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The Bénard–Conway invariant of links in the 3-sphere is a Casson–Lin type invariant defined by counting irreducible SU(2)-representations of the link group with fixed meridional traces. For two-component links with linking number one, the invariant has been shown to equal a symmetrized multivariable link signature. We extend this result to all two-component links with nonzero linking number. A key ingredient in the proof is an explicit calculation of the Bénard–Conway invariant for $(2\,,2\ell)$-torus links with the help of Chebyshev polynomials.

1. Introduction

The practice of defining invariants of links in 3-manifolds using unitary representations of the link group has a long history. Xiao-Song Lin [13] defined an invariant of knots in S^3 by counting irreducible SU(2)-representations of the knot group sending the meridians to zero-trace matrices. Herald [10] and Heusener and Kroll [11] extended this construction by allowing matrices of a fixed trace which is not necessarily zero. The construction was further extended to links of more than one component by Harper and Saveliev [9] and Boden and Harper [2] by counting *projective* unitary representations. These invariants are closely related to gauge theory: for example, Floer homology theories of Daemi–Scaduto [7] and Kronheimer–Mrowka [12] can be viewed as categorifying the invariants of Lin and Harper–Saveliev, respectively.

The latest in this line of link invariants is the multivariable Casson–Lin type invariant h_L of Bénard and Conway [1]. It is defined for colored links $L \subset S^3$ by counting irreducible SU(2)-representations of the link group sending the meridians to matrices of a fixed trace away from the roots of the multivariable Alexander polynomial. While this invariant is defined for links L of any number of components, it is best studied for two-component links. In particular, for an oriented ordered link $L = L_1 \cup L_2 \subset S^3$ with linking number $\operatorname{lk}(L_1, L_2) = 1$, Bénard and Conway

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[1, Theorem 1.1] identify h_L with a symmetrized multivariable link signature σ_L of Cimasoni and Florens [5].

The purpose of this paper is to extend the result of Bénard and Conway to arbitrary oriented ordered links $L_1 \cup L_2 \subset S^3$ of two components with linking number $\operatorname{lk}(L_1, L_2) \neq 0$. To be precise, we prove the following theorem.

Theorem 1.1. *Let* $L = L_1 \cup L_2 \subset S^3$ *be a two-component oriented ordered link with* $\operatorname{lk}(L_1, L_2) \neq 0$ *and, for any* $(\alpha_1, \alpha_2) \in (0, \pi)^2$, *write* $(\omega_1, \omega_2) = (e^{2i\alpha_1}, e^{2i\alpha_2})$. *If the multivariable Alexander polynomial of* L *satisfies* $\Delta_L(\omega_1^{\epsilon_1}, \omega_2^{\epsilon_2}) \neq 0$ *for all possible* $\epsilon_1, \epsilon_2 = \pm 1$ *then*

$$h_L(\alpha_1, \alpha_2) = -\tfrac{1}{2}\big(\sigma_L(\omega_1, \omega_2) + \sigma_L(\omega_1, \omega_2^{-1})\big). \tag{1}$$

The proof consists of two parts, just like the proof of [1, Theorem 1.1]. The first part shows that a crossing change within an individual component of L changes both sides of the formula of Theorem 1.1 by the same amount. This fact was only proved in [1, Theorem 1.1] for links with $\operatorname{lk}(L_1, L_2) = 1$, but that proof easily extends to links with $\operatorname{lk}(L_1, L_2) \neq 0$, as we explain in Section 3. After changing enough crossings within individual components of L, we only need to check that (1) holds for just one representative in each link homotopy class of L. According to Milnor [14], link homotopy classes of two component links are completely characterized by $\operatorname{lk}(L_1, L_2)$; therefore, it is sufficient to prove Theorem 1.1 for $(2, 2\ell)$-torus links L_ℓ with $\operatorname{lk}(L_1, L_2) = \ell \neq 0$.

This second part of the proof occupies Sections 4 and 5 of the paper, where we compute the Bénard–Conway invariant h_{L_ℓ} directly from its definition. The invariant h_{L_ℓ}, whose definition we recall in Section 2, is in an intersection number of two oriented curves in a 2-dimensional orbifold, traditionally referred to as a pillowcase. We come up with a parameterization of the pillowcase, in which the intersecting curves are given by explicit equations in terms of the Chebyshev polynomials; see Theorem 5.4. Checking the transversality and computing the intersection signs are then accomplished by a straightforward calculation.

This argument fails for links $L = L_1 \cup L_2$ with $\operatorname{lk}(L_1, L_2) = 0$ because the base case, which is the link L_ℓ with $\ell = 0$, has a vanishing Alexander polynomial and hence its Bénard–Conway invariant is not defined.

The next result follows from Theorem 1.1 and the properties of the Cimasoni–Florens signature [5]. It is proved in Section 6 together with Theorem 1.1.

Theorem 1.2. *For any two-component link* $L = L_1 \cup L_2$ *as in the statement of Theorem 1.1, the invariant* $h_L(\alpha_1, \alpha_2)$ *is independent of the orientation of the link* L. *Moreover,* $h_L(\pi/2, \pi/2)$ *equals minus the Murasugi signature* [17] *of the link* L.

Finally, we wish to mention how our work is related to that of Daemi and Scaduto [8]. Let L be a link in S^3 of any number of components with nonzero determinant.

Daemi and Scaduto [8] define irreducible instanton homology $I(L)$ as a $\mathbb{Z}/4$ graded abelian group which, in favorable circumstances, is generated at the chain level by the conjugacy classes of irreducible SU(2)-representations of $\pi_1(S^3 - L)$ sending meridians to zero-trace matrices. In general, a perturbation may be necessary to achieve transversality. One expects that the Euler characteristic of $I(L)$ equals, up to an overall constant, the Bénard–Conway invariant $h_L(\pi/2, \dots, \pi/2)$. Daemi and Scaduto [8] further show that the Euler characteristic of $I(L)$ is proportional to the Murasugi signature of the link L, which matches our Theorem 1.2 in the special case of two-component links L with $\alpha_1 = \alpha_2 = \pi/2$.

2. Preliminaries

In this section, we will recall the definition of the Bénard–Conway invariant [1] for oriented links in the 3-sphere.

2.1. ***Colored links and colored braids.*** A *μ-colored link* $L \subset S^3$ is an oriented link whose components are partitioned into sublinks $L = L_1 \cup \dots \cup L_\mu$. Following Murakami [16], we will interpret these as the closures of colored braids.

Recall that the (Artin) braid group B_n on n strands is the finitely presented group with $n-1$ generators $\sigma_1, \sigma_2, \dots, \sigma_{n-1}$ subject to the relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for each i, and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| > 1$. Geometrically, a generator σ_i can be viewed as the isotopy class of the braid whose $(i+1)$-st strand crosses over the i-th strand. The *closure* $\widehat{\beta}$ of a braid $\beta \in B_n$ is the link obtained from β by connecting the lower endpoints of the braid and its upper endpoints with parallel strands. The link $\widehat{\beta}$ is canonically oriented by choosing the downward orientations on the strands of β.

A *μ-colored braid* is a braid $\beta \in B_n$ together with an assignment to each of its strands of an integer (called the *color*) in $\{1, 2, \dots, \mu\}$ via a surjective map. A μ-colored braid induces *μ-colorings* c and c' on the upper and lower endpoints of the braid, which are n-tuples of integers in $\{1, 2, \dots, \mu\}$. For any μ-coloring c, the μ-colored braids with $c' = c$ form a *colored braid group* $B_c \subset B_n$. For example, if $\mu = 1$, then $c = (1, \dots, 1)$ and B_c is simply the braid group B_n, and if $\mu = n$ and $c = (1, 2, \dots, n)$ then B_c is the pure braid group on n strands.

The *closure* $\widehat{\beta}$ of a μ-colored braid $\beta \in B_c$, obtained from β by connecting the lower endpoints of the braid with the upper endpoints with colored parallel strands, is a μ-colored link. A colored version of Alexander's theorem states that every μ-colored link is the closure $\widehat{\beta}$ of a μ-colored braid $\beta \in B_c$, and a colored version of Markov's theorem determines when two μ-colored braids have isotopic closures.

2.2. ***The Bénard–Conway invariant.*** Consider the Lie group SU(2) of two-by-two unitary matrices with determinant one. We will be identifying it with the Lie group

Sp(1) of unit quaternions via

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mapsto a + b\,\boldsymbol{j} \tag{2}$$

and using the language of matrices and unit quaternions interchangeably.

Let F_n be a free group on n generators $x_1, \dots, x_n$. The group B_n acts naturally on F_n via

$$x_j \sigma_i = \begin{cases} x_i x_{i+1} x_i^{-1} & \text{if } j = i, \\ x_i & \text{if } j = i+1, \\ x_j & \text{otherwise.} \end{cases} \tag{3}$$

This action induces an action on the representation space

$$R(F_n) = \mathrm{Hom}(F_n, \mathrm{SU}(2)) = \mathrm{SU}(2)^n.$$

Concretely, every braid $\beta \in B_n$ induces a homeomorphism $\beta : \mathrm{SU}(2)^n \to \mathrm{SU}(2)^n$ by the rule $(X_1, \dots, X_n)\beta = (X_1\beta, \dots, X_n\beta)$. Thus, for example, $(X_1, X_2)\sigma_1 = (X_1 X_2 X_1^{-1}, X_1)$ for the generator $\sigma_1 \in B_2$.

Let now $L = L_1 \cup \dots \cup L_\mu$ be an oriented μ-colored link. Represent it as the closure of a μ-colored braid $\beta \in B_c$ on n strands. Given a μ-tuple $\alpha = (\alpha_1, \dots, \alpha_\mu) \in (0, \pi)^\mu$, consider the representation space

$$R_n^{\alpha,c} = \big\{(X_1, X_2, \dots, X_n) \in \mathrm{SU}(2)^n \;\big|\; \mathrm{tr}(X_i) = 2\cos(\alpha_{c_i}) \quad \text{for } i = 1, \dots, n\big\}.$$

Since the trace is preserved by conjugation, the homeomorphism $\beta : \mathrm{SU}(2)^n \to \mathrm{SU}(2)^n$ constructed above restricts to a homeomorphism $\beta : R_n^{\alpha,c} \to R_n^{\alpha,c}$ with the graph

$$\Gamma_\beta^{\alpha,c} = \big\{(X_1, X_2, \dots, X_n, X_1\beta, X_2\beta, \dots, X_n\beta) \in R_n^{\alpha,c} \times R_n^{\alpha,c}\big\}.$$

Note that the trivial braid in B_c gives rise to the graph which is just the diagonal

$$\Lambda_n^{\alpha,c} = \big\{(X_1, X_2, \dots, X_n, X_1, X_2, \dots, X_n) \in R_n^{\alpha,c} \times R_n^{\alpha,c}\big\}.$$

Since the product $X_1 \cdots X_n$ is preserved by the action of β (see formula (3)), one immediately concludes that both $\Gamma_\beta^{\alpha,c}$ and $\Lambda_n^{\alpha,c}$ are subspaces of the ambient space

$$H_n^{\alpha,c} = \big\{(X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n) \in R_n^{\alpha,c} \times R_n^{\alpha,c} \;\big|\; X_1 \cdots X_n = Y_1 \cdots Y_n\big\}.$$

The group SO(3) acts via conjugation on the spaces $\Gamma_\beta^{\alpha,c}$, $\Lambda_n^{\alpha,c}$ and $H_n^{\alpha,c}$. We restrict this action to irreducible representations, where it is free, and denote the quotient spaces by $\widehat{\Gamma}_\beta^{\alpha,c}$, $\widehat{\Lambda}_n^{\alpha,c}$ and $\widehat{H}_n^{\alpha,c}$. These are smooth open manifolds of dimensions

$$\dim \widehat{\Gamma}_\beta^{\alpha,c} = 2n - 3, \quad \dim \widehat{\Lambda}_n^{\alpha,c} = 2n - 3, \quad \dim \widehat{H}_n^{\alpha,c} = 4n - 6;$$

see [1, Lemma 3.4].

The key observation now is that the points of the intersection $\widehat{\Lambda}_n^{\alpha,c} \cap \widehat{\Gamma}_\beta^{\alpha,c} \subset \widehat{H}_n^{\alpha,c}$ are precisely the conjugacy classes of irreducible SU(2)-representations of the link group

$$\pi_1(S^3 - L) = \langle x_1, \dots, x_n \mid x_i = x_i\beta \text{ for } i = 1, \dots, n\rangle \tag{4}$$

sending the meridians x_i to matrices $X_i \in \mathrm{SU}(2)$ of trace $\mathrm{tr}(X_i) = 2\cos(\alpha_{c_i})$. From this point on, the definition of the Bénard–Conway invariant proceeds by making sense of the intersection number of $\widehat{\Lambda}_n^{\alpha,c}$ and $\widehat{\Gamma}_\beta^{\alpha,c}$ in $\widehat{H}_n^{\alpha,c}$. We will briefly outline the procedure and refer to [1] for detailed proofs.

Let $\Delta_L(t_1, \dots, t_\mu)$ be the multivariable Alexander polynomial of L, and, given a μ-tuple $\alpha = (\alpha_1, \dots, \alpha_\mu) \in (0, \pi)^\mu$, consider the finite set

$$S(\alpha) = \left\{\left(e^{\epsilon_1 2\boldsymbol{i}\alpha_1}, \dots, e^{\epsilon_\mu 2\boldsymbol{i}\alpha_\mu}\right) \,\middle|\, \epsilon_i = \pm 1 \text{ for } i = 1, \dots, \mu\right\}. \tag{5}$$

Proposition 2.1 [1, Proposition 3.5]. *If $\Delta_{\widehat{\beta}}(\omega) \neq 0$ for all μ-tuples $\omega \in S(\alpha)$ then the intersection $\widehat{\Lambda}_n^{\alpha,c} \cap \widehat{\Gamma}_\beta^{\alpha,c} \subset \widehat{H}_n^{\alpha,c}$ is compact.*

Let $\alpha = (\alpha_1, \dots, \alpha_\mu)$ satisfy the condition of Proposition 2.1. Since $\widehat{\Lambda}_n^{\alpha,c} \cap \widehat{\Gamma}_\beta^{\alpha,c} \subset \widehat{H}_n^{\alpha,c}$ is compact, the graph $\widehat{\Gamma}_\beta^{\alpha,c}$ can be perturbed if necessary using a perturbation with compact support to make the intersection $\widehat{\Lambda}_n^{\alpha,c} \cap \widehat{\Gamma}_\beta^{\alpha,c} \subset \widehat{H}_n^{\alpha,c}$ transversal and hence a compact 0-dimensional manifold.

Next, we will orient all the manifolds in question. Denote by $\mathbb{S}(\theta)$ the conjugacy class of matrices in SU(2) with the trace $2\cos\theta$. Assuming that $0 < \theta < \pi$, the conjugacy classes $\mathbb{S}(\theta)$ are naturally homeomorphic to each other and to the standard 2-sphere S^2. Choose an (arbitrary) orientation on S^2. The space $R_n^{\alpha,c}$ is a product of the spheres $\mathbb{S}(\alpha_{c_i})$, $i = 1, \dots, n$, and we will endow it with the product orientation. The spaces $\Lambda_n^{\alpha,c}$ and $\Gamma_\beta^{\alpha,c}$, which are diffeomorphic to $R_n^{\alpha,c}$ via projection onto the first n factors, will be endowed with the induced orientations. To orient $H_n^{\alpha,c}$, consider the map $f_n : R_n^{\alpha,c} \times R_n^{\alpha,c} \to \mathrm{SU}(2)$ given by $f_n(X_1, \dots, X_n, Y_1, \dots, Y_n) = X_1 \cdots X_n (Y_1 \cdots Y_n)^{-1}$. Observe that $H_n^{\alpha,c} = f_n^{-1}(1)$, so that we can pull back the canonical orientation of SU(2) to obtain an orientation on $H_n^{\alpha,c}$. Since the adjoint action of SO(3) on each $\mathbb{S}(\theta)$ is orientation preserving, we can endow $\widehat{\Lambda}_n^{\alpha,c}$, $\widehat{\Gamma}_\beta^{\alpha,c}$, and $\widehat{H}_n^{\alpha,c}$ with the induced quotient orientation.

The intersection number of $\widehat{\Lambda}_n^{\alpha,c}$ and $\widehat{\Gamma}_\beta^{\alpha,c}$ will be denoted by $\langle \widehat{\Lambda}_n^{\alpha,c}, \widehat{\Gamma}_\beta^{\alpha,c} \rangle_{\widehat{H}_n^{\alpha,c}}$. The following result, which is proved in [1, Section 3.3], ensures that it only depends on the isotopy class of the closure of β.

Proposition 2.2. *Under the assumptions of Proposition 2.1, the intersection number $\langle \widehat{\Lambda}_n^{\alpha,c}, \widehat{\Gamma}_\beta^{\alpha,c} \rangle_{\widehat{H}_n^{\alpha,c}}$ is preserved by the Markov moves.*

We will summarize the above construction in the following definition.

Definition 2.3. Let L be a μ-colored link in S^3 and $\alpha \in (0,\pi)^\mu$ a μ-tuple. Let $\beta \in B_c$ be a colored braid of n strands whose closure is L. Suppose that $\Delta_L(\omega_\epsilon) \neq 0$ for all $\omega_\epsilon \in S(\alpha)$. Then the *Bénard–Conway invariant* of L is well-defined by the formula

$$h_L(\alpha) = \langle \widehat{\Lambda}_n^{\alpha,c}, \widehat{\Gamma}_\beta^{\alpha,c} \rangle_{\widehat{H}_n^{\alpha,c}}. \tag{6}$$

3. Inductive step for two-component links

From now on, we will restrict ourselves to 2-colored links L of two components, which are just ordered two-component links. In this section, we will investigate what happens to the two sides of the formula (1) under a crossing change within a single component of L.

Theorem 3.1. *Under the assumptions of Theorem 1.1, the two sides of formula (1) stay well-defined and change by the same amount when a crossing change occurs within one of the components of the link.*

The rest of this section will be devoted to the proof of Theorem 3.1. Given a pair $\alpha = (\alpha_1, \alpha_2) \in (0,\pi)^2$, the set defined in (5) becomes

$$S(\alpha) = \{ (e^{\epsilon_1 2 \boldsymbol{i} \alpha_1}, e^{\epsilon_2 2 \boldsymbol{i} \alpha_2}) \mid \epsilon_1 = \pm 1, \epsilon_2 = \pm 1 \}.$$

In addition, define the sets

$$S_j(\alpha) = \{ (e^{\epsilon_1 2 \boldsymbol{i} \alpha_1}, e^{\epsilon_2 2 \boldsymbol{i} \alpha_2}) \in S(\alpha) \mid \epsilon_j = 1 \}, \quad j = 1, 2,$$

and denote by $\nabla_L(t_1, t_2)$ the multivariable Conway potential function [4]. Recall that $\nabla_L(t_1, t_2)$ equals $\Delta_L(t_1^2, t_2^2)$ up to multiplication by the powers of $\pm t_1$ and $\pm t_2$.

Lemma 3.2 [1, Proposition 5.10 and Remark 5.11]. *Let $L = L_1 \cup L_2$ be a two-component oriented ordered link and denote by L_+ the link obtained from L by a negative crossing change within a component L_j of the link L. Suppose that $\alpha = (\alpha_1, \alpha_2) \in (0,\pi)^2$ is such that, for all $\omega = (\omega_1, \omega_2) \in S_j(\alpha)$, one has $\omega_1^2 \neq 1$, $\omega_2^2 \neq 1$ and $\omega_1\omega_2 \neq 1$. If $\nabla_L(\omega^{1/2}) \neq 0$ and $\nabla_{L_+}(\omega^{1/2}) \neq 0$ then*

$$h_{L_+}(\alpha) - h_L(\alpha) = \#\{ \omega \in S_j(\alpha) \mid \nabla_{L_+}(\omega^{1/2}) \nabla_L(\omega^{1/2}) < 0 \}.$$

Lemma 3.3 [1, Lemma 6.2]. *Let $L = L_1 \cup L_2$ be an ordered two-component link with $\operatorname{lk}(L_1, L_2) \neq 0$ and suppose that $\omega \in (S^1 \setminus \{1\})^2$ is not a root of $\Delta_L(t_1, t_2)$. Then the Cimasoni–Florens signature $\sigma_L(\omega)$ is well defined and*

$$\sigma_L(\omega) \equiv 2 + \operatorname{lk}(L_1, L_2) + \operatorname{sign}(\nabla_L(\omega^{1/2})) \bmod 4.$$

In addition, suppose that L_+ is obtained from L by a negative crossing change within one of the components of L, and that ω is not a root of either $\Delta_{L_+}(t_1, t_2)$ or $\Delta_L(t_1, t_2)$. Then $\sigma_{L_+}(\omega) - \sigma_L(\omega)$ is either 0 or -2.

Corollary 3.4. *Let L be an ordered two-component link with $\mathrm{lk}(L_1, L_2) \neq 0$ and L_+ a link obtained from L by a negative crossing change within one of its components. Suppose that $\omega \in (S^1 \setminus \{1\})^2$ is such that $\nabla_L(\omega^{1/2}) \neq 0$ and $\nabla_{L_+}(\omega^{1/2}) \neq 0$. Then*

$$\sigma_{L_+}(\omega) - \sigma_L(\omega) = \begin{cases} 0 & \text{if } \nabla_{L_+}(\omega^{1/2})\nabla_L(\omega^{1/2}) > 0, \\ -2 & \text{if } \nabla_{L_+}(\omega^{1/2})\nabla_L(\omega^{1/2}) < 0. \end{cases}$$

We are now ready to sketch a proof of Theorem 3.1, which will be a straightforward modification of the proof of Theorem 6.4 in [1]. That theorem is first proved under the additional assumption that α_1 and α_2 are transcendental. This assumption is then removed by showing that the invariant $h_L(\alpha)$ is locally constant in α. Only the first part of the proof needs to be modified.

Observe first that a crossing change within one component of L does not make the multivariable Alexander polynomial vanish, which ensures that the Bénard–Conway invariant of L_+ is well-defined. This follows from the Torres formula

$$\Delta_{L_1 \cup L_2}(t_1, 1) \doteq \frac{t_1^{\mathrm{lk}(L_1,L_2)} - 1}{t_1 - 1} \Delta_{L_1}(t_1)$$

(see [18]) and our assumption that $\mathrm{lk}(L_1, L_2) \neq 0$. Next, we will show that the two sides of (1) change by the same amount under the crossing change. Assume without loss of generality that L_+ is obtained from $L = L_1 \cup L_2$ by a negative crossing change in L_1. According to Lemma 3.2, we have

$$h_{L_+}(\alpha) - h_L(\alpha) = \#\big\{\omega \in S_1(\alpha) \;\big|\; \nabla_{L_+}(\omega^{1/2})\nabla_L(\omega^{1/2}) < 0\big\}.$$

Note that $S_1(\alpha)$ has exactly two elements, (ω_1, ω_2) and $(\omega_1, \omega_2^{-1})$, where $\omega_1 = e^{2\boldsymbol{i}\alpha_1}$ and $\omega_2 = e^{2\boldsymbol{i}\alpha_2}$. It now follows from Corollary 3.4 that

$$\begin{aligned} h_{L_+}(\alpha) - h_L(\alpha) &= -\tfrac{1}{2}(\sigma_{L_+}(\omega_1,\omega_2) - \sigma_L(\omega_1,\omega_2)) - \tfrac{1}{2}(\sigma_{L_+}(\omega_1,\omega_2^{-1}) - \sigma_L(\omega_1,\omega_2^{-1})) \\ &= -\tfrac{1}{2}(\sigma_{L_+}(\omega_1,\omega_2) + \sigma_{L_+}(\omega_1,\omega_2^{-1})) + \tfrac{1}{2}(\sigma_L(\omega_1,\omega_2) + \sigma_L(\omega_1,\omega_2^{-1})), \end{aligned}$$

which completes the proof.

4. SU(2)-representations of torus links

To complete the proof of Theorem 1.1, it is sufficient to verify the formula (1) for $(2, 2\ell)$-torus links L_ℓ with $\ell \neq 0$. Our convention here is that $\ell > 0$ gives the right-handed torus link and $\ell < 0$ the left-handed one. The verification will take up the rest of the paper. We begin in this section by describing the irreducible SU(2)-representations of the link group of L_ℓ with fixed meridional traces.

4.1. ***Geometry of* SU(2).** We will continue to identify SU(2) matrices with unit quaternions as in (2). Any $q \in \mathrm{SU}(2)$ can then be written in the form $q = \cos\alpha + \sin\alpha\ Q = e^{\alpha Q}$, where $\alpha \in [0, \pi]$ and Q is a purely imaginary unit quaternion. This expression is unique except when $q = \pm 1$. Using all unit quaternions, $q = \cos\alpha + \sin\alpha\ Q$ can be conjugated to $\cos\alpha + \sin\alpha\, \boldsymbol{i} = e^{\alpha \boldsymbol{i}}$. Using only unit complex numbers, $q = \cos\alpha + \sin\alpha\ Q$ can be conjugated to $\cos\alpha + \sin\alpha(\cos\beta\, \boldsymbol{i} + \sin\beta\, \boldsymbol{j})$ for some $\beta \in [0, \pi]$. Alternatively, it can be expressed as $\cos\gamma(\cos\beta + \sin\beta\, \boldsymbol{i}) + \sin\gamma\, \boldsymbol{j}$ for $\gamma \in [0, \pi]$ and $\beta \in [0, 2\pi]$ with $\cos\gamma\cos\beta = \cos\alpha$ since the real part is conjugation invariant.

4.2. ***Counting the representations.*** Let us first assume that $\ell > 0$ and consider the presentation of the link group given by

$$\pi_1(S^3 \setminus L_\ell) = \langle x_1, x_2 \mid (x_1 x_2)^\ell = (x_2 x_1)^\ell \rangle,$$

where x_1 and x_2 are the meridians of the two components of L_ℓ. For a fixed choice of $(\alpha_1, \alpha_2) \in (0, \pi)^2$, we wish to describe the conjugacy classes of irreducible representations $\rho : \pi_1(S^3 \setminus L_\ell) \to \mathrm{SU}(2)$ sending the meridians of the components of L_ℓ to unit quaternions with respective real parts $\cos(\alpha_j)$, $j = 1, 2$. Since

$$(x_2 x_1)^\ell = (x_2 x_1)^\ell x_2 x_2^{-1} = x_2 (x_1 x_2)^\ell x_2^{-1},$$

the relation $(x_1 x_2)^\ell = (x_2 x_1)^\ell$ is equivalent to $(x_1 x_2)^\ell$ commuting with x_2, and by symmetry, also with x_1. Therefore, $(x_1 x_2)^\ell$ belongs to the center of $\pi_1(S^3 \setminus L_\ell)$. Since ρ is irreducible, we must have $(\rho(x_1)\rho(x_2))^\ell = \pm 1$. We end up looking for noncommuting unit quaternions $\rho(x_1)$ and $\rho(x_2)$ with prescribed real parts such that $\rho(x_1)\rho(x_2)$ is an ℓ-th root of ± 1 different from ± 1 (because otherwise ρ is reducible). Conjugate ρ so that

$$\rho(x_1) = \cos\alpha_1 + \sin\alpha_1(\cos\varphi\, \boldsymbol{i} + \sin\varphi\, \boldsymbol{j}) \quad \text{and} \quad \rho(x_2) = \cos\alpha_2 + \sin\alpha_2\, \boldsymbol{i}$$

for some $\varphi \in (0, \pi)$. The condition above then means that the real part of $\rho(x_1)\rho(x_2)$ equals $\cos(\pi m/\ell)$ or, equivalently, that

$$\cos\alpha_1 \cos\alpha_2 - \sin\alpha_1 \sin\alpha_2 \cos\varphi = \cos(\pi m/\ell), \quad m = 1, 2, \dots, \ell - 1. \tag{7}$$

Proposition 4.1. *Let L_ℓ be a $(2, 2\ell)$-torus link, $\ell \neq 0$. Given $(\alpha_1, \alpha_2) \in (0, \pi)^2$, the number of conjugacy classes of irreducible representations $\pi_1(S^3 \setminus L_\ell) \to \mathrm{SU}(2)$ sending the meridians to unit quaternions with real parts $\cos\alpha_j$, $j = 1, 2$, equals the number of integers $m \in \{1, \dots, |\ell| - 1\}$ such that*

$$\cos(\pi m/\ell) \in (\cos(\alpha_1 + \alpha_2), \cos(\alpha_1 - \alpha_2)).$$

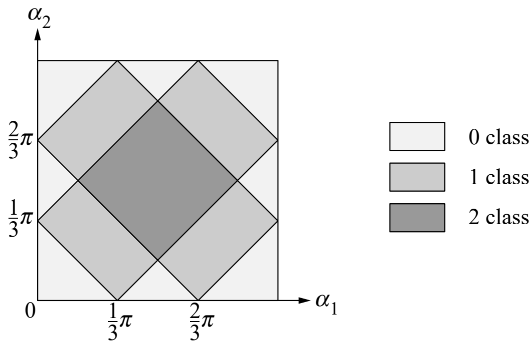

Figure 1. Counting conjugacy classes, $\ell = 3$.

Proof. Let us assume that $\ell > 0$ since the argument for $\ell < 0$ is similar. Use trigonometric identities to express (7) in terms of $\alpha_1 - \alpha_2$ and $\alpha_1 + \alpha_2$ as

$$\cos(\alpha_1 - \alpha_2)\sin^2(\varphi/2) + \cos(\alpha_1 + \alpha_2)\cos^2(\varphi/2) = \cos(\pi m/\ell),$$

or, equivalently,

$$\cos(\alpha_1 - \alpha_2)(1 - t) + \cos(\alpha_1 + \alpha_2)t = \cos(\pi m/\ell),$$

where $t = \cos^2(\varphi/2) \in [0, 1]$. The latter equation has a unique solution $t \in [0, 1]$, and therefore (7) has a unique solution $\varphi \in (0, \pi)$ if and only if $\cos(\pi m/\ell) \in \big(\cos(\alpha_1 + \alpha_2), \cos(\alpha_1 - \alpha_2)\big)$. □

Example 4.2. The cases of $\ell = 1$ and $\ell = -1$ correspond to the two oriented Hopf links. In both cases, the link group has no irreducible SU(2)-representations. The former case served as the base of induction in [1]. The count of conjugacy classes for $\ell = 3$ is depicted in Figure 1. The diamond shapes in the figure have to do with the Alexander polynomial of the link as discussed in detail in the following subsection.

4.3. ***The Alexander polynomial.*** The count of irreducible representations in Proposition 4.1 may jump as α_1 and α_2 change. The following proposition shows that these jumps occur at the roots of the multivariable Alexander polynomial, that is, exactly where the Bénard–Conway invariant is not defined.

Proposition 4.3. *Let L_ℓ be a $(2, 2\ell)$-torus link with $\ell \neq 0$ and $\Delta_{L_\ell}(t_1, t_2)$ its multivariable Alexander polynomial. The number of conjugacy classes of irreducible representations $\pi_1(S^3 \setminus L_\ell) \to \mathrm{SU}(2)$ sending the meridians to unit quaternions with real parts $\cos(\alpha_j)$, $j = 1, 2$, is a locally constant function on the complement of the set given by the equation $\Delta_{L_\ell}(\omega_1^{\pm 1}, \omega_2^{\pm 1}) = 0$, where $(\omega_1, \omega_2) = (e^{2i\alpha_1}, e^{2i\alpha_2})$.*

Proof. The jumps in the count of irreducible representations in Proposition 4.1 occur exactly when $\cos(\alpha_1 \pm \alpha_2) = \cos(\pi m/\ell)$ with $m = 1, \dots, |\ell| - 1$, that is, when

$$\alpha_1+\alpha_2=\frac{\pi m}{|\ell|} \quad \text{or} \quad \alpha_1+\alpha_2=\pi+\frac{\pi m}{|\ell|} \quad \text{or} \quad \alpha_1-\alpha_2=\pm\frac{\pi m}{|\ell|}. \tag{8}$$

These (α_1,α_2) are precisely the solutions of the equation $\Delta_{L_\ell}(\omega_1^{\pm 1},\omega_2^{\pm 1})=0$, which can be easily verify using the formula

$$\Delta_{L_\ell}(t_1,t_2)\doteq\frac{(t_1t_2)^{|\ell|}-1}{t_1t_2-1}$$

for the multivariable Alexander polynomial of L_ℓ due to Milnor [15]. □

Remark 4.4. When $\ell=0$, the torus link L_ℓ is just the unlink of two components. Its multivariable Alexander polynomial is identically zero hence its Bénard–Conway invariant is not defined. This is the reason behind our assumption that $\ell\neq 0$.

5. The pillowcase and intersection theory

In this section, we will compute the Bénard–Conway invariants of $(2,2\ell)$-torus links as the intersection number of the manifolds $\widehat{\Lambda}_n^{\alpha,c}$ and $\widehat{\Gamma}_\beta^{\alpha,c}$ inside of $\widehat{H}_n^{\alpha,c}$. This will involve counting the representations described in Proposition 4.1 with plus/minus signs, after making sure that the intersection in question is transversal.

5.1. *The setup.* Let L_ℓ be a $(2,2\ell)$-torus link and assume that ℓ is positive. The case of negative ℓ can be treated in a similar manner, and both cases will be discussed in detail when we perform explicit calculations in Section 5.5. Let $\alpha=(\alpha_1,\alpha_2)$ be an arbitrary point in $(0,\pi)^2$ away from the set (8). This guarantees that the Bénard–Conway invariant of L_ℓ is well-defined; see Proposition 4.3. We will view the link L_ℓ as a 2-colored link which is the closure of the 2-colored braid $\beta=\sigma_1^{2\ell}\in B_c$ with the 2-coloring $c=(1,2)$. The Bénard–Conway invariant of L_ℓ is then the intersection number of $\widehat{\Lambda}_2^{\alpha,c}$ and $\widehat{\Gamma}_\beta^{\alpha,c}$ inside of $\widehat{H}_2^{\alpha,c}$. Our first goal will be to parameterize these manifolds.

5.2. *The pillowcase.* Recall that $\widehat{H}_2^{\alpha,c}$ is an open 2-manifold obtained by removing the conjugacy classes of reducible representations from the orbifold $H_2^{\alpha,c}/\operatorname{SO}(3)$, where

$$H_2^{\alpha,c}=\{(X_1,X_2,Y_1,Y_2)\in R_2^{\alpha,c}\times R_2^{\alpha,c}\mid X_1X_2=Y_1Y_2\}.$$

In the special case of $\alpha_1=\alpha_2=\pi/2$, the orbifold $H_2^{\alpha,c}/\operatorname{SO}(3)$ is referred to as a pillowcase; see for instance Lin [13]. We will extend this terminology to the general case.

After conjugation, we may assume that $X_2=e^{\alpha_2\boldsymbol{i}}$ and $X_1=e^{\alpha_1P_1}$, where $P_1=\boldsymbol{i}\,e^{-\boldsymbol{k}\varphi}$ with $\varphi\in[0,\pi]$; see Section 4.2. Write $Y_1=e^{\alpha_1Q_1}$ and $Y_2=e^{\alpha_2Q_2}$, where Q_1 and Q_2 are purely imaginary unit quaternions. For any given Y_1, the equation $X_1 X_2 = Y_1 Y_2$ can be uniquely solved for Y_2 if and only if the real part of $Y_1^{-1} X_1 X_2$ matches that of Y_2, that is,

$$\operatorname{Re}\big(e^{-\alpha_1 Q_1} e^{\alpha_1 \boldsymbol{i} e^{-\boldsymbol{k}\varphi}} e^{\alpha_2 \boldsymbol{i}}\big) = \cos\alpha_2.$$

Write $Q_1 = u\,\boldsymbol{i} + v\,\boldsymbol{j} + w\,\boldsymbol{k}$, where $u^2 + v^2 + w^2 = 1$. The above equation is equivalent to

$$(9)\quad (\sin\alpha_1 \cos\alpha_2 \cos\varphi + \cos\alpha_1 \sin\alpha_2)u + \sin\alpha_1 \cos\alpha_2 \sin\varphi\, v - \sin\alpha_1 \sin\alpha_2 \sin\varphi\, w = \sin\alpha_1 \cos\alpha_2 + \cos\alpha_1 \sin\alpha_2 \cos\varphi.$$

We will handle the case of $\varphi \in (0, \pi)$ first. In this case, (9) has the form $\boldsymbol{n} \cdot (u, v, w) = d$, where $d = \sin\alpha_1 \cos\alpha_2 + \cos\alpha_1 \sin\alpha_2 \cos\varphi$ and

$$\boldsymbol{n} = \begin{pmatrix} \sin\alpha_1 \cos\alpha_2 \cos\varphi + \cos\alpha_1 \sin\alpha_2 \\ \sin\alpha_1 \cos\alpha_2 \sin\varphi \\ -\sin\alpha_1 \sin\alpha_2 \sin\varphi \end{pmatrix}$$

is a nonzero vector (because its third coordinate is never zero). Therefore, (9) describes a plane in the space $\mathbb{R}^3$ of purely imaginary quaternions, and our task becomes describing the intersection of this plane with the unit sphere S^2 of purely imaginary unit quaternions given by the equation $u^2 + v^2 + w^2 = 1$. Note that the point on the plane closest to the origin is given by $d\boldsymbol{n}/|\boldsymbol{n}|^2$. One can easily see that the distance from this point to the origin equals

$$(10)\qquad \frac{d}{|\boldsymbol{n}|} = \frac{d}{\sqrt{d^2 + \sin^2\alpha_2 \sin^2\varphi}}$$

and that this distance is strictly less than one, making the intersection of the plane with the unit sphere S^2 a circle S^1_φ for any choice of $\varphi \in (0, \pi)$. Therefore, away from the points with $\varphi = 0$ and $\varphi = \pi$, the orbifold $H_2^{\alpha,c} / \operatorname{SO}(3)$ is homeomorphic to a cylinder $S^1 \times (0, \pi)$. It can be parameterized as follows.

The circle S^1_φ contains the vector $P_1 = \boldsymbol{i}\, e^{-\boldsymbol{k}\varphi} = \cos\varphi\, \boldsymbol{i} + \sin\varphi\, \boldsymbol{j}$ corresponding to the solution with (X_1, X_2, X_1, X_2) of (9). Let

$$\boldsymbol{v}_1 = P_1 - \frac{d\boldsymbol{n}}{|\boldsymbol{n}|^2} \quad \text{and} \quad \boldsymbol{v}_2 = \frac{\boldsymbol{n}}{|\boldsymbol{n}|} \times \boldsymbol{v}_1.$$

Since the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are orthogonal to each other, we can write

$$(11)\qquad Q_1 = \frac{d\boldsymbol{n}}{|\boldsymbol{n}|^2} + \cos\theta\, \boldsymbol{v}_1 + \sin\theta\, \boldsymbol{v}_2$$

for some $\theta \in [0, 2\pi]$; see Figure 2. Therefore, away from the points with $\varphi = 0$ and $\varphi = \pi$, $H_2^{\alpha,c} / \operatorname{SO}(3)$ is a cylinder parameterized by $(\varphi, \theta) \in (0, \pi) \times [0, 2\pi]$ with the intervals $\theta = 0$ and $\theta = 2\pi$ identified.

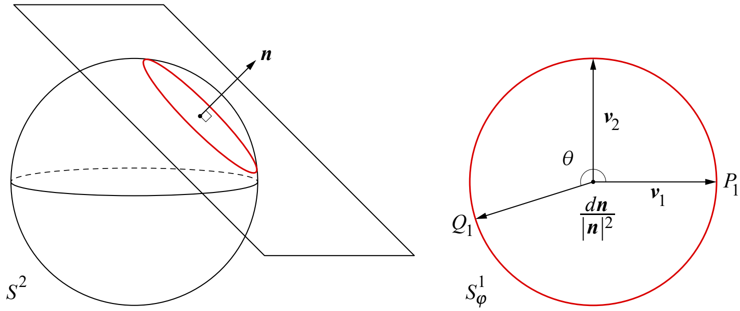

Figure 2. Vectors v_1, v_2 and points P_1, Q_1 on the circle S^1_φ.

We now turn to the full space $H_2^{\alpha,c} / \mathrm{SO}(3)$. It is a compactification of the cylinder described above by the points with $\varphi = 0$ and $\varphi = \pi$. The type of compactification one obtains depends on α_1 and α_2:

(1) The special case of $\alpha_1 = \alpha_2 = \pi/2$ is the original case investigated by Lin [13]. The traceless condition forces the representations with $\varphi = 0$ and $\varphi = \pi$ to be irreducible for all $\theta \in (0, 2\pi)$. The resulting $\widehat{H}_2^{\alpha,c}$ is a sphere with four points removed.

(2) The case of $\alpha_1 = \alpha_2 \neq \pi/2$, in which $\sin(\alpha_1 - \alpha_2) = 0$ but $\sin(\alpha_1 + \alpha_2) \neq 0$, was originally investigated by Heusener and Kroll [11] and used in [1]. In this case, all representations with $\varphi = \pi$ and $\theta \in (0, 2\pi)$ are irreducible and the points (π, θ) and $(\pi, 2\pi - \theta)$ are identified; see Figure 3. Topologically, $\widehat{H}_2^{\alpha,c}$ is a 2-sphere with three points removed.

(3) In the special case of $\alpha_1 \neq \alpha_2$ and $\alpha_1 + \alpha_2 = \pi$ we have $\sin(\alpha_1 + \alpha_2) = 0$ but $\sin(\alpha_1 - \alpha_2) \neq 0$. This time around, the representations with $\varphi = 0$ and $\theta \in (0, 2\pi)$ are irreducible and $\widehat{H}_2^{\alpha,c}$ is a mirror image of Figure 3.

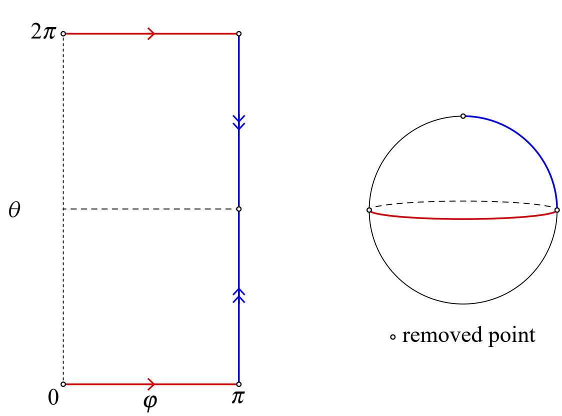

Figure 3. Special case of the parametrization when $\alpha_1 = \alpha_2 \neq \pi/2$.

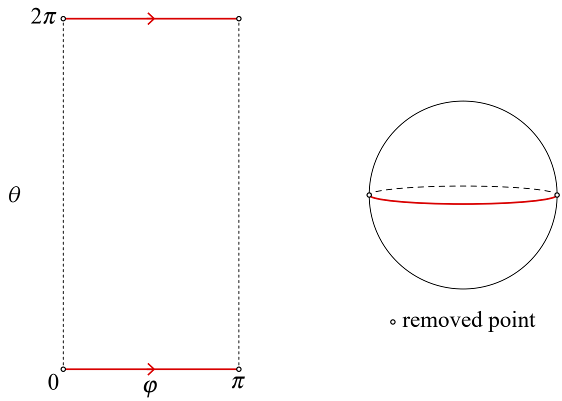

Figure 4. Generic case with $\sin(\alpha_1 \pm \alpha_2) \neq 0$.

(4) For a generic choice of α_1 and α_2 with $\sin(\alpha_1 \pm \alpha_2) \neq 0$, the above calculation extends to the points with $\varphi = 0$ and $\varphi = \pi$ giving rise in each case to a single reducible representation. Topologically, the cylinder compactifies to a 2-sphere, the reducible locus collapses along one direction, thus $\widehat{H}_2^{\alpha,c}$ is a 2-sphere with two points removed; see Figure 4.

All of the resulting orbifolds $H_2^{\alpha,c}/\,\mathrm{SO}(3)$ will be referred to as pillowcases. This includes the novel cases (3) and (4), which arise because we allow distinct α_1 and α_2. The calculation that follows will be the same in all four cases because it happens away from the points with $\varphi = 0$ and $\varphi = \pi$.

5.3. *Intersection theory in the pillowcase.* The diagonal $\widehat{\Lambda}_2^{\alpha,c}$ is a subspace of $\widehat{H}_2^{\alpha,c}$ given by the equation $X_1 = Y_1$. In our parameterization of the pillowcase, $\widehat{\Lambda}_2^{\alpha,c}$ is then exactly the subspace where $P_1 = Q_1$ or $\theta = 0$. It is shown in red in Figure 4.

Next, we will parametrize $\widehat{\Gamma}_\beta^{\alpha,c}$. Since θ is the angle between the vectors $P_1 - d\boldsymbol{n}/|\boldsymbol{n}|^2$ and $Q_1 - d\boldsymbol{n}/|\boldsymbol{n}|^2$, we have

$$\cos\theta = \frac{(P_1 - d\boldsymbol{n}/|\boldsymbol{n}|^2)\cdot(Q_1 - d\boldsymbol{n}/|\boldsymbol{n}|^2)}{1 - d^2/|\boldsymbol{n}|^2}.$$

Using formula (10), this simplifies to

$$\cos\theta = \frac{(|\boldsymbol{n}|^2 P_1 - d\boldsymbol{n})\cdot Q_1}{\sin^2\alpha_2 \sin^2\varphi}. \tag{12}$$

The lemmas that follow simplify this formula further and eventually lead to Theorem 5.4, which identifies the right-hand side of (12) as a Chebyshev polynomial.

Lemma 5.1. *The right-hand side of formula* (12) *is a polynomial in* $\cos\varphi$, *which will be denoted by* $P(\cos\varphi)$.

Proof. We will proceed by simplifying (12) while keeping track of its dependence on φ. By a direct calculation,

$$|\boldsymbol{n}|^2 P_1 - d\boldsymbol{n} = \begin{pmatrix} \sin^2\varphi\, A_1 \\ \sin\varphi\, A_2 \\ \sin\varphi\, A_3 \end{pmatrix}, \tag{13}$$

where A_1, A_2 and A_3 are real-valued polynomials in $\cos\varphi$ of degrees $\deg A_1 = 1$, $\deg A_2 = 2$ and $\deg A_3 = 1$. To compute Q_1, recall that Y_1 is the image of X_1 under the action of the braid $\beta = \sigma_1^{2\ell}$, where $(X_1, X_2)\sigma_1 = (X_1 X_2 X_1^{-1}, X_1)$; see Section 2.2. Therefore,

$$(X_1, X_2)\sigma_1^{2\ell} = \big((X_1X_2)^\ell X_1 (X_1X_2)^{-\ell}, (X_1X_2)^\ell X_2 (X_1X_2)^{-\ell}\big)$$

and

$$Q_1 = (X_1X_2)^\ell P_1 (X_1X_2)^{-\ell}.$$

Write

$$\begin{aligned} X_1X_2 &= (\cos\alpha_1 + \sin\alpha_1\cos\varphi\,\boldsymbol{i} + \sin\alpha_1\sin\varphi\,\boldsymbol{j})(\cos\alpha_2 + \sin\alpha_2\,\boldsymbol{i}) \\ &= (\cos\alpha_1\cos\alpha_2 - \sin\alpha_1\sin\alpha_2\cos\varphi) \\ &\quad + (\cos\alpha_1\sin\alpha_2 + \sin\alpha_1\cos\alpha_2\cos\varphi)\,\boldsymbol{i} \\ &\quad + \sin\alpha_1\cos\alpha_2\sin\varphi\,\boldsymbol{j} - \sin\alpha_1\sin\alpha_2\sin\varphi\,\boldsymbol{k} \\ &= U_1 + \sin\varphi\, V_1\,\boldsymbol{j}, \end{aligned}$$

where U_1 and V_1 are complex-valued polynomials in $\cos\varphi$ of degrees $\deg U_1 = 1$ and $\deg V_1 = 0$. Using induction on ℓ one can easily see that

$$(X_1X_2)^\ell = U_\ell + \sin\varphi\, V_\ell\,\boldsymbol{j},$$

where U_ℓ and V_ℓ are complex-valued polynomials in $\cos\varphi$ of degrees $\deg U_\ell = \ell$ and $\deg V_\ell = \ell - 1$. A straightforward calculation then shows that

$$Q_1 = \begin{pmatrix} B_1 \\ \sin\varphi\, B_2 \\ \sin\varphi\, B_3 \end{pmatrix}, \tag{14}$$

where B_1, B_2 and B_3 are real-valued polynomials in $\cos\varphi$ of degrees $\deg B_1 = 2\ell + 1$, $\deg B_2 = 2\ell$ and $\deg B_3 = 2\ell$. By taking the dot product of (13) and (14) we conclude that the numerator of (12) has the form $\sin^2\varphi$ times a polynomial in $\cos\varphi$ of degree at most $2\ell + 2$. The factors of $\sin^2\varphi$ in the numerator and the denominator of (12) cancel, thereby finishing the proof. □

Lemma 5.2. *The polynomial $P(\cos\varphi)$ has degree 2ℓ and its leading coefficient is* $2^{2\ell-1}\sin^{2\ell}\alpha_1\,\sin^{2\ell}\alpha_2$.

Proof. We will follow the proof of the previous lemma but make our calculations more precise. One can check using induction on ℓ that

$$\boldsymbol{i}\,\overline{U}_\ell = \cos\varphi\, V_\ell + \boldsymbol{i}\,\cot\alpha_1\, V_\ell + \cdots, \tag{15}$$

where the dots stand for a polynomial in $\cos\varphi$ of degree at most $\ell-2$. A tedious but direct calculation of the polynomials in formulas (13) and (14) then yields the formulas

$$\begin{aligned}
A_1 &= \sin^2\alpha_1\sin^2\alpha_2\cos\varphi - \sin\alpha_1\sin\alpha_2\cos\alpha_1\cos\alpha_2,\\
A_2 &= -\sin^2\alpha_1\sin^2\alpha_2\cos^2\varphi + \sin\alpha_1\sin\alpha_2\cos\alpha_1\cos\alpha_2\cos\varphi + \sin^2\alpha_2,\\
A_3 &= \sin\alpha_1\cos\alpha_1\sin^2\alpha_2\cos\varphi + \sin^2\alpha_1\sin\alpha_2\cos\alpha_2,
\end{aligned}$$

and

$$\begin{aligned}
B_1 &= \big[(|U_\ell|^2 + |V_\ell|^2\cos^2\varphi)\cos\varphi - 2\,\mathrm{Im}(U_\ell\overline{V}_\ell)\cos^2\varphi\big]_{2\ell+1}\\
&\qquad\qquad - \big[|V_\ell|^2\cos\varphi - 2\,\mathrm{Im}(U_\ell\overline{V}_\ell)\big]_{2\ell-1},\\
B_2 &= \big[-\mathrm{Re}\,V_\ell^2\cos^2\varphi + 2\,\mathrm{Im}(U_\ell V_\ell)\cos\varphi + \mathrm{Re}\,U_\ell^2\big]_{2\ell} + \big[\mathrm{Re}\,V_\ell^2\big]_{2\ell-2},\\
B_3 &= \big[-\mathrm{Im}\,V_\ell^2\cos^2\varphi - 2\,\mathrm{Re}(U_\ell V_\ell)\cos\varphi + \mathrm{Im}\,U_\ell^2\big]_{2\ell-1} + \big[\,\mathrm{Im}\,V_\ell^2\big]_{2\ell-2}.
\end{aligned}$$

The brackets in the formulas for B_1, B_2 and B_3 contain polynomials whose degrees are indicated by the subscripts. Note that the first bracket in the formula for B_3 has degree $2\ell-1$ rather than 2ℓ, which follows from (15). Another lengthy calculation shows that

$$\begin{aligned}
A_1B_1 + A_2B_2 + A_3B_3 &= 2\sin^2\alpha_1\sin^2\alpha_2\cos^2\varphi\;(\mathrm{Im}\,U_\ell - \cos\varphi\;\mathrm{Re}\,V_\ell)^2\\
&\quad -2\sin\alpha_1\sin\alpha_2\cos\alpha_1\cos\alpha_2\cos\varphi\;(\mathrm{Im}\,U_\ell - \cos\varphi\,\mathrm{Re}\,V_\ell)^2\\
&\quad +2\sin^2\alpha_1\cos^2\varphi\;(\mathrm{Im}\,V_\ell)^2 + \cdots,
\end{aligned}$$

where the dots stand for a polynomial in $\cos\varphi$ of degree at most $2\ell-1$. Using formula (15), this further simplifies to

$$A_1B_1 + A_2B_2 + A_3B_3 = 2\cos^2\varphi\sin^2\alpha_2\;(\mathrm{Im}\,V_\ell)^2 + \cdots,$$

which is a polynomial of degree 2ℓ. It follows that $P(\cos\varphi)$ is a polynomial of degree 2ℓ whose leading term matches that of the polynomial $2\cos^2\varphi\;(\mathrm{Im}\,V_\ell)^2$. Another induction shows that the leading term of $\mathrm{Im}\,V_\ell$ equals

$$(-1)^\ell 2^{\ell-1}\cos^{\ell-1}\varphi\,\sin^\ell\alpha_1\,\sin^\ell\alpha_2,$$

which completes the proof. □

Lemma 5.3. *The polynomial* $P(\cos\varphi)$ *evaluates to, respectively,* $\cos(2\ell(\alpha_1+\alpha_2))$ *and* $\cos(2\ell(\alpha_1-\alpha_2))$ *at* $\varphi = 0$ *and* $\varphi = \pi$.

Proof. We will use formula (12), which defines $P(\cos\varphi)$ for $\varphi \in (0,\pi)$, to show that $P(\cos\varphi)$ tends to, respectively, $\cos(2\ell(\alpha_1+\alpha_2))$ and $\cos(2\ell(\alpha_1-\alpha_2))$ as $\varphi \to 0$ and $\varphi \to \pi$. We first calculate the limit as $\varphi \to 0$. Once we eliminate the factors $\sin^2\varphi$ as in Lemma 5.1, the calculation reduces to evaluating the expression

$$\frac{A_1B_1 + A_2B_2 + A_3B_3}{\sin^2\alpha_2}$$

at $\varphi = 0$. Here, we use the notation from the proof of Lemma 5.2. It is easy to see that A_1, A_2, A_3 evaluate to

$$\begin{aligned} A_1 &= -\sin\alpha_1 \sin\alpha_2 \cos(\alpha_1+\alpha_2),\\ A_2 &= \cos\alpha_1 \sin\alpha_2 \sin(\alpha_1+\alpha_2),\\ A_3 &= \sin\alpha_1 \sin\alpha_2 \sin(\alpha_1+\alpha_2). \end{aligned}$$

To evaluate B_1, B_2 and B_3, we will keep track of $\sin\varphi$ in the formula $Q_1 = (X_1X_2)^\ell P_1 (X_1X_2)^{-\ell}$ while setting $\cos\varphi$ equal to one. An induction on ℓ can be used to show that

$$(X_1X_2)^\ell = q^\ell + \sin\varphi \sin\alpha_1\, e^{\boldsymbol{i}\alpha_1}\left(q^{-\ell}\frac{1-q^{2\ell}}{1-q^2}\right)\boldsymbol{j} + \cdots, \tag{16}$$

where $q = e^{\boldsymbol{i}(\alpha_1+\alpha_2)}$ and the dots stand for higher-degree polynomials in $\sin\varphi$. Using the identity

$$\frac{1-q^{2\ell}}{1-q^2} = e^{\boldsymbol{i}(\ell-1)(\alpha_1+\alpha_2)}\frac{\sin(\ell(\alpha_1+\alpha_2))}{\sin(\alpha_1+\alpha_2)},$$

the above formulas can be written in trigonometric form. It now follows from the formula

$$Q_1 = (X_1X_2)^\ell P_1 (X_1X_2)^{-\ell} = \begin{pmatrix} B_1 \\ \sin\varphi\, B_2 \\ \sin\varphi\, B_3 \end{pmatrix}$$

that, when $\varphi = 0$, we have

$$\begin{aligned} B_1 &= 1,\\ B_2 &= \cos(2\ell(\alpha_1+\alpha_2))\\ &\quad + 2\sin\alpha_1\big(-\sin\alpha_2\cos(\ell(\alpha_1+\alpha_2)) + \cos\alpha_2\sin(\ell(\alpha_1+\alpha_2))\big)\frac{\sin(\ell(\alpha_1+\alpha_2))}{\sin(\alpha_1+\alpha_2)},\\ B_3 &= \sin(2\ell(\alpha_1+\alpha_2))\\ &\quad - 2\sin\alpha_1\big(\cos\alpha_2\cos(\ell(\alpha_1+\alpha_2)) + \sin\alpha_2\sin(\ell(\alpha_1+\alpha_2))\big)\frac{\sin(\ell(\alpha_1+\alpha_2))}{\sin(\alpha_1+\alpha_2)}. \end{aligned}$$

Finally, a tedious but straightforward trigonometric calculation using the formulas above shows that

$$A_1 B_1 + A_2 B_2 + A_3 B_3 = \sin^2\alpha_2 \cos(2\ell(\alpha_1+\alpha_2)),$$

which immediately implies that $P(\cos\varphi)$ limits to $\cos(2\ell(\alpha_1+\alpha_2))$ as $\varphi\to 0$. The calculation of the limit as $\varphi\to\pi$ is similar. □

Recall that the Chebyshev polynomial of the first kind $T_m(x)$ is the unique polynomial of degree m satisfying $T_m(\cos(\psi)) = \cos(m\psi)$ for $m = 0, 1, 2, \ldots$. The following theorem is the main result of this section.

Theorem 5.4. *Let $T_{2\ell}(x)$ be the Chebyshev polynomial of the first kind of degree 2ℓ. Then the formula (12) is equivalent to*

$$\cos\theta = T_{2\ell}(\cos\alpha_1\cos\alpha_2 - \cos\varphi\sin\alpha_1\sin\alpha_2). \tag{17}$$

Proof. The solutions of the equation $P(\cos\varphi) = 1$ with $0<\varphi<\pi$ are precisely the intersection points of $\widehat{\Lambda}_2^{\alpha,c}$ and $\widehat{\Gamma}_\beta^{\alpha,c}$ in the pillowcase $\widehat{H}_2^{\alpha,c}$. According to (4), these correspond to the conjugacy classes of irreducible representations $\rho:\pi_1(S^3\setminus L_\ell)\to SU(2)$, which we described as the values of φ solving (7). The function $P(\cos\varphi)$ achieves its absolute maximum at all such φ; hence we also know that $P'(\cos\varphi)=0$.

Let us first assume that the values of (α_1,α_2) are in a sufficiently small neighborhood of $(\pi/2,\pi/2)$ chosen so that (7) has the maximal possible number of solutions, which is $\ell-1$. Consider the polynomial

$$R(x) = P(x) - T_{2\ell}(\cos\alpha_1\cos\alpha_2 - x\sin\alpha_1\sin\alpha_2).$$

It has $\ell-1$ roots $x=\cos(\pi m/\ell)$, $m=1,\ldots,\ell-1$, each of multiplicity at least two. In addition, it has roots $x=\pm 1$ by Lemma 5.3. On the other hand, the degree of $R(x)$ is at most 2ℓ by Lemma 5.2, while the leading coefficients of $P(x)$ and $T_{2\ell}(\cos\alpha_1\cos\alpha_2 - x\sin\alpha_1\sin\alpha_2)$ match by Lemma 5.2. Therefore, the degree of $R(x)$ is at most $2\ell-1$ so $R(x)$ must vanish.

To conclude, we observe that both $P(x)$ and $T_{2\ell}(\cos\alpha_1\cos\alpha_2 - x\sin\alpha_1\sin\alpha_2)$ are analytic functions in (α_1,α_2). Since we proved that they equal each other in an open neighborhood of $(\pi/2,\pi/2)$, they must equal each other for all (α_1,α_2). □

5.4. *Transversality.* In general, $\widehat{\Gamma}_\beta^{\alpha,c}$ needs to be perturbed for the intersection number (6) to make sense. However, in the case of $(2,2\ell)$-torus links, no perturbation is necessary as the intersection is automatically transversal.

Proposition 5.5. *Let $\beta=(\sigma_1)^{2\ell}$ be the braid whose closure is the $(2,2\ell)$-torus link with $\ell\neq 0$. Then the intersection of $\widehat{\Gamma}_\beta^{\alpha,c}$ and $\widehat{\Lambda}_2^{\alpha,c}$ in $\widehat{H}_2^{\alpha,c}$ is transversal, and all the intersections points contribute with the same sign into the algebraic count.*

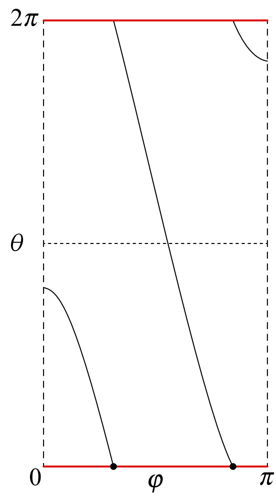

Figure 5. An example of $\widehat{\Gamma}^{\alpha,c}_{\beta}$.

Proof. We will assume that $\ell > 0$ since the case of $\ell < 0$ can be handled similarly. The curves $\widehat{\Gamma}^{\alpha,c}_{\beta}$ and $\widehat{\Lambda}^{\alpha,c}_{2}$ intersect exactly at the points $(\varphi, 0)$, where φ solves the equation

$$\cos(0) = 1 = T_{2\ell}(\cos\alpha_1 \cos\alpha_2 - \cos\varphi \sin\alpha_1 \sin\alpha_2),$$

which we have shown happens exactly when

$$\cos\alpha_1 \cos\alpha_2 - \cos\varphi \sin\alpha_1 \sin\alpha_2 = \cos\frac{m\pi}{\ell} \quad \text{for } m = 1, 2, \dots, \ell - 1.$$

The curve $\widehat{\Gamma}^{\alpha,c}_{\beta}$ is smooth near each of the intersection points. It is parameterized by (φ, θ), where θ is a smooth function of φ found by solving (11). The intersection points give local maxima of the Chebyshev polynomial $T_{2\ell}$ hence the curve $\widehat{\Gamma}^{\alpha,c}_{\beta}$ must be decreasing near these points as shown in Figure 5. Therefore, all the intersection numbers will be of the same sign once we prove that the intersections are transversal.

To prove transversality, it suffices to show that the derivative of θ with respect to φ is not zero near the intersection points. Differentiating the formula (17) and keeping in mind that $T_n'(x) = nU_{n-1}(x)$, where $U_{n-1}(x)$ is the Chebyshev polynomial of the second kind of degree $n-1$, we obtain

$$\frac{d\theta}{d\varphi} = -2\ell \sin\alpha_1 \sin\alpha_2 \sin\varphi \frac{U_{2\ell-1}(x)}{\sqrt{1 - T_{2\ell}^2(x)}}.$$

We wish to show that, as $x \to \cos(\pi m/\ell)$ from the left, the limit of the right-hand side of this equation is nonzero. This is true because of the well known fact that each $\cos(\pi m/\ell)$ is a simple root of $U_{2\ell-1}(x)$ and a double root of $1 - T_{2\ell}^2(x)$. □

5.5. *Fixing the overall sign.* An immediate corollary of Propositions 4.1 and 5.5 is that, up to an overall sign, the Bénard–Conway invariant $h_{L_\ell}(\alpha)$ of the $(2,2\ell)$-torus link equals the number of integers $m \in \{1,\dots,|\ell|-1\}$ such that $\cos(\pi m/\ell)$ lies in the interval $(\cos(\alpha_1+\alpha_2), \cos(\alpha_1-\alpha_2))$.

To determine the sign, we need to get specific about the orientations of $\widehat{\Gamma}_\beta^{\alpha,c}$, $\widehat{\Lambda}_2^{\alpha,c}$ and $\widehat{H}_2^{\alpha,c}$ and to compute at least one intersection number explicitly. We will only compute the intersection number in the special case of $\alpha = (\alpha_1,\alpha_2) = (\pi/2,\pi/2)$. The calculation for general $\ell > 0$ and α will be similar, and the case of negative ℓ will be addressed later in this section. The following argument is a modification of the argument of Boden and Herald [3].

In the case at hand, $\widehat{\Gamma}_\beta^{\alpha,c}$ intersects $\widehat{\Lambda}_2^{\alpha,c}$ at one point $(\varphi,\theta) = (\pi/2, 0)$, which corresponds to the conjugacy class of $(\boldsymbol{j},\boldsymbol{i},\boldsymbol{j},\boldsymbol{i})$ in $\widehat{H}_2^{\alpha,c}$. Consider the function $f : R_2^{\alpha,c} \times R_2^{\alpha,c} \to \mathrm{SU}(2)$ given by $f(X_1,X_2,Y_1,Y_2) = X_1X_2Y_2^{-1}Y_1^{-1}$ and the function $g : (0,\pi)\times[0,2\pi) \to R_2^{\alpha,c}\times R_2^{\alpha,c}$ given by

$$g(\varphi,\theta) = (\boldsymbol{i}\,e^{-\boldsymbol{k}\varphi}, \boldsymbol{i}, \boldsymbol{i}\,e^{-\boldsymbol{k}(\varphi-\theta)}, \boldsymbol{i}\,e^{\boldsymbol{k}\theta}).$$

The latter is exactly the quadruple $(e^{\alpha_1 P_1}, e^{\alpha_2 \boldsymbol{i}}, e^{\alpha_1 Q_1}, e^{\alpha_2 Q_2})$ used to parameterize the pillowcase $\widehat{H}_2^{\alpha,c}$, with $\alpha_1 = \alpha_2 = \pi/2$ substituted in the equation. Notice that $\widehat{H}_2^{\alpha,c}$ is the quotient of $f^{-1}(1)$ by conjugation. We will orient $f^{-1}(1)$ and $\widehat{H}_2^{\alpha,c}$ by applying the base-fiber rule.

First, we will consider the map g. Two tangent vectors that span the tangent space to $\widehat{H}_2^{\alpha,c}$ at $(\boldsymbol{j},\boldsymbol{i},\boldsymbol{j},\boldsymbol{i})$ are given by

$$u_1 = \frac{\partial g}{\partial\varphi}\Big|_{(\varphi,\theta)=(\pi/2,0)} = (-\boldsymbol{i},0,-\boldsymbol{i},0), \quad u_2 = \frac{\partial g}{\partial\theta}\Big|_{(\varphi,\theta)=(\pi/2,0)} = (0,0,\boldsymbol{i},-\boldsymbol{j}).$$

The tangent space to the orbit through $(\boldsymbol{j},\boldsymbol{i},\boldsymbol{j},\boldsymbol{i})$ is spanned by the vectors

$$v_1 = \frac{\partial}{\partial t} e^{\boldsymbol{i}t}(\boldsymbol{j},\boldsymbol{i},\boldsymbol{j},\boldsymbol{i})e^{-\boldsymbol{i}t}\Big|_{t=0} = (2\boldsymbol{k},0,2\boldsymbol{k},0),$$

$$v_2 = \frac{\partial}{\partial t} e^{\boldsymbol{j}t}(\boldsymbol{j},\boldsymbol{i},\boldsymbol{j},\boldsymbol{i})e^{-\boldsymbol{j}t}\Big|_{t=0} = (0,-2\boldsymbol{k},0,-2\boldsymbol{k}),$$

$$v_3 = \frac{\partial}{\partial t} e^{\boldsymbol{k}t}(\boldsymbol{j},\boldsymbol{i},\boldsymbol{j},\boldsymbol{i})e^{-\boldsymbol{k}t}\Big|_{t=0} = (-2\boldsymbol{i},2\boldsymbol{j},-2\boldsymbol{i},2\boldsymbol{j}).$$

The vectors $\{u_1,u_2,v_1,v_2,v_3\}$ form a basis of $T_{(\boldsymbol{j},\boldsymbol{i},\boldsymbol{j},\boldsymbol{i})}f^{-1}(1)$. Complete it to a basis for $T_{(\boldsymbol{j},\boldsymbol{i},\boldsymbol{j},\boldsymbol{i})}(R_2^{\alpha,c}\times R_2^{\alpha,c})$ using the vectors $\{w_1,w_2,w_3\}$, where we can choose $w_1 = (\boldsymbol{k},0,0,0)$, $w_2 = (0,\boldsymbol{k},0,0)$, and $w_3 = (0,\boldsymbol{j},0,0)$. Notice that the orientation of the ordered triple $\{df(w_1), df(w_2), df(w_3)\} = \{\boldsymbol{i},-\boldsymbol{j},-\boldsymbol{k}\}$ is consistent with that of the standard basis of Lie algebra $\mathfrak{su}(2)$.

The two oriented bases $\{w_1,w_2,w_3,u_1,u_2,v_1,v_2,v_3\}$ and

$$\{(\boldsymbol{i},0,0,0),(\boldsymbol{k},0,0,0),(0,\boldsymbol{j},0,0),(0,-\boldsymbol{k},0,0),(0,0,\boldsymbol{i},0),(0,0,\boldsymbol{k},0),(0,0,0,\boldsymbol{j}),(0,0,0,-\boldsymbol{k})\}$$

of the space $T_{(\boldsymbol{j},\boldsymbol{i},\boldsymbol{j},\boldsymbol{i})}\left(R_2^{\alpha,c}\times R_2^{\alpha,c}\right)$ are related by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

of determinant -8. This implies that the ordered pair $\{u_2,u_1\}$ gives a positively oriented basis in $T_{(\boldsymbol{j},\boldsymbol{i},\boldsymbol{j},\boldsymbol{i})}\widehat{H}_2^{\alpha,c}$.

Next, consider the one-variable parameterizations $\widehat{\Lambda}_2^{\alpha,c}=\{(\boldsymbol{i}\,e^{-\boldsymbol{k}\varphi},\boldsymbol{i},\boldsymbol{i}\,e^{-\boldsymbol{k}\varphi},\boldsymbol{i})\}$ and $\widehat{\Gamma}_\beta^{\alpha,c}=\{(\boldsymbol{i}\,e^{-\boldsymbol{k}\varphi},\boldsymbol{i},(\boldsymbol{i}\,e^{-\boldsymbol{k}\varphi},\boldsymbol{i}))\sigma_1^4\}$, where $(X_1,X_2)\sigma_1=(X_1X_2X_1^{-1},X_1)$ is the homeomorphism induced by the braid. Compute the tangent vector

$$\psi_1=\frac{\partial}{\partial\varphi}(\boldsymbol{i}\,e^{-\boldsymbol{k}\varphi},\boldsymbol{i},ie^{-\boldsymbol{k}\varphi},\boldsymbol{i})\Big|_{\varphi=\frac{\pi}{2}}=(-\boldsymbol{i},0,-\boldsymbol{i},0)$$

to $\widehat{\Lambda}_2^{\alpha,c}$ and the tangent vector

$$\psi_2=\frac{\partial}{\partial\varphi}(\boldsymbol{i}\,e^{-\boldsymbol{k}\varphi},\boldsymbol{i},\sigma_1^4(\boldsymbol{i}\,e^{-\boldsymbol{k}\varphi},\boldsymbol{i}))\Big|_{\varphi=\frac{\pi}{2}}=(-\boldsymbol{i},0,-5\boldsymbol{i},4\boldsymbol{j})$$

to $\widehat{\Gamma}_\beta^{\alpha,c}$. The bases $\{\psi_1,\psi_2\}$ and $\{u_2,u_1\}$ in the tangent space $T_{(\boldsymbol{j},\boldsymbol{i},\boldsymbol{j},\boldsymbol{i})}\widehat{H}_2^{\alpha,c}$ are related by the matrix

$$\begin{pmatrix} 0 & -4 \\ 1 & 1 \end{pmatrix}.$$

Since the determinant of this matrix is positive, we conclude that the intersection number $h_L(\alpha)=\langle\widehat{\Lambda}_2^{\alpha,c},\widehat{\Gamma}_\beta^{\alpha,c}\rangle_{\widehat{H}_2^{\alpha,c}}$ equals $+1$.

Corollary 5.6. *For any integer $\ell>0$ and for any choice of (α_1,α_2) away from the set (8), the Bénard–Conway invariant $h_{L_\ell}(\alpha)$ of the $(2,2\ell)$-torus link is well defined and is equal to the number of integers $m\in\{1,\dots,|\ell|-1\}$ such that*

$$\cos(\pi m/\ell)\in(\cos(\alpha_1+\alpha_2),\cos(\alpha_1-\alpha_2)).$$

For $\ell=1$, we obviously obtain $h_{L_1}(\alpha)=0$, as established by Bénard and Conway [1]. For the first few $\ell\geq 1$, the invariant $h_{L_\ell}(\alpha)$ equals m in each of the respective diamond-shaped regions R_m depicted in Figure 6. The boundaries of R_m consist of the values of (α_1,α_2) from the set (8), where the multivariable Alexander polynomial of L_ℓ vanishes and hence the Bénard–Conway invariant is not defined.

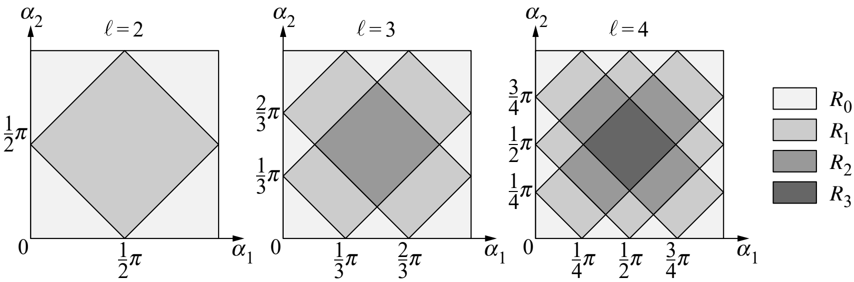

Figure 6. Regions R_m for small ℓ.

If ℓ is negative, we perform a similar calculation for $\ell = -2$ and $\alpha = (\alpha_1, \alpha_2) = (\pi/2, \pi/2)$ to determine the overall sign. The only difference is the parametrization of $\widehat{\Gamma}_{\beta}^{\alpha,c}$, which is now $\{(\boldsymbol{i}\, e^{-\kappa\varphi}, \boldsymbol{i}, (\boldsymbol{i}\, e^{-\kappa\varphi}, \boldsymbol{i}))\sigma_1^{-4}\}$, reflecting the change in braid representative from σ_1^4 to σ_1^{-4}. The new tangent vector ψ_2 is $(-\boldsymbol{i}, 0, 3\boldsymbol{i}, -4\boldsymbol{j})$ and the resulting change-of-basis matrix has negative determinant.

Corollary 5.7. *For any integer $\ell > 0$ and for any choice of (α_1, α_2) away from the set (8) the invariants $h_{L_\ell}(\alpha)$ and $h_{L_{-\ell}}(\alpha)$ are well defined and related by*

$$h_{L_{-\ell}}(\alpha) = -h_{L_\ell}(\alpha).$$

6. Proof of Theorems 1.1 and 1.2

According to [14], the linking number is a complete invariant of link homotopy for two-component links. Therefore, any two-component oriented link with linking number $\ell \neq 0$ can be obtained from the $(2, 2\ell)$-torus link via a sequence of crossing changes within individual components. The Cimasoni–Florens signature for $(2, 2\ell)$-torus links can be easily computed directly from its definition. Assuming that ℓ is positive, it is given by the formulas

$$\sigma_{L_\ell}(\omega_1, \omega_2) = \begin{cases} \ell - 2i - 1 & \text{if } i\pi/\ell < \alpha_1 + \alpha_2 < (i+1)\pi/\ell \text{ and } 0 \le i \le \ell - 1, \\ -3\ell + 2i + 1 & \text{if } i\pi/\ell < \alpha_1 + \alpha_2 < (i+1)\pi/\ell \text{ and } \ell \le i \le 2\ell - 1, \end{cases}$$

and $\sigma_{L_{-\ell}}(\omega_1, \omega_2) = -\sigma_{L_\ell}(\omega_1, \omega_2)$; see [6, Example 2.5]. From $\sigma_{L_\ell}(\omega_1, \omega_2)$ we obtain $\sigma_{L_\ell}(\omega_1, \omega_2^{-1})$ by a flip across the axis $\alpha_2 = \pi/2$, and the average of the two quantities is then easily calculated. Comparing the answer with the calculation of Section 4 and Corollary 5.6, we conclude that

$$h_{L_\ell}(\alpha) = h_{L_\ell}(\alpha_1, \alpha_2) = -\tfrac{1}{2}\big(\sigma_{L_\ell}(\omega_1, \omega_2) + \sigma_{L_\ell}(\omega_1, \omega_2^{-1})\big).$$

According to Theorem 3.1, the equality (1) remains true with each crossing change. Theorem 1.1 now follows.

In order to prove Theorem 1.2, we will use a formula of Cimasoni and Florens,

$$\sigma_{L_1 \cup L_2}(\omega_1, \omega_2^{-1}) = \sigma_{L_1 \cup -L_2}(\omega_1, \omega_2),$$

where $-L_2$ stands for the component L_2 with reversed orientation; see Proposition 2.8 in [5]. Together with the formula

$$\sigma_{L_1\cup L_2}(\omega_1^{-1},\omega_2^{-1}) = \sigma_{L_1\cup L_2}(\omega_1,\omega_2),$$

which easily follows from the definition of the Cimasoni–Florens signature, this implies that

$$h_L(\alpha_1,\alpha_2) = -\tfrac{1}{2}\big(\sigma_{L_1\cup L_2}(\omega_1,\omega_2)+\sigma_{L_1\cup -L_2}(\omega_1,\omega_2)\big)$$

is independent of the choice of orientation on the link L. If $\omega_1=\omega_2=\omega$, one can use the formula

$$\sigma_{L_1\cup L_2}(\omega) = \sigma_{L_1\cup L_2}(\omega,\omega) - \operatorname{lk}(L_1,L_2)$$

from [5, Proposition 2.5], relating the multivariable signature $\sigma_L(\omega,\omega)$ with the Levine–Tristram signature $\sigma_L(\omega)$, to obtain

$$\begin{aligned} h_L(\alpha,\alpha) &= -\tfrac{1}{2}\big(\sigma_{L_1\cup L_2}(\omega,\omega)+\sigma_{L_1\cup -L_2}(\omega,\omega)\big)\\ &= -\tfrac{1}{2}\big(\sigma_{L_1\cup L_2}(\omega)+\operatorname{lk}(L_1,L_2)+\sigma_{L_1\cup -L_2}(\omega)+\operatorname{lk}(L_1,-L_2)\big)\\ &= -\tfrac{1}{2}\big(\sigma_{L_1\cup L_2}(\omega)+\sigma_{L_1\cup -L_2}(\omega)\big). \end{aligned}$$

In the special case of $\omega=-1$, this implies that $h_L(\pi/2,\pi/2)$ equals minus the Murasugi signature of L.

Remark 6.1. There does not appear to be a name in the literature for the quantity $\frac{1}{2}\big(\sigma_{L_1\cup L_2}(\omega)+\sigma_{L_1\cup -L_2}(\omega)\big)$, nor for an analogous quantity for links with more than two components, when $\omega\neq -1$. Perhaps it should be called the equivariant Murasugi signature.

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ZEDAN LIU
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MIAMI
CORAL GABLES, FL
UNITED STATES
zedan.liu@miami.edu

NIKOLAI SAVELIEV
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MIAMI
CORAL GABLES, FL
UNITED STATES
saveliev@math.miami.edu

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