, we may encode the countable sequence of points by a single suitable equivalence class of regular open sets. By considering a sequence $\sigma$ of points in $M$, we may consider the odd and even index points in $\sigma$, thus obtaining a countable collection of points in $M \times M$. We then place definable conditions on such pairs to make sure the points occurring in each coordinate are dense in $M$, and so that these pairs actually arise from the graph of a homeomorphism of $M$. We have included some figures to aid the reader.

**Lemma 3.1.** *The group $\mathscr{H}$ admits a parameter-free interpretation of the sort* $\mathrm{seq}(M)$ *of countable sequences of points in $M$, which is uniform for all manifolds of dimension $d$. Moreover, the predicate $p \in \sigma$ expressing membership of a point $p$ in a sequence $\sigma$, and the predicate $\sigma(i) = p$ expressing that $p$ is the $i$-th term of $\sigma$, are both parameter-free definable.*

For technical reasons, we first prove the lemma in the case where $M$ is not the interval, and give an adapted proof for the interval later.

*Proof of Lemma 3.1 for $M \neq I$.* We retain the notation $n(d)$ from Proposition 2.1.

Choosing a cover $M$. We first fix a collection of regular open sets in $M$ of bounded cardinality (depending on $d$) which cover $M$, and which can be used as charts in an atlas for $M$. Fix a collar neighborhood $K$ of $\partial M$ in $M$. Since $M$ has dimension $d$ and $\partial M$ has dimension $d-1$, Proposition 2.1 shows that $M \setminus K$ can be covered by $n(d)$ regular open sets, each component of which is a collared open ball, and each component of $K$ can be cover by $n(d-1)$ such sets consisting of collared open half-balls. By Items (3), (4), and (8) of Theorem 2.3, we may express the existence of collections

$$\mathfrak{B} = \{U_1, \ldots, U_{n(d)}\} \quad \text{and} \quad \mathfrak{H} = \{V_1, \ldots, V_{n(d-1)}\}$$

such that:

(1) The sets $\mathfrak{B} \cup \mathfrak{H}$ cover $M$; this is expressible since we simply require every point of $M$ to lie in an element of $\mathfrak{B} \cup \mathfrak{H}$.
(2) For $W \in \mathfrak{B} \cup \mathfrak{H}$ and $W_0$ a component of $W$, the closure of $W_0$ is contained inside of a collared open ball or open half-ball depending on whether $W \in \mathfrak{B}$ or $W \in \mathfrak{H}$, respectively.

Observe that the components of the sets $U_i$ and $V_i$ need not themselves be balls or half-balls, only have their closures be contained inside of balls or half-balls. Since the collections $\mathfrak{B}$ and $\mathfrak{H}$ have bounded cardinality depending only on $d$, the parameter space of choices for $(\mathfrak{B}, \mathfrak{H})$ is itself parameter-free definable.

The number of charts required in the atlas is the only part of the proof which depends on the dimension of $M$. All other dependencies on dimension fundamentally arise from the number of charts in the atlas.

Initializing a scratchpad. A schematic illustration of the initialized scratchpad is given in Figure 1. Fix regular open sets $W$ and $W^\partial$ with the following properties:

(1) The closure of $W$ is contained in a collared open ball in $M$.
(2) For all components $W_0$ of $W^\partial$, the closure of $W_0$ is contained in a collared open half-ball in $M$.
(3) If $\hat{W}$ is an arbitrary regular open set whose closure is contained in a collared open half-ball in $M$, then there exists an element $g \in \mathscr{H}$ such that $g(\hat{W}) \subseteq W^\partial$.
(4) Each component of $\partial M$ meets at most one component of $W^\partial$.

It is straightforward to see that, in view of Theorem 2.3, the conditions defining $W$ and $W^\partial$ are expressible, and that such $W$ and $W^\partial$ always exist. Next, choose elements $\{g_U \mid U \in \mathfrak{B}\}$ and $\{g_V \mid V \in \mathfrak{H}\}$ such that:

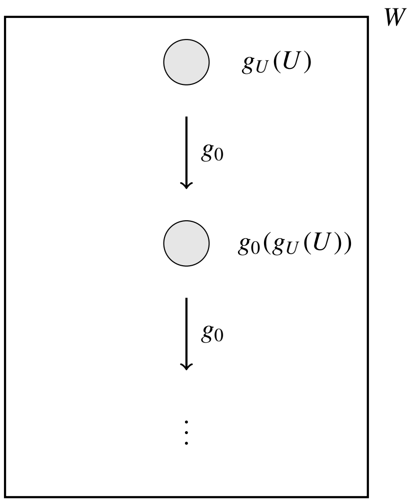


**Figure 1.** A schematic of the scratchpad; here we draw the image of one chart $U$ in the atlas (which need not actually be a disk) in $W$, and the image under $g_0$. The iterates under $g_0$ continue to infinity.

(1) For all $U \in \mathfrak{B}$ and $V \in \mathfrak{H}$, we have $g_U(U)$ has compact closure inside of $W$ and $g_V(V)$ has compact closure inside of $W^\partial$.

(2) For distinct $U_1, U_2 \in \mathfrak{B}$, the images $g_{U_1}(U_1)$ and $g_{U_2}(U_2)$ are disjoint; we place the same requirement on distinct elements of $\mathfrak{H}$. Let

$$U_0 = \bigcup_{U \in \mathfrak{B}} g_U(U) \quad \text{and} \quad V_0 = \bigcup_{V \in \mathfrak{H}} g_V(V).$$

(3) Choose elements $g_0 \in \mathscr{H}[W]$ and $g_0^\partial \in \mathscr{H}[W^\partial]$ such that for all distinct $i, j \geq 0$, we have

$$g_0^i(U_0) \cap g_0^j(U_0) = \varnothing,$$

and similarly

$$(g_0^\partial)^i(V_0) \cap (g_0^\partial)^j(V_0) = \varnothing.$$

Here, we are implicitly using the fact that we may quantify over the arguments of the (definable) exponentiation function.

We write $U^i = g_0^i(U_0)$ and $V^i = (g_0^\partial)^i(V_0)$, respectively. The reader may observe that this is the point where the argument fails for $M = I$, since in the case of the interval the homeomorphism $g_0^\partial$ may not exist.

Encoding countable sequences of points in $M$. For a schematic of this part, see Figure 2. We now choose a regular open set $P$, which together with the scratchpad will encode a countable sequence of points in $M$. Here, we require $P$ to satisfy the following conditions:

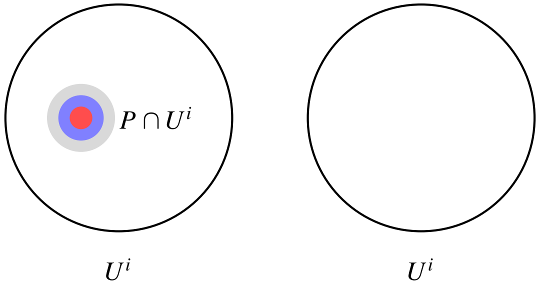


**Figure 2.** A schematic of two components in $U^i$. The sets $P$ meets $U^i$ and isolates a unique point in it.

(1) The set $P$ is contained in $\bigcup_i U^i \cup \bigcup_i V^i$. This can be expressed by requiring for each component of $P$, there is an $i$ so that the $-i$-th power of the relevant $g_0$ or $g_0^\partial$ is contained in $U_0$ or $V_0$, respectively.

(2) For each $i$, exactly one of the intersections $P \cap U^i$ and $P \cap V^i$ is nonempty and isolates a unique point $p_i$ in $U^i$ or $V^i$. From here on, write $q_i$ for the backwards image of $p_i$ under the $i$-th power of $g_0$ or $g_0^\partial$ respectively, followed by the relevant $g_U^{-1}$ or $g_V^{-1}$.

Via the set $P$, we have thus encoded (with parameters), in an unambiguous way, a countably infinite sequence of points $\{q_i\}_{i\in\mathbb{N}} \subseteq M$. This is the sort $\mathrm{seq}(M)$.

Since we can quantify over the arguments in the exponentiation function, it is straightforward to see from the construction that the membership predicate $p \in \sigma$ and $\sigma(i) = p$ are both definable, *a priori* with parameters.

Eliminating parameters. It is clear from the descriptions of the regular open sets chosen in the covers and the relevant homeomorphisms of $\mathscr{H}$ that are chosen, that the choices are made over definable sets of parameters. Given two choices of parameters, we simply declare two interpretations of two sequences of points to be equivalent if for each $i \in \mathbb{N}$, the $i$-th terms of the sequences represent the same point of $M$; this is possible in view of Item (6) of Theorem 2.3. This completes the proof of the lemma. □

We can now give a modified proof of Lemma 3.1 for the interval. Technically we will only interpret sequences of points in the interior $(0, 1)$ of $I$, which is all that will be needed. It is not difficult to add "dummy entries" of two varieties to sequences which stand for possible choices of endpoints of $I$.

*Proof of Lemma 3.1 for $M = I$.* We begin by defining the set of homeomorphisms of $I$ which attract to a point in the interior $(0, 1)$ of $I$. Fixing a point $p_0 \in (0, 1)$, we may define the set of elements $f \in \mathscr{H}$ such that for all $U$ containing $p_0$ and with closure contained in $(0, 1)$, and for all $q \in (0, 1)$, there exists an $n \in \mathbb{N}$ such that $f^n(q) \in U_0$. Call these elements of $\mathscr{H}$ the $p_0$-*attracting* homeomorphisms. In light of Theorem 2.3, the $p$-attracting homeomorphisms of $I$ are definable, with the point $p_0$ as the sole parameter.

Now, let $f \in \mathscr{H}$ be a $p_0$-attracting homeomorphism for some $p_0 \in (0, 1)$, let $U \subseteq (0, 1)$ be a regular open set whose closure is contained in $(0, 1)$, let $U_0 \subseteq U$ be a regular open set containing $p$ whose closure is contained in $U$, and let $g \in \mathscr{H}[U]$ have the property that for all distinct $i, j \in \mathbb{N}$, we have $g^i(U_0) \cap g^j(U_0) = \varnothing$. Write $U_i = g^i(U_0)$ for $i \in \mathbb{N}$. Up to now, we have carried out the interval analogue of initializing the scratchpad.

We now interpret countable sequences of points in $(0, 1)$. We do this by choosing a regular open set $P$ which isolates a unique point $p_i$ in each $U_i$. Defining $q_i = f^{-i}g^{-i}(p_i)$, we have unambiguously interpreted the sequence $\{q_i\}_{i \in \mathbb{N}}$ inside of $\mathscr{H}$. Moreover, every sequence of points in $(0, 1)$ arises as some such $\{q_i\}_{i \in \mathbb{N}}$, for various choices of $P$ and $f$. This defines sequences of points in $(0, 1)$ with parameters. We declare two sequences $\sigma_1$ and $\sigma_2$, with different choices of parameters, to be equivalent if for all $i \in \mathbb{N}$ the encoded points $\sigma_1(i)$ and $\sigma_2(i)$ represent the same point of $(0, 1)$. □

***Interpreting pregraphs.*** Armed with the interpretation of sort $\mathrm{seq}(M)$, we can interpret the sort of *pregraphs*; we define pregraphs to be countable subsets $\Gamma \subseteq M \times M$ such that the projection of $\Gamma$ to each factor is dense in $M$.

**Lemma 3.2.** *The sort of pregraphs is uniformly interpretable for manifolds in dimension $d$, from the sort* $\mathrm{seq}(M)$. *Moreover, the predicate* $(x, y) \in \Gamma$ *expressing that a pair* $(x, y) \in M \times M$ *is an element of* $\Gamma$ *is parameter-free interpretable.*

*Proof.* We may quantify over terms of a sequence $\sigma \in \mathrm{seq}(M)$ and thus encode a countable subset $\Gamma$ of $M \times M$ from $\sigma$ by declaring $(x, y) \in \Gamma$ if and only if there exists an $n \in \mathbb{N}$ such that $\sigma(2n) = x$ and $\sigma(2n+1) = y$. Density of the projections is expressed by saying that for each nonempty regular $U \in \mathrm{RO}(M)$, there is an odd index $i$ and an even index $j$ such that $\sigma(i), \sigma(j) \in U$. The set of $\Gamma$ encoded by this definable set of sequences clearly coincides with pregraphs. We finally put an equivalence relation on elements of $\mathrm{seq}(M)$ encoding pregraphs, which expresses that $\sigma_1$ and $\sigma_2$ are equivalent if and only if they encode pregraphs that are equal as subsets of $M \times M$; this is evidently a definable equivalence relation. This completes the parameter-free interpretation. □

***From pregraphs to graphs.*** We now pass to graphs of homeomorphisms of $M$.

**Lemma 3.3.** *Pregraphs in dimension $d$ admit a parameter-free interpretation of* $\mathrm{HS}_0(M)$.

*Proof.* We put definable conditions on pregraphs to guarantee that they define graphs of homeomorphisms of $M$. Since $M$ is compact, it suffices to require that a pregraph $\Gamma$ extend continuously to the graph of a continuous self-map of $M$ which is injective and surjective.

Continuity: We need only require for all $(x_0, y_0) \in \Gamma$ that for all open $V$ containing $y_0$, there is a $U$ containing $x_0$ such that for all $(x, y) \in \Gamma$ with $x \in U$, we have $y \in V$. This is clearly expressible. Any $\Gamma$ satisfying this continuity requirement automatically encodes a continuous map

$$f_\Gamma : M \to M.$$

Injectivity: We need only require that for all disjoint open $U_1$ and $U_2$ there exist disjoint open $V_1$ and $V_2$ such that if $(x_i, y_i) \in \Gamma$ for $i \in \{1, 2\}$ with $x_i \in U_i$, then $y_i \in V_i$.

Surjectivity: We need only require that the image of $f_\Gamma$ be dense in $M$. This can be achieved by requiring for all nonempty $V$ that there be an $(x, y) \in \Gamma$ with $y \in V$.

Any pregraph $\Gamma$ satisfying the foregoing conditions will automatically encode the graph of a homeomorphism of $M$. Moreover, every homeomorphism of $M$ is encoded by some pregraph, simply by taking a dense subset of the graph of the homeomorphism. To complete the interpretation of $\mathrm{HS}_0(M)$, we put an equivalence relation on pregraphs which expresses that two pregraphs $\Gamma_1$ and $\Gamma_2$ are equivalent if they encode the same homeomorphism of $M$. For this, it suffices to require that if $(x_1, y_1) \in \Gamma_1$ with $x_1 \in U$ and $y_1 \in V$ then there exists a pair $(x_2, y_2) \in \Gamma_2$ with $x_2 \in U$ and $y_2 \in V$. □

**3.2. *Interpreting $\mathscr{H}$ within* $\mathbf{HS_0}(M)$.** Recall that the initial given data is $\mathscr{H}$, whereas here we have interpreted elements of $\mathrm{Homeo}(M)$ via their graphs; *a priori*, $\mathrm{Homeo}(M)$ may be substantially larger than $\mathscr{H}$. We note that it is straightforward to interpret $\mathscr{H}$ as a set within $\mathrm{HS}_0(M)$: indeed, consider the association $g \mapsto \Gamma_g$, which sends an element $g \in \mathrm{Homeo}(M)$ to the graph of $g$ as a homeomorphism of $M$. We have $\Gamma_g$ corresponds to a graph of an element of $\mathscr{H}$ if and only if

$$(\exists \gamma)\big[\forall x \forall y((x, y) \in \Gamma_g \leftrightarrow \gamma(x) = y)\big].$$

Thus, we are justified in saying that $\mathscr{H}$ can interpret its own elements via graphs, and we are justified in saying we have interpreted elements of $\mathscr{H}$ inside of $\mathrm{HS}_0(M)$. We will interpret the group operation below. We summarize with the following corollary:

**Corollary 3.4.** *There is a definable predicate $R \subseteq \mathscr{H} \times \mathrm{HS}_0(M)$ defining the pairs $(g, \Gamma)$ such that $\Gamma = \Gamma_g$ encodes the graph of $g$.*

**3.3. *Interpreting the sorts* $\mathbf{HS}_n(M)$ *for* $n \geq 1$.** The interpretation of the sorts $\mathrm{HS}_n(M)$ for $n \geq 1$ is now straightforward, because of the existence of a computable bijection $\phi : \mathbb{N}^2 \to \mathbb{N}$.

**Lemma 3.5.** *For all $n \geq 1$, the sort* $\mathrm{HS}_n(M)$ *is parameter-free interpretable in* $\mathrm{seq}(M)$, *uniformly interpretable for manifolds of dimension $d$.*

*Proof.* We proceed by induction, $\mathrm{HS}_0(M)$ having been interpreted already. To interpret $\mathrm{HS}_{n+1}(M)$ once $\mathrm{HS}_n(M)$ has been parameter-free interpreted, we use the bijection $\phi$ as above to definably pass from $\mathbb{N}$-indexed sequences of points to $\mathbb{N}^2$-indexed points $\{q_{(i,j)}\}_{(i,j)\in\mathbb{N}^2}$. For $i$ fixed, we simply require that the (obviously parameter-free definable) subsequence $\{q_{(i,j)}\}_{j\in\mathbb{N}}$ encode an element of $\mathrm{HS}_n(M)$. It is clear that this furnishes a parameter-free interpretation of $\mathrm{HS}_{n+1}(M)$.

It is clear that the predicate $\in_n \subseteq \mathrm{HS}_n(M)\times\mathrm{HS}_{n+1}(M)$ is parameter-free definable, as is the predicate defining the $i$-th term in a sequence in $\mathrm{HS}_n(M)$. □

**3.4. *Predicates for manipulating* HS($M$).** Most predicates for manipulating sequences in $\mathrm{HS}_n(M)$ are easily seen to be interpretable, as follows from the fact that one can freely quantify over the arguments in the exponentiation function; we have argued concerning membership $\in_n$ and the predicate $s(i) = t$ for $s \in \mathrm{HS}_{n+1}(M)$ and $t \in \mathrm{HS}_n(M)$ already.

Let $f_1, f_2, f_3 \in \mathrm{HS}_0(M)$ be terms in a sequence $\sigma \in \mathrm{HS}_1(M)$. It is easy to see that there is a predicate expressing that $f_1 * f_2 = f_3$ in $\mathrm{Homeo}(M)$. Indeed, let $\Gamma_i$ be graphs of $f_i$ for $i \in \{1, 2, 3\}$. To express that $f_1 * f_2 = f_3$, it suffices to express that for all $(x, z) \in \Gamma_3$ and all open sets $U$ and $V$ such that $x \in U$ and $z \in V$, whenever $(x', y) \in \Gamma_1$ with $x' \in U$ and all open $W$ such that $y \in W$, there exists a $(y', z') \in \Gamma_2$ such that $y' \in W$ and $z' \in V$.

For homeomorphisms of $M$, extended supports are regular open sets which are interpretable via Rubin's interpretability theorem, and which is given by a purely first order group theoretic formula; see [45], and specifically Theorem 3.6.3 of [25] and Section 3.2 of [28]. It is clear then that we may interpret a new sort which represents the extended support of an element $f \in \mathrm{HS}_0(M)$, and which is canonically identified with the extended support of the homeomorphism $f$. This completes the proof of Theorem 1.1.

We have the following consequences of interpreting the sort $\mathrm{HS}_1(M)$ and the preceding predicates.

**Corollary 3.6.** (1) *The set of sequences $s \in \mathrm{HS}_1(M)$ which, via the identification of* $\mathrm{HS}_0(M)$ *with* $\mathrm{Homeo}(M)$, *form subgroups of* $\mathrm{Homeo}(M)$ *is parameter-free definable.*

(2) *If $X \subseteq \mathrm{HS}_0(M) = \mathrm{Homeo}(M)$ is arbitrary, then there is a predicate*

$$\mathrm{member}_X \subseteq \mathrm{Homeo}(M),$$

*using $X$ as a parameter, which expresses whether an arbitrary $f \in \mathrm{Homeo}(M)$ is a finite product of elements of $X$. In particular, if $X$ is parameter-free definable then* $\mathrm{member}_X$ *is parameter-free definable.*

*Proof.* The first part reduces to requiring for all $f, g \in s$, we have $f^{-1} \in s$ and $f \cdot g \in s$. The second part reduces to the existence of a sequence $s \in \mathrm{HS}_1(M)$ with $s(0) = 1$, with $s(n) = f$ for some $n \in \mathbb{N}$, and such that for all $0 < m \leq n$ we have $s(n-1)^{-1}s(n) \in X$. □

## 4. Intermediate subgroups, mapping class groups, and Theorem 1.4

We now use the interpretation of the sorts $\mathrm{HS}(M)$ to extract group-theoretic consequences. Observe first that $\mathscr{H}$ interprets $\mathrm{Homeo}(M)$. Indeed, this is part of the content of Theorem 1.1. Next, we can interpret $\mathrm{Homeo}_0(M)$. The key to interpreting $\mathrm{Homeo}_0(M)$ is the following result, which appears as Corollary 1.3 in [12].

**Theorem 4.1** (Edwards and Kirby). *Let $\mathscr{U}$ be an open cover of a compact manifold $M$. An arbitrary element $g \in \mathrm{Homeo}_0(M)$ admits a **fragmentation** subordinate to $\mathscr{U}$. That is, $g$ can be written as a composition of homeomorphisms that are supported in elements of $\mathscr{U}$.*

**Proposition 4.2.** *The group $\mathscr{H}$ interprets* $\mathrm{Homeo}_0(M) \subseteq \mathrm{HS}_0(M)$.

As always, the interpretation of $\mathrm{Homeo}_0(M)$ in $\mathscr{H}$ is uniform in manifolds of bounded dimension.

*Proof of Proposition 4.2.* It suffices to construct a formula $\mathrm{isotopy}_0(\gamma)$ that is satisfied by a homeomorphism $g$ if and only if $g$ is isotopic to the identity. We will carry out the construction for closed manifolds, with the general case being similar.

Consider $\Gamma_g$, the graph of a homeomorphism as obtained from interpreting the sort $\mathrm{HS}_1(M)$, and let $\mathfrak{B} = \{U_1, \ldots, U_{n(d)}\}$ be a cover of $M$, with each component of each $U_i$ having compact closure inside of a collared open ball.

By imposing suitable definable conditions on the data defining $\Gamma_g$, we may insist that there exists an $i$ and a component $\hat{U}_i$ of $U_i$ such for all $(p, q) \in \Gamma_g$, we have $p = q$ unless $p \in \hat{U}_i$. Specifically, we may write

$$\text{small-sup}(\Gamma) := (\forall (x, y) \in \Gamma)(\exists i \leq n(d))(\exists \hat{u} \in \pi_0(u_i))[x \notin \hat{u} \to x = y];$$

in this formula we are implicitly treating elements of $\mathfrak{B}$ as parameters.

This condition implies that the homeomorphism $g$ encoded by $\Gamma$ is the identity outside of $\hat{U}_i$. Since $\hat{U}_i$ is compactly contained in the interior of a collared ball in $M$ we have that $g$ is isotopic to the identity, as follows from the Alexander trick.

By quantifying over all such covers $\mathfrak{B}$ of $M$, we thus obtain a parameter-free definable set $X \subseteq \mathrm{HS}_0(M)$ consisting of graphs of elements of $\mathrm{Homeo}(M)$ which satisfy small-sup for some such cover.

By Theorem 4.1, we have that $g \in \mathrm{Homeo}(M)$ is isotopic to the identity if and only if $g$ is a product of a finite tuple of homeomorphisms lying in $X$. By

Corollary 3.6, it follows that $\mathrm{Homeo}_0(M)$ is parameter-free definable as a subset of the sort $\mathrm{HS}_0(M)$. □

An arbitrary subgroup $\mathrm{Homeo}_0(M) \leq \mathscr{H}' \leq \mathrm{Homeo}(M)$ is automatically of countable index in $\mathrm{Homeo}(M)$, as follows from the fact that for a compact manifold, $\mathrm{Homeo}(M)$ is separable and therefore has countably many connected components.

*Proof of Proposition 1.3.* A subgroup

$$\mathrm{Homeo}_0(M) \leq \mathscr{H}' \leq \mathrm{Homeo}(M)$$

can be encoded by a definable equivalence class of countable subsets of $\mathrm{Homeo}(M)$; indeed, if $\underline{g}$ is a sequence then we obtain a subgroup $\mathscr{H}_{\underline{g}}$ (viewed as a subset of $\mathrm{HS}_0(M)$) via

$$\mathscr{H}_{\underline{g}} = \{h \mid (\exists g \in \underline{g})[h \in g \cdot \mathrm{Homeo}_0(M)]\},$$

after adding the further condition that $\mathscr{H}_{\underline{g}}$ be a group (which can be guaranteed by imposing the first order condition that $\underline{g}$ be a group, for instance). Two sequences of homeomorphisms $\underline{g}$ and $\underline{h}$ are equivalent if $\mathscr{H}_{\underline{g}} = \mathscr{H}_{\underline{h}}$. Since the mapping class group of $M$ is countable, any such subgroup $\mathscr{H}'$ occurs as $\mathscr{H}_{\underline{g}}$ for some sequence $\underline{g}$. We thus obtain a canonical bijection between subgroups $\mathscr{H}'$ as above and suitable equivalence classes of sequences of homeomorphisms, as desired.

We have already shown that $\mathrm{Homeo}(M)$ and $\mathrm{Homeo}_0(M)$ are interpretable without parameters. The group $\mathscr{H}$ itself is also definable without parameters in the interpretation of $\mathrm{Homeo}(M) = \mathrm{HS}_0(M)$ in $\mathscr{H}$, as is part of the content of Theorem 1.1. □

It is not difficult to argue the conclusions of Theorem 1.4, and so we only sketch the arguments. Because $\mathrm{Homeo}(M)$ and $\mathrm{Homeo}_0(M)$ are parameter-free interpretable in $\mathscr{H}$, so is $\mathrm{Mod}(M)$. The sorts of countable subgroups of $\mathrm{Homeo}(M)$ and $\mathrm{Mod}(M)$ are parameter-free interpretable, by Corollary 3.6; it is immediate that one can quantify over arbitrary subsets of countable subgroups, since these subsets will always be countable. All of the countable algebra of groups can be formalized within ACA or slightly stronger systems, which is substantially weaker than full second order theory of countable groups to which we have access: see [46], page 14, and also Chapter III. Here and for the rest of the section, “subgroup" will refer to a subgroup of $\mathrm{Homeo}(M)$ or of $\mathrm{Mod}(M)$.

Membership in a fixed countable subgroup follows from Corollary 3.6. Finite generation asks whether for a countable subgroup, there exists a sequence $\sigma$ wherein all but finitely many terms are the identity, so that every element in the subgroup can be written as a finite product of entries in $\sigma$; this is clearly expressible: indeed, we think of the set $X$ in Corollary 3.6 as a sequence $\sigma$ where there exists an $n \in \mathbb{N}$

such that for all $i > n$ the term $\sigma(i)$ is the identity. Finite presentability is slightly more complicated but still straightforward.

Finite index subgroups of a given countable subgroup are easily defined, using the subgroup itself as a parameter; thus, residual finiteness is expressible. For finitely generated groups, linearity can be expressed via the Lubotzky Linearity Criterion [33], and in general by quantifying suitably over countable subgroups of $\mathrm{GL}_n(\mathbb{C})$; we omit the tedious details.

Isomorphism of a countable subgroup with a particular definable group is straightforward. Amenability can be encoded with the Følner Criterion; see Chapter 2 of [9]. Kazhdan's Property (T) for finitely generated subgroups can be encoded using Ozawa's Criterion, which is the main result of [42].

## 5. Descriptive set theory in Homeo($M$)

In this section, we show how to interpret the projective hierarchy in Homeo($M$) and characterize definability via Theorem 1.6. The interpretation is modeled on the fact that the projective hierarchy in Euclidean space is definable in second order arithmetic.

In this section, we emphasize that all interpretations are *uniform*; that is, there is a single formula, depending only on the dimension of the manifold $M$, which defines all open sets of Homeo($M$) (as subsets of $\mathrm{HS}_0(M)$) for various choices of parameters. The same holds for all sets at the various levels of the Borel hierarchy, analytic sets, and sets in the projective hierarchy.

**5.1.** ***Generalities on descriptive set theory.*** The reader is directed to [21; 37] for a more thorough background. Suppose that we are given a Polish (i.e., completely metrizable and separable) space $X$. We recall the definition of the Borel hierarchy. For every nonempty countable ordinal $\alpha$ one can define the families $\Sigma^0_\alpha(X)$ and $\Pi^0_\alpha(X)$ of subsets of $X$ as follows.

- The class $\Sigma^0_1(X)$ consists of all open subsets of $X$.
- For all $\alpha < \aleph_1$ the class $\Pi^0_\alpha$ is the collection of complements of subsets in $\Sigma^0_\alpha(X)$.
- For any limit ordinal $\alpha < \aleph_1$ we have $\Sigma^0_\alpha(X) = \bigcup_{\beta<\alpha} \Sigma^0_\beta(X)$.
- For all $\alpha < \aleph_1$ the family $\Sigma^0_{\alpha+1}(X)$ consists of all countable unions of sets in $\Pi^0_\alpha(X)$.

A subset of $X$ is called *Borel* if it belongs to $\Sigma^0_\alpha$ for some $\alpha < \aleph_1$.

A subset of $X$ is *analytic* if it is a continuous image of Baire space $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$; equivalently, a subset $A \subseteq X$ is analytic if and only if there is a closed subset of $C \subseteq X \times \mathcal{N}$ such that $A$ is the projection of $C$ to $X$. Observe that Baire space and its topology are parameter-free interpretable in second order arithmetic.

In the projective hierarchy, analytic sets are called $\Sigma^1_1$. One can extend the notation of the Borel hierarchy in order to define classes $\Sigma^1_\alpha(X)$ and $\Pi^1_\alpha(X)$ for all $\alpha < \aleph_1$; here we will need only concern ourselves with integer values of $\alpha$, for which the definition can be given by usual induction as follows:

- For all $n$ for which $\Sigma^1_n(X)$ has been defined, let $\Pi^1_n(X)$ be the class of complements of sets in $\Sigma^1_n$. In particular, for $n = 1$, we obtain the family $\Pi^1_1(X)$ of projective sets in $X$.
- A set $Z \subseteq X$ is in $\Sigma^1_{n+1}$ if there is a $\Pi^1_n$ subset $Y \subseteq X \times \mathscr{N}$ such that $Z$ is the projection of $Y$ to $X$.

The sets $\{\Sigma^1_n(X)\}_{n\geq 1}$ form the *projective hierarchy*; when $X$ is a Euclidean space, the projective hierarchy is easily seen to be definable in second order arithmetic. We note that in the definition of the projective hierarchy, the factor $\mathscr{N}$ can be replaced by an arbitrary uncountable Polish space (e.g., $\mathrm{Homeo}(M)$ itself).

It is a standard fact that a subset $X \subseteq \mathbb{R}$ is definable (with parameters) in second order arithmetic if and only if it lies in the projective hierarchy. Theorem 1.6 establishes the corresponding characterization of definable subsets of $\mathrm{Homeo}(M)$. We prove one direction first.

**Proposition 5.1.** *Let $X \subseteq \mathrm{Homeo}(M)$ be definable with parameters, as a subset of the sort $\mathrm{HS}_0(M)$. Then $X$ lies in the projective hierarchy. If $X \subseteq \mathscr{H}$ is definable with parameters in the language of group theory, then $X$ lies in the projective hierarchy.*

*Proof.* This follows by induction on the quantifier complexity $\phi$ of a formula defining $X$. Equalities and inequalities (with parameters) in $\mathrm{Homeo}(M)$ define closed and open sets respectively, and so if $X$ is defined by quantifier-free formula then it is certainly Borel.

Suppose that $X$ is defined by

$$\phi(x) = (\exists y)\psi(x, y, a),$$

where $\psi$ defines a set

$$Y \subseteq (\mathrm{Homeo}(M))^k$$

in the projective hierarchy for some $k \in \mathbb{N}$, and where $a$ is a tuple of parameters. Then $X$ is given by a projection of $Y$ to a smaller Cartesian power of copies of $\mathrm{Homeo}(M)$, and so $X$ lies at most one level higher than $Y$ in the projective hierarchy. If $X \subseteq \mathscr{H}$ then a canonical definable identification between $X$ and a subset of the sort $\mathrm{HS}_0(M)$ is given by Corollary 3.4. The proposition now follows easily. □

In the remainder of this section, we will interpret sorts for levels of the projective hierarchy together with the predicate $\in$, and show that every subset of $\mathrm{Homeo}(M)$ in the projective hierarchy is definable with parameters (as a subset of the sort

$\mathrm{HS}_0(M)$), and every subset of the home sort $\mathscr{H}$ in the projective hierarchy is definable with parameters.

**5.2.** ***Open sets.*** We interpret the compact-open topology on $\mathrm{Homeo}(M)$ directly. First, cover $M$ by regular open sets. Since regular open sets themselves are encoded by definable equivalence classes of homeomorphisms by associating their extended supports (cf. Theorem 2.3), finite covers of $M$ can be encoded by equivalence classes of finite tuples of homeomorphisms. To define finite tuples $\tau$ of homeomorphisms whose supports cover $M$ since one need only express that for all $p \in M$, there exists an $f \in \tau$ such that $p \in \mathrm{supp}^e(f)$.

From a finite cover $\mathscr{V}$ of $M$, one can define an open set $U_{\mathscr{V}} \subseteq \mathrm{Homeo}(M)$ by considering homeomorphisms $f$ such that for all $p \in M$, there exists a $V \in \mathscr{V}$ such that both $p$ and $f(p)$ lie in $V$. It is not so difficult to see that $U_{\mathscr{V}}$ is indeed open.

Now, if $f \in \mathrm{Homeo}(M)$ and $\mathscr{V}$ is a finite covering of $M$ then we set $U_{\mathscr{V}}(f)$ to be the set of homeomorphisms $g$ such that $g^{-1}f$ lies in $U_{\mathscr{V}}$ as defined above. Observe that for a given cover $\mathscr{V}$ and $f \in \mathrm{Homeo}(M)$, the subset of $\mathrm{HS}_0(M)$ contained $U_{\mathscr{V}}(f)$ is definable (with $\mathscr{V}$ as a parameter).

As $\mathscr{V}$ varies over finite covers of $M$ and $f$ varies over $\mathrm{Homeo}(M)$, we have that the sets $U_{\mathscr{V}}(f)$ form a basis for the compact-open topology of $\mathrm{Homeo}(M)$. Thus, a basis for the topology on $\mathrm{Homeo}(M)$ is encoded by certain equivalence classes of finite tuples of elements of $\mathrm{Homeo}(M)$, which is to say certain definable subsets of $\mathrm{HS}_1(M)$, up to definable equivalence.

More explicitly, for a tuple $(f_1, \ldots, f_n)$, we definably associate extended supports via $f_i \mapsto V_i = \mathrm{supp}^e f_i$ for $i \geq 2$, while requiring that

$$M \subseteq \bigcup_{i=2}^{n} \mathrm{supp}^e(f_i).$$

Setting $\mathscr{V} = \{V_2, \ldots, V_n\}$, such a tuple of homeomorphisms encodes the set $U_{\mathscr{V}}(f_1)$. This interpretation is clearly uniform in the sense described at the beginning of the section. The predicate $\in$ is trivial to interpret.

We see now that basic open sets are interpretable as a definable subset of $\mathrm{HS}_1(M)$, up to definable equivalence by setting two tuples to be equivalent if and only if the basic open sets they encode contain the same homeomorphisms (viewed as elements in the sort $\mathrm{HS}_0(M)$).

An arbitrary open set is then interpreted as a countable sequence of basic open sets, with a homeomorphism $f$ being a member of the open set if and only if it is a member of one of the elements in the sequence. Since basic open sets are parameter-free interpretable in $\mathrm{HS}_1(M)$, we see that open sets are parameter-free interpretable in $\mathrm{HS}_2(M)$. Closed sets are then simply complements of open sets. It is trivial to interpret the membership relation of homeomorphisms in an open or closed set.

**Corollary 5.2.** *The sorts of open and closed sets of* $\mathrm{Homeo}(M)$, *viewed as subsets of* $\mathrm{HS}_0(M)$, *are uniformly interpretable with parameters. Open and closed subsets of* $\mathscr{H}$ *are uniformly definable with parameters.*

*Proof.* We have argued that open sets in $\mathrm{Homeo}(M)$ are encoded by elements in $\mathrm{HS}_2(M)$. Thus, a particular open set $U$ is identified with a parameter-free definable equivalence class of elements $\tau \in \mathrm{HS}_2(M)$, and $f \in \mathrm{HS}_0(M)$ if and only if there exists a $\sigma \in \tau$ such that $f \in \sigma$. The case of closed sets is identical. The definability of open and closed sets in the home sort follows now from Corollary 3.4. □

With a minor variation on the preceding arguments, we can recover the topology on $\mathrm{Homeo}(M)^\ell \times \mathscr{N}^k$ for all $\ell, k \geq 0$. Note that for $k \geq 1$, we have $\mathrm{Homeo}(M)^\ell \times \mathscr{N}^k \cong \mathrm{Homeo}(M) \times \mathscr{N}$. We record the following corollary.

**Corollary 5.3.** *The sorts of open and closed subsets of* $\mathrm{Homeo}(M)^\ell \times \mathscr{N}^k$ *are parameter-free interpretable in* $\mathscr{H}$, *and open and closed sets in these spaces are uniformly interpretable with parameters.*

**5.3.** ***The Borel hierarchy.*** The Borel hierarchy of $\mathrm{Homeo}(M)$ and $\mathscr{H}$ is now straightforward to interpret. We first indicate an interpretation of finite levels of the Borel hierarchy, followed by the case of arbitrary countable ordinals using Borel codes; see Section 1.4 of [16].

We have already interpreted open and closed sets in $\mathrm{HS}_2(M)$, which corresponds to $\Sigma^0_1(\mathrm{Homeo}(M))$ and $\Pi^0_1(\mathrm{Homeo}(M))$, respectively; we will suppress the notation of $\mathrm{Homeo}(M)$ since it will not cause confusion.

By induction, $\Sigma^0_k$ and $\Pi^0_k$ are uniformly interpreted in $\mathrm{HS}_{k+1}(M)$. By definition, elements of $\Sigma^0_{k+1}$ are countable unions of elements of $\Pi^0_k$, which are then encoded by definable equivalence classes of elements in $\mathrm{HS}_{k+2}(M)$. Elements of $\Pi^0_{k+1}$ are just given by complementation. The proof of the following is nearly identical to that of Corollary 5.2.

**Corollary 5.4.** *Let* $X \subseteq \mathrm{Homeo}(M)$ *lie in a finite level of the Borel hierarchy. Then* $X$ *is uniformly interpretable with parameters, viewed as a subset of* $\mathrm{HS}_0(M)$. *If* $X \subseteq \mathscr{H}$ *lies in a finite level of the Borel hierarchy then* $X$ *is uniformly definable with parameters in the home sort.*

For the general Borel hierarchy, it is helpful to use *Borel codes*, which are a standard tool in descriptive set theory. For a Polish space $X$, one chooses a countable basis $\{U_\tau\}_{\tau \in \mathbb{N}^{<\mathbb{N}}}$ for the topology of $X$. A Borel set $Y \subseteq X$ is encoded by a *labeled*, *well-founded tree*, the definition of which we briefly recall here; cf. Section 1.4 of [16]. A tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a prefix-closed subset, where elements of $\mathbb{N}^{<\mathbb{N}}$ (also called nodes) are viewed as finite sequences; there is an obvious notion of length for a node. A tree $T$ is *well-founded* if there is no infinite sequence $\{\tau_i\}_{i \in \mathbb{N}}$ where $\tau_{i-1}$ is a prefix of $\tau_i$. An element $\tau \in T$ is *terminal* if it admits no proper extension in $T$.

If $\tau \in T$ then one writes $T_\tau$ for the set of suffixes of elements of $T$ which have $\tau$ as a prefix, so that $T_\tau$ is itself a tree. A well-founded tree $T \neq \varnothing$ together with a label function $\lambda : T \longrightarrow \mathbb{N}$ forms a Borel code provided that:

(1) If $\tau \in T$ is non-terminal then $\lambda(\tau) \in \{0, 1\}$.

(2) If $\tau \in T$ is non-terminal and $\lambda(\tau) = 0$ then there exists a unique $\sigma \in T$ extending $\tau$ by exactly one entry, i.e., of length exactly one more than $\tau$.

If $\tau \in T$ and $\lambda$ is a labeling of $T$ then there is an obvious labeling of $T_\tau$ which we also call $\lambda$.

The *rank* of $\tau \in T$ is defined recursively:

(1) If $\tau$ is terminal then the rank of $\tau$ is zero.

(2) If $\tau \in T$ is not terminal, then the rank of $\tau$ is one more than the supremum of the ranks of the one-entry extensions of $\tau$ in $T$, i.e., of length exactly one more than $\tau$.

(3) The rank of $T$ is the rank of the empty sequence $\varnothing \in T$.

Choose a bijection $\mathbb{N}^{<\mathbb{N}}$ with $\mathbb{N}$, which we write $\tau \mapsto \langle\tau\rangle$. A Borel set $B_{(T,\lambda)}$ in $X$ is encoded by the pair $(T, \lambda)$ as follows.

(1) If $\varnothing$ is the only node of $T$ then $B_{(T,\lambda)} = U_\tau$, where $\langle\tau\rangle = \lambda(\varnothing)$.

(2) If $\varnothing$ is non-terminal and $\lambda(\varnothing) = 0$ then there is a unique node $\sigma$ of length one extending $\varnothing$. We write $B_{(T,\lambda)} = X \setminus B_{(T_\sigma,\lambda)}$.

(3) If $\varnothing$ is non-terminal and $\lambda(\varnothing) = 1$, then write $\{\sigma_i\}_{i\in\mathbb{N}}$ for the nodes of length one in $T$ and define
$$B_{(T,\lambda)} = \bigcup_i B_{(T_{\sigma_i},\lambda)}.$$

This encoding makes sense because of the well-foundedness of $T$. A set in $X$ is Borel if and only if it admits a Borel code. Moreover, for a countable ordinal $\alpha$, a Borel set lies in $\Sigma^0_\alpha$ if and only if it is encoded by a Borel code encoded by $(T, \lambda)$ of rank at most $\alpha$ with $\lambda(\varnothing) \neq 0$. Similarly, a Borel set lies in $\Pi^0_\alpha$ if and only if it is encoded by a Borel code encoded by $(T, \lambda)$ of rank at most $\alpha$ with $\lambda(\varnothing) = 0$.

**Corollary 5.5.** *The following are uniformly parameter-free interpretable in* $\mathscr{H}$:

(1) *the Borel sets* $\mathscr{B}$ *of* $\mathrm{Homeo}(M)$, *viewed as subsets of* $\mathrm{HS}_0(M)$;

(2) *the membership predicate for Borel subsets*;

(3) *a rank predicate* $\mathrm{rk} \subseteq \mathscr{B} \times \aleph_1$, *consisting of pairs* $(A, \alpha)$ *with* $A \in \Sigma^0_\alpha$.

*Proof.* This is nearly immediate. First, countable ordinals are parameter-free definable in second order arithmetic. Moreover, there is a definable bijection between $\mathbb{N}^{<\mathbb{N}}$ and $\mathbb{N}$, so that in second order arithmetic we may define (without parameters) well-founded trees and hence Borel codes.

It is straightforward to see that, in light of Section 5.2, we may have direct access to countable bases for the topology on Homeo($M$). It is similarly straightforward to see that via Borel codes, we may encode:

(1) Borel sets;
(2) a parameter-free predicate that expresses when two Borel codes encode the same Borel set;
(3) the rank function rk;
(4) the membership predicate in members of the class of Borel sets.

Moreover, individual Borel sets are interpretable with parameters. We omit the remaining details. □

**5.4.** ***The projective hierarchy.*** We will now complete the proof of Theorem 1.6; precisely, we will show that the levels $\Sigma^1_n$ and $\Pi^1_n$ of the projective hierarchy of Homeo($M$) are uniformly interpretable sorts, and that a set in the projective hierarchy is definable with parameters, uniformly within a level of the hierarchy.

By definition, an analytic set in a Polish space $X$ is a continuous image of $\mathscr{N}$. Equivalently, an analytic set in $X$ is the projection of a closed subset of $X \times \mathscr{N}$ to $X$. By Corollary 5.3, we have interpreted closed subsets of Homeo$(M) \times \mathscr{N}$. More precisely, an open set in Homeo$(M) \times \mathscr{N}$ is a countable union of basic open sets in the product, which can be taken to be pairs of basic open sets in each factor. It is not difficult to see then that open sets in Homeo$(M) \times \mathscr{N}$ can be encoded in $\mathrm{HS}_3(M)$, and closed sets by complementation. If

$$C \subseteq \mathrm{Homeo}(M) \times \mathscr{N}$$

is a closed subset then the set

$$Y_C = \{f \mid (\exists x)[(f, x) \in C]\}$$

is analytic, and every analytic set arises this way. Thus, membership of a homeomorphism $f \in$ Homeo($M$) in an analytic (or co-analytic) set is expressible.

**Corollary 5.6.** *Let $X \subseteq$ Homeo($M$) be analytic or co-analytic. Then $X$ is uniformly interpretable with parameters, viewed as a subset of* $\mathrm{HS}_0(M)$*. If $X \subseteq \mathscr{H}$ is analytic or co-analytic then $X$ is uniformly definable with parameters in the home sort.*

It is trivial to extend this discussion to analytic and co-analytic subsets of finite Cartesian powers of Homeo($M$).

To interpret the higher levels of the projective hierarchy, suppose by induction that $X \subseteq \mathrm{Homeo}(M)^\ell$ is a $\Pi^1_n$ set that is definable with parameters. Then the set $Y \subseteq \mathrm{Homeo}(M)^{\ell-1}$ given by projecting $X$ to the first $\ell - 1$ factors is $\Sigma^1_{n+1}$, and every $\Sigma^1_{n+1}$ occurs this way. Thus, we have:

**Corollary 5.7.** *Let $X \subseteq \mathrm{Homeo}(M)$ lie in a fixed level the projective hierarchy. Then $X$ is uniformly interpretable with parameters, viewed as a subset of* $\mathrm{HS}_0(M)$. *If $X \subseteq \mathscr{H}$ lies in a fixed level of the projective hierarchy then $X$ is uniformly definable with parameters in the home sort.*

This completes the proof of Theorem 1.6.

## 6. Undefinability of sentences isolating manifolds

Throughout this section, we will limit ourselves to full homeomorphism groups of manifolds; it is easy to see that the entire discussion could be carried out for any subgroup between $\mathrm{Homeo}_0$ and Homeo, and we make this choice for the sake of concision.

In this section, we prove Theorems 1.7, 6.1, and 6.2; together, these results show that many natural sets of natural numbers associated to homeomorphism groups of manifolds are not definable in second order arithmetic.

We fix an arbitrary numbering of the symbols in the language of group theory, and thus obtain a computable Gödel numbering of strings of symbols in this language. As is standard, well-formed formulae and sentences are definable in arithmetic, which is to say the set of Gödel numberings of formulae and sentences are definable in first order arithmetic. For formulae and sentences $\psi$ in the language of group theory (and occasionally, by abuse of notation, in arithmetic), we will write $\#\psi$ for the corresponding Gödel numbers. For a class of sentences in group theory, the definability of the set of Gödel numbers of sentences in that class is independent of the Gödel numbering used. The proof of Theorem 1.7 will follow ultimately from Tarski's well-known undefinability of truth [3; 18; 34]. That is, there is no predicate True that is definable in arithmetic (first or second order) such that for all sentences $\phi$ in second order arithmetic, we have

$$\phi \longleftrightarrow \mathrm{True}(\#\phi).$$

See Theorem 12.7 of [20] for a general discussion.

For the remainder of this section, we will fix a uniform interpretation of second order arithmetic in homeomorphism groups of compact manifolds. If $\psi$ is an arithmetic sentence, we will write $\tilde{\psi}$ for the corresponding interpreted group theoretic statement. Thus, we have $\mathrm{Arith}_2 \models \psi$ if and only if $\mathrm{Homeo}(M) \models \tilde{\psi}$ for all compact manifolds $M$; here we use $\mathrm{Arith}_2$ to denote second order arithmetic, as opposed to $\mathbb{N}$ which usually denotes first order arithmetic. For a fixed Gödel numbering in arithmetic, the association $\#\psi \mapsto \#\tilde{\psi}$ is computable.

Let $M$ be a fixed compact manifold and let $\psi$ be a sentence in group theory. Recall that $\psi$ *isolates* $M$ if for all compact manifolds $N$, we have

$$\mathrm{Homeo}(N) \models \psi \longleftrightarrow M \cong N.$$

Notice that if $\psi$ isolates $M$ then $\mathrm{Homeo}(M) \models \psi$. Similarly, we will say $\psi$ *isolates a manifold* if there is a unique compact manifold $M$ such that $\mathrm{Homeo}(M) \models \psi$.

Recall that Rice's theorem from computability theory asserts that if $\mathscr{C}$ is a class of partial recursive functions then the set $\{n \mid \phi_n \in \mathscr{C}\}$ is computable if and only if $\mathscr{C}$ is empty or equal to the whole class of partial recursive functions. Here, we have adopted the standard notation $\phi_n$ for the $n$-th function computed by the universal Turing machine. See Corollary 1.6.14 in [47].

Here we will prove two analogues of Rice's theorem for homeomorphism groups of manifolds. Let $\mathscr{M}$ be a class of (homeomorphism classes) of compact manifolds. We will say that $\mathscr{M}$ is *finitely axiomatized* if there is a first order sentence $\phi_{\mathscr{M}}$ in the language of group theory such that for all compact manifolds $M$, we have

$$M \in \mathscr{M} \Leftrightarrow \mathrm{Homeo}(M) \models \phi_{\mathscr{M}};$$

in particular, $\phi_{\mathscr{M}}$ isolates precisely those manifolds $M$ which lie in $\mathscr{M}$.

**Theorem 6.1.** *Let $\mathscr{M}$ be a class of compact manifolds that is finitely axiomatized, and let*

$$\mathrm{axiom}(\mathscr{M}) := \{\#\phi \mid \phi \textit{ finitely axiomatizes } \mathscr{M}\}.$$

*Then* $\mathrm{axiom}(\mathscr{M})$ *is not definable in second order arithmetic.*

The reader may note that Theorem 6.1 implies that even the set of sentences which are *false* for all compact manifold homeomorphism groups (i.e., $\mathrm{axiom}(\varnothing)$) is so complicated as to be undefinable in second order arithmetic.

Even more generally, let $\mathscr{A}$ denote the set of all homeomorphism classes of compact manifolds, and let $\mathscr{F}$ denote the set of nonempty subsets of $\mathscr{A}$ that are finitely axiomatized by first order sentences in the language of group theory.

**Theorem 6.2.** *Let $A \subseteq \mathscr{F}$ be nonempty and proper. Then the set*

$$\chi(A) = \{\#\psi \mid \psi \textit{ finitely axiomatizes some } a \in A\}$$

*is not definable in second order arithmetic.*

Before giving the proof of Theorem 6.2, we note that it implies Theorem 6.1, as well as Theorem 1.7 from the introduction.

*Proof of Theorem 6.1.* Suppose first that $A = \mathscr{M}$ is nonempty and finitely axiomatized. We have $A \neq \mathscr{F}$ because $A$ is a subset of $\mathscr{A}$ and because each of the countably infinitely many singletons of $\mathscr{A}$ is finitely axiomatized; this is part of the content of Theorem 2.3. By Theorem 6.2, we have that $\chi(A) = \mathrm{axiom}(\mathscr{M})$ is not definable in second order arithmetic.

To see that $\mathrm{axiom}(\varnothing)$ is not definable in second order arithmetic, we simply note that for all arithmetic sentences $\psi$, we have $\#\tilde{\psi} \in \mathrm{axiom}(\varnothing)$ if and only if $\psi$ is false in $\mathrm{Arith}_2$. This violates the undefinability of truth. □

*Proof of Theorem 1.7.* Let $M$ be a fixed compact manifold. The undefinability of the set $\mathrm{Sent}_M$ is precisely the conclusion of Theorem 6.1 when $\mathscr{M} = \{M\}$.

For the undefinability of Sent, we note that if $\phi$ isolates some compact manifold $M$ then for all arithmetic sentences $\psi$, we have $\phi \wedge \tilde{\psi}$ isolates some compact manifold $M$ if and only $\mathrm{Arith}_2 \models \psi$; this is simply because $\neg\tilde{\psi}$ is always false in compact manifolds homeomorphism groups, and $\neg\phi$ isolates no compact manifold because there are at least three pairwise non-homeomorphic compact manifolds. Thus, if $\mathrm{Sent}_M$ were definable then we would be able to define truth in $\mathrm{Arith}_2$, a contradiction. □

To add to the complexity of the sets $\mathrm{Sent}_M$ and Sent, note that it is well-known that there is a Diophantine equation which does not admit a solution if and only if ZFC is consistent (or, if and only if PA is consistent); cf. Chapter 6 of [44]. For such an equation, we may express the nonexistence of a solution to a particular Diophantine equation as a sentence $\phi$ in first order arithmetic. Interpreting this sentence in $\mathrm{Homeo}(M)$ to get a group theoretic sentence $\tilde{\phi}$, we see that if $\psi$ isolates $M$ then $\psi \wedge \tilde{\phi}$ isolates $M$ if and only if ZFC is consistent (or, if and only if PA is consistent). A similar argument works for sentences isolating some manifold. Thus, for a particular Gödel numbering, there are numbers whose membership in $\mathrm{Sent}_M$ and Sent cannot be proved in ZFC.

We finally establish Theorem 6.2.

*Proof of Theorem 6.2.* Let $\phi \in \chi(A)$ finitely axiomatize some $a \in A$ and let $\theta$ finitely axiomatize some $\varnothing \neq b \notin A$; the sentence $\theta$ exists since $A$ is assumed to be proper. For each arithmetic sentence $\psi$, we let

$$\psi^* := (\tilde{\psi} \wedge \phi) \vee (\neg\tilde{\psi} \wedge \theta).$$

Notice that $\#\psi^* \in \chi(A)$ if and only if $\mathrm{Arith}_2 \models \psi$. Indeed, if $\psi$ is true in arithmetic then $\tilde{\psi}$ is true for all compact manifolds and $\neg\tilde{\psi} \wedge \theta$ is false for all compact manifolds. In this case, $\psi^*$ is true in $\mathrm{Homeo}(M)$ if and only if $\phi$ holds in $\mathrm{Homeo}(M)$, in which case $\#\psi^* \in \chi(A)$.

Conversely, suppose that $\psi$ is false in arithmetic. Then $\tilde{\psi} \wedge \phi$ is false for all compact manifolds, and so $\psi^*$ is true for $\mathrm{Homeo}(M)$ if and only if $\mathrm{Homeo}(M) \models \theta$, in which case $M \in b \notin A$. It follows that $\#\psi^* \notin \chi(A)$.

Thus, if $\chi(A)$ were definable in second order arithmetic then we could define truth, a contradiction. □

Theorem 6.2 has many other consequences regarding undefinability. As a single example, a finite list of compact manifolds is finitely axiomatized, in view of Theorem 2.3; the set of sentences axiomatizing finite collections of manifolds is itself undefinable.

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# EVALUATION 2-FUNCTORS FOR KAC–MOODY 2-CATEGORIES OF TYPE $A_2$

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**We construct a 2-functor from the Kac–Moody 2-category for the extended quantum affine $\mathfrak{sl}_3$ to the homotopy 2-category of bounded chain complexes with values in the Kac–Moody 2-category for quantum $\mathfrak{gl}_3$, categorifying the evaluation map between the corresponding quantum Kac–Moody algebras. Our approach establishes and exploits a categorical analogue of the well-known relation between the evaluation map and Lusztig's internal braid group action for quantum $\mathfrak{gl}_3$.**



## 1. Introduction

In the 90s, Chari and Pressley launched a systematic study of finite-dimensional representations of quantum affine algebras, starting with affine $\mathfrak{sl}_2$ in [4]. Since then, these representations have been studied intensively and continue to be an active research topic with important open questions and interesting links to other research areas, e.g., mathematical physics and cluster algebras; see [8], for example, for more information.

In affine type $A$, there is a special class of irreducible finite-dimensional representations, the so-called *evaluation representations*. These are obtained by pulling back irreducible representations of finite type $A$ through a so-called *evaluation map*, which is an algebra homomorphism $\mathrm{ev}_{a,n} : \mathbf{U}_{\triangle}(n) \to \mathbf{U}(n)$, where $a \in \mathbb{C}^{\times}$ is

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a scalar, $\mathbf{U}_{\triangle}(n)$ is the so-called *extended quantum affine* $\mathfrak{sl}_n$ and $\mathbf{U}(n) = \mathbf{U}_q(\mathfrak{gl}_n)$, see Section 2. As we will recall in more detail in that section, the level zero weight lattice of the former quantum affine Kac–Moody algebra can be identified with the $\mathfrak{gl}_n$-weight lattice. The fact that we have to pass from $\mathfrak{sl}_n$ to $\mathfrak{gl}_n$ is important and has a categorical counterpart, as we will explain below. For more information on evaluation maps and evaluation representations in general, see, e.g., [6; 5; 7]. Since we wish to categorify this construction, we must pass to the idempotented forms of these algebras, which can be considered as categories, and the evaluation map can therefore be considered as a functor.

Quantum Kac–Moody algebras were *categorified* by Khovanov and Lauda [10], and independently by Rouquier [18]. We call these 2-categories *Kac–Moody 2-categories* after [3]. The ones of interest to us in this paper are $\tilde{\mathcal{U}}_{\triangle}(n)$ and $\tilde{\mathcal{U}}(n)$, which categorify $\mathbf{U}_{\triangle}(n)$ and $\mathbf{U}(n)$, respectively. The tilde indicates that our choice of signs in their definition differs from Khovanov and Lauda's original choices; see below for more comments on this. In finite Dynkin types, all irreducible finite-dimensional representations can be categorified by certain quotients of the Kac–Moody 2-categories, which nowadays go under the name of *cyclotomic KLR algebras*. In other Dynkin types, such as affine Dynkin types, this is not true. In particular, evaluation representations in affine type $A$ cannot be categorified by cyclotomic KLR-algebras, because the latter categorify highest weight representations and evaluation representations do not have a highest weight. However, we conjecture that the evaluation map (considered as an evaluation functor) $\mathrm{ev}_n^t := \mathrm{ev}_{q^t,n}$, for any $t \in \mathbb{Z}$ and $n \in \mathbb{N}_{>2}$, can be categorified by an evaluation 2-functor $\mathcal{E}v_n^t \colon \tilde{\mathcal{U}}_{\triangle}(n) \to K^b(\tilde{\mathcal{U}}(n))$, which can be used to define evaluation 2-representations (i.e., categorified evaluation representations) of $\tilde{\mathcal{U}}_{\triangle}(n)$ by pulling back "irreducible" 2-representations (i.e., cyclotomic KLR algebras) of $\tilde{\mathcal{U}}(n)$. Here $K^b(\tilde{\mathcal{U}}(n))$ denotes the homotopy 2-category of bounded complexes in $\tilde{\mathcal{U}}(n)$, so the 1-morphisms of $\tilde{\mathcal{U}}_{\triangle}(n)$ act by composing with bounded complexes in $\tilde{\mathcal{U}}(n)$. As a matter of fact, we not only conjecture $\mathcal{E}v_n^t$ to exist but also an extension of it to $K^b(\tilde{\mathcal{U}}_{\triangle}(n))$.

In this paper, we prove the first conjecture for $\tilde{\mathcal{U}}_{\triangle}(3)$ and hope that it serves as the base case for an inductive proof for $\tilde{\mathcal{U}}_{\triangle}(n)$, when $n > 3$, in a forthcoming paper. Proving that there is no obstruction to extending $\mathcal{E}v_n^t$ to $K^b(\tilde{\mathcal{U}}_{\triangle}(n))$ is not easy and certainly beyond the scope of this paper and its sequel.

There are two good reasons for publishing the case $n = 3$ separately. Firstly, in this case there is a close relation with the categorification of the internal braid group action on $\mathbf{U}_q(\mathfrak{gl}_3)$ in [1] (strictly speaking, in that paper they consider $\mathbf{U}_q(\mathfrak{sl}_3)$, so part of our work consists in adapting their results to our setting - see the following paragraph for more details). This is the categorical analogue of a relation between the evaluation map and the braid group action on the decategorified level, which is certainly known to experts, although we couldn't find a reference in the literature. We therefore spell it out in Section 2.2.1, because it is not completely straightforward. Its categorification is conceptually clear, but requires solving multiple nontrivial sign problems, which we do by using certain 2-isomorphisms. This is also why we define two versions of the evaluation 2-functor, denoted $\mathcal{E}v$ and $\mathcal{E}v'$, respectively. The former uses relatively nice sign conventions, whereas the signs in the definition of $\mathcal{E}v'$ are much more complicated. However, the latter are easier to match with the signs in the categorified internal braid group action (for our choice of signs in $\tilde{\mathcal{U}}_{\triangle}(3)$ and $\tilde{\mathcal{U}}(3)$), which is necessary to prove that $\mathcal{E}v'$ is well-defined in our approach; see Theorem 4.3 and its proof in Section 6.4. The relation between $\mathcal{E}v$ and $\mathcal{E}v'$, given in Lemma 4.4, guarantees that well-definedness of the latter implies well-definedness of the former. In principle, all of this should also work for $n > 3$, but only if the categorified braid group action extends to $K^b(\tilde{\mathcal{U}}(n))$ (to include the action of longer braids), which has been conjectured to be the case but not yet proved (see [1, Conjecture 1.2]). This is why our approach for $n > 3$ will be completely different. We hope that presenting the base case $n = 3$ here will prepare the ground for the general case and also keep the size of the forthcoming paper within reasonable bounds.

The second reason for publishing this case separately, is that it reveals the need to pass from $\mathfrak{sl}_n$ to $\mathfrak{gl}_n$ once more, but now on the categorical level. Recall that the definition of a Kac–Moody 2-category depends on a choice of invertible scalars and compatible bubble parameters; see, e.g., [13]. In finite type $A$ all choices yield essentially the same 2-category, i.e., up to 2-isomorphism, but in affine type $A$ they don't. In particular, Khovanov and Lauda's original affine type $A$ *unsigned* Kac–Moody 2-category in [10], with all scalars and bubble parameters equal to one, and the Kac–Moody 2-category defined in [15], with nontrivial bubble parameters depending on level zero $\widehat{\mathfrak{gl}}_n$-weights (instead of level zero $\widehat{\mathfrak{sl}}_n$-weights), are not 2-isomorphic when $n$ is odd. This was mentioned in [11] without proof and, therefore, we prove it in Theorem A.4. Although this does not by itself imply that there is no evaluation 2-functor for trivial scalars and bubble parameters when $n = 3$, we failed to find one. More generally, it seems that one is forced to use the scalars and level zero $\widehat{\mathfrak{gl}}_n$-bubble parameters from [15] when $n$ is odd. When $n$ is even, everything is simpler because in that case both choices of scalars and bubble parameters yield essentially the same Kac–Moody 2-category, see Section 6.4.

There is an analogous story for the affine Hecke algebra and its finite-dimensional representations. The categorification of the corresponding evaluation map was carried out in [17] and was technically less challenging than the categorification of the evaluation map for the affine type A Kac–Moody algebra. In both cases, the target (2-)category of the evaluation (2-)functor is a homotopy category of bounded complexes and, as was argued in [17], one motivation for defining and studying evaluation 2-representations is that they might provide some important clues for the development of triangulated 2-representation theory, which at the moment is very poorly understood, even at the most basic level. For example, it was shown in [17] that every evaluation 2-representation of extended affine Soergel bimodules has a *finitary cover*, somehow relating finitary and triangulated 2-representations. The same might hold for the evaluation 2-representations of $\tilde{\mathcal{U}}_{\triangle}(n)$, but that question is outside the scope of this paper. Also in both cases, one would like to categorify tensor products of evaluation representations, which play a fundamental role in the finite-dimensional representation theory of the affine quantum algebras in question; see, e.g., [5, Chapter 12, Section 2C] for the case of $\tilde{\mathcal{U}}_{\triangle}(n)$. However, it is far from clear how to do that at this point. Perhaps it is possible to somehow adapt Webster's tensor algebras of Stendhal diagrams [19] in that case. We hope to address these and some other interesting questions about evaluation 2-representations in the future.

The structure of the paper is as follows. Section 2 reviews the evaluation map/functor $\mathrm{ev}_3^t$ and Section 3 presents the definitions of the affine and finite type A Kac–Moody 2-categories $\tilde{\mathcal{U}}_{\triangle}(3)$ and $\tilde{\mathcal{U}}(3)$ that we will be working with. In Section 4 we define the two evaluation 2-functors $\mathcal{E}v$ and $\mathcal{E}v'$ and prove their relationship to each other. We translate the categorified braid group actions to our choice of scalars in Section 5, and then in Section 6 we prove Theorem 4.3, saying that $\mathcal{E}v'$ is a well-defined 2-functor that decategorifies to $\mathrm{ev}_3^t$, from which Theorem 4.1 follows. We finish the paper with Section 6.4, where we justify our choice of the scalars and bubble parameters in the definition of $\tilde{\mathcal{U}}_{\triangle}(3)$ over a choice in [10] by proving in Theorem A.4 that the two choices are not related by a 2-isomorphism that fixes objects and 1-morphisms.

## 2. The decategorified setting

Our main reference for this section is [7], though the evaluation map was first considered in [9]. Note that we are interested in the idempotented version of some of the quantum algebras in that paper, so we have to adapt Du and Fu's definitions. We use the idempotented versions because these are the ones that are categorified by Kac–Moody 2-categories.

**2.1.** ***Finite type $\dot{\mathrm{U}}(n)$ and affine type $\dot{\mathrm{U}}_{\triangle}(n)$ of level zero.*** Throughout this paper we identify both the (integral) $\mathfrak{gl}_n$-weight lattice and the level-zero (integral) $\widehat{\mathfrak{gl}}_n$-weight lattice with $\mathbb{Z}^n$, denoting either sort of (integral) weight by, e.g., $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$. The simple $\widehat{\mathfrak{gl}}_n$-roots $\alpha_1, \dots, \alpha_n$ are then given by

$$\alpha_i = \begin{cases} (0, \dots, 0, 1, -1, 0, \dots, 0) & \text{if } 1 \le i \le n-1, \\ (-1, 0, \dots, 0, 1) & \text{if } i = n, \end{cases}$$

where the 1 is always the $i$-th entry. Note that $\alpha_1, \dots, \alpha_{n-1}$ are the simple $\mathfrak{gl}_n$-roots.

Under this identification, the bilinear form on these weight lattices corresponds to the Euclidean inner product on $\mathbb{Z}^n$. Its restriction to the root lattices then reads

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i \equiv j \pm 1 \bmod n, \\ 0 & \text{else,} \end{cases}$$

for $1 \leq i, j \leq n$. Note that, in the affine case, the indices $1, \ldots, n$ are interpreted as representatives of the residue classes modulo $n$. From now on, we will always tacitly use this interpretation of the indices of affine weights and roots. We also recall the standard notation $i \cdot j := (\alpha_i, \alpha_j)$, which we will often use below.

Finally, given $\lambda \in \mathbb{Z}^n$, define $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_n) \in \mathbb{Z}^n$, where $\bar{\lambda}_i = \lambda_i - \lambda_{i+1}$ for all $i = 1, \ldots, n$. By the above convention for affine weights, we have $\bar{\lambda}_n = \lambda_n - \lambda_1$, so $\bar{\lambda}_1 + \ldots + \bar{\lambda}_n = 0$. In other words, $\bar{\lambda}$ belongs to a rank $n-1$ sublattice of $\mathbb{Z}^n$, which can be identified with the level-zero integral $\widehat{\mathfrak{sl}}_n$-weight lattice. The element $(\bar{\lambda}_1, \ldots, \bar{\lambda}_{n-1}) \in \mathbb{Z}^{n-1}$ can then be identified with an (integral) $\mathfrak{sl}_n$-weight.

Recall that the quantum integer $[m]$, for $m \in \mathbb{Z}$, is defined as

$$[m] = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

**Definition 2.1.** The *idempotented extended quantum affine* $\mathfrak{sl}_n$, denoted by $\dot{\mathbf{U}}_{\triangle}(n)$, is the associative idempotented $\mathbb{Q}(q)$-algebra generated by $1_\lambda$, $E_i 1_\lambda$ and $F_i 1_\lambda$, for $\lambda \in \mathbb{Z}^n$ and $i = 1, \ldots, n$, subject to the relations

$$\begin{aligned}
1_\lambda 1_\mu &= \delta_{\lambda,\mu} 1_\lambda, \\
E_i 1_\lambda 1_{\lambda'} &= \delta_{\lambda,\lambda'} E_i 1_\lambda, \\
F_i 1_\lambda 1_{\lambda'} &= \delta_{\lambda,\lambda'} F_i 1_\lambda, \\
1_\mu E_i 1_\lambda &= \delta_{\mu,\lambda+\alpha_i} E_i 1_\lambda, \\
1_\mu F_i 1_\lambda &= \delta_{\mu,\lambda-\alpha_i} F_i 1_\lambda, \\
E_i F_j 1_\lambda - F_j E_i 1_\lambda &= \delta_{i,j} [\bar{\lambda}_i] 1_\lambda, \\
E_i E_j 1_\lambda &= E_j E_i 1_\lambda && \text{if } i \cdot j = 0, \\
F_i F_j 1_\lambda &= F_j F_i 1_\lambda && \text{if } i \cdot j = 0, \\
E_i^2 E_j 1_\lambda + E_j E_i^2 1_\lambda &= [2] E_i E_j E_i 1_\lambda && \text{if } i \cdot j = -1, \\
F_i^2 F_j 1_\lambda + F_j F_i^2 1_\lambda &= [2] F_i F_j F_i 1_\lambda && \text{if } i \cdot j = -1.
\end{aligned}$$

Note that $E_i 1_\lambda = 1_{\lambda+\alpha_i} E_i 1_\lambda$, so we can use the notation $E_i E_j 1_\lambda := E_i 1_{\lambda+\alpha_j} \cdot E_j 1_\lambda$ without ambiguity. Similarly, we will use the notation $1_\mu E_i = 1_\mu E_i 1_{\mu-\alpha_i}$ and $1_\mu F_i = 1_\mu F_i 1_{\mu+\alpha_i}$, so that $E_i 1_\lambda = 1_{\lambda+\alpha_i} E_i$ and $F_i 1_\lambda = 1_{\lambda-\alpha_i} F_i$.

**Definition 2.2.** The *idempotented quantum* $\mathfrak{gl}_n$, denoted by $\dot{\mathbf{U}}(n)$, is the idempotented subalgebra of $\dot{\mathbf{U}}_\triangle(n)$ generated by $1_\lambda$, $E_i 1_\lambda$ and $F_i 1_\lambda$, for $i = 1, \dots, n-1$ and $\lambda \in \mathbb{Z}^n$.

Note that $\dot{\mathbf{U}}_\triangle(n)$ and $\dot{\mathbf{U}}(n)$ share the same idempotents, but, whereas $\dot{\mathbf{U}}(n) = \dot{\mathbf{U}}(\mathfrak{gl}_n)$, the idempotented algebra $\dot{\mathbf{U}}_\triangle(n)$ is only an idempotented subalgebra of $\dot{\mathbf{U}}(\widehat{\mathfrak{gl}}_n)$, which is why it is called the idempotented *extended* quantum affine $\mathfrak{sl}_n$ and not the idempotented quantum affine $\mathfrak{gl}_n$. See [7, Section 2] for more details.

**Remark 2.3.** Recall that these idempotented algebras can be seen as linear categories whose object sets are given by the sets of weights and whose hom-spaces are given by, e.g.,

$$\mathrm{Hom}_{\dot{\mathbf{U}}(n)}(\lambda, \mu) = 1_\mu \dot{\mathbf{U}}(n) 1_\lambda$$

with composition corresponding to multiplication. This is why these idempotented algebras are categorified by 2-categories rather than categories.

**2.2. *Evaluation maps.*** Fix $t \in \mathbb{Z}$ and let $[X, Y]_{q^{\pm 1}} = XY - q^{\pm 1} YX$ be the $q^{\pm 1}$-commutator. From now on we will always assume that $n > 2$.

**Definition 2.4.** The *evaluation map* $\mathrm{ev}_n^t \colon \dot{\mathbf{U}}_\triangle(n) \to \dot{\mathbf{U}}(n)$ is the homomorphism of idempotented algebras defined by

(1) $\mathrm{ev}_n^t(1_\lambda) = 1_\lambda,$

(2) $\mathrm{ev}_n^t(E_i 1_\lambda) = E_i 1_\lambda \quad \text{for } i \neq n,$

(3) $\mathrm{ev}_n^t(F_i 1_\lambda) = F_i 1_\lambda \quad \text{for } i \neq n,$

(4) $\mathrm{ev}_n^t(E_n 1_\lambda) = q^{\lambda_1+\lambda_n+t-1}[\cdots[[F_1, F_2]_q, F_3]_q \cdots]_q, F_{n-1}]_q 1_\lambda,$

(5) $\mathrm{ev}_n^t(F_n 1_\lambda) = q^{-\lambda_1-\lambda_n-t+1}[E_{n-1}, [E_{n-2}, [\cdots[E_2, E_1]_{q^{-1}}]_{q^{-1}} \cdots]_{q^{-1}} 1_\lambda.$

**Remark 2.5.** (a) Note that

$$[\cdots[[F_1, F_2]_q, F_3]_q \cdots]_q, F_{n-1}]_q 1_\lambda = 1_{\lambda-\alpha_1-\cdots-\alpha_{n-1}}[\cdots[[F_1, F_2]_q, F_3]_q \cdots]_q, F_{n-1}]_q,$$

so $\mathrm{ev}_n^t(E_n 1_\lambda) = \mathrm{ev}_n^t(1_{\lambda+\alpha_n} E_n 1_\lambda)$ is well defined, because $\alpha_1 + \ldots + \alpha_{n-1} + \alpha_n = 0$. The same is true for $\mathrm{ev}_n^t(F_n 1_\lambda) = \mathrm{ev}_n^t(1_{\lambda-\alpha_n} F_n 1_\lambda)$.

(b) When we consider the idempotented algebras as categories as in Remark 2.3, $\mathrm{ev}_n^t$ becomes a linear functor. This is why it is categorified by a 2-functor rather than a functor.

The expressions for $\mathrm{ev}_n^t(E_n 1_\lambda)$ and $\mathrm{ev}_n^t(F_n 1_\lambda)$ in (4) and (5) can be written as alternating sums, which will be important later on. For $\xi = (\xi_1, \dots, \xi_{n-2}) \in$

$\{0,1\}^{n-2}$ set

$$E_\xi 1_\lambda := E_{n-1}^{1-\xi_{n-2}} E_{n-2}^{1-\xi_{n-3}} \cdots E_2^{1-\xi_1} E_1 E_2^{\xi_1} \cdots E_{n-2}^{\xi_{n-3}} E_{n-1}^{\xi_{n-2}} 1_\lambda, \tag{6}$$

$$F_\xi 1_\lambda := F_{n-1}^{\xi_{n-2}} F_{n-2}^{\xi_{n-3}} \cdots F_2^{\xi_1} F_1 F_2^{1-\xi_1} \cdots F_{n-2}^{1-\xi_{n-3}} F_{n-1}^{1-\xi_{n-2}} 1_\lambda. \tag{7}$$

and let $|\xi| = \xi_1 + \cdots + \xi_{n-2}$. The following can be obtained by direct computation.

**Lemma 2.6.** *We have*

$$\mathrm{ev}_n^t(E_n 1_\lambda) = q^{\lambda_1+\lambda_n+t-1} \sum_{\xi\in\{0,1\}^{n-2}} (-q)^{|\xi|} F_\xi 1_\lambda, \tag{8}$$

$$\mathrm{ev}_n^t(F_n 1_\lambda) = q^{-\lambda_1-\lambda_n-t+1} \sum_{\xi\in\{0,1\}^{n-2}} (-q)^{-|\xi|} E_\xi 1_\lambda. \tag{9}$$

For more details on the evaluation map, see [7, Section 5].

**2.2.1.** *Connection with the braid group action for $n = 3$.* For each $i = 1, \dots, n-1$ and $e = \pm 1$, Lusztig defined algebra automorphisms $T'_{i,e}$ and $T''_{i,e}$ of $\dot{\mathrm{U}}(\mathfrak{sl}_n)$; see, e.g., [14, Section 37.1] for their definition, which we can adapt to $\dot{\mathrm{U}}(n)$ without issue. The two automorphisms are related by the equation $(T'_{i,e})^{-1} = T''_{i,-e}$ (see [14, Proposition 37.1.2]) and, for a fixed choice of $e$, the $T'_{1,e}, \dots, T'_{n-1,e}$, resp. the $T''_{1,e}, \dots, T''_{n-1,e}$, satisfy the braid relations (see [14, Theorem 39.4.3]) and, therefore, define two actions of the braid group $B_n$ on $\dot{\mathrm{U}}(n)$, called the *internal braid group actions*.

Let $n = 3$ and $t \in \mathbb{Z}$, and set $\mathrm{ev} = \mathrm{ev}_3^t$. Comparison of the expressions in [14, Subsection 37.1.3] with the ones in Definition 2.4 shows that ev can be partially expressed in terms of the above algebra automorphisms. For $i = 1, 3$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$, we have

$$\begin{aligned}
\mathrm{ev}(E_3 1_\lambda) &= q^{\lambda_1+\lambda_3+t-1}\, T'_{1,-1}(F_2 1_{s_1(\lambda)}),\\
\mathrm{ev}(F_3 1_\lambda) &= q^{-\lambda_1-\lambda_3-t+1}\, T'_{1,-1}(E_2 1_{s_1(\lambda)}),\\
\mathrm{ev}(E_1 1_\lambda) &= -q^{\lambda_1-\lambda_2}\, T'_{1,-1}(F_1 1_{s_1(\lambda)}),\\
\mathrm{ev}(F_1 1_\lambda) &= -q^{-\lambda_1+\lambda_2+2}\, T'_{1,-1}(E_1 1_{s_1(\lambda)}),
\end{aligned}$$

where $s_1(\lambda) = (\lambda_2, \lambda_1, \lambda_3)$. For $i = 2, 3$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$, we have

$$\begin{aligned}
\mathrm{ev}(E_3 1_\lambda) &= q^{\lambda_1+\lambda_3+t-1}\, T''_{2,1}(F_1 1_{s_2(\lambda)}),\\
\mathrm{ev}(F_3 1_\lambda) &= q^{-\lambda_1-\lambda_3-t+1}\, T''_{2,1}(E_1 1_{s_2(\lambda)}),\\
\mathrm{ev}(E_2 1_\lambda) &= -q^{-\lambda_2+\lambda_3+2}\, T''_{2,1}(F_2 1_{s_2(\lambda)}),\\
\mathrm{ev}(F_2 1_\lambda) &= -q^{\lambda_2-\lambda_3}\, T''_{2,1}(E_2 1_{s_2(\lambda)}),
\end{aligned}$$

where $s_2(\lambda) = (\lambda_1, \lambda_3, \lambda_2)$. Using the fact that $T'_{1,-1}$ and $T'_{2,1}$ are well-defined algebra automorphisms of $\dot{\mathbf{U}}(3)$, it is easy to prove that ev: $\dot{\mathbf{U}}_{\triangle}(3) \to \dot{\mathbf{U}}(3)$ is a well-defined algebra homomorphism. Specifically, the fact that $T'_{1,-1}$ is an algebra automorphisms implies that ev preserves the relations in Definition 2.1 for $i = 1, 3$, the fact that $T''_{2,1}$ is an algebra automorphisms implies that ev preserves the relations in Definition 2.1 for $i = 2, 3$, and ev preserves the relations in Definition 2.1 for $i = 1, 2$ by definition, of course. Since all relations in Definition 2.1 involve either one colour $i$ or two colours $i, j$, and it's very easy to check that ev preserves the one-colour relations directly, we see that ev preserves all relations in $\dot{\mathbf{U}}_{\triangle}(3)$ and is therefore a well-defined algebra homomorphism.

Of course, one can also prove that ev preserves the relations in $\dot{\mathbf{U}}_{\triangle}(3)$ directly, but that is besides the point. To show that the evaluation 2-functor $\mathcal{E}v$ preserves the relations in $\tilde{\mathcal{U}}_{\triangle}(3)$, we will follow the same reasoning as above for all one- and two-colour KLR relations, taking advantage of the categorification of $T'_{i,1}$ in [1]. For the three-colour KLR relations, the results in that paper cannot be used and we will give a direct proof.

## 3. Kac–Moody 2-categories

We will move on to recalling in detail the 2-categories $\tilde{\mathcal{U}}(n)$ and $\tilde{\mathcal{U}}_{\triangle}(n)$ as defined in [16, Definition 3.1] and [15, Definition 3.19], respectively. These decategorify to $\dot{\mathbf{U}}(n)$ and $\dot{\mathbf{U}}_{\triangle}(n)$.

**3.1. *Definition.*** We define $\tilde{\mathcal{U}}_{\triangle}(n)$ and $\tilde{\mathcal{U}}(n)$ simultaneously, because only the range of the indices of the 1-morphisms and of the colours of the 2-morphisms differ. For concreteness, we will work over $\mathbb{Q}$, but any field of characteristic zero would serve.

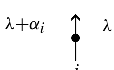
**Definition 3.1.** The 2-category $\tilde{\mathcal{U}}_{\triangle}(n)$ (resp. $\tilde{\mathcal{U}}(n)$) is the graded $\mathbb{Q}$-linear 2-category with:

- Objects: $\lambda \in \mathbb{Z}^n$.
- 1-morphisms: formal direct sums of shifts of

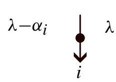
$$\mathbf{1}_\lambda, \quad \mathcal{E}_i\mathbf{1}_\lambda = \mathbf{1}_{\lambda+\alpha_i}\mathcal{E}_i\mathbf{1}_\lambda = \mathbf{1}_{\lambda+\alpha_i}\mathcal{E}_i, \quad \mathcal{E}_i\mathbf{1}_\lambda = \mathbf{1}_{\lambda-\alpha_i}\mathcal{E}_i\mathbf{1}_\lambda = \mathbf{1}_{\lambda-\alpha_i}\mathcal{E}_i,$$

for $\lambda \in \mathbb{Z}^n$ and for $i \in \{1, \ldots, n\}$ (resp. $i \in \{1, \ldots, n-1\}$).

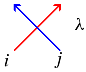
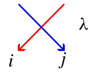
- 2-morphisms: equivalence classes of $\mathbb{Q}$-linear combinations of diagrams obtained by horizontally concatenating and vertically gluing the generators below. By convention, a 2-morphism $\alpha\colon X\langle r\rangle \to Y\langle s\rangle$, for $r, s \in \mathbb{Z}$, is given by a linear combination of homogeneous diagrams of degree $s - r$, as defined in [10].

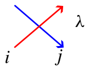
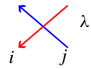


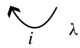
$: \mathcal{E}_i\mathbf{1}_\lambda \to \mathcal{E}_i\mathbf{1}_\lambda\langle 2\rangle\,,$

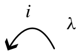


$: \mathcal{F}_i\mathbf{1}_\lambda \to \mathcal{F}_i\mathbf{1}_\lambda\langle 2\rangle\,,$

 $: \mathscr{E}_i\mathscr{E}_j\mathbf{1}_\lambda \to \mathscr{E}_j\mathscr{E}_i\mathbf{1}_\lambda\langle -i\cdot j\rangle\,,$  $: \mathscr{F}_i F_j\mathbf{1}_\lambda \to \mathscr{F}_i\mathscr{F}_j\mathbf{1}_\lambda\langle -i\cdot j\rangle\,,$

 $: \mathscr{E}_i\mathscr{F}_j\mathbf{1}_\lambda \to \mathscr{F}_j\mathscr{E}_i\mathbf{1}_\lambda,$  $: \mathscr{F}_i E_j\mathbf{1}_\lambda \to \mathscr{E}_i\mathscr{F}_j\mathbf{1}_\lambda,$

 $: \mathbf{1}_\lambda \to \mathscr{F}_i\mathscr{E}_i\mathbf{1}_\lambda\langle 1+\bar{\lambda}_i\rangle\,,$  $: \mathbf{1}_\lambda \to \mathscr{E}_i\mathscr{F}_j\mathbf{1}_\lambda\langle 1-\bar{\lambda}_i\rangle\,,$

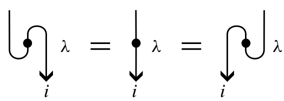
 $: \mathscr{F}_i\mathscr{E}_1\mathbf{1}_\lambda \to \mathbf{1}_\lambda\langle 1+\bar{\lambda}_i\rangle\,,$  $: \mathscr{E}_i F_i\mathbf{1}_\lambda \to \mathbf{1}_\lambda\langle 1-\bar{\lambda}_i\rangle\,.$

The equivalence relation is defined by the equations below.

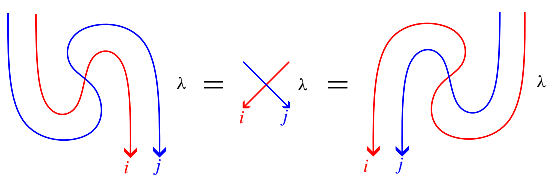
(KM1) Right and left adjunction:

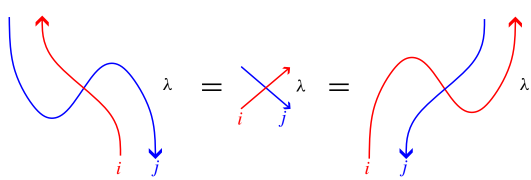
(10) 

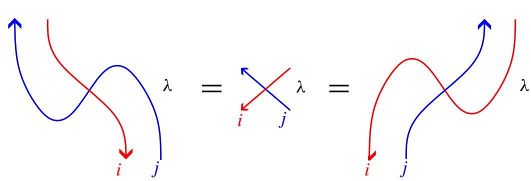
(KM2) Dot cyclicity:

(11) 

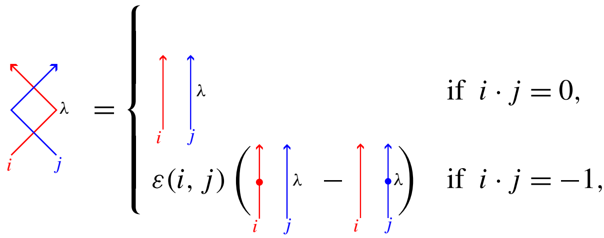
(KM3) Crossing cyclicity:

(12) 

(13) 

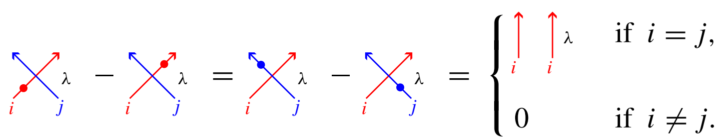
(14) 

(KM4) Quadratic KLR:

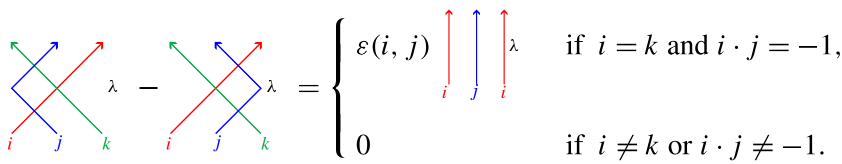
(15)



$$\text{where } \varepsilon(i,j) = \begin{cases} 1 & \text{if } i = j+1 \pmod n, \\ -1 & \text{if } i = j-1 \pmod n, \\ 0 & \text{else.} \end{cases}$$

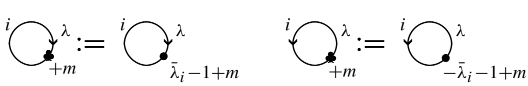
(KM5) Dot slide:

(16)



(KM6) Cubic KLR:

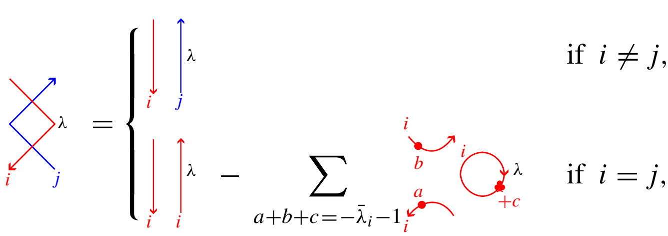
(17)


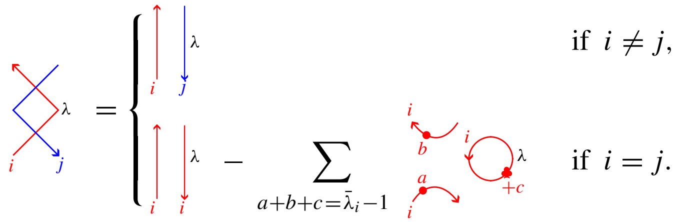

Before we list more relations, we introduce a shorthand using ♣ in lieu of •:

(18)


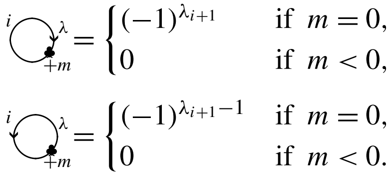

Using this notation, the other relations on diagrams are:

(KM7) Mixed EF:

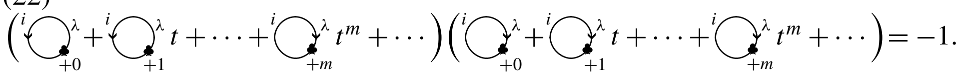
(19)


(20) 


(KM8) Bubble relations:

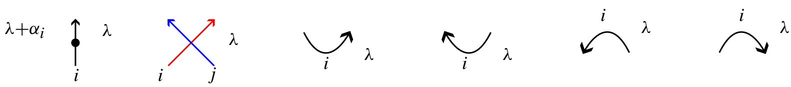
(21) 


(KM9) Infinite Grassmannian relation:

(22)


This ends the definition of the 2-category $\tilde{\mathcal{U}}_{\triangle}(n)$ (resp. $\tilde{\mathcal{U}}(n)$).

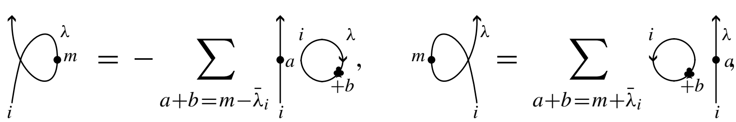
**Remark 3.2.** Thanks to adjunction and cyclicity (equations (10) through (14)), the 2-morphisms of $\tilde{\mathcal{U}}_{\triangle}(n)$ are already generated by

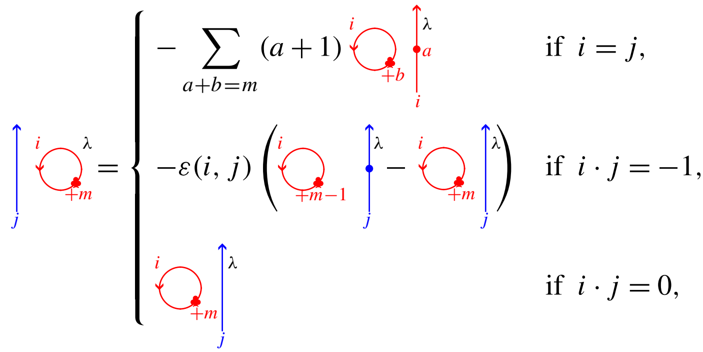


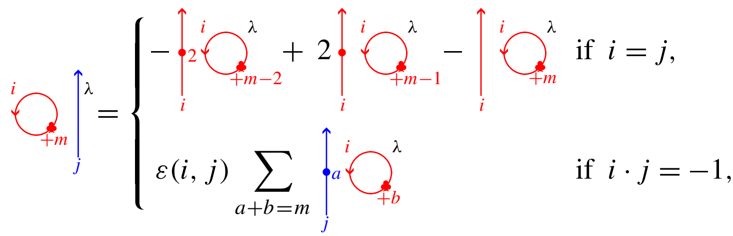
for $\lambda \in \mathbb{Z}^n$ and $i, j = 1, \dots, n$ (and similarly for $\tilde{\mathcal{U}}(n)$). It therefore suffices to define the evaluation functor on this smaller set of generators.

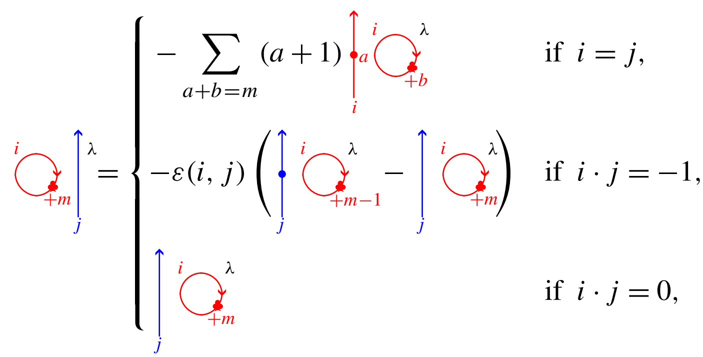
**Remark 3.3.** The choice of signs in Definition 3.1 is not covered by [1]. This choice of signs is referred to as a *choice of scalars* and *bubble parameters*; see [13] for an in-depth explanation. Since we will be adapting various proofs from [1] for the proof of our main result, we will therefore need to take care when translating them across the different sign conventions. We will discuss difference choices of scalars and bubble parameters further in Section 6.4, since they might have implications for the existence of an evaluation 2-functor.

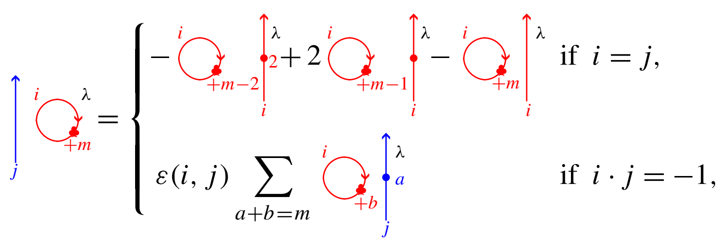
**3.2.** ***Some additional relations.*** Some well-known consequences of the above relations are listed below. For the proofs, see [2], for instance.

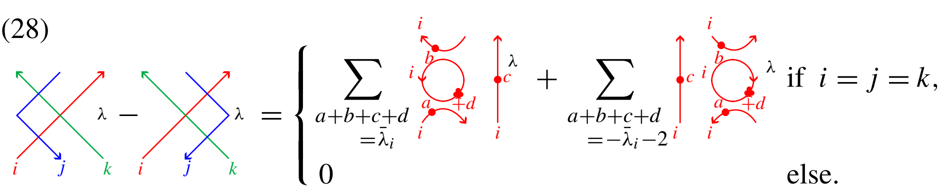
(23) 

(24) 
if $i = j$,
if $i \cdot j = -1$,
if $i \cdot j = 0$,

(25) 
if $i = j$,
if $i \cdot j = -1$,

(26) 
if $i = j$,
if $i \cdot j = -1$,
if $i \cdot j = 0$,

(27) 
if $i = j$,
if $i \cdot j = -1$,

(28) 
if $i = j = k$,
else.

## 4. Two versions of the evaluation 2-functor for $n = 3$

In this section, we will define the 2-functors $\mathcal{E}v$ and $\mathcal{E}v'$ discussed in the introduction, which are $\mathbb{Q}$-linear monoidal functors

$$\mathcal{E}v, \mathcal{E}v' : \tilde{\mathcal{U}}_{\triangle}(3) \to K^b(\tilde{\mathcal{U}}(3)),$$

defined in the next pages. Note that in this case, Definition 2.4 is particularly simple, because (4) and (5) only involve one $q$-commutator each:

$$\begin{aligned} \mathrm{ev}(E_3 1_\lambda) &= q^{\lambda_1+\lambda_3+t-1}(F_1F_2 1_\lambda - qF_2F_1 1_\lambda), \\ \mathrm{ev}(F_3 1_\lambda) &= q^{-\lambda_1-\lambda_3-t+1}(E_2E_1 1_\lambda - q^{-1}E_1E_2 1_\lambda), \end{aligned}$$

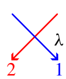
for $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$ (recall that $t \in \mathbb{Z}$ is arbitrary but fixed). For the remainder of this paper, we set $S(\lambda) = \lambda_1 + \lambda_3 + t - 1$, and we suppress the $\lambda$ when there is no confusion. We also use the notation $\mathcal{E}_{i_1 i_2 \dots i_k}\mathbf{1}_\lambda = \mathcal{E}_{i_1}\mathcal{E}_{i_2}\dots\mathcal{E}_{i_k}\mathbf{1}_\lambda$ and $\mathcal{F}_{i_1 i_2 \dots i_k}\mathbf{1}_\lambda = \mathcal{F}_{i_1}\mathcal{F}_{i_2}\dots\mathcal{F}_{i_k}\mathbf{1}_\lambda$.

**4.1.** ***The definition of*** $\mathcal{E}v$**.** In the definition below, we underline the 1-morphism in homological degree zero in each complex.

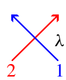
- For objects, we define $\mathcal{E}v(\lambda) = \lambda$, where $\lambda \in \mathbb{Z}^3$.
- For 1-morphisms, we define the action of $\mathcal{E}v$ on generating 1-morphisms and extend it to all 1-morphisms via composition and direct sums, using the standard composition of complexes:

  (1) $\mathcal{E}v(\mathcal{E}_1\mathbf{1}_\lambda) = \underline{\mathcal{E}_1\mathbf{1}_\lambda}$, $\mathcal{E}v(\mathcal{E}_2\mathbf{1}_\lambda) = \underline{\mathcal{E}_2\mathbf{1}_\lambda}$,

  (2) $\mathcal{E}v(\mathcal{F}_1\mathbf{1}_\lambda) = \underline{\mathcal{F}_1\mathbf{1}_\lambda}$, $\mathcal{E}v(\mathcal{F}_2\mathbf{1}_\lambda) = \underline{\mathcal{F}_2\mathbf{1}_\lambda}$,

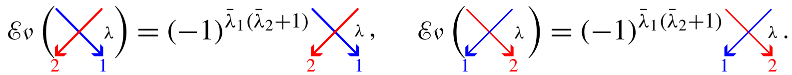
  (3) $\mathcal{E}v(\mathcal{E}_3\mathbf{1}_\lambda) = \underline{\mathcal{F}_{12}\mathbf{1}_\lambda}\langle S\rangle \longrightarrow \mathcal{F}_{21}\mathbf{1}_\lambda\langle S+1\rangle$ ,

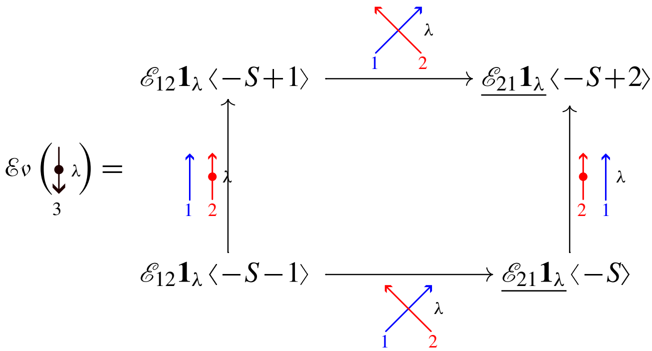
  (4) $\mathcal{E}v(\mathcal{F}_3\mathbf{1}_\lambda) = \mathcal{E}_{12}\mathbf{1}_\lambda\langle -S-1\rangle \longrightarrow \underline{\mathcal{E}_{21}\mathbf{1}_\lambda}\langle -S\rangle$ .

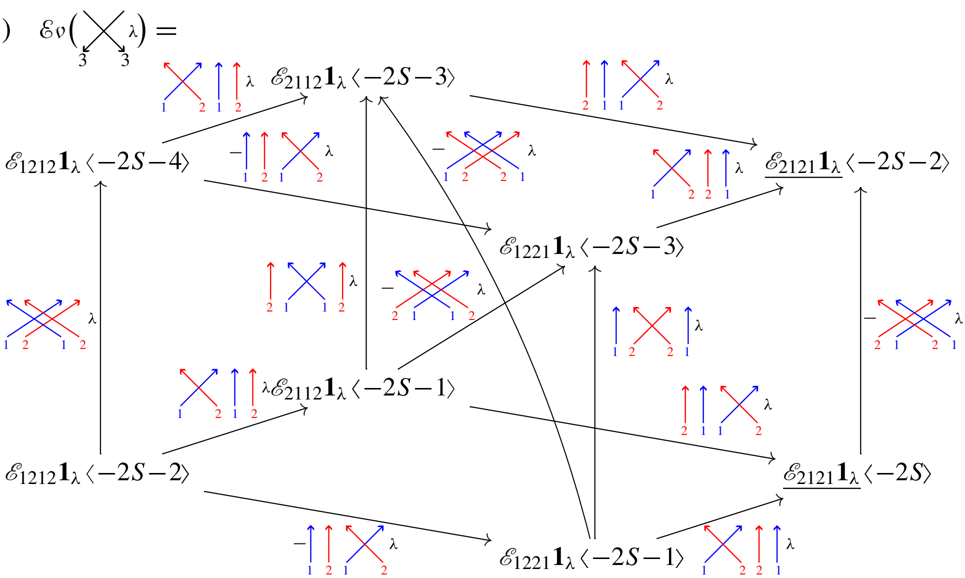
- We set $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3$ and call $|\lambda|$ the *Schur level* of $\lambda$.

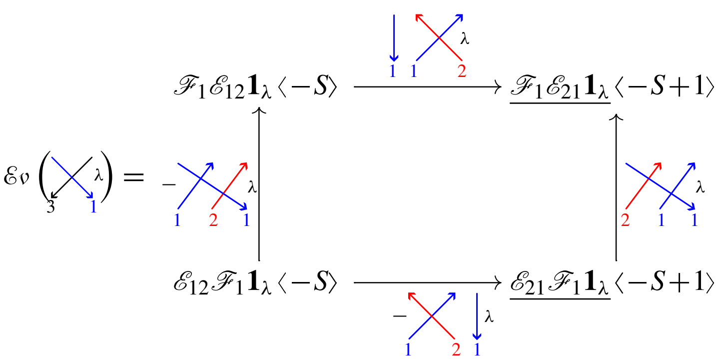
For compatibility with later proofs, we give the definition of $\mathcal{E}v$ on downwards-pointing generating 2-morphisms (that is, the horizontally mirrored versions of the 2-morphisms in Remark 3.2). For generating 2-morphisms consisting only of strands between $\mathcal{E}_1\mathbf{1}_\lambda$, $\mathcal{E}_2\mathbf{1}_\lambda$, $\mathcal{F}_1\mathbf{1}_\lambda$ and $\mathcal{F}_2\mathbf{1}_\lambda$, the 2-functor $\mathcal{E}v$ acts as the identity, with the following exceptions:

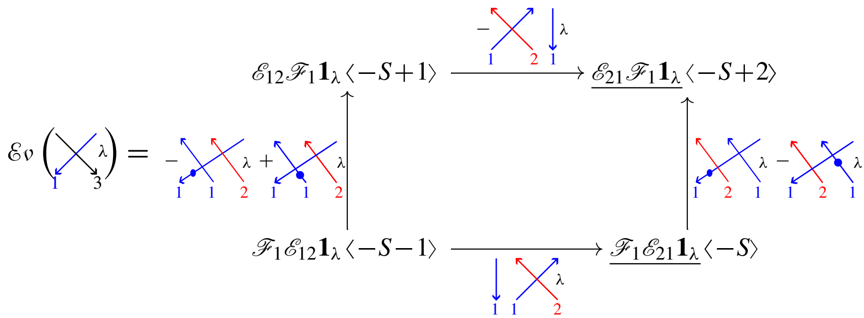
(29) 

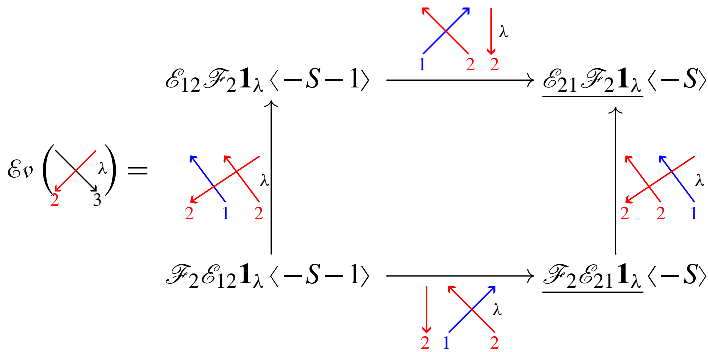
For the remaining generating 2-morphisms, we define the images as follows, emphasising these are commutative diagrams:

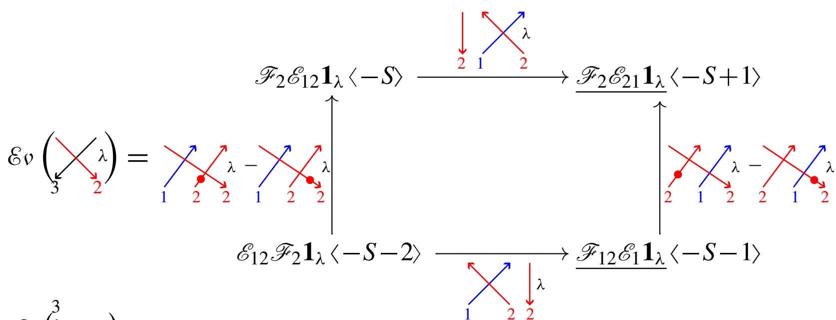
(30) 

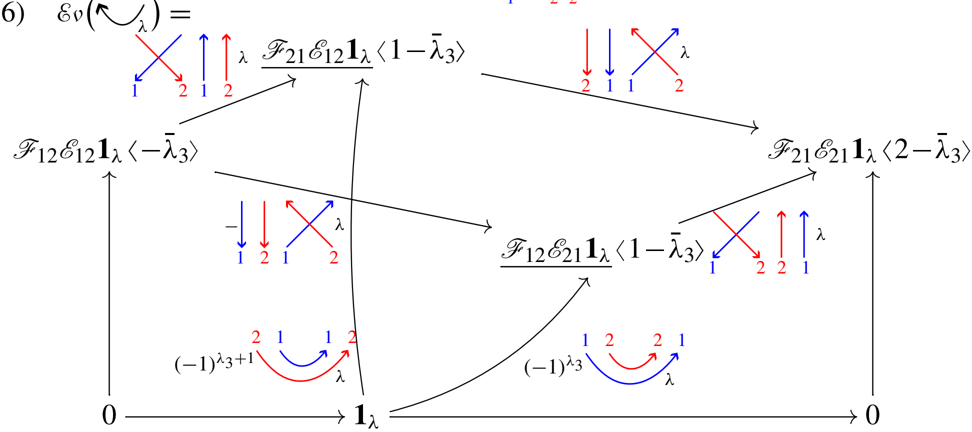

(31) 


(32) 

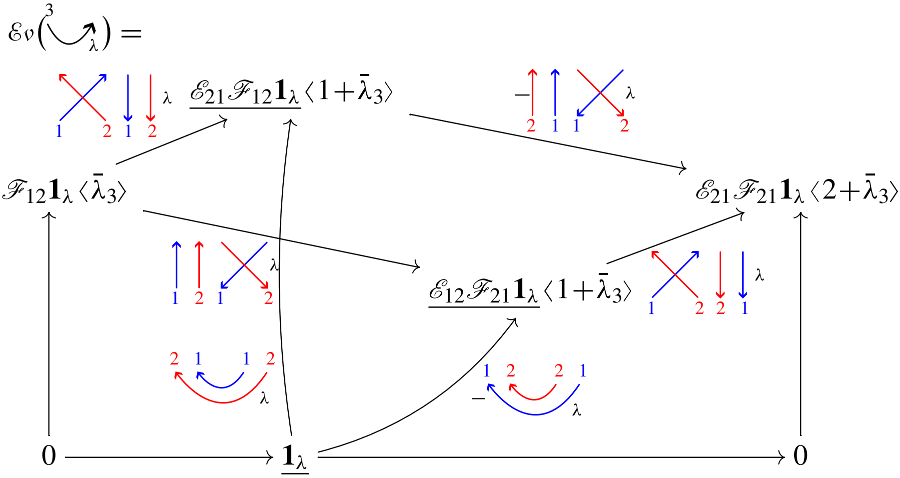
(33)

$\mathcal{E}v$ =
$\mathscr{E}_{12}\mathscr{F}_1\mathbf{1}_\lambda\langle -S+1\rangle$
$\mathscr{E}_{21}\mathscr{F}_1\mathbf{1}_\lambda\langle -S+2\rangle$
$\mathscr{F}_1\mathscr{E}_{12}\mathbf{1}_\lambda\langle -S-1\rangle$
$\mathscr{F}_1\mathscr{E}_{21}\mathbf{1}_\lambda\langle -S\rangle$

(34)

$\mathcal{E}v$ =
$\mathscr{E}_{12}\mathscr{F}_2\mathbf{1}_\lambda\langle -S-1\rangle$
$\mathscr{E}_{21}\mathscr{F}_2\mathbf{1}_\lambda\langle -S\rangle$
$\mathscr{F}_2\mathscr{E}_{12}\mathbf{1}_\lambda\langle -S-1\rangle$
$\mathscr{F}_2\mathscr{E}_{21}\mathbf{1}_\lambda\langle -S\rangle$

(35)

$\mathcal{E}v$ =
$\mathscr{F}_2\mathscr{E}_{12}\mathbf{1}_\lambda\langle -S\rangle$
$\mathscr{F}_2\mathscr{E}_{21}\mathbf{1}_\lambda\langle -S+1\rangle$
$\mathscr{E}_{12}\mathscr{F}_2\mathbf{1}_\lambda\langle -S-2\rangle$
$\mathscr{F}_{12}\mathscr{E}_1\mathbf{1}_\lambda\langle -S-1\rangle$

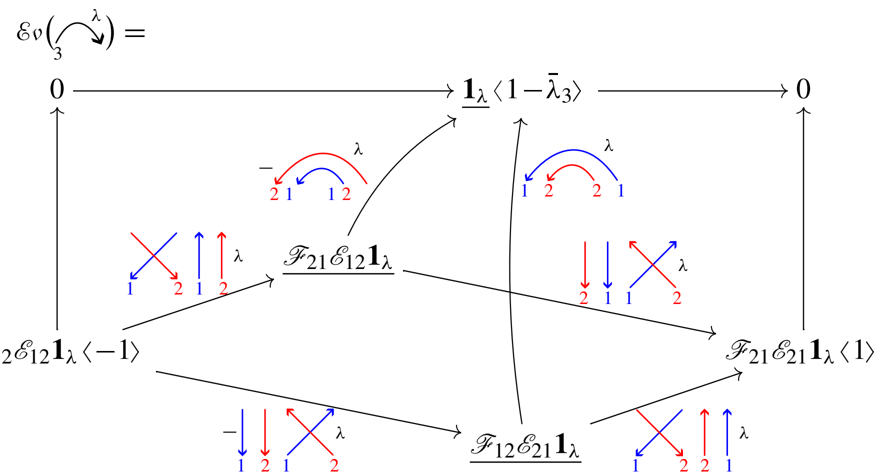
(36)

$\mathcal{E}v$ =
$\mathscr{F}_{21}\mathscr{E}_{12}\mathbf{1}_\lambda\langle 1-\bar{\lambda}_3\rangle$
$\mathscr{F}_{12}\mathscr{E}_{12}\mathbf{1}_\lambda\langle -\bar{\lambda}_3\rangle$
$\mathscr{F}_{21}\mathscr{E}_{21}\mathbf{1}_\lambda\langle 2-\bar{\lambda}_3\rangle$
$\mathscr{F}_{12}\mathscr{E}_{21}\mathbf{1}_\lambda\langle 1-\bar{\lambda}_3\rangle$
$(-1)^{\lambda_3+1}$
$(-1)^{\lambda_3}$
0
$\mathbf{1}_\lambda$
0

(37)



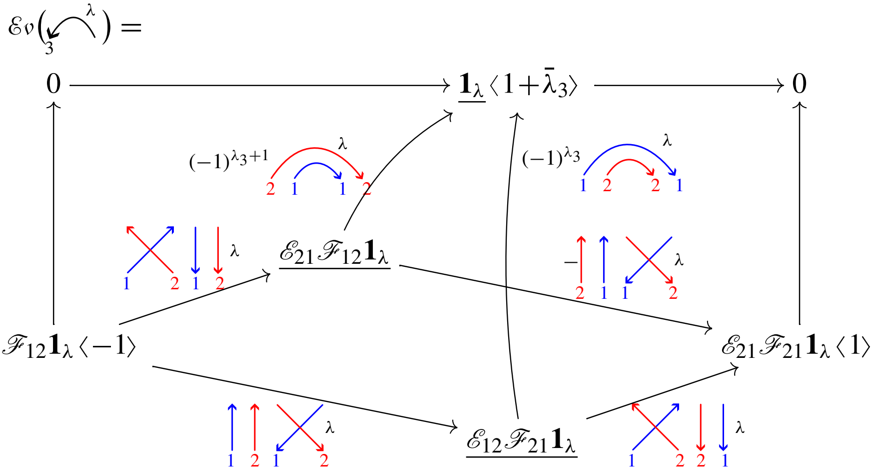
$\mathcal{E}v(\ldots) =$
$\mathcal{E}_{21}\mathcal{F}_{12}\mathbf{1}_\lambda\langle 1+\bar{\lambda}_3\rangle$
$\mathcal{E}_{12}\mathcal{F}_{12}\mathbf{1}_\lambda\langle \bar{\lambda}_3\rangle$
$\mathcal{E}_{21}\mathcal{F}_{21}\mathbf{1}_\lambda\langle 2+\bar{\lambda}_3\rangle$
$\mathcal{E}_{12}\mathcal{F}_{21}\mathbf{1}_\lambda\langle 1+\bar{\lambda}_3\rangle$
0
$\mathbf{1}_\lambda$
0


(38)



$\mathcal{E}v(\ldots) =$
0
$\mathbf{1}_\lambda\langle 1-\bar{\lambda}_3\rangle$
0
$\mathcal{F}_{21}\mathcal{E}_{12}\mathbf{1}_\lambda$
$\mathcal{F}_{12}\mathcal{E}_{12}\mathbf{1}_\lambda\langle -1\rangle$
$\mathcal{F}_{21}\mathcal{E}_{21}\mathbf{1}_\lambda\langle 1\rangle$
$\mathcal{F}_{12}\mathcal{E}_{21}\mathbf{1}_\lambda$


(39)



$\mathcal{E}v(\ldots) =$
0
$\mathbf{1}_\lambda\langle 1+\bar{\lambda}_3\rangle$
0
$(-1)^{\lambda_3+1}$
$(-1)^{\lambda_3}$
$\mathcal{E}_{21}\mathcal{F}_{12}\mathbf{1}_\lambda$
$\mathcal{E}_{12}\mathcal{F}_{12}\mathbf{1}_\lambda\langle -1\rangle$
$\mathcal{E}_{21}\mathcal{F}_{21}\mathbf{1}_\lambda\langle 1\rangle$
$\mathcal{E}_{12}\mathcal{F}_{21}\mathbf{1}_\lambda$

The following theorem is the main result of the paper and we will prove in Section 4.3 that it is a consequence of Theorem 4.3 and Lemma 4.4.

**Theorem 4.1.** *$\mathcal{E}v$ is a well-defined 2-functor that decategorifies to* ev.

**4.2.** ***The definition of $\mathcal{E}v'$.*** To define $\mathcal{E}v'$, we need some notation. For the rest of the paper, we let $\equiv$ denote $\equiv_4$, that is, congruence modulo 4. We define

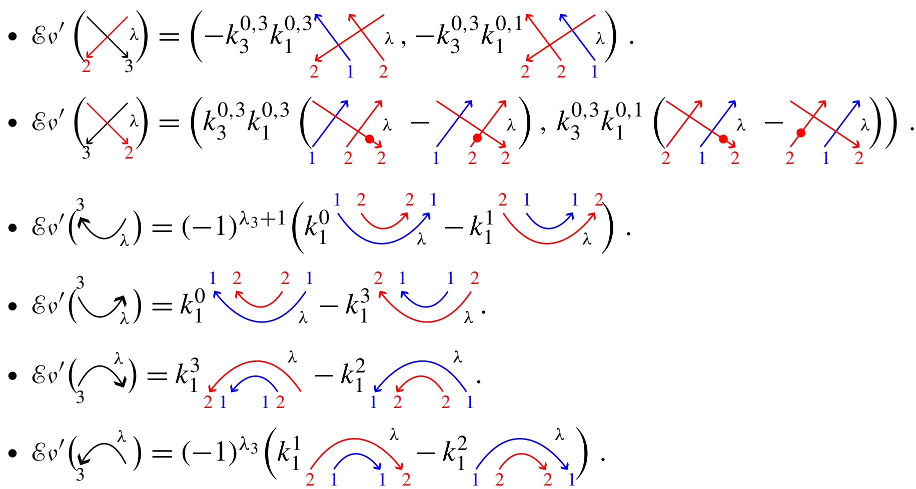
$$k_i^{a_1,\dots,a_n}(\lambda) := \begin{cases} 1 & \text{if } \bar{\lambda}_i \equiv a_1, \dots, a_n, \\ -1 & \text{if otherwise.} \end{cases} \tag{40}$$

We will often omit the argument when it is $\lambda$. For example, $k_1^{0,1} = 1$ (resp. $-1$) if $\bar{\lambda}_1 \equiv 0, 1$ (resp. $\equiv 2, 3$).

**Remark 4.2.** We will use the following relations involving the $k_i^{a_1,\dots,a_n}$ at various points: $k_i^0 k_i^2 = (-1)^{\bar{\lambda}_i+1}$, $k_i^1 k_i^3 = (-1)^{\bar{\lambda}_i}$, $k_1^a(\lambda+\alpha_2) = k_1^{a+1}(\lambda)$, $k_1^a(s_1(\lambda)) = k_1^{-a}(\lambda)$, and $k_1^a(s_2(\lambda)) = k_3^{-a}(\lambda)$.

We will also be using the more compact notation for 2-morphisms between complexes found in [1], where they are presented as ordered tuples (most commonly ordered pairs).

We now define $\mathcal{E}v'$ to be identical to $\mathcal{E}v$ on objects, 1-morphisms and all generating 2-morphisms except for the following:



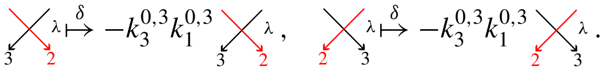
We will prove the following result in Section 6.4.

**Theorem 4.3.** *$\mathcal{E}v'$ is a well-defined 2-functor that decategorifies to* ev.

**4.3.** ***Relating $\mathcal{E}v$ and $\mathcal{E}v'$.*** We now show that $\mathcal{E}v$ can be given by composing $\mathcal{E}v'$ with 2-isomorphisms. The first such 2-isomorphism is $\gamma$, which acts as the identity on objects, 1-morphisms, and generating 2-morphisms with the exception of:

We abuse notation by also using $\gamma$ to refer to the 2-isomorphism of $K^b(\tilde{\mathcal{U}}(3))$ that acts in the same fashion. Similarly to $\beta$, it is straightforward to see that $\gamma$ preserves (KM1) through (KM9), and is therefore a (pair of) well-defined 2-isomorphism(s).

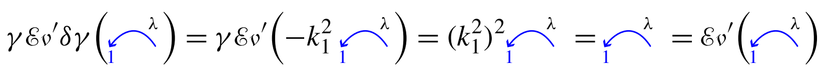
The second 2-isomorphism is $\delta$, which is again the identity on all objects, 1-morphisms and generating 2-morphisms except



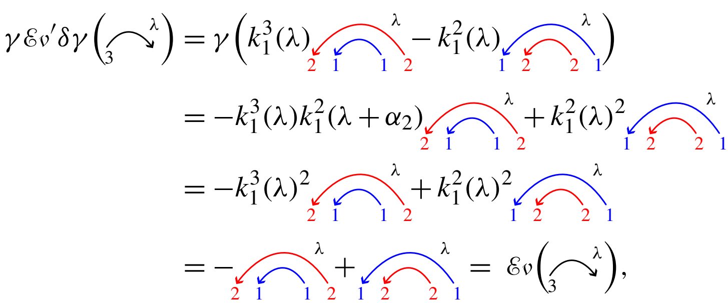
It is again an easy calculation that $\delta$ preserves (KM1) through (KM9) and is therefore a well-defined 2-isomorphism.

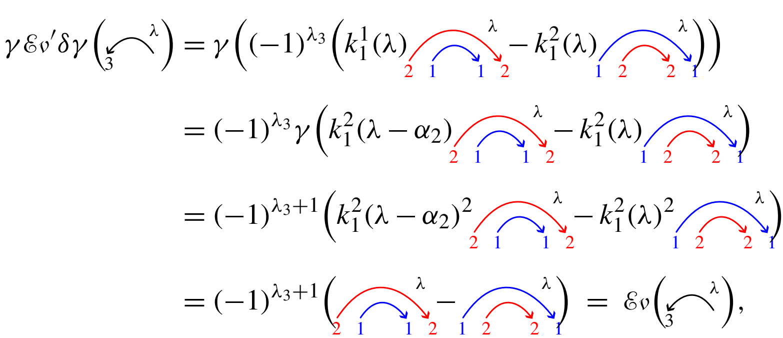
**Lemma 4.4.** $$\mathcal{E}\upsilon = \gamma\mathcal{E}\upsilon'\delta\gamma.$$

*Proof.* Outside of the generating 2-morphisms that $\mathcal{E}\upsilon'$ sends to a 2-morphism of complexes with a dependency on $\bar{\lambda}_i$ modulo 4, $\mathcal{E}\upsilon$ and $\gamma\mathcal{E}\upsilon'\delta\gamma$ agree with $\mathcal{E}\upsilon'$. In particular, we have, e.g.,



For the 2-morphisms that differ we have

(41)


(42)


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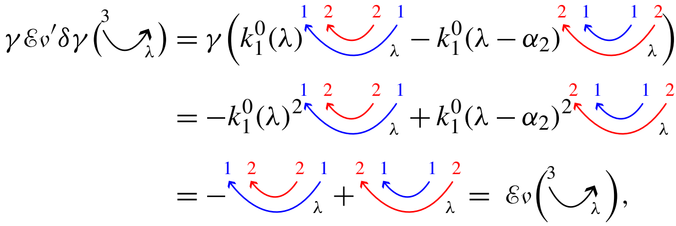
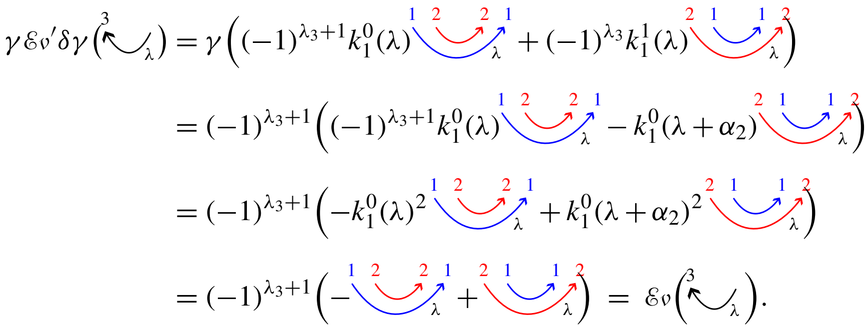
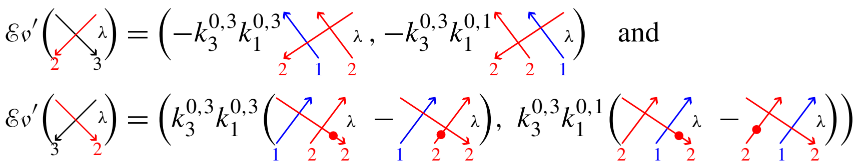


(44)


Recalling that



it is straightforward to see that

□

This shows that Theorem 4.3 implies Theorem 4.1, and so it suffices to prove the former.

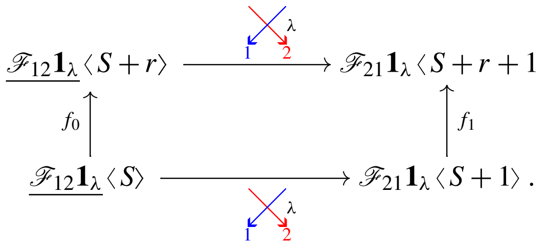
**4.4.** ***Essential uniqueness of the image of the dotted 3-strand.*** Finally, let us show that the above choice for the image of a dotted 3-strand under $\mathcal{E}v$ is the only one possible, up to multiplication by a scalar. We will be using this on occasion in the proof of Theorem 4.3 and elsewhere.

**Lemma 4.5.** $\mathrm{End}^*_{K^b(\tilde{\mathcal{U}}(3))}(\mathcal{E}v(\mathscr{E}_3\mathbf{1}_\lambda))$ *and* $\mathrm{End}^*_{K^b(\tilde{\mathcal{U}}(3))}(\mathcal{E}v(\mathscr{F}_3\mathbf{1}_\lambda))$ *are isomorphic to* $\mathbb{Q}[x]$, *where* $\deg x = 2$.

*Proof.* We tackle the case $\mathrm{End}^*_{K^b(\tilde{\mathcal{U}}(3))}(\mathcal{E}v(\mathscr{E}_3\mathbf{1}_\lambda))$, the other one being similar. We first work in the 2-category of bounded complexes $\mathcal{C}^b(\tilde{\mathcal{U}}(3))$. We claim that

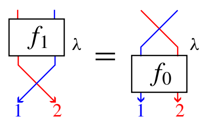
$$\mathrm{End}^*_{\mathcal{C}^b(\tilde{\mathcal{U}}(3))}(\mathcal{E}v(\mathscr{E}_3\mathbf{1}_\lambda)) \cong \mathbb{Q}[x_1, x_2],$$

where $\deg x_1 = \deg x_2 = 2$. An element of $\mathrm{End}^*_{\mathcal{C}^b(\tilde{\mathcal{U}}(3))}(\mathcal{E}v(\mathscr{E}_3\mathbf{1}_\lambda))$ is a commutative square of the form

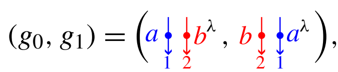


By [10, Theorem 2.7], the shift $r$ has to be even and $f_0$ is a linear combination of 2-morphisms of the form $g_0 = a \downarrow_1 \downarrow_2 b^{\lambda}$, where $a+b = r/2$.

The equality

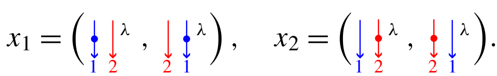


implies that, for each summand $g_0$ of $f_0$, there is a corresponding summand $g_1$ of $f_1$ that is determined by the choice of $g_0$, i.e.,



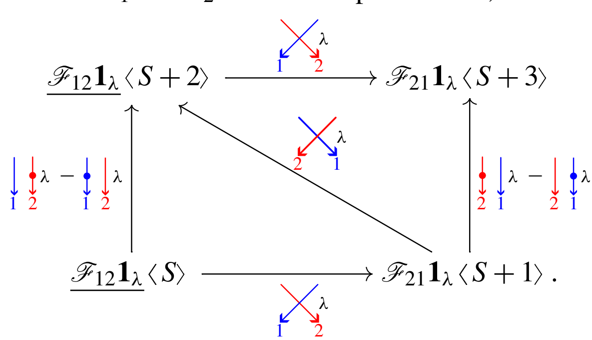
where we use the presentation used in [1] of only giving the vertical 2-morphisms as an ordered pair, for clarity of reading.

This proves the claim, with



We further claim that $x_1$ and $x_2$ are homotopic. Indeed, consider the diagram

(45)



One sees that this diagram is commutative, by the downward version of relation (15), and hence $x_1 - x_2 \simeq_h 0$, which proves the lemma. $\square$

A directly analogous result and proof hold for $\mathscr{E}\mathscr{V}'$.

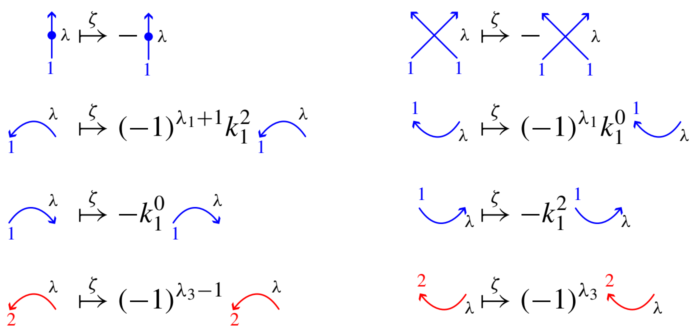
## 5. Categorified braid group action

To categorify the connection between our desired evaluation functor and Lusztig's algebra automorphisms $T'_{1,-1}$ and $T''_{2,1}$, discussed in Section 2.2.1, we need to introduce various 2-functors to deal with some complications. While the automorphisms have already been categorified in [1], that paper works over $\mathfrak{sl}_3$ and does not cover our choice of scalars and bubble parameters. We therefore adapt their constructions to our setup through composition with 2-isomorphisms.

**5.1.** ***The braid group actions.*** Denote by $\mathcal{U}(3)$ the $\mathfrak{gl}_3$ version of the (unsigned version of the) 2-category $\mathcal{U}_Q(\mathfrak{sl}_3)$ defined in [1, Definition 3.3] with the trivial choice of scalars and bubble parameters. For this section, we will be utilising the 2-functors $\mathcal{T}'_{1,1}$ and $\mathcal{T}''_{2,1}$ as defined in [1, Section 4], and the 2-isomorphisms $\omega$, $\psi$ defined in [11] as follows. The 2-isomorphism $\omega : \mathcal{U}(3) \to \mathcal{U}(3)$ is 1- and 2-covariant and degree-preserving, and sends a weight $\lambda$ to $-\lambda$, reverses the orientation of 2-morphisms, and scales the 1, 1- and 2, 2-crossings by a factor of $-1$. Similarly, $\psi : \mathcal{U}(3) \to \mathcal{U}(3)^{co}$ is a 1-covariant, 2-contravariant 2-isomorphism that is the identity on objects, scales weights of 1-morphisms by a factor of $-1$, and reflects diagrams of 2-morphisms in the horizontal axis and then reverses their orientation. We recall that $k_i^{a_1,\dots,a_n}(\mu)$, defined in (40), omits the argument $\mu$ when it is equal to $\lambda$, but retains it otherwise (generally when it is $s_1(\lambda)$ or $s_2(\lambda)$).

We also use the 2-isomorphism $\zeta : \mathcal{U}(3) \to \tilde{\mathcal{U}}(3)$, first defined as $\Sigma$ in [10, Section 4.2] and [12], which is the identity on objects and 1-morphisms, and the identity on 2-morphisms except for the following generating 2-morphisms (and hence the 2-morphisms derived from them):

(46)
(47)
(48)
(49)



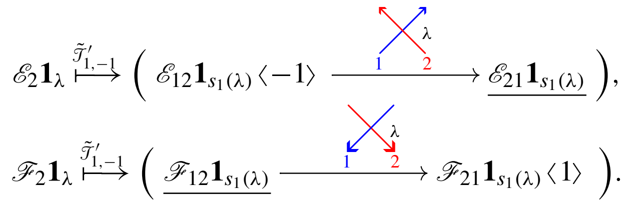

We now define two 2-functors $\tilde{\mathcal{T}}'_{1,-1}, \tilde{\mathcal{T}}''_{2,1} : \tilde{\mathcal{U}}(3) \to K^b(\tilde{\mathcal{U}}(3))$ using composites of the above 2-functors:

$$\tilde{\mathcal{T}}'_{1,-1} := \zeta\psi\mathcal{T}'_{1,1}\psi\zeta^{-1} = \zeta\mathcal{T}'_{1,-1}\zeta^{-1}, \quad \tilde{\mathcal{T}}''_{2,1} := \zeta\omega\mathcal{T}'_{2,1}\omega\zeta^{-1} = \zeta\mathcal{T}''_{2,1}\zeta^{-1}.$$

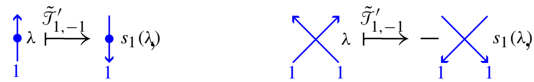
We let $X[y]\langle z\rangle$ denote the 1-term complex with the 1-morphism at homological degree $-y$ with internal degree shift of $z$. In detail, $\tilde{\mathcal{T}}'_{1,-1}$ acts as follows:

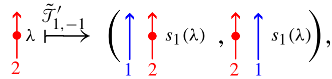
- On objects, $\lambda \overset{\tilde{\mathcal{T}}'_{1,-1}}{\longmapsto} s_1(\lambda)$.
- On 1-morphisms,

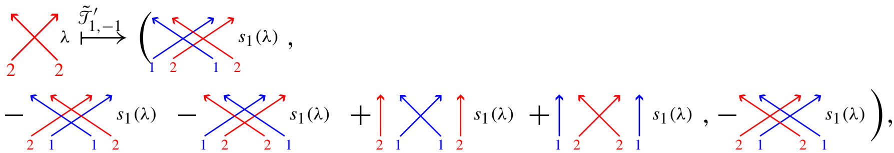
$$\mathcal{E}_1\mathbf{1}_\lambda \overset{\tilde{\mathcal{T}}'_{1,-1}}{\longmapsto} \mathcal{F}_1\mathbf{1}_{s_1(\lambda)}[-1]\langle 2+\bar{\lambda}_1\rangle\,, \qquad \mathcal{F}_1\mathbf{1}_\lambda \overset{\tilde{\mathcal{T}}'_{1,-1}}{\longmapsto} \mathcal{E}_1\mathbf{1}_{s_1(\lambda)}[1]\langle -\bar{\lambda}_1\rangle\,,$$

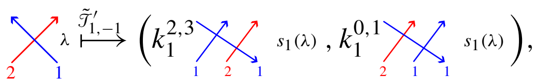


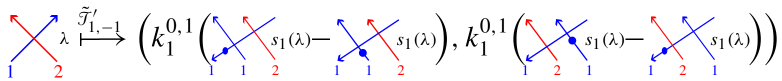
- On nonidentity generating 2-morphisms,

(50) 

(51) 

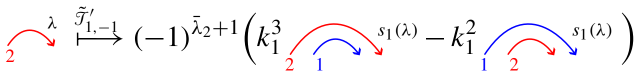
(52) 

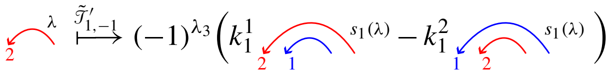
(53) 

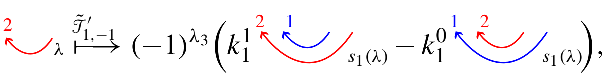
(54) 

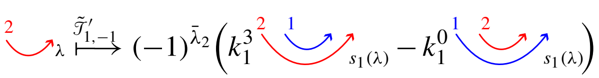
(55) 

(56) 

(57) 

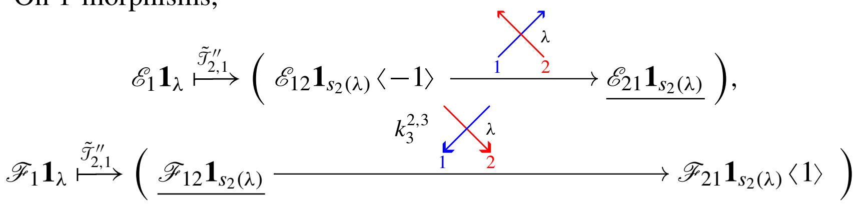
(58) 

(59) 

(60) 

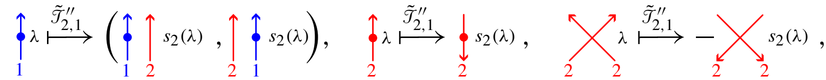
In detail, $\tilde{\mathcal{T}}''_{2,1}$ acts as follows:

- On objects, $\lambda \xmapsto{\tilde{\mathcal{T}}''_{2,1}} s_2(\lambda)$.
- On 1-morphisms,



$$\mathcal{E}_2\mathbf{1}_\lambda \xmapsto{\tilde{\mathcal{T}}''_{2,1}} \mathcal{F}_2\mathbf{1}_{s_2(\lambda)}[-1]\langle\bar{\lambda}_2\rangle\,, \qquad \mathcal{F}_2\mathbf{1}_\lambda \xmapsto{\tilde{\mathcal{T}}''_{2,1}} \mathcal{E}_2\mathbf{1}_{s_2(\lambda)}[1]\langle -2-\bar{\lambda}_2\rangle\,.$$

- On nonidentity generating 2-morphisms,

(61)


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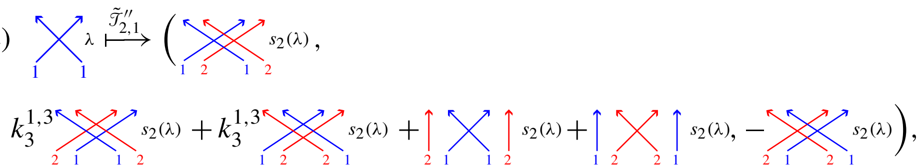

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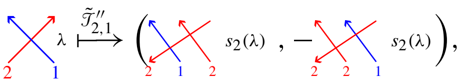

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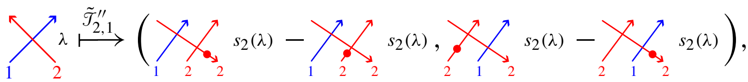

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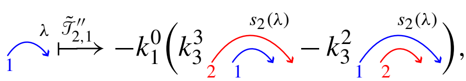

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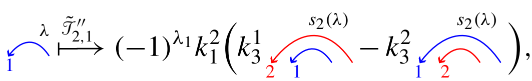

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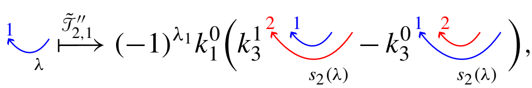

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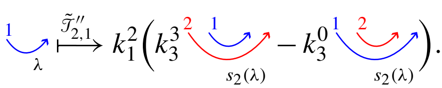
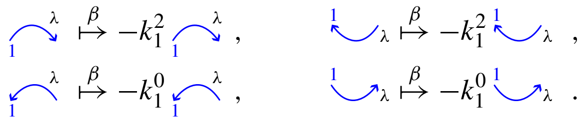


**5.2.** ***Relating the two actions.*** We define a 2-automorphism $\beta : \tilde{\mathcal{U}}(3) \to \tilde{\mathcal{U}}(3)$, which is the identity on objects, 1-morphisms and all generating 2-morphisms with the exception of




It is a straightforward calculation to confirm that this preserves the axioms (KM1)–(KM3) and (KM7)–(KM9) and is therefore a well-defined 2-automorphism.

We also define a 2-automorphism $\alpha : \tilde{\mathcal{U}}(3) \to \tilde{\mathcal{U}}(3)$ which is the identity on all objects and 1-morphisms and on all generating 2-morphisms except

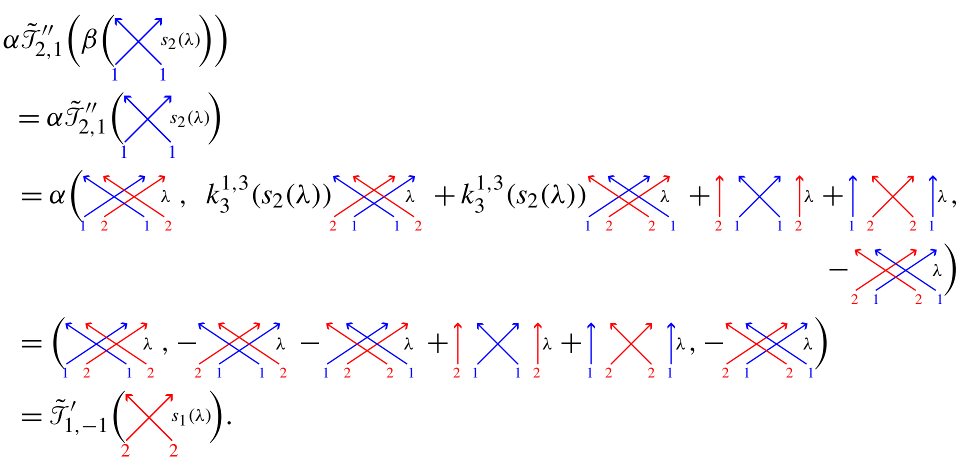

It is easy to see that $\alpha$ preserves axioms (KM3)–(KM7), and is therefore a 2-automorphism. We also use $\alpha$ to denote its extension to a 2-automorphism of $K^b(\tilde{\mathcal{U}}(3))$.

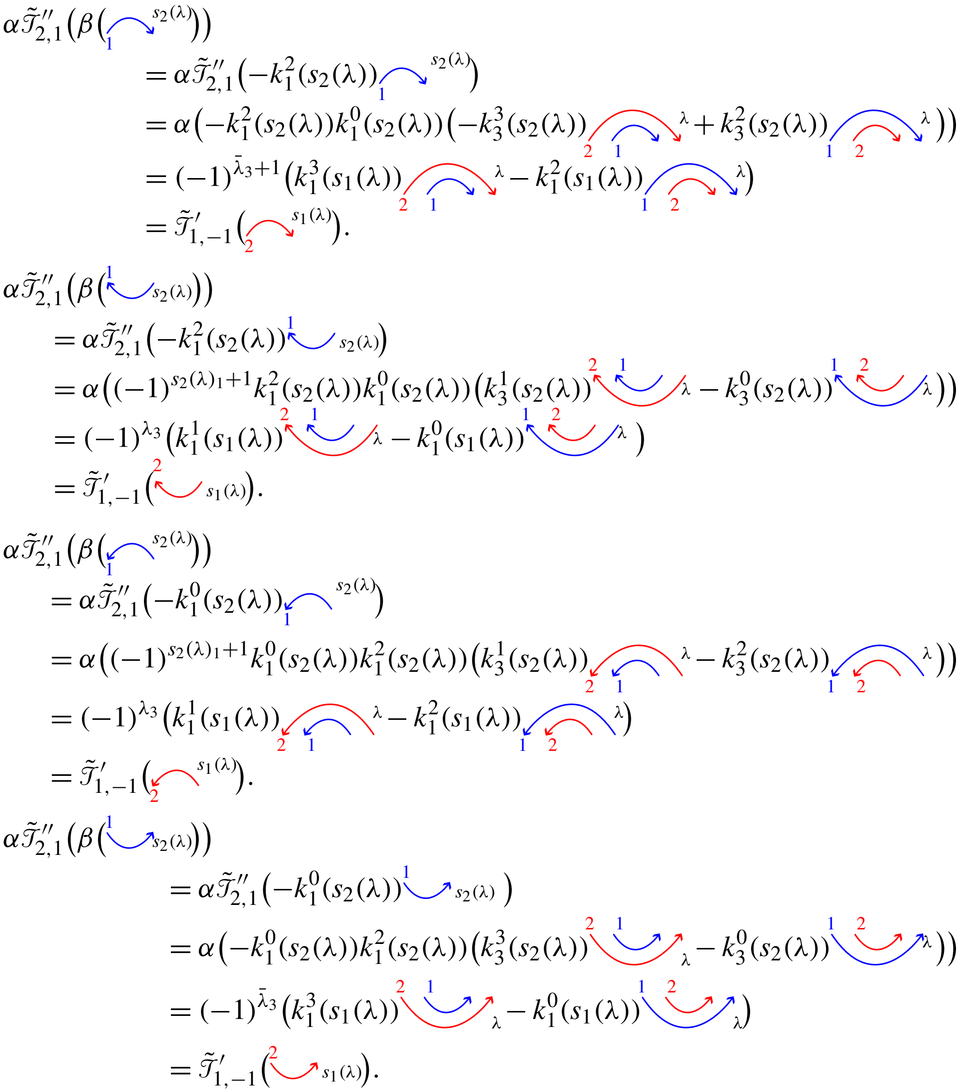
We now define some notation for ease of stating the following lemma. Let $D_i(\lambda)$ denote a 2-morphism diagram with strands mono-coloured in colour $i$ such that the weight to the right of the diagram is $\lambda$.

**Lemma 5.1.** *For any diagram $D_i(\lambda)$,*

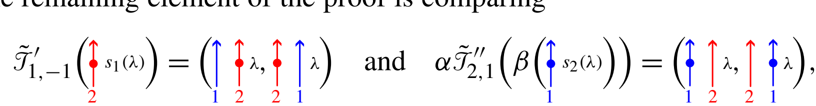
$$\tilde{\mathcal{T}}'_{1,-1}(D_2(s_1(\lambda))) \sim_h \alpha\tilde{\mathcal{T}}''_{2,1}(\beta(D_1(s_2(\lambda)))).$$

*Proof.* The proof follows from calculations that, while not complicated, are liable to confuse. We therefore present them below.





The remaining element of the proof is comparing



which are equal up to homotopy, by (45). □

The following result will often be used in proving the main theorem:

**Lemma 5.2.** $\tilde{\mathcal{T}}'_{1,-1}$ *and* $\alpha\tilde{\mathcal{T}}''_{2,1}\beta$ *preserve all the KM identities.*

*Proof.* All component 2-functors of these 2-functors are either 2-isomorphisms (which clearly preserve defining axioms) or preserve identities (KM1)–(KM9) by [1, Section 4]. □

## 6. Proof of Theorem 4.3

As a last bit of preparation before proving Theorem 4.3, we define some extra 2-functors to account for $\mathcal{E}\mathcal{v}'$ having a different domain 2-category to $\tilde{\mathcal{T}}'_{1,-1}$ and $\tilde{\mathcal{T}}''_{2,1}$, and for categorifying the powers of $q$ and signs found in Section 2.2.1. We remind the reader that $k_i^{a_1,\dots,a_n}(\mu)$ omits the argument $\mu$ when it is equal to $\lambda$, but retains it otherwise (generally when it is $s_1(\lambda)$ or $s_2(\lambda)$).

**6.1.** ***Embeddings.*** We define 2-embeddings $\iota, \iota' : \tilde{\mathcal{U}}(3) \to \tilde{\mathcal{U}}_{\triangle}(3)$ using other component 2-functors. First, we define $\tilde{\omega} = \zeta\omega\zeta^{-1} : \tilde{\mathcal{U}}(3) \to \tilde{\mathcal{U}}(3)$. Second, we define 2-functors $\eta, \eta' : \tilde{\mathcal{U}}(3) \to \tilde{\mathcal{U}}_{\triangle}(3)$ as follows.

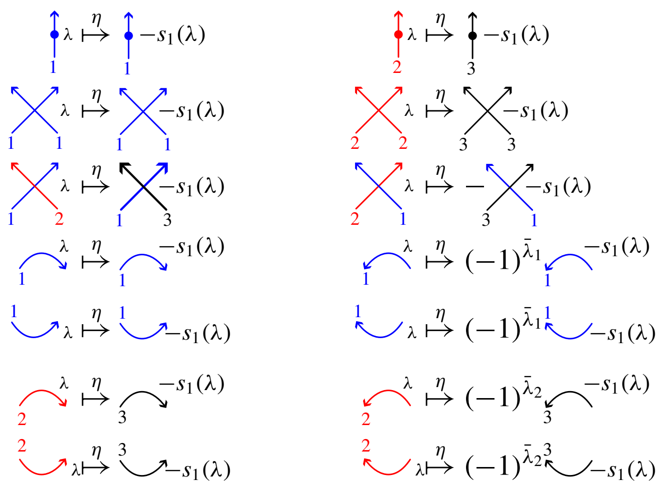
For $\eta$:

- On objects, $\lambda \overset{\eta}{\mapsto} -s_1(\lambda)$.
- On 1-morphisms,

$$\mathcal{E}_1\mathbf{1}_\lambda \overset{\eta}{\mapsto} \mathcal{E}_1\mathbf{1}_{-s_1(\lambda)}, \qquad \mathcal{F}_1\mathbf{1}_\lambda \overset{\eta}{\mapsto} \mathcal{F}_1\mathbf{1}_{-s_1(\lambda)},$$
$$\mathcal{E}_2\mathbf{1}_\lambda \overset{\eta}{\mapsto} \mathcal{E}_3\mathbf{1}_{-s_1(\lambda)}, \qquad \mathcal{F}_2\mathbf{1}_\lambda \overset{\eta}{\mapsto} \mathcal{F}_3\mathbf{1}_{-s_1(\lambda)}.$$

- On 2-morphisms,

(71)
(72)
(73)
(74)
(75)
(76)
(77)



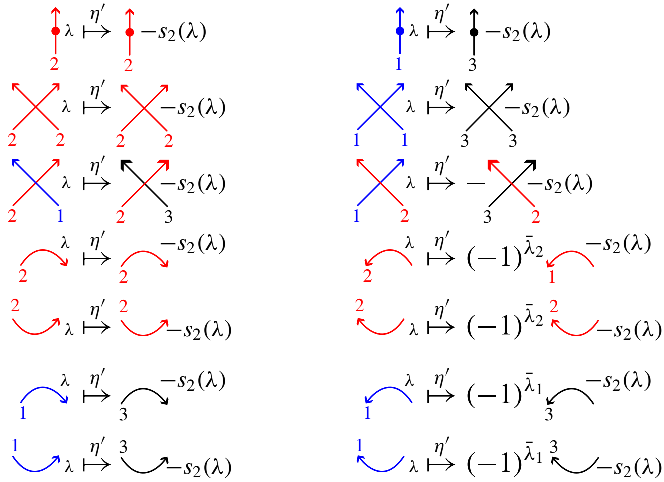

For $\eta'$:

- On objects, $\lambda \overset{\eta'}{\mapsto} -s_2(\lambda)$.
- On 1-morphisms,

$$\mathcal{E}_2\mathbf{1}_\lambda \overset{\eta'}{\mapsto} \mathcal{E}_2\mathbf{1}_{-s_2(\lambda)}, \qquad \mathcal{F}_2\mathbf{1}_\lambda \overset{\eta'}{\mapsto} \mathcal{F}_2\mathbf{1}_{-s_2(\lambda)},$$
$$\mathcal{E}_1\mathbf{1}_\lambda \overset{\eta'}{\mapsto} \mathcal{E}_3\mathbf{1}_{-s_2(\lambda)}, \qquad \mathcal{F}_1\mathbf{1}_\lambda \overset{\eta'}{\mapsto} \mathcal{F}_3\mathbf{1}_{-s_2(\lambda)}.$$

- On 2-morphisms,

(78)
(79)
(80)
(81)
(82)
(83)
(84)



It is straightforward to check that $\eta$ and $\eta'$ preserve (KM1)–(KM9) and are therefore are well-defined.

We now define $\iota = \eta\tilde{\omega}$ and $\iota' = \eta'\tilde{\omega}$. Explicitly, $\iota$ is given by:

- On objects, $\lambda \overset{\iota}{\mapsto} s_1(\lambda)$.
- On 1-morphisms,

$$\mathscr{E}_1\mathbf{1}_\lambda \overset{\iota}{\mapsto} \mathscr{F}_1\mathbf{1}_{s_1(\lambda)}, \qquad \mathscr{F}_1\mathbf{1}_\lambda \overset{\iota}{\mapsto} \mathscr{E}_1\mathbf{1}_{s_1(\lambda)},$$
$$\mathscr{E}_2\mathbf{1}_\lambda \overset{\iota}{\mapsto} \mathscr{F}_3\mathbf{1}_{s_1(\lambda)}, \qquad \mathscr{F}_2\mathbf{1}_\lambda \overset{\iota}{\mapsto} \mathscr{E}_3\mathbf{1}_{s_1(\lambda)}.$$

- On 2-morphisms,

(85)
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(87)
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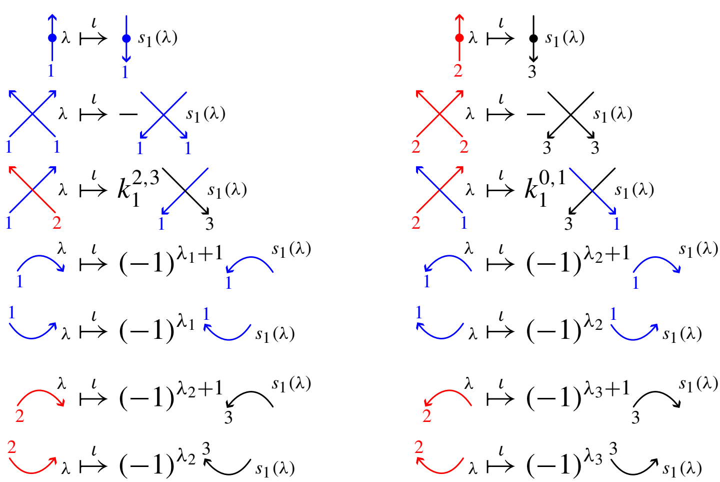
(89)
(90)
(91)



Explicitly, $\iota'$ is given by:

- On objects, $\lambda \overset{\iota'}{\mapsto} s_2(\lambda)$.
- On 1-morphisms,

$$\mathscr{E}_2\mathbf{1}_\lambda \overset{\iota'}{\mapsto} \mathscr{F}_2\mathbf{1}_{s_2(\lambda)}, \qquad \mathscr{F}_2\mathbf{1}_\lambda \overset{\iota'}{\mapsto} \mathscr{E}_2\mathbf{1}_{s_2(\lambda)},$$
$$\mathscr{E}_1\mathbf{1}_\lambda \overset{\iota'}{\mapsto} \mathscr{F}_3\mathbf{1}_{s_2(\lambda)}, \qquad \mathscr{F}_1\mathbf{1}_\lambda \overset{\iota'}{\mapsto} \mathscr{E}_3\mathbf{1}_{s_2(\lambda)}.$$

- On 2-morphisms,

(92)


(93)


(94)


(95)


(96)


(97)


(98)


**6.2.** ***Degree shifts.*** Finally, we introduce two "shift" 2-isomorphisms

$$\sigma_1, \sigma_2 : K^b(\tilde{\mathcal{U}}(3)) \to K^b(\tilde{\mathcal{U}}(3)).$$

They shift the homological degree and internal degree of complexes, but otherwise act as the identity. We recall that $X[y]\langle z\rangle$ is the 1-term complex with the 1-morphism at homological degree $-y$ with grade shift $z$. Specifically, we define $\sigma_1$ on the generating 1-morphisms by

- $\mathscr{E}_1\mathbf{1}_\lambda[y]\langle z\rangle \overset{\sigma_1}{\mapsto} \mathscr{E}_1\mathbf{1}_\lambda[y-1]\langle z-\bar{\lambda}_1\rangle$ ,
- $\mathscr{F}_1\mathbf{1}_\lambda[y]\langle z\rangle \overset{\sigma_1}{\mapsto} \mathscr{F}_1\mathbf{1}_\lambda[y+1]\langle z-2+\bar{\lambda}_1\rangle$ ,
- $\mathscr{E}_2\mathbf{1}_\lambda[y]\langle z\rangle \overset{\sigma_1}{\mapsto} \mathscr{E}_2\mathbf{1}_\lambda[y+1]\langle z-\lambda_2-\lambda_3-t\rangle$ ,
- $\mathscr{F}_2\mathbf{1}_\lambda[y]\langle z\rangle \overset{\sigma_1}{\mapsto} \mathscr{F}_2\mathbf{1}_\lambda[y-1]\langle z+\lambda_2+\lambda_3+t\rangle$ ,

and $\sigma_2$ on the generating 1-morphisms by

- $\mathscr{E}_1\mathbf{1}_\lambda[y]\langle z\rangle \overset{\sigma_2}{\mapsto} \mathscr{E}_1\mathbf{1}_\lambda[y+1]\langle z-2\lambda_3-\bar{\lambda}_1-t+4\rangle$ ,
- $\mathscr{F}_1\mathbf{1}_\lambda[y]\langle z\rangle \overset{\sigma_2}{\mapsto} \mathscr{F}_1\mathbf{1}_\lambda[y-1]\langle z+2\lambda_3+\bar{\lambda}_1+t-4\rangle$ ,

- $\mathscr{E}_2\mathbf{1}_\lambda[y]\langle z\rangle \overset{\sigma_2}{\mapsto} \mathscr{E}_2\mathbf{1}_\lambda[y+1]\langle z+2-\bar{\lambda}_2\rangle$ ,
- $\mathscr{F}_2\mathbf{1}_\lambda[y]\langle z\rangle \overset{\sigma_2}{\mapsto} \mathscr{F}_2\mathbf{1}_\lambda[y-1]\langle z+\bar{\lambda}_2\rangle$ ,

each extended to compositions and complexes in the obvious fashion. We will use these to match the homological degree and grading of the image of 1-morphisms under $\tilde{\mathcal{T}}'_{1,-1}$ and $\tilde{\mathcal{T}}''_{2,1}$ to their image under $\mathcal{E}v$.

**6.3. *Auxiliary result.*** We present here the categorification of Section 2.2.1.

**Proposition 6.1.** (1) *$\mathcal{E}v'\circ\iota$ and $\sigma_1\circ\tilde{\mathcal{T}}'_{1,-1}$ are equal on objects and generating* 1-*morphisms in $\tilde{\mathcal{U}}(3)$, and are equal up to homotopy on generating* 2-*morphisms in $\tilde{\mathcal{U}}(3)$.*

(2) *$\mathcal{E}v'\circ\iota'$ and $\sigma_2\circ\alpha\circ\tilde{\mathcal{T}}''_{2,1}\circ\beta$ are equal on objects and generating* 1-*morphisms in $\tilde{\mathcal{U}}(3)$, and are equal up to homotopy on generating* 2-*morphisms in $\tilde{\mathcal{U}}(3)$.*

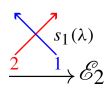
**Remark 6.2.** We phrase the proposition thus because we have not yet proved that $\mathcal{E}v'$ is a well-defined 2-functor (indeed, we will use this proposition to do so), and so it would not be accurate to claim that the two sides are equivalent 2-functors.

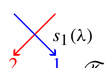
*Proof.* It is clear from the definitions that both sides of each equation agree on objects. Further, we note that $\sigma_1$ and $\sigma_2$ only affect 1-morphisms, and do not change the generating 2-morphisms. We recall that $S(\lambda)=\lambda_1+\lambda_3+t-1$.

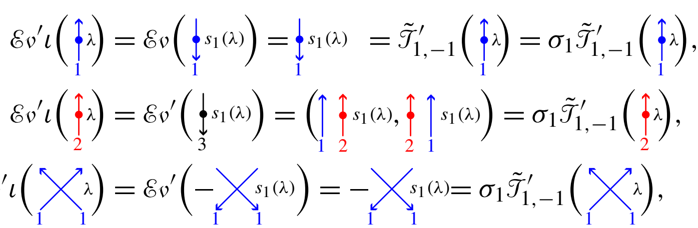
For $\mathcal{E}v'\circ\iota$:

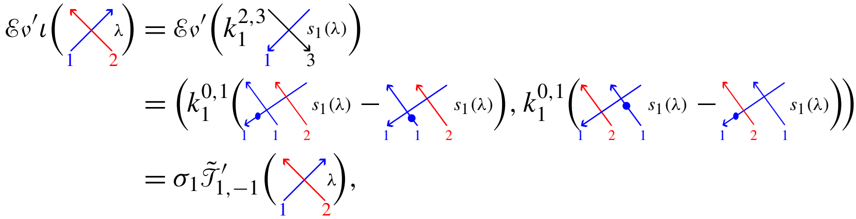
(99) $\mathcal{E}v'\iota(\mathscr{E}_1\mathbf{1}_\lambda)=\mathcal{E}v'(\mathscr{F}_1\mathbf{1}_{s_1(\lambda)})=\underline{\mathscr{F}_1\mathbf{1}_{s_1(\lambda)}}=\sigma_1\tilde{\mathcal{T}}'_{1,-1}(\mathscr{E}_1\mathbf{1}_\lambda)$,

(100) $\mathcal{E}v'\iota(\mathscr{F}_1\mathbf{1}_\lambda)=\mathcal{E}v'(\mathscr{E}_1\mathbf{1}_{s_1(\lambda)})=\underline{\mathscr{E}_1\mathbf{1}_{s_1(\lambda)}}=\sigma_1\tilde{\mathcal{T}}'_{1,-1}(\mathscr{F}_1\mathbf{1}_\lambda)$,

(101) $\mathcal{E}v'\iota(\mathscr{E}_2\mathbf{1}_\lambda)=\mathcal{E}v'(\mathscr{F}_3\mathbf{1}_{s_1(\lambda)})$
$=\mathscr{E}_{12}\mathbf{1}_{s_1(\lambda)}\langle -S(s_1(\lambda))-1\rangle \longrightarrow \underline{\mathscr{E}_{21}\mathbf{1}_{s_1(\lambda)}}\langle -S(s_1(\lambda))\rangle$

$=\sigma_1\tilde{\mathcal{T}}'_{1,-1}(\mathscr{E}_2\mathbf{1}_\lambda)$,

(102) $\mathcal{E}v'\iota(\mathscr{F}_2\mathbf{1}_\lambda)=\mathcal{E}v'(\mathscr{E}_3\mathbf{1}_{s_1(\lambda)})$
$=\underline{\mathscr{F}_{12}\mathbf{1}_{s_1(\lambda)}}\langle S(s_1(\lambda))\rangle \longrightarrow \mathscr{F}_{21}\mathbf{1}_{s_1(\lambda)}\langle S(s_1(\lambda))+1\rangle$

$=\sigma_1\tilde{\mathcal{T}}'_{1,-1}(\mathscr{F}_2\mathbf{1}_\lambda)$,

(103)
(104)
(105)


(106)


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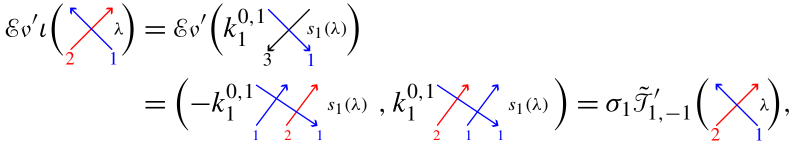
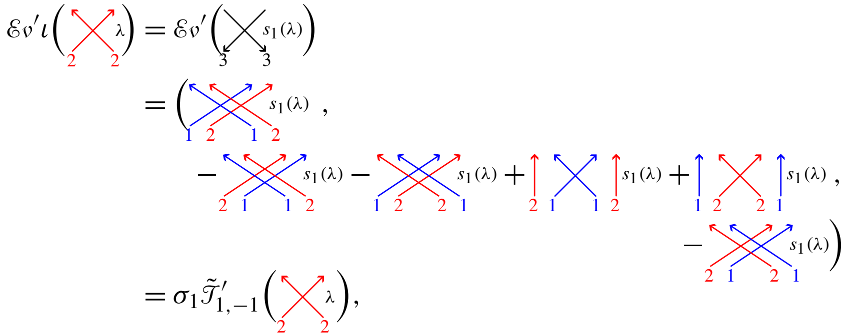


(108)


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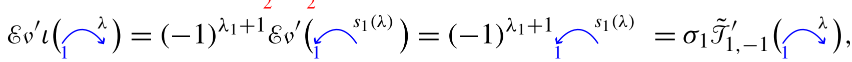
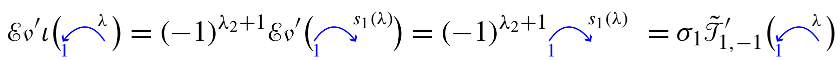
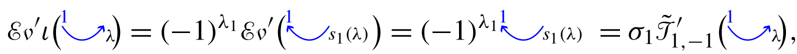


(110)


(111)


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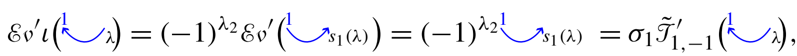
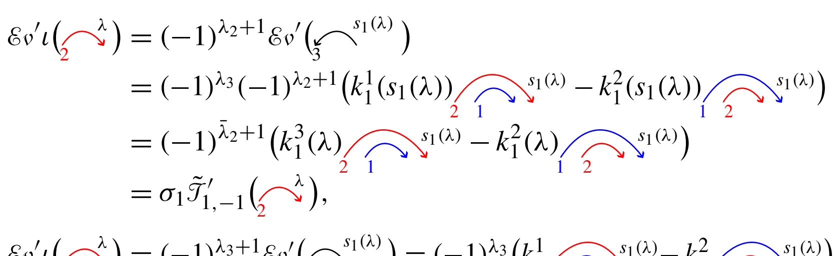
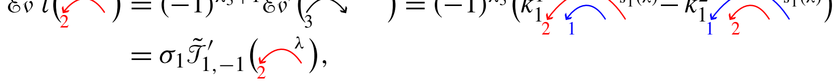


(113)


(114)


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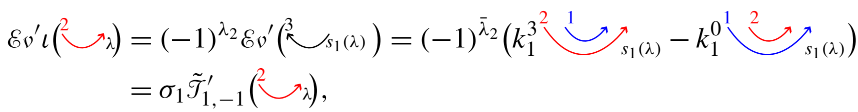
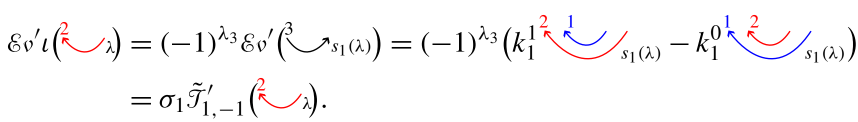
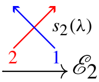


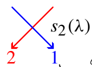
(116)


For $\mathcal{E}v' \circ \iota'$:

(117)
$$\begin{aligned}\mathcal{E}v'\iota'(\mathcal{E}_1\mathbf{1}_\lambda) &= \mathcal{E}v'(\mathcal{F}_3\mathbf{1}_{s_2(\lambda)})\\ &= \mathcal{E}_{12}\mathbf{1}_{s_2(\lambda)}\langle -S(s_2(\lambda))-1\rangle \longrightarrow \underline{\mathcal{E}_{21}\mathbf{1}_{s_2(\lambda)}}\langle -S(s_2(\lambda))\rangle\\ &= \sigma_2\alpha\tilde{\mathcal{T}}''_{2,1}\beta(\mathcal{E}_1\mathbf{1}_\lambda),\end{aligned}$$

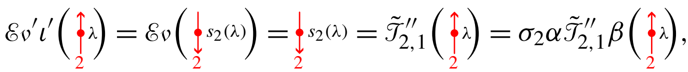


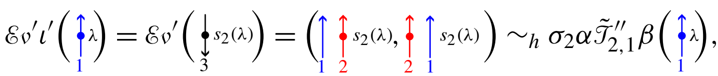
(118)
$$\begin{aligned}\mathcal{E}v'\iota'(\mathcal{F}_1\mathbf{1}_\lambda) &= \mathcal{E}v'(\mathcal{E}_3\mathbf{1}_{s_2(\lambda)})\\ &= \underline{\mathcal{F}_{12}\mathbf{1}_{s_2(\lambda)}}\langle S(s_2(\lambda))\rangle \longrightarrow \mathcal{F}_{21}\mathbf{1}_{s_2(\lambda)}\langle S(s_2(\lambda))+1\rangle\\ &= \sigma_2\alpha\tilde{\mathcal{T}}''_{2,1}\beta(\mathcal{F}_1\mathbf{1}_\lambda),\end{aligned}$$

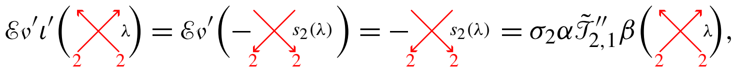


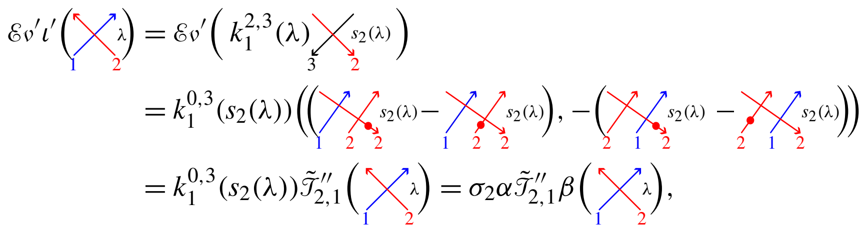
(119) $\mathcal{E}v'\iota'(\mathcal{E}_2\mathbf{1}_\lambda) = \mathcal{E}v'(\mathcal{F}_2\mathbf{1}_\lambda) = \underline{\mathcal{F}_2\mathbf{1}_{s_2(\lambda)}} = \sigma_2\alpha\tilde{\mathcal{T}}''_{2,1}\beta(\mathcal{E}_2\mathbf{1}_\lambda),$

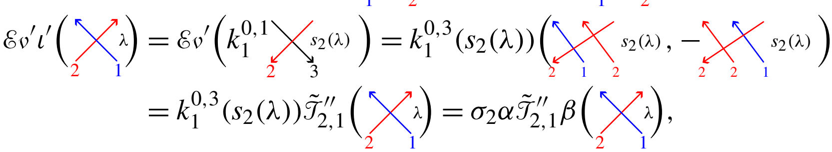
(120) $\mathcal{E}v'\iota'(\mathcal{F}_2\mathbf{1}_\lambda) = \mathcal{E}v'(\mathcal{E}_2\mathbf{1}_\lambda) = \underline{\mathcal{E}_2\mathbf{1}_{s_2(\lambda)}} = \sigma_2\alpha\tilde{\mathcal{T}}''_{2,1}\beta(\mathcal{F}_2\mathbf{1}_\lambda),$

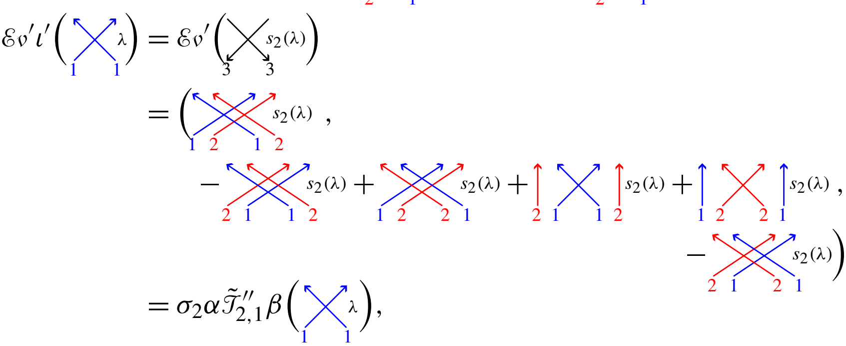
(121) 

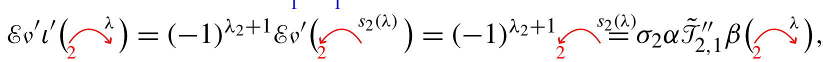
(122) 

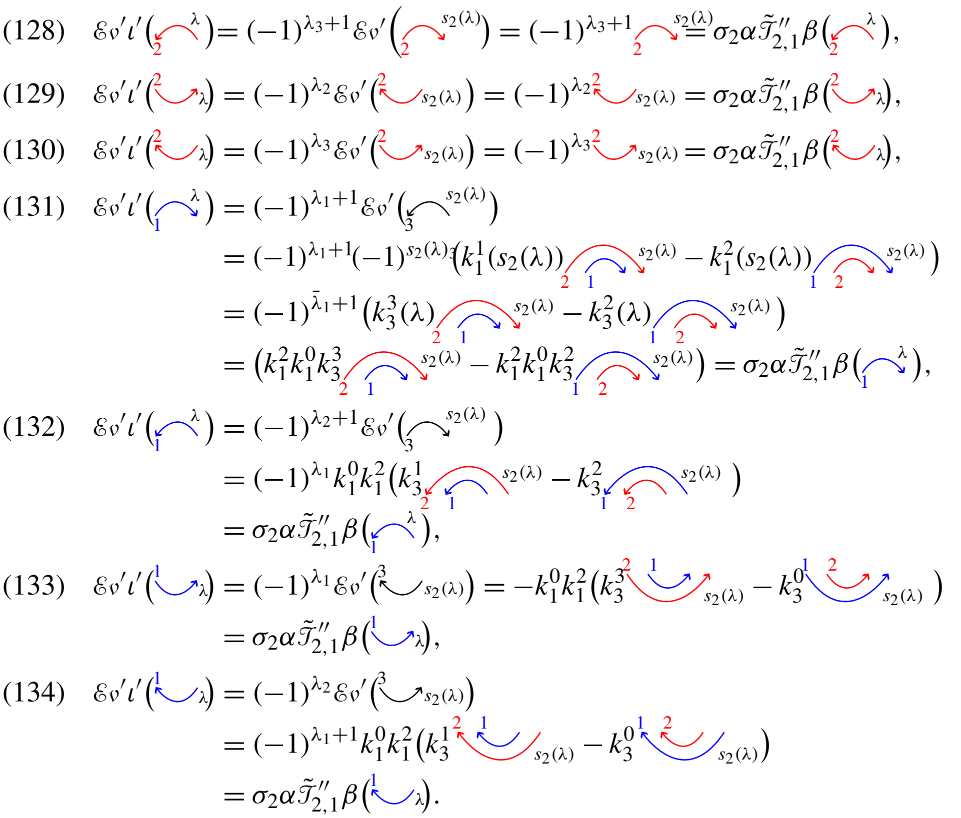
(123) 

(124) 

(125) 

(126) 

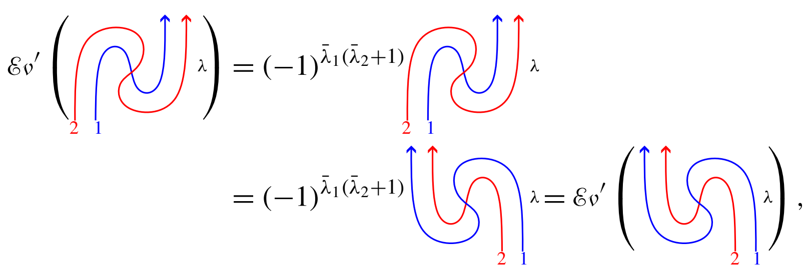
(127) 



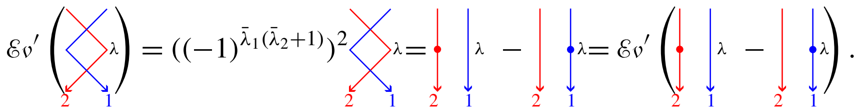

This finishes the proof. □

**6.4.** ***Proof of Theorem 4.3.*** Because $\iota$ reverses the orientation of the diagrams, we felt that this proof would be clearer to the reader if we proved that $\mathcal{E}\upsilon'$ preserved the 180 degree rotated versions of relations (10)-(22). By the cyclicity relations (KM2) and (KM3), this is equivalent to proving the original relations are preserved.

For any KM relation that only involves strands labelled 1 and 2 and does not involve a crossing of a 1-strand and a 2-strand, $\mathcal{E}\upsilon'$ acts as the identity and therefore trivially preserves the relation. For relations that do involve these crossings, the calculations are generally straightforward. For example,



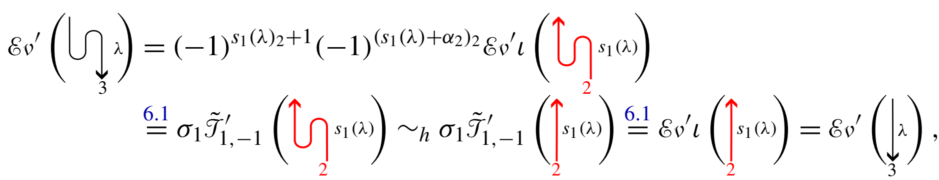
with the other cyclicity relations following similarly, as does the relevant (KM5) relation (since $(-1)^{\bar{\lambda}_1(\bar{\lambda}_2+1)} \cdot 0 = 0$). For (KM4), we have



Identity (KM7) is similar, as is (KM6); any (multicolour) cubic KLR diagram consisting only of strands labelled 1 and 2 will have precisely two multicoloured crossings, leading to a similar squaring of the sign. It therefore remains to consider only those diagrams with at least one strand labelled 3.

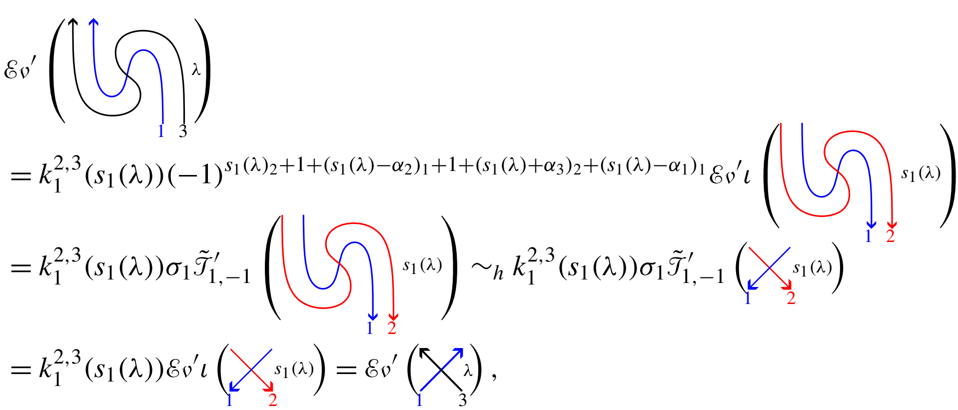
For most of the KM identities discussed below, we are able to use that $\iota$ and $\iota'$ are locally faithful 2-functors, and therefore we are able to consider the unique preimage of any 2-morphism in their images. The results will then follow from liberal use of Proposition 6.1 (we give an example in first equation below of where it is used). We also implicitly make use of Lemma 5.1 when there is a diagram in the image of both $\iota$ and $\iota'$. We will present a representative sampling of the identities of each KM axiom. The exception is the six instances of (KM6) where, using the notation of (17), $\{i, j, k\} = \{1, 2, 3\}$, since such 2-morphisms are not in the image of either $\iota$ or $\iota'$. In these cases, we will be proving directly that $\mathcal{E}v'$ preserves (KM6).

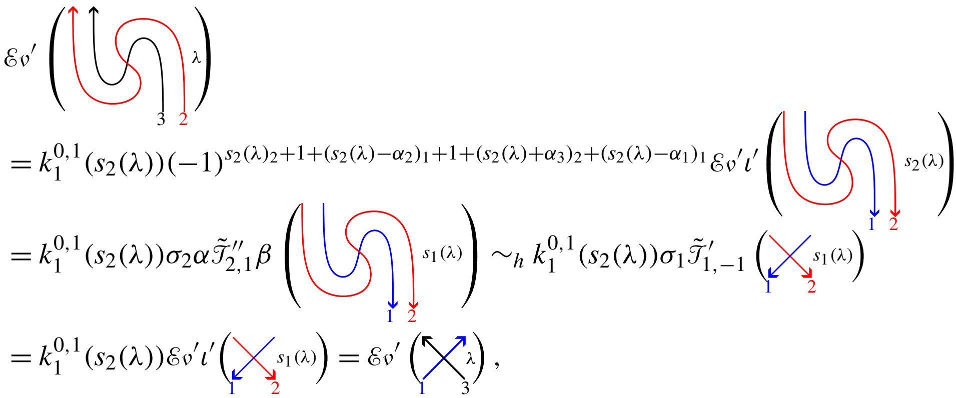
(KM1) and (KM2):

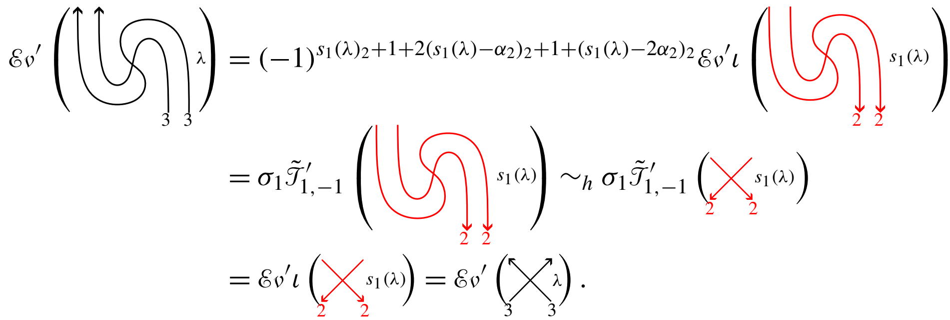


where the homotopy (and all future homotopies in this proof) follows from Lemma 5.2 and from the $\sigma_i$ being 2-isomorphisms. The other adjunction relation and dot cyclicity work similarly.

(KM3):

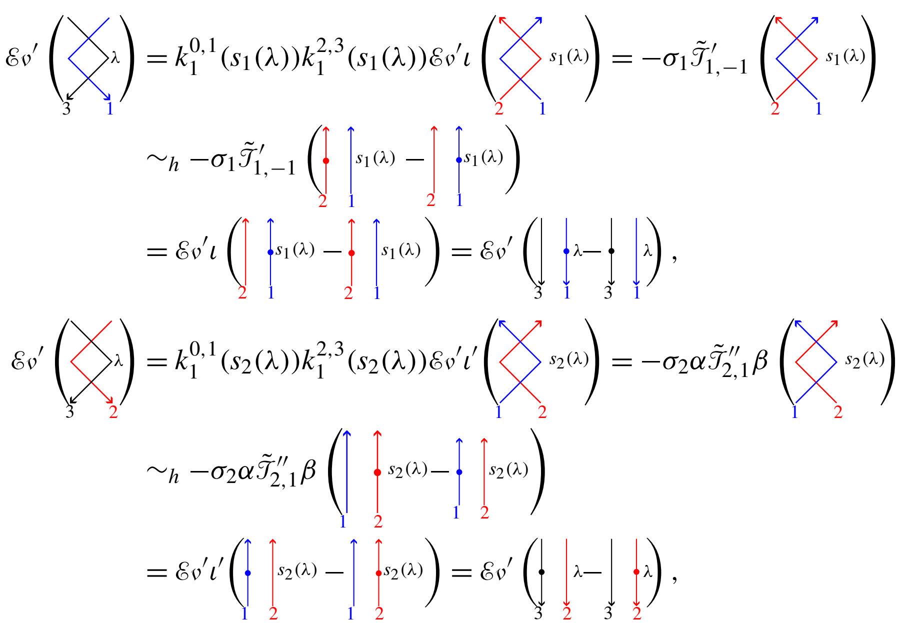


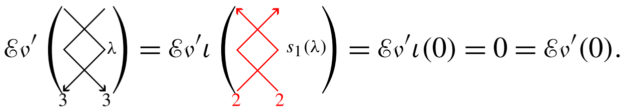




The other crossing cyclicity identities are similar.

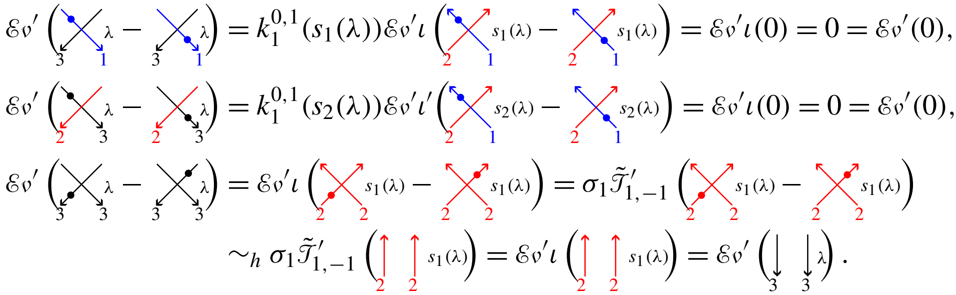
(KM4):





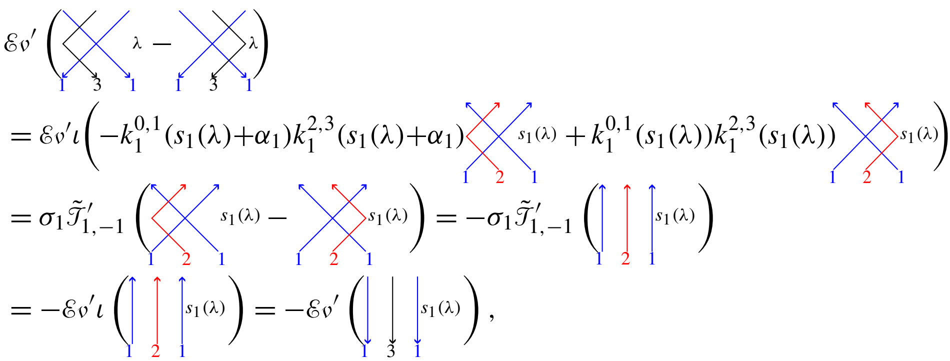
The other two identities are similar.

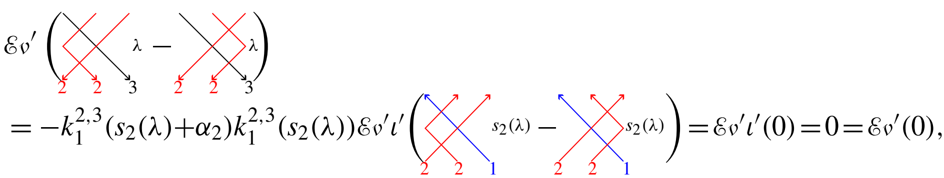
(KM5):

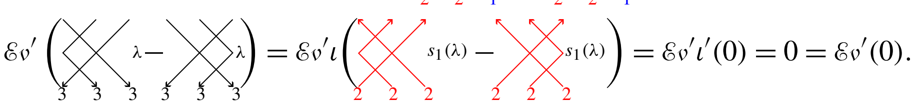


The other two identities are similar.

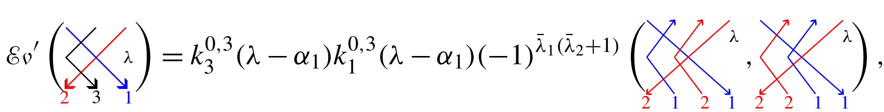
(KM6):

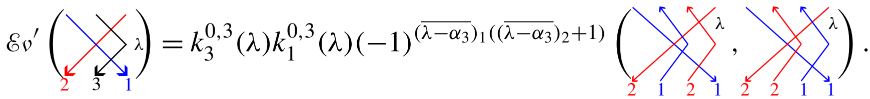






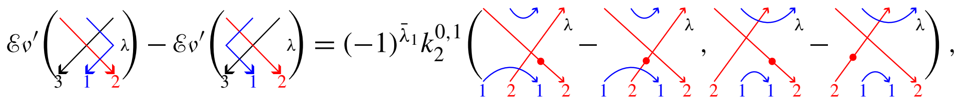
With the exception of the three-coloured identities discussed below, the other cubic KLR relations are similar. We prove three of these identities directly; the other three are similar. We have



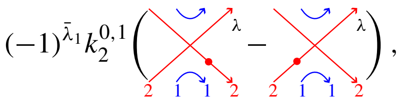


It is straightforward to check that the signs are both equal to $(-1)^{\bar{\lambda}_1} k_2^{0,1}$, and the diagrams themselves are clearly equal via mixed cubic KLR relations.

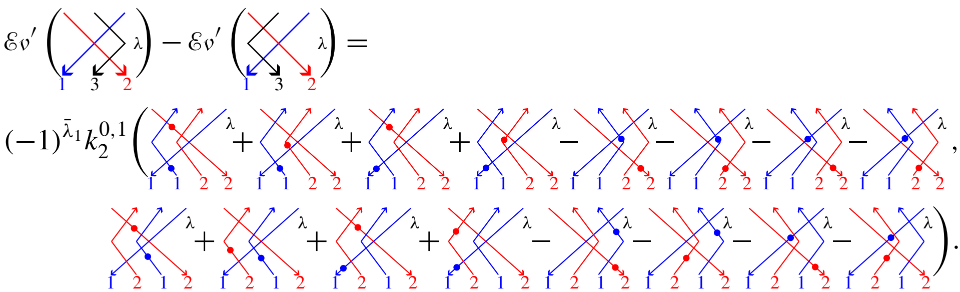
Using (KM6) and (KM5), one shows that



and that this is null-homotopic via the homotopy

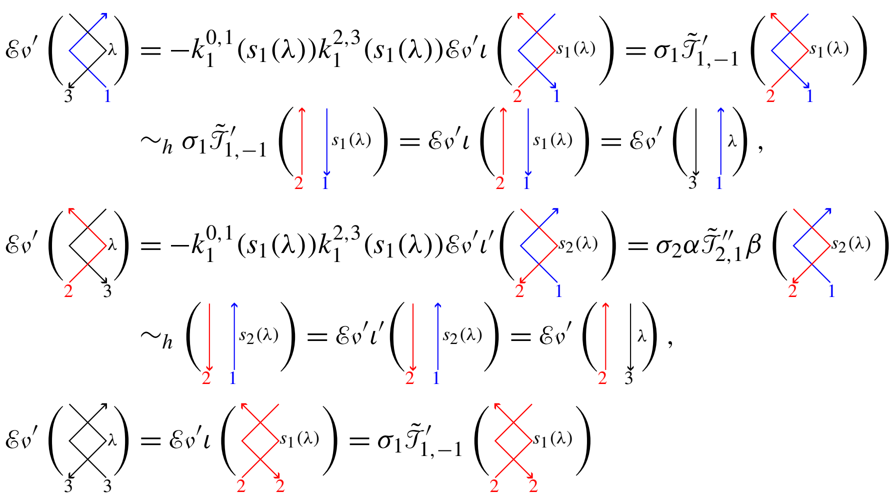


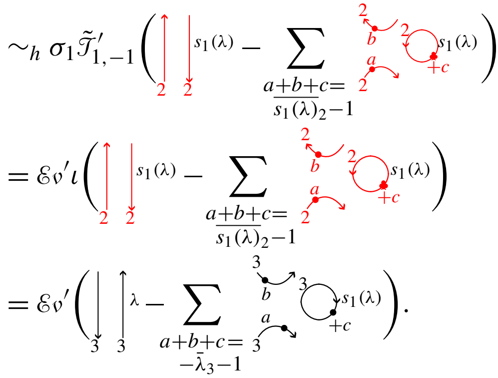
as required. Also,



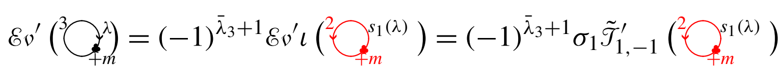
With generous use of (KM5), all the dots in the above can be moved to either the top or bottom of their arrow in a consistent fashion. It then follows from mixed direction cubic KLR that the difference is equal to zero, as required.

(KM7):





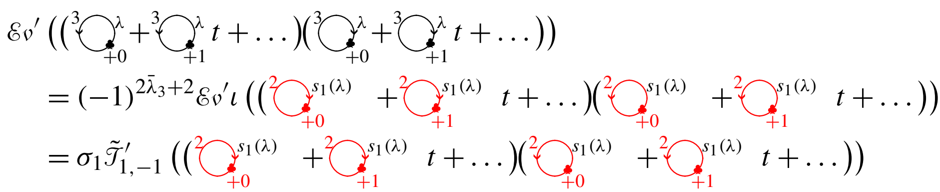
The other two identities are similar.

(KM8): 

$$= \sigma_1 \tilde{\mathcal{T}}'_{1,-1}\big((-\max(0, m+1))^{s_1(\lambda)_2}\big) = \mathcal{E}\mathfrak{v}'\big((-\max(0, m+1))^{\lambda_1}\big).$$

The anticlockwise bubble is similar.

(KM9):



$$= \sigma_1 \tilde{\mathcal{T}}'_{1,-1}(-1) = -1 = \mathcal{E}\mathfrak{v}'(-1).$$

## Appendix: A remark on 2-isomorphism classes

In Definition 3.1 of [10], Khovanov and Lauda chose a different set of scalars and bubble parameters for the Kac–Moody 2-categories than those we use in this paper. We were unable to define an evaluation 2-functor for their choice and we conjecture that no such evaluation 2-functor exists. Although we have yet to prove this conjecture, this would be consistent with the first paragraph of [11, p. 2699], which mentions the existence of a one-parameter family of mutually non-2-isomorphic sub-2-categories of the affine type A Kac–Moody 2-categories categorifying the Borel subalgebra in affine type A. This contrasts with the finite type A case, where [13, Theorem 3.5] proves that any two choices of scalars and bubble parameters yield 2-isomorphic Kac–Moody 2-categories. Since Khovanov and Lauda did not include a proof in the above paper, we prove here that the two different choices discussed do lead to non-2-isomorphic Kac–Moody 2-categories in affine type A.

First, a small comment on the choice of weights for the 2-categories in question. The objects of the cyclic 2-categories $\tilde{\mathcal{U}}(n)$ and $\tilde{\mathcal{U}}_\triangle(n)$ from Definition 3.1 are $\mathfrak{gl}_n$-weights and level-zero $\widehat{\mathfrak{gl}}_n$-weights, respectively, which coincide (as explained in 2.1). Moreover, the relations satisfied by the generating 2-morphisms of those two 2-categories really depend on those weights, and not on the induced $\mathfrak{sl}_n$-weights and level-zero $\widehat{\mathfrak{sl}}_n$-weights, respectively. Specifically, the degree-zero $i$-coloured bubbles, in a region labelled by $\lambda \in \mathbb{Z}^n$, are equal to $(-1)^{\lambda_{i+1}}$ or $(-1)^{\lambda_{i+1}-1}$ (depending on orientation), so they cannot be expressed in terms of $\bar{\lambda}$ (unless we choose and fix a certain Schur level).

On the other hand, the cyclic 2-categories $\mathcal{U}_Q(\mathfrak{sl}_n)$ and $\mathcal{U}_Q(\widehat{\mathfrak{sl}}_n)$, defined in [2, Definition 1.3] (generalizing [10, Definition 3.1]), trivially induce cyclic 2-categories $\mathcal{U}_Q(\mathfrak{gl}_n)$ and $\mathcal{U}_Q(\widehat{\mathfrak{gl}}'_n)$ whose objects are $\mathfrak{gl}_n$-weights and level-zero $\widehat{\mathfrak{gl}}_n$-weights, respectively: Simply label the regions of the string diagrams by $\lambda \in \mathbb{Z}^n$ and let the relations be those for $\bar{\lambda}$, see Section 2.1 for the notation. We write $\mathcal{U}_Q(\widehat{\mathfrak{gl}}'_n)$ to indicate that it is actually an extended version of $\mathcal{U}_Q(\widehat{\mathfrak{sl}}_n)$ rather than the full $\mathcal{U}_Q(\widehat{\mathfrak{gl}}_n)$ (whatever that would be), see remarks below Definition 2.2.

Recall that, following [13], $\mathcal{U}_Q(\mathfrak{sl}_n)$ and $\mathcal{U}_Q(\widehat{\mathfrak{sl}}_n)$ depend on a choice of scalars $t_{ij} \in \mathbb{Q}^\times$ satisfying $t_{ii} = 1$ and $t_{ij} = t_{ji}$ when $j \neq i \pm 1 \mod n$, and bubble parameters $\beta_i = \beta_{i,\lambda}, c^+_{i,\lambda}, c^-_{i,\lambda} \in \mathbb{Q}^\times$ satisfying

$$c^+_{i,\lambda} c^-_{i,\lambda} = -1/\beta_i = 1/t_{ii} \quad \text{and} \quad c^\pm_{i,\lambda+\alpha_j} = t_{ij} c^\pm_{i,\lambda}.$$

Here $i, j \in 1, \ldots, n-1$ and $\lambda \in \mathbb{Z}^{n-1}$, for $\mathfrak{sl}_n$, and $i, j \in 1, \ldots, n$ and $\lambda \in \mathbb{Z}^n$, for $\widehat{\mathfrak{sl}}_n$. For Khovanov and Lauda's original choice in [10, Definition 3.1], with all scalars and bubble parameters equal to one, we will follow their notation and denote the corresponding 2-categories by $\mathcal{U}(\mathfrak{sl}_n)$ and $\mathcal{U}(\widehat{\mathfrak{sl}}_n)$, and the trivially induced $\mathfrak{gl}_n$ versions of these by $\mathcal{U}(\mathfrak{gl}_n)$ and $\mathcal{U}(\widehat{\mathfrak{gl}}'_n)$, respectively. The 2-categories $\tilde{\mathcal{U}}(n)$ and $\tilde{\mathcal{U}}_\triangle(n)$ correspond to the choice $t_{ii} = -1 = t_{i,i+1} = -1$ and $t_{ij} = 1$ for all $i$ and $j \neq i, i+1$ in the respective ranges, and $c^+_{i,\lambda} = (-1)^{\lambda_{i+1}} = -c^-_{i,\lambda}$ for all $i$ in the respective ranges.

For any $n \in \mathbb{N}_{\geq 2}$, the 2-categories $\tilde{\mathcal{U}}(n)$ and $\mathcal{U}(\mathfrak{gl}_n)$ are 2-isomorphic, with the 2-isomorphism being obtained by composing the 2-isomorphism from [16, (6)] and the 2-isomorphism $\Sigma$ from [10, Section 4.2.1] (see also [12]). When $n \in \mathbb{N}_{\geq 2}$ is even, that composite 2-isomorphism extends to a 2-isomorphism between $\tilde{\mathcal{U}}_\triangle(n)$ and $\mathcal{U}_Q(\widehat{\mathfrak{gl}}'_n)$. When $n$ is odd, it does not extend to the affine 2-categories, because Khovanov and Lauda's 2-isomorphism $\Sigma$ is no longer well-defined in that case. The reason is that in the definition of $\Sigma$ occur factors like $(-1)^i$, for $i = 1, \ldots, n-1$, which are not well-defined for $i \in \mathbb{Z}/n\mathbb{Z}$ when $n$ is odd.

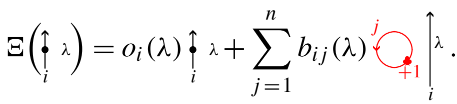
We show that there is no 2-isomorphism between $\tilde{\mathcal{U}}_\triangle(n)$ and $\mathcal{U}_Q(\widehat{\mathfrak{gl}}'_n)$ for odd $n$, for any choice of scalars and bubble parameters satisfying the above conditions.

**Lemma A.3.** *Let $Q$ be a choice of scalars and bubble parameters for $\widehat{\mathfrak{sl}}_n$ and let $\Xi : \mathcal{U}_Q(\widehat{\mathfrak{gl}}'_n) \to \tilde{\mathcal{U}}_\triangle(n)$ be a 2-isomorphism which is the identity on objects and 1-morphisms. Then*

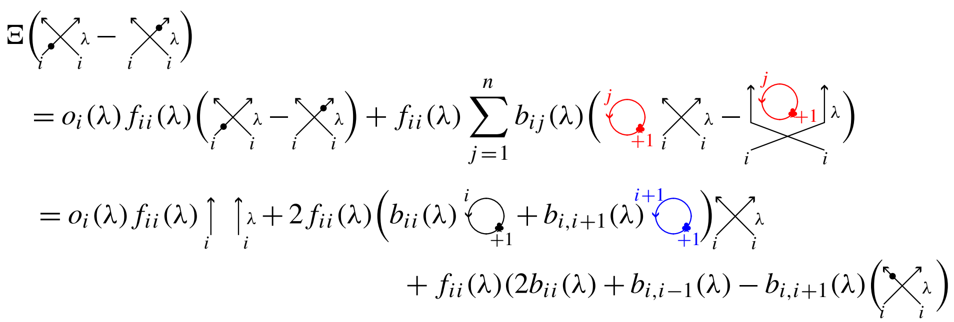
(135) 

*for some $o_i(\lambda), f_{ij}(\lambda) \in \mathbb{Q}^\times$ and for all $i, j \in \{1, \ldots, n\}$ and all $\lambda \in \mathbb{Z}^n$. Moreover, these scalars satisfy $o_i(\lambda) f_{ii}(\lambda) = 1$ for all $i \in \hat{I}$ and all $\lambda \in \mathbb{Z}^n$.*

*Proof.* For degree reasons, the second equality in (135) is immediate, but the first one requires an argument. A priori, we have

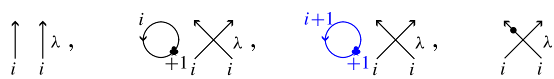


Now consider the image of the nil-Hecke relation:



The fact that $\Xi$ has to preserve the nil-Hecke relation implies that $o_i(\lambda) f_{ii}(\lambda) = 1$ and $b_{ii}(\lambda) = b_{i,i-1}(\lambda) = b_{i,i+1}(\lambda) = 0$ for all $i \in \hat{I}$ and all $\lambda \in \mathbb{Z}^n$.

To see this, first note that



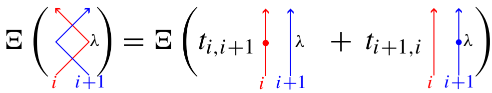
are linearly independent in $\mathrm{Hom}_{\tilde{\mathcal{U}}_\triangle(n)}(\mathcal{E}_{ii}\mathbf{1}_\lambda, \mathcal{E}_{ii}\mathbf{1}_\lambda)$. Just as in the proof of [10, Lemma 6.16], this follows from looking at their images under the 2-representation $\mathcal{F}_{\mathrm{Bim}}$ from [16, Section 4.2] (in particular, see (45) in that paper), and its extension for the affine case in [15, Definition 5.6].

The condition $o_i(\lambda) f_{ii}(\lambda) = 1$ is therefore immediate. Further, for each $i \in \hat{I}$, linear independence of the two degree-two bubbles above, coloured $i-1$ and $i$ implies that $b_{i,i+1} = b_{ii} = 0$. Using this and

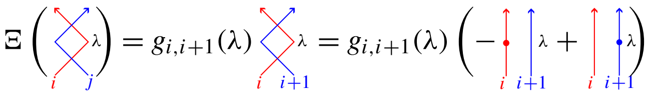
$$2b_{ii}(\lambda) + b_{i,i-1}(\lambda) - b_{i,i+1}(\lambda) = 0,$$

we see that $b_{i,i-1} = 0$ as well.

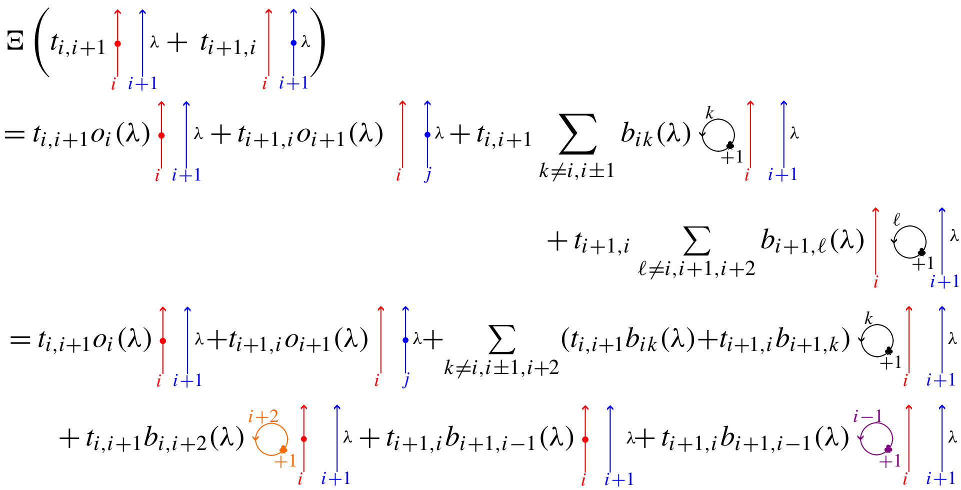
Next we show that $b_{ij}(\lambda) = 0$ for all $i, j \in \hat{I}$, using the fact that $\Xi$ has to satisfy



for $j = i + 1$. To shorten notation, put $g_{ij}(\lambda) := f_{ij}(\lambda) f_{ji}(\lambda)$ for all $i, j \in \hat{I}$. On the one hand, we have



and on the other hand, we have



By linear independence of the different terms of each expression and comparing corresponding terms in both expressions, as above, we get

$$b_{i,i+2}(\lambda) = 0, \quad b_{i+1,i-1}(\lambda) = 0, \quad t_{i,i+1} b_{ik}(\lambda) + t_{i+1,i} b_{i+1,k}(\lambda) = 0$$

for all $i, k \in \hat{I}$ such that $k \neq i, i \pm 1, i + 2$. Together with the previous results, these equations imply that $b_{ij}(\lambda) = 0$ for all $i, j \in \hat{I}$ and all $\lambda \in \mathbb{Z}^n$. □

This proof also shows that

$$g_{i,i+1}(\lambda) = -t_{i,i+1} o_i(\lambda) = t_{i+1,i} o_{i+1}(\lambda) \quad \text{for all } i \in \hat{I}. \tag{136}$$

**Theorem A.4.** *When $n$ is odd, there does not exist a 2-isomorphism $\Xi : \mathcal{U}_Q(\widehat{\mathfrak{gl}}'_n) \to \tilde{\mathcal{U}}_\triangle(n)$ which is the identity on objects and 1-morphisms, for any choice of scalars $Q$ with compatible bubble parameters.*

*Proof.* Assume for contradiction that such a 2-isomorphism $\Xi$ exists for some choice of scalars with compatible bubble parameters, with $\Xi$ having parameters

$\{o_i, f_{ij} | i, j = 1, \dots, n\}$ as above. Recall that $o_i(\lambda), g_{ij}(\lambda), t_{ij} \in \mathbb{Q}^\times$ and that $g_{ij} = g_{ji}$ for all $i, j \in \hat{I}$. Thus, suppressing $\lambda$ for readability, we get

$$\begin{aligned} o_1 &= -g_{12}t_{12}^{-1} = -o_2 t_{21} t_{12}^{-1} = (-1)^2 g_{23} t_{23}^{-1} t_{21} t_{12}^{-1} = \cdots = (-1)^n g_{n,1} t_{n,1}^{-1} \dots t_{21} t_{12}^{-1} \\ &= (-1)^n o_1 \prod_{i \in \hat{I}} t_{i+1,i} t_{i,i+1}^{-1}. \end{aligned}$$

This implies that $\prod_{i\in\hat{I}} t_{i+1,i}t_{i,i+1}^{-1} = (-1)^n$ has to hold. But by the definition of $Q$ and the fact that $\sum_{k=1}^n \alpha_k = 0$ in the (level zero) $\widehat{\mathfrak{sl}}_n$-root lattice, we have the following:

- $t_{ii} = 1$ for all $i = 1, \dots, n$, so in particular $\prod_{i=1}^n t_{ii} = 1$.
- $t_{ij} = t_{ji}$ whenever $|i - j| > 1 \bmod n$, so in particular $\prod_{\substack{i,j=1,\dots,n \\ |i-j|>1}} t_{ij} = x^2$ for some $x \in \mathbb{Q}^\times$.
- $1 = \dfrac{c_{i,\bar{\lambda}}}{c_{i,\bar{\lambda}}} = \dfrac{c_{i,\bar{\lambda}+\sum_{k=1}^n \alpha_k}}{c_{i,\bar{\lambda}}} = \prod_{j=1}^n \dfrac{c_{i,\bar{\lambda}+\sum_{k=1}^j \alpha_k}}{c_{i,\bar{\lambda}+\sum_{k=1}^{j-1} \alpha_k}} = \prod_{j=1}^n t_{ij}$ for any $\lambda \in \mathbb{Z}^n$ and any $i = 1, \dots, n$.

Therefore, for any $\lambda \in \mathbb{Z}^n$

$$\prod_{i,j=1,\dots,n} t_{ij} = \prod_{i=1}^n \frac{c_{i,\bar{\lambda}}}{c_{i,\bar{\lambda}}} = 1.$$

But

$$\prod_{i,j=1,\dots,n} t_{ij} = \Big(\prod_{i=1}^n t_{ii}\Big)\Big(\prod_{\substack{i,j=1,\dots,n \\ |i-j|>1}} t_{ij}\Big)\Big(\prod_{\substack{i=1,\dots,n \\ |i-j|=1}} t_{ij}\Big) = x^2 \prod_{\substack{i=1,\dots,n \\ |i-j|=1}} t_{ij},$$

for some $x \in \mathbb{Q}^\times$, by the above remarks. Multiplying this by $\prod_{i\in\hat{I}} t_{i+1,i}t_{i,i+1}^{-1}$ yields

$$\prod_{i=1}^n t_{i+1,i}^2 = (-1)^n x^{-2},$$

which implies $n$ has to be even, completing our proof. □

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# ON THE DEFINITION OF STABLE TRANSFER FACTORS

TIAN AN WONG

**We construct stable geometric and spectral transfer factors for a general reductive group and develop some of their basic properties, assuming the refined local Langlands correspondence. Using our definition of stable geometric transfer factors, we show that the stable transfer conjecture for orbital integrals implies the stable transfer of characters and vice versa. The latter is also implied by local Langlands, and in particular this establishes archimedean stable geometric transfer. Finally, we show how the stable geometric transfer factors can be used to define stable spectral transfer factors.**



## 1. Introduction

**1.1.** ***The stable transfer of distributions.*** Let $G$ be a connected reductive group over a local field $F$ of characteristic zero. The notion of stable transfer was first formulated by Langlands in [Lan13], primarily in the context of SL(2). By stable transfer, we really mean one of two things: On the spectral side, given a local $L$-parameter $\phi$ of $G(F)$ and suitable test function $f$, we write $f^G(\phi)$ for the associated stable character of $f$ at $\phi$. If $\phi'$ is a local $L$-parameter on another group $G'(F)$ that transfers to $\phi$, then the desired transfer $f'$ should satisfy an identity of stable characters

$$f^{G'}(\phi') = f^G(\phi).$$

*MSC2020:* primary 22E55; secondary 11R39.
*Keywords:* stable spectral transfer, stable geometric transfer.

This we call *stable spectral transfer*. Here we are identifying the test function $f'$ with the stable character $f^{G'}$, though $f' = f^{G'}$ should be determined only up to its stable character.

On the geometric side, given a strongly regular conjugacy class $\delta$ of $G(F)$, we write $f^G(\delta)$ for the stable orbital integral of $f$ at $\delta$. Then we also expect that the spectral transfer induces a transfer of stable orbital integrals,

$$f^G \to f^{G'}$$

hence the overlap of notation, which is standard following Arthur. This we call *stable geometric transfer*. Once again $f' = f^{G'}$ should be determined only up to its stable orbital integral. Implicit in the notation is the expectation that the stable geometric transfer and stable spectral transfers agree with each other.

Whereas in endoscopy, the endoscopic geometric transfer $f^e = f^{G^e}$ was given in terms of the endoscopic transfer factor, also called the Langlands–Shelstad transfer, roughly

$$f^e(\delta^e) = \sum_{\gamma} \Delta(\delta^e, \gamma) f^G(\gamma),$$

it is expected that the stable geometric transfer $f'$ requires a stable transfer factor

$$f'(\delta') = \int_{\Delta(G/Z)} \Theta(\delta', \delta) f^G(\delta)\, d\delta,$$

where now $\Theta(\delta', \delta)$ is a stable distribution on $G'$ and $G$ that is often referred to as the *stable transfer factor*, and $\Delta(G)$ is the set of stable regular semisimple conjugacy classes of $G$.

These stable transfer factors have only been constructed in special cases [Lan13; Joh17; JL20], while Shelstad has sketched a general approach in the archimedean case [She21]. In particular, we note that in [Lan13] and [She21], only in special cases of SL(2) is the stable transfer factor constructed explicitly, and in [Joh17; JL20] for special cases related to GL(2). In related work, Sakellaridis has advanced a theory of transfer operators in the relative setting, cf. [Sak21], we are motivated here by the so-called group case.

**1.2.** ***This paper.*** We give a general formulation of stable geometric and spectral transfer factors for general quasisplit connected reductive groups $G$ over a local field $F$. We will assume the refined local Langlands correspondence, for the most part, where by refined we mean that the correspondence is uniquely characterised by endoscopic character identities [Kal16]. We show that the latter implies the existence of these transfer factors abstractly, and we shall also propose an explicit formula for the geometric transfer. We prove properties related to it that, if the case of endoscopy is any indication, will be needed for the primitivisation of the trace formula. Our construction builds directly on Arthur's works, so that it is readily adapted for application to the stable trace formula in full generality.

More precisely, we first associate a mesoscopic datum $(G', \mathcal{G}', \xi')$ with auxiliary datum $(\tilde{G}', \tilde{\xi}')$, defined in Section 2.4 and following, which can be viewed as a weakened or 'beyond' endoscopic datum. In general, one expects a stable transfer

$$f \to f' = f^{\tilde{G}'}$$

of suitable test functions on $G$ to the space of stable orbital integrals on $\tilde{G}'$ satisfying the stable character identity

$$f'(\phi') = f^G(\tilde{\xi}' \circ \phi'), \tag{1-1}$$

where $\phi'$ is a bounded Langlands parameter for $\tilde{G}'$ and $f'(\phi')$ is the stable character of $f'$ at $\phi'$ (resp. $f^G(\phi)$). This can be rephrased in terms of the existence of a function $f'$ on $\tilde{G}'$ such that the character identity holds, where the function $f'$ is not uniquely determined but its stable orbital integral is. Taking the refined local Langlands conjecture for $G$, $G'$ as known (due to Shelstad in the archimedean case [She82]), it is possible to specify the transfer in terms of Paley–Wiener functions on the space of tempered Langlands parameters on $G$ and $G'$, denoted $\Phi(G)$ and $\Phi(G')$. The stable transfer for functions $f$ is then a consequence of the refined local Langlands conjecture, stated as Proposition 4.1. Note that we impose a simplifying Hypothesis 2.1 that $\xi'(\hat{G}')$ and $\hat{G}$ have equal rank, which can be removed with a more careful analysis by stratifying the transfer map according to Levi subgroups.

As noted by Arthur [Art24, §11], and stated differently by Langlands in [Lan13, §2], the stable spectral transfer should lead to the transfer of stable orbital integrals. That is, the transfer $f'$ is required to satisfy

$$f'(\delta') = \int_{\Delta(G/Z)} \Theta(\delta', \delta) f^G(\delta)\, d\delta, \tag{1-2}$$

where the distribution $\Theta(\delta', \delta)$ is the focus of our study. Here $f^G(\delta)$ is the stable orbital integral of $f$ at the stable conjugacy class $\delta$. The existence of the stable transfer factor follows as an application of the Schwartz kernel theorem and the stable spectral transfer (Corollary 4.3). For applications to the trace formula, it will likely be necessary to have explicit spectral and geometric characterisations of the transfer $f'$.

**1.3.** ***Stable geometric transfer factors.*** The key result of this paper is an explicit formula for $\Theta(\delta', \delta)$, and we distinguish our construction with a subscript $\Theta_{\tilde{\xi}'}(\delta', \delta)$. Our stable geometric transfer factors rely heavily on the local character relations satisfied by stable orbital integrals and stable characters developed by Arthur in [Art96], which we recall in Section 3. In particular, we show that they satisfy an adjoint relation similar to their unstable counterparts, a result that appears to be new. Our geometric transfer factors, defined in Section 4, take the form of a distribution

$$\Theta_{\tilde{\xi}'}(\delta', \delta) = \int_{\Phi(\tilde{G}', \tilde{\zeta}')} S'(\delta', \phi') S(\tilde{\xi}' \circ \phi', \delta)\, d\phi',$$

where $S'(\delta', \phi')$ and $S(\phi, \delta)$ are the stable character and the kernel of the Fourier inversion of stable orbital integrals respectively. As an illustration, consider the Fourier transform and its inversion

$$f(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{2\pi i (x-y)\xi} f(y)\, dy\, d\xi,$$

formally interchanging the order of integration, the inner integral becomes the delta distribution. The distribution $\Theta_{\tilde{\xi}'}(\delta', \delta)$ is essentially a generalisation of this observation. As we are working with stable objects, much of the theory of endoscopy is essential to our constructions. Our definition of the stable transfer factor is modelled after Shelstad's heuristics [She21]. We first formulate in Conjecture 4.4 that $\Theta_{\xi'}$ realises the kernel $\Theta$ of (1-2), then prove it at the end of Section 5.

**1.4.** ***Stable spectral transfer factors.*** With our stable geometric transfer factor in hand, in Section 6 we apply the preceding results to construct stable spectral transfer factors. Importantly, their definition depends on the stable geometric transfer factors and the surjectivity of a certain stable transfer map $\mathscr{T}^{\mathcal{F}}$ in Theorem 6.1, which we prove unconditionally in the archimedean case, and conditional on local Langlands in the nonarchimedean case. Assuming these, we show that the spectral transfer factors $\Theta_{\tilde{\xi}'}(\phi', \phi)$ are well-defined.

It is clear that our results in this paper rely heavily on local Langlands. It is an important question to ask how one might give an intrinsic definition of the stable geometric transfer factor $\Theta_{\tilde{\xi}'}(\delta', \delta)$ without recourse to it. (Though it should be noted that to even state the characterisation of the geometric transfer requires local Langlands.) For the moment, we only explore the surjectivity of $\mathscr{T}^{\mathcal{F}}$ without the use of local Langlands. We show in the Appendix that one can instead turn to a study of the descent of stable geometric transfer factors along the lines of Langlands and Shelstad in the endoscopic case [LS87], which we initiate but do not complete in Section A.1, and also a stable analogue (A-4) of Waldspurger's kernel formula for Fourier transforms on Lie algebras [Wal97]. Assuming these two identities instead, we can then also define the stable spectral transfer factors.

The broader goal, of course, is the primitivisation of the stable trace formula, following Arthur's formalisation of Langlands' beyond endoscopy proposal. As we hope to show in future work, this framework of stable transfer will lay the local foundations for work on the problem of primitivisation as outlined in [Art17].

## 2. Mesoscopic data: a simplified case

We begin first by fixing notation and recalling basic definitions in §2.1–2.3, and then introduce the notion of mesoscopic data and related constructions in §2.4–2.6.

**2.1.** ***Preliminaries.*** Let $F$ be a local field of characteristic zero, with an algebraic closure $\bar{F}$. Let $G$ be a quasisplit connected reductive group over $F$. We denote by $\mathscr{L}(M)$ the collection of Levi subgroups of $G$ containing $M$, $\mathscr{L}^0(M)$ the subset of proper Levi subgroups in $\mathscr{L}(M)$, and $\mathscr{P}(M)$ the collection of parabolic subgroups of $G$ containing $M$. Let $A_M$ be the maximal split torus in the centre of a Levi subgroup $M$ of $G$. We then identify the Weyl group of $(G, A_M)$ with the quotient of the normaliser of $M$ by $M$, thus

$$W(M) = W^G(M) = \mathrm{Norm}_G(M)/M.$$

If $M_0$ is a minimal Levi subgroup of $G$, which we shall assume to be fixed, and denote $\mathscr{L} = \mathscr{L}(M_0)$, $\mathscr{P} = \mathscr{P}(M_0)$, $\mathscr{L}^0 = \mathscr{L}^0(M_0)$, and $W_0^G = W^G(M_0)$. Also write $P_0$ for a minimal parabolic (i.e., Borel) subgroup containing $M_0$. Also, we fix a maximal compact subgroup $K$ of $G(F)$, which is hyperspecial when $F$ is nonarchimedean and $G$ unramified over $F$.

As usual, we form the real vector space $\mathfrak{a}_M = \mathrm{Hom}(X(M)_F, \mathbb{R})$ where $X(G)_F$ is the module of $F$-rational characters on $G$. We note by $H_G : G(F) \to \mathfrak{a}_G$ canonical homomorphism defined by $e^{\langle H_G(x),\chi\rangle} = |\chi(x)|$, for $x \in G(F)$, $\chi \in X(G)_F$, where $|\cdot|$ is the normalised valuation on $F$. We set $\mathfrak{a}_{G,F} = H_G(G(F))$ and $\tilde{\mathfrak{a}}_{G,F} = H_G(A_G(F))$, which are closed subgroups of $\mathfrak{a}_G$, with associated dual spaces $\mathfrak{a}^\vee_{G,F} = \mathrm{Hom}(\mathfrak{a}_{G,F}, 2\pi i\mathbb{Z})$ and $\tilde{\mathfrak{a}}^\vee_{G,F} = \mathrm{Hom}(\tilde{\mathfrak{a}}_{G,F}, 2\pi i\mathbb{Z})$, which are closed subgroups of $i\mathfrak{a}^*_G$. If $F$ is nonarchimedean, all four groups are lattices; if $F$ is archimedean, we have $\tilde{\mathfrak{a}}_{G,F} = \mathfrak{a}_{G,F} = \mathfrak{a}_G$ and $\tilde{\mathfrak{a}}^\vee_{G,F} = \mathfrak{a}^\vee_{G,F} = \{0\}$. Fixing a Haar measure on $\mathfrak{a}_G$, we obtain a dual Haar measure on the real vector space $i\mathfrak{a}^*_G$. If $F$ is nonarchimedean, we normalise measures so that $\mathfrak{a}_G/\tilde{\mathfrak{a}}_{G,F}$ and $i\mathfrak{a}^*_G/\tilde{\mathfrak{a}}^\vee_{G,F}$ have volume 1. It follows that the volume of the quotient $i\mathfrak{a}^*_{G,F} = i\mathfrak{a}^*_G/\mathfrak{a}^\vee_{G,F}$ equals the index $|\mathfrak{a}_{G,F}/\tilde{\mathfrak{a}}_{G,F}|$.

Let $\Gamma = \Gamma_F$ and $W_F$ be the Galois and Weil groups of $\bar{F}/F$. Let $G^*$ be a quasi-split inner form of $G$ with inner twist $\psi : G \to G^*$. In other words, $\psi$ is an isomorphism such that $\psi \circ \sigma(\psi)^{-1}$ is an inner automorphism for all $\sigma \in \Gamma$. Moreover, fix a bijection of canonical based root data $\Psi(G)^\vee \to \Psi(\hat{G})$, where $\hat{G}$ is the complex dual group of $G$. Let $(B, T)$ be a Borel pair of $G$ where $B$ is a Borel subgroup of $G$ and $T$ a maximal torus of $B$, not necessarily defined over $F$. For each pair $(B, T)$ in $G$ and $(B_1, T_1)$ in $\hat{G}$ we have a canonical isomorphism $\hat{T} \simeq T_1$. Define a pinning by $\mathcal{E} = (B, T, \{X_\alpha\})$ where $\{X_\alpha\}$ runs over simple roots $\alpha$ of $T$ acting on the Lie algebra of the unipotent radical of $B$, and $X_\alpha$ is an element of the eigenspace associated to $\alpha$. Any two pinnings are related by the adjoint action $\mathrm{ad}_g$

for some $g \in G_{\mathrm{sc}}$, the simply connected cover of $G$, unique up to translation by the centre $Z(G_{\mathrm{sc}})$. The restriction of $\mathrm{ad}_g$ to $B$ and $T$ are uniquely determined, so we may define a canonical pinning by taking the inductive limit over all pinnings of $(B, T)$. If we denote by $\rho(w)$ the action of $\Gamma$ on $\hat{G}$, we can define a new action $\rho_g(w) = \mathrm{ad}_g \rho(g) \mathrm{ad}_{g^{-1}}$ that fixes the original pinning, giving an exact sequence $W_F \to \Gamma \to \mathrm{Out}(\hat{G})$. Then ${}^L G$ is isomorphic to $\hat{G} \rtimes W_F$ under this action, sending $(h, w)$ to $(h\rho(w)(g)g^{-1}, w)$. The $L$-isomorphism ${}^L\psi : {}^L G \to {}^L G^*$ induced by $\psi$ allows us to identify the two groups. (We recall that an $L$-homomorphism here is a continuous homomorphism that is analytic on $\hat{G}$, semisimple on $W_F$ in the sense that the image of any $w \in W_F$ in $\hat{G}^*$ is semisimple, and commutes with the projections onto $W_F$.)

**2.2.** ***K-groups.*** We shall work with $K$-groups, following [Art99, §2], which streamlines endoscopy theory over archimedean local fields. It is an algebraic variety constructed in the following manner: If $F$ is $p$-adic, then $G$ is just an ordinary connected reductive group, whereas if $F$ is archimedean, then $G$ can have several connected components

$$G = \coprod_{\alpha} G_\alpha, \quad \alpha \in \pi_0(G),$$

a variety whose connected components $G_\alpha$ are reductive groups over $F$, equipped with an equivalence class of frames $(\psi, u) = \{(\psi_{\alpha\beta}, u_{\alpha\beta}) : \alpha, \beta \in \pi_0(G)\}$ satisfying natural compatibility conditions given in [Art99, §2]. Here $\psi_{\alpha\beta} : G_\alpha \to G_\beta$ is an isomorphism over $\bar{F}$, and $u_{\alpha\beta}$ is a locally constant function from $\Gamma = \mathrm{Gal}(\bar{F}/F)$ to the simply connected cover $G_{\alpha,\mathrm{sc}}$ of the derived group of $G_\alpha$. Any connected reductive group is a component of a $K$-group that is unique up to weak isomorphism.

We call $G^*$ a quasisplit inner twist of $G$ if $G^*$ is a connected, quasisplit group over $F$ equipped with a $G^*$-inner class of compatible inner twists $\psi_\alpha : G_\alpha \to G^*$ and a corresponding family of compatible, locally constant functions $u_\alpha : \Gamma \to G^*_{\alpha,\mathrm{sc}}$. We shall call a $K$-group $G$ quasisplit if one of the isomorphisms $\psi_\alpha$ is defined over $F$, i.e., $G_\alpha$ is quasisplit. Unless otherwise indicated, we shall assume $G$ to be a quasisplit $K$-group over $F$.

The usual definitions for connected groups extend to $K$-groups in a natural way. For example, there are similar notions of a Levi $K$-subgroup $M$ of $G$, with associated sets $\mathscr{L}(M)$ and $\mathscr{P}(M)$. The isomorphism $\psi_{\alpha\beta}$ induces a bijection of Borel pinnings from $G_\alpha$ to $G_\beta$, and taking inverse limits their canonical pinnings are thus equivalent and Galois equivariant. A central induced torus $Z$ of a $K$-group $G$ will have central embeddings $Z \simeq Z_\alpha \subset Z(G_\alpha)$ for each $\alpha$, where $Z(G_\alpha)$ is the centre of $G_\alpha$, and $\zeta$ determines a character $\zeta_\alpha$ for each $\alpha$. These isomorphisms are required to be compatible with the isomorphisms $\psi_{\alpha\beta}$ and $\psi_\alpha$ respectively.

We shall call such a pair $(Z, \zeta)$ a central datum for $G$. Finally, we note that $G(F)/Z(F) = G/Z(F)$.

**2.3.** ***Stable conjugacy classes.*** Let $G'$ be a reductive group with an embedding of semisimple conjugacy classes into that of $G$. We recall that a semisimple element $\gamma' \in G'(\bar{F})$ is called $G$-regular if the image of its conjugacy class in $G(\bar{F})$ consists of regular semisimple elements, and strongly $G$-regular if the image consists of strongly regular elements, that is, whose centralisers are tori. If $c$ is a semisimple conjugacy class of $G$, we write $G_{c,+}$ for the centraliser of a representative of $c$ in the component $G_\alpha$ that contains $c$, and write $G_c$ for the identity component of $G_{c,+}$. We call $c$ elliptic if it lies in an elliptic maximal torus in $G_\alpha$ modulo the split component $A_G \simeq A_{G_\alpha}$ of the centre of $G$.

We write $\Gamma_{\mathrm{ss}}(G) = \Gamma_{\mathrm{ss}}(G(F))$ for the set of semisimple conjugacy classes of $G(F)$, $\Gamma(G) = \Gamma_{\mathrm{reg}}(G(F))$ for the subset of strongly regular, semisimple conjugacy classes in $G(F)$, and $\Gamma_{\mathrm{ell}}(G) = \Gamma_{\mathrm{reg,ell}}(G(F))$ for the subset of regular elliptic conjugacy classes. That is,

$$\Gamma_{\mathrm{ell}}(G) \subset \Gamma_{\mathrm{reg}}(G) \subset \Gamma_{\mathrm{ss}}(G).$$

We also write $\Gamma_G(G') = \Gamma_{G\text{-reg}}(G'(F))$ and $\Gamma_{G,\mathrm{ell}}(G') = \Gamma_{G\text{-reg,ell}}(G'(F))$ for the set of $G$-regular (resp. $G$-regular elliptic) conjugacy classes in $G'(F)$. Clearly each of these sets are equal to the disjoint union over $\alpha$ of the corresponding sets for each connected component, e.g.,

$$\Gamma(G) = \coprod_{\alpha \in \pi_0(G)} \Gamma(G_\alpha)$$

and so on. The Weyl group $W(M) \simeq \prod_\alpha \mathrm{Norm}_{G_\alpha}(M_\alpha)/M_\alpha$ acts on $\Gamma_{G,\mathrm{ell}}(M)$, and we have a decomposition

$$\Gamma(G) = \bigoplus_{\{M\}} \Gamma_{G,\mathrm{ell}}(M)/W(M),$$

where the direct sum ranges over conjugacy classes of $K$-Levi subgroups $M$ in the sense of [Art99, p. 221].

We say that two semisimple elements $c_1 \in G_{\alpha_1}$ and $c_2 \in G_{\alpha_2}$ are stably conjugate if there is a $g_1 \in G_{\alpha_1}(\bar{F})$ such that the mapping

$$\varphi = \mathrm{Int}(g_1) \circ \psi_{\alpha_1 \alpha_2} : G_{\alpha_2} \to G_{\alpha_1}$$

maps $c_2$ to $c_1$, and has the property that for any $\sigma \in \Gamma$, the automorphism $\varphi \circ \sigma(\varphi)^{-1}$ of $G_{c_1}$ is inner. Let $\Delta_{\mathrm{ss}}(G) = \Delta_{\mathrm{ss}}(G(F))$ be the set of semisimple stable conjugacy classes in $G(F)$. There is a canonical injective mapping $\delta \to \delta^*$ from $\Delta_{\mathrm{ss}}(G)$ to

$\Delta_{\mathrm{ss}}(G^*)$, which is a bijection if $G$ is quasisplit. We also define subsets

$$\Delta_{\mathrm{ell}}(G) \subset \Delta_{\mathrm{reg}}(G) = \Delta(G) \subset \Delta_{\mathrm{ss}}(G)$$

as above, $\Delta_G(G') = \Delta_{G\text{-reg}}(G'(F))$ and $\Delta_{G,\mathrm{ell}}(G') = \Delta_{G\text{-reg},\mathrm{ell}}(G'(F))$ analogously. We also define

$$\Delta(G) = \bigoplus_{\{M\}} \Delta_{G,\mathrm{ell}}(M)/W(M) \tag{2-1}$$

for the $W(M)$-orbits as above. For any maximal torus $T$ of $G$ over $F$, we have the finite abelian group $\mathcal{K}(T) = \pi_0(\hat{T}^\Gamma/Z(\hat{G})^\Gamma)$. Given $\gamma \in \Gamma(G)$, there is a bijection between the set of $G(F)$ classes in the stable conjugacy class $\delta$ of $\gamma$ and the set of characters on the group $\mathcal{K}_\delta = \mathcal{K}_\gamma = \mathcal{K}(G_\gamma)$, so we set $n(\delta) = |\mathcal{K}_\delta|$. (When $F$ is archimedean, this is only true because we are taking $G$ to be a $K$-group.)

**2.4.** ***Mesoscopic datum.*** We caution that as was with endoscopy, it is likely necessary to refine the datum introduced here, which is simply a weakened version of endoscopic datum. Let us call a *mesoscopic datum* for a connective reductive group $G$ over $F$, or mesoscopic datum for short, a tuple $(G', \mathcal{G}', \xi')$, where

(1) $G'$ is a connected quasisplit reductive group over $F$,

(2) $\mathcal{G}'$ is a split extension

$$1 \to \hat{G}' \to \mathcal{G}' \to W_F \to 1$$

such that the homomorphism $W_F \to \mathrm{Out}(\mathcal{G}')$ given by this extension coincides with the homomorphism $W_F \to \mathrm{Out}(\hat{G}')$,

(3) $\xi'$ is an admissible $L$-embedding of $\mathcal{G}'$ into ${}^LG$.

As usual, we shall use $G'$ to stand in for the triple itself. Denote by $G'_{\xi'}$ the connected quasisplit reductive group whose dual group is equal to $\hat{G}_{\xi'} = \xi'(\mathcal{G}') \cap \hat{G}$. In contrast to the endoscopic setting, it will be important to distinguish between the groups $G'$ and $G'_{\xi'}$. We say that a mesoscopic datum $G'$ is *elliptic* if the connected component of $\Gamma$-invariants of the centres satisfy $(Z(\hat{G}'_{\xi'})^\Gamma)^0 = (Z(\hat{G})^\Gamma)^0$. The latter condition is also equivalent to the property that

$$\left|\mathrm{Cent}(\xi'(\mathcal{G}'), \hat{G})/Z(\hat{G})^\Gamma\right| < \infty,$$

where $\mathrm{Cent}(\xi'(\mathcal{G}'), \hat{G})$ is the centraliser of $\xi'(\mathcal{G}')$ in $\hat{G}$, and $Z(\hat{G})$ the centre of $\hat{G}$.

Given mesoscopic data $(G', \mathcal{G}', \xi')$ and $(G'_1, \mathcal{G}'_1, \xi'_1)$, we say they are isomorphic if there exist an $F$-isomorphism $\alpha : G'_1 \to G'$, an $L$-isomorphism $\beta : \mathcal{G}' \to \mathcal{G}'_1$, and an element $g \in \hat{G}'_1$ such that $\alpha : \Psi(G'_1) \to \Psi(G')$ and $\beta : \Psi(\hat{G}') \to \Psi(\hat{G}'_1)$ are dual, and $\mathrm{Int}(g) \circ \xi'_1 \circ \beta = \xi'$. We denote by $\mathcal{F}_{\mathrm{ell}}(G)$ the set of isomorphism classes of elliptic mesoscopic datum for $G$, and let $G'$ be a representative of such a class. Let

$\mathrm{Aut}_G(G')$ be the set of $g \in \hat{G}$ that induce an isomorphism of mesoscopic data, i.e., $\mathrm{Int}(g)$ is an $L$-isomorphism of $\mathcal{G}'$ onto itself, so that we may identify the group of outer automorphisms as $\mathrm{Out}_G(G') \simeq \mathrm{Aut}_G(G')/\xi'(\hat{G}')$. Any element in $\mathrm{Out}_G(G')$ can be identified with an outer automorphism of $G'$ which is defined over $F$.

The definitions extend to $K$-groups in a straightforward manner. If $G'$ is a mesoscopic datum for the component $G_\alpha$, then so it is also for $G_\beta$ for any $\beta \in \pi_0(G)$. We can therefore view $G'$ as a mesoscopic datum for the $K$-group $G$. We shall write $\mathcal{F}(G)$ for the set of isomorphism classes of mesoscopic data $G'$ for $G$ that are *relevant* to $G$, by which we mean that there is an element in $\Delta_{G\text{-reg}}(G')$ that is an image of some element in $\Delta_{\mathrm{reg}}(G)$ under $\xi'$, in the sense defined below in Section 2.5. We also write $\mathcal{F}_{\mathrm{ell}}(G)$ for the set of elliptic mesoscopic data.

**2.5.** ***Images of semisimple elements.*** The isomorphism $\hat{T} \simeq T_1$ sends the coroots of $T$ in $G$ to the roots of $T_1$ in $\hat{G}$, the $B$-simple coroots to the $B_1$-simple roots, and the Weyl group of $T$ with contragredient action to the Weyl group of $T_1$. We impose the following simplifying assumption for convenience:

**Hypothesis 2.1.** *Let $G'$ be a mesoscopic datum such that $\xi'(\hat{G}')$ and $\hat{G}$ have equal rank.*

If $(B_1', T_1')$ is a Borel pair in $\hat{G}'$ then there is an $x \in \hat{G}$ such that $\mathrm{Int}(x) \circ \xi'$ maps $T_1'$ to $T_1$ and $B_1'$ to $B_1$. If $(B', T')$ is a Borel pair in $G'$, then we have an isomorphism $\xi'(\hat{T}') \simeq \hat{T}$ defined by the composition

$$\xi'(\hat{T}') \to \xi'(T_1') \to T_1 \to \hat{T}.$$

Let us write $T'_{\xi'}$ for the torus dual to $\xi'(\hat{T}')$, so that $T'_{\xi'} \subset G'_{\xi'}$. Dualising the maps above gives an isomorphism $T'_{\xi'} \simeq T$. These isomorphisms map the coroots of $T'_{\xi'}$ in $G'_{\xi'}$ to a subsystem of coroots of $T$ in $G$, the Weyl group $W_{T'_{\xi'}}$ of $T'$ into a subgroup of the Weyl group $W_T$ of $T$, and the roots of $T'_{\xi'}$ into a subset of the roots of $T$. Then the map $T'_{\xi'}/W_{T'_{\xi'}} \to T/W_T$ of Weyl group orbits is independent of all choices. Since these orbits classify the conjugacy classes of semisimple elements in $G'_{\xi'}(\bar{F})$ and $G(\bar{F})$, we thus have a canonical map of semisimple conjugacy classes from $G'_{\xi'}(\bar{F})$, and in fact $G'(\bar{F})$, to $G(\bar{F})$.

Suppose that $T'$ is defined over $F$, and recall that we are assuming that $G$ is quasisplit. If $(B, T)$ is Borel pair in $G$ such that $T$ and $T'_{\xi'} \to T$ are defined over $F$ following Steinberg's theorem, then we say $T'_{\xi'} \to T$ is an *admissible embedding* of $T'_{\xi'}$ in $G$. It is determined up to conjugation by elements in

$$\{g \in G_{\mathrm{sc}}(\bar{F}) : g\sigma(g^{-1}) \in T(\bar{F}), \sigma \in \Gamma\},$$

where as usual $G_{\mathrm{sc}}$ denotes the simply connected cover of the derived group of $G$. We say a strongly $G$-regular $\gamma' \in G'(F)$ is an *image of* $\gamma \in G(F)$ if $\gamma$ lies in the image of the stable conjugacy class of $\gamma'$. For arbitrary $G$-regular semisimple $\gamma'$, we set $T' = G'_{\gamma'}$ and choose an admissible embedding of $T'$ in $G$. Then if $\gamma \in G(F)$ is regular semisimple and $T = G_\gamma$ then we say that $\gamma'$ is an *image of* $\gamma$ if there exists $x \in G$ such that $\mathrm{Int}(x)$ maps $\gamma$ to the image $\gamma$ of $\gamma'$ under $T' \to T$. The correspondence $(\gamma', \gamma)$ is independent of the choice of admissible embedding.

**Remark 2.2.** If $G'$ has the same rank as $G$, it follows from Borel-de Siebenthal theory regarding maximal reductive subgroups of complex reductive groups that $G'$ is simply an elliptic endoscopic group of $G$. Yet even in this case, the problem of stable transfer remains to be addressed unless $G'$ is isomorphic to $G$. On the other hand, if we do not require the ranks of $G'_{\xi'}$ and $G'$ to be equal, we have to stratify the stable transfer mapping according to Weyl orbits of Levi subgroups of $G$ in a manner similar to [Art96], but we shall avoid this situation for the time being. Note that Shelstad's work suggests that it is also possible to reduce to the equal rank case [She21].

**2.6.** ***Auxiliary data.*** The group $\mathcal{G}'$ need not be an $L$-group, so there might not be an $L$-isomorphism from $\mathcal{G}'$ to ${}^L G'$ which is the identity on $\hat{G}'$. Thus given any $G' \in \mathcal{F}(G)$, we shall fix an auxiliary datum $(\tilde{G}', \tilde{\xi}')$ where $\tilde{G}' \to G'$ is a $z$-extension, by which we mean a split central extension of $G'$ by an induced torus $\tilde{C}'$, and $\tilde{\xi}' : \mathcal{G}' \to {}^L\tilde{G}'$ is an $L$-embedding satisfying the conditions of [Art96, Lemma 2.1]. Namely, we require that the $z$-extension

$$1 \to \tilde{C}' \to \tilde{G}' \xrightarrow{r} G' \to 1$$

over $F$ satisfies:

(1) The central subgroup $\tilde{C}'$ is an induced torus.

(2) The dual exact sequence $1 \to \hat{G}' \to \hat{\tilde{G}}' \to \hat{\tilde{C}}' \to 1$ extends to a short exact sequence of $L$-homomorphisms

$$1 \to \mathcal{G}' \xrightarrow{\tilde{\xi}'} {}^L\tilde{G}' \to {}^L\tilde{C}' \to 1.$$

(3) Every element of $\mathrm{Out}_G(G')$ extends uniquely to an outer automorphism of $\tilde{G}'$ over $F$ which leaves $\tilde{C}'$ pointwise fixed.

As a $K$-group, the $z$-extension $\tilde{G}'$ satisfies $\pi_0(\tilde{G}') = \pi_0(G')$, and $\tilde{G}'_\alpha$ is a $z$-extension of $G'_\alpha$ by $\tilde{C}'$ for each $\alpha \in \pi_0(G)$. Moreover, for any frame $(\psi', u')$ of $G'$ there is a corresponding frame $(\tilde{\psi}', \tilde{u}')$ for $\tilde{G}'$ such that $r_\alpha \tilde{\psi}_{\alpha\beta} = \psi_{\alpha\beta} r_\beta$ and $\tilde{u}_{\alpha\beta} = u_{\alpha\beta}$ for all $\alpha, \beta \in \pi_0(G)$.

Fix a central datum $(Z, \zeta)$, where $Z$ is a central induced torus of $G$, and $\zeta$ is a admissible character of $Z(F)$ if $F$ is local and a character of $Z(F)\backslash Z(\mathbb{A})$ if $F$ is global. We also choose $\tilde{Z}$ and $\tilde{\zeta}$ on $\tilde{G}$ to be compatible with this datum. Let $\tilde{\eta}'$ be the character dual to the Langlands parameter induced by the composition

$$W_F \to \mathcal{G}' \xrightarrow{\tilde{\xi}'} {}^L\tilde{G}' \to {}^L\tilde{C}',$$

where $W_F \to \mathcal{G}'$ is any section. By condition (3), $\mathrm{Out}_G(G')$ can be identified with a finite group of $F$-rational outer automorphisms of $\tilde{G}'$ which leave $\tilde{C}'$ pointwise invariant, thus fixing the central character $\tilde{\eta}'$. We write $\tilde{Z}'$ for the preimage of $Z$ in $\tilde{G}'$, and $\tilde{\zeta}'$ for the product of $\tilde{\eta}'$ and the pullback of $\zeta$. We can assume that the choice of auxiliary datum $(\tilde{G}', \tilde{\xi}')$ is compatible under isomorphisms of mesoscopic data $G'$, and therefore depends only on the elements $G' \in \mathcal{F}(G)$.

Furthermore, we can also assume that $\tilde{\xi}'$ is of unitary type, in the sense that if $\phi' : W_F \to \mathcal{G}'$ is an $L$-homomorphism such that the image of $\xi' \circ \phi'$ projects to a relatively compact subset of $\hat{G}$, then the image of $\tilde{\xi}' \circ \phi'$ also projects to a relatively compact subset of $\hat{\tilde{G}}$. The analogous condition ensures that the relative endoscopic transfer factors, defined for $K$-groups in [Art99, §2], have absolute value 1.

**Remark 2.3.** In the endoscopic case, we have an endoscopic set $\mathcal{E}(G)$ consisting of ordinary endoscopic datum $(G^e, \mathcal{G}^e, s^e, \xi^e)$, where $s^e$ is a semisimple element in $\hat{G}$ satisfying certain assumptions. The associated auxiliary endoscopic datum $(\tilde{G}^e, \tilde{\xi}^e)$ is defined similarly, and also required to satisfy compatibility conditions, which we refer for example to [Art99, §2] for details. In this paper, we shall generally indicate endoscopic objects with the superscript $^e$, and 'stable' objects with $'$.

**Remark 2.4.** We shall show in Lemma 5.2 that the stable transfer is independent of choice of auxiliary datum. Recent work of Kaletha provides an alternative construction that amounts to a canonical choice of auxiliary datum [Kal22].

## 3. Stable kernels and adjoint relations

We now focus on the local setting. In preparation for the stable geometric transfer factors, we require several constructions related to the (inverse) Fourier transforms of stable orbital integrals and stable characters from [Art96], and develop some properties that we shall require. Most of this section is essentially review, aside from Lemma 3.2.

**3.1.** ***Stable orbital integrals.*** Let $\mathscr{C}(G)$ be the space of Harish-Chandra Schwartz functions on $G(F)$, and let $\mathscr{C}(G, \zeta)$ be the subspace of $\zeta^{-1}$ equivariant functions, i.e., such that

$$f(xz) = \zeta(z)^{-1} f(x), \quad x \in G(F), z \in Z(F).$$

First, if $G$ is a connective reductive group, we define the normalised orbital integral

$$f_G(\gamma) = |D(\gamma)|^{1/2} \int_{G_\gamma(F)\backslash G(F)} f(x^{-1}\gamma x)dx, \quad f \in \mathscr{C}(G, \zeta),$$

where $D(\gamma)=\det(1-\mathrm{Ad}(\gamma))_{\mathfrak{g}/\mathfrak{g}_\gamma}$ is the Weyl discriminant and $dx$ a fixed invariant measure on the orbit $G_\gamma(F)\backslash G(F)$. If $G$ is a $K$-group, we set $f_G(\gamma)=f_{\alpha,G_\alpha}$ where $G_\alpha$ is the component that contains $\gamma$. Define

$$\mathcal{I}(G,\zeta)=\{f_G : f\in\mathscr{C}(G,\zeta)\},$$

a topological space of functions on $\Gamma_{\mathrm{reg}}(G)$, topologised in a manner so that the map $f\to f_G$ is open and continuous. Denote by $\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$ the subspace of cuspidal functions, that is, functions that vanish on the complement of $\Gamma_{\mathrm{reg,ell}}(G)$. We may view it as the space of functions that are annihilated by the restriction mapping $a_G\to a_M$ from $\mathcal{I}(G,\zeta)$ to $\mathcal{I}(M,\zeta)$ for any proper Levi $M$ of $G$. More precisely, let $P\in\mathscr{P}(M)$ such that $B\subset P$ and $T\subset M$ for a fixed pinning $(B,T,\{X_\alpha\})$ of $G$. Then $(B\cap M,T,\{X_{\alpha_M}\})$ is a pinning of $M$, where $\alpha_M$ runs over simple roots of $T$ relative to $M$. Fixing Haar measures on $G(F)$ and $M(F)$, we obtain measures on the unipotent radical $N_P$ and maximal compact subgroup $K$ by the formula

$$\int_{G(F)} f(g)\,dg=\int_{M(F)}\int_{N_P(F)}\int_K f(muk)\,dk\,dn\,dm$$

for all $f\in\mathscr{C}(G)$. The map $f\to f_M$ is then given by

$$f_M(\gamma)=\int_{N_P(F)}\int_K f(k^{-1}n^{-1}\gamma nk)\,dn\,dk.$$

In particular, the map on the level of functions depends on the choice of $P$ and $K$, but choosing measures appropriately, it can be shown that the map induced from $\mathcal{I}(G,\zeta)$ to $\mathcal{I}(M,\zeta)$ indeed independent of these choices.

There is a natural measure on $\Gamma_{\mathrm{ell}}(G)$ given by

$$\int_{\Gamma_{\mathrm{ell}}(G)}\alpha(\gamma)\,d\gamma=\sum_{\{T\}}|W(G(F),T(F))|^{-1}\int_{T(F)}\alpha(t)\,dt,$$

for any $\alpha\in C_c(\Gamma(G))$, where $\{T\}$ is a set of representatives of $G(F)$-conjugacy classes of elliptic maximal tori in $G(F)$, $W(G(F),T(F))$ is the Weyl group of $(G(F),T(F))$, and $dt$ is a fixed Haar measure on $T(F)$. The corresponding measures on $\Gamma_{\mathrm{ell}}(M)$ determine a measure

$$\int_{\Gamma(G)}\alpha(\gamma)\,d\gamma=\sum_{\{M\}}|W(M)|^{-1}\int_{\Gamma_{\mathrm{ell}}(G)}\alpha(\gamma_M)\,d\gamma_M,$$

on $\Gamma(G)$.

The stable orbital integral of $f\in\mathscr{C}(G,\zeta)$ at $\delta\in\Delta_{\mathrm{reg}}(G)$ is given by

$$f^G(\delta)=\sum_{\gamma} f^G(\gamma),$$

where the sum is taken over the finite set of $\gamma \in \Gamma_{\text{reg}}(G)$ that lie in the stable class $\delta$. We then define the subspace of $\mathcal{I}(G, \zeta)$

$$S\mathcal{I}(G, \zeta) = \{f^G : f \in \mathscr{C}(G, \zeta)\},$$

and set

$$S\mathcal{I}_{\text{cusp}}(G, \zeta) = S\mathcal{I}(G, \zeta) \cap \mathcal{I}_{\text{cusp}}(G, \zeta).$$

We call a tempered, $\zeta$-equivariant distribution on $G(F)$ stable if its value at any $f \in \mathscr{C}(G, \zeta)$ depends only on $f^G$. We similarly define measures on $\Delta_{\text{ell}}(G)$ and $\Delta(G)$ by

$$\int_{\Delta_{\text{ell}}(G)} \beta(\delta)\, d\delta = \sum_{\{T\}_{\text{st}}} |W_F(G, T)|^{-1} \int_{T(F)} \beta(t)\, dt,$$

where $\beta \in C_c(\Delta(G), \{T\}_{\text{st}}$ is a set of representatives of stable conjugacy classes of elliptic maximal tori in $G$ over $F$, and $W_F(G, T)$ is the subgroup of elements in the absolute Weyl group of $(G, T)$ defined over $F$; and

$$\int_{\Delta(G)} \beta(\delta)\, d\delta = \sum_{\{M\}} |W(M)|^{-1} \int_{\Delta_{\text{ell}}(M)} \beta(\delta_M)\, d\delta_M.$$

Note that for the induced torus $Z$, we have that $\Gamma(G)/Z(F) = \Gamma(G/Z)$ and $\Delta(G)/Z(F) = \Delta(G/Z)$. Also, we have $\Delta(\tilde{G}')/\tilde{Z}'(F) = \Delta(\tilde{G}'/\tilde{Z}') = \Delta(G')$.

**3.2.** ***Conjugacy classes again.*** We shall construct certain 'functorial' sets that keep track of the stable transfer mappings, parallel to $\Delta(G)$. Given $G$, the orbits of $\text{Out}_G(G')$ on $\Delta_{\text{ell}}(G')$ depend only on the isomorphism class of $G'$ in $\mathcal{F}(G)$. It makes sense then to define the set

$$\Delta_{\text{ell}}^{\mathcal{F}}(G) = \coprod_{G' \in \mathcal{F}_{\text{ell}}(G)} \Delta_{G,\text{ell}}(G')/\text{Out}_G(G'),$$

which we can view as equivalence classes of pairs $(G', \delta')$. That is, if we write $\Delta_{G,\text{ell}}(G', G) = \Delta_{G,\text{ell}}(G')/\text{Out}_G(G')$, then

$$\Delta_{\text{ell}}^{\mathcal{F}}(G) = \{(G', \delta') : G' \in G' \in \mathcal{F}_{\text{ell}}(G), \delta' \in \Delta_{G,\text{ell}}(G')\}.$$

We can similarly define $\Delta_{G,\text{ell}}^{\mathcal{F}}(M)$ for any $M \in \mathscr{L}$. Taking the union over the $W(M)$-orbits, we set

$$\Delta_{\text{reg}}^{\mathcal{F}}(G) = \coprod_{\{M\}} \Delta_{G,\text{ell}}^{\mathcal{F}}(M)/W(M).$$

Fix an auxiliary datum $(\tilde{M}', \tilde{\xi}'_M)$ for $M' \in \mathcal{F}(M)$. We then define the set

$$\tilde{\Delta}_{G,\text{ell}}^{\mathcal{F}}(M) = \coprod_{M' \in \mathcal{F}_{\text{ell}}(M)} \Delta_{G,\text{ell}}(\tilde{M}'),$$

which fibres over $\Delta^{\mathcal{F}}_{G,\mathrm{ell}}(M)$, with the group $\prod_{M'}(\tilde{Z}'(F)\times \mathrm{Out}_M(M'))$ acting transitively on the fibres. We again take the union of $W(M)$ orbits of $\tilde{\Delta}^{\mathcal{F}}_{G,\mathrm{ell}}(M)$, and set

$$\tilde{\Delta}^{\mathcal{F}}_{\mathrm{reg}}(G)=\coprod_{\{M\}}\tilde{\Delta}^{\mathcal{F}}_{G,\mathrm{ell}}(M)/W(M),$$

which fibres over $\Delta^{\mathcal{F}}_{\mathrm{reg}}(G)$. For brevity, we write $\Delta^{\mathcal{F}}_{\mathrm{reg}}(G)=\Delta^{\mathcal{F}}(G)$ and $\tilde{\Delta}^{\mathcal{F}}_{\mathrm{reg}}(G)=\tilde{\Delta}^{\mathcal{F}}(G)$. We can also view elements of $\tilde{\Delta}^{\mathcal{F}}(G)$ as equivalence classes of tuples $(G',\tilde{G}',\tilde{\xi}',\delta')$. These constructions are readily seen to generalise the 'endoscopic' sets $\Delta^{\mathcal{E}}(G)$ and its variants in [Art02, §4], also denoted $\tilde{\Gamma}^{\mathcal{E}}(G)$ in [Art99, §2], which we shall use in this paper without comment.

**3.3.** ***Endoscopic geometric transfer factors.*** We briefly recall some basic facts about endoscopic transfer factors, such as in [Art02, §4–5]. Given an endoscopic datum $G^e\in\mathcal{E}(G)$, the geometric endoscopic transfer factor is a smooth function $\Delta(\cdot,\cdot)$ on $\Delta_G(\tilde{G}^e)\times\Gamma(G)$ such as defined in [Art99, §2]. The transfer factor determines a map

$$f\to f^e(\delta^e)=\sum_{\gamma\in\Gamma(G)}\Delta(\delta^e,\gamma)f_G(\gamma),\quad \delta^e\in\Delta_G(\tilde{G}^e),$$

from functions $f\in\mathscr{C}(G,\zeta)$ to $f^e=f^{G^e}$ on $\Delta_G(\tilde{G}^e)$. The Langlands–Shelstad transfer then implies that $f^e$ belongs to $S\mathcal{I}(\tilde{G}^e,\tilde{\zeta}^e)$. Fix an auxiliary endoscopic datum $(\tilde{G}^e,\tilde{\xi}^e)$ of $G^e$, so that $\tilde{G}^e$ is an extension of $G^e$ by a central induced torus $\tilde{C}^e$ with associated character $\tilde{\eta}^e$. The group $\tilde{C}^e(F)$ acts simply transitively on the fibres of the map $\Delta_G(\tilde{G}^e)\to\Delta_G(G^e)$, and $H^1(W_F,Z(\hat{\tilde{G}}^e))$ acts simply transitively on the set of $Z(\hat{\tilde{G}}^e)$-orbits of admissible embeddings $\tilde{\xi}^e$. Then if $az\delta$ is the image in $\Delta^{\mathcal{E}}(G)$ of a point $(G^e,\tilde{G}^e,a\tilde{\xi}^e,z\delta^e)$ with $a\in H^1(W_F,Z(\hat{\tilde{G}}^e))$ and $z\in\tilde{C}^e(F)$, then the transfer factor satisfies $\Delta(az\delta,\gamma)=\chi_a(\delta^e)\tilde{\eta}^e(z)\Delta(\delta,\gamma)$, where $\chi_a$ is a character on $\tilde{G}^e$ determined by $a$ and the local Langlands correspondence for tori.

The transfer factors and consequently the Langlands–Shelstad transfer depend only on the image of $\delta^e$ in $\tilde{\Delta}^{\mathcal{E}}(G)$, and we can extend the transfer factors to $\tilde{\Delta}^{\mathcal{E}}(G)\times\Gamma(G)$ and define the extended map

$$f\to f^{\mathcal{E}}_G=\bigoplus_{G^e\in\mathcal{E}_{\mathrm{ell}}(G)}f^e.$$

That is, we define $\Delta(\delta^e,\gamma)$ to be zero unless there is an $M$ such that $(\delta^e,\gamma)$ belongs to the Cartesian product of $\tilde{\Delta}^{\mathcal{E}}_{G,\mathrm{ell}}(M)/W(M)$ with $\Gamma_{G,\mathrm{ell}}(M)/W(M)$. If there is such an $M$, then $(\delta^e,\gamma)$ is the image of a pair $(\delta^e_M,\gamma_M)$ in $\tilde{\Delta}^{\mathcal{E}}_{G,\mathrm{ell}}(M)\times\Gamma_{G,\mathrm{ell}}(M)$, and we set

$$\Delta(\delta^e,\gamma)=\Delta_G(\delta^e,\gamma)=\sum_{w\in W(M)}\Delta_M(\delta^e_M,w\gamma_M).$$

Each sum contains at most one nonzero term, and depends only on $\delta^e$ and $\gamma$.

Define also the adjoint transfer factor

$$\Delta(\gamma, \delta^e) = |\mathcal{K}_\gamma|^{-1} \overline{\Delta(\delta^e, \gamma)}$$

on $\Gamma(G) \times \tilde{\Delta}^{\mathcal{E}}(G)$. Then by [Art99, Lemma 2.3], we have the adjoint relations

$$\sum_{\delta^e \in \Delta^{\mathcal{E}}_{\mathrm{reg}}(G)} \Delta(\gamma, \delta^e)\Delta(\delta^e, \gamma_1) = \delta(\gamma, \gamma_1), \quad \gamma, \gamma_1 \in \Gamma(G),$$

where $\delta(\cdot, \cdot)$ is the usual Kronecker delta, and

$$\sum_{\gamma \in \Gamma_{\mathrm{reg}}(G)} \Delta(\delta^e, \gamma)\Delta(\gamma, \delta^e_1) = \tilde{\delta}(\delta^e, \delta^e_1), \quad \delta^e, \delta^e_1 \in \tilde{\Delta}^{\mathcal{E}}(G),$$

where $\tilde{\delta}(\delta, \delta_1) = \tilde{\eta}^e(z)$ if $\delta_1 = z\delta$ for some $z \in \tilde{C}^e(F)$ (or equivalently, if $\delta, \delta_1$ have the same projection onto $\Delta^{\mathcal{E}}_{\mathrm{reg}}(G)$) and equal to zero otherwise. The adjoint relations imply that $f_G \to f^{\mathcal{E}}_G$ is an isomorphism from $\mathcal{I}(G, \zeta)$ onto its image.

**3.4.** ***Stable virtual characters.*** Recall from [Art93, §3] the set $T(G)$ of $W_0$-orbits of essential triples $\tau = (L, \pi, r)$ where $L \in \mathscr{L}$, $\pi \in \Pi_2(L)$, and $r \in R_\pi$, where $\Pi_2(L)$ is the set of equivalence classes of irreducible unitary representations of $L(F)$ which are square integrable mod centre, and $R_\pi$ is the $R$-group of $\pi$. Let $T_{\mathrm{ell}}(G)$ be the subset of $\tau$ such that the kernel of $(1-r)$ acting on $\mathfrak{a}_L$ is equal to $\mathfrak{a}_G$. We define

$$T(G) = \coprod_{\{M\}} T_{\mathrm{ell}}(M)/W(M).$$

We also have a decomposition with respect to any central induced torus $Z(F)$, which we assume contains the maximal $F$-split torus $A_G$,

$$T(G) = \coprod_{\zeta} T(G, \zeta),$$

where $\zeta$ runs over characters of $Z(F)$, and $T(G, \zeta)$ is the subset of elements of $T(G)$ whose central character on $Z(F)$ equals $\zeta$. We also write

$$T_{\mathrm{ell}}(G, \zeta) = T_{\mathrm{ell}}(G) \cap T(G, \zeta).$$

The set $T(G)$ parametrises a family of locally integrable functions

$$\gamma \to I(\tau, \gamma), \quad \gamma \in \Gamma(G),$$

such that for any $\zeta$, the functions $\overline{I(\tau, \gamma)}$ for $\tau \in T_{\mathrm{ell}}(G, \zeta)$ form an orthogonal basis of $\mathcal{I}_{\mathrm{cusp}}(G, \zeta)$. Also, we have that $I(\tau, \gamma z) = I(\tau, \gamma)\zeta(z)$ for any $z \in Z(F)$.

This family of functions has a stable analogue. If $F$ is nonarchimedean, by [Art96, Lemma 5.1] one can construct a set $\Phi_2(G,\zeta)$ parametrising a family of functions

$$\delta \to n(\delta)\overline{S(\phi,\delta)}, \quad \delta \in \Delta(G),$$

which forms an orthogonal basis of $S\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$, where $n(\delta) = |\mathcal{K}_\delta|$. Parallel to $T_{\mathrm{ell}}(M)$ above, the basis then provides constructions of the larger sets

$$\Phi_2(G) = \coprod_{\zeta} \Phi_2(G,\zeta), \quad \Phi(G) = \coprod_{\{M\}} \Phi_2(M)/W(M).$$

The set $\Phi(G)$ comes with an action $\phi \to \phi_\lambda = \phi \cdot \rho_\lambda$ where $\rho_\lambda$ in the nonarchimedean case is the unramified parameter which maps the Frobenius element to the image of $\lambda$ in $(Z(\hat{G})^\Gamma)^0$ under the exponential map, noting that $\mathfrak{a}^*_{G,\mathbb{C}}$ is equal to the Lie algebra of $(Z(\hat{G})^\Gamma)^0$. If $F$ is archimedean, we shall obtain this basis as a consequence of Lemma 3.1 below.

In any case, if we take the local Langlands correspondence as known, we can identify $\Phi_2(M,\zeta)$ with the set of equivalence classes of cuspidal Langlands parameters $\phi : L_F \to {}^LM$ that are compatible with $\zeta$ in the sense that the composition of $\phi$ with the projection ${}^LM \to {}^LZ$ is the Langlands parameter defined by $\zeta$. Here $L_F$ is $W_F$ or $W_F \times SL_2(\mathbb{C})$ depending on whether $F$ is real or $p$-adic. We can take instead

$$S(\phi,\delta) = \sum_{\pi \in \Pi_2(G)} \Delta(\phi,\pi) I(\pi,\gamma), \quad \gamma \in \Gamma(G),$$

where $\Delta(\phi,\pi)$ are the endoscopic spectral transfer factors defined below, and $\Pi_2(G)$ is the set of equivalence classes of irreducible unitary representations of $G(F)$ that are square integrable mod centre. The orthogonality relation for $n(\delta)\overline{S(\phi,\delta)}$ defined this way follows from that of $I(\pi,\gamma)$.

The measure on $T_{\mathrm{ell}}(G)$ is chosen to be

$$\int_{T_{\mathrm{ell}}(G)} \alpha(\tau)\,d\tau = \sum_{\tau \in T_{\mathrm{ell}}(G)/i\mathfrak{a}^*_{G,\tau}} \int_{i\mathfrak{a}^*_{G,\tau}} \alpha(\tau_\lambda)\,d\lambda \tag{3-1}$$

for any $\alpha \in C_c(T(G))$. Here we recall that $i\mathfrak{a}^*_{G,F} = i\mathfrak{a}^*_G/\mathfrak{a}^\vee_{G,F}$, and also $i\mathfrak{a}^*_{G,\tau} = i\mathfrak{a}^*_G/\mathfrak{a}^\vee_{G,\tau}$, where $\mathfrak{a}^\vee_{G,\tau}$ is the stabiliser of $\tau$ in $i\mathfrak{a}^*_G$, a lattice that lies between $\mathfrak{a}^\vee_{G,F}$ and $\tilde{\mathfrak{a}}^\vee_{G,F}$, and $d\lambda$ is a fixed measure on $i\mathfrak{a}_{G,F}$. We then define the measure on $T(G)$ to be

$$\int_{T(G)} \alpha(\tau)\,d\tau = \sum_{\{M\}} |W(M)|^{-1} \int_{T_{\mathrm{ell}}(M)} \alpha(\tau_M)\,d\tau_M.$$

Similarly, we define a measure on $\Phi_2(G)$ by setting

$$\int_{\Phi_2(G)} \beta(\phi)\, d\phi = \sum_{\phi \in \Phi_2(G)/i\mathfrak{a}_G^*} \int_{i\mathfrak{a}_{G,\phi}^*} \beta(\phi_\lambda)\, d\lambda, \tag{3-2}$$

for any $\beta \in C_c(\Phi_2(G))$, where $i\mathfrak{a}_{G,\phi}^* = i\mathfrak{a}_G^*/\mathfrak{a}_{G,\phi}^\vee$, where $\mathfrak{a}_{G,\phi}^\vee$ is the stabiliser of $\phi$ in $i\mathfrak{a}_G^*$. We then define the measure on $\Phi(G)$ to be

$$\int_{\Phi(G)} \beta(\phi)\, d\phi = \sum_{\{M\}} |W(M)|^{-1} \int_{\Phi_2(M)} \beta(\phi_M)\, d\phi_M \tag{3-3}$$

for any $\beta \in C_c(\Phi_2(G))$.

**3.5.** ***Endoscopic spectral transfer factors.*** Now for each elliptic endoscopic group $G^e \in \mathcal{E}_{\text{ell}}(G)$, we define

$$\Phi_2(\tilde{G}^e, G) = \Phi_2(\tilde{G}^e, \tilde{\zeta}^e)/\text{Out}_G(G^e).$$

The spectral transfer factors $\Delta(\phi^e, \tau)$ are then defined in [Art02, §5] to be uniquely determined functions on $\Phi_2(\tilde{G}^e, G) \times \tilde{T}_{\text{ell}}(G)$, satisfying

$$f^e(\phi^e) = \sum_{\tau \in T_{\text{ell}}(G)} \Delta(\phi^e, \tau) f_G(\tau),$$

and $\Delta(\phi^e, z_\tau \tau) = \chi_\tau(z_\tau)\Delta(\phi^e, \tau)$ for $z_\tau \in Z_\tau$, where $Z_\tau = Z_\pi$ is a central subgroup used to define a central extension $\tilde{R}_\pi$ of $R_\pi$. Define

$$T_{\text{ell}}^{\mathcal{E}}(G) = \{(G^e, \phi^e) : G^e \in \mathcal{E}_{\text{ell}}(G), \phi^e \in \Phi_2(\tilde{G}^e, G)\}$$

and

$$T^{\mathcal{E}}(G) = \coprod_{\{M\}} T_{\text{ell}}^{\mathcal{E}}(M)/W(M).$$

Then $\Delta(\phi^e, \tau)$ can be extended to a function on $T^{\mathcal{E}}(G) \times \tilde{T}(G)$ again as follows. We define $\Delta(\phi^e, \tau)$ to be zero unless there is an $M$ such that $(\phi^e, \tau)$ belongs to the Cartesian product of $T_{\text{ell}}^{\mathcal{E}}(M)/W(M)$ with $\tilde{T}_{\text{ell}}(M)/W(M)$. If there is such an $M$, then $(\phi^e, \tau)$ is the image of a pair $(\phi_M^e, \tau_M)$ in $T_{\text{ell}}^{\mathcal{E}}(M) \times \tilde{T}(M)$, and we set

$$\Delta(\phi^e, \tau) = \Delta_G(\phi^e, \tau) = \sum_{\tilde{\tau}_M} \Delta_M(\phi_M^e, \tilde{\tau}_M),$$

where the sum runs over Weyl orbit $W(M)\tau_M$.

The orthogonality relation of the stable virtual characters which is a consequence of [Art96, Lemma 5.1] and its extension to strongly regular classes $\Delta(G)$ by (2-1), is given by

$$\int_{\Delta_{\text{ell}}(G/Z)} n(\delta) S(\phi, \delta) \overline{S(\phi_1, \delta)}\, d\delta = \delta(\phi, \phi_1) n(\phi), \tag{3-4}$$

where $\delta$ is again the Kronecker delta, and $n(\phi)$ is simply defined to be the value

$$n(\phi) = \int_{\Delta_{\text{ell}}(G/Z)} n(\delta) S(\phi, \delta) \overline{S(\phi, \delta)}\, d\delta.$$

It is parallel to the formula for the invariant virtual characters

$$\int_{\Gamma_{\text{ell}}(G/Z)} I(\tau, \gamma) \overline{I(\tau_1, \gamma)}\, d\gamma = \delta(\tau, \tau_1) n(\tau),$$

where the constant $n(\tau)$ is defined by [Art93, Theorem 6.2]. We shall later derive a dual orthogonality relation for $S(\delta, \phi)$ in Lemma 4.7 below.

**3.6. *Fourier transforms.*** To define our stable transfer factors, we must recall some constructions relating to the (inverse) Fourier transforms of orbital integrals. We have for any $f \in \mathcal{H}(G, \zeta)$, the relations

$$f_G(\gamma) = \int_{T(G,\zeta)} I(\gamma, \tau) f_G(\tau)\, d\tau \tag{3-5}$$

and

$$f_G(\tau) = \int_{\Gamma(G/Z)} I(\tau, \gamma) f_G(\gamma)\, d\gamma,$$

where we denote by $I(\tau, \gamma) = |D(\gamma)|^{\frac{1}{2}} \Theta(\tau, \gamma)$ the normalised virtual character associated to $\tau$, and $I(\gamma, \tau)$ on the other hand can be viewed as the coefficient in the Fourier inversion of the orbital integral $f_G(\gamma)$. They are smooth functions in both variables, described in Theorems 4.1 and 4.3 of [Art94a]. We shall be interested in their stable analogues. By [Art96, Lemma 6.3] there exist smooth functions $S(\delta, \phi)$ and $S(\phi, \delta)$ of $\phi \in \Phi(G, \zeta)$ and $\delta \in \Delta(G/Z)$, which are respectively $\zeta$ and $\zeta^{-1}$-equivariant under translation by $Z(F)$, such that

$$f^G(\delta) = \int_{\Phi(G,\zeta)} S(\delta, \phi) f^G(\phi)\, d\phi \tag{3-6}$$

and

$$f^G(\phi) = \int_{\Delta(G/Z)} S(\phi, \delta) f^G(\delta)\, d\delta \tag{3-7}$$

for any $f \in \mathcal{H}(G, \zeta)$. The smooth functions are given by

$$S(\delta, \phi) = \sum_{\gamma \in \Gamma(G)} \sum_{\tau \in T(G,\zeta)} \Delta(\delta, \gamma) I(\gamma, \tau) \Delta(\tau, \phi) \tag{3-8}$$

and

$$S(\phi, \delta) = \sum_{\tau \in T(G,\zeta)} \sum_{\gamma \in \Gamma(G)} \Delta(\phi, \tau) I(\tau, \gamma) \Delta(\gamma, \delta). \tag{3-9}$$

Here $\Delta(\delta, \gamma)$ is the endoscopic geometric transfer factor with adjoint $\Delta(\gamma, \delta)$, and $\Delta(\phi, \tau)$ is the endoscopic spectral transfer factor with adjoint $\Delta(\tau, \phi)$, as recalled above. While it is probably best to renormalise these transfer factors according to the works of Kaletha (cf. [Kal16, §4]), we neglect to to do so here.

We record here the archimedean analogue.

**Lemma 3.1.** *Let $F$ be an archimedean local field. Then there exist smooth functions $S(\phi, \delta)$ and $S(\delta, \phi)$ of $\phi \in \Phi(G, \zeta)$ and $\delta \in \Delta(G)$, which are respectively $\zeta$ and $\zeta^{-1}$-equivariant under translation of $\delta$ by $Z(F)$, such that*

$$f^G(\delta) = \int_{\Phi(G,\zeta)} S(\delta, \phi) f^G(\phi)\, d\phi, \quad f^G(\phi) = \int_{\Delta(G/Z)} S(\phi, \delta) f^G(\delta)\, d\delta,$$

*for any $f \in \mathcal{H}(G, \zeta)$.*

*Proof.* As in the nonarchimedean case, the proof will follow in the same way as [Art96, Lemma 6.3] from the property that the linear mapping

$$f \to f^G_{\mathrm{gr}}(\phi) = \sum_{\tau \in T(G,\zeta)} \Delta(\phi, \tau) f_G(\tau), \quad \phi \in \Phi(G, \zeta),$$

is stable, and induces an isomorphism from $S\mathcal{I}(G, \zeta)$ to the graded vector space

$$S\mathcal{I}_{\mathrm{gr}}(G, \zeta) = \bigoplus_{\{M\}} S\mathcal{I}_{\mathrm{cusp}}(M, \zeta)^{W(M)},$$

(see also (4-1) below). These in turn can be deduced from [MW16, IV.2], in particular théorème IV.2.3(i) and corollaire IV.2.9. Moreover, the functions once again take the form (3-8) and (3-9). □

**3.7.** ***An adjoint relation for stable kernels.*** We derive the following identity which will play an important role in proving identities for our stable transfer factor. It is the stable analogue of the relation (3-13) in [Art94a, Theorem 4.5] relating $I(\tau, \gamma)$ to $I(\gamma, \tau)$ for elliptic virtual characters.

**Lemma 3.2.** *The stable kernels satisfy the adjoint relation*

$$n(\delta)\overline{S(\phi, \delta)} = n(\phi) S(\delta, \phi) \tag{3-10}$$

*for $\phi \in \Phi_2(G, \zeta)$ and $\delta \in \Delta_{\mathrm{ell}}(G)$.*

We can motivate the identity as follows. Applying the inversion formulae (3-7) and (3-6) consecutively to

$$f^G(\phi) = \int_{\Delta(G/Z)} S(\phi, \delta) \int_{\Phi(G,\zeta)} S(\delta, \phi_1) f^G(\phi_1)\, d\phi_1 d\delta,$$

and then interchanging the integrals, we have

$$\int_{\Phi(G,\zeta)} \int_{\Delta(G/Z)} S(\phi, \delta) S(\delta, \phi_1)\, d\delta\ f^G(\phi_1)\, d\phi_1.$$

Then if we had the relation (3-10), this is

$$\int_{\Phi(G,\zeta)} n(\phi)^{-1} \int_{\Delta(G/Z)} n(\delta) S(\phi,\delta) \overline{S(\phi_1,\delta)}\, d\delta\, f^G(\phi_1)\, d\phi_1,$$

so that the orthogonality relation (3-4) gives us the tautology

$$\int_{\Phi(G,\zeta)} \delta(\phi,\phi_1) f^G(\phi_1)\, d\phi_1 = f^G(\phi),$$

as expected.

*Proof.* For simplicity, we assume $(Z,\zeta)$ to be trivial, since it does not affect the proof. We would like to compare

$$S(\delta,\phi) = \sum_{\gamma\in\Gamma(G)} \sum_{\tau\in T(G,\zeta)} \Delta(\delta,\gamma) I(\gamma,\tau) \Delta(\tau,\phi)$$

with

$$\overline{S(\phi,\delta)} = \sum_{\tau\in T(G,\zeta)} \sum_{\gamma\in\Gamma(G)} \overline{\Delta(\phi,\tau) I(\tau,\gamma) \Delta(\gamma,\delta)}.$$

We recall the identities satisfied by the endoscopic geometric and spectral transfer factors, relating them to their adjoint functions from [Art96, (2.3) and (5.5)]:

(3-11) $$\Delta(\delta^e,\gamma) = n(\delta)\overline{\Delta(\gamma,\delta^e)}$$

and

(3-12) $$\Delta(\tau,\phi^e) = |Z(\hat{G}^e)^\Gamma / Z(\hat{G})^\Gamma|^{-1} n(\tau) n(\phi)^{-1} \overline{\Delta(\phi^e,\tau)},$$

where

$$n(\tau) = |R_{\pi,r}|\,|\det(1-r)_{\mathfrak{a}_L/\mathfrak{a}_G}|,$$

and $R_{\pi,r}$ is the centraliser of $r$ in the $R$-group $R_\pi$. Recall that $T(G)$ is the set of $W_0^G$-orbits of essential triplets $\tau = (L,\pi,r)$ where $L\in\mathscr{L}$, $\pi$ is an equivalence class of an irreducible unitary representation of $L(F)$ that is square integrable modulo centre, and $r\in R_\pi$. The $R$-group of $\pi$ is the quotient $R_\pi = W_\pi / W_\pi^0$, where $W_\pi^0$ is the subgroup of elements of $w\in W_\pi$ such that the normalised intertwining operator $R(\pi,w)$ acts by a scalar. The subset $T_{\text{ell}}(G)$ consists of $\tau$ for which the kernel of $(1-r)$ acting on $\mathfrak{a}_L$ equals $\mathfrak{a}_G$. Finally, we note that in the case at hand, we shall identify $\delta$ with its image $\delta^e = \delta^*$ in $\Delta_G(G^*)$, similarly $\phi$ with $\phi^e = \phi^*$ in $\Phi(G^*,\zeta^*)$.

If $\tau\in T_{\text{ell}}(G)$ we write $\tau^\vee = (L,\pi^\vee,r)$ for the contragredient and set

$$i^G(\tau) = |\det(1-r)_{\mathfrak{a}_L/\mathfrak{a}_G}|^{-1};$$

then it follows from the special case $M=G$ of [Art94a, Theorem 4.5] that

(3-13) $$I^{\text{old}}(\gamma,\tau) = i^G(\tau) I(\tau^\vee,\gamma).$$

for any $\gamma \in \Gamma_{\mathrm{ell}}(G)$ and $\tau \in T_{\mathrm{disc}}(G)$, which we define below. But it is crucial that the measures on $T(G)$ assigned in [Art94a, §4] differ from that of [Art96, §4] by a factor of $|R_{\pi,r}|$, which we must reconcile. This explains our notation $I^{\mathrm{old}}(\gamma,\tau)$ for the kernel used in [Art94a]. We first explain the measures. We write $T_{\mathrm{disc}}(G)$ for the subset of $W_0^G$-orbits for which the set of regular elements

$$W_\pi(r)_{\mathrm{reg}} = \{w \in W_\pi(r) : \mathfrak{a}_L^w = \mathfrak{a}_G\}$$

is nonempty. Here $W_\pi(r)$ is the subset of elements in $W(\mathfrak{a}_L) = W_\pi^0 \cdot r$ which stabilise $\pi$ and which have the same projection onto the $R$-group as $r$. For any $w$ in this set, we write $\varepsilon_\pi(w)$ for the sign of the element $wr^{-1}$ in the Weyl group $W_\pi^0$. The function $i(\tau) = i^G(\tau)$ is more generally defined on $T_{\mathrm{disc}}(G)$ as

$$i(\tau) = |W_\pi^0|^{-1} \sum_{w \in W_\pi(r)_{\mathrm{reg}}} \varepsilon_\pi(w)\,|\det(1-w)_{\mathfrak{a}_L/\mathfrak{a}_G}|^{-1}.$$

For $\tau \in T_{\mathrm{ell}}(G)$, the group $W_\pi^0$ is trivial and $i(\tau)$ specialises to the former expression. Now the measure on $T_{\mathrm{disc}}(G)$ is chosen in [Art94a] to be

$$\int_{T_{\mathrm{disc}}(G)} \alpha(\tau)\,d\tau = \sum_{\tau \in T_{\mathrm{disc}}(G)/i\mathfrak{a}_G^*} |R_{\pi,r}|^{-1} |\mathfrak{a}_{G,\tau}^\vee/\mathfrak{a}_{G,F}^\vee|^{-1} \int_{i\mathfrak{a}_{G,F}^*} \alpha(\tau_\lambda)\,d\lambda$$

for any $\alpha \in C_c(T_{\mathrm{disc}}(G))$. On the other hand, recalling the measure chosen in (3-1) compatibly with [Art96], it follows that the right-hand side can then be written as the sum over $\tau \in T_{\mathrm{ell}}(G)/i\mathfrak{a}_G^*$ of

$$|i\mathfrak{a}_G^*/\mathfrak{a}_{G,\tau}^\vee|^{-1} \int_{i\mathfrak{a}_{G,\tau}^*} \alpha(\tau_\lambda)\,d\lambda = |\mathfrak{a}_{G,\tau}^\vee/\mathfrak{a}_{G,F}^\vee|^{-1} \int_{i\mathfrak{a}_{G,F}^*} \alpha(\tau_\lambda)\,d\lambda.$$

In particular, we see that the measure conversion from [Art94a] to [Art96] is given by multiplication by $|R_{\pi,r}|^{-1}$.

We shall show that the factor $i(\tau)$ in (3-13) must therefore be multiplied by the same factor in order for the choice of measures to be consistent. We shall in fact prove a slightly stronger result, that is, for the weighted kernels $I_M(\gamma,\tau)$, by which $I_G(\gamma,\tau) = I(\gamma,\tau)$ is a special case. By our choice of measure, Theorem 4.1 of [Art94a] asserts the existence of a smooth function $I_M(\gamma,\tau)$ on $\gamma \in \Gamma(M) \cap G_{\mathrm{reg}}(F)$ and $\tau \in T_{\mathrm{disc}}(L)$ for $L \in \mathscr{L}$ such that

$$I_M(\gamma, f) = \sum_{L \in \mathscr{L}} |W_0^L|\,|W_0^G|^{-1} \int_{T_{\mathrm{disc}}(L)} |R_{\pi,r}|^{-1} I_M^{\mathrm{old}}(\gamma,\tau) f_L(\tau)\,d\tau,$$

where $I_M(\gamma, f)$ is the weighted orbital integral of $f \in \mathscr{C}(G)$, and $R_{\pi,r}$ is the group associated to $\tau = (L,\pi,r)$. In particular, the kernel $I_M^{\mathrm{old}}(\gamma,\tau)$ of [Art94a] relates to the kernel $I^M(\gamma,\tau) = I_M^{\mathrm{new}}(\gamma,\tau)$ of [Art96], which is the one we are using, by the renormalisation

$$I^M(\gamma,\tau)=|R_{\pi,r}|^{-1}I_M^{\text{old}}(\gamma,\tau).$$

We may as well verify the formula using Arthur's argument. Substituting this expression into the geometric side of the invariant local trace formula [Art94a, (5.1)],

$$\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M/A_G)}\int_{\Gamma_{\text{ell}}(M/Z)}I_M(\gamma,f)g_M(\gamma)\,d\gamma$$

for $g\in C_c^\infty(G_{\text{reg}}(F))$, we obtain the expression

$$\sum_{L\in\mathscr{L}}|W_0^L||W_0^G|^{-1}(-1)^{\dim(A_L/A_G)}\int_{T_{\text{disc}}(L)}|R_{\pi,r}|^{-1}I'_M(\tau,g)f_L(\tau^\vee)\,d\tau,$$

where

$$I'_L(\tau,g)=\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}(-1)^{\dim(A_M\times A_L)}\int_{\Gamma_{\text{ell}}(M/Z)}I_M^{\text{old}}(\gamma,\tau^\vee)g_M(\gamma)\,d\gamma.$$

By the local trace formula, this is equal to the spectral expansion

$$\sum_{L\in\mathscr{L}}|W_0^L||W_0^G|^{-1}(-1)^{\dim(A_L/A_G)}\int_{T_{\text{disc}}(L)}|R_{\pi,r}|^{-1}i^L(\tau)I_L(\tau,g)f_L(\tau^\vee)\,d\tau.$$

The remainder of the argument follows that of [Art94a, §6]. Namely, considering the difference of the spectral and geometric expansions as distributions in $f_G$, we see that the difference is a finite sum of smooth symmetric functions on the strata $T_{\text{disc}}(L)$ of $T(G)$ as $L$ varies. Since $f_G$ ranges over $\mathcal{I}(G)$, we can separate the contributions of the various strata, and it follows that

$$i^L(\tau)I_L(\tau,g)=I'_L(\tau,g),\quad L\in\mathscr{L},\tau\in T_{\text{disc}}(L).$$

From this, the same argument as Arthur's gives the parallel expansion

$$I_L(\tau,g)=\sum_{M\in\mathscr{L}}|W_0^M||W_0^G|^{-1}\int_{\Gamma_{\text{ell}}(M/Z)}I_L(\tau,\gamma)g_M(\gamma)\,d\gamma$$

of [Art94a, $(4.1)^\vee$]. Then comparing the expansions for $I_L(\tau,g)$ and $I'_L(\tau,g)$ that we have obtained, we have again that

$$(-1)^{\dim(A_M\times A_L)}I_M^{\text{old}}(\gamma,\tau^\vee)=i^L(\tau)I_L(\tau,\gamma).$$

Using the fact that $i^L(\tau^\vee)=i^L(\tau)$ and setting $M=L=G$, we conclude that the identity (3-13) should be indeed multiplied by $|R_{\pi,r}|^{-1}$ to give

$$I(\gamma,\tau)=|R_{\pi,r}|^{-1}i^G(\tau)I(\tau^\vee,\gamma)\tag{3-14}$$

by our choice of measures.

Now we can prove the proposition. Combining the identities (3-11), (3-12), and (3-14), it follows that $S(\delta,\phi)$ is equal to

$$\frac{n(\delta)}{|Z(\hat{G}^e)^\Gamma/Z(\hat{G})^\Gamma|n(\phi)}\sum_{\tau\in T(G)}\sum_{\gamma\in\Gamma(G)} n(\tau)|R_{\pi,r}|^{-1}i^G(\tau)\overline{\Delta(\gamma,\delta)}I(\tau^\vee,\gamma)\overline{\Delta(\phi,\tau)},$$

which simplifies to

$$n(\delta)n(\phi)^{-1}\sum_{\tau\in T(G)}\sum_{\gamma\in\Gamma(G)}\overline{\Delta(\gamma,\delta)}I(\tau^\vee,\gamma)\overline{\Delta(\phi,\tau)},$$

where we have used the fact that the quotient $Z(\hat{G}^e)^\Gamma/Z(\hat{G})^\Gamma$ is trivial for $G^e=G^*$. Finally, we see that if

$$I(\tau^\vee,\gamma)=\overline{I(\tau,\gamma)},$$

then the desired formula follows. We simply deduce this from the properties that

$$\overline{f_G(\gamma)}=\bar{f}_G(\gamma),\qquad \overline{f_G(\tau^\vee)}=\bar{f}_G(\tau),$$

(see for example, the proof of [Art93, Theorem 6.1]) and comparing the expansions on the either side of the first identity using (3-5),

$$\int_{T(G)}\overline{I(\gamma,\tau)f_G(\tau)}\,d\tau=\int_{T(G)}I(\gamma,\tau)\bar{f}_G(\tau)\,d\tau=\int_{T(G)}I(\gamma,\tau^\vee)\overline{f_G(\tau)}\,d\tau,$$

and again varying $f_G$ in $\mathcal{I}(G)$ accordingly. From this we have that

$$I(\gamma,\tau^\vee)=\overline{I(\gamma,\tau)},$$

then using the fact that $i_G(\tau^\vee)=i_G(\tau)$ and the relation (3-14), the claim follows.

Note that from the proof above we also find the parallel statement for the invariant kernels, namely, $I(\gamma,\tau)=\tilde{i}^G(\tau)\overline{I(\tau,\gamma)}$ where $\tilde{i}^G(\tau)=|R_{\pi,r}|^{-1}i^G(\tau)$. □

At various times in the following sections, we will want to interchange the order of integration of stable kernels in a manner similar to the heuristic above. We will give a justification for it in Section 5.3.

## 4. Stable geometric transfer factors

**4.1.** ***Local Langlands correspondence.*** Our construction of stable transfer factors relies on the transfer of $L$-parameters. Thus it is necessary to assume the local Langlands conjecture in order to formulate that definition. The local Langlands group $L_F$ is defined to be $W_F$ if $F$ is archimedean and $W_F\times SU(2)$ if $F$ is nonarchimedean. Recall that an $L$-homomorphism in this context is a homomorphism $\phi: L_F\to {}^LG$ that commutes with projections onto $W_F$ of its source and target. We say it is admissible if it is continuous and sends elements of $W_F$ to semisimple elements of ${}^LG$, and relevant if its image being contained in a Levi subgroup ${}^LM$ of ${}^LG$ implies that ${}^LM$ is the $L$-group of a Levi subgroup $M$ of $G$ over $F$.

Let $\Phi^+(G)$ be the set of $\hat{G}$-conjugacy classes of relevant, admissible $L$-homomorphisms $\phi$. By abuse of notation, let $\Phi(G) = \Phi_{\text{temp}}(G)$ be the subset of bounded or tempered Langlands parameters $\phi \in \Phi(G)$, that is, whose image projects onto a relatively compact subset of $\hat{G}$. Their corresponding $L$-packets $\Pi_\phi$ are expected to consist of tempered representations. We also let $\Phi_2(G)$ be the set of cuspidal parameters $\phi$, whose image does not lie in a proper parabolic subgroup ${}^LP$ of ${}^LG$. (The sets described above by the same notation were introduced by Arthur in order to avoid the use of the local Langlands correspondence.) The local Langlands conjecture, at the most basic level, asserts that the set $\Pi^+(G)$ of irreducible admissible representations of $G(F)$ can be written as a disjoint union of finite packets $\Pi_\phi$ as $\phi$ ranges over $\Phi^+(G)$. In other words, there exists a surjective map $\Pi(G)^+ \to \Phi^+(G)$ with finite fibres, which restricts to a surjective map of tempered representations to tempered parameters $\Pi(G) = \Pi_{\text{temp}}(G) \to \Phi_{\text{temp}}(G)$.

In the present context, given $G' \in \mathcal{F}(G)$ with auxiliary datum $(\tilde{G}', \tilde{\xi}')$, the associated Langlands parameters are $\hat{G}$-conjugacy classes of homomorphisms $W_F \to \mathcal{G}'$, and composing with $\tilde{\xi}'$ we have a map into an $L$-group ${}^L\tilde{G}'$. We shall write $\Pi(G, \zeta)$ for the subset of representations with central character equal to $\zeta$, and similarly $\Phi(G, \zeta)$, whereby

$$\Phi(G) = \coprod_{\zeta} \Phi(G, \zeta).$$

By construction, for any $G'$, the auxiliary data $(\tilde{G}', \tilde{\xi}')$ and central datum $(Z, \zeta)$ are chosen such that any $\phi' \in \Phi(\tilde{G}', \tilde{\zeta}')$ maps to a parameter $\phi \in \Phi(G, \zeta)$, again with parallel restrictions to tempered parameters.

The existence of this surjective map for $\Phi(G)$ (and the relevant endoscopic character identities) allows us to deduce the existence stable transfer mapping. In particular, as a result of the refined local Langlands correspondence, the set $\Phi(G)$ parametrises tempered $L$-packets of $G(F)$, and the space of stable orbital integrals on regular semisimple elements of $G(F)$ corresponds to the Paley–Wiener space on $\Phi(G)$ under the map given by taking stable characters.

The local Langlands correspondence implies stable transfer. The proof follows [Mok18, p. 999] and relies on the trace Paley–Wiener theorem for Schwartz functions [Art94b]. This result was recently established for $G'$ of general rank [Sun24], under the nonarchimedean local Langlands correspondence. Nonetheless, we include the proof below for the equal rank case to illustrate the main idea.

**Proposition 4.1.** *Assume the refined local Langlands correspondence for $G$ and $G'$ over $F$ nonarchimedean. Then for any local field $F$ of characteristic zero and $f \in \mathscr{C}(G, \zeta)$, there exists a unique $f' \in S\mathcal{I}(\tilde{G}', \tilde{\zeta}')$ characterised by (1-1).*

*Proof.* If $F$ is nonarchimedean, the action $\phi \to \phi_\lambda$ of $i\mathfrak{a}_G^*$ on $\Phi_2(G)$ makes it into a disjoint union of compact tori of the form $i\mathfrak{a}_{G,\phi}^* = i\mathfrak{a}_G^*/\mathfrak{a}_{G,\phi}^\vee$, where $\mathfrak{a}_{G,\phi}^\vee$ is the stabiliser of $\phi$ in $i\mathfrak{a}_G^*$. The orthogonal basis $n(\delta)\overline{S(\phi,\delta)}$ makes $S\mathcal{I}_{\text{cusp}}(G,\zeta)$ into the Paley–Wiener space on $\Phi_2(G,\zeta)$, in the sense that it is the space of functions on $\Phi_2(G,\zeta)$ supported on finitely many connected components, and which on the component of any $\phi$ pullback to a finite Fourier series on $i\mathfrak{a}_{G,\phi}^*$ [Art96, p. 541]. Moreover, the larger graded vector space

$$S\mathcal{I}_{\text{gr}}(G,\zeta) = \bigoplus_{\{M\}} S\mathcal{I}_{\text{cusp}}(M,\zeta)^{W(M)} \tag{4-1}$$

can be identified with the natural Paley–Wiener space on $\Phi(G,\zeta)$. It is a consequence of [Art96, Theorem 6.1] that we may identify $S\mathcal{I}_{\text{gr}}(G,\zeta)$ with $S\mathcal{I}(G,\zeta)$. If $F$ is archimedean, we recall that $\Phi(G,\zeta)$ is the space of $\zeta$-equivariant tempered Langlands parameters, and is a basis for $S\mathcal{I}(G,\zeta)$.

Now for general $F$, the space $S\mathcal{I}(G,\zeta)$ corresponds to the Paley–Wiener space of $\Phi(G,\zeta)$ by the map

$$\phi \to f^G(\phi), \quad f \in \mathscr{C}(G,\zeta).$$

Moreover, the $L$-embedding $\tilde{\xi}'$ induces isomorphisms on the corresponding maximal tori, and it follows that the function

$$\phi' \to f^G(\tilde{\xi}' \circ \phi'), \quad \phi' \in \Phi(\tilde{G}'),$$

belongs to the Paley–Wiener space on the set of tempered local Langlands parameters of $\tilde{G}'(F)$. Then there exists a function $f' \in \mathscr{C}(\tilde{G}',\tilde{\zeta}')$, uniquely determined up to its stable orbital integral, such that $f'(\phi') = f^G(\tilde{\xi}' \circ \phi')$. □

**Remark 4.2.** Suppose $F$ is a nonarchimedean field over which $G$ is unramified. Let $\mathcal{H}(G,K)$ denote the spherical Hecke algebra of $G(F)$, where $K$ is a hyperspecial maximal compact subgroup of $G(F)$. The Satake isomorphism implies an algebra homomorphism of Hecke algebras induced by the restriction of the embedding $\xi' : \mathcal{G}' \to {}^LG$, compatible with this spectral mapping $f \to f'$.

The goal of the stable transfer factors is to provide a parallel construction in terms of stable orbital integrals. We first note that the stable spectral transfer immediately implies an abstract stable geometric transfer.

**Corollary 4.3.** *Given (1-1), there exists a stable distribution $\Theta$ on $G \times G'$ such that*

$$f'(\delta') = \int_{\Delta(G/Z)} \Theta(\delta',\delta) f^G(\delta)\, d\delta.$$

*Proof.* The stable transfer mapping of Proposition 4.1 gives a continuous linear map from $\mathscr{C}(G,\zeta) \to S\mathcal{I}(G',\zeta')$. In particular, the $K$-invariant functions in the Harish-Chandra Schwartz space form a nuclear Fréchet space, so that $\mathscr{C}(G,\zeta)$ is an LF space, that is, a countable strict inductive limit of Fréchet spaces, in the case where $F$ is nonarchimedean. If $F$ is archimedean, it is simply nuclear Fréchet [Tre16, §51]. Then an application of the Schwartz kernel theorem [Gro55, II, §3, théorème 12] (also [Tre16, Theorem 51.7]) applied to nuclear LF spaces allows us to identify this mapping with the existence of the integral kernel as desired. □

One main goal of this paper is to propose an explicit construction for this kernel.

**4.2.** ***Stable geometric transfer factors.*** We write $S'(\cdot,\cdot) = S^{\tilde{G}'}(\cdot,\cdot)$ for the kernel functions associated to $G' \in \mathcal{F}(G)$. We can now introduce our stable geometric transfer factor as a stable distribution on $\Delta_G(\tilde{G}') \times \Delta(G)$,

$$\Theta_{\tilde{\xi}'}(\delta', \delta) = \int_{\Phi(\tilde{G}', \tilde{\zeta}')} S'(\delta', \phi') S(\tilde{\xi}' \circ \phi', \delta)\, d\phi', \tag{4-2}$$

where we identify $\tilde{\xi}' \circ \phi'$ with the image of $\phi'$ in $\Phi(G, \zeta)$ determined by $\xi'$. (Recall that the stable character $f^G(\phi)$ is independent of choice of auxiliary datum.) We distinguish this distribution from the abstract one in Corollary 4.3 by the subscript. Notice that unlike the endoscopic transfer factor, we cannot require that $\Theta_{\tilde{\xi}'}$ vanish if $\delta'$ is not an image (we are indebted to the referee for this observation). By construction, it is clear that the stable transfer factor depends only on the stable conjugacy classes of its inputs. It also implicitly depends on the normalisation of the Langlands–Shelstad transfer factors.

The stable transfer factors can be extended to distributions on $\tilde{\Delta}^{\mathcal{F}}(G) \times \Delta(G)$ as follows. Set $\Theta_{\tilde{\xi}'}(\delta', \delta)$ to be zero unless there is an $M$ such that $(\delta', \delta)$ belongs to the Cartesian product of $\tilde{\Delta}^{\mathcal{F}}_{G,\mathrm{ell}}(M)/W(M)$ with $\Delta_{G,\mathrm{ell}}(M)/W(M)$. If there is such an $M$, then $(\delta', \delta)$ is the image of a pair $(\delta'_M, \delta_M)$ in $\tilde{\Delta}^{\mathcal{F}}_{G,\mathrm{ell}}(M) \times \Delta_{G,\mathrm{ell}}(M)$, and we set

$$\Theta_{\tilde{\xi}'}(\delta', \delta) = \Theta_{\tilde{\xi}',G}(\delta', \delta) = \sum_{w \in W(M)} \Theta_{\tilde{\xi}',M}(\delta'_M, w\delta_M), \tag{4-3}$$

where again each sum contains at most one nonzero term, and depends only on $\delta'$ and $\delta$.

The candidate kernel is the main construction of this paper, which we conjecture realises the kernel of stable geometric transfer, and which we attribute to Shelstad [She21]. Note that as in the case of endoscopy, the stable transfer factor is not unique as it depends on various choices of data that we shall elaborate in the next section.

**Conjecture 4.4.** *The distribution (4-2) is a kernel satisfying Corollary 4.3. That is, we may take $\Theta = \Theta_{\tilde{\xi}'}$.*

We shall use our stable transfer factor (4-2) to study the stable transfer of orbital integrals (1-2), and relate it to the stable transfer of characters (1-1). But first we develop some basic properties that are parallel to that of the endoscopic transfer factors.

**Remark 4.5.** In [Tho20; She21] it is explicitly computed that the stable transfer factor for $G = \mathrm{SL}_2(\mathbb{R})$ and $G' = \mathrm{SO}_2(\mathbb{R})$ gives a divergent infinite sum, but can be used to produce the stable transfer of compactly supported functions.

**Lemma 4.6.** *Let $G' \in \mathcal{F}_{\mathrm{ell}}(G)$.*

(i) *For any $z \in \tilde{Z}'(F)$ whose image in $Z(F)$ equals $z_G$, we have*

$$\Theta_{\tilde{\xi}'}(\delta' z, \delta z_G) = \tilde{\zeta}'(z)^{-1}\Theta_{\tilde{\xi}'}(\delta', \delta)\zeta(z_G).$$

(ii) *Given the injective linear map $\lambda \to \lambda'$ from $\mathfrak{a}^*_{G,\mathbb{C}}$ to $\mathfrak{a}^*_{\tilde{G}',\mathbb{C}}$, we have*

$$e^{\lambda'(H_{\tilde{G}'}(\delta'))}\Theta_{\tilde{\xi}'}(\delta', \delta) = e^{\lambda(H_G(\delta))}\Theta_{\tilde{\xi}'}(\delta', \delta).$$

*whenever $\delta'$ is an image of $\delta$.*

*Proof.* From the definition, we first write

$$\Theta_{\tilde{\xi}'}(\delta' z, \delta z_G) = \int_{\Phi(\tilde{G}', \tilde{\zeta}')} S'(\delta' z, \phi') S(\tilde{\xi}' \circ \phi', \delta z_G)\, d\phi',$$

and the equivariance properties of the stable kernels (which follow from that of the endoscopic transfer factors and invariant kernels, cf. [Art96, Lemma 6.3]) yield

$$S'(\delta' z, \phi') S(\tilde{\xi}' \circ \phi', \delta z_G) = \tilde{\zeta}'(z)^{-1} S'(\delta', \phi') S(\tilde{\xi}' \circ \phi', \delta)\zeta(z_G).$$

The first result follows.

In the second place, the map from $\mathfrak{a}^*_{G,\mathbb{C}}$ to $\mathfrak{a}^*_{\tilde{G}',\mathbb{C}}$ can be viewed as a map of the complex Lie algebras of

$$(Z(\hat{G})^{\Gamma})^0 \xrightarrow{\sim} (Z(\hat{\tilde{G}}')^{\Gamma})^0,$$

where the isomorphism follows from the fact that $G'$ is elliptic. It follows then that there is an injection from $(Z(\hat{G}')^{\Gamma})^0$ to $(Z(\hat{\tilde{G}}')^{\Gamma})^0$ dual to the projection $\tilde{G}' \to G$ given by property (1) of the auxiliary datum. Then the second identity follows. □

**4.3.** ***Adjoint relations.*** Define the adjoint stable geometric transfer factor

$$\Theta^{\mathrm{ad}}_{\tilde{\xi}'}(\delta', \delta) = n(\delta')^{-2}\overline{\Theta_{\tilde{\xi}'}(\delta', \delta)}, \tag{4-4}$$

and its extension analogous to (4-3).The stable geometric transfer factors then satisfy adjoint relations parallel to those of the endoscopic geometric transfer factors in Section 3.3. But first, we derive an orthogonality relation for the $S(\delta, \phi)$ analogous to (3-4).

**Lemma 4.7.** *Let $\delta_1, \delta_2 \in \Delta_{\mathrm{ell}}(G)$ and $\phi \in \Phi_2(G, \zeta)$. Then*

$$\int_{\Phi_2(G,\zeta)} n(\phi) S(\delta_2, \phi)\overline{S(\delta_1, \phi)}\, d\phi = n(\delta_1)\delta(\delta_1, \delta_2).$$

*Proof.* The proof is based on an application of the simple local stable trace formula from [Art99, §9–10]. Let $f = f_1 \times \bar{f_2}$ with $f_i \in \mathscr{C}_{\text{cusp}}(G, \zeta)$, hence $f_{i,G}(\gamma)$ is supported on $\Gamma_{\text{ell}}(G)$ for $i = 1, 2$. Since $G$ is quasisplit, we have a stable linear form $\mathscr{C}_{\text{cusp}}(G, \zeta)$ given by

$$S^G_{\text{disc}}(f) = \int_{\Phi_2(G,\zeta)} n(\phi)^{-1} f_1^G(\phi) \overline{f_2^G(\phi)}\, d\phi$$

that is equal to

$$S^G(f) = \int_{\Delta_{\text{ell}}(G/Z)} n(\delta)^{-1} f_1^G(\delta) \overline{f_2^G(\delta)}\, d\delta.$$

We first consider the spectral expansion. Applying the relation (3-10) to (3-7), we may write

$$f_i^G(\phi) = \int_{\Delta_{\text{ell}}(G/Z)} S(\phi, \delta) f_i^G(\delta)\, d\delta = n(\phi) \int_{\Delta_{\text{ell}}(G/Z)} n(\delta)^{-1} \overline{S(\delta, \phi)} f_i^G(\delta)\, d\delta.$$

We then vary $f_i$ in a manner such that $f_i^G$ has compact support modulo $Z(F)$ on $\Delta(G)$ and so that $f_i^G$ approaches the $\zeta^{-1}$-equivariant Dirac measure at the image of $\delta_i Z(G)$ in $\Delta_{\text{ell}}(G)$ respectively for $i = 1, 2$. The function $f_i^G(\phi)$ thus approaches $n(\phi) n(\delta_i)^{-1} \overline{S(\delta_i, \phi)}$, and $S^G_{\text{disc}}(f)$ approaches

$$n(\delta_1)^{-1} n(\delta_2)^{-1} \int_{\Phi_2(G,\zeta)} n(\phi) S(\delta_2, \phi) \overline{S(\delta_1, \phi)}\, d\phi.$$

On the geometric side, we see that as $f_i$ approach the Dirac measures on $\delta_i Z(F)$ respectively, the geometric expansion

$$\int_{\Delta_{\text{ell}}(G/Z)} n(\delta)^{-1} f_1^G(\delta) \overline{f_2^G(\delta)}\, d\delta$$

approaches $n(\delta_1)^{-1} \delta(\delta_1, \delta_2)$. Equating both sides, the identity follows. □

We now state the orthogonality relation for the distributions $\Theta_{\xi'}(\delta', \delta)$ and $\Theta^{\text{ad}}_{\xi'}(\delta', \delta)$. Note that our proof involves an interchange of the order of integration of the stable kernels, understood distributionally. We delay the proof of this property to Section 5.3, where we shall require it again for (5-9).

**Proposition 4.8.** *Given $\delta', \delta'_1 \in \Delta^{\mathcal{F}}(G')$ for $G' \in \mathcal{F}_{\text{ell}}(G)$, we have*

$$\int_{\Delta(G/Z)} n(\delta) \Theta^{\text{ad}}_{\xi'}(\delta', \delta) \Theta_{\xi'}(\delta'_1, \delta) d\delta = n(\delta') \tilde{\delta}(\delta', \delta'_1). \tag{4-5}$$

*Similarly, given $\delta, \delta_1 \in \Delta(G)$, we have*

$$\int_{\Delta(G')} n(\delta') \Theta^{\text{ad}}_{\xi'}(\delta', \delta) \Theta_{\xi'}(\delta', \delta_1) d\delta' = n(\delta) \delta(\delta, \delta_1). \tag{4-6}$$

*Proof.* The first identity will be a consequence of the second. We start with the first. We will first show that for any $\delta', \delta'_1 \in \Delta^{\mathcal{F}}_{\text{ell}}(G')$, we have

$$\int_{\Delta_{\text{ell}}(G/Z)} n(\delta)\Theta^{\text{ad}}_{\tilde{\xi}'}(\delta', \delta)\Theta_{\tilde{\xi}'}(\delta'_1, \delta)d\delta = n(\delta')\tilde{\delta}(\delta', \delta'_1). \tag{4-7}$$

Then the required formula will follow from (4-3) and the decomposition of the integral over $\Delta(G)$ into

$$\sum_{\{M\}} |W(M)|^{-1} \int_{\Delta_{\text{ell}}(M)} n(\delta_M)\Theta^{\text{ad}}_{\tilde{\xi}'}(\delta', \delta_M)\Theta_{\tilde{\xi}'}(\delta'_1, \delta_M)d\delta_M,$$

where we note that the sum contains at most one nonzero term. From the definitions, we first write (4-7) as

$$\int_{\Delta_{\text{ell}}(G/Z)} n(\delta)\int_{\Phi(\tilde{G}',\tilde{\zeta}')} S'(\delta',\phi')S(\tilde{\xi}'\circ\phi',\delta)\,d\phi' \int_{\Phi(\tilde{G}',\tilde{\zeta}')} \overline{S'(\delta'_1,\phi'_1)S(\tilde{\xi}'\circ\phi'_1,\delta)}\,d\phi'_1\,d\delta.$$

Suppose first that the integration over $\Delta_{\text{ell}}(G/Z)$ and $\Phi(\tilde{G}', \tilde{\zeta}')$ can be interchanged. Then the integral over $\delta$ can be evaluated using the orthogonality relation (3-4) for $S(\phi, \delta)$ and (3-3). It follows then that the latter is equal to

$$\int_{\Phi(\tilde{G}',\tilde{\zeta}')} \delta(\tilde{\xi}'\circ\phi', \tilde{\xi}'\circ\phi'_1)n(\tilde{\xi}'\circ\phi')S'(\delta',\phi')\,d\phi' \int_{\Phi(\tilde{G}',\tilde{\zeta}')} \overline{S'(\delta'_1,\phi'_1)}\,d\phi'_1,$$

and reducing to the terms with $\phi = \phi_1$, the two integrals combine to

$$\int_{\Phi(\tilde{G}',\tilde{\zeta}')} n(\tilde{\xi}'\circ\phi')S'(\delta',\phi')\overline{S'(\delta'_1,\phi')}\,d\phi'.$$

We claim that $n(\tilde{\xi}' \circ \phi')$ equals $n(\phi')$, so that the orthogonality relation from Lemma 4.7 yields the required identity (4-7).

To prove the claim, let us compare

$$n(\tilde{\xi}'\circ\phi') = \int_{\Delta_{\text{ell}}(G/Z)} n(\delta)S(\tilde{\xi}'\circ\phi',\delta)\overline{S(\tilde{\xi}'\circ\phi',\delta)}\,d\delta$$

with

$$n(\phi') = \int_{\Delta_{\text{ell}}(G')} n(\delta')S'(\phi',\delta')\overline{S'(\phi',\delta')}\,d\delta',$$

where we note that the latter integrand depends only on the image of $\delta' \in \Delta_{\text{ell}}(\tilde{G}')$ in the set $\Delta_{\text{ell}}(\tilde{G}')/\tilde{Z}'(F) = \Delta_{\text{ell}}(\tilde{G}'/\tilde{Z}') = \Delta_{\text{ell}}(G')$. We can define an inner product on $S\mathcal{I}(G)$ by

$$(a^G, b^G) = \int_{\Delta(G/Z)} n(\delta)^{-1}a^G(\delta)\overline{b^G(\delta)}\,d\delta, \tag{4-8}$$

whose restriction to $S\mathcal{I}_{\text{cusp}}(G, \zeta)$ reduces to an integral over elliptic elements $\Delta_{\text{ell}}(G)$. Note that any function in $S\mathcal{I}(G)$ is bounded on $\Delta(G)$. Since the families of functions $\{n(\delta)\overline{S(\phi,\delta)}\}$ and $\{n(\delta')\overline{S'(\phi',\delta')}\}$ are orthogonal bases of $S\mathcal{I}_{\text{cusp}}(G, \zeta)$ and $S\mathcal{I}_{\text{cusp}}(\tilde{G}', \tilde{\zeta}')$, the identity will follow from showing that the stable transfer map is an isometry. Let us then consider

$$(a^{G'}, b^{G'}) = \int_{\Delta(G')} n(\delta')^{-1} a^{G'}(\delta') \overline{b^{G'}(\delta')}\, d\delta'.$$

Once again the integrand is

$$n(\delta')^{-1} \int_{\Delta(G/Z)} \int_{\Delta(G/Z)} \Theta_{\tilde{\xi}'}(\delta', \delta_1) \overline{\Theta_{\tilde{\xi}'}(\delta', \delta_2)} a^G(\delta_1) \overline{b^G(\delta_2)}\, d\delta_1\, d\delta_2,$$

then by (4-4) and (4-6) we see that the inner product is equal to

$$\int_{\Delta(G/Z)} \int_{\Delta(G/Z)} n(\delta_2)^{-1} \delta(\delta_1, \delta_2) a^G(\delta_1) \overline{b^G(\delta_2)}\, d\delta_1\, d\delta_2,$$

and evaluating at $\delta_1 = \delta_2$, we obtain $(a^G, b^G)$ as desired.

It remains to prove the second required identity (4-6). In this case, beginning the argument parallel to the above leads to

$$\int_{\Phi(\tilde{G}', \tilde{\zeta}')} \delta(\phi', \phi_1) n(\phi') S(\tilde{\xi}' \circ \phi', \delta)\, d\phi' \int_{\Phi(\tilde{G}', \tilde{\zeta}')} \overline{S(\tilde{\xi}' \circ \phi'_1, \delta_1)}\, d\phi'_1,$$

and hence

$$\int_{\Phi(\tilde{G}', \tilde{\zeta}')} n(\phi') S(\tilde{\xi}' \circ \phi', \delta) \overline{S(\tilde{\xi}' \circ \phi', \delta_1)}\, d\phi'.$$

We see that this closely resembles the orthogonality relation of Lemma 4.7, and indeed we shall use a variation on the proof of the latter. In particular, recall that we may choose a suitable family of test functions $f_i \in \mathscr{C}_{\text{cusp}}(G, \zeta)$ such that $f_i^G(\phi)$ approaches $n(\phi) n(\delta_i)^{-1} \overline{S(\delta_i, \phi)}$ for $i = 1, 2$. Replacing $\phi$ by $\tilde{\xi}' \circ \phi'$ and choosing $\tilde{\zeta}'$ compatibly, we thus obtain a family of functions on $\mathscr{C}_{\text{cusp}}(\tilde{G}', \tilde{\zeta}')$, which we write as $\tilde{f}_i$, so that the above equation is given as the limit of

$$\int_{\Phi(\tilde{G}', \tilde{\zeta}')} n(\phi')^{-1} f_1(\tilde{\xi}' \circ \phi') \overline{f_2(\tilde{\xi}' \circ \phi')}\, d\phi' = \int_{\Phi(\tilde{G}', \tilde{\zeta}')} n(\phi')^{-1} \tilde{f}_1(\phi') \overline{\tilde{f}_2(\phi')}\, d\phi'$$

as $f_1, f_2$ vary, where we note that the cuspidality of $\tilde{f}_i$ follows from that of $f_i$. That is, $f_i$ is supported on the set $\Gamma_{\text{ell}}(G)$ which we can identify with a subset of $\Gamma_{G,\text{ell}}(G')$ by Section 2.5. Applying the local stable trace formula in this case to $\tilde{G}'$, we have that the latter is equal to

$$\int_{\Delta_{\text{ell}}(G')} n(\delta')^{-1} \tilde{f}_1^G(\delta') \overline{\tilde{f}_2^G(\delta')}\, d\delta'.$$

Since $f_i$ is chosen so that $f_i^G$ approaches the $\zeta^{-1}$-equivariant Dirac measure at the image of $\delta_i Z(G)$ in $\Delta_{\text{ell}}(G)$, it follows that $\tilde{f}_i$ vanishes unless $\delta'$ is an image of some $\delta_i$. Moreover, for such $\delta'$ we have

$$\pi_0((\hat{G}'_{\delta'})^\Gamma / Z(\hat{G}')^\Gamma) = \pi_0(\hat{G}^\Gamma_{\delta_i} / Z(\hat{G})^\Gamma),$$

since $\hat{G}'_{\delta'}$ is isomorphic to $\hat{G}_{\delta_i}$ and $G'$ is elliptic, so that $n(\delta') = n(\delta_i)$, thus giving $n(\delta)\delta(\delta_1, \delta_2)$. □

## 5. Stable transfer: geometric and spectral

**5.1.** ***The stable transfer conjecture.*** Let us now return to our main Conjecture 4.4. We may reformulate it again as follows, with the proposed formula (4-2) for the transfer in Corollary 4.3.

**Conjecture 5.1.** *For every $f \in \mathscr{C}(G, \zeta)$, there exists an $f' \in S\mathcal{I}(\tilde{G}', \tilde{\zeta}')$ such that*

$$f'(\delta') = \int_{\Delta(G/Z)} \Theta_{\tilde{\xi}'}(\delta', \delta) f^G(\delta)\, d\delta, \quad \delta' \in \Delta_G(\tilde{G}'), \tag{5-1}$$

*from $G$ to $G'$, where $f^G(\delta)$ denotes the stable orbital integral of $f$ at a strongly regular stable conjugacy class $\delta$.*

The stable transfer depends on the choice of auxiliary data, transfer factors, and Haar measures. In particular, we can view the conjecture as a transfer of Haar measures from $G$ to $G'$. We shall establish this at the end of the section.

As in the case of endoscopy, our transfer factors are defined only up to normalisation, so it is more appropriate to speak of families of transfer factors. We shall say that $f \in \mathscr{C}(G)$ and $f' \in \mathscr{C}(G')$ have matching (stable) orbital integrals if there exists a distribution $\Theta_{\tilde{\xi}'}(\delta', \delta)$ on $\Delta(G') \times \Delta(G)$ such that (5-1) holds for all $\delta \in \Delta_G(\tilde{G}')$. Further, we call $\Theta_{\tilde{\xi}'}(\delta', \delta)$ a stable transfer factor if for each $f \in \mathscr{C}(G)$ there exists $f' \in \mathscr{C}(G)$ such that $f$ and $f'$ have matching orbital integrals. We may as well require that $\Theta_{\tilde{\xi}'}(\delta', \delta)$ be nonzero only if $\delta'$ is an image of $\delta$. Conjecture 5.1 then can be rephrased as the existence of a function $f' \in \mathscr{C}(\tilde{G}', \tilde{\zeta}')$ with matching stable orbital integrals and, implicitly, that our proposed distribution (4-2) is a stable transfer factor in the latter sense.

Of course, we remind the reader that this conjecture is by no means new. We discuss some known or simple cases.

(1) When $G' = \{1\}$, it is trivially verified in [Lan13, p. 178]. In that case, $f'$ is a constant, equal to the integral over $\Delta(G)$ of the product of $f^G(\delta)$ with the stable character $S(\phi, \delta)$, the latter being equal to $\Theta_{\tilde{\xi}'}(1, \delta)$.

(2) When $G = \mathrm{SL}(2)$ and $G'$ a torus, this is again verified in [Lan13, §2] and in the archimedean case, [She21, §27] (see also [Tho20, §8]). We explain the transfer factors here in brief. In the split case $G' = \mathrm{GL}(1)$, the representations are simply one-dimensional characters $\chi$. The stable character on $G$ is a stably invariant function on regular semisimple elements, evaluating on the split torus to

$$|D^G(\delta(t))|^{-1}(\chi(t) + \chi^{-1}(t)),$$

where we embed $\mathrm{GL}(1)$ by the usual $\delta(t) = \mathrm{diag}(t, t^{-1})$, and zero on elliptic classes. Then for $\delta \in G(F)$, the stable transfer factor can be computed by comparing stable characters on $G$ and $G'$,

$$\Theta(\delta', \delta) = |D^G(\delta)|^{-1}(\delta(\delta, \delta') + \delta(\delta^{-1}, \delta')),$$

where $\delta(\cdot, \cdot)$ is understood as the delta distribution as in [Lan13, (2.13)].

In the nonsplit case, for the real elliptic torus $G'(\mathbb{R}) = T(\mathbb{R})$, we parametrise its elements by $s(\theta)$ with $0 \le \theta < 2\pi$. The stable transfer factor $\Theta_{\xi'}(s(\theta), \delta)$ is given by

$$\sum_{n\in\mathbb{Z}} e^{in\theta} \frac{\mp e^{in\theta}}{|e^{i\theta} - e^{-i\theta}|} \quad \text{or} \quad \sum_{n\in\mathbb{Z}} e^{in\theta} \frac{e^{nt} + e^{-nt}}{|e^{t} - e^{-t}|},$$

depending on whether $\delta$ lies in the elliptic or the split torus of $G$ respectively. In the $p$-adic case, the cases separate into whether $G'$ is ramified, and in both cases the stable transfer is computed explicitly in [Lan13, §2.4], where we refer the reader for explicit formulas.

Now let us examine slightly more general cases, *without* using the local Langlands correspondence. We call a function $f \in \mathscr{C}(G, \zeta)$ cuspidal if $f_M$ vanishes for every proper Levi subgroup $M$ of $G$ (see also Section 6.1). We denote by $\mathscr{C}_{\mathrm{cusp}}(G, \zeta)$ the subspace of cuspidal functions. It is the subspace of $\mathscr{C}(G, \zeta)$ whose image in $S\mathcal{I}(G, \zeta)$ equals $S\mathcal{I}_{\mathrm{cusp}}(G, \zeta)$.

Let $f \in \mathscr{C}_{\mathrm{cusp}}(G, \zeta)$. The property that $f$ is cuspidal implies that the image $f'$, if it exists, must vanish unless there are elliptic maximal tori $T \subset G$ and $T' \subset G'$ with admissible $L$-embeddings ${}^LT \subset {}^LG$ and ${}^LT' \subset {}^LG'$ such that $\xi'({}^LT')$ is contained in ${}^LT$. The problem thus reduces to that of tori. Similarly, it is also possible to consider minimal Levi subgroups $M \subset G$ and $M' \subset G'$, which are maximal tori, and restricting to stable conjugacy classes in $M(F)$ and $M'(F)$, though we will not study this here.

**Lemma 5.2.** *The transfer $f \to f'$ is independent of choice of auxiliary datum.*

*Proof.* Suppose $(\tilde{G}'_1, \tilde{\xi}'_1)$ and $(\tilde{G}'_2, \tilde{\xi}'_2)$ are two auxiliary data with fixed central data $(Z_1, \zeta_1)$ and $(Z_2, \zeta_2)$, and associated stable transfer factors $\Theta_{\tilde{\xi}'_1}$ and $\Theta_{\tilde{\xi}'_2}$. Let $\tilde{G}'_{12}$ be the fibre product of $\tilde{G}'_1$ and $\tilde{G}'_2$ over $G'$. We have

$$Z(\hat{\tilde{G}}'_{12}) = (Z(\hat{\tilde{G}}'_1) \times Z(\hat{\tilde{G}}'_2))/\mathrm{diag}\_(Z(\tilde{G}'))$$

where $\mathrm{diag}\_(Z(\tilde{G}'))$ is the anti-diagonal embedding. Given $w \in W_F$, let $g_w = (g(w), w)$ be an element in $\mathcal{G}'$ such that $\mathrm{ad}_{g_w}$ acts by $\rho'$ on $\hat{G}'$. Also let

$$\tilde{\xi}'_i(g_w) = (\zeta_i(w), w), \quad \zeta_i(w) \in Z(\hat{\tilde{G}}'_i),$$

for $i = 1, 2$. Let $z_{12}(w)$ be the image of $(z_1(w), z_2(w)^{-1})$ in $Z(\hat{\tilde{G}}'_{12})$, which is a cocycle of $W_F$ valued in $Z(\hat{\tilde{G}}'_{12})$, and by duality determines a character $\tilde{\eta}'_{12}$ of $\tilde{G}'_{12}$. Its restriction to $\tilde{C}_1 \times \tilde{C}_2$ determining the fibre product $\tilde{G}'_1 \times \tilde{G}'_2$ over $G'$ is equal to

$\tilde{\eta}'_1 \times (\tilde{\eta}'_2)^{-1}$, and pulling back the central datum $(Z, \zeta)$ we obtain the character $\tilde{\zeta}'_{12}$ on $\tilde{G}'_{12}$. For $i = 1, 2$, let $\delta'_i \in \Delta_G(\tilde{G}'_i)$ be such that $(\delta'_1, \delta'_2) \in \tilde{G}'_{12}$. Then we have

$$\Theta_{\tilde{\xi}'_2}(\delta'_2, \delta) = \tilde{\zeta}'_{12}(\delta'_1, \delta'_2)\Theta_{\tilde{\xi}'_1}(\delta'_1, \delta). \tag{5-2}$$

from Lemma 4.6(i) and the definition of $\tilde{\zeta}'_{12}$.

The isomorphism $S\mathcal{I}(\tilde{G}'_1, \tilde{\zeta}_1)$ with $S\mathcal{I}(\tilde{G}'_2, \tilde{\zeta}_2)$ induced by the linear isomorphism $f_1 \to f_2$ from $\mathscr{C}(\tilde{G}'_1, \tilde{\zeta}_1)$ to $\mathscr{C}(\tilde{G}'_2, \tilde{\zeta}_2)$ defined by

$$f_2(\delta'_2) = \tilde{\zeta}_{12}(\delta'_1, \delta'_2) f_1(\delta'_1),$$

where $\delta_1$ is any element such that $(\delta_1, \delta_2) \in \tilde{G}'_{12}$. The isomorphism commutes with the transfer mappings $f \to f'_i = f^{\tilde{G}'_i}$. Then taking the inductive limit over such maps we see that the stable transfer mapping is independent of choice of auxiliary datum. □

As a consequence, the distribution $f'(\phi')$ depends on $\phi'$ rather than $\tilde{\phi}' = \tilde{\xi}' \circ \phi'$. However, $f'(\phi')$ does still depend on the choice of transfer factor. We shall think of $\Theta_{\tilde{\xi}'}$ as a family of transfer factors, one for each choice of $(\tilde{G}', \tilde{\xi}')$. Let us briefly indicate this. The (absolute) Langlands–Shelstad transfer factor that we have been discussing is based on the canonical relative transfer factor

$$\Delta(\delta^e, \gamma, \bar{\delta}^e, \bar{\gamma}), \quad \delta', \bar{\delta}' \in \Delta_G(\tilde{G}^e), \quad \gamma, \bar{\gamma} \in \Gamma_G(G),$$

associated to each $G$, $G^e$, and $(\tilde{G}^e, \tilde{\zeta}^e)$ (see for example [Art99, §2]). Recall that our assumption that $\tilde{\xi}'$ is of unitary type ensures that $|\Delta(\delta^e, \gamma, \bar{\delta}^e, \bar{\gamma})| = 1$. The pair $(\bar{\delta}^e, \bar{\gamma})$ are chosen base points used to define the absolute transfer factor

$$\Delta(\delta^e, \gamma) = \Delta(\delta^e, \gamma, \bar{\delta}^e, \bar{\gamma})\Delta(\bar{\delta}^e, \bar{\gamma}), \tag{5-3}$$

defined to be zero unless $\delta^e$ is an image of $\gamma$. If we call an absolute transfer factor any function $\Delta(\delta^e, \gamma)$ on $\Delta_G(\tilde{G}') \times \Gamma(G)$ such that (5-3) holds if $\delta^e$ is an image of $\gamma$, and is zero otherwise, then the space of absolute transfer factors forms a $U(1)$-torsor. Following [Art06, §2], we call $\Delta$ a transfer family for $(G, G^e)$ that varies according to $(\tilde{G}^e, \tilde{\zeta}^e)$, uniquely determined up to a multiplicative constant of absolute value one.

As the stable transfer factor $\Theta_{\tilde{\xi}'}$ depends on the transfer family $\Delta$, we can in particular define a stable transfer family depending on $\Delta$. The relation (5-2) allows us to relate stable transfer factors associated to different auxiliary data. Similarly, suppose $t : G'_1 \xrightarrow{\sim} G'_2$ is an isomorphism of mesoscopic data equipped with a dual $L$-isomorphism $\hat{t} : \mathcal{G}'_2 \xrightarrow{\sim} \mathcal{G}'_1$. and $(\tilde{G}'_1, \tilde{\xi}'_1)$ is an auxiliary datum for $G'_1$, we obtain an auxiliary datum

$$(\tilde{G}'_2, \tilde{\xi}'_2) = t(\tilde{G}'_1, \tilde{\xi}'_1)$$

for $G'_2$ such that $t$ canonically extends to an $F$-isomorphism $\tilde{G}'_1 \xrightarrow{\sim} \tilde{G}'_2$. We also have canonically a corresponding $L$-isomorphism ${}^L t : {}^L\tilde{G}'_1 \xrightarrow{\sim} {}^L\tilde{G}'_2$ such that

$$\tilde{\xi}'_2 = {}^L t \circ \tilde{\xi}'_1 \circ \hat{t}^{-1}.$$

Then if $\Theta_{\tilde{\xi}'_1}(\delta'_1, \delta)$ is a transfer factor for $(\tilde{G}'_1, \tilde{\xi}'_1)$, it follows that $\Theta_{\tilde{\xi}'_1}(t\delta'_1, \delta)$ is a transfer factor for $(\tilde{G}'_2, \tilde{\xi}'_2)$. A similar relation also holds in the endoscopic case.

As regards the general case, we have as usual the following reduction, which will allow us to work with $G'$ in place of $\tilde{G}'$ in many cases.

**Lemma 5.3.** *If Conjecture 5.1 holds for $G$ with $G_{\mathrm{der}}$ simply connected and $(Z, \zeta)$ trivial, then it holds for arbitrary $G$ and $(Z, \zeta)$.*

*Proof.* Suppose that $G$, $G'$ and $(Z, \zeta)$ are arbitrary. Let $\tilde{G}$ be a $z$-extension of $G$ by the central induced torus $\tilde{C}$, and let $(\tilde{Z}, \tilde{\zeta})$ be the pullback of $(Z, \zeta)$ to $\tilde{G}$. Since $G(F) = \tilde{G}(F)/\tilde{C}(F)$, we can identify $\mathcal{I}(\tilde{G}, \tilde{\zeta})$ with $\mathcal{I}(G, \zeta)$. Moreover, there is a bijection between isomorphism classes of mesoscopic data $\mathcal{F}(G)$ and $\mathcal{F}(\tilde{G})$. For any $G'$, we can find a extension $\tilde{\mathcal{G}}'$ equipped with an $L$-isomorphism $\tilde{\xi}' : \tilde{\mathcal{G}}' \to {}^L\tilde{G}'$ and a natural embedding of $\mathcal{G}'$ into $\tilde{\mathcal{G}}'$, so that $(\tilde{G}', \tilde{\zeta}')$ is an auxiliary datum for both $G'$ and $\tilde{G}'$. We have a natural projection $\pi$ of $\mathcal{I}(\tilde{G})$ onto $\mathcal{I}(\tilde{G}, \tilde{\zeta}) = \mathcal{I}(G, \zeta)$ given by

$$a(\gamma) \mapsto \int_{\tilde{Z}(F)} a(z\gamma)\tilde{\zeta}(z)\, dz$$

for any $a \in \mathcal{I}(\tilde{G})$. Similarly, we define a projection $\pi'$ of $S\mathcal{I}(\tilde{G}')$ onto $S\mathcal{I}(\tilde{G}', \tilde{\zeta}') = S\mathcal{I}(G', \zeta')$. It then follows from Lemma 4.6(i) that the projections commute with the stable transfer mappings from $\mathcal{I}(\tilde{G})$ to $S\mathcal{I}(\tilde{G}')$ and $\mathcal{I}(G, \zeta)$ to $S\mathcal{I}(G', \zeta')$,

$$\begin{array}{ccc} \mathcal{I}(\tilde{G}) & \longrightarrow & S\mathcal{I}(\tilde{G}') \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi'} \\ \mathcal{I}(G, \zeta) & \longrightarrow & S\mathcal{I}(G', \zeta') \end{array}$$

where the horizontal maps are the stable transfer mappings. The result follows. □

**5.2. *Geometric transfer and spectral transfer.*** The stable geometric transfer leads us naturally to a proof of the following stable character identity, which is simply a restatement of the conjectural formula [Lan13, (2.3)] in different terms.

**Corollary 5.4.** *The stable spectral transfer (1-1) implies that for any $\delta \in \Delta(G)$, we have*

$$S(\tilde{\xi}' \circ \phi', \delta) = \int_{\Delta(G')} \Theta_{\tilde{\xi}'}(\delta', \delta) S'(\phi', \delta')\, d\delta'. \tag{5-4}$$

*Proof.* As noted in [Lan13, §2.1], this follows from the stable spectral transfer (1-1), which follows by Propositions 4.1 and 4.3, and the fact that the functions $S(\phi, \delta)$

and $S'(\phi', \delta')$ are bases of the respective spaces $S\mathcal{I}_{\text{cusp}}(G, \zeta)$ and $S\mathcal{I}_{\text{cusp}}(\tilde{G}', \tilde{\zeta}')$ [Art96, Lemma 5.1], which then extend to the full spaces $S\mathcal{I}(G, \zeta)$ and $S\mathcal{I}(\tilde{G}', \tilde{\zeta}')$ as before.

We also note that this can also be seen directly from a combination of geometric and spectral transfer. Consider the stable spectral transfer (Proposition 4.1). Using the Fourier expansions for $f'(\phi')$ in (3-7), we have

$$\int_{\Delta(G')} S'(\phi', \delta') f'(\delta')\, d\delta' = \int_{\Delta(G/Z)} S(\tilde{\xi}' \circ \phi', \delta) f^G(\delta)\, d\delta. \tag{5-5}$$

Applying the stable geometric transfer, Corollary 4.3 which follows from (1-1), the left-hand side is equal to

$$\int_{\Delta(G')} S'(\phi', \delta') \int_{\Delta(G/Z)} \Theta_{\tilde{\xi}'}(\delta', \delta) f^G(\delta)\, d\delta\, d\delta'. \tag{5-6}$$

Letting $f$ approach the $\zeta^{-1}$-equivariant Dirac measure at a fixed $\delta_1 Z(F)$, we have

$$\int_{\Delta(G')} S'(\phi', \delta') \Theta_{\tilde{\xi}'}(\delta', \delta_1) d\delta' = S(\tilde{\xi}' \circ \phi', \delta_1)$$

as required. □

In [Lan13], the integration is taken over the Steinberg–Hitchin base, the variety of stable semisimple conjugacy classes of $G'(F)$. Its measure is determined by the Haar measure on $G'$, in particular the singular locus has measure zero, and coincides with $\Delta(G')$.

**Remark 5.5.** We can also give a heuristic derivation of the previous result using (4-2). By definition, the right hand side of (5-4) is

$$\int_{\Delta(G')} \int_{\Phi(\tilde{G}', \tilde{\zeta}')} S'(\delta', \phi'_1) S(\tilde{\xi}' \circ \phi'_1, \delta)\, d\phi'_1 S'(\phi', \delta')\, d\delta',$$

and supposing we may interchange integrals, we then have

$$\int_{\Phi(\tilde{G}', \tilde{\zeta}')} \int_{\Delta(G')} S'(\delta', \phi'_1) S'(\phi', \delta')\, d\delta' S(\tilde{\xi}' \circ \phi'_1, \delta)\, d\phi'_1.$$

Then applying Lemma 3.2 to the inner integral leads to the orthogonality relation (3-4) for the stable virtual characters $S(\phi, \delta)$,

$$\int_{\Delta_{\text{ell}}(G')} n(\delta') S(\phi', \delta') \overline{S'(\phi'_1, \delta')}\, d\delta' = \delta(\phi', \phi'_1) n(\phi'),$$

for a positive real number $n(\phi')$. Then evaluating the outer integral at $\phi' = \phi'_1$, we obtain $S(\tilde{\xi}' \circ \phi', \delta)$ as desired.

As was observed heuristically in [She21, §26], the stable transfer can be easily seen to be functorial in the following sense. Given $G$, let $G' \in \mathcal{F}(G)$ and $G'' \in \mathcal{F}(G')$ with accompanying auxiliary data.

**Corollary 5.6.** *With hypotheses as above, we have* $f^{\tilde{G}''} = (f^{\tilde{G}'})^{\tilde{G}''}$.

*Proof.* The result can be seen to hold for stable characters by composing maps of Paley–Wiener spaces using the argument in Proposition 4.1 and Corollary 5.4. □

**Remark 5.7.** To see this on the level of stable orbital integrals heuristically, assume for simplicity that $\mathcal{G}'$ and $\mathcal{G}''$ are $L$-groups, which we identify as ${}^L G'$ and ${}^L G''$. Then we have $L$-embeddings

$$ {}^L G'' \xrightarrow{\xi''} {}^L G' \xrightarrow{\xi'} {}^L G, $$

and denote by $\xi'''$ the composition. First, we claim that

$$ \Theta_{\xi'''}(\delta'', \delta) = \int_{\Delta(G')} \Theta_{\xi''}(\delta'', \delta')\Theta_{\tilde{\xi}'}(\delta', \delta)\, d\delta'. \tag{5-7} $$

To see this, we expand the transfer factor $\Theta_{\xi''}(\delta'', \delta')$ in the right-hand side

$$ \int_{\Delta(G')} \int_{\Phi(\tilde{G}'', \tilde{\zeta}'')} S''(\delta'', \phi'') S'(\tilde{\xi}'' \circ \phi'', \delta')\, d\phi'' \Theta_{\tilde{\xi}'}(\delta', \delta)\, d\delta'. $$

Formally interchanging the order of integration, this is

$$ \int_{\Phi(\tilde{G}'', \tilde{\zeta}'')} S''(\delta'', \phi'') \int_{\Delta(G')} S'(\tilde{\xi}'' \circ \phi'', \delta')\Theta_{\tilde{\xi}'}(\delta', \delta)\, d\delta' d\phi'', $$

and we can apply Corollary 5.4 to see that the inner integral is equal to $S(\tilde{\xi}' \circ \tilde{\xi}'' \circ \phi'', \delta) = S(\tilde{\xi}''' \circ \phi'', \delta)$. In other words, we have

$$ \int_{\Phi(\tilde{G}'', \tilde{\zeta}'')} S''(\delta'', \phi'') S(\tilde{\xi}''' \circ \phi'', \delta)\, d\phi'', $$

which is equal to $\Theta_{\xi'''}(\delta'', \delta)$, the left-hand side. Finally, writing $(f^{\tilde{G}'})^{\tilde{G}''}$ as

$$ \int_{\Delta(G')} \Theta_{\xi''}(\delta'', \delta') \int_{\Delta(G/Z)} \Theta_{\tilde{\xi}'}(\delta', \delta) f^G(\delta)\, d\delta\, d\delta', $$

then interchanging integrals and applying (5-7) gives the result.

**5.3.** ***Proof of Conjecture 5.1.*** By definition we may write (5-1) as

$$ \int_{\Delta(G/Z)} \int_{\Phi(\tilde{G}', \tilde{\zeta}')} S'(\delta', \phi') S(\tilde{\xi}' \circ \phi', \delta)\, d\phi' f^G(\delta)\, d\delta. \tag{5-8} $$

On the other hand, Let $f'(\phi')$ be the transfer given by (1-1). Applying the inversion formulas together with spectral transfer, we have

$$ \begin{aligned} f'(\delta') &= \int_{\Phi(\tilde{G}', \tilde{\zeta}')} S'(\delta', \phi') f'(\phi')\, d\phi \\ &= \int_{\Phi(\tilde{G}', \tilde{\zeta}')} S'(\delta', \phi') f(\tilde{\xi}' \circ \phi')\, d\phi \\ &= \int_{\Phi(\tilde{G}', \tilde{\zeta}')} S'(\delta', \phi') \int_{\Delta(G/Z)} S(\tilde{\xi}' \circ \phi', \delta) f^G(\delta)\, d\delta\, d\phi, \qquad (5\text{-}9) \end{aligned} $$

and in particular we note that this last expression is bounded. So we see that the result amounts to interchanging orders of integration. We shall consider this in two ways.

Firstly, consider (5-9). The stable kernels $S$, $S'$ are smooth and locally integrable on $G(F)$, and the inner integral converges absolutely for $f \in \mathscr{C}(G,\zeta)$. The outer integral, by (3-2) and (3-3), decomposes into

$$\sum_{\{M'\}} |W(M')|^{-1} \sum_{\phi' \in \Phi_2(\tilde{M}',\tilde{\zeta}')/i\mathfrak{a}^*_{\tilde{M}'}} \int_{i\mathfrak{a}^*_{\tilde{M}',\phi'}} S'(\delta',\phi'_{\lambda'}) S(\tilde{\xi}' \circ \phi'_{\lambda'},\delta)\, d\lambda',$$

where in the integrand we have only indicated the terms depending on $\phi'$ for now. Using the canonical projection of $i\mathfrak{a}^*_M$ onto $i\mathfrak{a}^*_G$, we can choose $\lambda \in i\mathfrak{a}^*_G$ such that $(\tilde{\xi}' \circ \phi)_\lambda = \tilde{\xi}' \circ \phi_{\lambda'}$, since $i\mathfrak{a}^*_{\tilde{G}'}$ acts on $\Phi(\tilde{G}',\tilde{\zeta}')$ and $i\mathfrak{a}^*_G$ on $\Phi(G,\zeta)$, and the embedding $\tilde{\xi}'$ induces an embedding of $\Phi(\tilde{G}',\tilde{\zeta}')$ into $\Phi(G,\zeta)$. Moreover, recall from [Art96, §5] the property that

$$S(\phi_\lambda,\delta) = S(\phi,\delta) e^{\lambda(H_G(\delta))}, \quad \lambda \in i\mathfrak{a}^*_G,\ \delta \in \Delta_{\mathrm{ell}}(G).$$

We see that by the adjoint relation (3-10) and the identity $n(\phi_\lambda) = n(\phi)$ which follows from the definition of $n(\phi)$, the same property also holds for $S(\delta,\phi_\lambda)$. The latter integral is therefore equal to a sum over $M$ and $\phi'$ of

$$S'(\delta',\phi') S(\tilde{\xi}' \circ \phi',\delta) \int_{i\mathfrak{a}^*_{\tilde{M}',\phi'}} e^{\lambda'(H_{\tilde{G}'}(\delta'))} e^{\lambda(H_G(\delta))}\, d\lambda. \tag{5-10}$$

If $F$ is nonarchimedean, the inner integral is taken over compact tori $i\mathfrak{a}^*_{\tilde{M}',\phi'}$, while in the archimedean case, absolute convergence follows from the growth conditions of the functions $I_M(\gamma,\pi)$ and $I_M(\pi,\gamma)$ from Theorems 4.1 and 4.3 of [Art94a]. In particular, the functions $S(\delta,\phi)$ and $S(\phi,\delta)$ are smooth and compactly supported on $\Phi(G,\zeta)$ when $F$ is nonarchimedean, and

$$|D_\lambda D_\delta S(\delta,\phi)| \le c(\delta)(1+\|\mu_\phi\|)^n$$

where $n$ is a positive integer and $c(\delta)$ is a locally bounded function on $\Delta_{\mathrm{reg}}(G)$, both depending on a given pair of invariant differential operators $D_\delta$ and $D_\lambda$ transferred from $A_M(F)$ and $i\mathfrak{a}^*_M$ respectively. Moreover, we write $\mu_\phi$ for the linear form that determines the infinitesimal character of $\phi$, which is a Weyl orbit of elements in the dual of a complex Cartan subalgebra of $G$, equipped with a suitable Hermitian norm $\|\cdot\|$. In particular, for every fixed $\phi'$, the given expression converges absolutely.

So write (5-9) as

$$\lim_{t\to\infty} \sum_{\{M'\}} |W(M')|^{-1} \sum_{\|\mu_{\phi'}\|\le t} \int_{i\mathfrak{a}^*_{\tilde{M}',\phi'}} S'(\delta',\phi'_{\lambda'}) \int_{\Delta(G/Z)} S(\tilde{\xi}' \circ \phi'_{\lambda'},\delta) f^G(\delta)\, d\delta\, d\lambda',$$

where the inner sum runs over $\phi' \in \Phi_2(\tilde{M}', \tilde{\zeta}')/i\mathfrak{a}^*_{\tilde{M}'}$ such that $\|\mu_{\phi'}\| \leq t$. The integrands now converging absolutely, the Fubini theorem for measurable functions (e.g., [Bil95, Theorem 18.3]) then allows for the change of order of integration. Finally, applying the dominated convergence theorem, we interchange the limit and integral to obtain the desired formula (5-1).

Alternatively, define functionals on $S\mathcal{I}(G, \zeta)$ by

$$T_{\phi'} : f^G(\delta) \to f^G(\tilde{\xi}' \circ \phi') = \int_{\Delta(G/Z)} S(\tilde{\xi}' \circ \phi', \delta) f^G(\delta)\, d\delta$$

and on $S\mathcal{I}(\tilde{G}', \tilde{\zeta}')$

$$T'_{\delta'} : f'(\phi') \to f'(\delta') = \int_{\Phi(\tilde{G}', \tilde{\zeta}')} S'(\delta', \phi') f'(\phi')\, d\phi,$$

which we may view as distributions on $\Delta(G/Z)$ and $\Phi(\tilde{G}', \tilde{\zeta}')$ respectively. Precomposing with the mapping $f \to f'$ of (1-1) we can consider the latter also as a distribution on $G$. Then using the property that the tensor product of distributions $T_{\phi'} \otimes T'_{\delta'}$ is commutative [Tre16, Theorem 40.4], so the order of integration can be interchanged, interpreted distributionally.

## 6. Stable transfer spaces and spectral transfer factors

**6.1.** ***Spaces of distributions.*** Let $\mathcal{D}(G, \zeta)$ be the space of $\zeta$-equivariant invariant distributions that are supported on the preimage in $G(F)$ of finitely many conjugacy classes in $\bar{G}(F) = G(F)/Z(F)$. For $F$ nonarchimedean, it is equal to the space of ordinary orbital integrals, whereas if $F$ is archimedean, it also includes radial derivatives of orbital integrals. Let $\mathcal{F}(G, \zeta)$ be the space of $\zeta$-equivariant invariant distributions spanned by the invariant characters of $G(F)$, hence generated by characters attached to the set $\Pi(G, \zeta)$ of irreducible representations of $G(F)$ whose central character restricts to $\zeta$ on $Z(F)$. A distribution $D$ in either $\mathcal{D}(G, \zeta)$ and $\mathcal{F}(G, \zeta)$ can be regarded as a linear form

$$D(f) = f_G(D)$$

on either $\mathcal{H}(G, \zeta)$ and $\mathcal{I}(G, \zeta)$.

Let $I$ be a continuous, invariant linear form on $\mathcal{H}(G, \zeta)$. We say that $I$ is supported on characters if $I(f) = 0$ for any $f$ such that $f_G = 0$. If so, then there is a continuous linear form $\hat{I}$ on $\mathcal{I}(G, \zeta)$ such that

$$\hat{I}(f_G) = I(f)$$

for all $f \in \mathcal{H}(G, \zeta)$. If $D \in \mathcal{F}(G, \zeta)$, it is clear that it is supported on characters. On the other hand, if $D \in \mathcal{D}(G, \zeta)$, it can be expressed in terms of strongly regular invariant orbital integrals, which are supported on characters by [Art88]. Together with the fact that characters are locally integrable functions, it follows that we can generate $\mathcal{I}(G, \zeta)$ by either irreducible tempered characters or strongly regular orbital integrals, both denoted $f_G$.

We denote by $\mathcal{I}_{\text{cusp}}(G, \zeta)$ the subspace of functions in $\mathcal{I}(G, \zeta)$ supported on $\Gamma_{\text{ell}}(G)$. If $Z$ contains the split component of the centre of $G$, there is a surjective linear map

$$\mathcal{F}(G, \zeta) \to \mathcal{I}_{\text{cusp}}(G, \zeta)$$

canonically given by the elliptic virtual characters $I(\tau, \gamma)$ associated to any $D \in \mathcal{F}(G, \zeta)$. There is also a canonical linear section defined by the set $T_{\text{ell}}(G, \zeta)$ in $\mathcal{F}(G, \zeta)$ whose image forms a basis in $\mathcal{I}_{\text{cusp}}(G, \zeta)$ [Art96, §4]. We also denote by $S\mathcal{I}_{\text{cusp}}(G, \zeta)$ the image of $\mathcal{I}_{\text{cusp}}(G, \zeta)$ in $S\mathcal{I}(G, \zeta)$, and $\mathcal{H}_{\text{cusp}}(G, \zeta)$ the preimage of $\mathcal{I}_{\text{cusp}}(G, \zeta)$ in $\mathcal{H}(G, \zeta)$.

Let $S\mathcal{D}(G, \zeta)$ and $S\mathcal{F}(G, \zeta)$ be the stable subspaces of stable distributions in $\mathcal{D}(G, \zeta)$ and $\mathcal{F}(G, \zeta)$. Any distribution $S$ in $S\mathcal{D}(G, \zeta)$ and $S\mathcal{F}(G, \zeta)$ can be identified with a linear form $f^G \mapsto f^G(S)$ on $S\mathcal{I}(G, \zeta)$. We say a linear form $S$ on $\mathcal{H}(G, \zeta)$ is stable if its value at $f$ depends only on the endoscopic transfer $f^e$ in the case $G^e = G^*$. If $G$ is quasisplit, there is a unique linear form $\hat{S}$ on $S\mathcal{I}(G^*, \zeta^*)$ associated to $S$ such that

$$\hat{S}(f^*) = S(f)$$

for any $f \in \mathcal{H}(G, \zeta)$. In general, for any stable distribution $S$ there is a unique continuous linear form $\hat{S}$ on $S\mathcal{I}(G, \zeta)$ such that

$$\hat{S}(f^G) = S(f).$$

One also has an alternative description of the cuspidal subspaces as follows. The restriction map $a^G \to a^M$ from $S\mathcal{I}(G(F))$ to $S\mathcal{I}(M(F))$ give a filtration

$$\mathcal{F}^M(S\mathcal{I}(G)) = \{a^G \in S\mathcal{I}(G) : a^L = 0, L \subsetneq M\}$$

of $S\mathcal{I}(G(F))$ over the partially ordered set $\mathscr{L}/W_0$. We can then identify $S\mathcal{I}_{\text{cusp}}(G)$ with $\mathcal{F}^G(S\mathcal{I}(G))$. Then the graded component

$$\mathcal{G}^M(S\mathcal{I}(G)) = \mathcal{F}^M(S\mathcal{I}(G)) / \sum_{L \supsetneq M} \mathcal{F}^L(S\mathcal{I}(G)) \tag{6-1}$$

attached to $\{M\}$ is canonically isomorphic to $S\mathcal{I}_{\text{cusp}}(M)^{W(M)}$.

**6.2.** ***Transfer spaces.*** For any $G' \in \mathcal{F}_{\text{ell}}(G)$, let $S\mathcal{I}(\tilde{G}', G)$ be the subspace of functions in $S\mathcal{I}(\tilde{G}', \zeta)$ which depend only on the image of $\Delta_G(\tilde{G}')$ in $\tilde{\Delta}^{\mathcal{F}}(G)$, which we denote by $\Delta(\tilde{G}', G)$. We define the cuspidal subspace $S\mathcal{I}_{\text{cusp}}(\tilde{G}', G)$ to be the intersection

$$S\mathcal{I}(\tilde{G}', G) \cap S\mathcal{I}_{\text{cusp}}(\tilde{G}', \tilde{\zeta}') = S\mathcal{I}_{\text{cusp}}(\tilde{G}', \tilde{\zeta}')^{\text{Out}_G(G')}.$$

Assuming the stable transfer conjecture, it follows from the definitions that $f \to f'$ maps $\mathscr{C}(G)$ continuously to $S\mathcal{I}(\tilde{G}', G)$ and $\mathscr{C}_{\text{cusp}}(G)$ continuously to $S\mathcal{I}_{\text{cusp}}(\tilde{G}', G)$. If we define a function

$$a'(\delta') = a^{G'}(\delta') = \int_{\Delta(G/Z)} \Theta_{\tilde{\xi}'}(\delta', \delta) a^G(\delta)\, d\delta \tag{6-2}$$

on $\Delta_G(\tilde{G}')$, the stable transfer gives a continuous map from $S\mathcal{I}(G)$ to $S\mathcal{I}(\tilde{G}', G)$ and $S\mathcal{I}_{\text{cusp}}(G)$ to $S\mathcal{I}(\tilde{G}', G)$. Define the topological vector space

$$S\mathcal{I}^{\mathcal{F}}_{\text{cusp}}(G) = \bigoplus_{G' \in \mathcal{F}_{\text{ell}}(G)} S\mathcal{I}_{\text{cusp}}(\tilde{G}', G)$$

of smooth functions on $\Delta^{\mathcal{F}}_{\text{ell}}(G)$. For any function $a^G \in S\mathcal{I}_{\text{cusp}}(G)$, we define the direct sum of images of $a^G$,

$$a^{\mathcal{F}} = a^{G,\mathcal{F}} = \bigoplus_{G' \in \mathcal{F}_{\text{ell}}(G)} a'.$$

Then the map

$$\mathscr{T}^{\mathcal{F}} : a^G \to a^{G,\mathcal{F}} \tag{6-3}$$

is a continuous linear map from $S\mathcal{I}_{\text{cusp}}(G)$ to $S\mathcal{I}^{\mathcal{F}}_{\text{cusp}}(G)$. The following is a natural analogue of the endoscopic mapping in [Art96, Proposition 3.5] and [MW16, I.4.11] in the nonarchimedean case and [MW16, I.4.12] in the archimedean case. The most difficult part of lies in proving the surjectivity, which we shall return to in the next section. For now we simply take it on as an assumption.

**Theorem 6.1.** *Assume that $\mathscr{T}^{\mathcal{F}}$ is surjective. Then it is an isometric isomorphism.*

*Proof.* It is straightforward to see using the adjoint relations (4-5) and (4-6) that $\mathscr{T}^{\mathcal{F}}$ is invertible on its image, with inverse $a^{G,\mathcal{F}} \to a^G$ given by

$$a^G(\delta) = n(\delta) \int_{\Delta^{\mathcal{F}}_{\text{ell}}(G)} n(\delta') \Theta_{\tilde{\xi}'}(\delta, \delta') a^{G,\mathcal{F}}(\delta')\, d\delta' \tag{6-4}$$

for any $\delta \in \Delta(G)$ and $a^{G,\mathcal{F}} \in S\mathcal{I}^{\mathcal{F}}_{\text{cusp}}(G)$. The map is moreover an isometry with respect to the inner product (4-8) on $S\mathcal{I}_{\text{cusp}}(G)$ and

$$(a^{\mathcal{F}}, b^{\mathcal{F}}) = \sum_{G' \in \mathcal{F}_{\text{ell}}(G)} \iota(G, G')(a', b') \tag{6-5}$$

on $S\mathcal{I}^{\mathcal{F}}_{\text{cusp}}(G)$, where $\iota(G, G') = |\text{Out}_G(G')|^{-1}$. The inner product can first be expanded as

$$\sum_{G' \in \mathcal{F}_{\text{ell}}(G)} \iota(G, G') \int_{\Delta_{G,\text{ell}}(G')} n(\delta')^{-1} a'(\delta') \overline{b'(\delta')}\, d\delta',$$

as $a'$, $b'$ are cuspidal hence supported on the elliptic set. Expanding $a'$, the integrand is equal to

$$n(\delta')^{-1}\int_{\Delta_{\mathrm{ell}}(G/Z)} \Theta_{\tilde{\xi}'}(\delta',\delta)a^G(\delta)\,d\delta\,\overline{b'(\delta')} = n(\delta')\int_{\Delta_{\mathrm{ell}}(G/Z)} a^G(\delta)\overline{\Theta_{\tilde{\xi}'}(\delta,\delta')b'(\delta')}\,d\delta.$$

Summing the integral over $G'$, we see that the constant $|\mathrm{Out}_G(G')|^{-1}$ normalises the measure on the quotient of $\Delta_{G,\mathrm{ell}}(G')$ by $\mathrm{Out}_G(G')$, then applying (6-4) to $b^G$ we have, as required,

$$\begin{aligned}(a^{\mathcal{F}}, b^{\mathcal{F}}) &= \int_{\Delta^{\mathcal{F}}_{\mathrm{ell}}(G)}\int_{\Delta_{\mathrm{ell}}(G/Z)} n(\delta')a^G(\delta)\overline{\Theta_{\tilde{\xi}'}(\delta,\delta')b^{\mathcal{F}}(\delta')}d\delta\,d\delta' \\ &= \int_{\Delta_{\mathrm{ell}}(G/Z)} n(\delta)^{-1}a^G(\delta)\overline{b^G(\delta)}\,d\delta \\ &= (a^G, b^G).\end{aligned}$$

□

**6.3.** ***Surjectivity.*** We now turn to the spectral analogue of our constructions so far. Our main goal will be to define stable spectral transfer factors. Assume that the spectral transfer (1-1) holds. We can then define a spectral basis parallel to $\Delta^{\mathcal{F}}_{\mathrm{ell}}(G)$, namely,

$$\Phi^{\mathcal{F}}_2(G) = \coprod_{G'\in\mathcal{F}_{\mathrm{ell}}(G)} \Phi_2(\tilde{G}',\tilde{\zeta}')/\mathrm{Out}_G(G'),$$

which can again be written as the set of pairs $(G',\phi')$. It parametrises a basis of $S\mathcal{I}^{\mathcal{F}}_{\mathrm{cusp}}(G)$. Also define

$$\Phi^{\mathcal{F}}(G) = \coprod_{\{M\}} \Phi^{\mathcal{F}}_2(M)/W(M), \tag{6-6}$$

which can also be described as the union over $W_0$-orbits $\{M\}$ in $\mathscr{L}$ and $W(M)$-orbits $\{M'\}$ in $\mathcal{F}_{\mathrm{ell}}(M)$ of the quotient of $\Phi_2(\tilde{M}',\tilde{\zeta}')$ by $\mathrm{Out}_M(M')\rtimes W(M)^{M'}$, where $W(M)^{M'}$ is the stabiliser of $M'$ in $W(M)$. We again have a decomposition according to central character,

$$\Phi^{\mathcal{F}}(G) = \coprod_{\zeta}\Phi^{\mathcal{F}}(G,\zeta). \tag{6-7}$$

With these definitions in place, we return in earnest to the surjectivity of the map $\mathscr{T}^{\mathcal{F}}$ in (6-3), which is required in order to define our stable spectral transfer factors. It can be obtained as a consequence of the refined local Langlands correspondence.

**Lemma 6.2.** *Suppose the refined local Langlands correspondence holds for $G$ over $F$. Then $\mathscr{T}^{\mathcal{F}}$ is surjective.*

*Proof.* Again relying on the local Langlands correspondence over $F$, we can identify $S\mathcal{I}(\tilde{G}',\tilde{\zeta}')$ with the natural Schwartz space on a basis $\Phi(\tilde{G}',\tilde{\zeta}')$ of the vector space spanned by tempered, stable, $\tilde{\zeta}'$-equivariant characters on $\tilde{G}'(F)$, and similarly for $S\mathcal{I}(G,\zeta)$ and $\Phi(G,\zeta)$. The elements in $\Phi(\tilde{G}',\tilde{\zeta}')$ are indexed by tempered Langlands parameters $\phi'$ of $\tilde{G}'$, and in particular decomposes into a disjoint union of cuspidal Langlands parameters attached to Levi subgroups $\tilde{M}'$ of $\tilde{G}'$. We define a space $\tilde{\Phi}^{\mathcal{F}}(G,\zeta)$ analogous to $\tilde{\Delta}^{\mathcal{F}}(G)$ in Section 3.2, which fibres over $\Phi^{\mathcal{F}}(G,\zeta)$, and $S\mathcal{I}^{\mathcal{F}}_{\text{cusp}}(G,\zeta)$ can be identified with the natural equivariant Schwartz space on it as a consequence of the trace Paley–Wiener theorem for Schwartz functions on $G$ [Art94b].

Suppose that $\phi_1$ is a finite linear combination of linear forms in $\tilde{\Phi}^{\mathcal{F}}(G,\zeta)$. We can assume that $\phi_1$ is the image of some $\phi'\in\Phi(\tilde{G}',\tilde{\zeta}')$ for some $G'\in\mathcal{F}_{\text{ell}}(G)$, such that $a^{\mathcal{F}}(\phi)=a'(\phi')$ for any $a^{\mathcal{F}}\in SI^{\mathcal{F}}(G,\zeta)$. The value at $\phi_1$ of any function $a^{\mathcal{F}}$ in $S\mathcal{I}^{\mathcal{F}}(G,\zeta)$ is then given by a finite linear combination

$$a^{\mathcal{F}}(\phi_1)=\sum_{\phi'}c_{\phi'}a'(\phi'),\quad \phi'\in\Phi(\tilde{G}',\tilde{\zeta}'),\,G'\in\mathcal{F}_{\text{ell}}(G).$$

As an invariant distribution on $\tilde{G}'(F)$, any $\phi'$ can be identified with a locally integrable function whose restriction to $\Delta_G(\tilde{G}')$ is smooth. The set $\Delta_G(\tilde{G}')$ maps onto an open subset of $\tilde{\Delta}^{\mathcal{F}}(G)$ with finite fibres, and we can thus write $a^{\mathcal{F}}(\phi_1)$ as

$$\int_{\Delta^{\mathcal{F}}(G/Z)}I^{\mathcal{F}}(\phi_1,\delta_1)a^{\mathcal{F}}(\delta_1)\,d\delta_1$$

for some smooth, $\zeta$-equivariant function $I^{\mathcal{F}}(\phi_1)$ on $\tilde{\Delta}^{\mathcal{F}}(G)$, whose integral against any $a^{\mathcal{F}}=a^{G,\mathcal{F}}$ converges with respect to the measure $d\delta_1$. Applying (6-2), we then have

$$\int_{\Delta(G/Z)}I(\phi_1,\delta)a^G(\delta)\,d\delta,\quad a^G\in S\mathcal{I}(G,\zeta),$$

where

$$I(\phi_1,\delta)=\int_{\Delta^{\mathcal{F}}(G/Z)}I^{\mathcal{F}}(\phi_1,\delta_1)\Theta_{\tilde{\xi}'}(\delta_1,\delta)\,d\delta_1$$

is again a smooth, $\zeta$-equivariant function on $\Delta(G)$, whose integral against any $a^G$ converges with respect to the measure $d\delta$. Letting $a^G$ now approximate the $\zeta^{-1}$-equivariant Dirac measure at $\delta Z(F)$, it follows that if the function

$$a\to a^{\mathcal{F}}(\phi_1),\quad a\in\mathscr{C}(G,\zeta),$$

induced by $\phi_1$ vanishes, then so does $I(\phi_1,\delta)$. By the inversion formula (4-5), it follows that $I^{\mathcal{F}}(\phi_1,\delta_1)$ also vanishes on $\tilde{\Delta}^{\mathcal{F}}(G)$, and hence $\phi_1$ itself vanishes. The mapping $\mathscr{T}^{\mathcal{F}}$ is thus locally surjective.

To see that it is surjective, we can define a spectral transfer factor describing the local mapping $a\to a^{\mathcal{F}}$,

$$a^{\mathcal{F}}(\phi_1)=a'(\phi')=\int_{\Phi_2(G,\zeta)}\Theta_{\tilde{\xi}'}(\phi_1,\phi)a(\phi)\,d\phi,\quad a\in\mathscr{C}(G,\zeta),$$

compatible with the decompositions (6-6) and (6-7) as below, hence with the characterisations of $S\mathcal{I}(G,\zeta)$ and $S\mathcal{I}^{\mathcal{F}}(G,\zeta)$ as Schwartz spaces of functions on $\Phi(G,\zeta)$ and $\tilde{\Phi}^{\mathcal{F}}(G,\zeta)$ respectively. Thus the surjectivity extends. □

**6.4. *Stable spectral transfer factors.*** Our discussion here now parallels the geometric case, so we may be brief. Let $f\in\mathscr{C}_{\text{cusp}}(G,\zeta)$. For any $G'\in\mathcal{F}_{\text{ell}}(G)$, the transfer $f'$ is a function in $S\mathcal{I}_{\text{cusp}}(\tilde{G}',\tilde{\zeta}')^{\text{Out}_G(G')}$ and $f'(\phi')$ is defined for every $\phi'\in\Phi_2(\tilde{G}',\tilde{\zeta}')$. If $\mathscr{T}^{\mathcal{F}}$ is surjective, we may define a stable spectral transfer factor $\Theta_{\tilde{\xi}'}(\phi',\phi)$ to be any distribution on $\Phi_2(\tilde{G}',\tilde{\zeta}')\times\Phi_2(G,\zeta)$ such that the identity

$$f'(\phi')=\int_{\Phi_2(G,\zeta)}\Theta_{\tilde{\xi}'}(\phi',\phi)f(\phi)\,d\phi$$

holds. The stable transfer factors can again be extended to distributions on $\Phi^{\mathcal{F}}(G)\times\Phi(G)$ as follows. Set $\Theta_{\tilde{\xi}'}(\delta',\delta)$ to be zero unless there is an $M$ such that $(\phi',\phi)$ belongs to the Cartesian product of $\tilde{\Delta}^{\mathcal{F}}_{G,\text{ell}}(M)/W(M)$ with $\Delta_{G,\text{ell}}(M)/W(M)$. If there is such an $M$, then $(\phi',\phi)$ is the image of a pair $(\phi'_M,\phi_M)$ in $\Phi^{\mathcal{F}}_2(M)\times\Phi_2(M)$, and we set

$$\Theta_{\tilde{\xi}'}(\phi',\phi)=\Theta_{\tilde{\xi}',G}(\phi',\phi)=\sum_{w\in W(M)}\Theta_{\tilde{\xi}',M}(\phi'_M,w\phi_M),\tag{6-8}$$

where again each sum contains at most one nonzero term, and depends only on $\phi'$ and $\phi$. Finally, define the adjoint spectral transfer factor

$$\Theta_{\tilde{\xi}'}(\phi,\phi')=n(\phi')^{-2}\overline{\Theta_{\tilde{\xi}'}(\phi',\phi)},$$

which complements the adjoint stable geometric transfer factors quite nicely. As in the geometric case, their definition is imposed upon us by the adjoint relations they satisfy, parallel to Proposition 4.8.

**Proposition 6.3.** *Given* $\phi',\phi'_1\in\Phi^{\mathcal{F}}(G')$ *for* $G'\in\mathcal{F}_{\text{ell}}(G)$, *we have*

$$\int_{\Phi(G)}n(\phi)\Theta_{\tilde{\xi}'}(\phi',\phi)\Theta_{\tilde{\xi}'}(\phi,\phi'_1)d\phi=n(\phi')\tilde{\delta}(\phi',\phi'_1).\tag{6-9}$$

*Similarly, given* $\phi,\phi_1\in\Phi(G)$, *we have*

$$\int_{\Phi(G')}n(\phi')\Theta_{\tilde{\xi}'}(\phi',\phi)\Theta_{\tilde{\xi}'}(\phi_1,\phi')d\phi'=n(\phi)\delta(\phi,\phi_1).\tag{6-10}$$

*Proof.* Since we do not have an explicit description of the spectral transfer factors, the proof will follow instead from interpreting the linear isometry $\mathscr{T}^{\mathcal{F}}$ spectrally. We recall the spectral form of the inner product on $S\mathcal{I}_{\text{cusp}}(G)$,

$$(a^G,b^G)=\int_{\Phi_2(G)}n(\phi)^{-1}a^G(\phi)\overline{b^G(\phi)}\,d\phi,\quad a^G,b^G\in S\mathcal{I}_{\text{cusp}}(G).$$

We then have the spectral form of the inner product on $S\mathcal{I}^{\mathcal{F}}_{\mathrm{cusp}}(G)$ in (6-5),

$$(a^{\mathcal{F}}, b^{\mathcal{F}}) = \sum_{G' \in \mathcal{F}_{\mathrm{ell}}(G)} \iota(G, G') \int_{\Phi_2(G')} n(\phi')^{-1} a'(\phi') \overline{b'(\phi')}\, d\phi'.$$

Then defining the spectral analogue of the inverse of $\mathscr{T}^{\mathcal{F}}$,

$$a^G(\phi) = n(\phi) \int_{\Phi_2^{\mathcal{F}}(G)} n(\phi') \Theta_{\tilde{\xi}'}(\phi, \phi') a^{G,\mathcal{F}}(\phi')\, d\phi',$$

and using the definition of $\Theta_{\tilde{\xi}'}(\phi', \phi)$ and its adjoint above, it follows by the same argument as in the proof of Theorem 6.1 that $(a^{\mathcal{F}}, b^{\mathcal{F}}) = (a^G, b^G)$. In particular, $\Theta_{\tilde{\xi}'}(\phi', \phi)$ and $\Theta_{\tilde{\xi}'}(\phi, \phi')$ represent kernels of inverse transforms of each other. □

## Appendix: On the surjectivity of $\mathscr{T}^{\mathcal{F}}$ (without local Langlands)

The proofs of surjectivity of the analogous endoscopic map for nonarchimedean fields [Art96, Lemma 3.4] and [MW16, I.4.11] both involve a reduction to the Lie algebra at the identity element, either by passing to germs of orbital integrals or Harish-Chandra descent to the Lie algebra of unipotent subgroups. In both cases, the descent of transfer factors and properties of the Fourier transform on the Lie algebra are employed. In order to discuss the latter, we need some preparations.

**A.1.** ***Descent principles.*** We explore the relationship between descent and transfer. The descent properties will be required later to define our stable spectral transfer factors. As in the case of endoscopy [LS90], this reduces to the descent of transfer factors, which we shall have to take on as a hypothesis in this paper. Given $G' \in \mathcal{F}(G)$ and $d' \in \Delta_{\mathrm{ss}}(G')$, let $\tilde{d}'$ be its preimage in $\Delta_{\mathrm{ss}}(\tilde{G}')$. We shall investigate the behaviour of

$$f'(\tilde{\delta}') = \int_{\Delta(G/Z)} \Theta_{\tilde{\xi}'}(\tilde{\delta}', \delta) f^G(\delta)\, d\delta$$

for $\tilde{\delta}'$ near to $\tilde{d}'$. If $d'$ is not the image of any semisimple element in $\Delta_{\mathrm{ss}}(G)$, then no strongly $G$-regular element in $G'_{d'}(F)$ can be the image of an element in $G(F)$, thus $f'$ vanishes on $\Delta_G(\tilde{G}'_{\tilde{d}'})$. It follows then that $f'$ vanishes for all $\tilde{d}'$ in a neighbourhood of $\tilde{d}'$ in $\tilde{G}'(F)$. We shall therefore assume that $d'$ is the image of some $d \in \Delta(G)$.

We can find a representative $d'_1 = x^{-1} d' x$ in the stable conjugacy class of $d'$ such that $G'_{d'_1}$ is quasisplit over $F$, and we can multiply $x$ by an element of $G'_{d'_1}$ if necessary so that

$$\mathrm{Int}(x^{-1}) : G'_{d'} \to G'_{d'_1}$$

is defined over $F$. Then $x$ acts on the preimage $\tilde{G}'_{d'}$ of $G'_{d'}$ in $\tilde{G}'$, so that $f'(x^{-1}\tilde{\delta}' x) = f'(\tilde{\delta}')$ for all $\tilde{\delta}' \in \Delta_G(\tilde{G}_{d'})$, which can also be seen to follow from the same property for stable orbital integrals [LS90, §1.3]. In particular, replacing $d'$ by $d'_1$ if necessary, we may assume that $G'_{d'}$ is quasisplit over $F$.

The group $G'_{d'}$ induces a mesoscopic datum for $G_d$ in the following sense. More generally, we call $d'$ a $T'$-image of $d$ in $G$ if there exists an admissible embedding of tori $T' \to T^*$ sending $d'$ to $d^*$ in $G^*$ and an $x \in G^*_{\mathrm{sc}}$ such that

$$(\mathrm{Int}(x) \circ \psi)(d) = d^*$$

and both $\mathrm{Int}(x) \circ \psi$ and the preimage of $T$ are defined over $F$. Varying over $T'$ we obtain all images of $d$. Let $d'$ be a $T'$-image of $d$ for some torus $T'$, and let $d^*$ be the image of $d'$ under an admissible embedding of $T'$ in $T^*$, which we may choose to be such that $G^*_{d^*}$ is quasisplit. Fixing $G' \in \mathcal{F}(G)$, we shall attach an extension $\mathcal{G}'_{d'}$ of $W_F$ by $\hat{G}'_{d'}$ and an admissible embedding $\xi'_{d'} : \mathcal{G}'_{d'} \to {}^L G'_d$ such that $(G'_{d'}, \mathcal{G}'_{d'}, \xi'_{d'})$ is a mesoscopic datum for $G_d$ and $T' \to T^*$ is admissible.

**Lemma A.1.** *The embedding $T' \to T^*$ can be chosen to be admissible for both $(G, G')$ and $(G_d, G'_{d'})$, unique up to isomorphism. Any admissible embedding of a maximal torus of $G'_{d'}$ in $G^*_{d^*}$ is admissible as an embedding of a maximal torus of $G'$ in $G^*$ and sends $d' \to d^*$.*

*Proof.* The proof is a simple modification of [LS90, 1.4]. We supply the details here for the sake of completeness. We first explain how the mesoscopic data is constructed. Given an embedding $T' \to T$, let $B'$ and $B^*$ be the associated Borel subgroups and $x \in G^*_{\mathrm{sc}}$ as above. The map $\psi_x = \mathrm{Int}(x) \circ \psi$ defines the quasisplit inner twist $G^*_{d^*}$ of $G_d$. The embedding $T' \to T^* \overset{\psi_x}{\leftarrow} T$ is dual to the diagram

$$\hat{G}' \leftarrow T'_1 \simeq T_1 \to \hat{T}^* \to \hat{T}, \tag{A-1}$$

by which we can identify the set of coroots $R(G, T)^\vee$ of $T$ in $G$ with the set of roots $R(\hat{G}, T_1)$ of $T_1$ in $\hat{G}$, and hence $R(G_d, T)^\vee$ with a subset of $R(\hat{G}, T_1)$.

Fix an $L$-group data, meaning a complex reductive group $\hat{G}_d$, an action $\rho_d$ of $\Gamma$ on $\hat{G}_d$, and a $\Gamma$-stable bijection $\Psi(G_d)^\vee \to \Psi(\hat{G}_d)$, by which we define ${}^L G_d = \hat{G}_d \rtimes W_F$. We may assume that $\hat{G}_d$ contains $T_1$ and that $R(\hat{G}_d, T_1)$ is equal to $R(G_d, T)^\vee$ as subsets of $R(\hat{G}, T_1)$. Let $B^*_d = B^* \cap G^*_{d^*}$, and let $\mathcal{B}_d$ be the Borel subgroup of $\hat{G}_d$ generated by $T_1$ and the $B_1$-positive roots of $T_1$ in $\hat{G}_d$. We can then identify the map $T_1 \to \hat{T}^*$ in (A-1) with the embedding $\hat{T}^* \to T_1$ in $\hat{G}_d$ given by $B^*_d$ and $\mathcal{B}_d$. The isomorphism

$$\hat{T} \overset{\psi_x}{\to} \hat{T}^* \to T_1$$

yields an embedding of $\hat{T}$ in $\hat{G}_d$ and extends to an admissible embedding of ${}^L T$ in ${}^L G_d$, whose image is independent of the choice of extension.

Given $G'$, the dual $\hat{G}'_{d'}$ of $G'_{d'}$ is a subgroup of $\hat{G}_d$ normalised by ${}^L T$. We define $\mathcal{G}'_{d'}$ to be the subgroup of ${}^L G$ generated by $\hat{G}'_{d'}$ and ${}^L T$, and note that it is contained in $\mathcal{G}'$. Then we can also define $\xi'_{d'}$ to be the map given by restriction of $\xi'$ from $\mathcal{G}'$ to $\mathcal{G}'_{d'}$. We have a split exact sequence

$$1 \to \hat{G}'_{d'} \to \mathcal{G}'_{d'} \to W_F \to 1,$$

and it follows that $(G'_{d'}, \mathcal{G}'_{d'}, \xi'_{d'})$ is a mesoscopic datum for $G_d$. Finally, we identify the embedding $\hat{T}' \to T_1$ given by $B' \cap G'_{d'}$ and $\mathcal{B}_d \cap \hat{G}'_{d'}$ with the restriction of the embedding $\hat{T} \to T_1$ above. This gives the admissible embedding $T' \to T^*$ as desired.

The choice of $\hat{G}_d$ is unique up to isomorphism of mesoscopic data. First suppose $B, B'$ are changed but $T' \to T^*$ remains fixed. Then the $L$-data $(\hat{G}_d, \rho_d)$ is replaced by another pair $(\hat{G}^1_d, \rho^1_d)$ that is $\Gamma$-isomorphic to it that sends the root datum $R(\hat{G}_d, T_1)$ to $R(\hat{G}^1_d, T_1)$, the image of ${}^LT$ in ${}^LG_d$ to its image in ${}^LG^1_d$, and $\hat{G}'_{d'}$ and $\mathcal{G}'_d$ to the new $\hat{G}^{1'}_{d'}$ and $\mathcal{G}^{1'}_d$ respectively. This gives an isomorphic mesoscopic datum for $G_d$.

If we replace $T' \to T^*$ with another admissible embedding of tori $T^{1'} \to T^{1*}$, such that $d'$ lies in $T', T^{1'}$ and $d^*$ in $T^*, T^{1*}$. Then we may assume that the new Borel subgroups are obtained from $B'$ and $B^*$ by conjugation in $G'_{d'}$ and $G^*_{d^*}$, and the new data is isomorphic. Finally, it is straightforward to see that the choice of $G' \in \mathcal{F}(G)$ within its isomorphism class does not affect the isomorphism class of $G'_{d'} \in \mathcal{F}(G_d)$. □

We shall say that $(G, G')$ admits $\Theta_{\tilde{\xi}'}$-transfer if for each $f \in C_c^\infty(G(F))$ there exists $f' = f^{\tilde{G}'} \in C_c^\infty(\tilde{G}'(F))$ such that $f, f'$ have $\Theta_{\tilde{\xi}'}$-matching orbital integrals, that is,

$$f'(\tilde{\delta}') = \int_{\Delta(G/Z)} \Theta_{\tilde{\xi}'}(\tilde{\delta}', \delta) f^G(\delta)\, d\delta.$$

As we are not in a position to prove the descent of stable transfer factors, we simply admit it as a hypothesis.

**Lemma A.2.** *Suppose that there exists some constant c such that*

$$\Theta_{\tilde{\xi}'}(\tilde{\delta}', \delta)\Theta_{\tilde{\xi}'_d}(\tilde{\delta}', \delta)^{-1} \to c \tag{A-2}$$

*as $\tilde{\delta} \to \tilde{d}'$ and $\delta \to d$. Then $f'(\tilde{\delta}')$ is equal to a finite linear combination of stable orbital integrals on $G'_{d'}$.*

*Proof.* According to the measure on $\Delta(G)$, the integral decomposes into

$$\sum_{\{M\}} |W(M)|^{-1} \sum_{\{T\}} |W_F(G, T)|^{-1} \int_{T(F)} \Theta_{\tilde{\xi}'}(\tilde{\delta}', t) f^G(t) dt,$$

where the inner sum is over stable conjugacy classes of elliptic maximal tori of $G$ over $F$, and $W_F(G, T)$ is the subgroup of elements in the absolute Weyl group of

$(G, T)$ defined over $F$. Moreover, our hypothesis (A-2) implies

$$\int_{T(F)} \Theta_{\tilde{\xi}'}(\tilde{\delta}', t) f^G(t)\, dt = c \int_{T(F)} \Theta_{\xi'_d}(\tilde{\delta}', t) f^G(t)\, dt.$$

Now by Lemma A.1, we may choose an embedding $T' \to T^*$ that is admissible for both $(G, G')$ and $(G_d, G'_{d'})$. If $\tilde{\delta}'$ is an element in the preimage of $T'(F)$ in $\Delta_G(\tilde{G}')$, then the integral in $f'(\tilde{\delta}')$ is taken over the stable conjugacy classes defined by the composition

$$\tilde{T}' \to T' \to T^* \to T$$

in $\Delta(G)$. Note that it is possible that $\tilde{\delta}'$ is a $T'$-image for more than one tori, and varying over equivalence classes tori in $\tilde{G}'_d$ we obtain all possible images. Then using the property that $\Theta_{\xi'_d}(\tilde{\delta}', t)$ vanishes unless $\tilde{\delta}'$ is an image, we conclude that the right-hand side can be written as a finite linear combination $f^{\tilde{G}'_{d'}}(\tilde{\delta}')$. □

It follows from Lemma 4.6(i) that for $\tilde{\delta}'$ close to the identity, $\Theta_{\tilde{\xi}'}(\tilde{\delta}', \delta)$ depends only on the image $\delta$ of $\delta'$ in $\Delta(G')$, so we write it as $\Theta^{\mathrm{loc}}_{\tilde{\xi}'}(\delta', \delta)$. We say that $(G, G')$ admits local $\Theta_{\tilde{\xi}'}$-transfer at the identity if for any $f \in C_c^\infty(G(F))$ we have

$$f'(\delta') = \int_{\Delta(G/Z)} \Theta^{\mathrm{loc}}_{\tilde{\xi}'}(\delta', \delta) f^G(\delta) d\delta \tag{A-3}$$

for all $\delta' \in \Delta_G(\tilde{G}')$ near to the identity.

**Corollary A.3.** *Let $F$ be nonarchimedean, and assume (A-2) holds. If $(G_d, G'_{d'})$ have local $\Theta_{\xi'_{d'}}$-transfer at the identity for all $d \in \Delta(G)$, then $(G, G')$ has $\Theta_{\tilde{\xi}'}$-transfer.*

*Proof.* Since the assumption continues to hold if $G$ is replaced by a $z$-extension $\tilde{G}$, we can assume that $G = \tilde{G}$ and $\mathcal{G}'$ is an $L$-group. By [LS90, Lemma 2.2A] it suffices to show that $f'(\delta')$ is a local stable orbital integral on $G'(F)$, in the sense that for every semisimple element $d$ in $G(F)$ there exists $f_d \in C_c^\infty(G(F))$ such that $f'(\delta) = f'_d(\delta)$ for all regular semisimple $\delta$ near to $d$. Again by Lemma 4.6(i), it follows that local $\Theta_{\xi'_{d'}}$-transfer at the identity implies local $\Theta_{\xi'_d}$-transfer at $d'$ in the sense that (A-3) holds for all $\delta' \in \Delta_G(\tilde{G}')$ near to $d'$. By assumption, we may apply the descent formula to express $f'(\tilde{\delta}')$ as a finite linear combination of stable orbital integrals on $G'_{d'}$. Then applying local transfer at $d'$ to each summand, the result follows. □

**A.2.** ***A stable kernel identity.*** Let $\mathfrak{g}$ be the Lie algebra of $G$, and similarly $\mathfrak{g}'$ of $G'$. Fix a symmetric, nondegenerate $G$-invariant bilinear form $B$ on $\mathfrak{g}$ and a nontrivial additive character $\psi_0$ on $F$. For any $\varphi \in C_c^\infty(\mathfrak{g}(F))$, we define the Fourier transform

$$\hat{\varphi}(Y) = \int_{\mathfrak{g}(F)} \varphi(X)\psi_0(B(X, Y))\, dX,$$

which acts as a linear isomorphism from $C_c^\infty(\mathfrak{g}(F))$ to itself. It a well-known result of Harish-Chandra that there exists a smooth, locally integrable function

$$i : \Gamma(\mathfrak{g}) \times \Gamma(\mathfrak{g}) \to \mathbb{C},$$

where $\Gamma(\mathfrak{g}) = \Gamma_{\text{reg}}(\mathfrak{g}(F))$ is the space of regular $G(F)$-orbits in $\mathfrak{g}(F)$, such that

$$\varphi_G(X) = \int_{\Gamma(\mathfrak{g})} i(X, Y)(\hat{\varphi})_G(Y)\, dY, \quad X \in \Gamma(\mathfrak{g}),$$

for a fixed Haar measure $dY$ on $\Gamma(\mathfrak{g})$. As with $\Gamma(G)$, we may decompose the integral into a sum of integrals over conjugacy classes of maximal tori in $G$. Taking $G$ to be quasisplit, we define the smooth function

$$s(S, T) = |\mathcal{K}_T|^{-1} \sum_{X \to S} \sum_{Y \to T} i(X, Y),$$

where $S, T$ are regular stable $G(F)$-orbits in $\mathfrak{g}_{\text{reg}}(F)$, the sums run over the distinct $G(F)$-orbits in each respective stable orbit, and $|\mathcal{K}_T|$ is equal to the number of $Y$ in the orbit of $T$. We also have the stable analogue

$$\varphi^G(S) = \int_{\Delta(\mathfrak{g})} s(S, T)(\hat{\varphi})^G(T)\, dT, \quad S \in \Delta(\mathfrak{g}),$$

which is a consequence of [Art96, (3.3)]. We define the Lie algebra analogue of the stable transfer factor

$$\Theta_{\xi'}(S', T) = \int_{\Delta_G(\mathfrak{g}')} s'(S', T')s(d\xi'(T'), T)\, dT', \quad S' \in \Delta_G(\mathfrak{g}'), T \in \Delta(\mathfrak{g}),$$

where $s'$ denotes the function associated to $G'$, and $d\xi'$ is the induced map on regular semisimple elements from $\mathfrak{g}(F)$ to $\mathfrak{g}'(F)$. Then suppose the following stable analogue of Waldspurger's kernel formula [Wal97, 1.2] holds for any $G' \in \mathcal{F}_{\text{ell}}(G)$,

$$\text{(A-4)} \qquad \int_{\Delta(\mathfrak{g})} \Theta_{\xi'}(S', S)s(S, T)\, dS = \delta_0 \int_{\Delta_G(\mathfrak{g}')} s'(S', T')\Theta_{\xi'}(T', T)\, dT',$$

for $T \in \Delta(\mathfrak{g}')$ and $S' \in \Delta_G(\mathfrak{g}')$, where the latter is the set of regular stable $G(F)$-orbits in $\mathfrak{g}'_{\text{reg}}(F)$ and $\delta_0$ is a constant depending only on $G, G'$. For any $\varphi \in C_c^\infty(\mathfrak{g}(F))$, we define the transfer

$$\varphi'(S') = \int_{\Delta(\mathfrak{g})} \Theta_{\xi'}(S', S)\varphi^G(S)\, dS,$$

and by the preceding formulas, it is equal to

$$\text{(A-5)} \qquad \begin{aligned} \varphi'(S') &= \int_{\Delta(\mathfrak{g})} \Theta_{\xi'}(S', S) \int_{\Delta(\mathfrak{g})} s(S, T)(\hat{\varphi})^G(T)\, dT \\ &= \delta_0 \int_{\Delta(\mathfrak{g})} \int_{\Delta_G(\mathfrak{g}')} s'(S', T')\Theta_{\xi'}(T', T)(\hat{\varphi})^G(T)\, dT'\, dT \\ &= \delta_0 \int_{\Delta_G(\mathfrak{g}')} s'(S', T')(\hat{\varphi})'(T')\, dT'. \end{aligned}$$

**A.3. *Surjectivity.*** With these considerations, the surjectivity of the map $\mathscr{T}^{\mathcal{F}}$ in the nonarchimedean case, without local Langlands, can then be shown to follow from the proposed kernel formula and the descent of transfer factors.

**Lemma A.4.** *Let $F$ be a non-archimedean local field, and assume (A-2) and (A-4). Then $\mathscr{T}^{\mathcal{F}}$ is surjective.*

*Proof.* Assume first that restriction of $\mathscr{T}^{\mathcal{F}}$ to $S\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$ maps onto the corresponding cuspidal subspace $S\mathcal{I}^{\mathcal{F}}_{\mathrm{cusp}}(G,\zeta)$ of $S\mathcal{I}^{\mathcal{F}}(G,\zeta)$. Recall that the filtration on $S\mathcal{I}(G,\zeta)$ with respect to $\mathscr{L}/W_0$ gives a grading (6-1) of the space, whereby

$$S\mathcal{I}(G,\zeta)=\bigoplus_{\{M\}} S\mathcal{I}_{\mathrm{cusp}}(M,\zeta)^{W(M)},$$

and similarly

$$S\mathcal{I}^{\mathcal{F}}(G,\zeta)=\bigoplus_{\{M\}} S\mathcal{I}^{\mathcal{F}}_{\mathrm{cusp}}(M,\zeta)^{W(M)}.$$

The map $\mathscr{T}^{\mathcal{F}}$ is compatible with these gradings, and the transfer mapping can be identified with the corresponding transfer mapping for cuspidal functions on each $M$.

The surjectivity of $\mathscr{T}^{\mathcal{F}}$ on $S\mathcal{I}_{\mathrm{cusp}}(G,\zeta)$ will follow from the corresponding surjectivity of germs on the Lie algebra. Assume for simplicity that $G'$ is an $L$-group, hence $\tilde{G}'=G'$, and moreover that the central datum $(Z,\zeta)$ is trivial. Given $a^{\mathcal{F}}\in S\mathcal{I}^{\mathcal{F}}_{\mathrm{cusp}}(G)$, by the relation (4-6) we see that the function

$$a^G(\delta)=\int_{\Delta^{\mathcal{F}}_{\mathrm{ell}}(G)}\Theta_{\xi'}(\delta,\delta')a^{\mathcal{F}}(\delta')\,d\delta',\quad \delta\in\Delta_{\mathrm{ell}}(G),$$

implies

$$a^{\mathcal{F}}(\delta')=\int_{\Delta_{\mathrm{ell}}(G)}\Theta_{\xi'}(\delta',\delta)a^G(\delta)\,d\delta=(\mathscr{T}^{\mathcal{F}}(a^G))(\delta').$$

Together with (4-5), we see that $a^{\mathcal{F}}$ lies in the image of $S\mathcal{I}_{\mathrm{cusp}}(G)$ if and only if $a^G\in S\mathcal{I}_{\mathrm{cusp}}(G)$. We may assume that the components $a'$ of $a^{\mathcal{F}}$ are nonzero for exactly one $G'\in\mathcal{F}_{\mathrm{ell}}(G)$. It suffices then to show that for $a^{\mathcal{F}}\in S\mathcal{I}^{\mathcal{F}}_{\mathrm{cusp}}(G)$, the function

$$a^G(\delta)=\int_{\Delta_{\mathrm{ell}}(G')}\Theta_{\xi'}(\delta,\delta')a'(\delta')\,d\delta',\quad \delta\in\Delta_{\mathrm{ell}}(G),\tag{A-6}$$

lies in $S\mathcal{I}_{\mathrm{cusp}}(G)$. Let $d$ be a fixed elliptic semisimple conjugacy class in $G(F)$, and let $S\mathcal{G}_{\mathrm{cusp}}(G,d)$ be the space of germs of functions in $S\mathcal{I}_{\mathrm{cusp}}(G)$ around $d$. Let $d'_1,\dots,d'_n$ be representatives of $\mathrm{Out}_G(G')$-orbits of stable conjugacy classes in $G'(F)$ that are images of $d$, chosen such that $G'_{d'_j}$ is quasisplit. The image of $a^G$ in $S\mathcal{G}_{\mathrm{cusp}}(G,d)$ depends only on the image of $a'$ in the spaces of germs of functions in $S\mathcal{I}_{\mathrm{cusp}}(G')$ around $d'_j$. We may assume that these images are nonzero for exactly one $j$, so that the integral (A-6) is supported on points close to $d'_j$. Then for any $\delta,\delta'$ close to $d,d'$ and assuming the descent of transfer factors, Lemma A.2 reduces

$a^G(\delta)$ and $a'(\delta')$ to orbital integrals on $G_d$ and $G'_{d'}$. Thus $d$ is central in $G_d$, and by Lemma 4.6(i) it suffices to show that $a^G(\delta)$ lies in $S\mathcal{G}_{\text{cusp}}(G, 1)$ for $\delta$ close to 1.

Let $C^\infty_{c,\text{cusp}}(\mathfrak{g}(F))$ be the subspace of cuspidal functions on $\mathfrak{g}(F)$, and let $S\mathcal{G}_{\text{cusp}}(\mathfrak{g})$ be the space of germs of stable orbital integrals of such functions around 0. It is a finite dimensional space of germs of functions on $\Delta(\mathfrak{g})$. The exponential map gives a linear bijection from $S\mathcal{G}_{\text{cusp}}(\mathfrak{g})$ to $S\mathcal{G}_{\text{cusp}}(G, 1)$. (This passage from orbital integrals on the group to germs on the Lie algebra can also be alternately described by Harish-Chandra descent; cf. [MW16, I.4.1].) Define also the finite dimensional vector space

$$S\mathcal{G}^{\mathcal{F}}_{\text{cusp}}(\mathfrak{g}) = \bigoplus_{G'\in\mathcal{F}_{\text{ell}}(G)} S\mathcal{G}_{\text{cusp}}(\mathfrak{g}')^{\text{Out}_G(G')},$$

and let

$$\tau^{\mathcal{F}} : S\mathcal{G}_{\text{cusp}}(\mathfrak{g}) \to S\mathcal{G}^{\mathcal{F}}_{\text{cusp}}(\mathfrak{g})$$

be the Lie algebra analogue of $\mathscr{T}^{\mathcal{F}}$ on germs. We would like to show that it is surjective. Fix $G' \in \mathcal{F}_{\text{ell}}(G)$ and consider the image of an arbitrary $g'(S') \in S\mathcal{G}_{\text{cusp}}(\mathfrak{g}')^{\text{Out}_G(G')}$ in $S\mathcal{G}^{\mathcal{F}}_{\text{cusp}}(\mathfrak{g})$, where $S'$ is belongs to the $G$-regular elliptic quotient $\Delta_{G,\text{ell}}(\mathfrak{g}')/\text{Out}_G(G')$. In particular, $g'$ is supported on the regular elliptic locus. By (A-5), we can express $g'(S')$ as a finite linear combination

$$g'(S') = \sum_{i=1}^{n} c_i s'(S', T'_i), \qquad T_i \in \Delta_{G,\text{ell}}(\mathfrak{g}').$$

We may choose the bilinear form $B'$ on $\mathfrak{g}'$ to be invariant under $\text{Out}_G(G')$, so that the coefficients $c_i$ are constant on $\text{Out}_G(G')$ orbits. By Howe's finiteness theorem applied to $\mathfrak{g}'(F)$ [How74, Theorem 2], we can choose a compact neighbourhood $V_i$ of $T_i$ in $\Delta_{G,\text{ell}}(\mathfrak{g}')$ for each $i$, such that $s'(S', T') = s'(S', T'_i)$ for all $T' \in V_i$ and $S'$ sufficiently small. Altogether, this implies that we can choose a $C^\infty_c$-function $\alpha'$ on $\Delta_{G,\text{ell}}(\mathfrak{g}')^{\text{Out}_G(G')}$ such that

$$g'(S') = \delta_0 \int_{\Delta_{G,\text{ell}(\mathfrak{g}')}} s'(S', T')\alpha'(T')\, dT'$$

for all $S'$ sufficiently close to 0. The adjoint relations of Proposition 4.8 have Lie algebra analogues. Recall that the proof in the group case relies on the stable local trace formula, and similarly one may employ a stable local trace formula for the Lie algebra, which is simpler, to deduce the necessary relations. This allows us to invert the map

$$C^\infty_c(\Delta_{\text{ell}}(\mathfrak{g})) \to \bigoplus_{G'_1\in\mathcal{F}_{\text{ell}}(G)} C^\infty_c(\Delta_{G,\text{ell}}(\mathfrak{g}')),$$

thereby giving a function $\varphi_0 \in C^\infty_c(\mathfrak{g}_{\text{reg,ell}}(F))$ such that for any $G'_1 \in \mathcal{F}_{\text{ell}}(G)$, we have $\varphi_0^{G'_1}$ equals $\alpha'$ if $G'_1 = G'$ and is trivial otherwise. Extending by zero, we can find a function $\varphi \in C_c^\infty(\mathfrak{g}(F))$ such that $\hat{\varphi} = \varphi_0$. By (A-5), it follows that

$$\varphi'(S') = \varphi^{G'}(S') = g'(S'),$$

and $\varphi^{G'_1} = 0$ for any $G'_1 \neq G'$. Thus the map $\tau^{\mathcal{F}}$ is surjective. □

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