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POLYNOMIAL IDENTITIES AND AZUMAYA LOCI FOR RATIONAL QUANTUM SPHERES

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We prove a number of structure and isomorphism results concerning the noncommutative Natsume–Olsen spheres $\mathbb{S}_\theta^{2n-1}$ deformed along a skew-symmetric matrix $\theta \in \mathbb{R}$. These include (a) the fact that two C^*-algebras of the form $\mathbb{S}_\theta^3 \otimes M_n$ are isomorphic precisely in the obvious cases; (b) the fact that m and n are recoverable from the isomorphism class of $C(\mathbb{S}_\theta^{2m-1}) \otimes M_n$; (c) the PI character, PI degree and Azumaya loci of $C(\mathbb{S}_\theta^{2m-1})$ for rational θ, along with a realization of their centers as (function algebras of) branched cover of $\mathbb{S}^{2n-1}$; and (d) for rational θ again, the topological finite generation of $C(\mathbb{S}_\theta^{2m-1})$ over their centers, with algebraic finite generation equivalent to being classical (equivalently, Azumaya).

Introduction

Consider a skew-symmetric real matrix $\theta \in M_n(\mathbb{R})$. We will be working extensively with the following noncommutative-geometric constructs.

- The *noncommutative* (or *quantum*) *tori* $\mathbb{T}_\theta^n$ [57, §1; 32, §12.2; 40, §1.1.5] defined as objects dual to the generator-and-relation C^*-algebras

$$A_\theta^n = C(\mathbb{T}_\theta^n) := \left\langle \text{unitaries } u_j,\ j \in [n] := \{1..n\} \;\middle|\; u_k u_j = e(\theta_{jk}) u_j u_k \right\rangle$$

for $e(-) := \exp(2\pi i -)$.

- The *noncommutative spheres* [48, Definition 2.1] analogously defined by

$$C(\mathbb{S}_\theta^{2n-1}) := \left\langle \text{normal } t_j,\ j \in [n] \;\middle|\; t_k t_j = e(\theta_{jk}) t_j t_k,\ \sum t_j^* t_j = 1 \right\rangle.$$

The former "glue" to produce the latter in ways reminiscent of classical topology (e.g., the *standard genus*-1 (*Heegaard*) *splitting* $\mathbb{S}^3 = (\mathbb{T}^2 \times [0,1]) \cup_{\mathbb{T}^2} (\mathbb{T}^2 \times [0,1])$ of [61, Proposition 3.3]): per [48, Theorem 2.5], we have

$$C(\mathbb{S}_\theta^{2n-1}) \cong \mathrm{Cont}_\partial\left(\mathbb{S}_+^{n-1} \to C(\mathbb{T}_\theta^n)\right), \tag{0-1}$$

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where

$$\mathbb{S}_{+}^{n-1} := \{(s_1, \dots, s_n) \in \mathbb{S}^{n-1} \subseteq \mathbb{R}^n \ : \ s_i \ge 0\},$$
$$\mathbb{S}_{+,F\subseteq[n]}^{n-1} := \{(s_i) \in \mathbb{S}_{+}^{n-1} \ : \ s_i = 0,\ \forall i \notin F\}, \tag{0-2a}$$

and '∂' means

$$f(\mathbb{S}_{+,F}^{n-1}) \subseteq C^*(u_i,\ i \in F) \subseteq A_\theta^n, \quad \forall F \subseteq [n]. \tag{0-2b}$$

Section 1 is concerned with reconstruction/isomorphism problems, i.e., the extent to which initial data (the deformation parameter $\theta \in M_n(\mathbb{R})$, or perhaps its size n) are determined by the isomorphism or stable isomorphism classes of the quantum-sphere algebras. A paraphrased aggregate of Theorems 1.1 and 1.2 reads as follows.

Theorem A. (1) *For $n, n' \in \mathbb{Z}_{>0}$ and $\theta, \theta' \in \mathbb{R}$ identified with skew-symmetric 2×2 matrices we have*

$$C(\mathbb{S}_\theta^3) \otimes M_n \cong C(\mathbb{S}_{\theta'}^3) \otimes M_{n'} \iff C(\mathbb{T}_\theta^2) \otimes M_n \cong C(\mathbb{T}_{\theta'}^2) \otimes M_{n'}$$
$$\iff n = n' \ \text{ and } \ \theta \in \pm\theta' + \mathbb{Z}.$$

(2) *For positive integers m, n with $m \ge 2$ and skew-symmetric $\theta \in M_m(\mathbb{R})$ the isomorphism class of either $C(\mathbb{T}_\theta^m) \otimes M_n$ or $C(\mathbb{S}_\theta^{2m-1}) \otimes M_n$ determines m and n.*

In Section 2 and onward we assume the deformation parameter $\theta \in M_n(\mathbb{Q})$ rational and examine consequent *polynomial-identity* (*PI*) phenomena (on which background the main text offers a brief reminder). The focus is on

- *Azumaya algebras* [16, §4.4]: one way to formalize a purely algebraic analogue of the section-space $\Gamma(\mathcal{E} \otimes \mathcal{E}^*)$ for a vector bundle $\mathcal{E}$ over a compact Hausdorff space; and
- the extent to which algebras fail to qualify, as measured by the *Azumaya locus* [11, §III.1.7].

Again summarizing for brevity, one rendition of Theorems 2.8 and 2.13 below is this:

Theorem B. *Let $\theta \in M_n(\mathbb{Q})$ be a skew-symmetric matrix for $n \in \mathbb{Z}_{\ge 2}$.*

(1) *The center $Z_\theta \le C_\theta := C(\mathbb{S}_\theta^{2n-1})$ is the function algebra of a branched cover of $\mathbb{S}^{2n-1}$ which is not, in general, a topological manifold.*

(2) *The smallest n^2 admitting a Z_θ-algebra embedding $C_\theta \le M_n(Z_\theta)$ is*

$$h_\theta := [(\mathbb{Z}^n + \operatorname{im}\theta) \ : \ \mathbb{Z}^n],$$

*so in particular C_θ is a PI algebra of **PI-degree*** (see [29, §1]) $\sqrt{h_\theta}$.

(3) *The Azumaya locus of C_θ consists precisely of those maximal ideals $p \in \mathrm{Max}(Z_\theta)$ whose restriction to $\mathbb{S}_+^{n-1}$ along equation (0-1) belongs to some face supported by $F \subseteq [n]$ with*

$h_\theta > h_{\theta,F} :=$ *h attached to the submatrix of θ supported on F-indexed rows/columns.*

In Section 3 we turn to the intimate connection between polynomial identities and the condition that an algebra be finitely generated as a module over its center (see e.g. [54, Chapter 6] and the motivating discussion on [3, p. 532]) in the context of studying the quantum spheres $\mathbb{S}_\theta^{2n-1}$. It will turn out that said finite generation (mostly) fails, but its weaker topological analogue (requiring that $C(\mathbb{S}_\theta^{2n-1})$ contain a dense finitely generated module over its center) always holds; per Theorem 3.3:

Theorem C. *For $n \in \mathbb{Z}_{\ge 2}$ and rational skew-symmetric $\theta \in M_n(\mathbb{Q})$ the algebra $C(\mathbb{S}_\theta^{2n-1})$ is*

- *always a topologically finitely generated module over its center Z_θ, but*
- ***algebraically** finitely generated as such precisely when θ is integral.*

Theorems B and C both link naturally to the theory of *Banach*, *Hilbert* and *C^* bundles* [27, §II.13; 21, pp. 7–9; 20, §1] over compact Hausdorff spaces and satellite topics: the theory of *noncommutative branched covers* initiated in [51] and phrased in the language of *finite-index conditional expectations* [28, Definition 2] is germane to the discussion below, which relies directly or indirectly on material from [7; 13; 51].

1. Isomorphisms of rationally deformed quantum 3-spheres

Throughout, unqualified (typically vector or algebra) *bundles* are assumed *locally trivial* [37, Definition 1.1.8]; they are to be distinguished from more general constructs termed *(F) Banach bundles* on [21, pp. 7–8], which will also make an appearance in Section 3.

The isomorphism problem for noncommutative 3-spheres is not difficult to resolve, given its 2-torus analogue. The following is very much in the spirit of [55, Theorem 3] for *irrational* tori. The *rational* torus version [33, Theorem 1.1] (recovered also as [56, Theorem 3.12]) typically does not involve matrix tensorands.

Theorem 1.1. *Consider $n, n' \in \mathbb{Z}_{>0}$, $\theta, \theta' \in \mathbb{R}$, and set*

$$C_{\theta,n} := C(\mathbb{S}_\theta^3) \otimes M_n, \quad A_{\theta,n} := C(\mathbb{T}_\theta^2) \otimes M_n,$$

and similarly for the primed parameters. The following conditions are equivalent.

(a) *We have an isomorphism*

$$C_{\theta,n} \cong C_{\theta',n'}. \tag{1-1}$$

(b) *We have an isomorphism*

$$A_{\theta,n} \cong A_{\theta',n'}. \tag{1-2}$$

(c) *The parameters coincide save for trivial modifications, in the sense that*

$$n = n' \quad \textit{and} \quad \theta \in \pm\theta' + \mathbb{Z}.$$

It might be worthwhile to first observe separately that the size n of the tensorand M_n can be recovered from the isomorphism class of either $C_{\theta,n}$ or $A_{\theta,n}$ and in fact, more generally, for noncommutative spheres/tori of arbitrary dimension.

Theorem 1.2. *Let $\theta \in M_m(\mathbb{R})$ be a skew-symmetric matrix, $m \ge 2$ and $n \in \mathbb{Z}_{>0}$.*

(1) *The isomorphism class of the C^*-algebra $C(\mathbb{T}^m_\theta) \otimes M_n$ determines m and n.*

(2) *The isomorphism class of the C^*-algebra $C(\mathbb{S}^{2m-1}_\theta) \otimes M_n$ determines m and n.*

Proof. We structure the argument conforming to the statement.

Statement (1) is a simple matter of unwinding some of the well-known K-theoretic results on noncommutative tori.

According to [22, Theorem 2.2], $K_0(C(\mathbb{T}^m_\theta))$ is free abelian of rank 2^{m-1} with the class of the identity as a generator. The usual isomorphism

$$K_0(A) \cong K_0(A \otimes M_n)$$

induced by the upper-corner embedding [67, Lemma 6.2.10] then makes it clear that in

$$K_0(C(\mathbb{T}^m_\theta) \otimes M_n) \cong \mathbb{Z}^{2^{m-1}}$$

the class of the identity is divisible precisely by n, hence the conclusion.

We turn to (2).

<u>Recovering n</u>: Equation (0-1) realizes $C(\mathbb{S}^{2m-1}_\theta)$ as a $C(X)$*-algebra* (i.e., a C^*-algebra receiving a central morphism from $C(X)$: see [39, Definition 1.5]) for $X = \mathbb{S}^{m-1}_+$. The *fibers*

$$C(\mathbb{S}^{2m-1}_\theta)|_p := C(\mathbb{S}^{2m-1}_\theta)/\big(C(\mathbb{S}^{2m-1}_\theta) \cdot \{f \in C(X) : f(p) = 0\}\big), \quad p \in \mathbb{S}^{n-1}_+,$$

are noncommutative-torus algebras A in general, and specifically to $C(\mathbb{S}^1)$ at the m vertices of the spherical simplex X. It follows that n can be recovered from the abstract C^*-algebra $C := C(\mathbb{S}^{2m-1}_\theta) \otimes M_n$ as the smallest dimension of an irreducible representation.

<u>Recovering m</u>: A small detour will first discern what can be recovered from the center

$$Z := Z(C(\mathbb{S}^{2m-1}_\theta) \otimes M_n) = Z(C(\mathbb{S}^{2m-1}_\theta)) = Z(C) \tag{1-3}$$

alone. The isomorphism (0-1) specializes to $Z \cong \mathrm{Cont}_{\partial}(\mathbb{S}_{+}^{m-1} \to Z(C(\mathbb{T}_{\theta}^{n})))$, with ∂ indicating the central analogue of (0-2b):

$$f(\mathbb{S}^{m-1}_{+,F}) \subseteq Z(C(\mathbb{T}^n_\theta)) \cap C^*(\{u_j \mid j \in F\}). \tag{1-4}$$

Recall [23, §2.2] that $C(\mathbb{T}^n_\theta)$ is a cocycle twist $C^*(\mathbb{Z}^n, \sigma)$ of the group algebra $C^*(\mathbb{Z}^n)$ (with the generators of $\mathbb{Z}^n$ mapping to the u_i). The proof of [22, Lemma 2.3] then describes the center $Z(C(\mathbb{T}^n_\theta))$ as

$$C^*(\Gamma) \subset C^*(\mathbb{Z}^n), \qquad \Gamma := \text{kernel of the bicharacter } \mathbb{Z}^n \wedge \mathbb{Z}^n \xrightarrow{e(\theta)=\exp(2\pi i\theta)} \mathbb{S}^1, \tag{1-5}$$

with the matrix θ regarded, as usual, as a bilinear form. The boundary condition (1-4) then reads

$$f(\mathbb{S}^{m-1}_{+,F}) \subseteq C^*(\Gamma_F) \cong C(\mathbb{T}_F),\ \mathbb{T}_F := \widehat{\Gamma_F},$$

where

$$\Gamma_F := \Gamma \cap \mathbb{Z}^F, \quad \mathbb{Z}^F := \text{sum of the } F\text{-indexed summands in } \mathbb{Z}^n. \tag{1-6}$$

We write $\mathrm{Max}(A)$ for the *maximal spectrum* [5, Exercise 1.26] of a commutative ring, typically applying the notion to commutative C^*-algebras (and occasionally omitting the qualifier "maximal"). $X := \mathrm{Max}(Z)$ can thus be described as follows:

- To each finite-set inclusion

  $$F \subseteq F' \subseteq [m]$$

  associate

  $$\text{an inclusion} \quad \Delta_F \hookrightarrow \Delta_{F'}, \quad \Delta_F := \mathbb{S}^{m-1}_{+,F},$$
  $$\text{and a quotient} \quad \mathbb{T}_F \twoheadrightarrow \mathbb{T}_{F'} \quad \text{dual to} \quad \Gamma_F \le \Gamma_{F'}.$$

- And then recover X as the *coend* [43, §IX.6]

$$X = \mathrm{Max}(Z(C(\mathbb{S}^{2m-1}_\theta))) \cong \int^F \Delta_F \times \mathbb{T}_F. \tag{1-7}$$

This is a particular instance of the *geometric realization* [58, Definition 3.8.1] of a *simplicial object* [43, §VII.5]: more precisely, $F \mapsto \mathbb{T}_F$ is a simplicial topological space (a simplicial object in the category of topological spaces), and (1-7) is its simplicial realization.

The space (1-7) does have *dimension*

$$\dim X = m - 1 + \dim \mathrm{Max}(Z(C(\mathbb{T}^m_\theta))) = m - 1 + \mathrm{rank}\ker\theta, \tag{1-8}$$

where

– "dimension" is used in any of three senses: the *small inductive*, *large inductive* and *covering* dimensions [24, Definitions 1.1.1, 1.6.1, 1.6.7], all coincident for separable metric spaces [24, Theorem 1.7.7];

– and the *rank* of an abelian group is [35, Definition A1.59]

$$\operatorname{rank}\Gamma := \dim_{\mathbb{Q}} \mathbb{Q}\otimes_{\mathbb{Z}}\Gamma.$$

By contrast to m (Example 1.3), this latter number (1-8) *is*, then, recoverable from the isomorphism class of $Z = Z(C)$ alone (so C itself is not needed for this).

The fibers of C at the various points of X are of the form $C^*(\mathbb{Z}^m/\Gamma_F, \sigma)\otimes M_n$ for subsets $F\subseteq[m]$ with the twist induced by the original cocycle σ on $\mathbb{Z}^n/\Gamma_F$ (possible, given that $\Gamma = \Gamma_{[m]}$ is by definition the kernel of θ).

The abelian groups $\mathbb{Z}^m/\Gamma_F$ have ranks

$$m - \operatorname{rank}\Gamma_F \ge m - \operatorname{rank}\Gamma_{[m]} = m - \operatorname{rank}\ker\theta, \tag{1-9}$$

with equality achieved. Since the sum between (1-8) and this minimum is $2m-1$, it will be enough to observe that the ranks on the left-hand side of (1-9) can be recovered from the corresponding fibers $C^*(\mathbb{Z}^m/\Gamma_F, \sigma)\otimes M_n$.

To see this,

- write

$$\mathbb{Z}^m/\Gamma_F \cong (\text{finite abelian group } H)\times\mathbb{Z}^r, \quad r := \operatorname{rank}\mathbb{Z}^m/\Gamma_F$$

  [10, p. VII.19, Theorem 2];
- so that $C^*(\mathbb{Z}^m/\Gamma_F, \sigma)$ can be expressed as an iterated, r-fold crossed product

$$C^*(H)\rtimes\mathbb{Z}\rtimes\cdots\rtimes\mathbb{Z}$$

  as in [32, Proposition 12.8];
- whence

$$K_0(C^*(\mathbb{Z}^m/\Gamma_F, \sigma)\otimes M_n)\cong K_0(C^*(\mathbb{Z}^m/\Gamma_F, \sigma))\cong\mathbb{Z}^{2^{r-1}}$$

  as for noncommutative tori, via the *Pimsner–Voiculescu sequence* [6, Theorem 10.2.1].

This completes the proof. ☐

Note that the isomorphism class of the center (1-3) alone does *not*, in general, determine m.

Example 1.3. We exhibit noncommutative-sphere algebras $C(\mathbb{S}^{2m-1}_\theta)$ with isomorphic centers and distinct m (so also, by necessity, distinct θ):

(I) Take first $m = 2$ and $\theta = 0$, so that

$$C(\mathbb{S}^{2m-1}_{\theta}) \cong C(\mathbb{S}^3)$$

and the spectrum of the center is the 3-sphere.

(II) On the other hand, take $m = 3$ and some θ for which the kernel Γ of (1-5) is of rank 1 (i.e., isomorphic to $\mathbb{Z}$) and not contained in any of the subgroups

$$\mathbb{Z}^F, \quad F \subset [3] \text{ properly}$$

of (1-6). The coend (1-7) is then

$$\mathbb{S}^2_+ \times \mathbb{S}^1 / ((x,t) = (x,t'),\ \forall x \in \partial S^2_+,\ t \in \mathbb{S}^1)$$

with ∂ denoting the boundary. Alternatively, in words: consider the product $\mathbb{D}^2 \times \mathbb{S}^1$ (where $\mathbb{D}^2$ is the closed 2-disk) and identify all copies

$$\mathbb{S}^1 \times \{t\} \subset \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{D}^2 \times \mathbb{S}^1$$

to each other in the obvious fashion. It is a simple exercise to see that this space is again (homeomorphic to) the 3-sphere.

Also in reference to Theorem 1.2, note that an isomorphism

$$A \otimes M_n \cong A' \otimes M_n \tag{1-10}$$

of unital C^*-algebras does not, in general, imply that A and A' themselves are isomorphic:

Example 1.4. Recall [45, §2] the unital C^*-algebras $A = U^{nc}_{m,n}$ generated by the mn elements of a unitary (possibly nonsquare) $m \times n$ matrix. As explained in loc.cit., we may as well assume that $m \le n$.

This is (by definition) the universal unital C^*-algebra A whose Hilbert modules A^m and A^n are isomorphic, so their endomorphism algebras (as Hilbert modules) will of course also be isomorphic:

$$A \otimes M_m \cong M_m(A) \cong M_n(A) \cong A \otimes M_n.$$

Now fix any $m \ge 2$, and consider the case $A = U^{nc}_{m,m^2}$. We then have

$$A \otimes M_m \cong A \otimes M_{m^2} \cong (A \otimes M_m) \otimes M_m,$$

which is (1-10) with $A' = A \otimes M_m$ and $n = m$. An isomorphism

$$A \cong A' \cong A \otimes M_m \tag{1-11}$$

does not exist though: according to the discussion immediately preceding [2, §6],

$$K_0(A) \cong \mathbb{Z}/(n-m) = \mathbb{Z}/(m^2-m) \text{ generated by the class of } 1 \in A,$$

so that class cannot be a multiple of m, as it would be, given an isomorphism (1-11).

The ensuing discussion freely references *Azumaya algebras* (in their various incarnations, such as those in [1, Definition 5.1.4; 46, §13.7.6; 11, §III.1.3]) and ancillary notions, including the following.

- *Polynomial identities* (or *PIs* for short), for (always unital) rings A, are noncommutative polynomials vanishing when substituting arbitrary elements of A for the variables. (See [1, Definition 2.2.1].)
- *PI algebras* are those algebras A over commutative rings (usually fields) $\Bbbk$ satisfying at least one nonzero PI that is *stable* [1, §2.2.3] in the sense that all $A \otimes_{\Bbbk} \Bbbk'$ for commutative-ring extensions $\Bbbk \subseteq \Bbbk'$ satisfy a common nonzero identity. (See [1, Definition 2.2.41].)
- An algebra A over a commutative ring R is *Azumaya* (*of rank* n^2) as such [1, Definition 5.1.4] if for some faithfully flat extension $R \to R'$ we have

$$A \otimes_R R' \cong M_n(R').$$

- Equivalently [1, Theorem 10.3.2], A is rank-n^2 Azumaya (unqualified, i.e., over its center) precisely when it satisfies the polynomial identities of $n \times n$ matrices but has no quotient satisfying the identities of $(n-1) \times (n-1)$ matrices.

We refer the reader to the sources just cited, with more specific citations accompanying the text where appropriate.

Remark 1.5. We will repeatedly (and henceforth tacitly) take it for granted that being Azumaya over a commutative ring is stable under scalar extension: if A is Azumaya over R then $A_S := A \otimes_R S$ is Azumaya over S for any commutative-ring morphism $R \to S$ [1, Proposition 5.4.28].

Proof of Theorem 1.1. That (c) implies the other conditions is clear enough: for tori transitioning between $\pm\theta$ is effected by interchanging the two unitary generators, and the realization (0-1) settles the matter for spheres. We thus focus on the converse implications, seeking to deduce (c) from each of the other conditions.

(I) <u>Rational vs. irrational parameters.</u> Given an isomorphism (1-1), the deformation parameters θ and θ' are simultaneously either rational or irrational.

Indeed, if one of them (θ, say) is irrational, then $C_{\theta,n}$ has infinite-dimensional *simple* quotients isomorphic to $A_{\theta,n}$ (simple, being a minimal tensor product of simple C^*-algebras: [32, Corollary 12.12] and [64, Corollary IV.4.21]). Furthermore, (0-1) makes it clear that *all* infinite-dimensional simple quotients are of this form.

On the other hand, if $\theta' \in \mathbb{Q}$ then the simple quotients of $C_{\theta',n'}$ are plainly finite-dimensional, again by (0-1).

The same argument also (more easily) handles (1-2).

(II) The irrational case. This is the simpler branch of the argument: we have already observed that if θ and θ' are both irrational then (1-1) implies (1-2), $A_{\theta,n}$ and $A_{\theta',n'}$ being the only infinite-dimensional simple quotients of $C_{\theta,n}$ and $C_{\theta',n'}$ respectively. We can then simply appeal to [56, Theorem 3.12].

(III) The rational case. We are now assuming that

$$\theta = \frac{p}{q} \quad \text{and} \quad \theta' = \frac{p'}{q'}$$

are rational (in lowest terms, as depicted).

We focus mostly on the implication (a) $\Longrightarrow$ (c), as it will turn out to be more elaborate, indicating along the way where and how the argument changes so as to also deliver (b) $\Longrightarrow$ (c). There is no harm in assuming $n = n'$ throughout, as Theorem 1.2 allows.

An isomorphism (1-1) induces isomorphisms between

- the centers $C(\mathbb{S}^3)$ of the two C^*-algebras [15, Proposition 4.5(1)],
- and hence also between the $q \times q$ matrix bundles over the Azumaya loci

$$\mathbb{S}^3 \setminus \left(\bigsqcup \text{two circles} \right) \cong \mathbb{T}^2 \times (0, 1)$$

of $C_{\theta,n}$ and $C_{\theta',n'}$ [15, Proposition 4.5(3)].

Over those Azumaya loci, $C_{\theta,n}$ and $C_{\theta',n'}$ are isomorphic $qn \times qn$ and $q'n' \times q'n'$ matrix bundles over $X = \mathbb{T}^2 \times (0, 1)$. We thus have $qn = q'n'$, and because

- matrix bundles are classifiable homotopically [37, §18.2.4],
- X deformation-retracts onto its slice

$$\mathbb{T}^2 \cong \mathbb{T}^2 \times \left\{ \tfrac{1}{2} \right\} \subset X,$$

- and over that slice the two algebras are $A_{\theta,n}$ and $A_{\theta',n'}$, respectively,

the implication (b) $\Longrightarrow$ (c) reduces to (a) $\Longrightarrow$ (c), on which we henceforth focus. We have been assuming that $n = n'$, so that $q = q'$ because $qn = q'n'$. The rest is simple conceptually, but requires a bit of unwinding of the attendant bundle-classification theory.

The $qn \times qn$-matrix bundles on $X = \mathbb{T}^2 \times (0, 1)$ are classified by homotopy classes of maps $X \to BPU(qn)$ into the *classifying space* [37, §7.2, Definition 2.7] of the projective $qn \times qn$ unitary group [37, §18.3, Assertion 3.2 and Remark 3.3]. Writing $[-, -]$ for homotopy classes of maps [37, §6.4, Definition 4.2], we have

$$[X, -] \cong [\mathbb{T}^2, -],$$

so we may as well assume we are working with the 2-torus.

We have an exact sequence

$$[\mathbb{T}^2,\ BSU(qn)] \to [\mathbb{T}^2,\ BPU(qn)] \to [\mathbb{T}^2,\ B^2\mathbb{Z}/qn] \cong H^2(\mathbb{T}^2,\ \mathbb{Z}/qn) \cong \mathbb{Z}/qn \tag{1-12}$$

[42, Corollary 3.2.7] attached to the top row of [37, §18.3.6], where B^2 denotes the iterated classifying-space construction discussed in [37, §7.4.6] for *abelian* groups (such as $\mathbb{Z}/qn$). The second arrow in (1-12) is an *embedding*, given that the leftmost term $[\mathbb{T}^2,\ BSU(qn)]$ is trivial: it classifies degree-0 rank-qn vector bundles on the 2-torus, and those are trivial [65, p. 2, Proposition]. In other words, $qn \times qn$-matrix-algebra bundles over the 2-torus are classified by the characteristic class denoted by β_{qn} in [37, §18.3.7] (our qn being that source's n).

Denote by $\widetilde{(-)}_q$ inverses modulo q and similarly for q'. In our case, focusing on the un-primed side, the $qn \times qn$-matrix bundle in question is of the form $\mathcal{E} \otimes \mathcal{E}^*$ for a vector bundle $\mathcal{E} \cong \mathcal{F} \otimes \mathbb{C}^n$ with rank-q degree-$\widetilde{p}_q$ $\mathcal{F}$, and the class β_{qn} is the image of the Chern class

$$n\widetilde{p}_q \in [\mathbb{T}^2,\ B^2\mathbb{Z}] \cong H^2(\mathbb{T}^2,\ \mathbb{Z}) \cong \mathbb{Z}$$

through the natural map $\mathbb{Z} \to \mathbb{Z}/nq$ (arising from the vertical maps in [37, §18.3.4]) so all in all, the isomorphism of matrix bundles simply amounts to

$$\widetilde{p}_q n = \pm \widetilde{p}'_q n \mod qn$$

(the $\pm$ depending on whether the homeomorphism between the two spheres preserves or reverses orientations). This is in turn equivalent to

$$qn \mid \widetilde{p}_q n \pm \widetilde{p}'_q n \iff q \mid \widetilde{p}_q \pm \widetilde{p}'_q \iff q \mid p \pm p' \iff \theta + \mathbb{Z} = \pm\theta' + \mathbb{Z}.$$

This concludes the proof. □

Remarks 1.6. (1) The obvious analogue of [55, Theorem 3] does in fact go through for rational deformation parameters, giving an extension of [33, Theorem 1.1].

(2) The literature on the classification of noncommutative tori up to isomorphism or Morita equivalence is extensive: the reader can consult [57; 12; 23], for instance.

(3) The proof of Theorem 1.1 takes for granted the following discussion on Chern classes and their relation to matrix-algebra-bundle characteristic classes.

The fibration sequence associated (as in [37, §18.3.6]) to the exact sequence

$$1 \to SU(n) \to U(n) \xrightarrow{\det} \mathbb{S}^1 \to 1$$

and (part of) the top portion of the diagram in [37, §18.3.6] fit together into a commutative diagram

$$
\begin{array}{ccccccc}
 & & BU(n) & \xrightarrow{B\,\det} & B\mathbb{S}^1 & \xrightarrow{\cong} & K(\mathbb{Z},2) \\
 & \nearrow & \downarrow & & \downarrow & & \downarrow \\
BSU(n) & \searrow & BPU(n) & \longrightarrow & B^2\mathbb{Z}/n & \xrightarrow{\cong} & K(\mathbb{Z}/n,2)
\end{array}
\tag{1-13}
$$

where

- $K(G,m)$ denotes, as usual [37, Definition 9.6.1], the m-th *Eilenberg–Mac Lane space* with homotopy group $\pi_n \cong G$ and vanishing homotopy in other degrees;
- the left-hand vertical map is the obvious one, resulting from the quotient $U(n) \to PU(n)$ that mods out the center;
- the middle vertical arrow is obtained by applying the classifying-space (homotopy) functor $B(-)$ [60, §3.7; 47, Proposition 8.1] to the map

$$\mathbb{S}^1 \to B\mathbb{Z}/n$$

classifying the $\mathbb{Z}/n$-bundle on the circle obtained via the n-fold cover

$$\mathbb{S}^1 \xhookrightarrow{\text{central embedding}} U(n) \xrightarrow{\det} \mathbb{S}^1;$$

- and the right-hand vertical map induced by $\mathbb{Z} \to \mathbb{Z}/n$, well-defined only upon choosing a generator for $\mathbb{Z}/n$ (so a choice is involved).

For a space X the upper composition

$$(n\text{-bundles on } X) \xrightarrow[\cong]{\text{[37, Assertion 18.3.2]}} [X,\ BU(n)] \to [X,\ K(\mathbb{Z},2)] \cong H^2(X,\mathbb{Z})$$

simply associates the first Chern class $c_1(\mathcal{E}) \in H^2(X,\mathbb{Z})$ to a vector bundle $\mathcal{E}$ on X, whereas the bottom composition

$$(n^2\text{-matrix bundles on } X) \xrightarrow{\cong} [X,\ BPU(n)] \to [X,\ K(\mathbb{Z}/n,2)] \cong H^2(X,\mathbb{Z}/n)$$

(where again the first isomorphism comes from [37, Assertion 18.3.2]) similarly associates the class $\beta_n(\mathcal{A})$ of [37, §18.3.7] to a matrix-algebra bundle $\mathcal{A}$.

In conclusion, the Chern class $c_1(\mathcal{E}) \in H^2(X,\mathbb{Z})$ of a vector bundle gets mapped to the $H^2(X,\mathbb{Z}/n)$-characteristic class of the corresponding matrix-algebra bundle $\mathcal{E} \otimes \mathcal{E}^*$.

(4) We will refer to the class $\beta_n \in H^2(X,\mathbb{Z}/n)$ associated in [37, §18.3.7] to an $n \times n$ matrix-algebra bundle on X as *the characteristic* 2*-class* of the matrix bundle, to distinguish it from the *Dixmier–Douady* (3-)class $\alpha \in H^3(X,\mathbb{Z})$ also discussed in [37, §18.3.7].

2. Azumaya theory for higher quantum spheres

Much can be said about higher-dimensional rational tori $C(\mathbb{T}^n_\theta) = A^n_\theta$ [32, §12.2] as well. These algebras are also Azumaya, (essentially) by [16, Proposition 7.2]. The setup there is more algebraically oriented so the discussion requires some translation, but nothing particularly problematic. The deformation-parameter data in [16, §7] consists of

- a primitive ℓ-th root of unity q in the ground field (which for us is $\mathbb{C}$); and
- a skew-symmetric $n \times n$ matrix $H = (h_{ij}) \in M_n(\mathbb{Z})$, which will turn out to be an integer multiple of our θ.

The n invertible generators x_i of loc. cit. are required to skew-commute in the sense that

$$x_j x_k = q^{h_{jk}} x_k x_j; \tag{2-1}$$

compare this with the usual skew-commutation relation [57, p. 193]

$$u_k u_j = e^{2\pi i \theta_{jk}} u_j u_k \tag{2-2}$$

for the unitary generators of A^n_θ. Now, if ℓ is a positive integer divisible by the denominators of all θ_{jk}, then

$$x_\bullet = u_\bullet, \quad q = e^{-\frac{2\pi i}{\ell}}, \quad H = \ell\theta \in M_n(\mathbb{Z})$$

will effect the transition between equations (2-1) and (2-2).

Proposition 7.2 of [16] (or rather the appropriately translated version) then shows that A^n_θ is Azumaya of rank

$$h = h_\theta := \text{cardinality of range}\big(\mathbb{Z}^n \xrightarrow{H=\ell\theta} (\mathbb{Z}/\ell)^n\big), \tag{2-3}$$

where a matrix is regarded as an operator in the usual way.

Note that although there was a choice in selecting ℓ, h itself will not depend on that choice: scaling ℓ to, say, $d\ell$ will also scale H by d, thus preserving the size of its image. In fact, the number (2-3) has the following alternative, direct description in terms of the matrix θ alone:

Lemma 2.1. *For a rational skew-symmetric $\theta \in M_n(\mathbb{Q})$ the number (2-3) can be recovered as the index*

$$h_\theta = [(\mathbb{Z}^n + \operatorname{im}\theta) : \mathbb{Z}^n]. \tag{2-4}$$

Proof. We can assume [9, §IX.5.1, Théorème 1] that θ is block-diagonal, consisting of a zero block and one of the form

$$\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \tag{2-5}$$

for a diagonal matrix

$$D = \mathrm{diag}(p_1/q_1,\ p_2/q_2,\ \dots)$$

with lowest-terms nonzero entries. The quantities in (2-3) and (2-4) are now easily seen to both be equal to $\prod_i q_i^2$. □

Remark 2.2. The proof of Lemma 2.1 also makes it plain that (2-3) is indeed a square, as implicit in the discussion: it is the common dimension of the matrix-algebra fibers of an Azumaya algebra.

Still assuming θ rational, we have:

Notation 2.3. Consider a skew-symmetric matrix $\theta \in M_n(\mathbb{R})$ (usually rational).

(1) For a tuple $\boldsymbol{u} := (u_i)_{i=1}^n$ of generators of A_θ^n and an integer tuple $\boldsymbol{m} = (m_i)_{i=1}^n$ we write

$$\boldsymbol{u}^{\boldsymbol{m}} := \prod_{i=1}^{n} u_i^{m_i}.$$

The product is to be understood as ordering the indices increasingly unless otherwise specified, but at no point will the ordering in fact matter: we will mostly be interested in C^*-subalgebras of A_θ^n generated by such products, and a reordering simply scales by a modulus-1 complex number.

(2) The *integral kernel* of θ is

$$\theta^\perp := \{\boldsymbol{m} \in \mathbb{Z}^n \ :\ \theta\boldsymbol{m} \in \mathbb{Z}^n\},$$

regarding θ as a linear endomorphism on $\mathbb{R}^n$.

(3) For $F \subseteq [n]$ write

- $\theta|_F$ for the $|F| \times |F|$ matrix consisting of θ-entries in F-labeled rows and columns;
- $h_{\theta,F} := h_{\theta|_F}$, with $h_\bullet$ as in (2-4).

The realization of A_θ^n as a cocycle twist of the group algebra of $\mathbb{Z}^n$ [23, §2.2], coupled with the proof of [22, Lemma 2.3], describing the center of such a twisted group algebra, gives

$$Z(A_\theta^n) = C^*(\boldsymbol{u}^{\theta^\perp}) := C^*(\boldsymbol{u}^{\boldsymbol{m}} \ :\ \boldsymbol{m} \in \theta^\perp) \overset{\theta \in M_n(\mathbb{Q})}{\cong} C^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n). \tag{2-6}$$

By the Azumaya claim we can once more realize A_θ^n as sections of a bundle of dimension-h_θ matrix algebras over $\mathbb{T}^n$ for the h_θ of (2-4).

Remark 2.4. In dealing with vector bundles over higher-dimensional tori, one additional complication not visible for $n = 2$ (or even $n = 3$) is that the cohomological classification recalled in [15, Remark 4.4(1)] no longer goes through: as observed in the introductory remarks (p. 247) of [49] and elaborated in § 3 of the same work, there are nonflat rank-2 vector bundles on complex 2-dimensional tori (*four*-dimensional when regarded as real tori) with vanishing Chern classes.

Such pathologies thus occur as soon as one can reasonably expect them, namely whenever the torus and bundle are large enough for both first Chern classes c_1 and c_2 to come into play, meaning the bundle has rank ≥ 2 and the torus dimension ≥ 4.

Incidentally, the proof of Lemma 2.1 also shows that rational noncommutative tori are decomposable into small-dimensional pieces:

Lemma 2.5. *For a skew-symmetric matrix $\theta \in M_n(\mathbb{Q})$ the corresponding noncommutative torus algebra A^n_θ decomposes as*

$$A^n_\theta \cong C(\mathbb{T}^k) \otimes \bigotimes_{i=1}^{t} A^2_{\theta_i}$$

for rationals $\theta_i \in \mathbb{Q}$ and $n = k + 2t$.

Proof. Immediate from the observation made in passing in the proof of Lemma 2.1 that after a change of (integer) basis θ is a direct sum of a ($k \times k$, say) zero matrix and a ($2t \times 2t$) matrix of the form (2-5). ☐

It also follows that, notwithstanding Remark 2.4 (and [15, Remark 4.4(1)]), the bundles relevant to the Azumaya structure of A^n_θ are relatively well-behaved. To make sense of the statement, recall [41, §§I.2 and I.4] the notion of a *projectively flat* vector bundle over a space X (either plain or *Hermitian*, i.e., with structure group $G = \mathrm{GL}(r)$ [41, §I.4] or $G = U(r)$), respectively. Then:

- Consider the *principal* G-bundle $P \to X$ associated to the vector bundle E in question, via the usual correspondence [37, §18.2, Assertion 2.3].
- The quotient $\overline{P} := P/Z(G)$ by the center of G (scalar matrices, so either $\mathbb{C}^\times$ when $G = \mathrm{GL}(r)$ or $\mathbb{S}^1$ for $G = U(r)$) is then a principal bundle over the *projective* group
$$PG := G/Z(G) = \mathrm{PGL}(r) \text{ or } PU(r);$$
- The original bundle is projectively flat if $\overline{P}$ is flat in the usual sense [41, Proposition I.2.6].

Proposition 2.6. *For skew-symmetric $\theta \in M_n(\mathbb{Q})$ we have*

$$A^n_\theta \cong \mathrm{End}(\mathcal{E}) = \Gamma(\mathcal{E} \otimes \mathcal{E}^*) := (\textit{continuous sections of the bundle } \mathcal{E} \otimes \mathcal{E}^*)$$

for a projectively flat bundle $\mathcal{E}$ on $\mathbb{T}^n \cong \mathrm{Max}(Z(A^n_\theta))$ of rank $\sqrt{h_\theta}$ for $h_\theta =$ (2-4).

In particular, $\mathcal{E} \otimes \mathcal{E}^*$ itself is a flat matrix-algebra bundle.

Proof. That the rank is as claimed has already been noted above, as a consequence of [16, Proposition 7.2]. Lemma 2.5 reduces the problem to noncommutative 2-tori. Indeed, it shows that $\mathcal{E}$ decomposes as an *external* (*or exterior*) *tensor product* [4, §2.6], and external tensor products preserve projective flatness: the latter is definable [41, Proposition I.2.8] as the existence of a connection whose curvature takes scalar values, and there is a simple formula for the curvature of the tensor product of the natural two connections [41, §I.5, (5.15)].

The conclusion now follows from the fact that *all* complex vector bundles on a 2-torus (and indeed a compact orientable surface) are projectively flat: being completely classified by rank and Chern class ([65, Proposition, p. 2] again) a rank-q bundle splits as the sum between a line bundle and a trivial rank-$(q-1)$ bundle, etc.

As for the last claim on flatness, it follows from the general remark that $\mathcal{E}\otimes\mathcal{E}^*$ is flat whenever $\mathcal{E}$ is projectively flat [41, Propositions I.2.9 and/or I.4.23]. □

Remark 2.7. The flatness of the relevant $q\times q$ matrix bundle for $n=2$ and $\theta = p/q$ is plain from its direct construction in the proof of [32, Proposition 12.2] (or [33, §2], on which the latter account is based): the bundle is obtained as the (total space of the) quotient

$$(\mathbb{T}^2\times M_q)/(\mathbb{Z}/q)^2$$

through the diagonal action, where

$$(\mathbb{Z}/q)^2\subset \mathbb{T}^2$$

acts on the torus via translation by the q-torsion subgroup and on M_q as described in loc. cit. (In particular, the two generators act by conjugation by two order-q unitary matrices that commute up to scalars.)

A few reminders will help make sense of the statement of Theorem 2.8, a higher-dimensional generalization of [15, Proposition 4.5].

- The *PI-degree* [19, Definitions A.7.1.8 and B.4.15; 29, §1] of a PI algebra is the largest n for which the polynomial identities of A are among those of M_n. Rank-n^2 Azumaya algebras, for instance, have PI-degree n (as observed just after Definition B.4.15 of [19]).

The slight definition variations one typically encounters in the literature will not make a difference here, so the above will do.

- The *Azumaya locus* of an algebra A consists of those maximal ideals

$$\mathfrak{m}\in \mathrm{Max}(Z := Z(A)) := \text{maximal spectrum of the center } Z$$

for which the *localization* [5, Example (1) post Corollary 3.2] $A_{\mathfrak{m}}$ is Azumaya over $Z_{\mathfrak{m}}$.

There are again minor departures from this setup in the literature (one might, for instance, consider all *prime* ideals with the requisite property [66, §1], impose additional finiteness/primality constraints on the algebra A [11, §III.1.7]).

Set

$$I_Y := \{f|_Y \equiv 0\} \trianglelefteq \mathrm{Cont}_\partial(\mathbb{S}_+^{n-1} \to C(\mathbb{T}_\theta^n)) \overset{(0\text{-}1)}{\cong} C_\theta, \quad Y = \overline{Y} \subseteq \mathbb{S}_+^{n-1} \tag{2-7}$$

and

$$C_\theta|_Y := C_\theta/I_Y, \quad Z_\theta|_Y := Z_\theta/(I_Y \cap Z_\theta). \tag{2-8}$$

For subsets $S \subseteq \mathbb{Z}^n$ and $F \subseteq [n]$ we write

$$S_{\downarrow F} := \{\boldsymbol{s} = (s_i)_{i=1}^n \in S \;:\; s_i = 0,\ \forall i \notin F\};$$

the portion of S *supported on* F, in other words. This applies to the symbol $\theta^\perp_{\downarrow F}$ employed below.

Theorem 2.8. *Let $\theta \in M_n(\mathbb{Q})$ be an $n \times n$ skew-symmetric rational matrix for some fixed $n \geq 2$.*

(1) *The center of the noncommutative sphere algebra $C_\theta := C(\mathbb{S}_\theta^{2n-1})$ is*

$$\begin{aligned} Z_\theta &:= Z(C(\mathbb{S}_\theta^{2n-1})) \\ &\cong \mathrm{Cont}_\partial(\mathbb{S}_+^{n-1} \to Z(C(\mathbb{T}_\theta^n))) \\ &= \{\mathbb{S}_+^{n-1} \xrightarrow[\text{cont.}]{f} C^*(\boldsymbol{u}^{\theta^\perp}) \;:\; f(\mathbb{S}_{+,F}^{n-1}) \in C^*(\boldsymbol{u}^{\theta^\perp_{\downarrow F}}),\ \forall F \subseteq [n]\}. \end{aligned} \tag{2-9}$$

(2) *The spectrum $X_\theta := \mathrm{Max}(Z_\theta)$ branch-covers the classical sphere*

$$\mathbb{S}^{2n-1} \cong \mathrm{Max}\{\mathbb{S}_+^{n-1} \xrightarrow[\text{cont.}]{f} C^*(u_i^{q_i}) : f(\mathbb{S}_{+,F}^{n-1}) \in C^*(u_i^{q_i},\ i \in F), \forall F \subseteq [n]\} \tag{2-10}$$

for

$$q_i := \mathrm{lcm}(\textit{lowest-term denominators of } \theta_{ij},\ j \in [n]).$$

(3) *C_θ embeds into $M_{\sqrt{h_\theta}}(Z_\theta)$ with h_θ as in (2-4) and is not a Z_θ-subalgebra of $M_n(Z_\theta)$ for any smaller n. In particular, C_θ is a PI algebra of PI-degree $\sqrt{h_\theta}$.*

Proof. (1) We have already recalled (2-6) that the centers $Z(C(\mathbb{T}_\theta^n))$ of the noncommutative torus algebras are as claimed, so the conclusion follows from the identification (0-1).

(2) That the right-hand side of (2-10) is indeed a sphere is a simple topology exercise (the classical counterpart to (0-1)), given that the $u_i^{q_i} \in A_\theta^n$ are central and hence the generators of a $C(\mathbb{T}^n)$. The claim is that the map

$$X_\theta \xrightarrow[\text{onto}]{\pi} \mathbb{S}^{2n-1}$$

dualizing the embedding

$$C(\mathbb{S}^{2n-1}) \cong \mathrm{Cont}_{\partial}\big(\mathbb{S}_{+}^{n-1} \to Z_{\downarrow}(A_{\theta}^{n}) := C^{*}(u_i^{q_i})\big) \tag{2-11}$$
$$:= \big\{\mathbb{S}_{+}^{n-1} \xrightarrow[\text{cont.}]{f} Z_{\downarrow}(A_{\theta}^{n}) : f(\mathbb{S}_{+,F}^{n-1}) \in C^{*}(u_i^{q_i}, i \in F), \forall F \subseteq [n]\big\}$$
$$\hookrightarrow \text{(2-9)} =: C(X_{\theta})$$

is a *branched cover* in the sense of [51, §1]: an open surjection of compact Hausdorff spaces, with a finite upper bound on the cardinalities of the fibers $\pi^{-1}(p)$, $p \in \mathbb{S}^{2n-1}$. Surjectivity is not at issue, so it is the two other requirements that require attention. Rather than attack that classical statement directly, we appeal to the theory of *noncommutative branched covers* developed in [51; 7], revolving around the notion of a C^{*} *conditional expectation* [6, Definition II.6.10.1].

We have embeddings

$$Z_{\downarrow}(A_{\theta}^{n}) \hookrightarrow Z(A_{\theta}^{n}) \hookrightarrow C^{*}(\mathbb{Z}^{n}) \cong C^{*}(t_i, i \in [n]), \quad t_i^{q_i} = u_i^{q_i}.$$

The usual [53, Proposition 8.5] (plainly of *finite index* [28, Definition 2]) expectation

$$C^{*}(t_i, i \in [n]) \xrightarrow{E} Z_{\downarrow}(A_{\theta}^{n})$$

restricts to another such (also E) on the intermediate $Z(A_{\theta}^{n})$, compatible with the inclusions of the subgroups generated by t_i, $i \in F$ for $F \subseteq [n]$.

We then have a finite-index

$$\mathrm{Cont}(\mathbb{S}_{+}^{n-1} \to Z(A_{\theta}^{n})) \xrightarrow{E\circ} \mathrm{Cont}(\mathbb{S}_{+}^{n-1} \to Z_{\downarrow}(A_{\theta}^{n})),$$

hence also (because of the noted F-compatibility) an analogous expectation between Cont_{∂} function algebras. These, however, are exactly the two extremes of (2-11), whence the conclusion by [7, Theorem 1.4].

(3) The isomorphism (0-1) and Proposition 2.6 ensure that the irreducible C_{θ}-representations are all at most $\sqrt{h_{\theta}}$-dimensional. That estimate is the best possible: in the language of [6, Definition IV.1.4.1], C_{θ} is sharply $\sqrt{h_{\theta}}$*-subhomogeneous*. To verify this last sharpness assertion, quotient out the ideal $I_p := I_{\{p\}}$ of (2-7) in (0-1) to obtain a surjection

$$C_{\theta} \twoheadrightarrow A_{\theta}^{n} \xrightarrow{\text{Proposition 2.6}} M_{\sqrt{h_{\theta}}}.$$

C_{θ} thus embeds [18, §2.7.3] into a product of matrix algebras M_n, $n \leq \sqrt{h_{\theta}}$ with at least one $M_{\sqrt{h_{\theta}}}$ quotient, and the conclusion follows. ☐

Corollary 2.9. *A noncommutative sphere algebra C_{θ} is Azumaya if and only if it is commutative, i.e., $\theta \in M_n(\mathbb{Z})$.*

Proof. This is immediate from Theorem 2.8 and its proof: $\theta \notin M_n(\mathbb{Z})$ is equivalent to $h_\theta > 1$, and it remains to observe that if that condition holds then C_θ cannot be Azumaya, for on the one hand it has an h_θ-dimensional matrix algebra quotient, while on the other hand $C_\theta|_p$ is abelian for any of the n vertices $p \in \mathbb{S}^{n-1}_+$. □

The branch-covering qualification in Theorem 2.8(2) is not redundant: as soon as $n \geq 3$ the center Z_θ of (2-9) need not be (the function algebra of) a sphere.

Example 2.10. Set

$$\theta := \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \in M_3(\mathbb{Q}).$$

Denoting by u_1, u_2, u_3 the generators of A^3_θ, the sphere $\mathbb{S}^{2n-1} = \mathbb{S}^5$ of Theorem 2.8(2) is

$$\mathbb{S}^5 \cong \operatorname{Max}\operatorname{Cont}_\partial(\mathbb{S}^2_+ \to C^*(u_i^2,\ i \in [3])),$$

while

$$X_\theta \cong \operatorname{Max}\operatorname{Cont}_\partial(\mathbb{S}^2_+ \to Z(A^3_\theta) = C^*(u_i^2,\ i \in [3],\ u_1u_2u_3)).$$

The latter space thus consists of two 5-spheres glued along the 3-dimensional complex

$$\operatorname{Max}\operatorname{Cont}_\partial(\partial\mathbb{S}^2_+ \to Z(A^3_\theta)) \tag{2-12}$$

(one copy of which is embedded in each of the two 5-spheres). Said complex can be described as follows:

- Consider three 3-spheres $\mathbb{S}^3_i$ indexed by $i \in \mathbb{Z}/3$, regarded as total spaces of the *Hopf fibration* [17, §14.1.9] $\mathbb{S}^3 \xrightarrow{\pi} \mathbb{S}^2$.
- In each $\mathbb{S}^3_i$ set $\mathbb{S}^1_{i,\pm} := \pi^{-1}(p_\pm)$ for antipodes $p_\pm \in \mathbb{S}^2$.
- Glue $\mathbb{S}^3_i$ to $\mathbb{S}^3_{i+1}$ (indices modulo 3) by identifying $\mathbb{S}^1_{i,+} \cong \mathbb{S}^1_{i+1,-}$ to obtain (2-12).

X_θ is now easily seen not to be homeomorphic to $\mathbb{S}^5$, and indeed, not a topological manifold: points in $\mathbb{S}^3_0$ but off the exceptional circles $\mathbb{S}^1_{0,\pm}$ have neighborhoods homeomorphic to two copies of $\mathbb{R}^5$ glued along a closed $\mathbb{R}^3$. The removal of that $\mathbb{R}^3$ disconnects such a neighborhood, so it cannot [25, Theorem 1.8.13] be homeomorphic to a Euclidean space.

Remark 2.11. Example 2.10 illustrates a qualitative distinction between quantum tori and spheres: while *equivalent* $\theta, \theta' \in M_n(\mathbb{Q})$, in the sense [9, §IX.1.6] (appropriate for bilinear forms) that

$$\exists(T \in \mathrm{GL}_n(\mathbb{Z})) \quad : \quad \theta' = T\theta T^t,$$

will produce isomorphic torus algebras $A^n_\theta \cong A^n_{\theta'}$, it may well be that $C_\theta \not\cong C_{\theta'}$. In fact, not even the centers of the latter two algebras need be isomorphic.

Indeed, Z_θ will be (the function algebra of) a $(2n-1)$-sphere whenever θ is block-diagonal with blocks (2-5), for in that case, in the notation of Theorem 2.8,

$$Z(A^n_\theta) = C^*(u_i^{q_i},\ i \in [n]).$$

Per Example 2.10, then, with that choice of θ and

$$\theta' := \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = T\theta T^t, \quad T := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

The respective centers $Z_{\theta,\theta'} = Z(C_{\theta,\theta'})$ are nonisomorphic.

To follow up on Corollary 2.9, determining the Azumaya locus of C_θ is now also. We again need some vocabulary and notation preliminaries.

Definition 2.12. Let $\theta \in M_n(\mathbb{Q})$ be a rational skew-symmetric matrix and recall Notation 2.3.

The *jump (sub)complex* JMP_θ of θ (just plain JMP when θ is understood) is

$$\mathrm{JMP} = \mathrm{JMP}_\theta := \{F \in 2^{[n]}\ :\ h_{\theta,F} < h_\theta\}.$$

Identifying *simplicial complexes* and their respective *geometric realizations* [17, §8.1], JMP_θ is a *subcomplex* [17, § 8.1; 62, §3.1] of the simplex $\Delta^{n-1} \cong \mathbb{S}^{n-1}_+$ of (0-2a) with vertex set $[n]$; this justifies the term.

Theorem 2.13. *The Azumaya locus of C_θ is the complement*

$$(X_\theta = \mathrm{Max}(Z_\theta)) \setminus \mathrm{Max}(Z_\theta|_{\mathrm{JMP}}) \tag{2-13}$$

in the notation of (2-8).

Proof. (2-13) contains the Azumaya locus: Suppose $p \in \mathrm{Max}(Z_\theta)$ belongs precisely to the subspaces

$$X_{\theta,\hat{\imath}} := \mathrm{Max}\big(Z_\theta|_{\mathbb{S}^{n-1}_{+,\hat{\imath}}}\big) \subseteq X_\theta, \quad i \in F \in 2^{[n]}.$$

The intersection

$$p \cap C(\mathbb{S}^{n-1}_+) \in \mathrm{Max}\, C(\mathbb{S}^{n-1}_+) \cong \mathbb{S}^{n-1}_+$$

will then belong to the interior of the face $\mathbb{S}^{n-1}_{+,\widehat{F}}$, and (0-1) specializes to

$$C_\theta/\big(C_\theta \cdot (p \cap C(\mathbb{S}^{n-1}_+))\big) \cong C^*(u_i,\ i \in \widehat{F}) \cong A^{|\widehat{F}|}_{\theta|_F} \subseteq A^n_\theta.$$

If $\widehat{F} \in \mathrm{JMP}$ then (by the very definition of the jump complex) this further specializes, at every maximal ideal of the center $Z(A^n_\theta) \cong C(\mathbb{T}^n)$, to a *proper* inclusion

$$C_\theta/(C_\theta \cdot p) \ \cong\ \big(M_{\sqrt{h_{\theta,\widehat{F}}}}\big)^{[(\theta|_{\widehat{F}})^\perp : \theta^\perp_{\downarrow \widehat{F}}]} \ \precneqq\ M_{\sqrt{h_\theta}}$$

of a nonmatrix algebra, with $(\theta|_{\widehat{F}})^{\perp} \le \mathbb{Z}^{|\widehat{F}|}$ regarded as a subgroup of $\mathbb{Z}^n$ by padding vectors with zero entries. C_θ, then, cannot be Azumaya at p, for its localization $(C_\theta)_p$ does not satisfy the polynomial identities of any M_n, $n < \sqrt{h_\theta}$.

(2-13) is contained in the Azumaya locus: Every maximal ideal $p \in$ (2-13) is contained in the interior of some

$$\mathrm{Max}(Z_\theta|_Y) \subset \mathrm{Max}(Z_\theta), \quad Y = \overline{Y} \subseteq \mathbb{S}^{n-1}_{+} \setminus \mathrm{JMP},$$

because the intersection $p \cap C(\mathbb{S}^{n-1}_{+})$ belongs to the latter open set, and it suffices to take for Y a closed neighborhood of that point still contained in that open. The conclusion now follows from the fact that for such Y the algebras $C_\theta|_Y$ are Azumaya:

$$C_\theta|Y \cong \mathrm{End}(\pi^*\mathcal{E}), \quad Y \times \mathbb{T}^n \xrightarrow{\pi:=\text{2nd projection}} \mathbb{T}^n \cong \mathrm{Max}\big(Z(A^n_\theta)\big). \qquad \square$$

3. On and around center-finiteness

Finiteness (i.e., being a finitely generated module) over the center is well-known not to be automatic for PI algebras: [54, Example post Proposition 5.1.3], say, is of a finitely generated algebra satisfying the identities of M_4 but not embeddable in a ring finite over its center. The noncommutative sphere algebras C_θ themselves, as will become apparent, are another case in point.

Subhomogeneous C^*-algebras A (such as C_θ) admit (unital) embeddings

$$A \le M_n^I \cong M_n(C(\beta I)), \qquad \beta(-) := \textit{Stone-Čech compactification } [30, \S 6.5] \text{ of } I, \tag{3-1}$$

as noted in the proof of Theorem 2.8 (essentially by [18, §2.7.3]). But this will also not (generally) suffice to ensure center-finiteness, per Example 3.1: the latter will indeed not even be *topologically* center-finite, in the sense of containing a (norm-topology-)dense module over its center. Naturally, rings sandwiched as $R \le A \le M_n(R)$ for *Noetherian* commutative R are center-finite, so Example 3.1 is in a sense a manifestation of the non-Noetherianness of the continuous-function algebras $C(X)$ involved.

Example 3.1. [52, Example 3.5] (also [7, Example 3.6]) provides a C^*-algebra A equipped with a central morphism $C(X) \to A$ (a $C(X)$-algebra) for

$$X := \Big(X_0 := \bigsqcup_{n\ge 1} \mathbb{CP}^n\Big)^+ := \textit{one-point compactification } [68, \text{Problem 19A}] \text{ of } X_0$$

with fiber $\mathbb{C}$ at the distinguished point $\infty \in X$ and fibers $M_2(\mathbb{C})$ over X_0. Because the associated M_2-bundle over X is by construction not *of finite type* (that is, [36, Definition 3.5.7] trivializable over a finite open cover of X_0), A cannot be topologically finitely generated as a $C(X)$-module by [31, Theorem 1.1] (or [14, Theorem A]) and [7, Proposition 3.7].

Remarks 3.2. (1) Example 3.1 is one instance of the following general setup.

- Consider compact Hausdorff spaces Y_n respectively equipped with (complex) vector d-bundles $\mathcal{E}_n$; we abuse notation and conflate these [37, Remark 15.1.2] with their corresponding principal $U(d)$-bundles.
- Assume the attached set

$$\{\operatorname{ind}_{U(d)}(\mathcal{E}_n)\}_n \subseteq \mathbb{Z}_{\ge 0}$$

of $U(d)$-*indices* [44, Definition 6.2.3] is unbounded: there is no finite upper bound on the cardinality of an open cover of Y_n that will trivialize $\mathcal{E}_n$.

- The $\mathcal{E}_n$ glue to a vector d-bundle over $X_0 := \bigsqcup_n Y_n$.
- Form the bundle $\mathcal{F} := \mathcal{E} \oplus (X_0 \times \mathbb{C})$ (i.e., add a trivial 1-dimensional summand to $\mathcal{E} \twoheadrightarrow X_0$).
- Construct the corresponding endomorphism bundle $\mathcal{F} \otimes \mathcal{F}^*$ (a $(d+1)\times(d+1)$-matrix bundle over X_0).
- And finally, take for the C^*-algebra A (supposed to play the same role as in Example 3.1) the *unitization* [6, §II.1.2.1] $\Gamma_0(\mathcal{F} \otimes \mathcal{F}^*)^+$, the subscript 0 indicating sections on X_0 vanishing at ∞.

(2) For path-connected (compact Hausdorff) Y_n in (1) above the unboundedness

$$\sup_n \dim Y_n, \quad \dim := \textit{covering dimension } [25, \text{Definition } 1.6.7]$$

is an essential feature of this family of examples: per [38, Proposition 2.1] (or as an immediate consequence of [50, Lemma 2.4], say) for any *paracompact* [26, §5.1] path-connected space Y there is an open cover

$$Y = \bigcup_{i=0}^{\dim Y} U_i, \quad U_i \text{ contractible in } Y.$$

The restriction of a bundle on Y will be trivializable [17, Theorem 14.3.3] over every U_i, rendering (1) inoperative.

(3) In reference to (3-1), observe that endomorphism algebras $\operatorname{End}(\mathcal{E}) = \Gamma(\mathcal{E} \otimes \mathcal{E}^*)$ for vector-bundles $\mathcal{E} \twoheadrightarrow X$ over compact Hausdorff X can always be embedded unitally in some $M_n(C(X))$ (same space X: one need not involve the typically nonmetrizable $\beta(\text{discrete } X)$).

To see this, consider a decomposition $\mathcal{E} \oplus \mathcal{F} \cong \mathbf{1}^m$ of a trivial bundle over X, with $\mathbf{1}$ denoting the trivial rank-1 bundle (one such exists by [63, Theorems 1 and 2]).

We then have

$$\mathcal{E}^d \oplus \mathcal{F}^d \cong \mathbf{1}^{md}, \quad d := \operatorname{rank} \mathcal{E},$$

and obtain an embedding

$$\operatorname{End}(\mathcal{E}) \xhookrightarrow{\iota:=\mathrm{id}^d \oplus (\mathrm{id}\otimes \mathrm{ev}_x)} \operatorname{End}(\mathcal{E}^d) \times \operatorname{End}(\mathcal{F}^d) \le \operatorname{End}(\mathbf{1}^{md}) \cong M_{md}(C(X)),$$

where the second component, $\mathrm{id}\otimes \mathrm{ev}_x$, means

- identifying $\mathcal{F}^d$ with $\mathcal{F}\otimes\mathbb{C}^d$ (focusing on the complex case to fix ideas), and
- mapping a global endomorphism $s \in \operatorname{End}(\mathcal{E})$ to the endomorphism of $\mathcal{F}\otimes\mathbb{C}^d$ operating trivially on the $\mathcal{F}$ tensorand and as the restriction $s_x \in \operatorname{End}(\mathbb{C}^d) \cong \operatorname{End}(\mathcal{E}_x)$ of x to the fiber at some fixed $x \in X$.

Returning to the C_θ: while (mostly) not center-finite, they nevertheless exhibit less pathology in that regard than Example 3.1 and the like.

Theorem 3.3. *Let $\theta \in M_n(\mathbb{Q})$ be a skew-symmetric matrix for some $n \in \mathbb{Z}_{\ge 2}$.*

(1) *The quantum sphere C^*-algebra C_θ is center-finite precisely when it is commutative, i.e., $\theta \in M_n(\mathbb{Z})$.*

(2) *On the other hand, C_θ is always topologically center-finite.*

Proof. (1) Theorem 2.8 makes C_θ into a subhomogeneous Z_θ-algebra, which can thus [13, Theorem A] be regarded as a Z_θ-*Hilbert module* [67, Definition 15.1.5]. Finite generation would imply [67, Corollary 15.4.8] that C_θ is also *projective* [1, Definition 5.3.1].

The Hilbert-module-to-Hilbert-bundle correspondence of [34, Scholium 6.7] and Swan's celebrated [59, Theorem 1.6.3] (originally [63, Theorems 1 and 2]) then imply that the (*F*) *Hilbert bundle* [21, p. 9] over $X := \operatorname{Max}(Z_\theta)$ with fibers

$$X \ni p \mapsto C_\theta|_p$$

is locally trivial, so (the sphere being connected) of constant rank. This, in turn, is equivalent (Corollary 2.9) to the commutativity of C_θ.

(2) Once more regarding C_θ, via Theorem 2.8(1), as the section-space $\Gamma(\mathcal{E})$ of a subhomogeneous (F) Banach bundle $\mathcal{E} \twoheadrightarrow \mathbb{S}^{2n-1} \cong \operatorname{Max}(Z_\theta)$, observe that the *strata*

$$X_d := \{p \in X \ : \ \dim \text{fiber } \mathcal{E}_p = \dim C_\theta / C_\theta \cdot p = d\}$$

are all members of the *set ring* [8, I, Definition 1.2.13] generated by the spaces

$$\operatorname{Max}(Z_\theta|_{\mathbb{S}^{n-1}_{+,F}}) \subseteq \operatorname{Max}(Z_\theta) = X, \quad F \in 2^{[n]}.$$

Thus they are all finite unions of path-connected paracompact spaces of finite (covering) dimension, so by [38, Proposition 2.1] they admit respective covers

$$X_d = \bigcup_{j=0}^{N} U_j, \quad U_j = \mathring{U}_j \text{ contractible in } \mathbb{S}_d^{2n-1} \quad (\text{some } N \in \mathbb{Z}_{\ge 0}).$$

The restrictions $\mathcal{E}|_{X_d}$ are thus all trivializable by finite covers, hence topological finite generation via [31, Theorem 1.1]. □

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