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Let $\lambda \in \mathbb{R}$ and let $H = -\frac{1}{2}\partial_x^2 + q\delta_0$ be the one-dimensional Schrödinger equation with a repulsive delta potential. We study the Cauchy problem for the nonlinear equation

$$\begin{cases} i\,\partial_t u(t,x) = Hu(t,x) + \lambda|u(t,x)|^2 u(t,x), & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = u_0(x), \end{cases}$$

in L^p-based spaces. Using the boundedness of wave operators, a characterization of Besov space adapted to H, and the cancellation property of the trilinear form $\mathcal{T}(v_1(\tau), v_2(\tau), v_3(\tau)) = \mathcal{U}(-\tau)\big(\mathcal{U}(-\tau)v_1(\tau)\mathcal{U}(\tau)v_2(\tau)\mathcal{U}(\tau)v_3(\tau)\big)$ with $\mathcal{U}(\tau) = e^{-itH}$, we demonstrate that under the linear transformation $v(t) = \mathcal{U}(-t)u(t)$, the problem is locally well-posed in $L^p(\mathbb{R})$ for $1 < p < 2$ and in the homogeneous Besov space $\dot{B}^s_{p,1}(\mathbb{R})$ with $1 < p < 2$ and $s = 1 - \frac{1}{p}$.

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1. Introduction

This paper is devoted to the Cauchy problem for the cubic nonlinear Schrödinger equation (NLSE) with a repulsive delta potential in dimension one. The equation is introduced as

(1-1)
$$\begin{cases} i\partial_t u(t,x) = Hu(t,x) + \lambda|u(t,x)|^2 u(t,x), & (t,x)\in\mathbb{R}\times\mathbb{R}, \quad \lambda\in\mathbb{R}, \\ u(0,x) = u_0(x). \end{cases}$$

Here H is the delta perturbation of the self-adjoint operator $H_0 = -\frac{1}{2}\partial_x^2$, that is,

$$H = -\tfrac{1}{2}\partial_x^2 + q\delta_0(x),$$

where δ_0 is the Dirac delta measure supported at the origin and $q\in\mathbb{R}$ is the strength of the perturbation. A more detailed introduction to the operator H will be presented in Section 2. For convenience, we will refer to the case $q>0$ as repulsive, and $q<0$ as attractive throughout the paper. We will speak of the focusing nonlinear Schrödinger equation (1-1) when $\lambda<0$ and the defocusing case when $\lambda>0$.

Equation (1-1) is of great interest in both mathematics and physics. It is frequently used to model the interaction between a quantum mechanical wave or particle and an impurity or a localized defect. See Adami, Golse and Teta [2] for the interaction of a one dimensional Bose condensate with an impurity, as well as Bergé [7] for the propagation of an optical wave pulse in an optical fiber in the presence of defects or junctions. A different occurrence of (1-1) is found in the study of soliton-soliton collisions within the framework of the coupled NLSEs, when considering the interaction between a narrow soliton and a wider one, which are governed by different equations, a suitable limiting process reduces the system of two coupled NLSEs to a single equation, in which the narrow soliton in the mate mode is effectively represented by a delta function; see Cao and Malomed [13].

In mathematical physics, Hilbert spaces provide a rigorous and versatile framework for describing the behavior of physical systems. For (1-1), Hilbert spaces such as L^2-spaces, L^2-based Sobolev spaces $W^{s,2}$ and weighted L^2-spaces are used to study local and global well-posedness, as well as the long-time behavior of the solutions. A briefly review of the relevant details will be provided later.

Much less is known about the properties of the solution to (1-1), when the initial data u_0 does not belong to L^2-based spaces. Mathematically, the L^p-based spaces ($p\neq 2$) would be popular and interesting to work with. In other words, from the viewpoint of mathematics, it is of great interest to consider the local and global well-posedness for evolution equations like (1-1) in L^p-based spaces, with the initial data u_0 also residing in these spaces. But this is not the whole story since the study of NLSE in L^p-based spaces is not merely a mathematical generalization. If we go back to the works of NLSEs with the initial data that is periodic, or localized perturbation of a periodic function, or quasi-periodic, we find that the initial data is usually not in L^2, but rather in L^∞ in many cases, see [9; 10; 12; 17; 26; 54; 64] and so on. For example, Dodson, Soffer and Spencer [26] and Oh [64] considered the local well-posedness of NLSE in dimension one with quasi-periodic initial data, they allow the initial data to be of the form

$$u_0(x) = \cos(x) + \cos(\sqrt{2}x),$$

which is obviously in $L^\infty \setminus L^2$. We note that the above models have their origins in mathematical physics. For instance, the periodic case appears naturally in the context of periodic signals propagating through fibers, while localized perturbations may be related to noise. Moreover, as mentioned by Vargas and Vega [77] that the LIA model for the vortex filament can be reduced to equation (1-1), where the L^2 theory no longer works, it leads to seek solution of (1-1) with infinite L^2 data. Further studies on NLSE in L^p-based spaces ($p \neq 2$) will be introduced in next section.

1.1. ***Background.*** This paper is concerned with the well-posedness of equation (1-1) with the initial data residing in either $L^p(\mathbb{R})$ or $\dot{B}^s_{p,r}(\mathbb{R})$. In this section, we will briefly review the known results for (1-1), and our attention is primarily restricted to the one-dimensional case.

***The case q* = 0.** Let $H_0 = -\frac{1}{2}\partial_x^2$. We are concerned with the following cubic NLSE in dimension one:

$$i\partial_t u(t,x) = H_0 u(t,x) + \lambda |u(t,x)|^2 u(t,x), \quad u(0,x) = u_0(x). \tag{1-2}$$

It is well known that the Cauchy problem (1-2) is globally well-posed in L^2 and $W^{1,2}$; see, e.g., Kato [55] and Tsutsumi [75]. An elementary problem that needs to be considered is the long-time behavior of global solutions. Notice that the cubic nonlinearity in dimension one is the borderline for short range and long range problems. The main reason lies in the fact that the free wave $e^{-itH_0}u_0$ decays at the rate $t^{-\frac{1}{2}}$, which causes the cubic nonlinearity $|u|^2u$ to behave like $t^{-1}u$, a term that is not integrable in t. In fact, it was proved by Barab [6] and Tsutsumi and Yajima [76] that the scattering in L^2 only occurs for the trivial zero solution. Consequently, a phase correction is required, which is known as modified scattering. When $\lambda > 0$, Deift and Zhou [23] obtained the long-time asymptotic behavior for all solutions based on the fact that equation (1-2) is completely integrable. The sign of λ does not make any difference if one considers the Cauchy problem (1-2) with initial data being small in L^2-based weighted Sobolev spaces. One can see Ozawa [67] for the existence of modified wave operators, as well as Hayashi and Naumkin [37], Lindblad and Soffer [59], Kato and Pusateri [57] and Ifrim and Tataru [52] the results for modified scattering.

The theory of equation (1-2) in L^p-based spaces has been investigated in the literature. In the following, we will omit specifying the spatial dimension in this circumstance, as some of the known results hold in the higher-dimensional setting. By using the Duhamel formula, the solution to (1-2) can be expressed as

$$u(t) = \mathcal{U}_0(t)u_0 - i\lambda \int_0^t \mathcal{U}_0(t-s)|u(s)|^2 u(s)\, ds, \tag{1-3}$$

where $\mathcal{U}_0(t)$ denotes the linear propagator e^{-itH_0}, and we will use the simplified notation $u(t)$ to denote $u(t,x)$ from time to time, whenever the spatial variable x is clear from the context. It is well-known from Hörmander [44] that if $u_0 \in L^p$, $\mathcal{U}_0(t)u_0$ is not necessarily in L^p unless $p = 2$, which is a basic obstacle to study equation the in L^p-based spaces. The linear propagator $\mathcal{U}_0(t)$ has the factorization

$$\mathcal{U}_0(t) = M(t)D(t)\mathcal{F}M(t), \tag{1-4}$$

where $M(t)f = e^{i\frac{|x|^2}{2t}} f(x)$ is multiplication, $D(t)f = (2\pi i t)^{-\frac{1}{2}} f\left(\frac{x}{2\pi i t}\right)$ is dilation and $\mathcal{F}$ is the Fourier transform. This implies that the boundedness properties of $\mathcal{U}_0(t)$ are analogous to those of the Fourier transform $\mathcal{F}$, which is bounded from L^p to $L^{p'}$ with $p \in [1,2]$ by Young's inequality, where p' is the conjugate of p. This indicates that one cannot expect (1-3) to be well-posed in L^p-based spaces with initial data belonging to the same spaces.

Zhou [82] found a way around this by considering the well-posedness of the integral equation

$$v(t) = u_0 - i\lambda \int_0^t \mathcal{U}_0(-s)|\mathcal{U}_0(s)v(s)|^2 \mathcal{U}_0(s)v(s)\, ds \tag{1-5}$$

in L^p and Besov space $\dot{B}^s_{p,r}$ with $1 < p < 2$, where $v(t) = \mathcal{U}_0(-t)u(t)$. A key observation lies in the cancellation of the multilinear form

$$\mathcal{T}(v_1, v_2, v_3; s) = \mathcal{U}_0(-s)\big(\mathcal{U}_0(s)v_1(s)\mathcal{U}_0(-s)\bar{v}_2(s)\mathcal{U}_0(s)v_3(s)\big),$$

which is arisen in the Duhamel term of (1-5). The cancellation for $\mathcal{T}$ leads to the $L^1(\mathbb{R}^d)$ estimate

$$\|\mathcal{T}(v_1, v_2, v_3; s)\|_{L^1(\mathbb{R}^d)} \lesssim s^{-d} \|v_1\|_{L^1(\mathbb{R}^d)} \|v_2\|_{L^1(\mathbb{R}^d)} \|v_3\|_{L^1(\mathbb{R}^d)} \tag{1-6}$$

which combined with a frequency localized L^2 estimate implies the multilinear estimates for $\mathcal{T}$ in both L^p and Besov spaces, based on these estimates, the author established the local well-posedness for $v(t)$, the solution of (1-5) in L^p and Besov spaces. The work [82] gave an efficient way to solve the Cauchy problem in L^p-based spaces by considering the integral equation for $\mathcal{U}_0(-t)u(t)$, even if only local theory is understood. Such an idea has been extended to study other NLSEs in non L^2-based spaces. Hoshino and Hyakuna [45] studied the local well-posedness of

NLSE in Sobolev spaces $W^{s,p}$ and Besov spaces $\dot{B}^s_{p,r}$ with Hartree type nonlinearity $(|x|^{-\gamma} * |u|^2)u$, the authors considered the integral equation similarly to (1-5) for $\mathcal{U}_0(-t)u(t)$ and the key point in their work is the new multilinear estimates for

$$\mathcal{U}_0(-s)\big[\big(|\cdot|^{-\gamma} * \mathcal{U}_0(s)v_1(s)\mathcal{U}_0(-s)\bar{v}_2(s)\big)\mathcal{U}_0(s)v_3(s)\big]$$

in Sobolev spaces, which compared to [82], this result is obtained by using the factorization for $\mathcal{U}_0(s)$ (see (1-4)) instead. Hyakuna [47] obtained the local and global well-posedness for Hartree type NLSE in $L^2 \cap L^p$ by using a similar approach, along with the blowup alternative argument. The same author [49] studied the well-posedness of the cubic NLSE in L^p for $p > 2$, establishing the multilinear estimates for $\mathcal{T}$ defined by (1-6) in terms of the factorization for $\mathcal{U}_0(s)$. Finally, we mention that in Hyakuna [46] and [48], the solvability of NLSEs for $\mathcal{U}_0(-t)u(t)$ was considered, with general nonlinearity $N(u)$ and Hartree type instead, respectively. Nevertheless, the author did not invoke the integral equation (1-5) nor exploit the cancellation property inherent to the mult-linear operator.

Notice that the work of [82] concerns the integral equation (1-5) for $v(t) = \mathcal{U}_0(-t)u(t)$ in L^p-based spaces. One can also study the solution $u(t)$ itself straightforwardly by using Strichartz estimates for initial data in non L^2-based spaces. The existing results in this direction are primarily based on the homogeneous Strichartz-type estimates

$$\|\mathcal{U}_0(t)f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^p} \quad \text{and} \quad \|\mathcal{U}_0(t)f\|_{L^q_t L^r_x} \lesssim \|f\|_{\widehat{L}^p},$$

where $\hat{f}$ is the Fourier transform of f and $\widehat{L}^p$ is defined by

$$\widehat{L}^p := \{f : \hat{f} \in L^{p'}\},$$

it follows from the Hausdorff–Young inequality that $L^p \subset \widehat{L}^p$ if $p \le 2$ and $\widehat{L}^p \subset L^p$ if $p \ge 2$. The triplet (q, r, p) in above estimates, called a p-admissible pair in dimension d, from the scaling point of view satisfies

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{p}$$

and some further restrictions. The generalized and variant Strichartz estimates for the Duhamel term

$$\int_0^t \mathcal{U}_0(t-s)N(u)(s)\,ds$$

are also needed; see [36; 45; 48; 56; 78]. The Strichartz estimates have been extensively applied in the study of well-posedness and long-time behavior of NLSE in L^2-based spaces, we refer to the books by Cazenave [15] and Tao [74] to see the basic philosophy. For the local and global well-posedness of (1-2) in non-L^2-based spaces, Vargas and Vega [77] established local well-posedness in the space $L^3_{t,\text{loc}}L^6_x$

in one spatial dimension, for initial data belonging to certain function spaces such that $\|\mathcal{U}_0(t)u_0\|_{L_t^3 L_x^6(I\times\mathbb{R})} < \infty$. They also obtained global solutions by employing a splitting argument, originally developed by Bourgain [11], which was used to establish global existence for large data in H^s-critical NLSEs with s near one. Such an idea has been generalized to the study of NLSEs with different nonlinearities $N(u)$ and function spaces X for the initial data. Here $N(u)$ could be one of

$$|u|^\alpha u, \quad (|\cdot|^{-\gamma} * |u|^2)u, \quad \partial_x(|u|^2 u),$$

and the function space X can be taken as any of

$$L^{p,\infty},\ \widehat{L}^p,\ \widehat{W}^{s,p},\ M^s_{p,\sigma},\ \dot{W}^{1,2}\cap L^p \ \text{ and the Wiener algebra } W.$$

($\widehat{W}^{s,p} = \{f : (1+|\xi|^2)^{\frac{s}{2}}\hat{f} \in L^{p'}\}$ and $M^s_{p,\sigma}$ are modulation spaces.) One can see related results in [14; 16; 17; 19; 25; 36; 46; 50; 51; 60; 70]. We note that one advantage of employing Strichartz estimates in combination with the splitting method is that it allows for the development of a global theory for NLSEs in L^p-based spaces, whereas the approach in [82] is primarily suited for establishing local results. Finally, there are different approaches to study the global well-posedness for NLSE (1-2) in L^p-based spaces; see Dodson, Soffer and Spencer [26; 27] for results in L^p with $2 < p \le \infty$, where use is made of Strichartz estimates and the energy method, as well as Ru and Chen [68] and Wang and Hudzik [79] for results in modulation spaces, where the authors applied the blow-up criterion and the smallness condition on the initial data, respectively.

The case $q \neq 0$. Returning to equation (1-1), the well-posedness of the solution in L^2-based spaces was studied by Adami and Noja [1] and Fukuizumu, Ohta and Ozawa [32]. The asymptotic behavior of solutions, the scattering theory in H^1, and the blow-up phenomenon have been studied by Banica and Visciglia [5] and Tang and Xu [73]. One can also see Segata [71] for the construction of the modified wave operator for (1-2) with $q > 0$. Conversely, the modified scattering was given by Masaki, Murphy and Segata [61] and Chen and Pusateri [18]. Results of nonlinear dynamics around solitons can be found in [20; 31; 32; 34; 42; 43; 41; 53; 58; 62; 63; 65; 72].

As for the results in L^p-based spaces, to the best of our knowledge, Angulo Pava and Ferreira [4] considered the local well-posedness of NLSE with double-well potential and a general nonlinearity of the form $|u|^{\rho-1}u$ in the spaces $L^{p,\infty}$. Obviously, the result in [4] can be applied to (1-1) by selecting the locations of two wells at zero and setting $\rho = 3$.

1.2. *The main results.* Before presenting our main results, concerning the local and global well-posedness of (1-1) in $L^p(\mathbb{R})$ and $\dot{B}^s_{p,r}(\mathbb{R})$ spaces, we recall relevant definitions and notation.

Denote by $\mathcal{U}(t)=e^{-itH}$ the linear propagator of $H=H_0+q\delta_0$ with $H_0=-\frac{1}{2}\partial_x^2$. It follows from the Duhamel formula that the solution u of (1-1) can be written as

$$u(t)=\mathcal{U}(t)u_0-i\lambda\int_0^t \mathcal{U}(t-s)|u(s)|^2u(s)\,ds. \tag{1-7}$$

We introduce the linear transformation

$$v(t)=\mathcal{U}(-t)u(t), \quad \text{or equivalently,} \quad u(t)=\mathcal{U}(t)v(t).$$

By combining these two identities with (1-7), we have

$$v(t)=u_0-i\lambda\int_0^t \mathcal{U}(-s)\big(\mathcal{U}(s)v(s)\,\overline{\mathcal{U}(s)v(s)}\,\mathcal{U}(s)v(s)\big)\,ds, \tag{1-8}$$

where we use the fact that $\overline{\mathcal{U}(t)}=\mathcal{U}(-t)$. Let us start with the local well-posedness of the integral equation (1-8) in the Besov space $\dot{B}^s_{p,r}(\mathbb{R})$.

Theorem 1.1. *Assume that $u_0\in\dot{B}^s_{p,1}(\mathbb{R})$ with $1<p<2$ and $s=1-\frac{1}{p}$. There exists a time T depending only on $\|u_0\|_{\dot{B}^s_{p,1}(\mathbb{R})}$ and such that the integral equation* (1-8) *has a unique solution*

$$v\in C\big([0,T),\dot{B}^s_{p,1}(\mathbb{R})\big)$$

satisfying, for all $t\in[0,T)$,

$$\|v(t)\|_{\dot{B}^s_{p,1}(\mathbb{R})}\le 2\|u_0\|_{\dot{B}^s_{p,1}(\mathbb{R})}.$$

If v_1 and v_2 are two solutions for (1-8) *with initial data u_{01} and u_{02}, then*

$$\|v_1(t)-v_2(t)\|_{\dot{B}^s_{p,1}(\mathbb{R})}\le 2\,\|u_{01}-u_{02}\|_{\dot{B}^s_{p,1}(\mathbb{R})}.$$

As mentioned previously, by factorization (1-4), $\mathcal{U}_0(t)$ is bounded from $L^p(\mathbb{R})$ to $L^{p'}(\mathbb{R})$ with $p\in[1,2]$ by Young's inequality, which combined with the identity

$$u(t)=\mathcal{U}_0(t)\mathcal{U}_0(-t)u(t)=\mathcal{U}_0(t)v(t),$$

implies that the solution $u(t)$ to the original equation (1-2) with $q=0$ stays in $L^{p'}(\mathbb{R})$ when $t\neq 0$, see Hyakuna [47] the results for Hartree type nonlinearity. In what follows, we establish a similar result for (1-1) within the framework of Besov spaces.

Corollary 1.2. *Let $v(t)=\mathcal{U}(-t)u(t)$ given in Theorem 1.1 be the solution to the integral equation* (1-8)*. Then*

$$u\in C\big((0,T);\dot{B}^s_{p',1}(\mathbb{R})\big).$$

Next, in the case where the initial data lies in $L^p(\mathbb{R})$, an appropriate space-time norm is needed to control the evolution of the solution $v(t)$. The following result provides such an estimate.

Theorem 1.3. *Suppose that $u_0 \in L^p(\mathbb{R})$ for $1 < p < 2$. Then there exists T depending only on $\|u_0\|_{L^p(\mathbb{R})}$ such that the integral equation* (1-8) *has a unique solution $v \in C\big([0,T), L^p(\mathbb{R})\big)$ satisfying*

$$\|v(t)\|_{L^p(\mathbb{R})} \le C\|u_0\|_{L^p(\mathbb{R})}, \quad \text{and} \quad \|t^{\frac{2}{p}-1}\partial_t v(t)\|_{L_t^{p'}L_x^p([0,T)\times\mathbb{R})} \le C\|u_0\|^3_{L^p(\mathbb{R})}$$

for all $t \in [0,T)$. If $v_1(t)$ and $v_2(t)$ are solutions for (1-8) *with initial data u_{01} and u_{02}, then, for all $t \in [0,T)$,*

$$\|v_1(t) - v_2(t)\|_{L^p(\mathbb{R})} \le C\|u_{01} - u_{02}\|_{L^p(\mathbb{R})},$$

where C is a absolute positive constant and $\frac{1}{p} + \frac{1}{p'} = 1$.

Corollary 1.4. *Let $v(t) = \mathcal{U}(-t)u(t)$ given in Theorem 1.3 be the solution to the integral equation* (1-8)*. Then*

$$u \in C\big((0,T); L^{p'}(\mathbb{R})\big).$$

Outline of proofs. The starting point in the proofs of Theorems 1.1 and 1.3 is the integral equation (1-8), where the central task is to analyze the associated trilinear form

$$\mathcal{T}(v_1(\tau), v_2(\tau), v_3(\tau)) = \mathcal{U}(-\tau)\big(\mathcal{U}(-\tau)v_1(\tau)\mathcal{U}(\tau)v_2(\tau)\mathcal{U}(\tau)v_3(\tau)\big)$$

in the Besov space $\dot{B}^s_{p,1}(\mathbb{R})$, if the well-posedness of equation (1-8) in $\dot{B}^s_{p,1}(\mathbb{R})$ is considered. If $\mathcal{U}(t)$ is replaced by $\mathcal{U}_0(t)$ in this form, by exploiting a remarkable cancellation property, one obtains an estimate of $\mathcal{T}$ in L^1; see (1-6). The case of general $\mathcal{U}$ presents two obstacles: the cancellation for the trilinear form and the noncommutativity of $\mathcal{U}(t)$ with the classic Littlewood–Paley projection $\varphi_j(\sqrt{2H_0})$ (see Section 2.1 for the definition).

The cancellation for the trilinear form $\mathcal{T}(v_1(\tau), v_2(\tau), v_3(\tau))$ is based on the explicit formula for the propagator $\mathcal{U}(t)$. In fact, in Proposition 2.2, we obtain the formula for the general multiplier $m(\sqrt{2H})$ in terms of the distorted Fourier transform $\mathcal{F}_q$ (see (2-10) for the definition), which can be applied to $\mathcal{U}(t)$ to obtain

$$e^{-itH}f(x) = \chi_+(x)K_t * \mathcal{L}_+(f)(x) + \chi_-(x)K_t * \mathcal{L}_-(f)(x)$$

where $K_t(x) = e^{-\frac{i\pi}{4}}(2\pi t)^{-1/2}e^{\frac{i|x|^2}{2t}}$. Notice that $\chi_\pm(x)$ are nonsmooth cutoff functions, the Hilbert transform shows up in the expansion of $\mathcal{T}(v_1(\tau), v_2(\tau), v_3(\tau))$. Thus we can't obtain the favorable L^1 estimate just like (1-6). To proceed, we will work in the Hardy space $H^1(\mathbb{R})$ instead of $L^1(\mathbb{R})$ and then estimate $\|\mathcal{T}\|_{L^1}$ via the H^1-norm. The details for cancellation have been established in Section 4.

The noncommutativity between $\mathcal{U}(t)$ and $\varphi_j(\sqrt{2H_0})$ can be circumvented by constructing the Besov spaces $\dot{B}^{s,H}_{p,r}(\mathbb{R})$ associated to the Schrödinger operator H.

The space $\dot{B}^{s,H}_{p,r}(\mathbb{R})$ is defined as the completion of the Schwartz space under

$$\|f\|_{\dot{B}^{s,H}_{p,r}(\mathbb{R})} = \Big(\sum_{j\in\mathbb{Z}} \big(2^{js}\|\varphi_j(\sqrt{2H})f\|_{L^p(\mathbb{R})}\big)^r\Big)^{\frac{1}{r}},$$

where $\varphi_j(\sqrt{2H})$ is the Littlewood–Paley projection associated with H and can be defined in terms of the distorted Fourier transform $\mathcal{F}_q$ (see Proposition 2.2). The advantage of using $\dot{B}^{s,H}_{p,r}(\mathbb{R})$ is that one has the commutation relation

$$\mathcal{U}(t)\varphi_j(\sqrt{2H}) = \varphi_j(\sqrt{2H})\mathcal{U}(t),$$

which is crucial for the estimate of the trilinear form $\mathcal{T}$ in $\dot{B}^{s,H}_{p,r}(\mathbb{R})$. The next step is to establish the equivalence between the two types Besov spaces under consideration. That is, we show that

$$\dot{B}^{s,H}_{p,r}(\mathbb{R}) = \dot{B}^{s}_{p,r}(\mathbb{R})$$

for some s, p and r. Similar results have been obtained by Georgieva and Giammetta [33] in the case $r = 2$, and by Cuccagna, Visciglia and Georgiev [21], where Sobolev spaces were considered instead of Besov spaces, for Schrödinger operators $-\Delta + V$ under different assumptions on the potential V. To prove the equivalence, in Section 3.1, we introduce wave operators $W_\pm$, which are defined by

$$W_\pm f = s - \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}f,$$

and investigate the boundedness on various function spaces. And then we use the results (see Proposition 3.2, Proposition 3.4 and Corollary 3.5) that the wave operator and its conjugate are bounded in $L^p(\mathbb{R})$ and $\dot{B}^{s}_{p,r}(\mathbb{R})$ to get the embedding

$$\dot{B}^{s}_{p,r}(\mathbb{R}) \subset \dot{B}^{s,H}_{p,r}(\mathbb{R}).$$

To establish inverse inclusion, we make use of an estimate of the type

$$\big\|\varphi_j(\sqrt{2H_0})\varphi_k(\sqrt{2H})f\big\|_{L^p(\mathbb{R})} \lesssim 2^{-|j-k|s}\|f\|_{L^p(\mathbb{R})}$$

for some $s > 0$ and certain restrictions on j, k. We note that since the potential in our setting is singular, we can't apply the perturbation argument used in [21; 33].

Outline. Section 2 introduces some notions, including the distorted Fourier transform associated with H, the explicit formulas for the multipliers $m(\sqrt{2H})$ and some linear estimates for the propagator e^{-itH}. Section 3 is devoted to the equivalence between Besov spaces associated to H and the classical Besov spaces, where the boundedness of wave operators will be involved. In Section 4, we explore the cancellation property for the trilinear form $\mathcal{T}$ and prove Theorems 1.1 and 1.3.

2. Preliminaries

2.1. ***Notation.*** In this section, we introduce notation for several function spaces. We use $A \lesssim B$ to mean that $A \le CB$ for some $C > 0$ that changes from line to line, independent of the main parameters. For $r \in [1, \infty]$, we let $r' \in [1, \infty]$ denote the Hölder dual of r, given by $\frac{1}{r} + \frac{1}{r'} = 1$. By χ_+ and χ_- we mean the characteristic functions of the intervals $[0, +\infty)$ and $(-\infty, 0)$, respectively.

The Hilbert transform of a function $f \in \mathcal{S}(\mathbb{R})$ is defined by

$$\mathcal{H}f(x) = p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy, \quad x \in \mathbb{R},$$

where $p.v.$ denotes the principal value, which is equivalent to the definition in terms of a or else "multipliers" in the plural Fourier multiplier,

$$\mathcal{H}f(x) = \mathcal{F}^{-1}\big(-i \operatorname{sgn} \xi \ \hat{f}(\xi)\big)(x), \quad f \in \mathcal{S}(\mathbb{R}).$$

As usual, $L^p(\mathbb{R})$ denotes the space of measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\|f\|_{L^p(\mathbb{R})} = \Big(\int_{\mathbb{R}} |f(x)|^p dx\Big)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty,$$

or, for $p = \infty$,

$$\|f\|_{L^\infty(\mathbb{R})} := \operatorname{ess\,sup}_{x\in\mathbb{R}}|f(x)| < \infty.$$

Let $I \subset \mathbb{R}$ be an interval and $1 \le p, r \le \infty$. The space $L^p_t L^r_x(I \times \mathbb{R})$ contains all measurable functions u on $I \times \mathbb{R}$ with $\|u\|_{L^p_t L^r_x(I\times\mathbb{R})} = \big\| \|u(t)\|_{L^r_x(\mathbb{R})} \big\|_{L^p_t(I)} < \infty$. We use $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ to denote the Schwartz space and its dual.

Denote by $\mathcal{F}$ (resp. $\hat{}$) and $\mathcal{F}^{-1}$ (resp. $\check{}$) the standard Fourier transform and its inverse. That is,

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi} f(x)\, dx,$$

$$\mathcal{F}^{-1}(f)(x) = \check{f}(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi} f(\xi)\, d\xi.$$

For a given real-valued measurable function m on the real line, we define the Fourier multiplier operator $m(i\nabla) = \mathcal{F}^{-1}m(\xi)\mathcal{F}$. Let φ be a smooth even function on $\mathbb{R}$ such that

$$\operatorname{supp}\varphi \subset \big\{\xi \,\big|\, \tfrac{1}{2} \le |\xi| \le 2\big\} \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \varphi_j(\xi) := \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad (\xi \ne 0).$$

The classic Littlewood–Paley projections $\varphi_j(\sqrt{2H_0})$ $(j \in \mathbb{Z})$ associated to $H_0 = -\frac{1}{2}\partial_x^2$ are defined in terms of Fourier multipliers by

$$\varphi_j(\sqrt{2H_0})f(x) = \mathcal{F}^{-1}\big(\varphi_j(\xi)\hat{f}(\xi)\big)(x). \tag{2-1}$$

The classic homogeneous Besov spaces $\dot{B}^s_{p,r}(\mathbb{R})$ for $1 \le p, r \le \infty$ can be defined as the closure of $\mathcal{S}(\mathbb{R})$ functions f with respect to the norm

$$\|f\|_{\dot{B}^s_{p,r}(\mathbb{R})} = \Big(\sum_{j\in\mathbb{Z}} 2^{jsr} \|\varphi_j(\sqrt{2H_0})f\|^r_{L^p(\mathbb{R})}\Big)^{1/r}.$$

We use $H^1(\mathbb{R})$ to denote the Hardy space. The space $H^1(\mathbb{R})$ is a proper subspace of $L^1(\mathbb{R})$, which is usually used as a substitution of $L^1(\mathbb{R})$ when one considers the boundedness of operators at the endpoint. One way to define the Hardy space $H^1(\mathbb{R})$ is as

$$H^1(\mathbb{R}) = \{f \in L^1(\mathbb{R}) \mid \mathcal{H}f \in L^1(\mathbb{R})\}.$$

2.2. ***The Schrödinger operator with delta potential.*** The Hamiltonian

$$H = -\tfrac{1}{2}\partial_x^2 + q\delta_0(x) \tag{2-2}$$

associated with the linear Schrödinger equation with a delta potential describes a δ-interaction of strength q centered at $x = 0$. This kind of interaction, also known as Fermi pseudopotential, gives rise to a variety of models that are widely used in contemporary physics. Throughout this paper, we will restrict our attention to the case of a repulsive delta potential, that is, $q > 0$ in (2-2).

The domain of H is given by

$$\mathcal{D}(H) = \{f \in H^1(\mathbb{R}) \cap H^2(\mathbb{R}\backslash\{0\}) : \partial_x f(0+) - \partial_x f(0-) = 2qf(0)\},$$

where $\pm$ denote limits from the right or left, and $H = -\frac{1}{2}\partial_x^2$ on its domain. Then H is a self-adjoint operator on $L^2(\mathbb{R})$, the Stone theorem yields that it generates a strongly continuous L^2-unitary group e^{-itH} for $t \in \mathbb{R}$. The spectrum of H is well understood, it is known that the essential spectrum and the absolutely continuous spectrum are identical, and $\sigma_{ess}(H) = \sigma_{ac}(H) = [0, +\infty)$, $\sigma_{sc}(H) = \varnothing$. The eigenvalue of H depends on the sign of q. In the repulsive case $q > 0$, the operator H has no eigenvalues, whereas in the attractive case $q < 0$, H has only one simple negative eigenvalue $-\frac{1}{2}q^2$, see for example Albeverio et al. [3] for more details.

2.3. ***The distorted Fourier transform.*** The *Jost functions* associated with our problem are the solutions $f_\pm = f_\pm(x, \xi)$ to the equation

$$Hf = \tfrac{1}{2}\xi^2 f \tag{2-3}$$

with boundary conditions

$$f_\pm(x, \xi) - e^{\pm ix\xi} \to 0 \quad \text{as } x \to \pm\infty.$$

Their explicit expressions are

$$f_+(x,\xi)=\begin{cases} e^{ix\xi} & \text{if } x\ge 0,\\ \frac{1}{t_q(\xi)}e^{ix\xi}+\frac{r_q(\xi)}{t_q(\xi)}e^{-ix\xi} & \text{if } x<0,\end{cases} \tag{2-4}$$

$$f_-(x,\xi)=\begin{cases} \frac{1}{t_q(\xi)}e^{-ix\xi}+\frac{r_q(\xi)}{t_q(\xi)}e^{ix\xi} & \text{if } x\ge 0,\\ e^{-ix\xi} & \text{if } x<0,\end{cases} \tag{2-5}$$

where the so-called transmission and reflection coefficients $t_q(\xi)$ and $r_q(\xi)$, $\xi\in\mathbb{R}$, are given by

$$t_q(\xi)=\frac{i\xi}{i\xi-q} \quad\text{and}\quad r_q(\xi)=\frac{q}{i\xi-q}.$$

The transmission and reflection coefficients enjoy the identities

$$\overline{t_q(\xi)}=t_q(-\xi),\quad \overline{r_q(\xi)}=r_q(-\xi),\ \ t_q(\xi)=r_q(\xi)+1, \tag{2-6}$$

$$|t_q(\xi)|^2+|r_q(\xi)|^2=1,\ \ t_q(\xi)\overline{r_q(\xi)}+\overline{t_q(\xi)}r_q(\xi)=0. \tag{2-7}$$

(Jost functions also arise when H is associated with more general potentials than the delta of our study. In this generality they may not be easy to write out explicitly, but equation (2-7) still holds.)

The distorted Fourier transform associated with H can be constructed via Jost functions, which serve as generalized eigenfunctions of H. To do so, we define the distorted plane wave by Jost functions:

$$e_+(x,\xi)=t_q(\xi)f_+(x,\xi),\quad e_-(x,\xi)=t_q(\xi)f_-(x,\xi). \tag{2-8}$$

Define

$$\Psi(x,\xi)=\begin{cases} (2\pi)^{-\frac12}e_+(x,\xi) & \text{if } \xi\ge 0,\\ (2\pi)^{-\frac12}e_-(x,-\xi) & \text{if } \xi<0.\end{cases} \tag{2-9}$$

Note that $\Psi(x,0)=0$ and $\Psi(x,\cdot)$ is continuous at $k=0$ provided $q>0$. The distorted Fourier transform $\mathcal{F}_q$ and its inverse $\mathcal{F}_q^{-1}$ associated with H are defined by

$$\mathcal{F}_q(f)(\xi)=\int_{\mathbb{R}}\overline{\Psi(x,\xi)}f(x)\,dx,\ \ \mathcal{F}_q^{-1}(f)(x)=\int_{\mathbb{R}}\Psi(x,\xi)f(\xi)\,d\xi. \tag{2-10}$$

Just as the classic Fourier transform can diagonalize H_0 by $H_0=\mathcal{F}^{-1}\big(\frac12\xi^2\big)\mathcal{F}$, the distorted Fourier can be used to diagonalize the Schrödinger operator H by $H=\mathcal{F}_q^{-1}\big(\frac12\xi^2\big)\mathcal{F}_q$. Consequently, the multipliers $m(2H)=\mathcal{F}_q^{-1}m(\xi^2)\mathcal{F}_q$ are well defined for some bounded measurable functions m. Below, we will derive the explicit expression for $m(\sqrt{2H})$.

The distorted Fourier transform and its inverse maintain a close structural resemblance to the classical Fourier transform. In fact, it follows from the results in Segata [71] that

$$\mathcal{F}_q(\phi)(\xi)=\begin{cases}\mathcal{F}(\phi)(\xi)+r_q(\xi)\mathcal{F}(\chi_+\phi)(-\xi)+r_q(\xi)\mathcal{F}(\chi_-\phi)(\xi) & \text{if } \xi\ge 0,\\ \mathcal{F}(\phi)(\xi)+\overline{r_q(\xi)}\mathcal{F}(\chi_-\phi)(-\xi)+\overline{r_q(\xi)}\mathcal{F}(\chi_+\phi)(\xi) & \text{if } \xi<0,\end{cases} \tag{2-11}$$

$$\mathcal{F}_q^{-1}(\phi)(x)=\begin{cases}\mathcal{F}^{-1}(\phi)(x)+\mathcal{F}^{-1}(\chi_+\bar{r}_q\phi)(-x)+\mathcal{F}^{-1}(\chi_- r_q\phi)(x) & \text{if } x\ge 0,\\ \mathcal{F}^{-1}(\phi)(x)+\mathcal{F}^{-1}(\chi_- r_q\phi)(-x)+\mathcal{F}^{-1}(\chi_+\bar{r}_q\phi)(x) & \text{if } x<0.\end{cases} \tag{2-12}$$

For self-containedness, we give a brief proof of (2-11) when $\xi\ge 0$; the other cases can be dealt with in a similar manner. It follows from the definition (2-10) and the identities (2-6)–(2-7) that

$$\begin{aligned}\mathcal{F}_q(\phi)(\xi)&=\int_{\mathbb{R}}\overline{\Psi(x,\xi)}\phi(x)\,dx\\ &=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{t_q(\xi)f_+(x,\xi)}\phi(x)\,dx\\ &=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\overline{t_q(\xi)}e^{-ix\xi}\phi(x)\,dx+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^0\big(e^{-ix\xi}+\overline{r_q(\xi)}e^{ix\xi}\big)\phi(x)\,dx\\ &=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\big(1+\overline{r_q(\xi)}\big)e^{-ix\xi}\phi(x)\,dx+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^0\big(e^{-ix\xi}+\overline{r_q(\xi)}e^{ix\xi}\big)\phi(x)\,dx\\ &=\mathcal{F}(\chi_+\phi)(\xi)+\overline{r_q(\xi)}\mathcal{F}(\chi_+\phi)(\xi)+\mathcal{F}(\chi_-\phi)(\xi)+\overline{r_q(\xi)}\mathcal{F}(\chi_-\phi)(-\xi)\\ &=\mathcal{F}(\phi)(\xi)+r_q(\xi)\mathcal{F}(\chi_+\phi)(-\xi)+r_q(\xi)\mathcal{F}(\chi_-\phi)(\xi),\end{aligned}$$

which gives (2-11) for $\xi\ge 0$.

We collect some basic properties of $\mathcal{F}_q$ and $\mathcal{F}_q^{-1}$. For more details, please refer to Segata [71] and Masaki, Murphy and Segata [61].

Lemma 2.1. *Assume that H is the Schrödinger operator given by* (2-2) *with $q>0$. Let $\mathcal{F}_q$ and $\mathcal{F}_q^{-1}$ be defined by* (2-10).

(i) *$\mathcal{F}_q$ and $\mathcal{F}_q^{-1}$ are unitary on $L^2(\mathbb{R})$, and*

$$\mathcal{F}_q^{-1}\mathcal{F}_q=\mathcal{F}_q\mathcal{F}_q^{-1}=I \quad \text{on } L^2(\mathbb{R}).$$

(ii) $\|\mathcal{F}_q(f)\|_{L^\infty(\mathbb{R})}\lesssim\|f\|_{L^1(\mathbb{R})}$. *The same estimate is true for $\mathcal{F}_q^{-1}$ as well.*

(iii) $\mathcal{F}_q(f)(0)=0$ *whenever* $\langle x\rangle f\in L^2(\mathbb{R})$.

Proof. Statements (i) and (iii) were established in [71] and [61]. Statement (ii) follows from (2-11), (2-12) and the fact that $r_q\in L^\infty(\mathbb{R})$. □

Next we give an explicit formula for the multipliers associated with H, which will be used to investigate the linear estimates for e^{-itH}, as well as the formula for the Littlewood–Paley projection $\varphi_j(\sqrt{2H})$.

Proposition 2.2. *If m is a bounded radial measurable function, $m(\sqrt{2H})f(x)$ is given by*

$$\begin{cases}\mathcal{F}^{-1}\big(m(\xi)(\mathcal{F}(f)(\xi)+r_q(\xi)\mathcal{F}(\chi_+ f)(-\xi)+r_q(\xi)\mathcal{F}(\chi_- f)(\xi))\big)(x) & \text{if } x\geq 0,\\ \mathcal{F}^{-1}\big(m(\xi)(\mathcal{F}(f)(\xi)+\bar{r}_q(\xi)\mathcal{F}(\chi_- f)(-\xi)+\bar{r}_q(\xi)\mathcal{F}(\chi_+ f)(\xi))\big)(x) & \text{if } x<0.\end{cases}$$

Proof. The explicit representation for the spectral projection implies

$$m(\sqrt{2H})f(x)=\\ \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\int_0^\infty m(\xi)|t_q(\xi)|^2(f_+(x,\xi)\overline{f_+(y,\xi)}+f_-(x,\xi)\overline{f_-(y,\xi)})\,d\xi\Big)f(y)\,dy,$$

where $f_\pm$ are Jost functions and t_q is the transmission coefficient. We first consider the case $x\geq 0$. We write

$$m(\sqrt{2H})f(x)=F_1(x)+F_2(x)+F_3(x)+F_4(x),$$

with

$$\begin{aligned}F_1(x)&:=\frac{1}{2\pi}\int_0^\infty m(\xi)|t_q(\xi)|^2 f_+(x,\xi)\Big(\int_0^\infty \overline{f_+(y,\xi)}f(y)\,dy\Big)\,d\xi,\\ F_2(x)&:=\frac{1}{2\pi}\int_0^\infty m(\xi)|t_q(\xi)|^2 f_+(x,\xi)\Big(\int_{-\infty}^0 \overline{f_+(y,\xi)}f(y)\,dy\Big)\,d\xi,\\ F_3(x)&:=\frac{1}{2\pi}\int_0^\infty m(\xi)|t_q(\xi)|^2 f_-(x,\xi)\Big(\int_0^\infty \overline{f_-(y,\xi)}f(y)\,dy\Big)\,d\xi,\\ F_4(x)&:=\frac{1}{2\pi}\int_0^\infty m(\xi)|t_q(\xi)|^2 f_-(x,\xi)\Big(\int_{-\infty}^0 \overline{f_-(y,\xi)}f(y)\,dy\Big)\,d\xi.\end{aligned}$$

It follows from the expressions (2-4)–(2-5) for $f_\pm$ that

$$\begin{aligned}F_1(x)=&\frac{1}{2\pi}\int_0^\infty m(\xi)|t_q(\xi)|^2e^{ix\xi}\Big(\int_0^\infty e^{-iy\xi}f(y)\,dy\Big)\,d\xi\\ =&\frac{1}{\sqrt{2\pi}}\int_0^\infty m(\xi)e^{ix\xi}|t_q(\xi)|^2\mathcal{F}(\chi_+ f)(\xi)\,d\xi\end{aligned}$$

and

$$F_2(x)=F_{21}(x)+F_{22}(x),$$

with

$$\begin{aligned}F_{21}(x)&:=\frac{1}{\sqrt{2\pi}}\int_0^\infty m(\xi)t_q(\xi)e^{ix\xi}\mathcal{F}(\chi_- f)(\xi)\,d\xi,\\ F_{22}(x)&:=+\frac{1}{\sqrt{2\pi}}\int_0^\infty m(\xi)t_q(\xi)e^{ix\xi}\overline{r_q(\xi)}\mathcal{F}(\chi_- f)(-\xi)\,d\xi.\end{aligned}$$

Notice that $\overline{t_q(\xi)}=t_q(-\xi)$; we further have

$$F_3(x)=F_{31}(x)+F_{32}(x)+F_{33}(x)+F_{34}(x),$$

with

$$
\begin{aligned}
F_{31}(x) &:= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} m(\xi) e^{ix\xi} \mathcal{F}(\chi_+ f)(\xi)\, d\xi, \\
F_{32}(x) &:= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} m(\xi) r_q(\xi) e^{ix\xi} \mathcal{F}(\chi_+ f)(-\xi)\, d\xi, \\
F_{33}(x) &:= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} m(\xi) e^{ix\xi} r_q(\xi) \mathcal{F}(\chi_+ f)(-\xi)\, d\xi, \\
F_{34}(x) &:= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} m(\xi) e^{ix\xi} |r_q(\xi)|^2 \mathcal{F}(\chi_+ f)(\xi)\, d\xi.
\end{aligned}
$$

Similarly,

$$
F_4(x) = F_{41}(x) + F_{42}(x),
$$

with

$$
\begin{aligned}
F_{41}(x) &:= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} m(\xi) e^{ix\xi} t_q(\xi) \mathcal{F}(\chi_- f)(\xi)\, d\xi, \\
F_{42}(x) &:= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} m(\xi) e^{ix\xi} \overline{t_q(\xi)} r_q(\xi) \mathcal{F}(\chi_- f)(-\xi)\, d\xi.
\end{aligned}
$$

Now by using identities (2-6)–(2-7), we have

$$
\begin{aligned}
&F_1(x) + F_{31}(x) + F_{34}(x) = \mathcal{F}^{-1}\big(m(\xi)\mathcal{F}(\chi_+ f)(\xi)\big)(x), \\
&F_{21}(x) + F_{41}(x) = \mathcal{F}^{-1}\big(m(\xi)\mathcal{F}(\chi_- f)(\xi)\big)(x) + \mathcal{F}^{-1}\big(m(\xi) r_q(\xi) \mathcal{F}(\chi_- f)(\xi)\big)(x), \\
&F_{22}(x) + F_{42}(x) = 0, \\
&F_{32}(x) + F_{33}(x) = \mathcal{F}^{-1}\big(m(\xi) r_q(\xi) \mathcal{F}(\chi_+ f)(-\xi)\big)(x),
\end{aligned}
$$

which imply that for $x \geq 0$,

$$
\begin{aligned}
&m(\sqrt{2H}) f(x) = \\
&\qquad \mathcal{F}^{-1}\big(m(\xi)(\mathcal{F}(f)(\xi) + r_q(\xi)\mathcal{F}(\chi_+ f)(-\xi) + r_q(\xi)\mathcal{F}(\chi_- f)(\xi))\big)(x).
\end{aligned}
$$

As for the case $x < 0$, noticing that $f_+(x,\xi) = f_-(-x,\xi)$, it follows that

$$
\begin{aligned}
&m(\sqrt{2H}) f(x) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big(\int_0^{\infty} m(\xi) |t_q(\xi)|^2 (f_-(-x,\xi) \overline{f_-(-y,\xi)} + f_+(-x,\xi) d\xi \Big) \overline{f_+(-y,\xi)}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times f(y)\, dy \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big(\int_0^{\infty} m(\xi) |t_q(\xi)|^2 (f_-(-x,\xi) \overline{f_-(y,\xi)} + f_+(-x,\xi) \overline{f_+(y,\xi)})\, d\xi \Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times f(-y)\, dy \\
&= m(\sqrt{2H}) f(-\cdot)(-x),
\end{aligned}
$$

concluding the proof. ☐

We now present a set of corollaries that provide a representation of the linear propagator e^{-itH} and its associated Strichartz estimates. These are essential tools in the analysis of the Cauchy problem (1-1) with initial data in $L^p(\mathbb{R})$.

First, choosing $m(\lambda) = e^{-\frac{1}{2}it\lambda^2}$ in Proposition 2.2, we obtain:

Corollary 2.3. *Let H be defined by* (2-2). *Then*

$$e^{-itH} f(x) = \chi_+(x) K_t * \mathcal{L}_+(f)(x) + \chi_-(x) K_t * \mathcal{L}_-(f)(x)$$

where $K_t(x) = e^{-\frac{i\pi}{4}}(2\pi t)^{-1/2} e^{\frac{i|x|^2}{2t}}$ *and*

$$\mathcal{L}_+(f) = f + \mathcal{F}^{-1}\big(r_q \mathcal{F}(\chi_+ f)(-\,\cdot)\big) + \mathcal{F}^{-1}\big(r_q \mathcal{F}(\chi_- f)\big),$$
$$\mathcal{L}_-(f) = f + \mathcal{F}^{-1}\big(\overline{r_q} \mathcal{F}(\chi_- f)(-\,\cdot)\big) + \mathcal{F}^{-1}\big(\overline{r_q} \mathcal{F}(\chi_+ f)\big).$$

The next result appears in [42] and [71], but we include the proof for convenience.

Corollary 2.4. *Let H be defined by* (2-2) *and*

$$\frac{2}{r_j} + \frac{1}{p_j} = \frac{1}{2}, \quad 4 \le r_j \le \infty, \quad j = 1, 2.$$

Then for any interval I and $s \in \bar{I}$, we have $\|e^{-itH} f\|_{L_t^{r_1} L_x^{q_1}(I\times\mathbb{R})} \lesssim \|f\|_{L_x^2(\mathbb{R})}$ *and*

$$\Big\| \int_s^t e^{-i(t-\tau)H} F(\tau)\, d\tau \Big\|_{L_t^{r_1} L_x^{q_1}(I\times\mathbb{R})} \lesssim \|F\|_{L_t^{r_2'} L_x^{q_2'}(I\times\mathbb{R})}.$$

Proof. The operators $\mathcal{L}_\pm$ are bounded on $L^p(\mathbb{R})$ for all $1 \le p \le \infty$, since $r_q \in L^\infty$. It then follows from Corollary 2.3 that

$$\|e^{-itH} f\|_{L^\infty(\mathbb{R})} \lesssim |t|^{-\frac{1}{2}} \|f\|_{L^1(\mathbb{R})};$$

combined with

$$\|e^{-itH} f\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$$

and the TT^* argument, this implies the desired estimates. □

Remark 2.5. The formula for e^{-itH} in Corollary 2.3 can be expressed in terms of e^{-itH_0}:

$$e^{-itH} f = \chi_+ e^{-itH_0} \mathcal{L}_+(f) + \chi_- e^{-itH_0} \mathcal{L}_-(f).$$

The dispersive estimates and Strichartz estimates for e^{-itH} can also be derived from those for e^{-itH_0} and the L^p-boundedness of $\mathcal{L}_\pm$, where $1 \le p \le \infty$.

3. Homogeneous Besov spaces associated to H

The function space theory associated to operator H is an important topic in harmonic analysis and has been extensively studied in recent years, one can see for example [24; 29; 30; 38; 40; 39], the theories of Hardy spaces, BMO spaces and Sobolev spaces associated with operators. On the one hand, they generalize the classic theories of corresponding function spaces. On the other hand, they are used to investigate the boundedness of singular integrals such as the Riesz transform, square function and area integral associated to operators at some endpoint. Such theories primarily rely on the point-wise estimates or off-diagonal estimates for the heat semigroup e^{-tH}, and can be applied to Schrödinger operator $H = -\Delta + V$, where V is an unbounded positive potential or V has small negative part.

However, if the potential V has large negative component, the Schrödinger operator $H = -\Delta + V$ may admit eigenvalue and resonance at zero energy. Under such circumstance, the development of function spaces adapted to $H = -\Delta + V$ is primarily grounded in spectral analysis and the theory of distorted Fourier transforms. Ólafsson and Zheng [66] studied the Triebel–Lizorkin spaces and Besov spaces associated to H, where V is taken to be the Pöschl–Teller potential. Cuccagna, Visciglia and Georgiev [21] considered the Sobolev spaces associated to H under the assumptions that $V \in \mathcal{S}(\mathbb{R})$ being both generic and exceptional. Moreover, Georgiev and Giammetta [33] studied the homogeneous Besov spaces associated to H with short range potential V. It is noteworthy that in [21] and [33], the authors proved the equivalence between the classical function spaces and the corresponding spaces associated to H.

The homogeneous Besov space $\dot{B}^{s,H}_{p,r}(\mathbb{R})$ associated with H for some $1 < p < \infty$, $1 \le r \le \infty$ and $s \in \mathbb{R}$ is defined as follows. Let $\varphi_j(\sqrt{2H})$ be the Littlewood–Paley associated with H, which can be defined by Proposition 2.2 with $m(\lambda) = \varphi_j(\lambda)$, where φ_j is defined as in (2-1). Let $1 < p < \infty$, $1 \le r \le \infty$ and $s \in \mathbb{R}$, the homogeneous Besov space $\dot{B}^{s,H}_{p,r}(\mathbb{R})$ associated with the perturbed Hamiltonian H is defined as the closure of $\mathcal{S}(\mathbb{R})$ function f with respect to the norm

$$\|f\|_{\dot{B}^{s,H}_{p,r}(\mathbb{R})} = \Big(\sum_{j\in\mathbb{Z}} 2^{jrs} \|\varphi_j(\sqrt{2H}) f\|^r_{L^p(\mathbb{R})}\Big)^{1/r}. \tag{3-1}$$

The space $\dot{B}^{s,H}_{p,r}(\mathbb{R})$ is independent of the choice of the Littlewood–Paley function φ. In fact, if ψ is of the same type as φ in Section 2.1, it follows that $\varphi_j \psi_k = 0$ if $|j-k| > 3$. Then by using the distorted Fourier transform and the uniform boundedness of $\varphi_j(\sqrt{2H})$ on $L^p(\mathbb{R})$ for $1 < p < \infty$ (see Remark 3.3(ii)), we have

$$\begin{aligned}
\|\varphi_j(\sqrt{2H}) f\|_{L^p(\mathbb{R})} &\le \sum_{k\in\mathbb{Z}} \|\varphi_j(\sqrt{2H})\psi_k(\sqrt{2H}) f\|_{L^p(\mathbb{R})} \\
&= \sum_{k\in\mathbb{Z}} \|\mathcal{F}_q^{-1}(\varphi_j(\xi)\psi_k(\xi)\mathcal{F}_q f(\xi))\|_{L^p(\mathbb{R})} \\
&\lesssim \sum_{k=j-3}^{j+3} \|\psi_k(\sqrt{2H}) f\|_{L^p(\mathbb{R})},
\end{aligned}$$

which means that $\dot{B}^{s,H}_{p,r}(\mathbb{R})$ is well defined.

We are now in a position to state the main theorem of this section.

Theorem 3.1. *Assume that* $1 < p < \infty$, $1 \le r \le \infty$ *and* $-\frac{1}{p'} < s < \frac{1}{p}$. *We have*

$$\dot{B}^s_{p,r}(\mathbb{R}) = \dot{B}^{s,H}_{p,r}(\mathbb{R})$$

with equivalent norms.

The proof, given in Section 3.2, depends on the boundedness of wave operators on Besov spaces and the cancellation property of the operator $\varphi_j(\sqrt{2H_0})\varphi_k(\sqrt{2H})$.

3.1. ***Wave operators.*** Wave operator methods are fundamental in the study of the evolution flow generated by the Hamiltonian H, typically considered as perturbations of the free Hamiltonian H_0. They are often used to deal with the behavior of particles interacting with each other or external potentials.

Namely, we have the free Hamiltonian $H_0 = -\frac{1}{2}\partial_x^2$ and the perturbed Hamiltonian $H = H_0 + q\delta_0$ with $q > 0$. The corresponding wave operators are defined by

$$W_\pm f = s - \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}f,$$

and their conjugates

$$W_\pm^* f = s - \lim_{t\to\pm\infty} e^{itH_0}e^{-itH}f.$$

If the potential $q\delta_0$ is replaced by a general potential, $W_\pm^*$ is well-defined for $P_c(H)f$, where $P_c(H)$ denotes the projection onto the continuous spectrum. The wave operators enjoy the splitting property

$$HW_\pm = W_\pm H_0,$$

and can lead to the functional calculus for H,

$$g(H) = W_+g(H_0)W_+^* = W_-g(H_0)W_-^* \tag{3-2}$$

for any function $g \in L^\infty_{\mathrm{loc}}(\mathbb{R})$.

There are several ways to represent the wave operators $W_\pm$. It follows from Schechter [69] and (2-12) that

$$W_+f(x) = \mathcal{F}_q^{-1}\mathcal{F}(f)(x) = I_d f(x) + \sum_{\ell=1}^{4} T_\ell f(x), \tag{3-3}$$

where I_d is the identity operator and

$$T_1 f(x) = \chi_+(x)\mathcal{F}^{-1}\big(\chi_+\bar{r}_q\mathcal{F}(f)\big)(-x),$$
$$T_2 f(x) = \chi_+(x)\mathcal{F}^{-1}\big(\chi_- r_q\mathcal{F}(f)\big)(x),$$
$$T_3 f(x) = \chi_-(x)\mathcal{F}^{-1}\big(\chi_- r_q\mathcal{F}(f)\big)(-x),$$
$$T_4 f(x) = \chi_-(x)\mathcal{F}^{-1}\big(\chi_+\bar{r}_q\mathcal{F}(f)\big)(x).$$

We will prove the boundedness of wave operators on various function spaces, which can be used to prove the equivalence between $\dot{B}^{s,H}_{p,r}(\mathbb{R})$ and $\dot{B}^{s}_{p,r}(\mathbb{R})$. The boundedness of wave operators under different assumptions regarding the potential V in Lebesgue spaces and Sobolev spaces has been extensively studied, so we will not go further on this topic; for more information, one can see for example D'Ancona and Fanelli [22], Weder [80] and Yajima [81].

Denote by $\mathbb{B}(X,Y)$ the space of bounded linear operators from X to Y.

Proposition 3.2. *The wave operators $W_\pm$ lie in $\mathbb{B}(X,Y)$, where*

(i) $X = Y = L^p(\mathbb{R})$ *with* $1 < p < \infty$, *or*

(ii) $X = L^1(\mathbb{R})$ *and* $Y = L^{1,\infty}(\mathbb{R})$, *or*

(iii) $X = H^1(\mathbb{R})$ *and* $Y = L^1(\mathbb{R})$.

Proof. By using the identity (3-3), we only prove $T_2 \in \mathbb{B}(X,Y)$. Write

$$T_2 f(x) = \chi_+(x)\mathcal{F}^{-1}\big(m_2(\xi)\mathcal{F}(f)(\xi)\big)(x)$$

with

$$m_2(\xi) = \chi_-(\xi) r_q(\xi) = \frac{\chi_-(\xi)q}{i\xi - q}.$$

Notice that m_2 is smooth away from the origin and

$$\big|\partial_\xi^k m_2(\xi)\big| \le C\langle\xi\rangle^{-k-1}, \quad \xi \in \mathbb{R}\backslash\{0\} \quad \text{and} \quad k \in \mathbb{N}_+,$$

where the constant $C > 0$ depends on k, q. Then it follows from a standard multiplier theorem (see Grafakos [35]) that W_+ is bounded from X to Y with X, Y satisfying any of the sets of conditions above. □

Remark 3.3. (1) By duality, the operators $W^*_\pm$ belong to $\mathbb{B}(L^p(\mathbb{R}), L^p(\mathbb{R}))$ with $1 < p < \infty$ and $\mathbb{B}(L^\infty(\mathbb{R}), BMO(\mathbb{R}))$.

(2) For $1 < p < \infty$, the functional calculus (3-2) and Proposition 3.2 show that if the classic multiplier $m(H_0)$ is bounded on $L^p(\mathbb{R})$, then so is $m(H)$.

(3) Duchêne, Marzuola and Weinstein [28] considered the boundedness of wave operators $W_\pm$ on Sobolev spaces $W^{1,p}(\mathbb{R})$ for singular potentials in dimension one. Their results also apply to the Schrödinger operator with delta potential.

(4) The wave operators $W_\pm$ may not map the Hardy space $H^1(\mathbb{R})$ to itself. This is because the projection $\chi_\pm$ will break the cancellation property for the atoms of $H^1(\mathbb{R})$.

Next we investigate the boundedness of the wave operators on homogeneous Besov spaces.

Proposition 3.4. *Assume that* $1 < p < \infty$, $1 \le r \le \infty$ *and* $-\frac{1}{p'} < s < \frac{1}{p}$. *Then*

$$W_\pm \in \mathbb{B}\big(\dot{B}^s_{p,r}(\mathbb{R}), \dot{B}^s_{p,r}(\mathbb{R})\big).$$

Proof. We prove that W_+ is bounded on $\dot{B}^s_{p,r}(\mathbb{R})$; the same argument can be applied for the proof on the boundedness of W_-. We will establish the inequality

$$\|W_+ f\|_{\dot{B}^s_{p,r}(\mathbb{R})} = \Big(\sum_{j\in\mathbb{Z}} 2^{jrs}\|\varphi_j(\sqrt{2H_0})W_+ f\|^r_{L^p(\mathbb{R})}\Big)^{1/r} \lesssim \|f\|_{\dot{B}^s_{p,r}(\mathbb{R})}, \tag{3-4}$$

where φ is defined in Section 2.1 and $\varphi_j(\xi) = \varphi(2^{-j}\xi)$. Let $\psi \in C_0^\infty(\mathbb{R})$ be an even function such that $\psi = 1$ on the support of φ and $0 \le \psi \le 1$. We write

$$f = \sum_{k\in\mathbb{Z}} \varphi_k(\sqrt{2H_0})\psi_k(\sqrt{2H_0})f = \sum_{k\in\mathbb{Z}} \varphi_k(\sqrt{2H_0})f_k$$

with $f_k = \psi_k(\sqrt{2H_0})f$. Then

$$\begin{aligned}\varphi_j(\sqrt{2H_0})W_+ f &= \sum_{k\in\mathbb{Z}} \varphi_j(\sqrt{2H_0})W_+\big(\varphi_k(\sqrt{2H_0})f_k\big)\\ &= \sum_{k\in\mathbb{Z}} \varphi_j(\sqrt{2H_0})\varphi_k(\sqrt{2H_0})f_k + \sum_{\ell=1}^{4}\sum_{k\in\mathbb{Z}} \varphi_j(\sqrt{2H_0})T_\ell\big(\varphi_k(\sqrt{2H_0})f_k\big)\\ &=: \sum_{\ell=0}^{4}\sum_{k\in\mathbb{Z}} R_{\ell,j,k}.\end{aligned} \tag{3-5}$$

Notice that for $j \in \mathbb{Z}$, $\varphi_j\varphi_k$ is nonzero when $|j-k| \le 3$ and otherwise it is zero. Thus

$$\sum_{k\in\mathbb{Z}} \|R_{0,j,k}\|_{L^p(\mathbb{R})} \lesssim \sum_{k=j-3}^{j+3} \|f_k\|_{L^p(\mathbb{R})}. \tag{3-6}$$

As for $R_{\ell,j,k}$ ($\ell = 1,\dots,4$) in (3-5), it follows from the proof of Proposition 3.2 that T_ℓ ($\ell = 1,\dots,4$) are bounded on L^p for all $1 < p < \infty$. Then for fixed $j \in \mathbb{Z}$ and $|j-k| \le 3$,

$$\sum_{\ell=1}^{4}\sum_{k=j-3}^{j+3} \|R_{\ell,j,k}\|_{L^p(\mathbb{R})} \lesssim \sum_{k=j-3}^{j+3} \|f_k\|_{L^p(\mathbb{R})}. \tag{3-7}$$

It remains to consider the estimates for $R_{\ell,j,k}$ ($\ell = 1,\dots,4$) in (3-5) with $|j-k| > 3$. We deal with the case $\ell = 2$. Notice that

$$\begin{aligned}&\mathcal{F}(T_2\varphi_k(\sqrt{2H_0})f_k)(\xi)\\ &\quad= \mathcal{F}\big(\chi_+\mathcal{F}^{-1}(\chi_- r_q\mathcal{F}(\varphi_k(\sqrt{2H_0})f_k))\big)(\xi)\\ &\quad= \mathcal{F}\big(\tfrac{1}{2}(1+\operatorname{sgn}\cdot)\mathcal{F}^{-1}(\chi_- r_q\varphi_k\mathcal{F}(f_k))\big)(\xi)\end{aligned}$$

$$= \tfrac{1}{2}\chi_-(\xi) r_q(\xi)\varphi_k(\xi)\mathcal{F}(f_k)(\xi) + \tfrac{1}{2}(2\pi)^{-\frac{1}{2}}\mathcal{F}(\operatorname{sgn}\cdot) * (r_q\chi_-\varphi_k\mathcal{F}(f_k))(\xi)$$
$$= \tfrac{1}{2}\chi_-(\xi) r_q(\xi)\varphi_k(\xi)\mathcal{F}(f_k)(\xi) + i(2\pi)^{-\frac{3}{2}}\int \frac{1}{y-\xi}\chi_-(y) r_q(y)\varphi_k(y)\int e^{-izy} f_k(z)\,dz\,dy.$$

Thus we can represent each $R_{2,j,k}$ as

(3-8)
$$\begin{aligned} R_{2,j,k} &= \mathcal{F}^{-1}\big(\varphi_j(\xi)\mathcal{F}(T_2\varphi_k(\sqrt{2H_0})f_k)\big)(x) \\ &= \tfrac{1}{2}\mathcal{F}^{-1}\big(\varphi_j(\xi)\varphi_k(\xi)\chi_-(\xi)r_q(\xi)\mathcal{F}(f_k)(\xi)\big)(x) \\ &\quad + i(2\pi)^{-2}\iiint e^{ix\xi}\varphi_j(\xi)\frac{1}{y-\xi}\chi_-(y)r_q(y)\varphi_k(y)e^{-izy} f_k(z)\,dz\,dy\,d\xi \\ &= \frac{1}{2}\mathcal{F}^{-1}\big(\varphi_j(\xi)\varphi_k(\xi)\chi_-(\xi)r_q(\xi)\mathcal{F}(f_k)(\xi)\big)(x) + \int \mathcal{K}_{j,k}(x,z) f_k(z)\,dz \\ &=: R_{21,j,k} + R_{22,j,k}, \end{aligned}$$

where the kernel $\mathcal{K}_{j,k}(x,z)$ is given by

$$\mathcal{K}_{j,k}(x,z) = i(2\pi)^{-2}\iint e^{ix\xi}e^{-izy}\varphi_j(\xi)\varphi_k(y)\frac{1}{y-\xi}\chi_-(y)r_q(y)\,dy\,d\xi.$$

By using the fact that $\varphi_j\varphi_k = 0$ when $|j-k| > 3$, the term $R_{21,j,k}$ in (3-8) vanishes, which means that

$$\|R_{2,j,k}\|_{L^p(\mathbb{R})} \le \|R_{22,j,k}\|_{L^p(\mathbb{R})} \lesssim \Big\|\int \mathcal{K}_{j,k}(x,z) f_k(z)\,dz\Big\|_{L^p(\mathbb{R})}. \tag{3-9}$$

By a change of variables, we have

$$\mathcal{K}_{j,k}(x,z) = i(2\pi)^{-2}2^j2^k \iint e^{i2^jx\xi}e^{-i2^kzy}\varphi(\xi)\varphi(y)\frac{1}{2^ky-2^j\xi}\chi_-(2^ky)r_q(2^ky)\,dy\,d\xi.$$

By integration by parts in y and ξ, it follows

$$\begin{aligned} \big|\mathcal{K}_{j,k}(x,z)\big| &\lesssim \frac{2^j2^k}{\langle 2^jx\rangle^2\langle 2^kz\rangle^2}\iint \Big|\partial_y^2\partial_\xi^2\frac{\varphi(\xi)\varphi(y)\chi_-(2^ky)r_q(2^ky)}{2^ky-2^j\xi}\Big|\,dy\,d\xi \\ &\lesssim \frac{2^j2^k}{\langle 2^jx\rangle^2\langle 2^kz\rangle^2}\frac{1}{\max(2^k,2^j)}. \end{aligned}$$

Now for $k < j-3$, it follows from Hölder's inequality that

$$\begin{aligned} \Big\|\int \mathcal{K}_{j,k}(x,z)f_k(z)\,dz\Big\|_{L^p(\mathbb{R})} &\lesssim \frac{2^k}{(2^j)^{\frac{1}{p}}(2^k)^{1-\frac{1}{p}}}\|f_k\|_{L^p(\mathbb{R})} \\ &\lesssim 2^{-|j-k|\frac{1}{p}}\|f_k\|_{L^p(\mathbb{R})}. \end{aligned} \tag{3-10}$$

As for $k > j+3$, by Hölder's inequality again, we have

$$\Big\| \int \mathcal{K}_{j,k}(x,z) f_k(z)\,dz \Big\|_{L^p(\mathbb{R})} \lesssim \frac{2^j}{(2^j)^{\frac{1}{p}}(2^k)^{(1-\frac{1}{p})}} \|f_k\|_{L^p(\mathbb{R})} \lesssim 2^{-|j-k|(1-\frac{1}{p})} \|f_k\|_{L^p(\mathbb{R})}. \tag{3-11}$$

For fixed $j \in \mathbb{Z}$, combining (3-7), (3-9), (3-10) and (3-11), we have

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \|R_{2,j,k}\|_{L^p(\mathbb{R})} &\lesssim \sum_{k\in\mathbb{Z}} \big(\chi_{\le 3}(|k-j|) + \chi_{>3}(k-j) + \chi_{<-3}(k-j)\big) \|R_{2,j,k}\|_{L^p(\mathbb{R})} \\ &\lesssim \sum_{k=j-3}^{j+3} \|f_k\|_{L^p(\mathbb{R})} \\ &\quad + \sum_{k\in\mathbb{Z}} \big(\chi_{>3}(k-j)2^{-|j-k|(1-\frac{1}{p})} + \chi_{<-3}(k-j)2^{-|j-k|\frac{1}{p}}\big) \|f_k\|_{L^p(\mathbb{R})}, \end{aligned} \tag{3-12}$$

where $\chi_{\le 3}(n)$ is the characteristic function of the set $(-\infty, 3]$ on $\mathbb{Z}$, and $\chi_{>3}$ and $\chi_{<-3}$ are defined similarly. The same estimate (3-12) is also true if the operator $R_{2,j,k}$ is replaced by $R_{\ell,j,k}$ with $(\ell \ne 2)$. Writing $\|a_n\|^r_{\ell^r_n(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} |a_n|^r$, notice that

$$\big\|\chi_{>3}(n)2^{-n(1-\frac{1}{p})-ns}\big\|_{\ell^1_n(\mathbb{Z})} \lesssim 1 \quad \text{for } s > -\big(1-\tfrac{1}{p}\big) = -\tfrac{1}{p'},$$
$$\big\|\chi_{<-3}(n)2^{-\frac{|n|}{p}-ns}\big\|_{\ell^1_n(\mathbb{Z})} \lesssim 1 \quad \text{for } s < \tfrac{1}{p}.$$

Then it follows from (3-5), (3-6), (3-12) and Young's inequality that for $-\frac{1}{p'} < s < \frac{1}{p}$,

$$\begin{aligned} \Big\| 2^{js} \|\varphi_j(\sqrt{2H_0})W_+ f\|_{L^p(\mathbb{R})} \Big\|_{\ell^r_j(\mathbb{Z})} &\lesssim \Big\| 2^{js} \sum_{\ell=0}^{4} \sum_{k\in\mathbb{Z}} \|R_{\ell,j,k}\|_{L^p(\mathbb{R})} \Big\|_{\ell^r_j(\mathbb{Z})} \\ &\lesssim \Big\| 2^{js} \sum_{|k-j|\le 3} \|f_k\|_{L^p(\mathbb{R})} \Big\|_{\ell^r_j(\mathbb{Z})} \\ &\quad + \Big\| \sum_{k\in\mathbb{Z}} \chi_{>3}(k-j) 2^{-|j-k|(1-\frac{1}{p})+(j-k)s} 2^{ks} \|f_k\|_{L^p(\mathbb{R})} \Big\|_{\ell^r_j(\mathbb{Z})} \\ &\quad + \Big\| \sum_{k\in\mathbb{Z}} \chi_{<-3}(k-j) 2^{-|j-k|\frac{1}{p}+(j-k)s} 2^{ks} \|f_k\|_{L^p(\mathbb{R})} \Big\|_{\ell^r_j(\mathbb{Z})} \\ &\lesssim \Big\| 2^{ks} \|f_k\|_{L^p(\mathbb{R})} \Big\|_{\ell^r_k(\mathbb{Z})}, \end{aligned} \tag{3-13}$$

which implies (3-4). □

Corollary 3.5. *Assume that* $1 < p < \infty$, $1 \le r < \infty$ *and* $-\frac{1}{p'} < s < \frac{1}{p}$. *Then*

$$W_\pm^* \in \mathbb{B}(\dot{B}^s_{p,r}(\mathbb{R}), \dot{B}^s_{p,r}(\mathbb{R})).$$

Proof. For given $1 < p < \infty$, $1 \le r < \infty$ and $-\frac{1}{p'} < s < \frac{1}{p}$, it is easy to see that $1 < p' < \infty$, $1 < r' \le \infty$ and $-s \in \left(-\frac{1}{p}, \frac{1}{p'}\right) = \left(-\frac{1}{(p')'}, \frac{1}{p'}\right)$. Then for any $f \in \dot{B}^s_{p,r}(\mathbb{R})$ and $g \in \dot{B}^{-s}_{p',r'}(\mathbb{R})$, by Proposition 3.4, we have

$$\left|\langle W_\pm^* f, g\rangle\right| \lesssim \|f\|_{\dot{B}^s_{p,r}(\mathbb{R})} \|W_\pm g\|_{\dot{B}^{-s}_{p',r'}(\mathbb{R})} \lesssim \|f\|_{\dot{B}^s_{p,r}(\mathbb{R})} \|g\|_{\dot{B}^{-s}_{p',r'}(\mathbb{R})}, \tag{3-14}$$

which implies the desired result. □

Remark 3.6. We didn't prove the boundedness of $W_\pm^*$ at the endpoint $r = \infty$. This doesn't mean the result is not true for $r = \infty$. In fact, we can represent $W_\pm^*$ similarly to (3-3) in terms of the Fourier transform, and use the same argument used in the proof of Proposition 3.4 to obtain the boundedness at $r = \infty$. However, Corollary 3.5 is enough for further applications.

3.2. ***Proof of Theorem 3.1.*** In this section, we prove Theorem 3.1, the equivalence between $\dot{B}^s_{p,r}(\mathbb{R})$ and $\dot{B}^{s,H}_{p,r}(\mathbb{R})$ for $1 < p < \infty$, $1 \le r < \infty$ and $-\frac{1}{p'} < s < \frac{1}{p}$.

Step I. We prove the embedding $\dot{B}^s_{p,r}(\mathbb{R}) \hookrightarrow \dot{B}^{s,H}_{p,r}(\mathbb{R})$, where the boundedness of the wave operators on Lebesgue spaces and Besov spaces obtained in Section 3.1 will be involved. In fact, using (3-1), the identity (3-2), Proposition 3.2 and Corollary 3.5, we have

$$\begin{aligned}
\|f\|^r_{\dot{B}^{s,H}_{p,r}(\mathbb{R})} &= \sum_{j\in\mathbb{Z}} 2^{jrs} \|\varphi_j(\sqrt{2H}) f\|^r_{L^p(\mathbb{R})} \\
&= \sum_{j\in\mathbb{Z}} 2^{jrs} \|W_+ \varphi_j(\sqrt{2H_0}) W_+^* f\|^r_{L^p(\mathbb{R})} \\
&\le \sum_{j\in\mathbb{Z}} 2^{jrs} \|\varphi_j(\sqrt{2H_0}) W_+^* f\|^r_{L^p(\mathbb{R})} = \|W_+^* f\|^r_{\dot{B}^s_{p,r}(\mathbb{R})} \lesssim \|f\|^r_{\dot{B}^s_{p,r}(\mathbb{R})},
\end{aligned}$$

which implies the desired conclusion.

Step II. We prove the inverse embedding, $\dot{B}^{s,H}_{p,r}(\mathbb{R}) \hookrightarrow \dot{B}^s_{p,r}(\mathbb{R})$. Let φ be as defined in Section 2.1, let $\varphi_j(\xi) = \varphi(2^{-j}\xi)$, and let $\psi \in C_0^\infty(\mathbb{R})$ be an even function such that $\psi = 1$ on the support of φ and $0 \le \psi \le 1$. We write

$$f = \sum_{k\in\mathbb{Z}} \varphi_k(\sqrt{2H}) f = \sum_{k\in\mathbb{Z}} \tilde{f}_k,$$

and then if we set $f_k = \psi_k(\sqrt{2H}) f$, we further have

$$\begin{aligned}
\varphi_j(\sqrt{2H_0}) f = \sum_{k\in\mathbb{Z}} \varphi_j(\sqrt{2H_0}) \tilde{f}_k &= \sum_{k\in\mathbb{Z}} \varphi_j(\sqrt{2H_0}) \psi_k(\sqrt{2H}) \tilde{f}_k \\
&= \sum_{k\in\mathbb{Z}} \varphi_j(\sqrt{2H_0}) \varphi_k(\sqrt{2H}) f_k.
\end{aligned}$$

We next estimate $\varphi_j(\sqrt{2H_0})\varphi_k(\sqrt{2H})$ in $L^p(\mathbb{R})$. For given $j \in \mathbb{Z}$ and any k with $|j-k| \le 3$, using the boundedness of the Littlewood–Paley projection and Remark 3.3(ii), we have

$$\|\varphi_j(\sqrt{2H_0})\varphi_k(\sqrt{2H}) f_k\|_{L^p(\mathbb{R})} \lesssim \|f_k\|_{L^p(\mathbb{R})}. \tag{3-15}$$

It suffices to consider the estimates for $|j-k| > 3$. By applying Proposition 2.2 to $m(\lambda) = \varphi_k(\lambda)$, we have

$$\varphi_j(\sqrt{2H_0})\varphi_k(\sqrt{2H}) f_k(x) = \mathcal{F}^{-1}\big(\varphi_j \mathcal{F}(\varphi_k(\sqrt{2H}) f_k)\big)(x) = \sum_{\ell=1}^{5} J_{\ell,j,k}, \tag{3-16}$$

where

$$\begin{aligned}
J_{1,j,k} &:= \mathcal{F}^{-1}(\varphi_j \varphi_k \mathcal{F}(f_k))(x),\\
J_{2,j,k} &:= \mathcal{F}^{-1}\big[\varphi_j \mathcal{F}\big(\chi_+ \mathcal{F}^{-1}(\varphi_k r_q \mathcal{F}(\chi_- f_k))\big)\big](x),\\
J_{3,j,k} &:= \mathcal{F}^{-1}\big[\varphi_j \mathcal{F}\big(\chi_+ \mathcal{F}^{-1}(\varphi_k r_q \mathcal{F}(\chi_+ f_k)(-\cdot))\big)\big](x),\\
J_{4,j,k} &:= \mathcal{F}^{-1}\big[\varphi_j \mathcal{F}\big(\chi_- \mathcal{F}^{-1}(\varphi_k \bar{r}_q \mathcal{F}(\chi_+ f_k))\big)\big](x),\\
J_{5,j,k} &:= \mathcal{F}^{-1}\big[\varphi_j \mathcal{F}\big(\chi_- \mathcal{F}^{-1}(\varphi_k \bar{r}_q \mathcal{F}(\chi_- f_k)(-\cdot))\big)\big](x).
\end{aligned}$$

Notice that $J_{1,j,k}$ vanishes, since $\varphi_j(\xi)\varphi_k(\xi) = 0$ when $|k-j| > 3$, and the other four summands $J_{\ell,j,k}$ ($\ell \neq 1$) are of the same type, so we only estimate $J_{2,j,k}$. We write

$$\begin{aligned}
&\mathcal{F}\big(\chi_+ \mathcal{F}^{-1}(\varphi_k r_q \mathcal{F}(\chi_- f_k))\big)(\xi)\\
&\quad= \mathcal{F}\Big(\frac{1}{2}(1+\operatorname{sgn}\cdot)\mathcal{F}^{-1}(r_q \varphi_k \mathcal{F}(\chi_- f_k))\Big)(\xi)\\
&\quad= \tfrac{1}{2} r_q(\xi)\varphi_k(\xi)\mathcal{F}(\chi_- f_k)(\xi) + \frac{1}{2\sqrt{2\pi}}\mathcal{F}(\operatorname{sgn}\cdot) * (r_q \varphi_k \mathcal{F}(\chi_- f_k))(\xi)\\
&\quad= \tfrac{1}{2} r_q(\xi)\varphi_k(\xi)\mathcal{F}(\chi_- f_k)(\xi) + \frac{1}{2\pi}\frac{1}{i\cdot} * (r_q \varphi_k \mathcal{F}(\chi_- f_k))(\xi)\\
&\quad= \tfrac{1}{2} r_q(\xi)\varphi_k(\xi)\mathcal{F}(\chi_- f_k)(\xi) + i(2\pi)^{-\frac{3}{2}}\\
&\qquad\qquad \times \int \frac{1}{y-\xi} r_q(y)\varphi_k(y) \int e^{-izy}\chi_-(z) f_k(z)\,dz\,dy,
\end{aligned} \tag{3-17}$$

which implies that

$$\begin{aligned}
J_{2,j,k} &= \tfrac{1}{2}\mathcal{F}^{-1}\big(\varphi_j(\xi)\varphi_k(\xi) r_q(\xi)\mathcal{F}(\chi_- f_k)(\xi)\big)(x) + \int \widetilde{\mathcal{K}}_{j,k}(x,z) f_k(z)\,dz\\
&=: J_{21,j,k} + J_{22,j,k},
\end{aligned}$$

with the kernel $\widetilde{\mathcal{K}}_{j,k}(x,z)$ given by

$$\widetilde{\mathcal{K}}_{j,k}(x,z) = i(2\pi)^{-2}\chi_-(z) \iint e^{ix\xi} e^{-izy} \varphi_j(\xi)\varphi_k(y)\frac{1}{y-\xi} r_q(y)\,dy\,d\xi.$$

Similarly to the proof of Proposition 3.4, by changing variables and integrating by parts, we have

$$\begin{aligned}\big|\widetilde{\mathcal{K}}_{j,k}(x,z)\big| &\lesssim \frac{2^j 2^k}{\langle 2^j x\rangle^2 \langle 2^k z\rangle^2} \iint \left|\partial_y^2 \partial_\xi^2 \frac{\varphi(\xi)\varphi(y)}{2^k y - 2^j \xi}\right| dy\, d\xi \\ &\lesssim \frac{2^j 2^k}{\langle 2^j x\rangle^2 \langle 2^k z\rangle^2} \frac{1}{\max(2^k, 2^j)},\end{aligned}$$

which leads to the estimates

$$\|J_{22,j,k}\|_{L^p(\mathbb{R})} \le \left\| \int \widetilde{\mathcal{K}}_{j,k}(x,z) f_k(z)\, dz \right\|_{L^p(\mathbb{R})} \lesssim 2^{-|j-k|\frac{1}{p}} \|f_k\|_{L^p(\mathbb{R})} \quad (k<j-3),$$

$$\|J_{22,j,k}\|_{L^p(\mathbb{R})} \le \left\| \int \widetilde{\mathcal{K}}_{j,k}(x,z) f_k(z)\, dz \right\|_{L^p(\mathbb{R})} \lesssim 2^{-|k-j|(1-\frac{1}{p})} \|f_k\|_{L^p(\mathbb{R})} \quad (k>j-3).$$

Notice that $J_{21,j,k}$ vanishes, since $\varphi_j(\xi)\varphi_k(\xi) = 0$ when $|k-j| > 3$. With the same meaning for the χ's as in (3-12), we have

$$\|J_{2,j,k}\|_{L^p(\mathbb{R})} \lesssim \big(\chi_{<-3}(k-j) 2^{-|j-k|\frac{1}{p}} + \chi_{>3}(k-j) 2^{-|k-j|(1-\frac{1}{p})}\big) \|f_k\|_{L^p(\mathbb{R})}. \tag{3-18}$$

The same estimate holds for $J_{\ell,j,k}$ with $\ell = 3, 4, 5$. Now combining (3-15)–(3-18) with the inequalities

$$\begin{aligned}\big\|\chi_{>3}(n) 2^{-n(1-\frac{1}{p})-ns}\big\|_{\ell^1_n(\mathbb{Z})} &\lesssim 1, \quad s > -(1-\tfrac{1}{p}) = -\tfrac{1}{p'}, \\ \big\|\chi_{<-3}(n) 2^{-\frac{|n|}{p}-ns}\big\|_{\ell^1_n(\mathbb{Z})} &\lesssim 1, \quad s < \tfrac{1}{p},\end{aligned}$$

and Young's inequality, we obtain

$$\begin{aligned}\|f\|_{\dot{B}^s_{p,r}(\mathbb{R})} &= \big\|2^{js} \|\varphi_j(\sqrt{2H_0}) f\|_{L^p(\mathbb{R})}\big\|_{\ell^r_j(\mathbb{Z})} \\ &\lesssim \Big\|2^{js} \sum_{k\in\mathbb{Z}} \|\varphi_j(\sqrt{2H_0})\varphi_k(\sqrt{2H}) f_k\|_{L^p(\mathbb{R})}\Big\|_{\ell^r_j(\mathbb{Z})} \\ &\lesssim \Big\|2^{js} \sum_{|k-j|\le 3} \|f_k\|_{L^p}\Big\|_{\ell^r_j(\mathbb{Z})} \\ &\quad + \Big\|\sum_{k\in\mathbb{Z}} \chi_{>3}(k-j) 2^{-|j-k|(1-\frac{1}{p})+(j-k)s} 2^{ks} \|f_k\|_{L^p(\mathbb{R})}\Big\|_{\ell^r_j(\mathbb{Z})} \\ &\quad + \Big\|\sum_{k\in\mathbb{Z}} \chi_{<-3}(k-j) 2^{-|j-k|\frac{1}{p}+(j-k)s} 2^{ks} \|f_k\|_{L^p(\mathbb{R})}\Big\|_{\ell^r_j(\mathbb{Z})} \\ &\lesssim \big\|2^{ks} \|f_k\|_{L^p(\mathbb{R})}\big\|_{\ell^r_k(\mathbb{Z})} = \|f\|_{\dot{B}^{s,H}_{p,r}(\mathbb{R})}.\end{aligned}$$

This concludes the proof of Theorem 3.1.

4. Proofs of Theorems of 1.1 and 1.3

We recall the integral equation

$$v(t) = u_0 - i\lambda \int_0^t \mathcal{U}(-s)\big(\mathcal{U}(s)v(s)\,\overline{\mathcal{U}(s)v(s)}\,\mathcal{U}(s)v(s)\big)\,ds,$$

where $v(t) = \mathcal{U}(-t)u(t)$, $u(t)$ is the solution to the original nonlinear Schrödinger equation (1-1) and $\mathcal{U}(t) = e^{-itH}$ is the linear propagator with $H = -\frac{1}{2}\partial_x^2 + q\delta_0$. In this section, we will prove Theorems 1.1 and 1.3, the local well-posedness of $v(t)$ in L^p-based spaces.

We start with the estimate for trilinear form

$$\mathcal{T}(v_1(\tau), v_2(\tau), v_3(\tau)) = \mathcal{U}(-\tau)\big(\mathcal{U}(-\tau)v_1(\tau)\mathcal{U}(\tau)v_2(\tau)\mathcal{U}(\tau)v_3(\tau)\big). \tag{4-1}$$

We will exploit the cancellation of $\mathcal{T}$, which is analogous to (1-6), with $\mathcal{U}(t)$ replaced by $e^{it\Delta}$. Set

$$M(t)f(x) = e^{i\frac{|x|^2}{2t}} f(x) \quad \text{and} \quad Jf(x) = f(-x).$$

Lemma 4.1. *Let the trilinear form $\mathcal{T}$ be defined by* (4-1). *We have*, *up to a constant*,

$$\begin{aligned}\mathcal{T}(v_1(t), v_2(t), v_3(t)) = t^{-1} \textstyle\sum_{\ell=1}^2 \big(&\mathcal{Q}_\ell^{(2)}(v_1(t), v_2(t), v_3(t)) \\ &+ \mathcal{Q}_{\ell+2}^{(1)}(v_1(t), v_2(t), v_3(t))\big) \\ &+ t^{-1}\chi_+\mathcal{F}^{-1}(r_q) * \textstyle\sum_{\ell=1}^4 \mathcal{Q}_\ell^{(1)}(v_1(t), v_2(t), v_3(t)) \\ &+ t^{-1}\chi_-\mathcal{F}^{-1}(\bar{r}_q) * \textstyle\sum_{\ell=1}^4 \mathcal{Q}_\ell^{(2)}(v_1(t), v_2(t), v_3(t)),\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{Q}_1^{(1)}(v_1(t), v_2(t), v_3(t)) \\
&\quad = M(t)\big\{(\overline{M(t)}\mathcal{L}_+(v_1(t))) * (JM(t)\mathcal{L}_+\big(v_2(t))) * (JM(t)\mathcal{L}_+(v_3(t)))\big\},\\
&\mathcal{Q}_2^{(1)}(v_1(t), v_2(t), v_3(t)) \\
&\quad = M(t)\big\{(\mathcal{H}\overline{M(t)}\mathcal{L}_+(v_1(t))) * (JM(t)\mathcal{L}_+\big(v_2(t))) * (JM(t)\mathcal{L}_+(v_3(t)))\big\},\\
&\mathcal{Q}_3^{(1)}(v_1(t), v_2(t), v_3(t)) \\
&\quad = M(t)\big\{(J\overline{M(t)}\mathcal{L}_-(v_1(t))) * (M(t)\mathcal{L}_-\big(v_2(t))) * (M(t)\mathcal{L}_-(v_3(t)))\big\},\\
&\mathcal{Q}_4^{(1)}(v_1(t), v_2(t), v_3(t)) \\
&\quad = M(t)\big\{(J\mathcal{H}\overline{M(t)}\mathcal{L}_-(v_1(t))) * (M(t)\mathcal{L}_-\big(v_2(t))) * (M(t)\mathcal{L}_-(v_3(t)))\big\},\\
&\mathcal{Q}_1^{(2)}(v_1(t), v_2(t), v_3(t)) \\
&\quad = M(t)\big\{(J\overline{M(t)}\mathcal{L}_+(v_1(t))) * (M(t)\mathcal{L}_+\big(v_2(t))) * (M(t)\mathcal{L}_+(v_3(t)))\big\},\\
&\mathcal{Q}_2^{(2)}(v_1(t), v_2(t), v_3(t)) \\
&\quad = M(t)\big\{(J\mathcal{H}\overline{M(t)}\mathcal{L}_+(v_1(t))) * (M(t)\mathcal{L}_+\big(v_2(t))) * (M(t)\mathcal{L}_+(v_3(t)))\big\},
\end{aligned}$$

$$
\begin{aligned}
&\mathcal{Q}_3^{(2)}(v_1(t), v_2(t), v_3(t)) \\
&\qquad = M(t)\big\{(\overline{M(t)}\mathcal{L}_-(v_1(t))) * (JM(t)\mathcal{L}_-\big(v_2(t))) * (JM(t)\mathcal{L}_-(v_3(t)))\big\}, \\
&\mathcal{Q}_4^{(2)}(v_1(t), v_2(t), v_3(t)) \\
&\qquad = M(t)\big\{(\mathcal{H}\overline{M(t)}\mathcal{L}_-(v_1(t))) * (JM(t)\mathcal{L}_-\big(v_2(t))) * (JM(t)\mathcal{L}_-(v_3(t)))\big\}.
\end{aligned}
$$

Proof. Set

$$v(t) = \mathcal{U}(-t)v_1(t)\mathcal{U}(t)v_2(t)\mathcal{U}(t)v_3(t).$$

It follows from Corollary 2.3 that

$$
\begin{aligned}
\text{(4-2)}\quad \mathcal{T}\big((v_1)(t), v_2(t), v_3(t)\big)(x)
&= \big(\mathcal{U}(-t)v(t)\big)(x) \\
&= \chi_+(x)\overline{K}_t * \mathcal{L}_+(v(t))(x) + \chi_-(x)\overline{K}_t * \mathcal{L}_-(v(t))(x) \\
&=: \chi_+(x)\mathcal{T}^+(t)(x) + \chi_-(x)\mathcal{T}^-(t)(x),
\end{aligned}
$$

where $K_t(x) = e^{-\frac{i\pi}{4}}(2\pi t)^{-1/2}e^{\frac{i|x|^2}{2t}}$ and

$$
\begin{aligned}
\mathcal{L}_+v(t) &= v(t) + \mathcal{F}^{-1}\big(r_q J\mathcal{F}\big(\chi_+v(t)\big)\big) + \mathcal{F}^{-1}\big(r_q\mathcal{F}\big(\chi_-v(t)\big)\big), \\
\mathcal{L}_-v(t) &= v(t) + \mathcal{F}^{-1}\big(\overline{r_q} J\mathcal{F}\big(\chi_-v(t)\big)\big) + \mathcal{F}^{-1}\big(\overline{r_q}\mathcal{F}\big(\chi_+v(t)\big)\big).
\end{aligned}
$$

In the following, we will omit absolute constants in some identities.

Let us turn to the expansion of $v(t)$. By using Corollary 2.3 again, we have

$$
\begin{aligned}
\text{(4-3)}\quad & v(t)(y) \\
&= \big(\mathcal{U}(-t)v_1(t)\big)(y)\big(\mathcal{U}(t)v_2(t)\big)(y)\big(\mathcal{U}(t)v_3(t)\big)(y) \\
&= \chi_+(y)\overline{K}_t * \mathcal{L}_+(v_1(t))(y)K_t * \mathcal{L}_+(v_2(t))(y)K_t * \mathcal{L}_+(v_3(t))(y) \\
&\qquad + \chi_-(y)\overline{K}_t * \mathcal{L}_-(v_1(t))(y)K_t * \mathcal{L}_-(v_2(t))(y)K_t * \mathcal{L}_-(v_3(t))(y) \\
&=: \chi_+(y)v_+(t)(y) + \chi_-(y)v_-(t)(y).
\end{aligned}
$$

Then it follows that

$$
\begin{aligned}
\text{(4-4)}\quad v_+(t)(y) &= \iiint \overline{K}_t(y-\alpha)\mathcal{L}_+(v_1(t))(\alpha)K_t(y-\beta)\mathcal{L}_+(v_2(t))(\beta) \\
&\qquad\qquad \times K_t(y-\gamma)\mathcal{L}_+(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma \\
&= t^{-\frac{3}{2}}\iiint e^{\frac{-i(|y-\alpha|^2-|y-\beta|^2-|y-\gamma|^2)}{2t}} \\
&\qquad\qquad \times \mathcal{L}_+(v_1(t))(\alpha)\mathcal{L}_+(v_2(t))(\beta)\mathcal{L}_+(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma,
\end{aligned}
$$

$$
\begin{aligned}
\text{(4-5)}\quad v_-(t)(y) &= t^{-\frac{3}{2}}\iiint e^{\frac{-i(|y-\alpha|^2-|y-\beta|^2-|y-\gamma|^2)}{2t}} \\
&\qquad\qquad \times \mathcal{L}_-(v_1(t))(\alpha)\mathcal{L}_-(v_2(t))(\beta)\mathcal{L}_-(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma.
\end{aligned}
$$

Now by (4-2), we have

$$\mathcal{T}^+(t)(x) = \overline{K}_t * \big(v(t) + \mathcal{F}^{-1}\big(r_q J\mathcal{F}(\chi_+ v(t))\big) + \mathcal{F}^{-1}\big(r_q\mathcal{F}(\chi_- v(t))\big)\big)(x)$$
$$=: \mathcal{T}_1^+(t)(x) + \mathcal{T}_2^+(t)(x) + \mathcal{T}_3^+(t)(x).$$

Set $\tilde{e}_1(t,x,y,\alpha,\beta,\gamma) = e^{\frac{-i(|x-y|^2+|y-\alpha|^2-|y-\beta|^2-|y-\gamma|^2)}{2t}}$. It is easy to see that

$$|x-y|^2+|y-\alpha|^2-|y-\beta|^2-|y-\gamma|^2 = (x^2+\alpha^2-\beta^2-\gamma^2)+2y(\beta+\gamma-\alpha-x),$$
$$\int \tilde{e}_1(t,x,y,\alpha,\beta,\gamma)\chi_+(y)\,dy$$
$$= te^{-i\frac{(x^2+\alpha^2-\gamma^2-\beta^2)}{2t}}\Big(\sqrt{\tfrac{\pi}{2}}\delta_0(\beta+\gamma-\alpha-x) + \tfrac{1}{i\sqrt{2\pi}}\tfrac{1}{\beta+\gamma-\alpha-x}\Big),$$
$$\int \tilde{e}_1(t,x,y,\alpha,\beta,\gamma)\chi_-(y)\,dy$$
$$= te^{-i\frac{(x^2+\alpha^2-\gamma^2-\beta^2)}{2t}}\Big(\sqrt{\tfrac{\pi}{2}}\delta_0(\beta+\gamma-\alpha-x) - \tfrac{1}{i\sqrt{2\pi}}\tfrac{1}{\beta+\gamma-\alpha-x}\Big),$$

which combined with (4-4) and (4-5) imply that

$$\text{(4-6)}\quad \overline{K}_t * \big(\chi_+ v_+(t)\big)(x)$$
$$= t^{-2}\iiiint \tilde{e}_1(t,x,y,\alpha,\beta,\gamma)$$
$$\times\chi_+(y)\mathcal{L}_+(v_1(t))(\alpha)\mathcal{L}_+(v_2(t))(\beta)\mathcal{L}_+(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma\,dy$$
$$= t^{-1}e^{-i\frac{x^2}{2t}}\iiint\Big(\delta_0(\beta+\gamma-\alpha-x)+\tfrac{1}{\beta+\gamma-\alpha-x}\Big)\overline{M(t)}\mathcal{L}_+(v_1(t))(\alpha)$$
$$\times M(t)\mathcal{L}_+(v_2(t))(\beta)M(t)\mathcal{L}_+(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma$$
$$= t^{-1}M(t)\big\{\big(J\overline{M(t)}\mathcal{L}_+(v_1(t))\big)*\big(M(t)\mathcal{L}_+\big(v_2(t)\big)\big)*\big(M(t)\mathcal{L}_+(v_3(t))\big)\big\}(x)$$
$$+t^{-1}M(t)\big\{\big(J\mathcal{H}\overline{M(t)}\mathcal{L}_+(v_1(t))\big)*\big(M(t)\mathcal{L}_+(v_2(t))\big)*\big(M(t)\mathcal{L}_+(v_3(t))\big)\big\}(x)$$

and

$$\overline{K}_t * \big(\chi_- v_-(t)\big)(x)$$
$$= t^{-2}\iiiint \tilde{e}_1(t,x,y,\alpha,\beta,\gamma)$$
$$\times\chi_-(y)\mathcal{L}_-(v_1(t))(\alpha)\mathcal{L}_-(v_2(t))(\beta)\mathcal{L}_-(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma\,dy$$
$$= t^{-1}e^{-i\frac{x^2}{2t}}\iiint\big(\delta_0(\beta+\gamma-\alpha-x)-\tfrac{1}{\beta+\gamma-\alpha-x}\Big)\overline{M(t)}\mathcal{L}_-(v_1(t))(\alpha)$$
$$\times M(t)\mathcal{L}_-(v_2(t))(\beta)M(t)\mathcal{L}_-(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma$$
$$= t^{-1}M(t)\big\{\big(J\overline{M(t)}\mathcal{L}_-(v_1(t))\big)*\big(M(t)\mathcal{L}_-\big(v_2(t)\big)\big)*\big(M(t)\mathcal{L}_-(v_3(t))\big)\big\}(x)$$
$$-t^{-1}M(t)\big\{\big(J\mathcal{H}\overline{M(t)}\mathcal{L}_-(v_1(t))\big)*\big(M(t)\mathcal{L}_-(v_2(t))\big)*\big(M(t)\mathcal{L}_-(v_3(t))\big)\big\}(x).$$

Combining these formulas with (4-3), we further have

$$\begin{aligned}(4\text{-}7)\quad &\mathcal{T}_1^+(t)(x)\\ &= \overline{K}_t * (\chi_+ v_+(t))(x) + \overline{K}_t * (\chi_- v_-(t))(x)\\ &=: t^{-1}\sum_{\ell=1}^{2}\mathcal{Q}_\ell^{(2)}(v_1(t), v_2(t), v_3(t)) + t^{-1}\sum_{\ell=3}^{4}\mathcal{Q}_\ell^{(1)}(v_1(t), v_2(t), v_3(t)).\end{aligned}$$

Let us turn to $T_2^+(t)$. Set $\tilde{e}_2(t,z,y,\alpha,\beta,\gamma) = e^{\frac{-i(|z+y|^2+|y-\alpha|^2-|y-\beta|^2-|y-\gamma|^2)}{2t}}$. Similarly as above, we have

$$|z+y|^2+|y-\alpha|^2-|y-\beta|^2-|y-\gamma|^2 = (z^2+\alpha^2-\beta^2-\gamma^2)+2y(\beta+\gamma+z-\alpha),$$

$$\begin{aligned}&\int \tilde{e}_2(t,z,y,\alpha,\beta,\gamma)\chi_+(y)\,dy\\ &\qquad = te^{-i\frac{(z^2+\alpha^2-\gamma^2-\beta^2)}{2t}}\Big(\sqrt{\tfrac{\pi}{2}}\delta_0(\beta+\gamma+z-\alpha) + \tfrac{1}{i\sqrt{2\pi}}\tfrac{1}{\beta+\gamma+z-\alpha}\Big),\end{aligned}$$

which combined with (4-4) and (4-5) imply that

$$\begin{aligned}&\overline{K}_t * \big(J\chi_+ v_+(t)\big)(z)\\ &= t^{-2}\iiiint \tilde{e}_2(t,z,y,\alpha,\beta,\gamma)\\ &\qquad\qquad \times\chi_+(y)\mathcal{L}_+(v_1(t))(\alpha)\mathcal{L}_+(v_2(t))(\beta)\mathcal{L}_+(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma\,dy\\ &= t^{-1}e^{-i\frac{z^2}{2t}}\iiint\Big(\delta_0(\beta+\gamma+z-\alpha)+\tfrac{1}{\beta+\gamma+z-\alpha}\Big)\overline{M(t)}\mathcal{L}_+(v_1(t))(\alpha)\\ &\qquad\qquad \times M(t)\mathcal{L}_+(v_2(t))(\beta)M(t)\mathcal{L}_+(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma\\ &= t^{-1}M(t)\big\{\big(\overline{M(t)}\mathcal{L}_+(v_1(t))\big)*\big(JM(t)\mathcal{L}_+(v_2(t))\big)*\big(JM(t)\mathcal{L}_+(v_3(t))\big)\big\}(z)\\ &\quad + t^{-1}M(t)\big\{\big(\mathcal{H}\overline{M(t)}\mathcal{L}_+(v_1(t))\big)*\big(JM(t)\mathcal{L}_+(v_2(t))\big)*\big(JM(t)\mathcal{L}_+(v_3(t))\big)\big\}(z).\end{aligned}$$

Combining this with (4-3), we have

$$\begin{aligned}(4\text{-}8)\qquad \mathcal{T}_2^+(t)(x) &= \mathcal{F}^{-1}(r_q) * \overline{K}_t * \big(J\chi_+ v_+(t)\big)(x)\\ &= t^{-1}\mathcal{F}^{-1}(r_q) * \sum_{\ell=1}^{2}\mathcal{Q}_\ell^{(1)}(v_1(t), v_2(t), v_3(t)).\end{aligned}$$

The expansion for $\mathcal{T}_3^+(t)$ follows from the expansion of $\mathcal{T}_1^+(t)$. Indeed, we have

$$\begin{aligned}(4\text{-}9)\qquad \mathcal{T}_3^+(t)(x) &= \mathcal{F}^{-1}(r_q) * \overline{K}_t * \big(\chi_- v_-(t)\big)(x)\\ &= t^{-1}\mathcal{F}^{-1}(r_q) * \sum_{\ell=3}^{4}\mathcal{Q}_\ell^{(1)}(v_1(t), v_2(t), v_3(t)).\end{aligned}$$

Turning to $\mathcal{T}^-(t)$, notice that

$$\begin{aligned}\mathcal{T}^-(t)(x) &= \overline{K}_t * \big(v(t) + \mathcal{F}^{-1}\big(\bar{r}_q \mathcal{F}(\chi_+ v(t))\big) + \mathcal{F}^{-1}\big(\bar{r}_q \mathcal{F}(J\chi_- v(t))\big)\big)(x)\\ &=: \mathcal{T}_1^-(t)(x) + \mathcal{T}_2^-(t)(x) + \mathcal{T}_3^-(t)(x).\end{aligned}$$

Obviously, $\mathcal{T}_1^-(t)(x) = \mathcal{T}_1^+(t)(x)$. From (4-6) and the identity

$$\begin{aligned}&\overline{K}_t * \big(J\chi_- v_-(t)\big)(z)\\ &= t^{-2} \iiiint \tilde{e}_2(t,z,y,\alpha,\beta,\gamma)\\ &\qquad\qquad \times \chi_-(y)\mathcal{L}_-(v_1(t))(\alpha)\mathcal{L}_-(v_2(t))(\beta)\mathcal{L}_-(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma\,dy\\ &= t^{-1} e^{-i\frac{z^2}{2t}} \iiint \Big(\delta_0(\beta+\gamma+z-\alpha) - \tfrac{1}{\beta+\gamma+z-\alpha}\Big)\overline{M(t)}\mathcal{L}_-(v_1(t))(\alpha)\\ &\qquad\qquad \times M(t)\mathcal{L}_-(v_2(t))(\beta)M(t)\mathcal{L}_-(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma\\ &= t^{-1}M(t)\big\{\big(\overline{M(t)}\mathcal{L}_-(v_1(t))\big) * \big(JM(t)\mathcal{L}_-(v_2(t))\big) * \big(JM(t)\mathcal{L}_-(v_3(t))\big)\big\}(z)\\ &\quad - t^{-1}M(t)\big\{\big(\mathcal{H}\overline{M(t)}\mathcal{L}_-(v_1(t))\big) * \big(JM(t)\mathcal{L}_-(v_2(t))\big) * \big(JM(t)\mathcal{L}_-(v_3(t))\big)\big\}(z)\end{aligned}$$

it follows that

$$\begin{aligned}\mathcal{T}_2^-(t)(x) &= \mathcal{F}^{-1}(\bar{r}_q) * \overline{K}_t * \big(\chi_+ v_+(t)\big)(x)\\ &= t^{-1}\mathcal{F}^{-1}(\bar{r}_q) * \textstyle\sum_{\ell=1}^{2} \mathcal{Q}_\ell^{(2)}(v_1(t),v_2(t),v_3(t)),\end{aligned}\tag{4-10}$$

$$\begin{aligned}\mathcal{T}_3^-(t)(x) &= \mathcal{F}^{-1}(\bar{r}_q) * \overline{K}_t * \big(J\chi_- v_-(t)\big)(x)\\ &= t^{-1}\mathcal{F}^{-1}(\bar{r}_q) * \textstyle\sum_{\ell=3}^{4} \mathcal{Q}_\ell^{(2)}(v_1(t),v_2(t),v_3(t)).\end{aligned}\tag{4-11}$$

Hence, by (4-2) and (4-8)–(4-11), we have

$$\begin{aligned}&\mathcal{T}(v_1(t),v_2(t),v_3(t))\\ &\quad =: \chi_+\mathcal{T}^+(t) + \chi_-\mathcal{T}^-(t)\\ &\quad = \mathcal{T}_1^+(t) + \chi_+\big(\mathcal{T}_2^+(t) + \mathcal{T}_3^+(t)\big) + \chi_-\big(\mathcal{T}_2^-(t) + \mathcal{T}_3^-(t)\big)\\ &\quad = t^{-1}\textstyle\sum_{\ell=1}^{2}\big(\mathcal{Q}_\ell^{(2)}(v_1(t),v_2(t),v_3(t)) + \mathcal{Q}_{\ell+2}^{(1)}(v_1(t),v_2(t),v_3(t))\big)\\ &\qquad\qquad + t^{-1}\chi_+\mathcal{F}^{-1}(r_q) * \textstyle\sum_{\ell=1}^{4} \mathcal{Q}_\ell^{(1)}(v_1(t),v_2(t),v_3(t))\\ &\qquad\qquad + t^{-1}\chi_-\mathcal{F}^{-1}(\bar{r}_q) * \textstyle\sum_{\ell=1}^{4} \mathcal{Q}_\ell^{(2)}(v_1(t),v_2(t),v_3(t)),\end{aligned}$$

which finishes the proof. □

4.1. *Proof of Theorem 1.1.* We next prove Theorem 1.1 and Corollary 1.2, the well-posedness of the integral equation (1-8) in homogeneous Besov spaces $\dot{B}^s_{p,1}(\mathbb{R})$. The following lemma will be used to deal with the nonlinear term in (1-8).

Lemma 4.2. *Let $1 < p \le 2$ and $\mathcal{T}$ be the trilinear form defined by* (4-1). *Assume that $v_2(t) = \varphi_k(\sqrt{2H})v_2(t)$ and $v_3(t) = \varphi_j(\sqrt{2H})v_3(t)$. Then we have*

$$\big\|\mathcal{T}(v_1(t), v_2(t), v_3(t))\big\|_{L^p(\mathbb{R})} \lesssim t^{1-\frac{2}{p}} 2^{(1-\frac{1}{p})\frac{j+k}{2}} \|v_1(t)\|_{L^p(\mathbb{R})} \|v_2(t)\|_{L^p(\mathbb{R})} \|v_3(t)\|_{L^p(\mathbb{R})}.$$

Proof. We use interpolation. Consider the estimate of $\mathcal{T}$ for $p = 1$. Notice that $\mathcal{F}^{-1}(r_q)(x) = -\sqrt{2\pi} q \chi_-(x) e^{qx} \in L^1(\mathbb{R})$, $\mathcal{L}_\pm$ are bounded on $L^p(\mathbb{R})$ for all $1 \le p \le \infty$ and $H^1(\mathbb{R}) \hookrightarrow L^1(\mathbb{R})$. It follows from Lemma 4.1 and Young's inequality that

$$\begin{aligned}
&\big\|\mathcal{T}(v_1(t), v_2(t), v_3(t))\big\|_{L^1(\mathbb{R})} \\
&\lesssim t^{-1} \sum_{\ell=1}^{4} \Big(\big\|\mathcal{Q}_\ell^{(1)}(v_1(t), v_2(t), v_3(t))\big\|_{L^1(\mathbb{R})} + \big\|\mathcal{Q}_\ell^{(2)}(v_1(t), v_2(t), v_3(t))\big\|_{L^1(\mathbb{R})} \Big) \\
&\lesssim t^{-1} \big(\|\mathcal{L} v_1(t)\|_{L^1(\mathbb{R})} + \|\mathcal{H} M(t) \mathcal{L} v_1(t)\|_{L^1(\mathbb{R})} \big) \|v_2(t)\|_{L^1(\mathbb{R})} \|v_3(t)\|_{L^1(\mathbb{R})} \\
&\lesssim t^{-1} \|M(t)\mathcal{L} v_1(t)\|_{H^1(\mathbb{R})} \|v_2(t)\|_{L^1(\mathbb{R})} \|v_3(t)\|_{L^1(\mathbb{R})},
\end{aligned}$$

where we write $\mathcal{L} = (\mathcal{L}_-, \mathcal{L}_+)$ and $\|\mathcal{L} f\|_{L^p(\mathbb{R})} = \|\mathcal{L}_+ f\|_{L^p(\mathbb{R})} + \|\mathcal{L}_- f\|_{L^p(\mathbb{R})}$.

On the other hand, it follows from Corollary 2.3 that

$$\begin{aligned}
\|\mathcal{U}(-t) v_1(t)\|_{L^2(\mathbb{R})} &= \|\chi_+ \overline{K}_t * \mathcal{L}_+(v_1(t)) + \chi_- \overline{K}_t * \mathcal{L}_-(v_1(t))\|_{L^2(\mathbb{R})} \\
&\le \|M(t)\mathcal{L} v_1(t)\|_{L^2(\mathbb{R})}.
\end{aligned} \tag{4-12}$$

By Lemma 2.1 and Hölder's inequality, we have

$$\|\mathcal{U}(t) v_2(t)\|_{L^\infty(\mathbb{R})} \lesssim \|\mathcal{F}_q(\mathcal{U}(t) v_2(t))\|_{L^1(\mathbb{R})} \lesssim 2^{\frac{k}{2}} \|v_2(t)\|_{L^2(\mathbb{R})},$$

which combined with (4-12) leads to

$$\begin{aligned}
&\|\mathcal{T}(v_1(t), v_2(t), v_3(t))\|_{L^2(\mathbb{R})} \\
&\quad = \|\mathcal{U}(-t)\big(\mathcal{U}(-t) v_1(t) \mathcal{U}(t) v_2(t) \mathcal{U}(t) v_3(t)\big)\|_{L^2(\mathbb{R})} \\
&\quad \lesssim \|\mathcal{U}(-t) v_1(t)\|_{L^2(\mathbb{R})} \|\mathcal{U}(t) v_2(t)\|_{L^\infty(\mathbb{R})} \|\mathcal{U}(t) v_3(t)\|_{L^\infty(\mathbb{R})} \\
&\quad \lesssim 2^{\frac{j+k}{2}} \|M(t)\mathcal{L} v_1(t)\|_{L^2(\mathbb{R})} \|v_2(t)\|_{L^2(\mathbb{R})} \|v_3(t)\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Now we achieve the result by using multilinear interpolation and the fact that both $M(t)$ and $\mathcal{L}$ are bounded on $L^p(\mathbb{R})$ for all $1 \le p \le \infty$. □

Proof of Theorem 1.1. Let us recall the integral equation

$$v(t) = u_0 - i\lambda \int_0^t \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}(s)(\mathcal{U}(s)v(s))^2\big)\, ds.$$

For fixed $1 < p < 2$, we define the space

$$X_T = \big\{ v \in C\big([0, T], \dot{B}^\sigma_{p,1}(\mathbb{R})\big) \;\big|\; \sup_{0 \le t \le T} \|v(t)\|_{\dot{B}^{s,H}_{p,1}(\mathbb{R})} \le 2\|u_0\|_{\dot{B}^s_{p,1}(\mathbb{R})} \big\},$$

with $\sigma = 1 - \frac{1}{p}$, and the map

$$Tv(t) = u_0 - i\lambda \int_0^t \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}(s)(\mathcal{U}(s)v(s))^2\big)\,ds.$$

We show that T maps X_T to itself and is a contraction.

• Given $v \in X_T$, we aim to prove that $Tv \in X_T$. By Theorem 3.1, we know that $\dot{B}^{s,H}_{p,1}(\mathbb{R}) = \dot{B}^{s}_{p,1}(\mathbb{R})$ for $1 < p < \infty$ and $-\frac{1}{p'} < s < \frac{1}{p}$. In the proof of Theorem 1.1, for $1 < p < 2$ and $\sigma = 1 - \frac{1}{p}$, one has $0 < \sigma < \frac{1}{p}$ and thus $\dot{B}^{\sigma,H}_{p,1}(\mathbb{R}) = \dot{B}^{\sigma}_{p,1}(\mathbb{R})$, which means that we can use $\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})$ instead of $\dot{B}^{\sigma}_{p,1}(\mathbb{R})$ when dealing with the nonlinear term.

We make the decomposition

$$\sum_{j\in\mathbb{Z}} \varphi_j(\sqrt{2H})v(s) = \sum_{j\in\mathbb{Z}} v_j(s) = v(s),$$

and estimate

$$\begin{aligned}
F(s) &= \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}(s)\mathcal{U}(s)v(s)\mathcal{U}(s)v(s)\big)\\
&= \sum_{j,k,l\in\mathbb{Z}} \mathcal{U}(-s)\big(\mathcal{U}(-s)\varphi_j(\sqrt{2H})\bar{v}(s)\mathcal{U}(s)\varphi_k(\sqrt{2H})v(s)\mathcal{U}(s)\varphi_l(\sqrt{2H})v(s)\big)\\
&= \sum_{j,k,l\in\mathbb{Z}} \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}_j(s)\mathcal{U}(s)v_k(s)\mathcal{U}(s)v_l(s)\big)
\end{aligned}$$

in $\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})$. By symmetry, it suffices to estimate

$$F_1(s) = \sum_{j\ge k\ge l} \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}_j(s)\,\mathcal{U}(s)v_k(s)\,\mathcal{U}(s)v_l(s)\big).$$

By Remark 3.3(ii), $\varphi_m(\sqrt{2H})$ are uniformly bounded on $L^p(\mathbb{R})$ for all $1 \le p \le \infty$; thus it follows from the definition of $\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})$ that, setting $\sigma = 1 - \frac{1}{p}$,

$$\begin{aligned}
&\|F_1(s)\|_{\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})}\\
&\quad\le \sum_{j\in\mathbb{Z}} \sum_{m=-\infty}^{j+4} 2^{m\sigma} \Big\| \varphi_m(\sqrt{2H})\\
&\qquad\qquad \times \sum_{k,l=-\infty}^{j} \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}_j(s)\,\mathcal{U}(s)v_k(s)\,\mathcal{U}(s)v_l(s)\big)\Big\|_{L^p(\mathbb{R})}\\
&\quad\le \sum_{j\in\mathbb{Z}} \sum_{m=-\infty}^{j+4} 2^{m\sigma} \Big\| \sum_{k,l=-\infty}^{j} \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}_j(s)\,\mathcal{U}(s)v_k(s)\,\big)\Big\|_{L^p(\mathbb{R})}\\
&\quad\lesssim \sum_{j\in\mathbb{Z}} \sum_{m=-\infty}^{j+k} 2^{j\sigma}\\
&\qquad\qquad \times \sum_{k\ge l\ge -\infty}^{j} s^{-(\frac{2}{p}-1)} 2^{(1-\frac{1}{p})(k+l)} \|v_j(s)\|_{L^p(\mathbb{R})} \|v_k(s)\|_{L^p(\mathbb{R})} \|v_l(s)\|_{L^p(\mathbb{R})}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim s^{-(\frac{2}{p}-1)} \sum_{j\in\mathbb{Z}} \sum_{k,l}^{j} 2^{j\sigma} 2^{\sigma(k+l)} \|v_j(s)\|_{L^p(\mathbb{R})} \|v_k(s)\|_{L^p(\mathbb{R})} \|v_l(s)\|_{L^p(\mathbb{R})} \\
&\lesssim s^{-(\frac{2}{p}-1)} \big(\sum_{j\in\mathbb{Z}} 2^{j\sigma} \|v_j(s)\|_{L^p} \big)^3 \\
&\lesssim s^{-(\frac{2}{p}-1)} \|v\|^3_{\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})}.
\end{aligned}
$$

Noting that $0 < \frac{2}{p} - 1 < 1$ when $1 < p < 2$, we have

$$
\begin{aligned}
\|Tv(t)\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} &\le \|u_0\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} + \int_0^t \big\|\mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}(s)((\mathcal{U}(s)v(s))^2)\big)\big\|_{\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})} ds \\
&\le \|u_0\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} + \int_0^t s^{-(\frac{2}{p}-1)} \|v(s)\|^3_{\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})} ds \\
&\le \|u_0\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} + CT^{1-(\frac{2}{p}-1)} \big(\sup_{0\le t\le T} \|v(s)\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})}\big)^3 \\
&\le \|u_0\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} + CT^{2-\frac{2}{p}} \|u_0\|^3_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} \\
&\le 2\|u_0\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})},
\end{aligned}
$$

where T is chosen small enough that $CT^{2-\frac{2}{p}} \|u_0\|^2_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} \le 1$.

- To prove that T is a contraction on X_T, let $v_1, v_2 \in X_T$. One shows easily that

$$
\begin{aligned}
&Tv_1(t) - Tv_2(t) \\
&\quad = -i\lambda \int_0^t \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}_1(s)\mathcal{U}(s)(v_1(s)-v_2(s))\mathcal{U}(s)(v_1(s)+v_2(s))\big)\, ds \\
&\qquad\qquad - i\lambda \int_0^t \mathcal{U}(-s)\big(\mathcal{U}(-s)(\bar{v}_1(s)-\bar{v}_2(s))(\mathcal{U}(s)v_2(s))^2\big)\, ds.
\end{aligned}
$$

Set $v^*(t) = Tv_1(t) - Tv_2(t)$. Without loss of generality we choose $\lambda = \pm 1$. By using the same procedure as in the estimation of F_1, we obtain

$$
\begin{aligned}
&\|v^*(t)\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} \\
&\quad \le \int_0^t \big\|\mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}_1(s)\mathcal{U}(s)(v_1(s)-v_2(s))\mathcal{U}(s)(v_1(s)+v_2(s))\big)\big\|_{\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})} ds \\
&\qquad\qquad + \int_0^t \big\|\mathcal{U}(-s)\big(\mathcal{U}(-s)(\bar{v}_1(s)-\bar{v}_2(s))(\mathcal{U}(s)v_2(s))^2\big)\big\|_{\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})} ds \\
&\quad \le C \int_0^t s^{-\frac{2}{p}+1} \|v_1(s)-v_2(s)\|_{\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})} \big(\|v_1(s)\|_{\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})} + \|v_2(s)\|_{\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})}\big)^2 ds \\
&\quad \le CT^{2-\frac{2}{p}} \sup_{0\le t\le T} \|v_1(s)-v_2(s)\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} \|u_0\|^2_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} \\
&\quad \le \tfrac{1}{2} \sup_{0\le t\le T} \|v_1(t)-v_2(t)\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})},
\end{aligned}
$$

where we choose T small enough that

$$
CT^{2-\frac{2}{p}} \|u_0\|^2_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} \le \tfrac{1}{2}.
$$

Thus we have proved that T is a contraction from X_T to itself, and then by a stand fixed point argument, we conclude that there exists a unique solution $v(t)$ to the integral equation (1-8).

It remains to prove the continuous dependence of the solution $v(t)$. Let $v_1(t)$ and $v_2(t)$ be two solutions of the integral equation (1-8), corresponding to the initial data u_{01} and u_{02}. Then

$$\begin{aligned} v_1(t)-v_2(t) &= (u_{01}-u_{02}) - i\lambda \int_0^t \mathcal{U}(-s)\big(\mathcal{U}(-s)(\bar v_1(s)-\bar v_2(s))(\mathcal{U}(s)v_1(s))^2\big)\,ds \\ &\quad - i\lambda \int_0^t \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar v_2(s)\mathcal{U}(s)(v_1(s)+v_2(s))\mathcal{U}(s)(v_1(s)-v_2(s))\big)\,ds. \end{aligned}$$

Using the same argument as in the proof of contraction just above, we further have

$$\begin{aligned} &\|v_1(t)-v_2(t)\|_{\dot B^\sigma_{p,1}(\mathbb{R})} \\ &\quad\le \|u_{01}-u_{02}\|_{\dot B^\sigma_{p,1}(\mathbb{R})} + CT^{2-\frac{2}{p}} \sup\nolimits_{0\le t\le T} \|v_1(t)-v_2(t)\|_{\dot B^{\sigma,H}_{p,1}} \|u_0\|^2_{\dot B^\sigma_{p,1}(\mathbb{R})} \\ &\quad\le \|u_{01}-u_{02}\|_{\dot B^\sigma_{p,1}(\mathbb{R})} + \tfrac{1}{2} \sup\nolimits_{0\le t\le T} \|v_1(t)-v_2(t)\|_{\dot B^\sigma_{p,1}(\mathbb{R})}, \end{aligned}$$

where T is chosen small enough that $CT^{2-\frac{2}{p}}\|u_0\|^2_{\dot B^\sigma_{p,1}(\mathbb{R})} \le \frac{1}{2}$. The inequality above further implies

$$\sup_{0\le t\le T} \|v_1(t)-v_2(t)\|_{\dot B^\sigma_{p,1}(\mathbb{R})} \leqslant 2\,\|u_{01}-u_{02}\|_{\dot B^\sigma_{p,1}(\mathbb{R})},$$

which finishes the proof. □

Proof of Corollary 1.2. Let $v(t)$ $(t\in[0,T))$ be the solution to the integral equation (1-8) given by Theorem 1.1 and $u(t)=\mathcal{U}(t)v(t)$. We will show that $u\in C\big((0,T),\dot B^s_{p',1}(\mathbb{R})\big)$ with $1<p<2$ and $s=1-\frac{1}{p}$. We proceed in three steps.

Step I. For $1<p<2$, $s=1-\frac{1}{p}$ and any $\phi\in\mathcal{S}(\mathbb{R})$, we have

(4-13)
$$\begin{aligned} \big\|\big(\mathcal{U}(t_1)-\mathcal{U}(t_2)\big)\phi\big\|_{\dot B^s_{p',1}(\mathbb{R})} &= \sum_{j\in\mathbb{Z}} 2^{js}\big\|\varphi_j(\sqrt{2H})\big(\mathcal{U}(t_1)-\mathcal{U}(t_2)\big)\phi\big\|_{L^{p'}(\mathbb{R})} \\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{js}\big\|\mathcal{F}_q\big((\mathcal{U}(t_1)-\mathcal{U}(t_2))\varphi_j(\sqrt{2H})\phi\big)\big\|_{L^p(\mathbb{R})} \\ &= \sum_{j\in\mathbb{Z}} 2^{js}\big\|(e^{-\frac{i}{2}t_1\xi^2}-e^{-\frac{i}{2}t_2\xi^2})\varphi_j(\xi)\mathcal{F}_q(\phi)\big\|_{L^p(\mathbb{R})} \\ &\lesssim |t_1-t_2|^{\frac14} \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac12}\varphi_j(\xi)\mathcal{F}_q(\phi)\big\|_{L^p(\mathbb{R})}, \end{aligned}$$

where we used the estimate $\|\mathcal{F}_q^{-1}\phi\|_{L^{p'}(\mathbb{R})} \lesssim \|\phi\|_{L^p(\mathbb{R})}$, which derives from Lemma 2.1. Next we prove that the right side of (4-13) is finite:

$$\sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac12}\varphi_j(\xi)\mathcal{F}_q(\phi)(\xi)\big\|_{L^p(\mathbb{R})} < \infty. \tag{4-14}$$

By the definition of the distorted Fourier transform, we have

$$\begin{aligned}\mathcal{F}_q(\phi)(\xi) = \mathcal{F}(\phi)(\xi) + \chi_+(\xi)\big(r_q(\xi)J\mathcal{F}(\chi_+\phi)(\xi) + r_q(\xi)\mathcal{F}(\chi_-\phi)(\xi)\big) \\ + \chi_-(\xi)\big(\overline{r_q(\xi)}J\mathcal{F}(\chi_-\phi)(\xi) + \overline{r_q(\xi)}\mathcal{F}(\chi_+\phi)(\xi)\big),\end{aligned}$$

where $Jf(x) = f(-x)$. Then it follows that

$$\begin{aligned}(4\text{-}15)\quad & \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac{1}{2}}\varphi_j(\xi)\mathcal{F}_q(\phi)(\xi)\big\|_{L^p(\mathbb{R})} \\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac{1}{2}}\varphi_j(\xi)\mathcal{F}(\phi)(\xi)\big\|_{L^p(\mathbb{R})} \\ &\quad + \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac{1}{2}}\varphi_j(\xi)\chi_+(\xi)r_q(\xi)J\mathcal{F}(\chi_+\phi)(\xi)\big\|_{L^p(\mathbb{R})} \\ &\quad + \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac{1}{2}}\varphi_j(\xi)\chi_+(\xi)r_q(\xi)\mathcal{F}(\chi_-\phi)(\xi)\big\|_{L^p(\mathbb{R})} \\ &\quad + \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac{1}{2}}\varphi_j(\xi)\chi_-(\xi)\overline{r_q(\xi)}\mathcal{F}(\chi_+\phi)(\xi)\big\|_{L^p(\mathbb{R})} \\ &\quad + \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac{1}{2}}\varphi_j(\xi)\chi_-(\xi)\overline{r_q(\xi)}J\mathcal{F}(\chi_-\phi)(\xi)\big\|_{L^p(\mathbb{R})}.\end{aligned}$$

For the first term on the right side, setting $\widetilde{\varphi}_j(\xi) = |\xi/2^j|^{\frac{1}{2}}\varphi_j(\xi)$, we have

$$\begin{aligned}(4\text{-}16)\quad \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac{1}{2}}\varphi_j(\xi)\mathcal{F}(\phi)(\xi)\big\|_{L^p(\mathbb{R})} &= \sum_{j\in\mathbb{Z}} 2^{j(s+\frac{1}{2})}\|\widetilde{\varphi}_j(\xi)\mathcal{F}(\phi)\|_{L^p(\mathbb{R})} \\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{j(s+\frac{1}{p})}\|\widetilde{\varphi}_j(\xi)\mathcal{F}(\phi)\|_{L^2(\mathbb{R})} \\ &= C\|\phi\|_{\dot{B}^{s+\frac{1}{p}}_{2,1}(\mathbb{R})}.\end{aligned}$$

The remaining terms on the right side of (4-15) are of the same type, so we only treat the third one. We write

$$\begin{aligned}(4\text{-}17)\quad & \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac{1}{2}}\varphi_j(\xi)\chi_+(\xi)r_q(\xi)\mathcal{F}(\chi_-\phi)(\xi)\big\|_{L^p(\mathbb{R})} \\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac{1}{2}}\varphi_j(\xi)r_q(\xi)\mathcal{F}\big((1-\operatorname{sgn}\cdot)\phi\big)(\xi)\big\|_{L^p(\mathbb{R})} \\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{j(s+\frac{1}{2})}\|\widetilde{\varphi}_j(\xi)\mathcal{F}(\phi)(\xi)\|_{L^p(\mathbb{R})} \\ &\qquad + \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac{1}{2}}\varphi_j(\xi)r_q(\xi)(\tfrac{1}{\cdot} * \mathcal{F}(\phi))(\xi)\big\|_{L^p(\mathbb{R})} \\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{j(s+\frac{1}{p})}\|\widetilde{\varphi}_j(\xi)\mathcal{F}(\phi)(\xi)\|_{L^2(\mathbb{R})} \\ &\qquad + \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac{1}{2}}\varphi_j(\xi)r_q(\xi)(\tfrac{1}{\cdot} * \mathcal{F}(\phi))(\xi)\big\|_{L^p(\mathbb{R})} \\ &\lesssim \|\phi\|_{\dot{B}^{s+\frac{1}{p}}_{2,1}(\mathbb{R})} + \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac{1}{2}}\varphi_j(\xi)r_q(\xi)(\frac{1}{\cdot} * \varphi_k\hat{\phi}_k)(\xi)\big\|_{L^p(\mathbb{R})},\end{aligned}$$

where $\widetilde{\varphi}_j(\xi)=|\xi/2^j|^{\frac{1}{2}}\varphi_j(\xi)$, and make the decomposition

$$\phi=\sum_{k\in\mathbb{Z}}\varphi_k(\sqrt{2H_0})\phi=\sum_{k\in\mathbb{Z}}\varphi_k(\sqrt{2H_0})\widetilde{\psi}_k(\sqrt{2H_0})\phi=\sum_{k\in\mathbb{Z}}\varphi_k(\sqrt{2H_0})\phi_k$$

with

$$\phi_k=\widetilde{\psi}_k(\sqrt{2H_0})\phi.$$

Setting $R_{j,k}(\xi)=|\xi|^{\frac{1}{2}}\varphi_j(\xi)r_q(\xi)(\frac{1}{\cdot}*\varphi_k\hat{\phi}_k)(\xi)$ in (4-17), we have

$$\begin{aligned}R_{j,k}(\xi)&=|\xi|^{\frac{1}{2}}\varphi_j(\xi)r_q(\xi)\int\frac{1}{\xi-y}\varphi_k(y)\hat{\phi}_k(y)\,dy\\&=\frac{1}{\sqrt{2\pi}}|\xi|^{\frac{1}{2}}\varphi_j(\xi)r_q(\xi)\iint\frac{1}{\xi-y}\varphi_k(y)e^{-iyz}\,dy\phi_k(z)\,dz\\&=:|\xi|^{\frac{1}{2}}\varphi_j(\xi)r_q(\xi)\int\mathcal{K}(\xi,z)\phi_k(z)\,dz\end{aligned}$$

with

$$\mathcal{K}(\xi,z)=\frac{1}{\sqrt{2\pi}}\int\frac{1}{\xi-y}\varphi_k(y)e^{-iyz}\,dy.$$

When $|k-j|\le 3$, since $|\cdot|^{\frac{1}{2}}r_q\in L^\infty(\mathbb{R})$, it follows from Hölder's inequality and the $L^2(\mathbb{R})$ boundedness of the Hilbert transform $\mathcal{H}$ that

$$\begin{aligned}\|R_{j,k}\|_{L^p(\mathbb{R})}&\lesssim 2^{(\frac{1}{p}-\frac{1}{2})j}\big\|\mathcal{H}(\varphi_k\hat{\phi}_k)\big\|_{L^2(\mathbb{R})}\lesssim 2^{(\frac{1}{p}-\frac{1}{2})j}\big\|\varphi_k\hat{\phi}_k\big\|_{L^2(\mathbb{R})}\\&\lesssim 2^{(\frac{1}{p}-\frac{1}{2})j}\|\phi_k\|_{L^2(\mathbb{R})}.\end{aligned}\tag{4-18}$$

When $k>j+3$, by integration by parts, we have, for any integer $N>0$,

$$\begin{aligned}|\mathcal{K}(\xi,z)|&\lesssim 2^k\left|\int\frac{1}{\xi-2^ky}e^{-i2^kz\cdot y}\varphi(y)\,dy\right|\\&\lesssim\frac{2^k}{\langle 2^kz\rangle^N}\int\left|\partial_y^N\Big(\frac{1}{\xi-2^ky}\varphi(y)\Big)\right|dy\lesssim\frac{1}{\langle 2^kz\rangle^N}.\end{aligned}$$

Combined with the fact that $|\cdot|^{\frac{1}{2}}r_q\in L^\infty(\mathbb{R})$ and Hölder's inequality, this implies

$$\begin{aligned}\|R_{j,k}\|_{L^p(\mathbb{R})}&\le\|\varphi_j\|_{L^p(\mathbb{R})}\left\|\int\mathcal{K}(\cdot,z)\phi_k(z)\,dz\right\|_{L^\infty(\mathbb{R})}\\&\lesssim 2^{\frac{1}{p}(j-k)}\|\phi_k\|_{L^{p'}(\mathbb{R})}.\end{aligned}\tag{4-19}$$

When $k<j-3$, similarly as above, we have

$$\begin{aligned}|\mathcal{K}(\xi,z)|&\lesssim 2^k\left|\int\frac{1}{\xi-2^ky}\varphi(y)e^{-i2^kzy}\,dy\right|\\&\lesssim 2^k\langle 2^kz\rangle^{-N}\int\left|\partial_y^N\Big(\frac{\varphi(y)}{\xi-2^ky}\Big)\right|dy\lesssim\frac{2^{k-j}}{\langle 2^kz\rangle^N},\end{aligned}$$

which, combined with the inequality $\big||\xi|^{\frac{1}{2}} r_q(\xi)\big| \lesssim (1+|\xi|)^{-\frac{1}{2}}$, leads to

$$\|R_{j,k}\|_{L^p(\mathbb{R})} \le \big\||\cdot|^{\frac{1}{2}}\varphi_j r_q\big\|_{L^p(\mathbb{R})} \Big\|\int \mathcal{K}(\cdot,z)\phi_k(z)\,dz\Big\|_{L^\infty(\mathbb{R})} \lesssim 2^{-\frac{1}{p'}(j-k)-\frac{j}{2}}\|\phi_k\|_{L^{p'}(\mathbb{R})}. \tag{4-20}$$

Combining (4-18)–(4-20), we have

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{js}\|R_{j,k}\|_{L^p(\mathbb{R})} \\
&\le \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{js}\big(\chi_{\le 3}(|k-j|)+\chi_{>3}(k-j)+\chi_{<-3}(k-j)\big)\|R_{j,k}\|_{L^p(\mathbb{R})} \\
&\lesssim \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{js}\Big(\chi_{\le 3}(|k-j|)2^{(\frac{1}{p}-\frac{1}{2})j}\|\phi_k\|_{L^2} \\
&\qquad\qquad + \chi_{>3}(k-j)2^{\frac{1}{p}(j-k)}\|\phi_k\|_{L^{p'}(\mathbb{R})} \\
&\qquad\qquad + \chi_{<-3}(k-j)2^{-\frac{1}{p'}(j-k)-\frac{j}{2}}\|\phi_k\|_{L^{p'}(\mathbb{R})}\Big) \\
&\lesssim \|\phi\|_{\dot B^{s+\frac{2-p}{2p}}_{2,1}(\mathbb{R})} + \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{-(k-j)(s+\frac{1}{p})}\chi_{>3}(k-j)2^{ks}\|\phi_k\|_{L^{p'}(\mathbb{R})} \\
&\qquad + \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{-(\frac{1}{2}+\frac{1}{p'}-s)(j-k)}\chi_{<-3}(k-j)2^{k(s-\frac{1}{2})}\|\phi_k\|_{L^{p'}(\mathbb{R})} \\
&\lesssim \|\phi\|_{\dot B^{s+\frac{2-p}{2p}}_{2,1}(\mathbb{R})} + \|\phi\|_{\dot B^s_{p',1}(\mathbb{R})} + \|\phi\|_{\dot B^{s-\frac{1}{2}}_{p',1}(\mathbb{R})},
\end{aligned} \tag{4-21}$$

where the discrete Young's inequality is used in the last inequality. By (4-16), (4-17) and (4-21), we show that (4-15) is finite, which verifies (4-14). Hence for $1<p<2$, $s=1-\frac{1}{p}$ and any $\phi\in\mathcal{S}(\mathbb{R})$, by (4-13), we have

$$\|\big(\mathcal{U}(t_1)-\mathcal{U}(t_2)\big)\phi\|_{\dot B^s_{p',1}(\mathbb{R})} \lesssim |t_1-t_2|^{\frac{1}{4}}. \tag{4-22}$$

<u>Step II.</u> For general $\phi\in\dot B^s_{p',1}(\mathbb{R})$ with $1<p<2$ and $s=1-\frac{1}{p}$, let $\tilde\phi\in\mathcal{S}(\mathbb{R})$ and $t_1,t_2\in\mathbb{R}$, by the decay estimates of $\mathcal{U}(t)$ (see the proof of Corollary 2.4), we have

$$\begin{aligned}
&\big\|\big(\mathcal{U}(t_1)-\mathcal{U}(t_2)\big)\phi\big\|_{\dot B^s_{p',1}(\mathbb{R})} \\
&\lesssim \big\|\mathcal{U}(t_1)\phi-\mathcal{U}(t_1)\tilde\phi\big\|_{\dot B^s_{p',1}(\mathbb{R})} + \big\|\mathcal{U}(t_1)\tilde\phi-\mathcal{U}(t_2)\tilde\phi\big\|_{\dot B^s_{p',1}(\mathbb{R})} + \big\|\mathcal{U}(t_2)\phi-\mathcal{U}(t_2)\tilde\phi\big\|_{\dot B^s_{p',1}(\mathbb{R})} \\
&\le |t_1|^{-(\frac{1}{p}-\frac{1}{2})}\big\|\phi-\tilde\phi\big\|_{\dot B^s_{p',1}(\mathbb{R})} + \big\|\mathcal{U}(t_1)\tilde\phi-\mathcal{U}(t_2)\tilde\phi\big\|_{\dot B^s_{p',1}(\mathbb{R})} + |t_2|^{-(\frac{1}{p}-\frac{1}{2})}\big\|\phi-\tilde\phi\big\|_{\dot B^s_{p',1}(\mathbb{R})},
\end{aligned}$$

which, combined with (4-22) and the standard $\frac{\varepsilon}{3}$-argument, implies that for any $\phi\in L^p(\mathbb{R})$ with $1<p<2$ and $s=1-\frac{1}{p}$, the map $t\to\mathcal{U}(t)\phi$ is continuous from $\mathbb{R}\setminus\{0\}$ to $\dot B^s_{p',1}(\mathbb{R})$.

<u>Step III.</u> Let $v(t)$ be the solution of the integral equation (1-8) given by Theorem 1.1 and let $u(t)=\mathcal{U}(t)v(t)$, for any $t,t_0\in(0,T)$. It follows from the decay estimates

of $\mathcal{U}(t)$ that

$$\begin{aligned}
&\|u(t)-u(t_0)\|_{\dot{B}^s_{p',1}(\mathbb{R})}\\
&\quad\lesssim \|\mathcal{U}(t)\big(\mathcal{U}(-t)u(t)-\mathcal{U}(-t_0)u(t_0)\big)\|_{\dot{B}^s_{p',1}(\mathbb{R})}\\
&\qquad\qquad\qquad + \|\mathcal{U}(t)\mathcal{U}(-t_0)u(t_0)-\mathcal{U}(t_0)\mathcal{U}(-t_0)u(t_0)\|_{\dot{B}^s_{p',1}(\mathbb{R})}\\
&\quad\lesssim |t|^{-(\frac{1}{p}-\frac{1}{2})}\|v(t)-v(t_0)\|_{\dot{B}^s_{p',1}(\mathbb{R})}+\|\mathcal{U}(t)v(t_0)-\mathcal{U}(t_0)v(t_0)\|_{\dot{B}^s_{p',1}(\mathbb{R})}.
\end{aligned}$$

Letting $t\to t_0$, the first term of the right side tends to 0 by the assumption, and the second term goes to 0 by Step II. This concludes the proof. □

4.2. ***Proof of Theorem 1.3.*** We now prove Theorem 1.3 and Corollary 1.4, the well-posedness of the integral equation (1-8) in $L^p(\mathbb{R})$. Set

$$v_0(t)=\mathcal{U}(-t)\big(\mathcal{U}(-t)\bar{v}_1(t)\mathcal{U}(t)v_2(t)\mathcal{U}(t)v_3(t)\big).$$

The next lemma will be used to treat the nonlinear term.

Lemma 4.3. *Let $1<p<2$. For any $T>0$,*

$$\big\|t^{\frac{2}{p}-1}v_0(t)\big\|_{L_t^{p'}L_x^p([0,T]\times\mathbb{R})}\lesssim\prod_{j=1}^{3}\big(\|v_j(0)\|_{L^p(\mathbb{R})}+\|\partial_t v_j(t)\|_{L_t^1L_x^p([0,T]\times\mathbb{R})}\big). \tag{4-23}$$

Proof. Notice that

$$v_j(t)=v_j(0)+\int_0^t\partial_\tau v_j(\tau)\,d\tau.$$

By the L^1 estimate for the trilinear form in the proof of Lemma 4.2, we have

$$\begin{aligned}
&\|tv_0(t)\|_{L_t^\infty L_x^1([0,T]\times\mathbb{R})}\\
&\quad\lesssim \|M(t)\mathcal{L}v_1(t)\|_{H_x^1(\mathbb{R})}\|v_2(t)\|_{L_x^1(\mathbb{R})}\|v_3(t)\|_{L_x^1(\mathbb{R})}\\
&\quad\lesssim \Big(\|M(t)\mathcal{L}v_1(0)\|_{H_x^1(\mathbb{R})}+\Big\|M(t)\mathcal{L}\int_0^t\partial_\tau v_i(\tau)\,d\tau\Big\|_{H_x^1(\mathbb{R})}\Big)\\
&\qquad\qquad\times\Big(\|v_2(0)\|_{L_x^1(\mathbb{R})}+\Big\|\int_0^t\partial_\tau v_2(\tau)\,d\tau\Big\|_{L_x^1(\mathbb{R})}\Big)\\
&\qquad\qquad\times\Big(\|v_3(0)\|_{L_x^1(\mathbb{R})}+\Big\|\int_0^t\partial_\tau v_3(\tau)\,d\tau\Big\|_{L_x^1(\mathbb{R})}\Big).
\end{aligned}$$

Next we show that (4-23) holds for $p=2$. To see this, let

$$u_j(t)=\mathcal{U}(t)v_j(t)\quad(j=1,2,3).$$

It is easy to see that

$$i\partial_t u_j(t)=Hu_j(t)+i\mathcal{U}(t)\partial_t v_j(t)\quad\text{and}\quad u_j(0)=v_j(0).$$

By Duhamel's formula, we have for $j=1,2,3$,

$$u_j(t) = \mathcal{U}(t)v_j(0) - i\int_0^t \mathcal{U}(t-s)\mathcal{U}(s)\partial_s v_j(s)\,ds = \mathcal{U}(t)\Big(v_j(0) - i\int_0^t \partial_s v_j(s)\,ds\Big), \tag{4-24}$$

which, combined with the Strichartz estimates from Corollary 2.4, implies

$$\|u_j(t)\|_{L_t^6L_x^6([0,T]\times\mathbb{R})} \lesssim \|v_j(0)\|_{L^2(\mathbb{R})} + \Big\|\int_0^t \partial_s v_j(s)\,ds\Big\|_{L^2(\mathbb{R})}.$$

On the other hand, it follows from Remark 2.5, (4-24) and the Strichartz estimates of $\mathcal{U}_0(t)$ that

$$\begin{aligned}\|u_1(t)\|_{L_t^6L_x^6([0,T]\times\mathbb{R})} &= \Big\|\chi_+\mathcal{U}_0(t)\mathcal{L}_+\Big(v_1(0) - i\int_0^t \partial_s v_1(s)\,ds\Big) \\ &\qquad + \chi_-\mathcal{U}_0(t)\mathcal{L}_-\Big(v_1(0) - i\int_0^t \partial_s v_1(s)\,ds\Big)\Big\|_{L_t^6L_x^6([0,T]\times\mathbb{R})} \\ &\lesssim \|M(t)\mathcal{L}v_1(0)\|_{L_x^2(\mathbb{R})} + \Big\|M(t)\mathcal{L}\int_0^t \partial_\tau v_i(\tau)\,d\tau\Big\|_{L_x^2(\mathbb{R})}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\|v_0(t)\|_{L_t^2L_x^2([0,T]\times\mathbb{R})} \\ &\quad\le \prod_{i=1}^3 \|u_i(t)\|_{L_t^6L_x^6([0,T]\times\mathbb{R})} \\ &\quad\lesssim \Big(\|M(t)\mathcal{L}v_1(0)\|_{L_x^2(\mathbb{R})} + \|M(t)\mathcal{L}\int_0^t \partial_\tau v_i(\tau)\,d\tau\|_{L_x^2(\mathbb{R})}\Big) \\ &\qquad\times\Big(\|v_2(0)\|_{L_x^2} + \Big\|\int_0^t \partial_s v_2(s)\,ds\Big\|_{L_x^2(\mathbb{R})}\Big)\Big(\|v_3(0)\|_{L_x^2} + \Big\|\int_0^t \partial_s v_3(s)\,ds\Big\|_{L_x^2(\mathbb{R})}\Big).\end{aligned}$$

By the interpolation theorem on the multilinear functionals (see Bergh and Löfström [8, Theorme 4.4.1]) and the fact that both $M(t)$ and $\mathcal{L}_\pm$ are bounded on $L^p(\mathbb{R})$ for all $1 \le p \le \infty$, we obtain the desired estimate. □

Proof of Theorem 1.3. Recall the integral equation

$$v(t) = u_0 - i\lambda\int_0^t \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}(s)(\mathcal{U}(s)v(s))^2\big)\,ds.$$

For fixed $1 < p < 2$, we define the space

$$X_T = \big\{v \mid v(0) = u_0,\ \|t^{\frac{2}{p}-1}\partial_t v(t)\|_{L_t^{p'}L_x^p([0,T]\times\mathbb{R})} \le C_1\|u_0\|_{L^p(\mathbb{R})}^3\big\},$$

where $C_1 > 0$ is a large constant independent of the initial data u_0, and a map

$$Tv(t) = u_0 - i\lambda\int_0^t \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}(s)(\mathcal{U}(s)v(s))^2\big)\,ds.$$

We show that T maps X_T to itself. For any $v \in X_T$, we have $Tv(0) = u_0$ and

$$\partial_t(Tv(t)) = -i\lambda\mathcal{U}(-t)\big(\mathcal{U}(-t)\bar{v}(t)(\mathcal{U}(t)v(t))^2\big) = -i\lambda v_0(t).$$

Then it follows from Lemma 4.3 and Hölder's inequality that

$$
\begin{aligned}
\left\|t^{\frac{2}{p}-1}\partial_t(Tv(t))\right\|_{L_t^{p'}L_x^p([0,T]\times\mathbb{R})}
&\le C\big(\|u_0\|_{L^p(\mathbb{R})}+\|\partial_t v(t)\|_{L_t^1L_x^p([0,T]\times\mathbb{R})}\big)^3\\
&\le C\big(\|u_0\|_{L^p(\mathbb{R})}+\|t^{-\frac{2}{p}+1}\|_{L_t^p([0,T])}\,\|t^{\frac{2}{p}-1}\partial_t v(t)\|_{L_t^{p'}L_x^p([0,T]\times\mathbb{R})}\big)^3\\
&\le C\big(\|u_0\|_{L^p(\mathbb{R})}+CT^{\frac{1}{p'}}C_1\|u_0\|_{L^p(\mathbb{R})}^3\big)^3\\
&\le C_1\|u_0\|_{L^p(\mathbb{R})}^3,
\end{aligned}
$$

where we choose T small enough that $CC_1T^{\frac{1}{p'}}\|u_0\|_{L^p(\mathbb{R})}^2\le 1$.

Now we show that T is a contraction on X_T; that is, for any $v_1, v_2\in X_T$,

$$
\left\|t^{\frac{2}{p}-1}\partial_t(Tv_1(t)-Tv_2(t))\right\|_{L_t^{p'}L_x^p([0,T]\times\mathbb{R})}\le \tfrac{1}{2}\left\|t^{\frac{2}{p}-1}\partial_t(v_1(t)-v_2(t))\right\|_{L_t^{p'}L_x^p([0,T]\times\mathbb{R})}.
$$

Note that

$$
\begin{aligned}
&\partial_t\big(Tv_1(t)-Tv_2(t)\big)\\
&\qquad=-i\lambda\mathcal{U}(-t)\big(\mathcal{U}(-t)(\bar v_1(t)-\bar v_2(t))(\mathcal{U}(t)v_1(t))^2\big)\\
&\qquad\quad-i\lambda\mathcal{U}(-t)\big(\mathcal{U}(-t)\bar v_2(t)\mathcal{U}(t)(v_1(t)+v_2(t))\mathcal{U}(t)(v_1(t)-v_2(t))\big).
\end{aligned}
$$

The two terms on the right side of this identity are of the same type as $v_0(t)$. We apply the same argument used in the proof of Lemma 4.3 to get (henceforth we abbreviate $L_t^1L_x^p([0,T]\times\mathbb{R})$ to $L_t^1L_x^p$)

$$
\begin{aligned}
&\left\|t^{\frac{2}{p}-1}\partial_t(Tv_1(t)-Tv_2(t))\right\|_{L_t^{p'}L_x^p}\\
&\quad\le C\big(\|v_1(0)\|_{L^p(\mathbb{R})}+\|\partial_t v_1(t)\|_{L_t^1L_x^p}\big)^2\|\partial_t(v_1(t)-v_2(t))\|_{L_t^1L_x^p}\\
&\qquad+C\big(\|(v_1+v_2)(0)\|_{L^p(\mathbb{R})}+\|\partial_t(v_1+v_2)(t)\|_{L_t^1L_x^p}\big)\\
&\qquad\qquad\times\big(\|v_2(0)\|_{L^p(\mathbb{R})}+\|\partial_t v_2(t)\|_{L_t^1L_x^p}\big)\|\partial_t(v_1(t)-v_2(t))\|_{L_t^1L_x^p}\\
&\quad\le C\big(\|v_1(0)\|_{L^p(\mathbb{R})}+\|v_2(0)\|_{L^p(\mathbb{R})}+\|\partial_t v_1(t)\|_{L_t^1L_x^p}+\|\partial_t v_2(t)\|_{L_t^1L_x^p}\big)^2\\
&\qquad\qquad\times\|\partial_t(v_1(t)-v_2(t))\|_{L_t^1L_x^p}.
\end{aligned}
\tag{4-25}
$$

For $v_i\in X_T$ $(i=1,2)$, by Hölder's inequality, we have

$$
\begin{aligned}
\|\partial_t v_i(t)\|_{L_t^1L_x^p}&\le\|t^{-\frac{2}{p}+1}\|_{L_t^p[0,T]}\|t^{\frac{2}{p}-1}\partial_t v(t)\|_{L_t^{p'}L_x^p}\\
&\le CT^{\frac{1}{p'}}C_1\|u_0\|_{L^p(\mathbb{R})}^3.
\end{aligned}
$$

Therefore,

$$\begin{aligned}\big\|t^{\frac{2}{p}-1}\partial_t\big(Tv_1(t)-Tv_2(t)\big)\big\|_{L_t^{p'}L_x^p}&\\ \le CT^{\frac{1}{p'}}\big(\|u_0\|_{L^p(\mathbb{R})}+CT^{\frac{1}{p'}}C_1\|u_0\|^3_{L^p(\mathbb{R})}\big)^2\big\|t^{\frac{2}{p}-1}\partial_t(v_1(t)-v_2(t))\big\|_{L_t^{p'}L_x^p}&\\ \le \tfrac{1}{2}\big\|t^{\frac{2}{p}-1}\partial_t(v_1(t)-v_2(t))\big\|_{L_t^{p'}L_x^p},&\end{aligned}$$

provided that T of order $\|u_0\|^{-2}_{L^p(\mathbb{R})}$ is small enough.

It remains to prove stability. Let $u_{01}, u_{02}\in L^p(\mathbb{R})$ and let $v_1(t), v_2(t)$ be the corresponding solutions of the integral equation (1-8). Then

$$\begin{aligned}v_1(t)-v_2(t)(u_{01}-u_{02})-i\lambda\int_0^t\mathcal{U}(-s)\big(\mathcal{U}(-s)(\bar v_1(s)-\bar v_2(s))(\mathcal{U}(s)v_1(s))^2\big)\,ds&\\ -i\lambda\int_0^t\mathcal{U}(-s)\big(\mathcal{U}(-s)\bar v_2(s)\mathcal{U}(s)(v_1(s)+v_2(s))\mathcal{U}(s)(v_1(s)-v_2(s))\big)\,ds.&\end{aligned}$$

Similarly to (4-25), we have

$$\begin{aligned}\big\|t^{\frac{2}{p}-1}\partial_t(v_1(t)-v_2(t))\big\|_{L_t^{p'}L_x^p}&\\ \le\big(\|u_{01}\|_{L^p(\mathbb{R})}+\|u_{02}\|_{L^p(\mathbb{R})}+CT^{\frac{1}{p'}}C_1\|u_{01}\|^3_{L^p(\mathbb{R})}+CT^{\frac{1}{p'}}C_1\|u_{02}\|^3_{L^p(\mathbb{R})}\big)^2&\\ \times\big(\|u_{01}-u_{02}\|_{L^p(\mathbb{R})}+CT^{\frac{1}{p'}}\big\|t^{\frac{2}{p}-1}\partial_t(v_1(t)-v_2(t))\big\|_{L_t^{p'}L_x^p}\big),&\end{aligned}$$

which implies

$$\big\|t^{\frac{2}{p}-1}\partial_t(v_1(t)-v_2(t))\big\|_{L_t^{p'}L_x^p}\le C\big(\|u_{01}\|_{L^p(\mathbb{R})}+\|u_{02}\|_{L^p(\mathbb{R})}\big)^2\|u_{01}-u_{02}\|_{L^p(\mathbb{R})},$$

by choosing T of order $(\|u_{01}\|_{L^p(\mathbb{R})}+|u_{02}\|_{L^p(\mathbb{R})})^{-2}$ to be sufficiently small. It follows from the last inequality that

$$\begin{aligned}\|v_1(t)-v_2(t)\|_{L^p(\mathbb{R})}&\\ \le\|u_{01}-u_{02}\|_{L^p(\mathbb{R})}+\Big(\int_0^T t^{-(\frac{2}{p}-1)p}\,dt\Big)^{\frac{1}{p}}\big\|t^{\frac{2}{p}-1}\partial_t(v_1(t)-v_2(t))\big\|_{L_t^{p'}L_x^p}&\\ \le\|u_{01}-u_{02}\|_{L^p(\mathbb{R})}+CT^{\frac{1}{p'}}\big(\|u_{01}\|_{L^p(\mathbb{R})}+\|u_{02}\|_{L^p(\mathbb{R})}\big)^2\|u_{01}-u_{02}\|_{L^p(\mathbb{R})}.&\end{aligned}$$

Provided that T of order $(\|u_{01}\|_{L^p(\mathbb{R})}+|u_{02}\|_{L^p(\mathbb{R})})^{-2}$ is sufficiently small, this gives the stability of the solution.

To conclude, the same procedure as above also implies that for the solution $v(t)$ with initial data u_0,

$$\|v(t)\|_{L^p(\mathbb{R})}\le C\|u_0\|_{L^p(\mathbb{R})},\quad t\in[0,T),$$

where T of order $\|u_0\|^{-2}_{L^p(\mathbb{R})}$ is small enough. □

Proof of Corollary 1.4. Let $v(t)$ ($t\in(0,T)$) be the solution to the integral equation (1-8) given by Theorem 1.3 and let $u(t)=U(t)v(t)$. We will show that $u\in C\big((0,T),L^{p'}(\mathbb{R})\big)$ with $1<p<2$.

We claim that for any $\phi \in L^p(\mathbb{R})$ with $1 < p < 2$, the map $t \to \mathcal{U}(t)\phi$ is continuous from $\mathbb{R} \setminus \{0\}$ to $L^{p'}(\mathbb{R})$. For any $\phi \in \mathcal{S}(\mathbb{R})$, we write

$$\phi = um_{k\in\mathbb{Z}}\varphi_k(\sqrt{2H})\phi,$$

where $\varphi_k(\sqrt{2H})$ is the standard Littlewood–Paley projection. Then for any $t_1, t_2 \in (0, T)$, the same procedure used in Step I of the proof of Corollary 1.2 leads to

$$\begin{aligned} \big\|\big(\mathcal{U}(t_1)-\mathcal{U}(t_2)\big)\phi\big\|_{L^{p'}(\mathbb{R})} &\lesssim \sum_{k\in\mathbb{Z}} \big\|\big(\mathcal{U}(t_1)-\mathcal{U}(t_2)\big)\varphi_k(\sqrt{2H})\phi\big\|_{L^{p'}(\mathbb{R})} \\ &\le C|t_1-t_2|^{\frac{1}{4}}, \end{aligned} \tag{4-26}$$

where the constant C depends on some Besov norm of ϕ. For general $\phi \in L^p(\mathbb{R})$ with $1 < p < 2$, we choose $\tilde{\phi} \in \mathcal{S}(\mathbb{R})$ such that

$$\begin{aligned} &\big\|\mathcal{U}(t_1)\phi - \mathcal{U}(t_2)\phi\big\|_{L^{p'}(\mathbb{R})} \\ &\quad\le \big\|\mathcal{U}(t_1)\phi-\mathcal{U}(t_1)\tilde{\phi}\big\|_{L^{p'}(\mathbb{R})} + \big\|\mathcal{U}(t_1)\tilde{\phi}-\mathcal{U}(t_2)\tilde{\phi}\big\|_{L^{p'}(\mathbb{R})} + \big\|\mathcal{U}(t_2)\tilde{\phi}-\mathcal{U}(t_2)\phi\big\|_{L^{p'}(\mathbb{R})} \\ &\quad\lesssim |t_1|^{-(\frac{1}{p}-\frac{1}{2})}\|\phi-\tilde{\phi}\|_{L^p(\mathbb{R})} + \big\|\mathcal{U}(t_1)\tilde{\phi}-\mathcal{U}(t_2)\tilde{\phi}\big\|_{L^{p'}(\mathbb{R})} + |t_2|^{-(\frac{1}{p}-\frac{1}{2})}\|\phi-\tilde{\phi}\|_{L^p(\mathbb{R})}, \end{aligned}$$

where we use the decay estimates of $\mathcal{U}(t)$ (see the proof of Corollary 2.4). From this estimate and (4-26), via the standard $\frac{\varepsilon}{3}$-argument, the claim follows.

For any $t_0, t \in (0, T)$, it follows from the decay estimate of $\mathcal{U}(t)$ that

$$\begin{aligned} &\|u(t)-u(t_0)\|_{L^{p'}(\mathbb{R})} \\ &\quad\le \big\|\mathcal{U}(t)(\mathcal{U}(-t)u(t)-\mathcal{U}(-t_0)u(t_0))\big\|_{L^{p'}(\mathbb{R})} \\ &\qquad\qquad + \big\|\mathcal{U}(t)\mathcal{U}(-t_0)u(t_0)-\mathcal{U}(t_0)\mathcal{U}(-t_0)u(t_0)\big\|_{L^{p'}(\mathbb{R})} \\ &\quad\lesssim |t|^{-(\frac{1}{p}-\frac{1}{2})}\|v(t)-v(t_0)\|_{L^p(\mathbb{R})} + \big\|(\mathcal{U}(t)-\mathcal{U}(t_0))\mathcal{U}(-t_0)u(t_0)\big\|_{L^{p'}(\mathbb{R})}. \end{aligned}$$

Letting $t \to t_0$, the first term on the last line tends to 0 by assumption, and the second term goes to 0 by the above claim. □

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