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CHAINMAIL LINKS AND L -SPACES

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We prove that alternating chainmail links are L -space links. The proof is inspired by corresponding proofs for double branched covers of alternating links. More generally, we show that flat augmented chainmail links are generalized L -space links. Some other properties of these links are also considered.

1. Introduction

In the process of exploring lens space Dehn fillings on knots, Ozsváth and Szabó introduced the notion of L -spaces [OS3, Definition 1.1] and proved that double branched covers of nonsplit alternating links are L -spaces [OS4, Lemma 3.2, Proposition 3.3]. L -spaces are closed rational homology 3-spheres Y whose Heegaard Floer homology is minimal (like a lens space), in the sense that $\text{rank}(\widehat{HF}(Y)) = |H_1(Y; \mathbb{Z})|$. L -spaces are interesting manifolds since they do not admit taut orientable foliations [OS1, Theorem 1.4; KR, Corollary 1.6]; the converse has been conjectured for irreducible closed 3-manifolds by Juhász [Juh, Conjecture 5]. Related concepts have been defined for other versions of Floer homology, for example an *instanton L -space* Y satisfies $\dim I^\#(Y) = |H_1(Y; \mathbb{Z})|$ [BS, Definition 1.13]. It is conjectured that $\dim I^\#(Y) = \dim \widehat{HF}(Y; \mathbb{C})$ [KM, Conjecture 7.24].

Quasi-alternating links [OS4, Definition 3.1] are a class of links whose double branched covers are L -spaces, leading to the question of whether there are L -spaces which are not double branched covers of links? Examples of hyperbolic L -spaces were given which do not admit any symmetries, and hence cannot be double branched covers of links [DHL, Theorem 1.2]. Later asymmetric hyperbolic L -space knots were discovered leading to further examples [BL; ABGK+]. Meanwhile, Boyer, Gordon and Watson conjectured that L -spaces do not have orderable fundamental group [BGW, Conjecture 1]; they proved this for branched double covers of alternating links [BGW, Theorem 4], with a subsequent simpler proof given by Greene [Gre2, Theorem 2.1]. Greene's proof relies on a presentation of the fundamental group associated to a certain Dehn filling on a link described by Ozsváth and Szabó which gives a Kirby diagram for a double branched cover of a link (described in [Gre1, Section 3.1]).

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Polyak introduced the notion of chainmail graphs and corresponding chainmail links, such that surgery on this class of links gives all closed 3-manifolds [Pol]. He indicated that there exists a finite set of moves on chainmail graphs (analogous to Reidemeister moves) generating the equivalence relation of chainmail graphs giving the same 3-manifold. This is just a restricted class of Kirby diagrams, motivated by an alternative proof of Kirby's theorem given by Matveev and Polyak using a tangle presentation for the mapping class group [MP]. The Ozsváth–Szabó surgery diagrams of for double branched covers of links (mentioned in the last paragraph) are certain surgeries on chainmail links. We learned about these topics from a MathOverflow post by Lucas Culler [hc].

Gorsky and Némethi introduced the terminology of L -space link (links such that sufficiently large positive surgeries are L -spaces) and proved that algebraic links are L -space links [GN2, Theorem 2]. Subsequently Yajing Liu explored the properties of L -space links and introduced the terminology of *generalized L -space links* which also admit infinitely many L -space fillings [Liu, Definition 2.9]. He asked whether every 3-manifold is a surgery on a generalized L -space link [Liu, Question 1.19]? The main result of this paper is that alternating chainmail links are L -space links. Thus each double branched cover associated to an alternating diagram of a link fits into infinite families of L -spaces.

More generally, we prove that fully augmented chainmail links are generalized L -space links, hence answering Liu's question positively by Polyak's observation that every 3-manifold is surgery on a chainmail link (and hence on a fully augmented chainmail link). Since the symmetries of flat augmented links are understood, we may also therefore conclude the existence of asymmetric L -space links with arbitrarily many components.

We show that certain positive surgeries on negative alternating chainmail links do not have orderable fundamental group. The proof follows closely the strategy of Greene, using a related presentation of the fundamental group. Finally, we note that all known examples in the literature of L -space links have complements that fiber over the circle (this is true for L -space knots by [Ni1]). We observe that alternating chainmail links have fibered complements, continuing this trend. In the final section we ask some natural questions stemming from these examples.

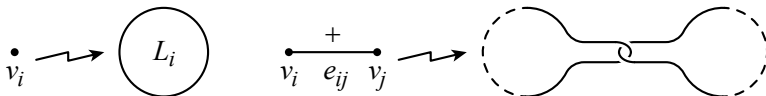
Background. Different sections of this paper assume different background knowledge. Because some of the proofs are fairly short modifications of results in the literature, we will use standard terminology as defined in the primary sources for each section to avoid a plethora of definitions which are only used once. Hopefully the reader will be able to black-box as much as possible and refer to the sources for more details when necessary.

2. Chainmail links

2.1. Chainmail graphs. Polyak introduced the terminology of *chainmail link* [Pol] and pointed out that a result of Matveev and Polyak implies that every closed connected orientable 3-manifold is realized by surgery on a chainmail link [MP]. Since Polyak's results have not been published, we review some of his notation and terminology which, although not strictly necessary, will be a useful way to formalize the proofs in this paper.

Definition 2.1. A *chainmail graph* is a finite planar graph $G \subset S^2$ whose edges and vertices are decorated with weights in \mathbb{Z} . Note that G may have loops and multiedges. Each vertex $v \in V(G)$ is decorated with a weight $v(v) \in \mathbb{Z}$, and each edge $e \in E(G)$ is given a weight $\epsilon(e) \in \mathbb{Z}$, where $V(G)$ and $E(G)$ are the set of vertices and edges of G respectively. Thus a chainmail graph consists of a triple (G, v, ϵ) , but much of the time we will abbreviate this just as G with the weights implicit (see also [AWvW] for this notation).

Given a chainmail graph G , one may define a *chainmail link* L_G in two steps in the following way. For each $v_i \in V(G)$, place a planar unknot L_i . Orient L_i counterclockwise. For each edge e connecting vertices v_i, v_j , introduce a $\epsilon(e)$ -clasped ribbon linking L_i and L_j . Let w_{ij} be the sum of edge weights of edges connecting v_i and v_j , with the convention that if distinct vertices v_i and v_j are not connected by an edge, then define $w_{ij} = 0$.



Now we assign framings to the components L_i with framing $w_{ii} = v(v_i) - \sum_{k \neq i} w_{ik}$. This framing is chosen in such a way that the linking number of the framing slope of L_i with the chainmail link is w_{ii} . In turn this gives a framing matrix $\Lambda(G) = (w_{ij})$. Surgery on L_G with these framings gives a 3-manifold M_G . Moreover, $\det(\Lambda(G)) \neq 0$ if and only if M_G is a rational homology 3-sphere, in which case $|H_1(M_G; \mathbb{Z})| = |\det(\Lambda(G))|$. We may also view $\Lambda(G)$ as the intersection form of $H_2(W_G)$ of the 4-manifold W_G obtained by adding two handles to the 4-ball along the link L_G with given framings.

One special property of chainmail links associated to simplicial graphs is that the crossing number is the minimal possible given the linking numbers between components. Chainmail graphs generalize plumbing diagrams for graph manifolds studied by Scharf [Sch] and Fintushel and Stern [FS], who considered the case of trees with edges of weight ± 1 and allowing rational weights on vertices (note however that the vertex labeling convention here may be different).

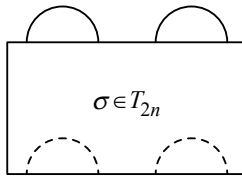


Figure 1. Closure of a tangle.

We briefly review some of the notation from [MP]. Consider framed tangles T_{2n} in $\mathbb{R}^2 \times [0, 1]$ with $2n$ framed endpoints on $\mathbb{R}^2 \times \{1\}$ and on $\mathbb{R}^2 \times \{0\}$. We may draw these tangles with blackboard framing, and compose the tangles by stacking.

One may close up a tangle in T_{2n} to get a framed link as in Figure 1 by adding strands on top of $\mathbb{R}^2 \times \{1\}$ and removing the strands meeting the bottom $\mathbb{R}^2 \times \{0\}$, and hence a Kirby diagram for a 3-manifold. Matveev and Polyak prove that one may obtain all genus 2 manifolds by stacking the tangles and their mirror images in Figure 2, then closing. (In the first diagram of figure 2, a positive twist, which increases the framing by 1 in the blackboard convention, is denoted by a little circle to simplify the notation.) In fact they show that these tangles generate the genus two mapping class group when one mods out by Kirby moves, such that the generators in Figure 2 correspond to the Dehn twists about curves in Figure 11. For example, the manifold associated to $\beta_2^{-1}\alpha_1\alpha_2\beta_1^{-1}\delta_2\beta_2^{-1}\alpha_1\alpha_2\beta_1^{-1}\beta_2^{-1}$ is shown in Figure 3.

Theorem 2.2 [Pol]. Any closed oriented 3-manifold $Y \cong M_G$ for some chainmail graph G .

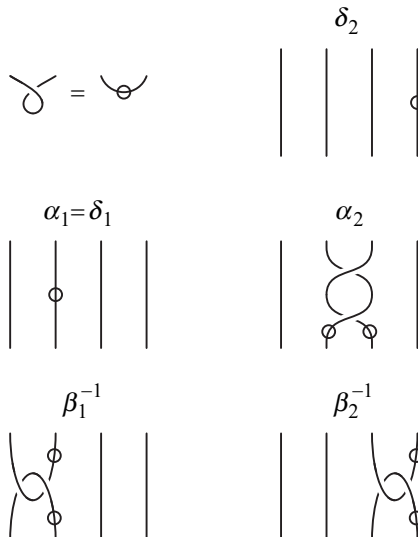


Figure 2. Generators for the genus 2 mapping class group.

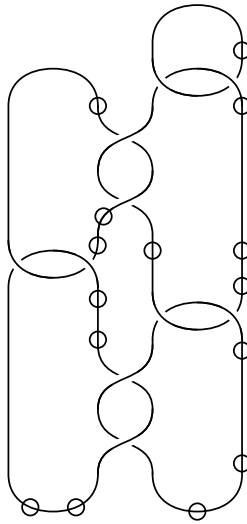


Figure 3. A plat closure associated to the tangle $\beta_2^{-1}\alpha_1\alpha_2\beta_1^{-1}\delta_2\beta_2^{-1}\alpha_1\alpha_2\beta_1^{-1}\beta_2^{-1}$.

Proof. The plat representation of a Kirby diagram for a three-manifold Y given in [MP, Theorem 7.1] is easily seen to be an integral surgery on a chainmail link [Pol] as in Figure 3. \square

We remark on a few moves on chainmail graphs that give rise to the same manifold. These may be verified by using flypes of the corresponding diagram keeping track of framings or Kirby calculus. A zero-weighted edge may be erased. Two edges e_1, e_2 connecting the same pair of vertices may be redrawn as one edge of weight $\epsilon(e_1) + \epsilon(e_2)$. A loop may be erased. A vertex v with weight $\nu(v) = 0$ of degree 1 adjacent to the edge e with $\epsilon(e) = \pm 1$ may be removed along with the edge e (see Figure 4; see also [EN, Theorem 18.3]). Even though we may get rid of loops and multiedges, it is convenient to allow them in chainmail graphs since they can get created under edge contraction.

2.2. Alternating chainmail links. When all of the edge weights of a chainmail graph G are of the same sign, then the associated chainmail link is alternating, see Figure 5. If the edge weights are all negative, then we call the associated

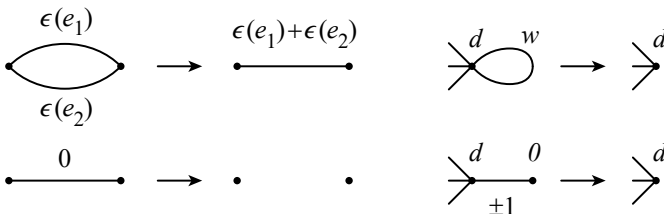


Figure 4. Moves on chainmail graphs G preserving the 3-manifold M_G .

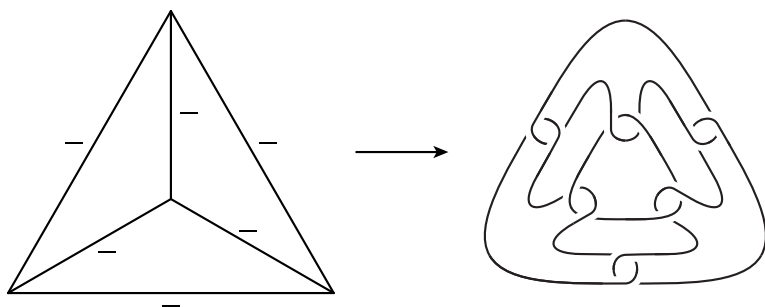


Figure 5. From a graph with negative edge weights to an alternating chainmail link.

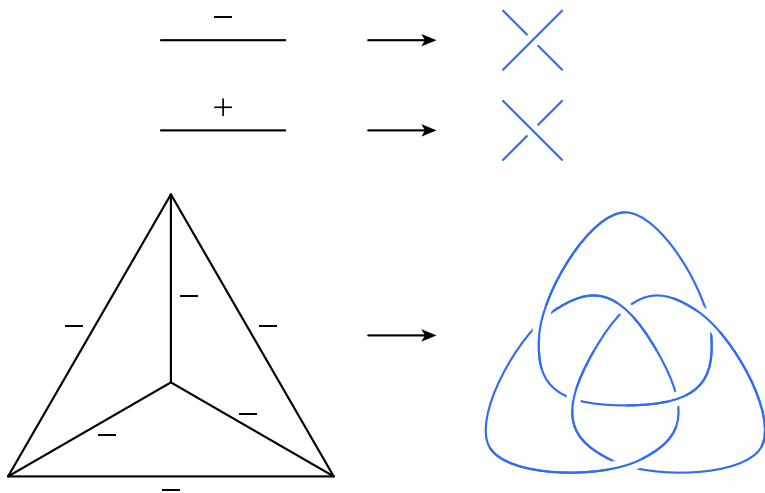


Figure 6. From an edge-weighted graph G to a link K_G .

link a *negative alternating chainmail link*. If the graph is balanced and connected ($\nu(v) = 0$ for all $v \in V(G)$), then it was shown by Ozsváth and Szabó that M_G is homeomorphic to the connected sum of the 2-fold branched cover of the link K_G and $S^2 \times S^1$, where K_G is obtained from G as in Figure 6 [Gre1, §3.1].

2.3. Augmented chainmail links. Assume the graph G is simplicial (so that there are no loops or edges with the same endpoints). We may insert crossing circles into L_G then twist to get an augmented link (Figure 7), possibly at only some of the edges. If we insert crossing circles associated to every edge, then it is called a *flat fully augmented (chainmail) link*; otherwise it is a partially augmented link. Thus one obtains a flat fully augmented chainmail link for every planar graph (Figure 8). The labels on a chainmail graph then correspond to framing coefficients for the loops corresponding to vertices and the reciprocals edge weights of the flat fully

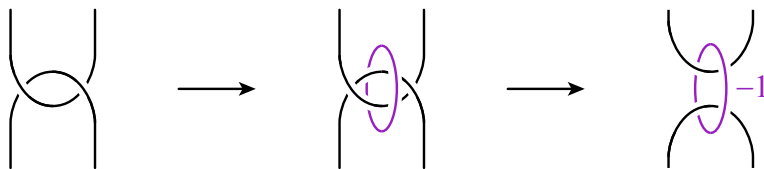


Figure 7. Augmenting a chainmail link.

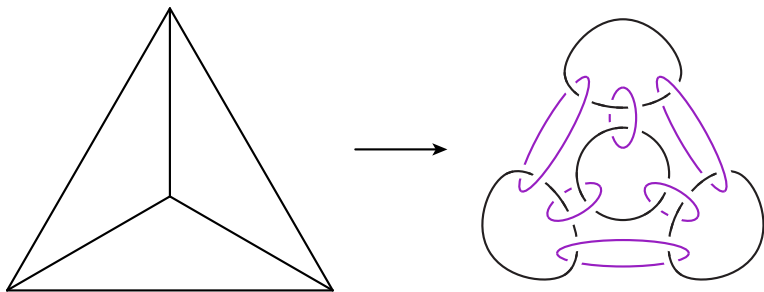


Figure 8. Going from a planar graph to a flat augmented link.

augmented chainmail link. Thus every 3-manifold is surgery on a fully augmented (flat) link by [Theorem 2.2](#), since every chainmail link is surgery on such a link.

The following theorem may be deduced from the main result of [\[MT\]](#):

Theorem 2.3. *A symmetry of a flat fully augmented link complement which sends crossing loops to crossing loops is induced by a symmetry of the link in S^3 .*

This will be used in the proof of [Theorem 6.1](#).

3. Alternating chainmail links are L -space links

In this section, we prove that alternating chainmail links are L -space links. Although [Theorem 3.2](#) is a corollary of [Theorem 4.2](#) by taking the coefficients on each crossing loop to be -1 , we prove it separately as it serves as a base case of the induction for the more general theorem.

Definition 3.1. [\[GN1, Definition 2.2\]](#) An l -component link $L \subset S^3$ is called an (instanton) L -space link if there exist integers p_1, \dots, p_l such that $S^3_{n_1, \dots, n_l}(L)$ is an (instanton) L -space for all n_1, \dots, n_l with $n_i > p_i$ for all $1 \leq i \leq l$.

Given a chainmail graph (G, ν, ϵ) , we may form minors. For an edge $e \in E(G)$ with $\epsilon(e) = -1$, let $G - e$ be the graph obtained by deleting the edge e and keeping the weight functions the same on vertices and edges. If e is not a loop, and e has endpoints v_1, v_2 , let G/e be the graph obtained by quotienting e to a point (which preserves planarity). The weight function on vertices $V(G) - \{v_1, v_2\}$ is the restriction of ν , and the weight function ϵ on edges are the same. There is a new

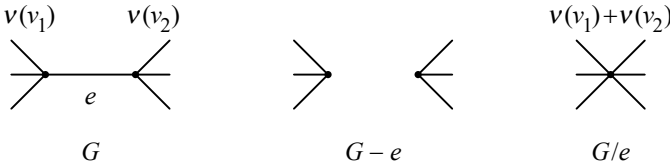


Figure 9. Minors of G obtained by deleting and contracting the edge e .

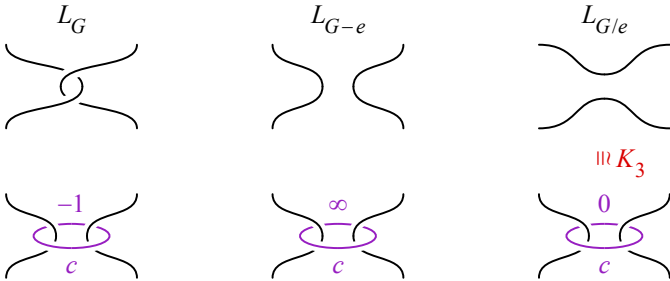


Figure 10. Links and surgeries on augmented link corresponding to minors of G .

vertex v' of G/e obtained by identifying v_1 and v_2 . We define $v(v') = v(v_1) + v(v_2)$ (Figure 9).

Let us see how the framed links / 3-manifolds associated to the graphs G , $G - e$, and G/e differ, with $\epsilon(e) = -1$. Consider the augmented link obtained by inserting a crossing loop c about L_1 and L_2 , the components corresponding to v_1 and v_2 , then do a twist to remove the double crossing. Then L_G corresponds to doing -1 surgery on c , L_{G-e} corresponds to ∞ surgery on c (which corresponds to erasing c), and $L_{G/e}$ corresponds to 0 surgery on c (see Figure 10). This last claim follows from a Kirby calculus operation K_3 described in [MP, Figure 3, p. 539].

Thus we see that M_G , M_{G-e} , and $M_{G/e}$ are related by surgeries on the knot c with slopes which have pairwise intersection number 1. Hence we have an exact triangle

$$\widehat{HF}(M_G) \rightarrow \widehat{HF}(M_{G-e}) \rightarrow \widehat{HF}(M_{G/e}) \rightarrow \widehat{HF}(M_G)$$

[OS2, Theorem 9.16].

The following theorem generalizes Example 3.15 of [Liu].

Theorem 3.2. *Negative alternating chainmail links are L -space links. More specifically, for a chainmail graph G with all $v(v) \geq 0$, $v \in V(G)$, and at least one $v(v) > 0$ in each component of G , and $\epsilon(e) < 0$ for all $e \in E(G)$, M_G is an L -space and an instanton L -space.*

Proof. The proof is similar to the proof that double branched covers of (quasi-)alternating links are L -spaces in [OS3, Proposition 3.3]. By replacing edges e

with weight $\epsilon(e) < -1$ with parallel copies, we may assume that $\epsilon(e) = -1$ for every edge $e \in E(G)$, and we may assume that there are no loops by removing them without changing the manifold.

If G is a chainmail graph and $\nu(v) = 0$ for all vertices and $\epsilon(e) = -1$ for all edges, then the linking matrix $\Lambda(G)$ is the matrix of the graph Laplacian of the underlying unweighted graph, hence all eigenvalues are ≥ 0 , and the 0-eigenspace has dimension the number of components of G (essentially the combinatorial Hodge theorem for 0-cocycles). If G has $\nu(v) \geq 0$, and at least one $\nu(v) > 0$ in each component of G and $\epsilon(e) = -1$, then when we add the diagonal matrix with diagonal entries $\nu(v_i)$ to the Laplacian matrix to get $\Lambda(G)$, we get a symmetric positive definite matrix. Thus $\det(\Lambda(G)) > 0$ for such graphs.

The proof is by induction on $|E(G)|$, the number of edges of G . The base case is an edgeless graph G , in which case all vertices have positive weight by hypothesis and M_G is a connected sum of lens spaces.

Now assume that $E(G) = n$, and all such chainmail graphs with fewer edges represent L -spaces. Choose an edge e of G , and delete the edge to get $G - e$ and contract the edge to get G/e , both of which have fewer edges. If there are parallel edges to e (with the same vertices), then G/e may have loops, but we may delete them in order to satisfy the induction hypothesis while still representing the same manifold (see [Figure 4](#)).

We may assume that e is not an isthmus, unless G is a tree (equivalently every edge is an isthmus). If G has at least two vertices with $\nu(v) > 0$, then we may choose e so that both components of $G - e$ satisfy the hypothesis that there is at least one vertex v with $\nu(v) > 0$. Otherwise, there is a single vertex v with $\nu(v) > 0$. Choose any leaf vertex v' with $\nu(v') = 0$ (a tree with nonempty edge set must have at least two leaf vertices, so such a vertex exists). Then we may delete v' keeping the same manifold $M_{G-v'}$ by the last move in [Figure 4](#). Thus in the latter case $M_G \cong M_{G-v'}$ will be an L -space by induction, and thus we may assume we are in the former case in which case $G - e$ is a chainmail graph satisfying the hypotheses. We will assume that the endpoints of e are labeled v_1 and v_2 .

Claim. $\det \Lambda(G) = \det \Lambda(G - e) + \det \Lambda(G/e)$.

Proof. Consider $\Lambda(G)$ and $\Lambda(G - e)$ as linear operators on functions on $V(G) = V(G - e)$. We may write the symmetric matrices for these as

$$\Lambda(G) = \begin{pmatrix} w_{11} & w_{12} & x_1^T \\ w_{12} & w_{22} & x_2^T \\ x_1 & x_2 & A \end{pmatrix}, \quad \Lambda(G - e) = \begin{pmatrix} w_{11}-1 & w_{12}+1 & x_1^T \\ w_{12}+1 & w_{22}-1 & x_2^T \\ x_1 & x_2 & A \end{pmatrix},$$

differing only in the upper left block corresponding to vertices v_1 and v_2 . Since we are only interested in the determinant, we may do elementary row operations

keeping the determinants the same. Add the first row to the second row and first column to the second column (in either order) of $\Lambda(G)$, $\Lambda(G - e)$ to get matrices with the same determinant:

$$\Lambda(G)' = \begin{pmatrix} w_{11} & w_{11}+w_{12} & x_1^T \\ w_{11}+w_{12} & w_{11}+w_{22}+2w_{12} & x_1^T+x_2^T \\ x_1 & x_1+x_2 & A \end{pmatrix},$$

$$\Lambda(G - e)' = \begin{pmatrix} w_{11}-1 & w_{11}+w_{12} & x_1^T \\ w_{11}+w_{12} & w_{11}+w_{22}+2w_{12} & x_1^T+x_2^T \\ x_1 & x_1+x_2 & A \end{pmatrix}.$$

Then we see that

$$\Lambda(G)' - \Lambda(G - e)' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By the linearity of determinants (say with respect to the first row), we see that

$$\det(\Lambda(G)) - \det(\Lambda(G - e)) = \begin{vmatrix} 1 & 0 & 0 \\ w_{11}+w_{12} & w_{11}+w_{22}+2w_{12} & x_1^T+x_2^T \\ x_1 & x_1+x_2 & A \end{vmatrix} = \det(\Lambda(G/e)),$$

since the lower block matrix is $\Lambda(G/e)$.

This establishes the claim. One may also prove it using [AWvW, Theorem 4.4]. □

Hence $|H_1(M_G)| = |H_1(M_{G-e})| + |H_1(M_{G/e})|$. By our induction hypothesis, M_{G-e} and $M_{G/e}$ are L -spaces, and they are related by an exact triangle. Then by [OS3, Proposition 2.1], M_G is also an L -space, where the knot is the crossing loop c . The proof is very similar to the proof of [OS4, Lemma 3.2, Proposition 3.3].

For the framed instanton homology case, the base case is the same (lens spaces are instanton L -spaces). We have $\dim I^\#(Y) \geq |H_1(Y)|$ by [Sca, Corollary 1.4]. There is an exact triangle (with $\lambda = 0$) given in [Sca, Section 7.5]. Applied to the case at hand, we get

$$I^\#(M_G) \rightarrow I^\#(M_{G-e}; \mu) \rightarrow I^\#(M_{G/e}) \rightarrow I^\#(M_G)$$

where μ is a loop representing the core of the Dehn filling of M_{G-e} on the crossing loop c (Figure 10), and similar exact sequences by permuting the role of μ for M_G and $M_{G/e}$. But in $M_G - c$, two out of the three slopes $\{0, -1, \infty\}$ will be nontrivial in $H_1(M_G - c; \mathbb{F}_2)$ and the third will be trivial by half-lives, half-dies. Filling along a homologically nontrivial slope, the homologically trivial slope will be isotopic to the core, and hence $[\mu] = 0 \in H_1(M_{G'}; \mathbb{F}_2)$, where G' is G , $G - e$, or

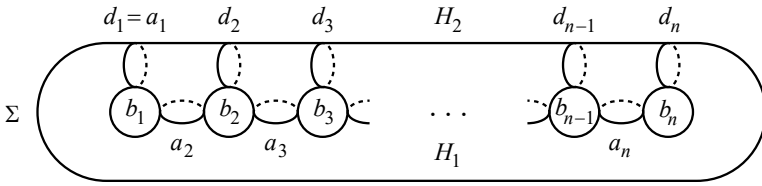


Figure 11. Generators for the mapping class group of a Heegaard surface Σ for S^3 .

G/e depending on the filling slope. So in this case we get an exact triangle

$$I^\#(M_G) \rightarrow I^\#(M_{G-e}) \rightarrow I^\#(M_{G/e}) \rightarrow I^\#(M_G).$$

Now we apply the same argument as for Heegaard Floer homology to conclude that

$$\begin{aligned} |H_1(M_G)| &\leq \dim I^\#(M_G) \leq \dim I^\#(M_{G-e}) + \dim I^\#(M_{G/e}) \\ &= |H_1(M_{G-e})| + |H_1(M_{G/e})| = |H_1(M_G)|, \end{aligned}$$

and hence M_G is an instanton L -space. □

Remark. Zhenkun Li pointed out that one can take a connected sum of any component of a negative alternating chainmail link with an L -space knot to get another L -space link. The point is that the proof goes through as before, with the vertex label adjusted larger, with the base case of the induction including a split link with one component an L -space knot so that surgery still gives an L -space for large enough coefficients.

One may give a slightly different characterization of some of the L -spaces described in [Theorem 3.2](#) in terms of Heegaard splittings. Consider a set of generators $\alpha_i, \beta_i, \delta_i$ for the mapping class group $\text{Mod}(\Sigma)$ of the Heegaard surface Σ by right-hand Dehn twists about curves a_i, b_i, d_i respectively shown in [Figure 11](#), which represents a standard Heegaard splitting $H_1 \cup_\Sigma H_2$ of S^3 .

Corollary 3.3. *Choose $\sigma \in \{\alpha_i, \beta_i^{-1}, \delta_i\}^*$, the submonoid of products of these elements. Then the resulting manifold $H_1 \cup_\sigma H_2$ is a connected sum of an L -space and $\#^k S^2 \times S^1$ for some k .*

Proof. The Kirby diagram produced from the products of these generators by [\[MP, Theorem 7.1\]](#) is a negative alternating chainmail link corresponding to a chainmail graph with nonnegative vertex weights (see [Figure 3](#) for an example). In each component of the chainmail graph with vertex weights 0, there will be an $S^2 \times S^1$ summand connected sum with a double-branched cover of an alternating link in the corresponding component of the Kirby diagram. Hence the manifold will be of the claimed type in each connect summand, and therefore in total. □

4. Augmented negative alternating chainmail links are generalized L -space links

Next we consider augmented chainmail links.

Definition 4.1. [Liu, Definition 2.9] Let $L = \cup_{i=1}^l L_i \subset S^3$ be a link, and for each i choose a sign $\epsilon_i \in \{\pm\}$. Then L is called a *generalized (instanton) $(\epsilon_1 \cdots \epsilon_l)$ L -space link* if there exist integers p_1, \dots, p_l such that $S_{\epsilon_1 n_1, \dots, \epsilon_l n_l}^3(L)$ is a (instanton) L -space for all n_1, \dots, n_l with $n_i > p_i$ for all $1 \leq i \leq l$. If we do not specify the sign, then L is denoted a *generalized L -space link*.

Theorem 4.2. *Partially augmented negative alternating chainmail links are generalized L -space links with a $+$ sign associated to each vertex loop and $-$ to each edge (crossing) loop.*

We answer a question of Yajing Liu [Liu, Question 1.9]:

Corollary 4.3. *Any closed orientable connected 3-manifold is obtained by surgery on a generalized L -space link.*

Proof. By Theorem 2.2 any 3-manifold is obtained by Dehn surgery on a chainmail link, which in turn arises from surgery on a flat fully augmented chainmail link. By Theorem 4.2, such a link is a generalized L -space link. \square

Proof of Theorem 4.2. First, we need a lemma that allows us to relate the homologies of manifolds in a Floer exact triangle.

Lemma 4.4. *Consider a partially augmented chainmail link simplicial graph G , with negative weights on the vertices $V_c(G) \subset V(G)$ corresponding to crossing loops and nonnegative weights on the other vertices $V(G) - V_c(G)$. Moreover, the edges between vertices of $V(G) - V_c(G)$ have weight -1 . Then the sign of the determinant of $\Lambda(G)$ is $(-1)^{|V_c(G)|}$ if nonzero.*

Proof. The proof is by induction on the sum of the negative crossing loop weights. If there are no crossing loops, then $\Lambda(G)$ is the linking matrix of a negative alternating chainmail graph, and hence the sign of the determinant is nonnegative by the proof of Theorem 3.2, establishing the base case.

Now consider a single crossing loop of the flat augmented link with weight $-c$, $c > 0$, and its two adjacent loops having framings a' and b' . The matrix $\Lambda(G)$ looks like

$$\begin{pmatrix} -c & 1 & -1 & 0 \\ 1 & a' & 0 & * \\ -1 & 0 & b' & * \\ 0 & * & * & A \end{pmatrix}.$$

When $c = 1$, we do column operations adding the first column to the second and

subtracting the first column from the third to get

$$\begin{vmatrix} -1 & 0 & 0 & 0 \\ 1 & a'+1 & -1 & * \\ -1 & -1 & b'+1 & * \\ 0 & * & * & A \end{vmatrix} = (-1) \begin{vmatrix} a'+1 & -1 & * \\ -1 & b'+1 & * \\ * & * & A \end{vmatrix}.$$

The last determinant is that of a partially augmented chainmail link by doing a -1 twist along the crossing loop (see the left diagrams of [Figure 10](#)). So the sign of this determinant is $(-1)^{|V_c(G)|-1}$ by induction. Thus it is true by induction for $c = 1$.

When $c > 1$, we have

$$\begin{vmatrix} -c-1 & 1 & -1 & 0 \\ 1 & a' & 0 & * \\ -1 & 0 & b' & * \\ 0 & * & * & A \end{vmatrix} = \begin{vmatrix} -c & 1 & -1 & 0 \\ 1 & a' & 0 & * \\ -1 & 0 & b' & * \\ 0 & * & * & A \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & a' & 0 & * \\ -1 & 0 & b' & * \\ 0 & * & * & A \end{vmatrix}.$$

The matrix on the right has determinant

$$\begin{vmatrix} a' & 0 & * \\ 0 & b' & * \\ * & * & A \end{vmatrix}$$

which is $\det \Lambda(G - v)$. By induction, this matrix has the opposite sign, so we get by induction on c that the sign with weight $-c - 1$ is $(-1)^{|V_c(G)|}$. \square

Returning to the proof of [4.2](#), let v be a vertex of the weighted graph G corresponding to a partially augmented link. Denote by G the graph with the vertex v with weight $v(v) = -c$ and by G' the graph with the weight $v(v) = -c - 1$. Assume that all vertices of $V(G) - V_c(C)$ have positive weight. The claim is that M_G is an L -space. We prove this by induction on c and on $|V_c(G)|$. If $c = 0$, then this corresponds to the graph $G - v$, and hence by induction on $|V_c(G)|$ this is an L -space. Suppose that M_G is an L -space. There is an exact triangle of Floer homology of M_G , $M_{G'}$ and M_{G-v} . Moreover, we have

$$\det \Lambda(G') = \det \Lambda(G) - \det \Lambda(G - v).$$

By [Lemma 4.4](#), $\text{sgn}(\det \Lambda(G')) = \text{sgn}(\det \Lambda(G)) = -\text{sgn}(\det \Lambda(G - v))$, so $|H_1(M'_G)| = |H_1(M_G)| + |H_1(M_{G-v})|$. It follows by induction that $|H_1(M_{G'})| = \text{rank } \widehat{HF}(M_{G'})$, and we see that $M_{G'}$ is an L -space. Again by induction, the augmented chainmail link associated to G is a generalized L -space link. The framed instanton case is similar to the argument at the end of the proof of [Theorem 3.2](#). \square

5. Nonorderable positive fillings on negative alternating chainmail links

We are not able to show that the L -spaces described in [Theorem 4.2](#) have nonorderable fundamental group, addressing [[BGW](#), Conjecture 1], but we can show nonorderability in special cases.

Theorem 5.1. *For a chainmail graph G with edge weights -1 , and nonnegative vertex weights with at most one vertex weight nonzero, $\pi_1(M_G)$ is nonorderable.*

Proof. The proof is modeled on the proof of Greene that double-branched covers of alternating links have nonorderable fundamental group [[Gre2](#), Theorem 2.1] (proved originally by Boyer, Gordon and Watson [[BGW](#), Theorem 4]).

Each edge of G with a chosen orientation corresponds to an element of the fundamental group of $S^3 - L_G$, as shown in [Figure 12](#), bottom left. If a vertex v_i has weight $v(v_i) = 0$, then the relation associated to Dehn filling will be the product of the incoming edges (see [Figure 13](#)).

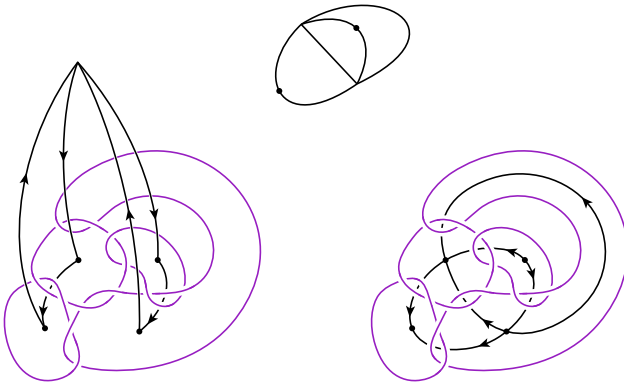


Figure 12. Top: a planar graph. Bottom left: loops associated to oriented edges. Bottom right: acyclic orientation from group ordering.

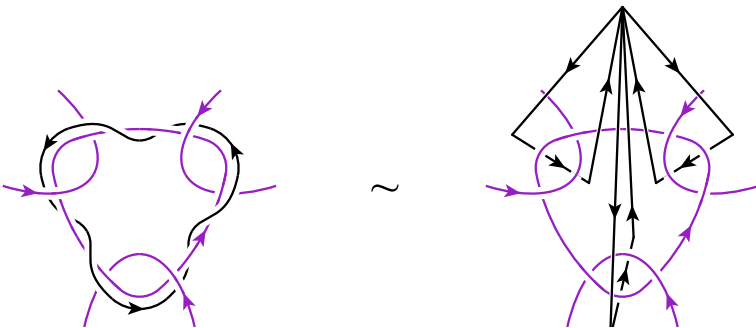


Figure 13. The zero-framing loop (left) is a product of incoming edge loops.

Suppose that $\pi_1(M_G)$ admits a left ordering. Then we may orient each edge so that the associated element is > 1 (assuming all of these elements are nontrivial first). This gives an acyclic orientation of the graph, seen in [Figure 12](#), bottom right; any oriented loop corresponds to the trivial element but is also a product of elements > 1 , a contradiction. Then there must be a sink vertex and a source vertex. Since only one vertex v has weight $\nu(v) > 0$ by assumption, we may choose a sink or source vertex v' with $\nu(v') = 0$. In this case, the product of the edges oriented into this will be trivial but also > 1 (or < 1), a contradiction.

In the case that some of the edges are trivial, we may still locate a sink or source vertex, allowing that some of the adjacent edges are trivial. If all edges are trivial, then either all vertex weights are zero, and we are in the case considered by Greene [\[Gre2\]](#), or one vertex has nonzero weight. In this case the relator from Dehn filling will give a power of the meridian of that loop is trivial. If the meridian is torsion, then the group cannot be orderable. If the meridian is trivial, the one can see that all meridians are trivial (since the edge loops are products of meridians), and hence the fundamental group is trivial. But by definition the trivial group is not left-orderable. \square

A theorem of Tao Li implies that a 3-manifold Y with Heegaard genus 2 that is an L -space does not have orderable fundamental group [\[Li, Theorem 1.2\]](#). One may apply this to conclude that the manifolds in [Corollary 3.3](#) whose gluing map lies in the monoid $\{\alpha_1, \alpha_2, \beta_1^{-1}, \beta_2^{-1}, \delta_2\}^*$ do not have orderable fundamental group unless it is a connected sum of copies of $S^1 \times S^2$. Such manifolds are realized by surgery on the framed links made by composing the tangles in [Figure 2](#), such as in [Figure 3](#).

6. Asymmetric L -space links

It was shown that there exist asymmetric L -spaces and asymmetric L -space knots [\[DHL; BL, Theorem 1.2\]](#). Here we show the existence of asymmetric L -space links.

Theorem 6.1. *There are alternating hyperbolic chainmail links of arbitrarily many components which have trivial symmetry group.*

Proof. Start with a flat augmented chainmail link associated to a graph G such as in [Figure 8](#) with hyperbolic complement (so no parallel crossing loops). Perform $-1/n_i$ surgeries with n_i large on each of the crossing loops c_i to make a negative alternating chainmail link. By choosing the n_i sufficiently large, we may assume that the cores of the Dehn fillings on each of the crossing loops c_i give the $|E(G)|$ shortest loops in the hyperbolic metric on the manifold, and they have distinct lengths [\[Thu, Theorem 5.8.2\]](#). Thus, any isometry of this link complement must preserve the Margulis tubes about these short loops. Thus it also preserves the

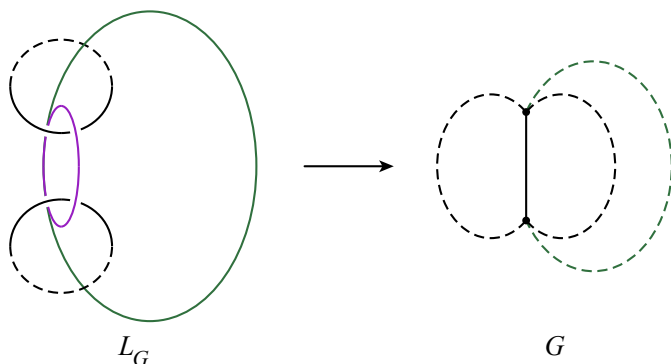


Figure 14. A flat augmented chainlink with two crossing disks must have a separating edge.

complements of these loops which is the initial flat augmented chainmail link complement. By [MT], any homeomorphism of this link complement must be induced by a homeomorphism of the link. Moreover, cusps are taken to cusps, and hence cusps corresponding to crossing loops are fixed.

Choose a simplicial planar graph G which contains no triangles and no vertices of degree 2 and no edge that separates the graph. By [MT, Proposition 2.7], the only thrice punctured spheres in the fully augmented chainmail link complement associated to G are crossing disks. Moreover, each crossing circle bounds a unique twice-punctured disk intersecting two knot loops (one can see that for a flat augmented chainmail link that a second crossing circle would give a separating edge; see Figure 14). We may also assume that these have a unique reflection surface, since our assumptions rule out the examples in that paper with multiple reflection surfaces [MT, Theorem 1.2]. Thus, any isometry of $S^3 - L$ must send the reflection surface to the reflection surface and the crossing disks to crossing disks. Each loop in the reflection surface bounds a disk punctured by crossing loops and intersecting the crossing disks in arcs. One may recover the graph G from this pattern of intersections, as well as its planar embedding from the reflection surface. So the only orientation preserving symmetries of the link complement must preserve the graph fixing each edge and hence are trivial. There is an orientation reversing symmetry by reflecting in the reflection surface, but this will not preserve the slopes of the fillings of the crossing loops, and hence does not extend to a symmetry of the alternating link. Thus we have shown the existence of negative alternating chainmail links which admit no nontrivial symmetry. \square

Remark. Further Dehn fillings on asymmetric negative alternating chain links give rise to more examples of asymmetric L -spaces. Compare to [DHL, Theorem 1.2] and [ABGK+] which require computation to verify the results, and [BL] which uses an intricate construction to give asymmetric L -space knots.

7. Alternating chainmail links are fibered

It was proved that L -space knots are fibered [Ni1]; in fact, all of the examples in the literature of L -space links have fibered complement. Example 3.9 of [Liu] shows that the links do not admit fibered Seifert surfaces; nevertheless the complements fiber.

Theorem 7.1. *Nonsplit negative alternating chainmail links have fibered complement.*

Proof. Take the checkerboard surface for the alternating chainmail link L_G which is a Seifert surface by applying Seifert's algorithm to the orientation of the link in which all of the loops are oriented counterclockwise (Figure 15). The complement of this surface is a sutured handlebody (H, γ) , where γ will be a union of curves parallel to the boundary components of the Seifert surface. Gabai proved that (H, γ) is disc-decomposable [Gab, Theorem 6.1]. Moreover, one sees that the restriction maps $H^1(S^3 - L_G) \rightarrow H^1(H)$ is surjective. Choose a cutset of oriented disks $D \subset H$ giving a taut sutured decomposition, and a cohomology class $\alpha' \in H^1(H)$ Poincaré dual to $[D, \partial D] \subset H_2(H, \partial H)$. Then there exists a cohomology class $\alpha \in H^1(S^3 - L_G)$ so that $\alpha|_H = \alpha'$. Let $\mu \in H^1(S^3 - L_G)$ denote the cohomology class dual to the Seifert surface. Then $k\mu + \alpha$ will lie in a cone over a face of the Thurston norm ball adjacent to that containing μ for $k \in \mathbb{N}$ large enough. By [AZ, Theorem 1.4] the guts $\Gamma(k\mu + \alpha)$ is obtained from $\Gamma(\mu)$ by decomposing along α' represented by D . (See [AZ, Definition 2.17] for the notion of guts.) Since (H, γ) is disc-decomposable, this is a product sutured manifold, and hence $k\mu + \alpha$ is a fibered class. \square

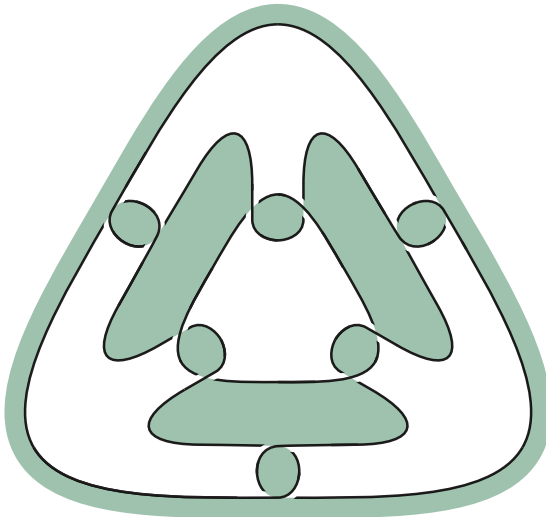


Figure 15. A Seifert surface for a chainmail link.

8. Conclusion

We conclude with some questions and challenges stemming from the examples in this paper.

- (1) Are any of the new examples of L -spaces described in this paper strong L -spaces [GL]?
- (2) Are there nonfibered L -space link complements?
- (3) Describe all L -space surgeries on alternating chainmail and augmented chainmail links. Since this includes all 3-manifolds, such a characterization will include a description of all L -spaces and thus is likely to be intricate [GN3].
- (4) Find the foliar surgeries on these links. If the L -space conjecture is true, then this should consist of the complement of the reducible and L -space fillings [Juh, Conjecture 5].
- (5) Prove that the L -space surgeries on alternating and augmented chainmail links described in these theorems have nonorderable fundamental group (Question (1) is relevant if there is a positive answer by [LL]). If the L -space conjecture is true, then one would expect the fundamental groups to be nonorderable.
- (6) Presumably, one could prove [Theorem 7.1](#) by computing the multivariable Alexander polynomial of negative alternating chainmail links, which allows one to detect if the link complement fibers [Ni1, Corollary 1.2; Ni2; OS5, Theorem 1.2]. It might be interesting to find a formula for these polynomials and use it to reprove [Theorem 7.1](#). It would also be interesting to know which augmented alternating links have fibered complement?
- (7) Are there more L -space links that may be found using this approach? Which chainmail links are (generalized) L -space links? Complete this analogy:

alternating links: alternating chainmail links :: quasi-alternating links: ?

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