Pacific Journal of Mathematics

FINITE BASIS PROBLEM FOR INVOLUTION SEMIGROUPS OF ORDER FOUR

MENG GAO, EDMOND W. H. LEE, YAN FENG LUO AND WEN TING ZHANG

Volume 342 No. 1 May 2026

PACIFIC JOURNAL OF MATHEMATICS
Vol. 342, No. 1, 2026
https://doi.org/10.2140/pjm.2026.342.163

FINITE BASIS PROBLEM FOR INVOLUTION SEMIGROUPS OF ORDER FOUR

MENG GAO, EDMOND W. H. LEE, YAN FENG LUO AND WEN TING ZHANG

Since the 1980s, it has been known that the smallest non-finitely based semigroups are of order six. Surprisingly, for involution semigroups, a non-finitely based example of order five was recently discovered. In this article, it is confirmed that every involution semigroup of order four is finitely based. Since every involution semigroup of order three or less is already known to be finitely based, it follows that the smallest non-finitely based involution semigroups are of order five.

1. Introduction

1.1. ***Minimal non-finitely based involution semigroups.*** An *identity basis* for an algebra A is a set of identities of A that axiomatizes all the identities of A. An algebra is *finitely based* if it has some finite identity basis; otherwise, it is *non-finitely based*. A prominent research problem in universal algebra is the *finite basis problem*: determine which finite algebras are finitely based. Finite groups [31], finite associative rings [3], finite Lie rings [13; 27], and finite lattices [29] are finitely based, but in general, not all finite algebras are finitely based. For instance, the multiplicative matrix semigroup

$$B_2^1 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\},$$

published by Perkins in 1969, is non-finitely based [32]. The discovery of this example focused much attention upon the finite basis problem for small semigroups. Decades of cumulative work that followed has shown that every semigroup of order five or less is finitely based [15; 33; 34], and among all semigroups of order six — 28,634 of them up to isomorphism [5] — only four are non-finitely based [23; 25; 26]. The four non-finitely based semigroups of order six, which include B_2^1, are thus *minimal non-finitely based*.

This research was partially supported by the National Natural Science Foundation of China (nos. 12271224, 12571018, and 12171213) and the Fundamental Research Funds for the Central Universities (no. lzujbky-2023-ey06).
Zhang is the corresponding author.
MSC2020: 20M05.
Keywords: semigroup, involution semigroup, identity basis, finitely based, finite basis problem.

The present article is concerned with *involution semigroups* $(S,{}^*)$, that is, semigroups S with a unary operation * that satisfy the identities

$$(x^*)^* \approx x, \quad (xy)^* \approx y^*x^*; \tag{1-1}$$

the unary operation * is called an *involution* of S. An *inverse semigroup* is an involution semigroup $(S,{}^*)$ that satisfies the additional identities

$$xx^*x \approx x, \quad xx^*yy^* \approx yy^*xx^*.$$

Examples of inverse semigroups include any group $(G,{}^{-1})$ with inversion $^{-1}$ and the Perkins semigroup $(B_2^1,{}^{\mathsf{T}})$ under the usual matrix transposition $^{\mathsf{T}}$. Examples of involution semigroups that are not inverse semigroups include the multiplicative $n\times n$ matrix semigroup $(M_n(\mathbb{F}),{}^{\mathsf{T}})$ over any field $\mathbb{F}$ with the usual transposition $^{\mathsf{T}}$ and the Perkins semigroup $(B_2^1,{}^{\mathsf{S}})$ under the *skew transposition* $^{\mathsf{S}}$ across the secondary diagonal, that is,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{\mathsf{S}} = \begin{bmatrix} d & b \\ c & a \end{bmatrix}.$$

Given how close involution semigroups are to semigroups, it seems reasonable to conjecture that a finite involution semigroup $(S,{}^*)$ and its semigroup *reduct* S are always simultaneously finitely based. For instance, the involution semigroups $(B_2^1,{}^{\mathsf{T}})$ and $(B_2^1,{}^{\mathsf{S}})$ are both non-finitely based [2; 12], while their reduct B_2^1 is also non-finitely based [32]. However, this conjecture has been refuted by several counterexamples [8; 11; 16; 19], the smallest of which is the multiplicative matrix semigroup

$$A_0^1 = \left\{ \begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 1&1&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0 \\ 0&0&1 \\ 0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix} \right\}$$

under the skew transposition $^{\mathsf{S}}$. As noted above, all semigroups of order five or less — which include A_0^1 — are finitely based, but the involution semigroup $(A_0^1,{}^{\mathsf{S}})$ is non-finitely based [8]. It follows that minimal non-finitely based involution semigroups are of order at most five; this result is quite unexpected given that minimal non-finitely based semigroups are of order six [26].

It is of fundamental importance to examine if there exists a non-finitely based involution semigroup that is smaller than $(A_0^1,{}^{\mathsf{S}})$. Since every involution semigroup of order three or less is finitely based [20], the answer would require addressing the finite basis problem for those of order four. Solving this problem is the objective of the present article.

Theorem 1.1. *Every involution semigroup of order four is finitely based.*

Consequently, minimal non-finitely based involution semigroups are of order five, and $(A_0^1,{}^{\mathsf{S}})$ is one such example. An obvious next step in the investigation is to question the uniqueness of this example.

Question 1.2. Is there a non-finitely based involution semigroup of order five that is not isomorphic to $(A_0^1, {}^{\mathsf{S}})$?

It has recently been confirmed that up to isomorphism, $(A_0^1, {}^{\mathsf{S}})$ is the unique smallest non-finitely based involution semigroup within the class of all involution semigroups with a unit element [9]. Therefore, to answer Question 1.2, it suffices to only examine involution semigroups without a unit element.

1.2. ***Finite basis problem for finite (involution) semigroups.*** Let $\mathfrak{M}_n$ denote the set of all subsemigroups of $M_n(\mathbb{R})$ consisting of binary matrices and let $\mathfrak{M}_\infty = \bigcup_{n\geq 1} \mathfrak{M}_n$. It is not a coincidence that all explicit examples of finite involution semigroups given so far are semigroups from $\mathfrak{M}_\infty$ with transpositions ${}^{\mathsf{T}}$ or ${}^{\mathsf{S}}$, given that every finite semigroup is isomorphic to some semigroup in $\mathfrak{M}_\infty$ and every finite inverse semigroup is isomorphic to some semigroup in $\mathfrak{M}_\infty$ with the usual transposition ${}^{\mathsf{T}}$; see, for instance, Howie [10, Theorems 1.1.2 and 5.1.7].

Regarding finite involution semigroups in general, it turns out that every one of them is isomorphic to some semigroup in $\mathfrak{M}_\infty$ with the skew transposition ${}^{\mathsf{S}}$ but not necessarily the usual transposition ${}^{\mathsf{T}}$ [22]. Therefore, when addressing the finite basis problem for finite semigroups (with involution) — which is currently open — it is equivalent to focus on finite semigroups in $\mathfrak{M}_\infty$ (with the skew transposition ${}^{\mathsf{S}}$). Refer to the survey by Volkov [35] for more information on the finite basis problem for finite semigroups.

On the other hand, the finite basis problem for finite algebras is undecidable in general [30].

1.3. ***Organization.*** Notation and background information are first given in Section 2. An outline of the proof of Theorem 1.1 is then given in Section 3, while the finer details of the proof are deferred to Sections 4–6. Multiplication tables of all involution semigroups of order up to four are listed in Section 7.

2. Preliminaries

Most of the notation and background results of this article are given in this section. Refer to the monograph of Burris and Sankappanavar [4] for any undefined notation and terminology of universal algebra in general.

2.1. ***Words.*** Let $\preceq$ be a total order on a countably infinite alphabet $\mathcal{X}$ that excludes the symbol 0; write $x \prec y$ to indicate that $x \preceq y$ and $x \neq y$.

Let $\mathcal{X}^* = \{x^* \mid x \in \mathcal{X}\}$ be a disjoint copy of $\mathcal{X}$. Elements of $\mathcal{X} \cup \mathcal{X}^*$ are called *variables*. The *free involution semigroup* over $\mathcal{X}$ is the free semigroup $F_{\mathsf{inv}}(\mathcal{X}) = (\mathcal{X} \cup \mathcal{X}^*)^+$ with unary operation given by $(x^*)^* = x$ for all $x \in \mathcal{X}$ and

$$(x_1 x_2 \cdots x_{n-1} x_n)^* = x_n^* x_{n-1}^* \cdots x_2^* x_1^*$$

for all $x_1, x_2, \ldots, x_{n-1}, x_n \in \mathcal{X} \cup \mathcal{X}^*$. The *free involution monoid* over $\mathcal{X}$ is $F^1_{\mathsf{inv}}(\mathcal{X}) = F_{\mathsf{inv}}(\mathcal{X}) \cup \{1\}$, where 1 is the empty word with $1^* = 1$. Elements of $F^1_{\mathsf{inv}}(\mathcal{X})$ are called *words* and words in $\mathcal{X}^+ \cup \{1\}$ are said to be *plain*.

Any word $\boldsymbol{w} \in F^1_{\mathsf{inv}}(\mathcal{X})$ can be written in the form

$$\boldsymbol{w} = x_1^{\circledast_1} x_2^{\circledast_2} \cdots x_n^{\circledast_n}$$

for some $x_1, x_2, \ldots, x_n \in \mathcal{X}$ and $\circledast_1, \circledast_2, \ldots, \circledast_n \in \{1, *\}$ with $n \geq 0$; the *plain projection* of such a word is the plain word

$$\overline{\boldsymbol{w}} = x_1 x_2 \cdots x_n.$$

For any words $\boldsymbol{u}, \boldsymbol{v} \in F^1_{\mathsf{inv}}(\mathcal{X})$, write $\boldsymbol{u} \hookrightarrow \boldsymbol{v}$ to indicate that $\boldsymbol{u}$ is a *subsequence* of $\boldsymbol{v}$, that is, $\boldsymbol{u} = x_1 x_2 \cdots x_n$ for some $x_1, x_2, \ldots, x_n \in \mathcal{X} \cup \mathcal{X}^*$ and

$$\boldsymbol{v} = \boldsymbol{v}_0 x_1 \boldsymbol{v}_1 x_2 \boldsymbol{v}_2 \cdots x_n \boldsymbol{v}_n$$

for some $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in F^1_{\mathsf{inv}}(\mathcal{X})$. Specifically, a subsequence $\boldsymbol{u}$ of $\boldsymbol{v}$ such that $\mathsf{con}(\overline{\boldsymbol{u}}) = \{x, y\}$ for some $x, y \in \mathcal{X}$ is called an $\{x, y\}$-*subsequence* of $\boldsymbol{v}$.

For any word $\boldsymbol{w} \in F_{\mathsf{inv}}(\mathcal{X})$, the *content* of $\boldsymbol{w}$, denoted by $\mathsf{con}(\boldsymbol{w})$, is the set of variables occurring in $\boldsymbol{w}$; the number of times that a variable $x \in \mathcal{X} \cup \mathcal{X}^*$ occurs in $\boldsymbol{w}$ is denoted by $\mathsf{occ}(x, \boldsymbol{w})$; the *head* of $\boldsymbol{w}$, denoted by $\mathsf{h}(\boldsymbol{w})$, is the first variable occurring in $\boldsymbol{w}$; the *tail* of $\boldsymbol{w}$, denoted by $\mathsf{t}(\boldsymbol{w})$, is the last variable occurring in $\boldsymbol{w}$; and the length of $\boldsymbol{w}$ is denoted by $|\boldsymbol{w}|$.

Example 2.1. If $\boldsymbol{w} = x^* x y^* x^2 y z^* x^* y$ for some $x, y, z \in \mathcal{X}$, then

- $\overline{\boldsymbol{w}} = x^2 y x^2 y z x y$;
- $\mathsf{con}(\boldsymbol{w}) = \{x, x^*, y, y^*, z^*\}$;
- $\mathsf{occ}(x, \boldsymbol{w}) = 3,\ \mathsf{occ}(x^*, \boldsymbol{w}) = 2,\ \mathsf{occ}(y, \boldsymbol{w}) = 2,\ \mathsf{occ}(y^*, \boldsymbol{w}) = \mathsf{occ}(z^*, \boldsymbol{w}) = 1$;
- $\mathsf{h}(\boldsymbol{w}) = x^*,\ \mathsf{t}(\boldsymbol{w}) = y$; and
- $|\boldsymbol{w}| = 9$.

For any word $\boldsymbol{w} \in F_{\mathsf{inv}}(\mathcal{X})$, a variable $x \in \mathsf{con}(\boldsymbol{w})$ is *simple* if $\mathsf{occ}(\bar{x}, \overline{\boldsymbol{w}}) = 1$. A word $\boldsymbol{w}$ is *simple* if every variable in $\boldsymbol{w}$ is simple. If $x, x^* \in \mathsf{con}(\boldsymbol{w})$ for some $x \in \mathcal{X}$, then $\{x, x^*\}$ is a *mixed pair* of $\boldsymbol{w}$. A word $\boldsymbol{w}$ is *mixed* if it has some mixed pair; otherwise, $\boldsymbol{w}$ is *bipartite*.

Two words $\boldsymbol{w}_1, \boldsymbol{w}_2 \in F^1_{\mathsf{inv}}(\mathcal{X})$ are *disjoint* if $\mathsf{con}(\overline{\boldsymbol{w}}_1) \cap \mathsf{con}(\overline{\boldsymbol{w}}_2) = \varnothing$. A non-simple word $\boldsymbol{w}$ is *connected* if it cannot be decomposed into a product of two disjoint nonempty words.

2.2. *Identities.* An *identity* is an expression $\boldsymbol{u} \approx \boldsymbol{v}$ formed by words $\boldsymbol{u}, \boldsymbol{v} \in F_{\mathsf{inv}}(\mathcal{X})$; it is *nontrivial* if $\boldsymbol{u} \neq \boldsymbol{v}$. An identity $\boldsymbol{u} \approx \boldsymbol{v}$ is *bipartite* if both $\boldsymbol{u}$ and $\boldsymbol{v}$ are bipartite words. A bipartite identity $\boldsymbol{u} \approx \boldsymbol{v}$ is *plain* if $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{X}^+$.

An involution semigroup $(S,{}^*)$ *satisfies* an identity $\boldsymbol{s} \approx \boldsymbol{t}$, or $\boldsymbol{s} \approx \boldsymbol{t}$ is *satisfied by* $(S,{}^*)$, if for any substitution $\varphi : \mathcal{X} \to S$, the elements $\varphi(\boldsymbol{s})$ and $\varphi(\boldsymbol{t})$ of S coincide; in this case, $\boldsymbol{s} \approx \boldsymbol{t}$ is also said to be an *identity of* $(S,{}^*)$.

Lemma 2.2 (Lee [17, Lemma 9]). *An involution semigroup satisfies a bipartite identity* $\boldsymbol{u} \approx \boldsymbol{v}$ *with* $\mathsf{con}(\boldsymbol{u}) = \mathsf{con}(\boldsymbol{v})$ *if and only if it satisfies* $\bar{\boldsymbol{u}} \approx \bar{\boldsymbol{v}}$.

Lemma 2.3 (Lee [21, Lemma 2.12]). *An involution semigroup satisfies an identity* $\boldsymbol{u} \approx \boldsymbol{v}$ *if and only if it satisfies the identity* $\overleftarrow{\boldsymbol{u}} \approx \overleftarrow{\boldsymbol{v}}$, *where* $\overleftarrow{\boldsymbol{u}}$ *and* $\overleftarrow{\boldsymbol{v}}$ *are the words* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *written in reverse order.*

Recall that a *semilattice* is a semigroup that is commutative and idempotent. Up to isomorphism, the smallest semilattice with nontrivial involution is the multiplicative matrix semigroup

$$S\ell_3 = \left\{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]\right\}$$

under the skew transposition ${}^{\mathsf{S}}$.

Lemma 2.4 (Lee [17, Lemma 8]). *Let* $\boldsymbol{u} \approx \boldsymbol{v}$ *be any identity of* $(S\ell_3, {}^{\mathsf{S}})$. *Then* $\boldsymbol{u}$ *is bipartite if and only if* $\boldsymbol{v}$ *is bipartite*; *in this case*, $\mathsf{con}(\boldsymbol{u}) = \mathsf{con}(\boldsymbol{v})$.

A set Σ of identities of $(S,{}^*)$ is an *identity basis* for $(S,{}^*)$ if every identity of $(S,{}^*)$ is deducible from Σ. An involution semigroup is *finitely based* if it possesses a finite identity basis. It is unambiguous and sometimes convenient to take the involution axioms (1-1) for granted and omit them from an identity basis for an involution semigroup.

3. Proof of Theorem 1.1

Every finite commutative involution semigroup is finitely based [9, Proposition 2.2]. Since every involution semigroup of order three or less is commutative [20], they are all finitely based. Therefore, it remains to consider only noncommutative involution semigroups of order four. With the help of a computer, it is routine to check that up to isomorphism, there exist 83 involution semigroups of order four; see Section 7. Only six of these 83 involution semigroups are noncommutative; see Table 1.

Since $(S_1,{}^*)$ and $(S_3,{}^*)$ satisfy the identity $x_1x_2x_3 \approx y_1y_2y_3$, their identities can be axiomatized by those formed by words of length at most four, whence they are finitely based. The involution semigroups $(S_2,{}^*)$, $(S_5,{}^*)$, and $(S_4,{}^*)$ are shown to be finitely based in Sections 4, 5, and 6, respectively. The identities of $(S_6,{}^*)$ — a rectangular band with involution — is long known to be axiomatized by $\{x^2 \approx x,\ xyz \approx xz\}$ [7, Lemma 2]. Consequently, every involution semigroup of order four is finitely based.

S_1	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	1
4	1	1	2	1
x	1	2	3	4
x^*	1	2	4	3

S_2	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	3
4	1	2	1	4
x	1	2	3	4
x^*	1	3	2	4

S_3	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	2	1
4	1	1	2	2
x	1	2	3	4
x^*	1	2	4	3

S_4	1	2	3	4
1	1	1	1	1
2	1	1	1	2
3	1	2	3	1
4	1	1	1	4
x	1	2	3	4
x^*	1	2	4	3

S_5	1	2	3	4
1	1	1	1	1
2	1	1	1	2
3	1	2	3	2
4	1	1	1	4
x	1	2	3	4
x^*	1	2	4	3

S_6	1	2	3	4
1	1	1	3	3
2	2	2	4	4
3	1	1	3	3
4	2	2	4	4
x	1	2	3	4
x^*	1	3	2	4

Table 1. The noncommutative involution semigroups of order four.

4. The involution semigroup $(S_2,{}^*)$

For any word $\boldsymbol{w} \in F_{\mathsf{inv}}(\mathcal{X})$, the *interior* of $\boldsymbol{w}$, denoted by $\mathsf{int}(\boldsymbol{w})$, is the word obtained from $\boldsymbol{w}$ by removing its first and last variables. Specifically, if $\boldsymbol{w} = h\boldsymbol{w}_0 t$ for some $h, t \in \mathcal{X} \cup \mathcal{X}^*$ and $\boldsymbol{w}_0 \in F^1_{\mathsf{inv}}(\mathcal{X})$, then $\mathsf{int}(\boldsymbol{w}) = \boldsymbol{w}_0$. Note that if $|\boldsymbol{w}| \leq 2$, then $\mathsf{int}(\boldsymbol{w}) = 1$.

Proposition 4.1. *The identities*

(4-1a) $\quad x^3 \approx x^2, \quad xyx \approx x^2y^2, \quad xyx \approx y^2x^2, \quad xy^2z \approx xyz,$

(4-1b) $\quad x^*x \approx x^2, \quad xx^* \approx x^2, \quad x^*yx \approx xyx, \quad xyx^* \approx xyx, \quad xy^*z \approx xyz$

constitute an identity basis for the involution semigroup $(S_2,{}^*)$.

Proof. In this proof, a word $\boldsymbol{w}$ is in *canonical form* if every variable in $\mathsf{int}(\boldsymbol{w})$ and every non-simple variable in $\boldsymbol{w}$ are plain. It is easy to see that the identities (4-1b) can be used to convert any word into one in canonical form.

It is routine to check that $(S_2,{}^*)$ satisfies the identities (4-1). Hence, there exists some set Σ of identities of $(S_2,{}^*)$, formed by words in canonical form, such that $\{(4\text{-}1)\} \cup \Sigma$ is an identity basis for $(S_2,{}^*)$. Now the identities (4-1a) in fact constitute an identity basis for the semigroup S_2 [1, Variety 24], so every plain identity of $(S_2,{}^*)$ is deducible from (4-1a). Therefore, the identities in Σ can be assumed non-plain. Let $\boldsymbol{u} \approx \boldsymbol{v}$ be any identity in Σ, say with $\boldsymbol{u}$ non-plain. If $|\boldsymbol{u}| = 1$, then $\boldsymbol{u} = x^*$ for some $x \in \mathcal{X}$, whence it is easy to show that the identity $\boldsymbol{u} \approx \boldsymbol{v}$ is trivial and so is clearly deducible from (4-1).

It remains to consider the case $|\boldsymbol{u}| \geq 2$. Since $\boldsymbol{u}$ is in canonical form and contains a non-plain variable, say x^* with $x \in \mathcal{X}$, the variable x^* is simple and so is either the head or the tail of $\boldsymbol{u}$. In view of Lemma 2.3, it suffices to assume the former, so that $\boldsymbol{u} = x^* \operatorname{int}(\boldsymbol{u})\operatorname{t}(\boldsymbol{u})$ with $\operatorname{int}(\boldsymbol{u}) \in \mathcal{X}^+ \cup \{1\}$ and $\operatorname{t}(\boldsymbol{u}) \in \mathcal{X} \cup \mathcal{X}^*$ such that $x \notin \operatorname{con}(\operatorname{int}(\boldsymbol{u})\overline{\operatorname{t}(\boldsymbol{u})})$. If $\operatorname{h}(\boldsymbol{v}) \neq x^*$, then letting $\varphi_1 : \mathcal{X} \to S_2$ be the substitution that maps x to 2 and every other variable to 4, the contradiction $\varphi_1(\boldsymbol{u}) = 3 \neq \varphi_1(\boldsymbol{v})$ is obtained. Therefore, $\operatorname{h}(\boldsymbol{v}) = x^*$, so that $\boldsymbol{v} = x^* \operatorname{int}(\boldsymbol{v})\operatorname{t}(\boldsymbol{v})$ with $\operatorname{int}(\boldsymbol{v}) \in \mathcal{X}^+ \cup \{1\}$ and $\operatorname{t}(\boldsymbol{v}) \in \mathcal{X} \cup \mathcal{X}^*$ such that $x \notin \operatorname{con}(\operatorname{int}(\boldsymbol{v})\overline{\operatorname{t}(\boldsymbol{v})})$. There are two cases.

Case 1: $\operatorname{t}(\boldsymbol{u}), \operatorname{t}(\boldsymbol{v}) \in \mathcal{X}$. Then $\operatorname{int}(\boldsymbol{u})\operatorname{t}(\boldsymbol{u})$ and $\operatorname{int}(\boldsymbol{v})\operatorname{t}(\boldsymbol{v})$ are plain, so that $\bar{\boldsymbol{u}} = x \operatorname{int}(\boldsymbol{u})\operatorname{t}(\boldsymbol{u})$ and $\bar{\boldsymbol{v}} = x \operatorname{int}(\boldsymbol{v})\operatorname{t}(\boldsymbol{v})$.

Case 2: $\operatorname{t}(\boldsymbol{u}) \notin \mathcal{X}$ or $\operatorname{t}(\boldsymbol{v}) \notin \mathcal{X}$. By symmetry, suppose that $\operatorname{t}(\boldsymbol{u}) \in \mathcal{X}^*$, say $\operatorname{t}(\boldsymbol{u}) = y^*$ for some $y \in \mathcal{X}\backslash\{x\}$. Then $\boldsymbol{u} = x^* \operatorname{int}(\boldsymbol{u})y^*$ with $\operatorname{int}(\boldsymbol{u}) \in \mathcal{X}^+ \cup \{1\}$ such that $x, y \notin \operatorname{con}(\operatorname{int}(\boldsymbol{u}))$. If $\operatorname{t}(\boldsymbol{v}) \neq y^*$, then letting $\varphi_2 : \mathcal{X} \to S_2$ be the substitution that maps y to 3 and every other variable to 4, the contradiction $\varphi_2(\boldsymbol{u}) = 2 \neq \varphi_2(\boldsymbol{v})$ is obtained. Therefore, $\operatorname{t}(\boldsymbol{v}) = y^*$, so that $\boldsymbol{v} = x^* \operatorname{int}(\boldsymbol{v})y^*$ with $\operatorname{int}(\boldsymbol{v}) \in \mathcal{X}^+ \cup \{1\}$ such that $x, y \notin \operatorname{con}(\operatorname{int}(\boldsymbol{v}))$. It follows that $\bar{\boldsymbol{u}} = x \operatorname{int}(\boldsymbol{u})y$ and $\bar{\boldsymbol{v}} = x \operatorname{int}(\boldsymbol{v})y$.

It is clear that in both cases, the identities $\{(1\text{-}1), \boldsymbol{u} \approx \boldsymbol{v}\}$ and $\{(1\text{-}1), \bar{\boldsymbol{u}} \approx \bar{\boldsymbol{v}}\}$ are deducible from one another. Since $\bar{\boldsymbol{u}} \approx \bar{\boldsymbol{v}}$ is an identity of S_2, it is deducible from (4-1a). It follows that $\boldsymbol{u} \approx \boldsymbol{v}$ is deducible from $\{(1\text{-}1), (4\text{-}1)\}$. Consequently, every identity in Σ is deducible from $\{(1\text{-}1), (4\text{-}1)\}$, so that (4-1) is an identity basis for $(S_2, {}^*)$. □

5. The involution semigroup $(S_5, {}^*)$

The involution semigroup $(S_5, {}^*)$ is isomorphic to the semigroup

$$A_0 = \langle \mathrm{E}, \mathrm{F} \mid \mathrm{E}^2 = \mathrm{E},\ \mathrm{F}^2 = \mathrm{F},\ \mathrm{FE} = 0 \rangle = \{0, \mathrm{E}, \mathrm{F}, \mathrm{EF}\}$$

with the operation * that interchanges E and F and fixes every other element.

A_0	0	EF	E	F
0	0	0	0	0
EF	0	0	0	EF
E	0	EF	E	EF
F	0	0	0	F
x	0	EF	E	F
x^*	0	EF	F	E

The involution semigroup $(A_0, {}^*)$ is isomorphic to the involution subsemigroup of $(A_0^1, {}^{\mathsf{S}})$ that consists of its non-unit elements.

The semigroup A_0 has been known to be finitely based for over 40 years [6]. The finite basis problem for the involution semigroup $(A_0, {}^*)$ has not been considered and has only recently been questioned [18, Question 6.4]. The present section addresses this problem by showing that $(A_0, {}^*)$ is finitely based.

Proposition 5.1. *The identities*

$$xx^*xy \approx xx^*x, \quad yxx^*x \approx xx^*x, \quad xx^*x \approx yy^*y, \tag{5-1a}$$

$$xyx^* \approx xy^*x^*, \tag{5-1b}$$

$$x^2\mathsf{H}x^* \approx x\mathsf{H}x^*, \quad x\mathsf{H}(x^*)^2 \approx x\mathsf{H}x^*, \tag{5-1c}$$

$$xy\mathsf{H}y^* \approx yx\mathsf{H}x^*, \tag{5-1d}$$

$$x\mathsf{H}x^*y \approx y^*x\mathsf{H}x^*, \tag{5-1e}$$

$$x^2\mathsf{H}y\mathsf{T}y^* \approx x\mathsf{H}y\mathsf{T}y^*, \tag{5-1f}$$

$$xy\mathsf{H}z\mathsf{T}z^* \approx yx\mathsf{H}z\mathsf{T}z^*, \tag{5-1g}$$

$$x^3 \approx x^2, \quad x^2yx \approx xyx, \quad xyx^2 \approx xyx, \tag{5-1h}$$

$$xyx \approx yxy, \tag{5-1i}$$

$$x\mathsf{H}yz\mathsf{T}x \approx x\mathsf{H}zy\mathsf{T}x, \tag{5-1j}$$

where $\mathsf{H} \in \{1, h\}$ *and* $\mathsf{T} \in \{1, t\}$, *constitute an identity basis for* $(A_0,^*)$.

It is easily checked that $(A_0,^*)$ satisfies the identities (5-1). In Section 5.1, some information on identities of $(A_0,^*)$ are given. In Section 5.2, it is shown that the identities of $(A_0,^*)$ can be used to convert every mixed word into one of two specific forms. Based on these results, it is shown in Section 5.3 that every identity of $(A_0,^*)$ is deducible from {(1-1), (5-1)}. This completes the proof of Proposition 5.1.

Corollary 5.2. *The identities*

$$\begin{aligned} x^3 \approx x^2, \quad xyxy \approx xyx, \quad x^2x^* \approx xx^*, \quad x^2yx^* \approx xyx^*, \\ xy^*x^* \approx xyx^*, \quad xx^*x \approx yy^*y, \quad xyx^*z \approx z^*xyx^* \end{aligned} \tag{5-2}$$

constitute an identity basis for $(A_0,^*)$.

Proof. It is routine to check, say with Prover9 [28], that the identities {(1-1), (5-1)} and {(1-1), (5-2)} are deducible from one another. □

Remarks 5.3. (i) Not only is the semigroup A_0 finitely based, it is *hereditarily finitely based* in the sense that every semigroup in the variety $\mathsf{Var}\,A_0$ is finitely based [14, Corollary 4.3].

(ii) In contrast, the finitely based involution semigroup $(A_0,^*)$ is not hereditarily finitely based because the variety $\mathsf{Var}(A_0,^*)$ contains a non-finitely based involution semigroup [20, Proposition 3.8].

5.1. *Some identities of* $(A_0,^*)$.

Lemma 5.4. *The identities* {(5-1h), (5-1i)} *constitute an identity basis for the semigroup* A_0.

Proof. The identities $\Sigma = \{x^3 \approx x^2,\ xyxy \approx xyx,\ xyxy \approx yxy\}$ constitute an identity basis for A_0 [24, Theorem 4.1]; it is routine to verify that Σ and {(5-1h), (5-1i)} are deducible from one another. □

Lemma 5.5. *Let* $\boldsymbol{u} \approx \boldsymbol{v}$ *be any identity of* $(A_0,{}^*)$ *such that either* $\boldsymbol{u}$ *or* $\boldsymbol{v}$ *is bipartite. Then* $\boldsymbol{u} \approx \boldsymbol{v}$ *is deducible from* {(1-1), (5-1)}.

Proof. Since $(S\ell_3,{}^{\mathsf{S}})$ is isomorphic to $(A_0,{}^*)$ modulo the ideal $\{0, \text{EF}\}$, the identity $\boldsymbol{u} \approx \boldsymbol{v}$ is satisfied by $(S\ell_3,{}^{\mathsf{S}})$. Then since either $\boldsymbol{u}$ or $\boldsymbol{v}$ is bipartite, by Lemma 2.4, both $\boldsymbol{u}$ and $\boldsymbol{v}$ are bipartite with $\mathsf{con}(\boldsymbol{u}) = \mathsf{con}(\boldsymbol{v})$. It follows from Lemma 2.2 that $(A_0,{}^*)$ satisfies the plain identity $\bar{\boldsymbol{u}} \approx \bar{\boldsymbol{v}}$. By Lemma 5.4, the identities {(5-1h), (5-1i)} constitute an identity basis for A_0, so that $\bar{\boldsymbol{u}} \approx \bar{\boldsymbol{v}}$ is deducible from {(5-1h), (5-1i)}. It then follows from Lemma 2.2 that $\boldsymbol{u} \approx \boldsymbol{v}$ is deducible from {(1-1), (5-1)}. □

An *ordered A_0-block* is a word of the form

$$\boldsymbol{c} = (y_1 y_2 \cdots y_k)^2,$$

where $y_1, y_2, \ldots, y_k \in \mathcal{X} \cup \mathcal{X}^*$ are such that $\bar{y}_1 \prec \bar{y}_2 \prec \cdots \prec \bar{y}_k$ in $\mathcal{X}$ and $k \geq 1$. Note that every ordered A_0-block is bipartite and connected.

Lemma 5.6 (Lee [21, Lemma 2.1]). *Let* $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{X}^+$ *be any plain connected words such that* $\mathsf{con}(\boldsymbol{u}) = \mathsf{con}(\boldsymbol{v})$. *Then* $\boldsymbol{u} \approx \boldsymbol{v}$ *is an identity of the semigroup* A_0.

Lemma 5.7. *Let* $\boldsymbol{w} \in F_{\mathsf{inv}}(\mathcal{X})$ *be any bipartite connected word such that* $\mathsf{con}(\boldsymbol{w}) = \{y_1, y_2, \ldots, y_k\}$ *and* $\bar{y}_1 \prec \bar{y}_2 \prec \cdots \prec \bar{y}_k$ *in* $\mathcal{X}$. *Then the identities* {(5-1h), (5-1i)} *can be used to convert* $\boldsymbol{w}$ *into the ordered* A_0*-block* $\boldsymbol{c} = (y_1 y_2 \cdots y_k)^2$.

Proof. Since $\boldsymbol{w}$ is bipartite, there exists a partition $I \cup J = \{1, 2, \ldots, k\}$ such that $y_i \in \mathcal{X}$ for all $i \in I$ and $y_j \in \mathcal{X}^*$ for all $j \in J$. Let $\varphi : \mathcal{X} \to \mathcal{X} \cup \mathcal{X}^*$ denote the substitution

$$\varphi(t) = \begin{cases} t^* & \text{if } t \in \{y_j, y_j^* \mid j \in J\}, \\ t & \text{otherwise.} \end{cases}$$

Then it is easy to check that for any word $\boldsymbol{v} \in \mathsf{con}(\boldsymbol{w})^+$, we have $\varphi(\boldsymbol{v}) = \bar{\boldsymbol{v}}$ and $\varphi(\bar{\boldsymbol{v}}) = \boldsymbol{v}$. Since $\bar{\boldsymbol{w}}$ and $\bar{\boldsymbol{c}}$ are plain words such that $\mathsf{con}(\bar{\boldsymbol{w}}) = \mathsf{con}(\bar{\boldsymbol{c}})$, by Lemma 5.6, the identity $\bar{\boldsymbol{w}} \approx \bar{\boldsymbol{c}}$ is satisfied by A_0. Then by Lemma 5.4, $\bar{\boldsymbol{w}} \approx \bar{\boldsymbol{c}}$ is deducible from {(5-1h), (5-1i)}. Consequently,

$$\boldsymbol{w} = \varphi(\bar{\boldsymbol{w}}) \overset{\text{(5-1h),(5-1i)}}{\approx} \varphi(\bar{\boldsymbol{c}}) = \boldsymbol{c}.$$ □

5.2. ***Some special forms of words.*** It is easily checked that for any substitution $\varphi : \mathcal{X} \to A_0$ and any variable $z \in \mathcal{X}$, we have $\varphi(zz^*z) = 0$ in A_0. Therefore, in the $\mathsf{Var}(A_0,{}^*)$-free algebra over $\mathcal{X}$, the class $[zz^*z]$ containing zz^*z is its zero element. This phenomenon can also be seen from the identities (5-1a) of $(A_0,{}^*)$.

Words of other possible forms in the class $[zz^*z]$ are given in the following result.

Lemma 5.8. *Let $\boldsymbol{w} \in F_{\mathsf{inv}}(\mathcal{X})$. Suppose that either $xx^*x \hookrightarrow \boldsymbol{w}$ for some $x \in \mathcal{X} \cup \mathcal{X}^*$ or $xx^*yy^* \hookrightarrow \boldsymbol{w}$ for some $x, y \in \mathcal{X} \cup \mathcal{X}^*$. Then the identities* (5-1) *can be used to convert $\boldsymbol{w}$ into the word zz^*z for any $z \in \mathcal{X} \cup \mathcal{X}^*$.*

Proof. If $\boldsymbol{w} = \boldsymbol{w}_0 x \boldsymbol{w}_1 x^* \boldsymbol{w}_2 x \boldsymbol{w}_3$ for some $\boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3 \in F^1_{\mathsf{inv}}(\mathcal{X})$, then

$$\boldsymbol{w} \overset{\text{(5-1j)}}{\approx} \boldsymbol{w}_0 x \boldsymbol{w}_1 \boldsymbol{w}_2 x^* x \boldsymbol{w}_3 \overset{\text{(5-1g)}}{\approx} \boldsymbol{w}_0 \boldsymbol{w}_1 \boldsymbol{w}_2 x x^* x \boldsymbol{w}_3 \overset{\text{(5-1a)}}{\approx} zz^*z.$$

If $\boldsymbol{w} = \boldsymbol{w}_0 x \boldsymbol{w}_1 x^* \boldsymbol{w}_2 y \boldsymbol{w}_3 y^* \boldsymbol{w}_4$ for some $\boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3, \boldsymbol{w}_4 \in F^1_{\mathsf{inv}}(\mathcal{X})$, then

$$\boldsymbol{w} \overset{\text{(5-1c)}}{\approx} \boldsymbol{w}_0 x^2 \boldsymbol{w}_1 x^* \boldsymbol{w}_2 y \boldsymbol{w}_3 y^* \boldsymbol{w}_4 \overset{\text{(5-1g)}}{\approx} \boldsymbol{w}_0 x x^* x \boldsymbol{w}_1 \boldsymbol{w}_2 y \boldsymbol{w}_3 y^* \boldsymbol{w}_4 \overset{\text{(5-1a)}}{\approx} zz^*z. \qquad \square$$

A word $\boldsymbol{w} \in F_{\mathsf{inv}}(\mathcal{X})$ is in A_0-*standard form* if

$$\boldsymbol{w} = \boldsymbol{w}_1 x \boldsymbol{w}_2 x^*, \tag{5-3}$$

where $x \in \mathcal{X} \cup \mathcal{X}^*$, $\boldsymbol{w}_1 = x_1 x_2 \cdots x_m$, and $\boldsymbol{w}_2 = \boldsymbol{s}_0 \prod_{i=1}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i)$ for some $m, p \geq 0$ such that the following conditions hold:

(A1) $x_1, x_2, \ldots, x_m \in \mathcal{X} \cup \mathcal{X}^*$ are such that $\bar{x}_1 \prec \bar{x}_2 \prec \cdots \prec \bar{x}_m \prec \bar{x}$;

(A2) $\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p \in F^1_{\mathsf{inv}}(\mathcal{X})$ are simple and $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_p \in F_{\mathsf{inv}}(\mathcal{X})$ are ordered A_0-blocks;

(A3) $x_1, x_2, \ldots, x_m, x, \boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p, \boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_p$ are pairwise disjoint;

(A4) if $\boldsymbol{w}_2 \neq 1$, then
   - (a) $p = 0$ with $\boldsymbol{w}_2 = \boldsymbol{s}_0$ and $\boldsymbol{s}_0 \in \mathcal{X}$; or
   - (b) $p = 1$ with $\boldsymbol{w}_2 = \boldsymbol{c}_1$, $\boldsymbol{s}_0 = \boldsymbol{s}_1 = 1$, and $\mathsf{h}(\boldsymbol{c}_1) \in \mathcal{X}$; or
   - (c) $\overline{\mathsf{h}(\boldsymbol{s}_0)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$ when $\boldsymbol{s}_0 \neq 1$; or
   - (d) $\overline{\mathsf{t}(\boldsymbol{c}_1)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$ when $\boldsymbol{s}_0 = 1$.

Remark 5.9. The following holds for the word $\boldsymbol{w}$ in (5-3) in A_0-standard form:

(i) If $m = 0$, then $\boldsymbol{w}_1 = 1$.

(ii) If $p = 0$, then $\boldsymbol{w}_2 = \boldsymbol{s}_0$.

(iii) $\{x, x^*\}$ is the only mixed pair of $\boldsymbol{w}$ and $x, x^* \notin \mathsf{con}(\boldsymbol{w}_1 \boldsymbol{w}_2)$.

(iv) $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are bipartite words such that $\mathsf{con}(\overline{\boldsymbol{w}}_1) \cap \mathsf{con}(\overline{\boldsymbol{w}}_2) = \varnothing$.

(v) Each variable in $\mathcal{X}$ occurs at most twice in $\overline{\boldsymbol{w}}$.

(vi) If $\boldsymbol{w}_2 \neq 1$ and $|\mathsf{con}(\overline{\boldsymbol{w}}_2)| \geq 2$, then $\overline{\mathsf{h}(\boldsymbol{w}_2)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$ by condition (A4). This is because
   - if $\boldsymbol{s}_0 \neq 1$, then $\overline{\mathsf{h}(\boldsymbol{w}_2)} = \overline{\mathsf{h}(\boldsymbol{s}_0)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$ by condition (A4)(c);
   - if $\boldsymbol{s}_0 = 1$, then since $\overline{\mathsf{h}(\boldsymbol{w}_2)} = \overline{\mathsf{h}(\boldsymbol{c}_1)} \preceq \overline{\mathsf{t}(\boldsymbol{c}_1)}$ due to $\boldsymbol{c}_1$ being an ordered A_0-block, we have $\overline{\mathsf{h}(\boldsymbol{w}_2)} \preceq \overline{\mathsf{t}(\boldsymbol{c}_1)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$ by condition (A4)(d).

Lemma 5.10. *Let $\boldsymbol{w} = \boldsymbol{w}_1 x \boldsymbol{w}_2 x^*$ be the word in* (5-3) *in A_0-standard form. Then there exist substitutions $\alpha_{\boldsymbol{w}}, \beta_{\boldsymbol{w}} : \mathcal{X} \to A_0$ such that*

(i) $\alpha_{\boldsymbol{w}}(\boldsymbol{w}_1 x \boldsymbol{w}_2) = \mathrm{E}$ *and* $\alpha_{\boldsymbol{w}}(x^*) = \mathrm{F}$, *so that* $\alpha_{\boldsymbol{w}}(\boldsymbol{w}) = \mathrm{EF}$;

(ii) $\beta_{\boldsymbol{w}}(\boldsymbol{w}_1 x) = \mathrm{E}$ *and* $\beta_{\boldsymbol{w}}(\boldsymbol{w}_2 x^*) = \mathrm{F}$, *so that* $\beta_{\boldsymbol{w}}(\boldsymbol{w}) = \mathrm{EF}$;

(iii) $\alpha_{\boldsymbol{w}}(t) = \beta_{\boldsymbol{w}}(t) = 0$ *for all $t \in \mathcal{X}$ such that $t \notin \mathsf{con}(\overline{\boldsymbol{w}})$.*

Proof. It follows from Remark 5.9(iii),(iv) that $\mathsf{con}(\boldsymbol{w}_1) = \mathcal{H}_1 \cup \mathcal{K}_1^*$ and $\mathsf{con}(\boldsymbol{w}_2) = \mathcal{H}_2 \cup \mathcal{K}_2^*$ for some $\mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{X}$ such that $\mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_1, \mathcal{K}_2, \{x, x^*\}$ are pairwise disjoint sets. By symmetry, it suffices to assume that $x \in \mathcal{X}$, so that $\mathsf{con}(\overline{\boldsymbol{w}}) = \mathcal{H}_1 \cup \mathcal{K}_1 \cup \mathcal{H}_2 \cup \mathcal{K}_2 \cup \{x\}$. Define

$$\alpha_{\boldsymbol{w}}(t) = \begin{cases} \mathrm{E} & \text{if } t \in \mathcal{H}_1 \cup \mathcal{H}_2 \cup \{x\}, \\ \mathrm{F} & \text{if } t \in \mathcal{K}_1 \cup \mathcal{K}_2, \\ 0 & \text{otherwise;} \end{cases} \qquad \beta_{\boldsymbol{w}}(t) = \begin{cases} \mathrm{E} & \text{if } t \in \mathcal{H}_1 \cup \mathcal{K}_2 \cup \{x\}, \\ \mathrm{F} & \text{if } t \in \mathcal{K}_1 \cup \mathcal{H}_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is routinely checked that the substitutions $\alpha_{\boldsymbol{w}}$ and $\beta_{\boldsymbol{w}}$ satisfy (i)–(iii). □

Corollary 5.11. *For any word $\boldsymbol{w}$ in A_0-standard form and any $z \in \mathcal{X} \cup \mathcal{X}^*$, the identity $\boldsymbol{w} \approx zz^*z$ is not satisfied by $(A_0, {}^*)$.*

Proof. Under the substitution $\alpha_{\boldsymbol{w}} : \mathcal{X} \to A_0$ in Lemma 5.10, we have $\alpha_{\boldsymbol{w}}(\boldsymbol{w}) = \mathrm{EF}$ and $\alpha_{\boldsymbol{w}}(zz^*z) = 0$. □

Lemma 5.12. *Let $\boldsymbol{w}$ be any mixed word. Then the identities* {(1-1), (5-1)} *can be used to convert $\boldsymbol{w}$ into exactly one of the following*:

(i) *the word zz^*z for any $z \in \mathcal{X} \cup \mathcal{X}^*$*;

(ii) *some word in A_0-standard form.*

Proof. In view of the identities {(1-1), (5-1c), (5-1e), (5-1g)}, we may assume that $\boldsymbol{w} = \boldsymbol{w}_1 x \boldsymbol{w}_2 x^*$, where $x \in \mathcal{X} \cup \mathcal{X}^*$ and $\boldsymbol{w}_1, \boldsymbol{w}_2 \in F^1_{\mathsf{inv}}(\mathcal{X})$ are such that $x \notin \mathsf{con}(\boldsymbol{w}_1)$, $x, x^* \notin \mathsf{con}(\boldsymbol{w}_2)$, and $\boldsymbol{w}_2$ is bipartite. If either $x^* \in \mathsf{con}(\boldsymbol{w}_1)$ or $\boldsymbol{w}_1$ contains some mixed pair, then by Lemma 5.8, the identities (5-1) can be used to convert $\boldsymbol{w}$ into the word zz^*z for any $z \in \mathcal{X} \cup \mathcal{X}^*$. Therefore, suppose that $x^* \notin \mathsf{con}(\boldsymbol{w}_1)$ and $\boldsymbol{w}_1$ is bipartite. In summary, we may assume that

(a) $x, x^* \notin \mathsf{con}(\boldsymbol{w}_1 \boldsymbol{w}_2)$ and

(b) $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are bipartite.

Suppose that $\mathsf{con}(\boldsymbol{w}_1) \cap \mathsf{con}(\boldsymbol{w}_2) \neq \varnothing$. Then the x in $\boldsymbol{w}$ is sandwiched between two occurrences of the same variable, one occurring in $\boldsymbol{w}_1$ and one in $\boldsymbol{w}_2$. Therefore, $\boldsymbol{w}_1 = \boldsymbol{a} y \boldsymbol{b}$ and $\boldsymbol{w}_2 = \boldsymbol{e} y \boldsymbol{f}$ for some $y \in \mathcal{X} \cup \mathcal{X}^*$ and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}, \boldsymbol{f} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that

$x \notin \mathsf{con}(\boldsymbol{abef})$ and $y \notin \mathsf{con}(\boldsymbol{bf})$, whence

$$\begin{aligned} \boldsymbol{w} &= \boldsymbol{a}y\boldsymbol{b}x\boldsymbol{e}y\boldsymbol{f}x^* \overset{(5\text{-}1h)}{\approx} \boldsymbol{a}y\boldsymbol{b}x\boldsymbol{e}y^2\boldsymbol{f}x^* \overset{(5\text{-}1j)}{\approx} \boldsymbol{a}y\boldsymbol{b}\boldsymbol{e}yxy\boldsymbol{f}x^* \\ &\overset{(5\text{-}1i)}{\approx} \boldsymbol{a}y\boldsymbol{b}\boldsymbol{e}xyx\boldsymbol{f}x^* \overset{(5\text{-}1g)}{\approx} \boldsymbol{a}y\boldsymbol{b}\boldsymbol{e}yx^2\boldsymbol{f}x^* \overset{(5\text{-}1c)}{\approx} \boldsymbol{a}y\boldsymbol{b}\boldsymbol{e}yx\boldsymbol{f}x^*; \end{aligned}$$

in other words, x can be moved to the right until it is no longer sandwiched by any two occurrences of y. This process can be repeated until x is not sandwiched by any two occurrences of the same variable. Therefore, we may further assume that

(c) $\mathsf{con}(\boldsymbol{w}_1) \cap \mathsf{con}(\boldsymbol{w}_2) = \varnothing$.

Suppose that $\mathsf{con}(\overline{\boldsymbol{w}}_1) \cap \mathsf{con}(\overline{\boldsymbol{w}}_2) \neq \varnothing$. Then the x in $\boldsymbol{w}$ is sandwiched by some mixed pair $\{y, y^*\}$ with $y \in \mathsf{con}(\boldsymbol{w}_1)$ and $y^* \in \mathsf{con}(\boldsymbol{w}_2)$. Therefore, $\boldsymbol{w}_1 = \boldsymbol{a}y\boldsymbol{b}$ and $\boldsymbol{w}_2 = \boldsymbol{e}y^*\boldsymbol{f}$ for some $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}, \boldsymbol{f} \in F^1_{\mathsf{inv}}(\mathscr{X})$ such that $x \notin \mathsf{con}(\boldsymbol{abef})$ and $y^* \notin \mathsf{con}(\boldsymbol{e})$. By (c), we also have $y \notin \mathsf{con}(\boldsymbol{e})$. Then

$$\begin{aligned} \boldsymbol{w} &= \boldsymbol{a}y\boldsymbol{b}x\boldsymbol{e}y^*\boldsymbol{f}x^* \overset{(5\text{-}1b)}{\approx} \boldsymbol{a}y\boldsymbol{b}x(\boldsymbol{e}y^*\boldsymbol{f})^*x^* \overset{(1\text{-}1)}{\approx} \boldsymbol{a}y\boldsymbol{b}x\boldsymbol{f}^*y\boldsymbol{e}^*x^* \\ &\overset{(5\text{-}1h)}{\approx} \boldsymbol{a}y\boldsymbol{b}x\boldsymbol{f}^*y^2\boldsymbol{e}^*x^* \overset{(5\text{-}1j)}{\approx} \boldsymbol{a}y\boldsymbol{b}\boldsymbol{f}^*yxy\boldsymbol{e}^*x^* \overset{(5\text{-}1i)}{\approx} \boldsymbol{a}y\boldsymbol{b}\boldsymbol{f}^*xyx\boldsymbol{e}^*x^* \\ &\overset{(5\text{-}1g)}{\approx} \boldsymbol{a}y\boldsymbol{b}\boldsymbol{f}^*yx^2\boldsymbol{e}^*x^* \overset{(5\text{-}1c)}{\approx} \boldsymbol{a}y\boldsymbol{b}\boldsymbol{f}^*yx\boldsymbol{e}^*x^* \overset{(5\text{-}1b)}{\approx} \boldsymbol{a}y\boldsymbol{b}\boldsymbol{f}^*yx\boldsymbol{e}x^*. \end{aligned}$$

Since $y, y^* \notin \mathsf{con}(\boldsymbol{e})$, the variable x is no longer sandwiched by the mixed pair $\{y, y^*\}$. This process can be repeated until x is not sandwiched by any mixed pair. Therefore, we may further assume that

(d) $\mathsf{con}(\overline{\boldsymbol{w}}_1) \cap \mathsf{con}(\overline{\boldsymbol{w}}_2) = \varnothing$.

Since the prefix $\boldsymbol{w}_1$ of $\boldsymbol{w}$ is bipartite by (b), the identities (5-1g) can be used to put the variables in $\boldsymbol{w}_1$ in order, and the identities (5-1f) can be used to reduce the exponent of any non-simple variable to 1. Hence, we may assume that $\boldsymbol{w}_1 = x_1x_2\cdots x_m$ for some $x_1, x_2, \ldots, x_m \in \mathscr{X} \cup \mathscr{X}^*$ with $m \geq 0$ such that $\bar{x}_1 \prec \bar{x}_2 \prec \cdots \prec \bar{x}_m$. By (a), we have $\bar{x} \notin \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m\}$. If $\bar{x}_m \not\prec \bar{x}$, then

$$\boldsymbol{w} = x_1x_2\cdots x_{m-1}x_mx\boldsymbol{w}_2x^* \overset{(5\text{-}1d)}{\approx} x_1x_2\cdots x_{n-1}xx_m\boldsymbol{w}_2x_m^*,$$

and the identities (5-1g) can be used to put the variables in the prefix $x_1x_2\cdots x_{m-1}x$ in an order such that condition (A1) is satisfied.

Since $\boldsymbol{w}_2$ is bipartite by (b), it can be written in the form $\boldsymbol{w}_2 = \boldsymbol{s}_0\prod_{i=1}^{p}(\boldsymbol{c}_i\boldsymbol{s}_i)$, where $\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p \in F^1_{\mathsf{inv}}(\mathscr{X})$ are simple words and $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_p \in F_{\mathsf{inv}}(\mathscr{X})$ are connected words with $p \geq 0$ such that

(e) $\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p, \boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_p$ are pairwise disjoint.

Then by Lemma 5.7, the identities {(5-1h), (5-1i)} can be used to convert each $\boldsymbol{c}_i$ into the ordered A_0-block word $\widehat{\boldsymbol{c}}_i$ with $\mathsf{con}(\boldsymbol{c}_i) = \mathsf{con}(\widehat{\boldsymbol{c}}_i)$. Therefore, condition (A2) is satisfied, and it follows from (d) and (e) that condition (A3) is also satisfied.

It remains to address condition (A4). Suppose that $\boldsymbol{w}_2 = \boldsymbol{s}_0 \prod_{i=1}^{p}(\boldsymbol{c}_i\boldsymbol{s}_i) \neq 1$. There are five cases.

Case 1: $p = 0$. Then $\boldsymbol{w}_2 = \boldsymbol{s}_0$.

SUBCASE 1.1: $|\mathsf{con}(\bar{\boldsymbol{s}}_0)| = 1$. Then $\boldsymbol{w}_2 = \boldsymbol{s}_0 \in \{y, y^*\}$ for some $y \in \mathcal{X}$. If $\boldsymbol{w}_2 = y$, then condition (A4)(a) is satisfied. If $\boldsymbol{w}_2 = y^*$, then

$$\boldsymbol{w} = \boldsymbol{w}_1 x y^* x^* \overset{(5\text{-}1\text{b})}{\approx} \boldsymbol{w}_1 x (y^*)^* x^* \overset{(1\text{-}1)}{\approx} \boldsymbol{w}_1 x y x^*;$$

in other words, the identities {(1-1), (5-1b)} can be used to convert the y^* in $\boldsymbol{w}$ to $y \in \mathcal{X}$, so that condition (A4)(a) is satisfied.

SUBCASE 1.2: $|\mathsf{con}(\bar{\boldsymbol{s}}_0)| \geq 2$. Then $\boldsymbol{w}_2 = \boldsymbol{s}_0 = y_1 y_2 \cdots y_r$ for some distinct $y_1, y_2, \ldots, y_r \in \mathcal{X} \cup \mathcal{X}^*$ with $r \geq 2$. If $\overline{\mathsf{h}(\boldsymbol{s}_0)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, then condition (A4)(c) is satisfied. If $\overline{\mathsf{h}(\boldsymbol{s}_0)} \not\prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, so that $\bar{y}_1 = \overline{\mathsf{h}(\boldsymbol{s}_0)} \succ \overline{\mathsf{t}(\boldsymbol{s}_0)} = \bar{y}_r$, then

$$\boldsymbol{w} \overset{(5\text{-}1\text{b})}{\approx} \boldsymbol{w}_1 x \boldsymbol{w}_2^* x^* = \boldsymbol{w}_1 x \underbrace{y_r^* y_{r-1}^* \cdots y_1^*}_{\boldsymbol{s}_0^* = \boldsymbol{w}_2^*} x^*,$$

where $\overline{\mathsf{h}(\boldsymbol{s}_0^*)} = \bar{y}_r \prec \bar{y}_1 = \overline{\mathsf{t}(\boldsymbol{s}_0^*)} = \overline{\mathsf{t}(\boldsymbol{w}_2^*)}$, whence condition (A4)(c) is satisfied.

Case 2: $p \geq 1$ and $\boldsymbol{s}_0 \neq 1 \neq \boldsymbol{s}_p$. Then $\boldsymbol{w}_2 = \boldsymbol{s}_0 \prod_{i=1}^{p}(\boldsymbol{c}_i\boldsymbol{s}_i)$, where $\boldsymbol{s}_0 = y_1 y_2 \cdots y_r$ and $\boldsymbol{s}_p = z_1 z_2 \cdots z_s$ for some distinct $y_1, y_2, \ldots, y_r, z_1, z_2, \ldots, z_s \in \mathcal{X} \cup \mathcal{X}^*$ with $r, s \geq 1$. If $\overline{\mathsf{h}(\boldsymbol{s}_0)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, then condition (A4)(c) is satisfied. Hence suppose that $\overline{\mathsf{h}(\boldsymbol{s}_0)} \not\prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, so that $\bar{y}_1 = \overline{\mathsf{h}(\boldsymbol{s}_0)} \succ \overline{\mathsf{t}(\boldsymbol{s}_p)} = \bar{z}_s$. Then

$$\boldsymbol{w} \overset{(5\text{-}1\text{b})}{\approx} \boldsymbol{w}_1 x \boldsymbol{w}_2^* x^* \overset{(1\text{-}1)}{\approx} \boldsymbol{w}_1 x \cdot \boldsymbol{s}_p^* \boldsymbol{c}_p^* \boldsymbol{s}_{p-1}^* \boldsymbol{c}_{p-1}^* \cdots \boldsymbol{s}_1^* \boldsymbol{c}_1^* \boldsymbol{s}_0^* \cdot x^*,$$

where $\overline{\mathsf{h}(\boldsymbol{s}_p^*)} = \bar{z}_s \prec \bar{y}_1 = \overline{\mathsf{t}(\boldsymbol{s}_0^*)}$, and the identities {(1-1), (5-1h), (5-1i)} can be used to convert each $\boldsymbol{c}_i^*$ into an ordered A_0-block (see Lemma 5.7). Therefore, condition (A4)(c) is satisfied.

Case 3: $p \geq 1$ and $\boldsymbol{s}_0 \neq 1 = \boldsymbol{s}_p$. Then $\boldsymbol{w}_2 = \boldsymbol{s}_0 \prod_{i=1}^{p-1}(\boldsymbol{c}_i\boldsymbol{s}_i)\boldsymbol{c}_p$, where $\boldsymbol{s}_0 = z_1 z_2 \cdots z_s$ and $\boldsymbol{c}_p = (y_1 y_2 \cdots y_r)^2$ for some distinct $y_1, y_2, \ldots, y_r, z_1, z_2, \ldots, z_s \in \mathcal{X} \cup \mathcal{X}^*$ with $r, s \geq 1$ such that $\bar{y}_1 \prec \bar{y}_2 \prec \cdots \prec \bar{y}_r$. If $\overline{\mathsf{h}(\boldsymbol{s}_0)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, then condition (A4)(c) is satisfied. If $\overline{\mathsf{h}(\boldsymbol{s}_0)} \not\prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, so that $\bar{z}_1 = \overline{\mathsf{h}(\boldsymbol{s}_0)} \succ \overline{\mathsf{t}(\boldsymbol{c}_p)} = \bar{y}_r$, then

$$\boldsymbol{w} \overset{(5\text{-}1\text{b})}{\approx} \boldsymbol{w}_1 x \boldsymbol{w}_2^* x^* \overset{(1\text{-}1)}{\approx} \boldsymbol{w}_1 x \cdot \boldsymbol{c}_p^* \boldsymbol{s}_{p-1}^* \boldsymbol{c}_{p-1}^* \cdots \boldsymbol{s}_1^* \boldsymbol{c}_1^* \boldsymbol{s}_0^* \cdot x^*,$$

and the identities {(1-1), (5-1h), (5-1i)} can be used to convert each $\boldsymbol{c}_i^*$ into an ordered A_0-block (Lemma 5.7); specifically, $\boldsymbol{c}_p^*$ is converted into $\widehat{\boldsymbol{c}}_p^* = (y_1^* y_2^* \cdots y_r^*)^2$. Since $\overline{\mathsf{t}(\boldsymbol{c}_p^*)} = \bar{y}_r \prec \bar{z}_1 = \overline{\mathsf{t}(\boldsymbol{s}_0^*)}$, condition (A4)(d) is satisfied.

Case 4: $p \geq 1$ and $\boldsymbol{s}_0 = 1 \neq \boldsymbol{s}_p$. Then $\boldsymbol{w}_2 = \prod_{i=1}^{p}(\boldsymbol{c}_i\boldsymbol{s}_i)$, where $\boldsymbol{c}_1 = (y_1 y_2 \cdots y_r)^2$ and $\boldsymbol{s}_p = z_1 z_2 \cdots z_s$ for some distinct $y_1, y_2, \ldots, y_r, z_1, z_2, \ldots, z_s \in \mathcal{X} \cup \mathcal{X}^*$ with

$r, s \geq 1$ such that $\bar{y}_1 \prec \bar{y}_2 \prec \cdots \prec \bar{y}_r$. If $\overline{\mathsf{t}(\boldsymbol{c}_1)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, then condition (A4)(d) is satisfied. If $\overline{\mathsf{t}(\boldsymbol{c}_1)} \not\prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, so that $\bar{y}_r = \overline{\mathsf{t}(\boldsymbol{c}_1)} \succ \overline{\mathsf{t}(\boldsymbol{s}_p)} = \bar{z}_s$, then

$$\boldsymbol{w} \overset{\text{(5-1b)}}{\approx} \boldsymbol{w}_1 x \boldsymbol{w}_2^* x^* \overset{\text{(1-1)}}{\approx} \boldsymbol{w}_1 x \cdot \boldsymbol{s}_p^* \boldsymbol{c}_p^* \boldsymbol{s}_{p-1}^* \boldsymbol{c}_{p-1}^* \cdots \boldsymbol{s}_1^* \boldsymbol{c}_1^* \cdot x^*,$$

and the identities {(1-1), (5-1h), (5-1i)} can be used to convert each $\boldsymbol{c}_i^*$ into an ordered A_0-block (Lemma 5.7); specifically, $\boldsymbol{c}_1^*$ is converted into $\widehat{\boldsymbol{c}}_1^* = (y_1^* y_2^* \cdots y_r^*)^2$. Since $\overline{\mathsf{h}(\boldsymbol{s}_p^*)} = \bar{z}_s \prec \bar{y}_r = \overline{\mathsf{t}(\widehat{\boldsymbol{c}}_1^*)}$, condition (A4)(c) is satisfied.

<u>Case 5</u>: $p \geq 1$ and $\boldsymbol{s}_0 = 1 = \boldsymbol{s}_p$.

SUBCASE 5.1: $p = 1$. Then $\boldsymbol{w}_2 = \boldsymbol{c}_1 = (y_1 y_2 \cdots y_k)^2$ for some $y_1, y_2, \ldots, y_k \in \mathcal{X} \cup \mathcal{X}^*$ with $k \geq 1$ such that $\bar{y}_1 \prec \bar{y}_2 \prec \cdots \prec \bar{y}_k$. If $y_1 \in \mathcal{X}$, then condition (A4)(b) is satisfied. Hence suppose that $y_1 \in \mathcal{X}^*$. Then

$$\boldsymbol{w} \overset{\text{(5-1b)}}{\approx} \boldsymbol{w}_1 x \boldsymbol{w}_2^* x^* = \boldsymbol{w}_1 x \cdot \boldsymbol{c}_1^* \cdot x^*,$$

and the identities {(1-1), (5-1h), (5-1i)} can be used to convert $\boldsymbol{c}_1^*$ into an ordered A_0-block $\widehat{\boldsymbol{c}}_1^* = (y_1^* y_2^* \cdots y_k^*)^2$ (see Lemma 5.7). Now since $y_1^* \in \mathcal{X}$, condition (A4)(b) is satisfied.

SUBCASE 5.2: $p \geq 2$. Then $\boldsymbol{w}_2 = \big(\prod_{i=1}^{p-1} (\boldsymbol{c}_i \boldsymbol{s}_i)\big) \boldsymbol{c}_p$, where $\boldsymbol{c}_1 = (y_1 y_2 \cdots y_r)^2$ and $\boldsymbol{c}_p = (z_1 z_2 \cdots z_s)^2$ for some distinct $y_1, y_2, \ldots, y_r, z_1, z_2, \ldots, z_s \in \mathcal{X} \cup \mathcal{X}^*$ with $r, s \geq 1$ such that $\bar{y}_1 \prec \bar{y}_2 \prec \cdots \prec \bar{y}_r$ and $\bar{z}_1 \prec \bar{z}_2 \prec \cdots \prec \bar{z}_s$. If $\overline{\mathsf{t}(\boldsymbol{c}_1)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, then condition (A4)(d) is satisfied. Hence suppose that $\overline{\mathsf{t}(\boldsymbol{c}_1)} \not\prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, so that $\bar{z}_s = \overline{\mathsf{t}(\boldsymbol{c}_p)} \succ \overline{\mathsf{t}(\boldsymbol{c}_1)} = \bar{y}_r$. Then

$$\boldsymbol{w} \overset{\text{(5-1b)}}{\approx} \boldsymbol{w}_1 x \boldsymbol{w}_2^* x^* \overset{\text{(1-1)}}{\approx} \boldsymbol{w}_1 x \cdot \boldsymbol{c}_p^* \boldsymbol{s}_{p-1}^* \boldsymbol{c}_{p-1}^* \cdots \boldsymbol{s}_1^* \boldsymbol{c}_1^* \cdot x^*,$$

and the identities {(1-1), (5-1h), (5-1i)} can be used to convert each $\boldsymbol{c}_i^*$ into an ordered A_0-block (Lemma 5.7); specifically, $\boldsymbol{c}_1^*$ is converted into $\widehat{\boldsymbol{c}}_1^* = (y_1^* y_2^* \cdots y_r^*)^2$ and $\boldsymbol{c}_p^*$ is converted into $\widehat{\boldsymbol{c}}_p^* = (z_1^* z_2^* \cdots z_s^*)^2$. Since $\overline{\mathsf{t}(\widehat{\boldsymbol{c}}_p^*)} = \bar{z}_s \prec \bar{y}_r = \overline{\mathsf{t}(\widehat{\boldsymbol{c}}_1^*)}$, condition (A4)(d) is satisfied.

Consequently, the identities {(1-1), (5-1)} can be used to convert $\boldsymbol{w}$ into either zz^*z or some word $\widetilde{\boldsymbol{w}}$ in A_0-standard form. But if the identities {(1-1), (5-1)} can be used to convert $\boldsymbol{w}$ into both zz^*z and $\widetilde{\boldsymbol{w}}$, then that would imply that $(A_0, {}^*)$ satisfies the identity $\widetilde{\boldsymbol{w}} \approx zz^*z$, which is impossible by Corollary 5.11. □

5.3. ***Proof of Proposition 5.1.*** Consider any identity

$$\boldsymbol{u} \approx \boldsymbol{v}$$

satisfied by $(A_0, {}^*)$. It suffices to show that $\boldsymbol{u} \approx \boldsymbol{v}$ is deducible from {(1-1), (5-1)}. By Lemma 5.5, this result holds if either $\boldsymbol{u}$ or $\boldsymbol{v}$ is bipartite. Therefore, suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are both mixed. By Corollary 5.11 and Lemma 5.12, the identities {(1-1), (5-1)} can be used to convert $\boldsymbol{u}$ and $\boldsymbol{v}$ simultaneously to either zz^*z or words in A_0-standard form. In the former case, $\boldsymbol{u} \approx \boldsymbol{v}$ is deducible from {(1-1), (5-1)}, whence the proof is complete. Therefore, it remains to consider the latter case, whence we may assume that $\boldsymbol{u}$ and $\boldsymbol{v}$ are in A_0-standard form, say

$$\boldsymbol{u} = \boldsymbol{u}_1 x \boldsymbol{u}_2 x^* \quad \text{and} \quad \boldsymbol{v} = \boldsymbol{v}_1 y \boldsymbol{v}_2 y^*,$$

where $x, y \in \mathcal{X} \cup \mathcal{X}^*$, $\boldsymbol{u}_1 = x_1 x_2 \cdots x_m$, $\boldsymbol{u}_2 = \boldsymbol{s}_0 \prod_{i=1}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i)$, $\boldsymbol{v}_1 = y_1 y_2 \cdots y_n$, and $\boldsymbol{v}_2 = \boldsymbol{t}_0 \prod_{i=1}^{q} (\boldsymbol{d}_i \boldsymbol{t}_i)$ satisfy conditions (A1)–(A4).

Lemma 5.13. *The following holds for the words* $\boldsymbol{u}$ *and* $\boldsymbol{v}$:

(i) $\mathsf{con}(\bar{\boldsymbol{u}}) = \mathsf{con}(\bar{\boldsymbol{v}})$;

(ii) $\boldsymbol{u}_1 x = \boldsymbol{v}_1 y$;

(iii) $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$.

Proof. (i) Suppose that $\mathsf{con}(\bar{\boldsymbol{u}}) \neq \mathsf{con}(\bar{\boldsymbol{v}})$, say there exists a variable $z \in \mathsf{con}(\bar{\boldsymbol{u}})$ such that $z \notin \mathsf{con}(\bar{\boldsymbol{v}})$. Then under the substitution $\alpha_{\boldsymbol{v}} : \mathcal{X} \to A_0$ in Lemma 5.10, the contradiction $\alpha_{\boldsymbol{v}}(\boldsymbol{u}) = 0 \neq \mathrm{EF} = \alpha_{\boldsymbol{v}}(\boldsymbol{v})$ is deduced.

(ii) Due to condition (A1), the equality $\boldsymbol{u}_1 x = \boldsymbol{v}_1 y$ follows from $\mathsf{con}(\boldsymbol{u}_1 x) = \mathsf{con}(\boldsymbol{v}_1 y)$; to establish the latter, by symmetry, it suffices to verify the inclusion $\mathsf{con}(\boldsymbol{u}_1 x) \subseteq \mathsf{con}(\boldsymbol{v}_1 y)$. To this end, we need to first show that $y \in \mathsf{con}(\boldsymbol{u}_1 x)$. Since $\bar{y} \in \mathsf{con}(\bar{\boldsymbol{v}}) = \mathsf{con}(\bar{\boldsymbol{u}})$ by part (i),

(a) either $y \in \mathsf{con}(\boldsymbol{u})$ or $y^* \in \mathsf{con}(\boldsymbol{u})$.

If $y^* \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$, then by Lemma 5.10, we have $\alpha_{\boldsymbol{u}}(\boldsymbol{u}) = \mathrm{EF}$ and

$$\alpha_{\boldsymbol{u}}(\boldsymbol{v}) = \alpha_{\boldsymbol{u}}(\boldsymbol{v}_1) \cdot \alpha_{\boldsymbol{u}}(y) \cdot \alpha_{\boldsymbol{u}}(\boldsymbol{v}_2) \cdot \alpha_{\boldsymbol{u}}(y^*) = \alpha_{\boldsymbol{u}}(\boldsymbol{v}_1) \cdot \mathrm{F} \cdot \alpha_{\boldsymbol{u}}(\boldsymbol{v}_2) \cdot \mathrm{E} = 0,$$

which is impossible. Therefore,

(b) $y^* \notin \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$, which implies that $y \neq x^*$.

If $y \neq x$, then together with (b), we have $y^* \notin \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2 x^*) = \mathsf{con}(\boldsymbol{u})$, so that $y \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$ by (a) and (b). On the other hand, if $y = x$, then clearly $y \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$. Therefore, $y \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$ either way. Now if $y \in \mathsf{con}(\boldsymbol{u}_2)$, then by Lemma 5.10, we have $\beta_{\boldsymbol{u}}(\boldsymbol{u}) = \mathrm{EF}$ and

$$\beta_{\boldsymbol{u}}(\boldsymbol{v}) = \beta_{\boldsymbol{u}}(\boldsymbol{v}_1) \cdot \beta_{\boldsymbol{u}}(y) \cdot \beta_{\boldsymbol{u}}(\boldsymbol{v}_2) \cdot \beta_{\boldsymbol{u}}(y^*) = \beta_{\boldsymbol{u}}(\boldsymbol{v}_1) \cdot \mathrm{F} \cdot \beta_{\boldsymbol{u}}(\boldsymbol{v}_2) \cdot \mathrm{E} = 0,$$

which is impossible. Hence $y \notin \mathsf{con}(\boldsymbol{u}_2)$; but since $y \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$, we in fact have

(c) $y \in \mathsf{con}(\boldsymbol{u}_1 x)$.

Now we are ready to establish the inclusion $\mathsf{con}(\boldsymbol{u}_1 x) \subseteq \mathsf{con}(\boldsymbol{v}_1 y)$. Suppose there exists some variable $z \in \mathsf{con}(\boldsymbol{u}_1 x)$ such that $z \notin \mathsf{con}(\boldsymbol{v}_1 y)$. Then clearly $z \neq y$. If $z = y^*$, then $y, y^* \in \mathsf{con}(\boldsymbol{u}_1 x)$ by (c), so that condition (A1) is contradicted. Hence

(d) $z \notin \{y, y^*\}$.

Since $\bar{z} \in \mathsf{con}(\bar{\boldsymbol{u}}) = \mathsf{con}(\bar{\boldsymbol{v}})$ by part (i), it follows from (d) that either $z \in \mathsf{con}(\boldsymbol{v}_1\boldsymbol{v}_2)$ or $z^* \in \mathsf{con}(\boldsymbol{v}_1\boldsymbol{v}_2)$. But since $z \notin \mathsf{con}(\boldsymbol{v}_1)$ by assumption, we have $z \in \mathsf{con}(\boldsymbol{v}_2)$ or $z^* \in \mathsf{con}(\boldsymbol{v}_1)$ or $z^* \in \mathsf{con}(\boldsymbol{v}_2)$. These three cases are shown in the following to be impossible. Therefore, the variable z does not exist, whence the required inclusion $\mathsf{con}(\boldsymbol{u}_1x) \subseteq \mathsf{con}(\boldsymbol{v}_1y)$ is established.

Case 1: $z \in \mathsf{con}(\boldsymbol{v}_2)$. By Lemma 5.10, we have $\beta_{\boldsymbol{v}}(\boldsymbol{v}_1y) = \mathrm{E}$ and $\beta_{\boldsymbol{v}}(\boldsymbol{v}_2y^*) = \mathrm{F}$, so that $\beta_{\boldsymbol{v}}(\boldsymbol{v}) = \mathrm{EF}$. Since $z \in \mathsf{con}(\boldsymbol{v}_2)$, we also have $\beta_{\boldsymbol{v}}(z) = \mathrm{F}$. Note that

$$\mathrm{EF} = \beta_{\boldsymbol{v}}(\boldsymbol{v}) = \beta_{\boldsymbol{v}}(\boldsymbol{u}) = \beta_{\boldsymbol{v}}(\boldsymbol{u}_1) \cdot \beta_{\boldsymbol{v}}(x) \cdot \beta_{\boldsymbol{v}}(\boldsymbol{u}_2) \cdot \beta_{\boldsymbol{v}}(x^*),$$

so we must have $\beta_{\boldsymbol{v}}(x^*) = \mathrm{F}$ and $\beta_{\boldsymbol{v}}(x) = \mathrm{E}$, so that $z \neq x$. But since $z \in \mathsf{con}(\boldsymbol{u}_1x)$ by assumption, it follows that $\beta_{\boldsymbol{v}}(\boldsymbol{u}_1x) = \cdots \mathrm{F} \cdots \mathrm{E} = 0$, whence the contradiction $\beta_{\boldsymbol{v}}(\boldsymbol{u}) = 0$ is deduced.

Case 2: $z^* \in \mathsf{con}(\boldsymbol{v}_1)$. By Lemma 5.10, we have $\alpha_{\boldsymbol{u}}(\boldsymbol{u}_1x\boldsymbol{u}_2) = \mathrm{E}$ and $\alpha_{\boldsymbol{u}}(x^*) = \mathrm{F}$, so that $\alpha_{\boldsymbol{u}}(\boldsymbol{u}) = \mathrm{EF}$. Since $z \in \mathsf{con}(\boldsymbol{u}_1x)$ by assumption, we also have $\alpha_{\boldsymbol{u}}(z) = \mathrm{E}$. Note that

$$\mathrm{EF} = \alpha_{\boldsymbol{u}}(\boldsymbol{u}) = \alpha_{\boldsymbol{u}}(\boldsymbol{v}) = \alpha_{\boldsymbol{u}}(\boldsymbol{v}_1) \cdot \alpha_{\boldsymbol{u}}(y) \cdot \alpha_{\boldsymbol{u}}(\boldsymbol{v}_2) \cdot \alpha_{\boldsymbol{u}}(y^*),$$

so we must have $\alpha_{\boldsymbol{u}}(y^*) = \mathrm{F}$ and $\alpha_{\boldsymbol{u}}(y) = \mathrm{E}$. But since $z^* \in \mathsf{con}(\boldsymbol{v}_1)$, it follows that $\alpha_{\boldsymbol{u}}(\boldsymbol{v}_1y) = \cdots \mathrm{F} \cdots \mathrm{E} = 0$, whence the contradiction $\alpha_{\boldsymbol{u}}(\boldsymbol{v}) = 0$ is deduced.

Case 3: $z^* \in \mathsf{con}(\boldsymbol{v}_2)$. By Lemma 5.10, we have $\alpha_{\boldsymbol{v}}(\boldsymbol{v}_1y\boldsymbol{v}_2) = \mathrm{E}$ and $\alpha_{\boldsymbol{v}}(y^*) = \mathrm{F}$, so that $\alpha_{\boldsymbol{v}}(\boldsymbol{v}) = \mathrm{EF}$. Since $z^* \in \mathsf{con}(\boldsymbol{v}_2)$, we also have $\alpha_{\boldsymbol{v}}(z) = \mathrm{F}$. Note that

$$\mathrm{EF} = \alpha_{\boldsymbol{v}}(\boldsymbol{v}) = \alpha_{\boldsymbol{v}}(\boldsymbol{u}) = \alpha_{\boldsymbol{v}}(\boldsymbol{u}_1) \cdot \alpha_{\boldsymbol{v}}(x) \cdot \alpha_{\boldsymbol{v}}(\boldsymbol{u}_2) \cdot \alpha_{\boldsymbol{v}}(x^*),$$

so we must have $\alpha_{\boldsymbol{v}}(x^*) = \mathrm{F}$ and $\alpha_{\boldsymbol{v}}(x) = \mathrm{E}$. But since $z \in \mathsf{con}(\boldsymbol{u}_1x)$ by assumption, it follows that $\alpha_{\boldsymbol{v}}(\boldsymbol{u}_1x) = \cdots \mathrm{F} \cdots \mathrm{E} = 0$, whence the contradiction $\alpha_{\boldsymbol{v}}(\boldsymbol{u}) = 0$ is deduced.

(iii) This is a consequence of parts (i) and (ii). □

Therefore, by Lemma 5.13, we now have

$$\boldsymbol{u} = \underbrace{x_1x_2\cdots x_m}_{\boldsymbol{u}_1} \cdot x \cdot \underbrace{\boldsymbol{s}_0 \prod_{i=1}^{p} (\boldsymbol{c}_i\boldsymbol{s}_i) \cdot x^*}_{\boldsymbol{u}_2} \quad \text{and} \quad \boldsymbol{v} = \underbrace{x_1x_2\cdots x_m}_{\boldsymbol{u}_1} \cdot x \cdot \underbrace{\boldsymbol{t}_0 \prod_{i=1}^{q} (\boldsymbol{d}_i\boldsymbol{t}_i)}_{\boldsymbol{v}_2} \cdot x^*,$$

where conditions (A1)–(A4) are satisfied.

Lemma 5.14. (i) $\mathsf{con}(\overline{\boldsymbol{s}_0\boldsymbol{s}_1\cdots\boldsymbol{s}_p}) = \mathsf{con}(\overline{\boldsymbol{t}_0\boldsymbol{t}_1\cdots\boldsymbol{t}_q})$.

(ii) $\mathsf{con}(\overline{\boldsymbol{c}_1\boldsymbol{c}_2\cdots\boldsymbol{c}_p}) = \mathsf{con}(\overline{\boldsymbol{d}_1\boldsymbol{d}_2\cdots\boldsymbol{d}_q})$.

Proof. (i) Let $\boldsymbol{s} = \boldsymbol{s}_0\boldsymbol{s}_1\cdots\boldsymbol{s}_p$ and $\boldsymbol{t} = \boldsymbol{t}_0\boldsymbol{t}_1\cdots\boldsymbol{t}_q$. Suppose there exists some variable $z \in \mathsf{con}(\bar{\boldsymbol{s}})$ such that $z \notin \mathsf{con}(\bar{\boldsymbol{t}})$. Then

$$\boldsymbol{u} = x_1x_2\cdots x_m \cdot x \cdot \boldsymbol{a}z^{\circledast}\boldsymbol{b} \cdot x^*$$

for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$ and $\circledast \in \{1, *\}$. By the definition of A_0-standard form, $\mathsf{occ}(z, \bar{\boldsymbol{u}}) = 1$ and the sets $\mathsf{con}(\overline{x_1 x_2 \cdots x_m \cdot x \cdot \boldsymbol{a}})$, $\{z\}$, $\mathsf{con}(\bar{\boldsymbol{b}})$ are pairwise disjoint. Therefore, if $\varphi : \mathcal{X} \cup \mathcal{X}^* \to A_0$ is the substitution given by

$$\varphi(t) = \begin{cases} \mathrm{E} & \text{if } t \in \mathsf{con}(x_1 x_2 \cdots x_m \cdot x \cdot \boldsymbol{a}), \\ \mathrm{EF} & \text{if } t = z, \\ \mathrm{F} & \text{otherwise,} \end{cases}$$

then $\varphi(\boldsymbol{u}) = \mathrm{E}^m \cdot \mathrm{E} \cdot \mathrm{E}^{|\boldsymbol{a}|}(\mathrm{EF})^{\circledast}\mathrm{F}^{|\boldsymbol{b}|} \cdot \mathrm{E}^* = \mathrm{EF}$. Now since $z \in \mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$ by Lemma 5.13(iii) and $z \notin \mathsf{con}(\bar{\boldsymbol{t}})$, there exists some $j \in \{1, 2, \ldots, q\}$ such that $z \in \mathsf{con}(\bar{\boldsymbol{d}}_j)$; further, since $\boldsymbol{d}_j$ is an ordered A_0-block, $\mathsf{occ}(z, \bar{\boldsymbol{d}}_j) = 2$. It follows that $\varphi(\boldsymbol{v}) = \cdots \mathrm{EF} \cdots \mathrm{EF} \cdots = 0 \neq \varphi(\boldsymbol{u})$, which is a contradiction. Consequently, the variable z does not exist, so that the inclusion $\mathsf{con}(\bar{\boldsymbol{s}}) \subseteq \mathsf{con}(\bar{\boldsymbol{t}})$ holds. The reverse inclusion $\mathsf{con}(\bar{\boldsymbol{s}}) \supseteq \mathsf{con}(\bar{\boldsymbol{t}})$ holds by a symmetrical argument.

(ii) This follows from part (i) since $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$ by Lemma 5.13(iii). □

Lemma 5.15. (i) *Suppose that* $yzyz \hookrightarrow \boldsymbol{u}_2$ *for some* $y, z \in \mathsf{con}(\boldsymbol{c}_i)$ *with* $1 \leq i \leq p$. *Then the longest* $\{\bar{y}, \bar{z}\}$*-subsequence of* $\boldsymbol{v}_2$ *can only be* $yzyz$ *or* $y^*z^*y^*z^*$.

(ii) *Suppose that* $yzyz \hookrightarrow \boldsymbol{v}_2$ *for some* $y, z \in \mathsf{con}(\boldsymbol{d}_i)$ *with* $1 \leq i \leq q$. *Then the longest* $\{\bar{y}, \bar{z}\}$*-subsequence of* $\boldsymbol{u}_2$ *can only be* $yzyz$ *or* $y^*z^*y^*z^*$.

Proof. (i) By Remark 5.9(v), $yzyz$ is the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{u}$. Further, $\bar{y} \prec \bar{z}$ because $\boldsymbol{c}_i$ is an ordered A_0-block. Hence $\bar{y}, \bar{z} \in \mathsf{con}(\bar{\boldsymbol{c}}_i) \subseteq \mathsf{con}(\overline{\boldsymbol{d}_1 \boldsymbol{d}_2 \cdots \boldsymbol{d}_q})$ by Lemma 5.14(ii). If $\bar{y}, \bar{z} \in \mathsf{con}(\bar{\boldsymbol{d}}_j)$ for some $j \in \{1, 2, \ldots, q\}$, then the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ is one of

$$yzyz, \quad y^*zy^*z, \quad yz^*yz^*, \quad y^*z^*y^*z^*;$$

and if $\bar{y} \in \mathsf{con}(\bar{\boldsymbol{d}}_j)$ and $\bar{z} \in \mathsf{con}(\bar{\boldsymbol{d}}_k)$ for some distinct $j, k \in \{1, 2, \ldots, q\}$, then the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ is one of

$$yyzz, \quad y^*y^*zz, \quad yyz^*z^*, \quad y^*y^*z^*z^*, \quad zzyy, \quad z^*z^*yy, \quad zzy^*y^*, \quad z^*z^*y^*y^*.$$

There are four cases to consider.

<u>Case 1</u>: y^*zy^*z or yz^*yz^* or y^*y^*zz or z^*z^*yy is a subsequence of $\boldsymbol{v}_2$. Then under the substitution $\alpha_{\boldsymbol{u}} : \mathcal{X} \to A_0$ in Lemma 5.10, we have $\alpha_{\boldsymbol{u}}(\boldsymbol{u}_1 x \boldsymbol{u}_2) = \mathrm{E}$ and $\alpha_{\boldsymbol{u}}(x^*) = \mathrm{F}$, so that $\alpha_{\boldsymbol{u}}(\boldsymbol{u}) = \mathrm{EF}$. Specifically, $\alpha_{\boldsymbol{u}}(y) = \alpha_{\boldsymbol{u}}(z) = \mathrm{E}$ because $y, z \in \mathsf{con}(\boldsymbol{u}_2)$. But this implies the contradiction $\alpha_{\boldsymbol{u}}(\boldsymbol{v}) = 0$.

<u>Case 2</u>: either $yyz^*z^* \hookrightarrow \boldsymbol{v}_2$ or $zzy^*y^* \hookrightarrow \boldsymbol{v}_2$. Then under the substitution $\beta_{\boldsymbol{u}} : \mathcal{X} \to A_0$ in Lemma 5.10, we have $\beta_{\boldsymbol{u}}(\boldsymbol{u}_1 x) = \mathrm{E}$ and $\beta_{\boldsymbol{u}}(\boldsymbol{u}_2 x^*) = \mathrm{F}$, so that $\beta_{\boldsymbol{u}}(\boldsymbol{u}) = \mathrm{EF}$. Specifically, $\beta_{\boldsymbol{u}}(y) = \beta_{\boldsymbol{u}}(z) = \mathrm{F}$ because $y, z \in \mathsf{con}(\boldsymbol{u}_2)$. But this implies the contradiction $\beta_{\boldsymbol{u}}(\boldsymbol{v}) = 0$.

Case 3: either $yyzz \hookrightarrow \boldsymbol{v}_2$ or $y^*y^*z^*z^* \hookrightarrow \boldsymbol{v}_2$. Then we can write $\boldsymbol{v}_2 = \boldsymbol{ab}$ such that $\mathsf{occ}(\bar{y}, \bar{\boldsymbol{a}}) = 2$ and $\mathsf{occ}(\bar{z}, \bar{\boldsymbol{b}}) = 2$. Let $\varphi : \mathcal{X} \cup \mathcal{X}^* \to A_0$ be any substitution that maps each variable in $\mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{a})$ to E and each variable in $\mathsf{con}(\boldsymbol{b})$ to F. Then

$$\varphi(\boldsymbol{v}) = \varphi(\boldsymbol{u}_1 x \boldsymbol{a}) \cdot \varphi(\boldsymbol{b}) \cdot \varphi(x^*) = \mathrm{E} \cdot \mathrm{E} \cdot \mathrm{E}^* = \mathrm{EF}.$$

Now depending on whether $yyzz \hookrightarrow \boldsymbol{v}_2$ or $y^*y^*z^*z^* \hookrightarrow \boldsymbol{v}_2$, the pair $(\varphi(y), \varphi(z))$ is either (E, F) or (F, E); but in either case,

$$\varphi(\boldsymbol{u}) = \cdots \varphi(y) \cdots \varphi(z) \cdots \varphi(y) \cdots \varphi(z) \cdots = 0,$$

which is a contradiction.

Case 4: either $zzyy \hookrightarrow \boldsymbol{v}_2$ or $z^*z^*y^*y^* \hookrightarrow \boldsymbol{v}_2$. This is symmetrical to the previous case and so also leads to a contradiction.

Since none of the four cases is possible, the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ can only be $yzyz$ or $y^*z^*y^*z^*$.

(ii) This is symmetrical to part (i). □

Lemma 5.16. (i) *Suppose that $yz \hookrightarrow \boldsymbol{u}_2$ for some $y, z \in \mathcal{X} \cup \mathcal{X}^*$ that are not in the same ordered A_0-block. Then the $\{\bar{y}, \bar{z}\}$-subsequences of $\boldsymbol{v}_2$ of length two can only be yz or z^*y^*.*

(ii) *Suppose that $yz \hookrightarrow \boldsymbol{v}_2$ for some $y, z \in \mathcal{X} \cup \mathcal{X}^*$ that are not in the same ordered A_0-block. Then the $\{\bar{y}, \bar{z}\}$-subsequences of $\boldsymbol{u}_2$ of length two can only be yz or z^*y^*.*

Proof. (i) By symmetry, it suffices to assume that within $\boldsymbol{u}_2$, the first y appears before the first z. Then by assumption, depending on which of y and z is simple or in an ordered A_0-block, the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{u}_2$ is $y^r z^s$ for some $r, s \in \{1, 2\}$. Since $\boldsymbol{u}$ is in A_0-standard form, $\boldsymbol{u}_2 = \boldsymbol{a} y \boldsymbol{b} z \boldsymbol{e}$ for some pairwise disjoint $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $y \notin \mathsf{con}(\boldsymbol{be})$ and $z \notin \mathsf{con}(\boldsymbol{ab})$. Since $\bar{y}, \bar{z} \in \mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$ by Lemma 5.14, the $\{\bar{y}, \bar{z}\}$-subsequences of $\boldsymbol{v}_2$ of length two can only be

$$yz, \quad y^*z, \quad yz^*, \quad y^*z^*, \quad zy, \quad z^*y, \quad zy^*, \quad z^*y^*.$$

There are three cases to consider.

Case 1: either $y^*z \hookrightarrow \boldsymbol{v}_2$ or $z^*y \hookrightarrow \boldsymbol{v}_2$. Then under the substitution $\alpha_{\boldsymbol{u}} : \mathcal{X} \to A_0$ in Lemma 5.10, we have $\alpha_{\boldsymbol{u}}(\boldsymbol{u}_1 x \boldsymbol{u}_2) = \mathrm{E}$ and $\alpha_{\boldsymbol{u}}(x^*) = \mathrm{F}$, so that $\alpha_{\boldsymbol{u}}(\boldsymbol{u}) = \mathrm{EF}$. Specifically, $\alpha_{\boldsymbol{u}}(y) = \alpha_{\boldsymbol{u}}(z) = \mathrm{E}$ because $y, z \in \mathsf{con}(\boldsymbol{u}_2)$. But this implies the contradiction $\alpha_{\boldsymbol{u}}(\boldsymbol{v}) = 0$.

Case 2: either $yz^* \hookrightarrow \boldsymbol{v}_2$ or $zy^* \hookrightarrow \boldsymbol{v}_2$. Then under the substitution $\beta_{\boldsymbol{u}} : \mathcal{X} \to A_0$ in Lemma 5.10, we have $\beta_{\boldsymbol{u}}(\boldsymbol{u}_1 x) = \mathrm{E}$ and $\beta_{\boldsymbol{u}}(\boldsymbol{u}_2 x^*) = \mathrm{F}$, so that $\beta_{\boldsymbol{u}}(\boldsymbol{u}) = \mathrm{EF}$. Specifically, $\beta_{\boldsymbol{u}}(y) = \beta_{\boldsymbol{u}}(z) = \mathrm{F}$ because $y, z \in \mathsf{con}(\boldsymbol{u}_2)$. But this implies the contradiction $\beta_{\boldsymbol{u}}(\boldsymbol{v}) = 0$.

Case 3: either $y^*z^* \hookrightarrow \boldsymbol{v}_2$ or $zy \hookrightarrow \boldsymbol{v}_2$. Then under any substitution $\varphi : \mathcal{X} \cup \mathcal{X}^* \to A_0$

that maps each variable in $\mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{a} y \boldsymbol{b})$ to E and each variable in $\mathsf{con}(z\boldsymbol{e})$ to F, we have $\varphi(\boldsymbol{u}) = \varphi(\boldsymbol{u}_1 x \boldsymbol{a} y \boldsymbol{b}) \cdot \varphi(z\boldsymbol{e}) \cdot \varphi(x^*) = \mathrm{E} \cdot \mathrm{F} \cdot \mathrm{E}^* = \mathrm{EF}$. But $\varphi(\boldsymbol{v}) = 0$ is a contradiction.

Since none of the three cases is possible, the $\{\bar{y}, \bar{z}\}$-subsequences of $\boldsymbol{v}_2$ of length two can only be yz or $z^* y^*$.

(ii) This is symmetrical to part (i). □

Lemma 5.17. *Suppose that* $\boldsymbol{u}_2, \boldsymbol{v}_2 \neq 1$. *Then* $\mathsf{h}(\boldsymbol{u}_2) = \mathsf{h}(\boldsymbol{v}_2)$ *and* $\mathsf{t}(\boldsymbol{u}_2) = \mathsf{t}(\boldsymbol{v}_2)$.

Proof. Recall from Lemma 5.13(iii) that $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$. First assume that $|\mathsf{con}(\bar{\boldsymbol{u}}_2)| = |\mathsf{con}(\bar{\boldsymbol{v}}_2)| = 1$, say $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2) = \{z\}$ for some $z \in \mathcal{X}$. Then each of $\boldsymbol{u}_2$ and $\boldsymbol{v}_2$ can only be simple or an ordered A_0-block, so that $\boldsymbol{u}_2, \boldsymbol{v}_2 \in \{z, z^2\}$ by conditions (A4)(a) and (A4)(b). Hence $\mathsf{h}(\boldsymbol{u}_2) = z = \mathsf{h}(\boldsymbol{v}_2)$ and $\mathsf{t}(\boldsymbol{u}_2) = z = \mathsf{t}(\boldsymbol{v}_2)$.

Now assume that $|\mathsf{con}(\bar{\boldsymbol{u}}_2)| = |\mathsf{con}(\bar{\boldsymbol{v}}_2)| \geq 2$. Let $h = \mathsf{h}(\boldsymbol{u}_2)$, $t = \mathsf{t}(\boldsymbol{u}_2)$, $H = \mathsf{h}(\boldsymbol{v}_2)$, and $T = \mathsf{t}(\boldsymbol{v}_2)$, so that

$$\bar{h} \prec \bar{t} \quad \text{and} \quad \bar{H} \prec \bar{T} \tag{5-4}$$

by conditions (A4)(c) and (A4)(d). Recall that $\boldsymbol{u}_2$ and $\boldsymbol{v}_2$ are bipartite words such that $\mathsf{occ}(z, \boldsymbol{u}_2), \mathsf{occ}(z, \boldsymbol{v}_2) \leq 2$ for all $z \in \mathcal{X} \cup \mathcal{X}^*$. There are five cases to consider; in each case, several intermediate results are established to eventually show that $h = H$ and $t = T$.

Case 1: $\mathsf{occ}(h, \boldsymbol{u}_2) = \mathsf{occ}(t, \boldsymbol{u}_2) = 1$. Then $h = \mathsf{h}(\boldsymbol{s}_0)$ and $t = \mathsf{t}(\boldsymbol{s}_p)$, so that $\boldsymbol{s}_0 = h\boldsymbol{a}$ and $\boldsymbol{s}_p = \boldsymbol{b}t$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$:

$$\boldsymbol{u} = x_1 x_2 \cdots x_m \cdot x \cdot \underbrace{h\boldsymbol{a} \cdot \boldsymbol{c}_1 \boldsymbol{s}_1 \cdot \boldsymbol{c}_2 \boldsymbol{s}_2 \cdots \boldsymbol{c}_{p-1} \boldsymbol{s}_{p-1} \cdot \boldsymbol{c}_p \boldsymbol{b} t}_{\boldsymbol{u}_2} \cdot x^*. \tag{5-5}$$

Result A. $\mathsf{occ}(\bar{h}, \bar{\boldsymbol{v}}_2) = \mathsf{occ}(\bar{t}, \bar{\boldsymbol{v}}_2) = 1$.

Proof. This holds because $\bar{h}, \bar{t} \in \mathsf{con}(\overline{\boldsymbol{s}_0 \boldsymbol{s}_p}) \subseteq \mathsf{con}(\overline{\boldsymbol{t}_0 \boldsymbol{t}_1 \cdots \boldsymbol{t}_q})$ by Lemma 5.14(i). □

Result B. The longest $\{\bar{h}, \bar{t}\}$*-subsequence of* $\boldsymbol{v}_2$ *is* ht.

Proof. Since $ht \hookrightarrow \boldsymbol{u}_2$, it follows from Lemma 5.16(i) and Result A that the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$ is either ht or $t^* h^*$. Seeking a contradiction, suppose that $t^* h^*$ is the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$. Note that t^* does not occur in any ordered A_0-block in $\boldsymbol{v}_2$ due to $\mathsf{occ}(\bar{t}, \bar{\boldsymbol{v}}_2) = 1$ by Result A. It follows that if $H \neq t^*$, so that $Ht^* \hookrightarrow \boldsymbol{v}_2$, then by Lemma 5.16(ii), either $Ht^* \hookrightarrow \boldsymbol{u}_2$ or $tH^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-5). By a symmetrical argument, it is impossible for $T \neq h^*$. Therefore, $H = t^*$ and $T = h^*$. It follows that $\mathsf{occ}(\bar{H}, \bar{\boldsymbol{v}}_2) = \mathsf{occ}(\bar{t}, \bar{\boldsymbol{v}}_2) = 1$ and $H = \mathsf{h}(\boldsymbol{v}_2) = \mathsf{h}(\boldsymbol{t}_0)$, so that $\boldsymbol{t}_0 \neq 1$. Given that $\boldsymbol{v}$ is in A_0-standard form, we have $\bar{H} \prec \bar{T}$ by condition (A4)(c); but this implies that $\bar{t} = \bar{H} \prec \bar{T} = \bar{h}$, which contradicts (5-4). □

Result C. $h = H$ *and* $t = T$.

Proof. Suppose that $h \neq H$. Then it follows from Result B that $Hh \hookrightarrow \boldsymbol{v}_2$, and h is not in any ordered A_0-block in $\boldsymbol{v}_2$ due to Result A. Therefore, by Lemma 5.16(ii), either $Hh \hookrightarrow \boldsymbol{u}_2$ or $h^*H^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-5). By a symmetrical argument, it is impossible for $t \neq T$. □

Case 2: $\mathsf{occ}(h, \boldsymbol{u}_2) = 2$ and $\mathsf{occ}(t, \boldsymbol{u}_2) = 1$. Then $\boldsymbol{s}_0 = 1$, $h = \mathsf{h}(\boldsymbol{c}_1)$, and $t = \mathsf{t}(\boldsymbol{s}_p)$, so that $\boldsymbol{c}_1 = \boldsymbol{haha}$ and $\boldsymbol{s}_p = \boldsymbol{bt}$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$:

$$\boldsymbol{u} = x_1x_2 \cdots x_m \cdot x \cdot \underbrace{\boldsymbol{haha}\, \boldsymbol{s}_1 \cdot \boldsymbol{c}_2\boldsymbol{s}_2 \cdots \boldsymbol{c}_{p-1}\boldsymbol{s}_{p-1} \cdot \boldsymbol{c}_p\boldsymbol{bt}}_{\boldsymbol{u}_2} \cdot x^*. \tag{5-6}$$

Result D. $\mathsf{occ}(\bar{h}, \bar{\boldsymbol{v}}_2) = 2$ *and* $\mathsf{occ}(\bar{t}, \bar{\boldsymbol{v}}_2) = 1$.

Proof. This holds because $\bar{h} \in \mathsf{con}(\bar{\boldsymbol{c}}_1) \subseteq \mathsf{con}(\overline{\boldsymbol{d}_1\boldsymbol{d}_2 \cdots \boldsymbol{d}_q})$ and $\bar{t} \in \mathsf{con}(\bar{\boldsymbol{s}}_p) \subseteq \mathsf{con}(\overline{\boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q})$ by Lemma 5.14. □

Result E. The longest $\{\bar{h}, \bar{t}\}$*-subsequence of* $\boldsymbol{v}_2$ *is* h^2t.

Proof. Since $h^2t \hookrightarrow \boldsymbol{u}_2$, it follows from Lemma 5.16(i) and Result D that the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$ is either h^2t or $t^*(h^*)^2$. Seeking a contradiction, suppose that $t^*(h^*)^2$ is the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$. Note that t^* does not occur in any ordered A_0-block in $\boldsymbol{v}_2$ due to $\mathsf{occ}(\bar{t}, \bar{\boldsymbol{v}}_2) = 1$ by Result D. It follows that if $H \neq t^*$, so that $Ht^* \hookrightarrow \boldsymbol{v}_2$, then by Lemma 5.16(ii), either $Ht^* \hookrightarrow \boldsymbol{u}_2$ or $tH^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-6). Hence $H = t^*$.

Now suppose that $T \neq h^*$, so that $h^*T \hookrightarrow \boldsymbol{v}_2$. Then since $\boldsymbol{v}_2$ is bipartite, we have $h, T^* \notin \mathsf{con}(\boldsymbol{v}_2)$. There are two cases.

(a) h^* and T are not in the same ordered A_0-block in $\boldsymbol{v}_2$. By Lemma 5.16(ii), either $h^*T \hookrightarrow \boldsymbol{u}_2$ or $T^*h \hookrightarrow \boldsymbol{u}_2$. We see from (5-6) that $h^*T \not\hookrightarrow \boldsymbol{u}_2$, so only $T^*h \hookrightarrow \boldsymbol{u}_2$ holds. Specifically, $T^*h \hookrightarrow \boldsymbol{c}_1 = \boldsymbol{haha}$, so that $hT^*hT^* \hookrightarrow \boldsymbol{u}_2$. Hence, by Lemma 5.15(i), either $hT^*hT^* \hookrightarrow \boldsymbol{v}_2$ or $h^*Th^*T \hookrightarrow \boldsymbol{v}_2$; the former contradicts $h, T^* \notin \mathsf{con}(\boldsymbol{v}_2)$, while the latter contradicts the assumption of the present case.

(b) h^* and T are in the same ordered A_0-block in $\boldsymbol{v}_2$. Specifically, since $T = \mathsf{t}(\boldsymbol{v}_2)$, we have $h^*, T \in \mathsf{con}(\boldsymbol{d}_q)$, so that $h^*Th^*T \hookrightarrow \boldsymbol{v}_2$ and $\bar{h} \prec \bar{T}$. By Lemma 5.15(ii), either $h^*Th^*T \hookrightarrow \boldsymbol{u}_2$ or $hT^*hT^* \hookrightarrow \boldsymbol{u}_2$. It follows from (5-6) that $h^*Th^*T \not\hookrightarrow \boldsymbol{u}_2$, so only $hT^*hT^* \hookrightarrow \boldsymbol{u}_2$ holds, whence $T^* \in \mathsf{con}(\boldsymbol{c}_1)$. Since $\boldsymbol{s}_0 = 1$ and $\boldsymbol{u}$ is in A_0-standard form, by condition (A4)(d), we have $\bar{T} \preceq \overline{\mathsf{t}(\boldsymbol{c}_1)} \prec \overline{\mathsf{t}(\boldsymbol{u}_2)} = \bar{t}$. Now the first variable in $\boldsymbol{v}_2$ is simple because $\mathsf{h}(\boldsymbol{v}_2) = H = t^*$ and $\mathsf{occ}(\bar{t}, \bar{\boldsymbol{v}}_2) = 1$; hence $\boldsymbol{t}_0 \neq 1$. Since $\boldsymbol{v}$ is in A_0-standard form, by condition (A4)(c), we have $\bar{t} = \bar{H} = \overline{\mathsf{h}(\boldsymbol{t}_0)} \prec \overline{\mathsf{t}(\boldsymbol{v}_2)} = \bar{T}$, which is a contradiction.

Since neither (a) nor (b) is possible, we have $T = h^*$. As observed in (b), the first variable in $\boldsymbol{v}_2$ is simple because $\mathsf{h}(\boldsymbol{v}_2) = H = t^*$ and $\mathsf{occ}(\bar{t}, \bar{\boldsymbol{v}}_2) = 1$, thus $\boldsymbol{t}_0 \neq 1$. Given that $\boldsymbol{v}$ is in A_0-standard form, by condition (A4)(c), we have $\bar{t} = \bar{H} = \overline{\mathsf{h}(\boldsymbol{t}_0)} \prec \overline{\mathsf{t}(\boldsymbol{v}_2)} = \bar{T} = \bar{h}$, which contradicts (5-4). □

Result F. $h = H$ *and* $t = T$.

Proof. First, suppose that $t \neq T$. Then $tT \hookrightarrow \boldsymbol{v}_2$ by Result E, and t is not in any ordered A_0-block in $\boldsymbol{v}_2$ due to Result D. Therefore, by Lemma 5.16(ii), either $tT \hookrightarrow \boldsymbol{u}_2$ or $T^*t^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-6). Hence $t = T$.

Now suppose that $h \neq H$. Then $Hh \hookrightarrow \boldsymbol{v}_2$ by Result E. Since $\boldsymbol{v}_2$ is bipartite, we have $h^*, H^* \notin \mathsf{con}(\boldsymbol{v}_2)$. There are two cases.

(a) H and h are not in the same ordered A_0-block in $\boldsymbol{v}_2$. Then by Lemma 5.16(ii), either $Hh \hookrightarrow \boldsymbol{u}_2$ or $h^*H^* \hookrightarrow \boldsymbol{u}_2$. But it is clear from (5-6) that $h^*H^* \not\hookrightarrow \boldsymbol{u}_2$, so only $Hh \hookrightarrow \boldsymbol{u}_2$ holds. Specifically, $Hh \hookrightarrow \boldsymbol{c}_1 = h\boldsymbol{a}h\boldsymbol{a}$, so that $hHhH \hookrightarrow \boldsymbol{u}_2$. Therefore, by Lemma 5.15(i), either $hHhH \hookrightarrow \boldsymbol{v}_2$ or $h^*H^*h^*H^* \hookrightarrow \boldsymbol{v}_2$; but the former contradicts the assumption of the present case, while the latter contradicts $h^*, H^* \notin \mathsf{con}(\boldsymbol{v}_2)$.

(b) H and h are in the same ordered A_0-block in $\boldsymbol{v}_2$. Then $HhHh \hookrightarrow \boldsymbol{v}_2$. Hence by Lemma 5.15(ii), either $HhHh \hookrightarrow \boldsymbol{u}_2$ or $H^*h^*H^*h^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-6).

Since neither (a) nor (b) is possible, we have $h = H$. □

<u>Case 3</u>: $\mathsf{occ}(h, \boldsymbol{u}_2) = 1$ and $\mathsf{occ}(t, \boldsymbol{u}_2) = 2$. Then $h = \mathsf{h}(\boldsymbol{s}_0)$, $t = \mathsf{t}(\boldsymbol{c}_p)$, and $\boldsymbol{s}_p = 1$, so that $\boldsymbol{s}_0 = h\boldsymbol{a}$ and $\boldsymbol{c}_p = \boldsymbol{b}t\boldsymbol{b}t$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$:

$$\boldsymbol{u} = x_1x_2\cdots x_m \cdot x \cdot \underbrace{h\boldsymbol{a} \cdot \boldsymbol{c}_1\boldsymbol{s}_1 \cdot \boldsymbol{c}_2\boldsymbol{s}_2 \cdots \boldsymbol{c}_{p-1}\boldsymbol{s}_{p-1} \cdot \boldsymbol{b}t\boldsymbol{b}t}_{\boldsymbol{u}_2} \cdot x^*. \tag{5-7}$$

Result G. $\mathsf{occ}(\bar{h}, \bar{\boldsymbol{v}}_2) = 1$ *and* $\mathsf{occ}(\bar{t}, \bar{\boldsymbol{v}}_2) = 2$.

Proof. This holds because $\bar{h} \in \mathsf{con}(\bar{\boldsymbol{s}}_0) \subseteq \mathsf{con}(\overline{\boldsymbol{t}_0\boldsymbol{t}_1\cdots\boldsymbol{t}_q})$ and $\bar{t} \in \mathsf{con}(\bar{\boldsymbol{c}}_p) \subseteq \mathsf{con}(\overline{\boldsymbol{d}_1\boldsymbol{d}_2\cdots\boldsymbol{d}_q})$ by Lemma 5.14. □

Result H. The longest $\{\bar{h}, \bar{t}\}$*-subsequence of* $\boldsymbol{v}_2$ *is* ht^2.

Proof. Since $ht^2 \hookrightarrow \boldsymbol{u}_2$, it follows from Lemma 5.16(i) and Result G that the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$ is either ht^2 or $(t^*)^2h^*$. Seeking a contradiction, suppose that $(t^*)^2h^*$ is the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$. Note that h^* does not occur in any ordered A_0-block in $\boldsymbol{v}_2$ due to $\mathsf{occ}(\bar{h}, \bar{\boldsymbol{v}}_2) = 1$ by Result G. It follows that if $T \neq h^*$, so that $h^*T \hookrightarrow \boldsymbol{v}_2$, then by Lemma 5.16(ii), either $h^*T \hookrightarrow \boldsymbol{u}_2$ or $T^*h \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-7). Hence $T = h^*$.

Now suppose that $H \neq t^*$, so that $Ht^* \hookrightarrow \boldsymbol{v}_2$. Then since $\boldsymbol{v}_2$ is bipartite, $H^*, t \notin \mathsf{con}(\boldsymbol{v}_2)$. There are two cases.

(a) H and t^* are not in the same ordered A_0-block in $\boldsymbol{v}_2$. Then by Lemma 5.16(ii), either $Ht^* \hookrightarrow \boldsymbol{u}_2$ or $tH^* \hookrightarrow \boldsymbol{u}_2$. But it is clear from (5-7) that $Ht^* \not\hookrightarrow \boldsymbol{u}_2$, so only $tH^* \hookrightarrow \boldsymbol{u}_2$ holds. Specifically, $tH^* \hookrightarrow \boldsymbol{c}_p = \boldsymbol{b}t\boldsymbol{b}t$, so that $H^*tH^*t \hookrightarrow \boldsymbol{u}_2$. Therefore, by Lemma 5.15(i), either $H^*tH^*t \hookrightarrow \boldsymbol{v}_2$ or $Ht^*Ht^* \hookrightarrow \boldsymbol{v}_2$; but the former contradicts $H^*, t \notin \mathsf{con}(\boldsymbol{v}_2)$, while the latter contradicts the assumption of the present case.

(b) H and t^* are in the same ordered A_0-block in $\boldsymbol{v}_2$. Specifically, since $H = \mathsf{h}(\boldsymbol{v}_2)$, we have $\boldsymbol{t}_0 = 1$ and $H, t^* \in \mathsf{con}(\boldsymbol{d}_1)$. Given that $\boldsymbol{v}$ is in A_0-standard form, by condition (A4)(d), we have $\bar{t} \preceq \overline{\mathsf{t}(\boldsymbol{d}_1)} \prec \overline{\mathsf{t}(\boldsymbol{v}_2)} = \overline{T} = \bar{h}$, but this contradicts (5-4).

Since neither (a) nor (b) is possible, we must have $H = t^*$. Now since $\overline{H} \prec \overline{T}$ by (5-4), we have $\bar{t} = \overline{H} \prec \overline{T} = \bar{h}$; but this contradicts $\bar{h} \prec \bar{t}$ in (5-4). □

Result I. $h = H$ *and* $t = T$.

Proof. First, suppose that $h \neq H$. Then $Hh \hookrightarrow \boldsymbol{v}_2$ by Result H, and h is not in any ordered A_0-block in $\boldsymbol{v}_2$ due to Result G. Therefore, by Lemma 5.16(ii), either $Hh \hookrightarrow \boldsymbol{u}_2$ or $h^*H^* \hookrightarrow \boldsymbol{u}_2$; but this is impossible in view of (5-7). Hence $h = H$.

Now suppose that $t \neq T$. Then $tT \hookrightarrow \boldsymbol{v}_2$ by Result H, and $t^*, T^* \notin \mathsf{con}(\boldsymbol{v}_2)$ due to $\boldsymbol{v}_2$ being bipartite. If t and T are in the same ordered A_0-block in $\boldsymbol{v}_2$, so that $tTtT \hookrightarrow \boldsymbol{v}_2$, then by Lemma 5.15(ii), either $tTtT \hookrightarrow \boldsymbol{u}_2$ or $t^*T^*t^*T^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-7). Therefore, t and T are not in the same ordered A_0-block in $\boldsymbol{v}_2$. Then by Lemma 5.16(ii), either $tT \hookrightarrow \boldsymbol{u}_2$ or $T^*t^* \hookrightarrow \boldsymbol{u}_2$. But it is clear from (5-7) that $T^*t^* \not\hookrightarrow \boldsymbol{u}_2$, so only $tT \hookrightarrow \boldsymbol{u}_2$ holds. Specifically, $tT \hookrightarrow \boldsymbol{c}_p = \boldsymbol{btbt}$, so that $TtTt \hookrightarrow \boldsymbol{u}_2$. Hence by Lemma 5.15(i), either $TtTt \hookrightarrow \boldsymbol{v}_2$ or $T^*t^*T^*t^* \hookrightarrow \boldsymbol{v}_2$; but the former contradicts t and T not being in the same ordered A_0-block in $\boldsymbol{v}_2$, while the latter contradicts $t^*, T^* \notin \mathsf{con}(\boldsymbol{v}_2)$. □

<u>Case 4</u>: $\mathsf{occ}(h, \boldsymbol{u}_2) = \mathsf{occ}(t, \boldsymbol{u}_2) = 2$ with $p \geq 2$. Then $\boldsymbol{s}_0 = 1$, $h = \mathsf{h}(\boldsymbol{c}_1)$, $t = \mathsf{t}(\boldsymbol{c}_p)$, and $\boldsymbol{s}_p = 1$, so that $\boldsymbol{c}_1 = h\boldsymbol{a}h\boldsymbol{a}$ and $\boldsymbol{c}_p = \boldsymbol{b}t\boldsymbol{b}t$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathscr{X})$:

$$\boldsymbol{u} = x_1x_2\cdots x_m \cdot x \cdot \underbrace{h\boldsymbol{a}h\boldsymbol{a}\,\boldsymbol{s}_1 \cdot \boldsymbol{c}_2\boldsymbol{s}_2 \cdots \boldsymbol{c}_{p-1}\boldsymbol{s}_{p-1} \cdot \boldsymbol{b}t\boldsymbol{b}t}_{\boldsymbol{u}_2} \cdot x^*. \tag{5-8}$$

Result J. $\mathsf{occ}(\bar{h}, \bar{\boldsymbol{v}}_2) = \mathsf{occ}(\bar{t}, \bar{\boldsymbol{v}}_2) = 2$.

Proof. This holds because $\bar{h}, \bar{t} \in \mathsf{con}(\overline{\boldsymbol{c}_1\boldsymbol{c}_p}) \subseteq \mathsf{con}(\overline{\boldsymbol{d}_1\boldsymbol{d}_2\cdots\boldsymbol{d}_q})$ by Lemma 5.14(ii). □

Result K. The longest $\{\bar{h}, \bar{t}\}$*-subsequence of* $\boldsymbol{v}_2$ *is either* h^2t^2 *or* $(t^*)^2(h^*)^2$.

Proof. Since $h^2t^2 \hookrightarrow \boldsymbol{u}_2$, it follows from Lemma 5.16(i) and Result J that the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$ is one of the following six words: h^2t^2, $htht$, $thth$, $(t^*)^2(h^*)^2$, $t^*h^*t^*h^*$, and $h^*t^*h^*t^*$. If either $htht \hookrightarrow \boldsymbol{v}_2$ or $h^*t^*h^*t^* \hookrightarrow \boldsymbol{v}_2$, then it follows from Lemma 5.15(ii) that either $htht \hookrightarrow \boldsymbol{u}_2$ or $h^*t^*h^*t^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-8). If $thth \hookrightarrow \boldsymbol{v}_2$, then t and h are in the same ordered A_0-block in $\boldsymbol{v}_2$, whence $\bar{t} \prec \bar{h}$; but this contradicts (5-4). A similar contradiction is obtained if $t^*h^*t^*h^* \hookrightarrow \boldsymbol{v}_2$. □

Result L. Suppose that $(t^*)^2(h^*)^2 \hookrightarrow \boldsymbol{v}_2$ *and* $H \neq t^*$. *Then* $\bar{t} \prec \overline{T}$.

Proof. By assumption, $Ht^* \hookrightarrow \boldsymbol{v}_2$. Since $\boldsymbol{v}_2$ is bipartite, we have $H^*, t \notin \mathsf{con}(\boldsymbol{v}_2)$. Suppose that H and t^* are not in the same ordered A_0-block in $\boldsymbol{v}_2$. Then by Lemma 5.16(ii), either $Ht^* \hookrightarrow \boldsymbol{u}_2$ or $tH^* \hookrightarrow \boldsymbol{u}_2$. But it is clear from (5-8) that $Ht^* \not\hookrightarrow \boldsymbol{u}_2$, so only $tH^* \hookrightarrow \boldsymbol{u}_2$ holds. Specifically, $tH^* \hookrightarrow \boldsymbol{c}_p = \boldsymbol{btbt}$, so that $H^*tH^*t \hookrightarrow \boldsymbol{u}_2$. Therefore, by Lemma 5.15(i), either $H^*tH^*t \hookrightarrow \boldsymbol{v}_2$ or $Ht^*Ht^* \hookrightarrow \boldsymbol{v}_2$; but the former contradicts $H^*, t \notin \mathsf{con}(\boldsymbol{v}_2)$, while the latter contradicts H and t^* not being in the same ordered A_0-block in $\boldsymbol{v}_2$.

Therefore, H and t^* are in the same ordered A_0-block in $\boldsymbol{v}_2$. Specifically, since $H = \mathsf{h}(\boldsymbol{v}_2)$, we have $H, t^* \in \mathsf{con}(\boldsymbol{d}_1)$ and $\boldsymbol{t}_0 = 1$. Given that $\boldsymbol{v}$ is in A_0-standard form, it follows from condition (A4)(d) that $\bar{t} \preceq \overline{\mathsf{t}(\boldsymbol{d}_1)} \prec \overline{\mathsf{t}(\boldsymbol{v}_2)} = \overline{T}$. □

Result M. Suppose that $(t^*)^2(h^*)^2 \hookrightarrow \boldsymbol{v}_2$ and $T \neq h^*$. Then $\overline{T} \prec \bar{t}$.

Proof. By assumption, $h^*T \hookrightarrow \boldsymbol{v}_2$. There are two cases.

(a) T and h^* are not in the same ordered A_0-block in $\boldsymbol{v}_2$. Then by Lemma 5.16(ii), either $h^*T \hookrightarrow \boldsymbol{u}_2$ or $T^*h \hookrightarrow \boldsymbol{u}_2$. It is clear from (5-8) that $h^*T \not\hookrightarrow \boldsymbol{u}_2$, so only $T^*h \hookrightarrow \boldsymbol{u}_2$ holds. Specifically, $T^*h \hookrightarrow \boldsymbol{c}_1 = h\boldsymbol{a}h\boldsymbol{a}$.

(b) T and h^* are in the same ordered A_0-block in $\boldsymbol{v}_2$. Then $h^*Th^*T \hookrightarrow \boldsymbol{v}_2$. By Lemma 5.15(ii), either $h^*Th^*T \hookrightarrow \boldsymbol{u}_2$ or $hT^*hT^* \hookrightarrow \boldsymbol{u}_2$. It is clear from (5-8) that $h^*Th^*T \not\hookrightarrow \boldsymbol{u}_2$, so only $hT^*hT^* \hookrightarrow \boldsymbol{u}_2$ holds, whence $hT^*hT^* \hookrightarrow \boldsymbol{c}_1 = h\boldsymbol{a}h\boldsymbol{a}$.

Therefore, in any case, we have $T^* \in \mathsf{con}(\boldsymbol{c}_1)$. Since $\boldsymbol{s}_0 = 1$ and $\boldsymbol{u}$ is in A_0-standard form, by condition (A4)(d), we have $\overline{T} \preceq \overline{\mathsf{t}(\boldsymbol{c}_1)} \prec \overline{\mathsf{t}(\boldsymbol{u}_2)} = \bar{t}$. □

Result N. The longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$ is h^2t^2.

Proof. By Result K, the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$ is either h^2t^2 or $(t^*)^2(h^*)^2$. Seeking a contradiction, suppose that $(t^*)^2(h^*)^2 \hookrightarrow \boldsymbol{v}_2$. If $H \neq t^*$, then $\bar{t} \prec \overline{T}$ by Result L, whence $T = h^*$ by Result M; but this implies that $\bar{t} \prec \overline{T} = \bar{h}$, which contradicts (5-4). If $T \neq h^*$, then $\overline{T} \prec \bar{t}$ by Result M, whence $H = t^*$ by Result L; but this implies that $\overline{T} \prec \bar{t} = \overline{H}$, which contradicts (5-4) again. Therefore, we must have $H = t^*$ and $T = h^*$. Now since $\overline{H} \prec \overline{T}$ by (5-4), we have $\bar{t} = \overline{H} \prec \overline{T} = \bar{h}$; but this contradicts $\bar{h} \prec \bar{t}$ in (5-4). □

Result O. $h = H$ and $t = T$.

Proof. Seeking a contradiction, suppose that $h \neq H$. Then $Hh \hookrightarrow \boldsymbol{v}_2$ by Result N, and $H^*, h^* \notin \mathsf{con}(\boldsymbol{v}_2)$ due to $\boldsymbol{v}_2$ being bipartite. If H and h are in the same ordered A_0-block in $\boldsymbol{v}_2$, so that $HhHh \hookrightarrow \boldsymbol{v}_2$, then by Lemma 5.15(ii), either $HhHh \hookrightarrow \boldsymbol{u}_2$ or $H^*h^*H^*h^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-8). Therefore, H and h are not in the same ordered A_0-block in $\boldsymbol{v}_2$. Then by Lemma 5.16(ii), either $Hh \hookrightarrow \boldsymbol{u}_2$ or $h^*H^* \hookrightarrow \boldsymbol{u}_2$. It is clear from (5-8) that $h^*H^* \not\hookrightarrow \boldsymbol{u}_2$, so only $Hh \hookrightarrow \boldsymbol{u}_2$ holds. Specifically, $Hh \hookrightarrow \boldsymbol{c}_1 = h\boldsymbol{a}h\boldsymbol{a}$, so that $hHhH \hookrightarrow \boldsymbol{u}_2$. Therefore, by Lemma 5.15(i), either $hHhH \hookrightarrow \boldsymbol{v}_2$ or $h^*H^*h^*H^* \hookrightarrow \boldsymbol{v}_2$; but the former contradicts H and h not being in the same ordered A_0-block in $\boldsymbol{v}_2$, while the latter contradicts $H^*, h^* \notin \mathsf{con}(\boldsymbol{v}_2)$.

A symmetrical argument shows that the assumption $t \neq T$ also leads to a contradiction. □

Case 5: $\mathsf{occ}(h, \boldsymbol{u}_2) = \mathsf{occ}(t, \boldsymbol{u}_2) = 2$ with $p = 1$. Then $\boldsymbol{s}_0 = 1$, $h = \mathsf{h}(\boldsymbol{c}_1)$, $t = \mathsf{t}(\boldsymbol{c}_1)$, and $\boldsymbol{s}_1 = 1$, so that $\boldsymbol{c}_1 = h\boldsymbol{a}th\boldsymbol{a}t$ for some $\boldsymbol{a} \in F^1_{\mathsf{inv}}(\mathcal{X})$:

$$\boldsymbol{u} = x_1 x_2 \cdots x_m \cdot x \cdot \underbrace{h\boldsymbol{a}th\boldsymbol{a}t}_{\boldsymbol{u}_2} \cdot x^*. \tag{5-9}$$

Note that due to condition (A4)(b), we have $h \in \mathcal{X}$.

Result P. $\mathsf{occ}(\bar{y}, \bar{\boldsymbol{v}}_2) = 2$ *for all* $y \in \mathsf{con}(\boldsymbol{v}_2)$.

Proof. Since $\boldsymbol{s}_0 = \boldsymbol{s}_1 = 1$ and $\mathsf{con}(\overline{\boldsymbol{t}_0 \boldsymbol{t}_1 \cdots \boldsymbol{t}_q}) = \mathsf{con}(\overline{\boldsymbol{s}_0 \boldsymbol{s}_1})$ by Lemma 5.14(i), we have $\boldsymbol{t}_0 = \boldsymbol{t}_1 = \cdots = \boldsymbol{t}_q = 1$. Therefore, $\boldsymbol{v}_2 = \boldsymbol{d}_1 \boldsymbol{d}_2 \cdots \boldsymbol{d}_q$ and the result follows. □

Result Q. The longest $\{\bar{h}, \bar{t}\}$*-subsequence of* $\boldsymbol{v}_2$ *is* $htht$.

Proof. Since $htht \hookrightarrow \boldsymbol{u}_2$, it follows from Lemma 5.15(i) that the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$ is either $htht$ or $h^*t^*h^*t^*$. Seeking a contradiction, suppose that $h^*t^*h^*t^* \hookrightarrow \boldsymbol{v}_2$.

First consider the case when $H \neq h^*$, so that $Hh^* \hookrightarrow \boldsymbol{v}_2$. Then $H^*, h \notin \mathsf{con}(\boldsymbol{v}_2)$ due to $\boldsymbol{v}_2$ being bipartite. If H and h^* are in the same ordered A_0-block in $\boldsymbol{v}_2$, so that $Hh^*Hh^* \hookrightarrow \boldsymbol{v}_2$, then it follows from Lemma 5.15(ii) that either $Hh^*Hh^* \hookrightarrow \boldsymbol{u}_2$ or $H^*hH^*h \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-9). Therefore, H and h^* are not in the same ordered A_0-block in $\boldsymbol{v}_2$. Then by Lemma 5.16(ii), either $Hh^* \hookrightarrow \boldsymbol{u}_2$ or $hH^* \hookrightarrow \boldsymbol{u}_2$. It is clear from (5-9) that $Hh^* \not\hookrightarrow \boldsymbol{u}_2$, so only $hH^* \hookrightarrow \boldsymbol{u}_2$ holds. It follows that $hH^*hH^* \hookrightarrow \boldsymbol{u}_2$, so that by Lemma 5.15(i), either $hH^*hH^* \hookrightarrow \boldsymbol{v}_2$ or $h^*Hh^*H \hookrightarrow \boldsymbol{v}_2$; but the former contradicts $H^*, h \notin \mathsf{con}(\boldsymbol{v}_2)$, while the latter contradicts H and h^* not being in the same ordered A_0-block in $\boldsymbol{v}_2$.

Therefore, $H = h^*$. By a symmetrical argument, we have $T = t^*$. It follows that $HTHT = h^*t^*h^*t^* \hookrightarrow \boldsymbol{v}_2$, so that $\boldsymbol{v}_2 = \boldsymbol{d}_1 = h^*\boldsymbol{b}t^*h^*\boldsymbol{b}t^*$ for some $\boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$ (with $q = 1$). As deduced in the proof of Result P, we have $\boldsymbol{t}_0 = \boldsymbol{t}_1 = 1$. Since $\boldsymbol{v}$ is in A_0-standard form, by condition (A4)(b), we have $h^* = \mathsf{h}(\boldsymbol{d}_1) \in \mathcal{X}$, which contradicts the observation $h \in \mathcal{X}$ made after (5-9). □

Result R. $h = H$ *and* $t = T$.

Proof. Seeking a contradiction, suppose that $h \neq H$. Then $Hh \hookrightarrow \boldsymbol{v}_2$ by Result Q, and $H^*, h^* \notin \mathsf{con}(\boldsymbol{v}_2)$ due to $\boldsymbol{v}_2$ being bipartite. If H and h are in the same ordered A_0-block in $\boldsymbol{v}_2$, so that $HhHh \hookrightarrow \boldsymbol{v}_2$, then it follows from Lemma 5.15(ii) that either $HhHh \hookrightarrow \boldsymbol{u}_2$ or $H^*h^*H^*h^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-9). Therefore, H and h are not in the same ordered A_0-block in $\boldsymbol{v}_2$. Then by Lemma 5.16(ii), either $Hh \hookrightarrow \boldsymbol{u}_2$ or $h^*H^* \hookrightarrow \boldsymbol{u}_2$. It is clear from

(5-9) that $h^*H^* \not\hookrightarrow \boldsymbol{u}_2$, so only $Hh \hookrightarrow \boldsymbol{u}_2$ holds. It follows that $hHhH \hookrightarrow \boldsymbol{u}_2$, so that by Lemma 5.15(i), either $hHhH \hookrightarrow \boldsymbol{v}_2$ or $h^*H^*h^*H^* \hookrightarrow \boldsymbol{v}_2$; but the former contradicts H and h not being in the same ordered A_0-block in $\boldsymbol{v}_2$, while the latter contradicts $H^*, h^* \notin \mathsf{con}(\boldsymbol{v}_2)$.

A contradiction can be similarly deduced if $t \neq T$. □

In conclusion, we have $h = H$ and $t = T$ in all five cases (Results C, F, I, O, and R). The proof of Lemma 5.17 is thus complete. □

Lemma 5.18. *The identity $\boldsymbol{u}_2 \approx \boldsymbol{v}_2$ is satisfied by $(A_0, {}^*)$.*

Proof. Recall from Lemma 5.13(iii) that $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$. As shown in the beginning of the proof of Lemma 5.17, if $|\mathsf{con}(\bar{\boldsymbol{u}}_2)| = |\mathsf{con}(\bar{\boldsymbol{v}}_2)| = 1$, then $\boldsymbol{u}_2, \boldsymbol{v}_2 \in \{z, z^2\}$ for some $z \in \mathcal{X}$, so that $\boldsymbol{u}_2 = \boldsymbol{v}_2$ by Lemma 5.14 and conditions (A4)(a) and (A4)(b). Therefore, it suffices to assume that $|\mathsf{con}(\bar{\boldsymbol{u}}_2)| = |\mathsf{con}(\bar{\boldsymbol{v}}_2)| \geq 2$, so that by Lemma 5.17, we have $h = \mathsf{h}(\boldsymbol{u}_2) = \mathsf{h}(\boldsymbol{v}_2)$ and $t = \mathsf{t}(\boldsymbol{u}_2) = \mathsf{t}(\boldsymbol{v}_2)$ with $\bar{h} \prec \bar{t}$. Specifically, $\boldsymbol{u}_2 = h\boldsymbol{a}t$ and $\boldsymbol{v}_2 = h\boldsymbol{b}t$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$.

Seeking a contradiction, suppose that $\boldsymbol{u}_2 \approx \boldsymbol{v}_2$ is not satisfied by $(A_0, {}^*)$. Then there exists a substitution $\psi : \mathsf{con}(\bar{\boldsymbol{u}}_2) \to A_0$ such that $\psi(\boldsymbol{u}_2) \neq \psi(\boldsymbol{v}_2)$, so that

$$\psi(h) \cdot \psi(\boldsymbol{a}) \cdot \psi(t) \neq \psi(h) \cdot \psi(\boldsymbol{b}) \cdot \psi(t). \tag{5-10}$$

Clearly, $\psi(h) \neq 0 \neq \psi(t)$. Further, it is routinely checked that

$$\begin{aligned} &\mathrm{E} \cdot A_0^1 \cdot \mathrm{E} = \{0, \mathrm{E}\}, && \mathrm{E} \cdot A_0^1 \cdot \mathrm{F} = \{0, \mathrm{EF}\}, && \mathrm{E} \cdot A_0^1 \cdot \mathrm{EF} = \{0, \mathrm{EF}\},\\ &\mathrm{F} \cdot A_0^1 \cdot \mathrm{E} = \{0\}, && \mathrm{F} \cdot A_0^1 \cdot \mathrm{F} = \{0, \mathrm{F}\}, && \mathrm{F} \cdot A_0^1 \cdot \mathrm{EF} = \{0\},\\ &\mathrm{EF} \cdot A_0^1 \cdot \mathrm{E} = \{0\}, && \mathrm{EF} \cdot A_0^1 \cdot \mathrm{F} = \{0, \mathrm{EF}\}, && \mathrm{EF} \cdot A_0^1 \cdot \mathrm{EF} = \{0\}. \end{aligned}$$

Thus for (5-10) to hold, we need $(\psi(h), \psi(t)) \in \{(\mathrm{E},\mathrm{E}), (\mathrm{E},\mathrm{F}), (\mathrm{E},\mathrm{EF}), (\mathrm{F},\mathrm{F}), (\mathrm{EF},\mathrm{F})\}$, whence $\{\psi(\boldsymbol{u}_2), \psi(\boldsymbol{v}_2)\}$ can be $\{0, \mathrm{E}\}$, $\{0, \mathrm{F}\}$, or $\{0, \mathrm{EF}\}$. Generality is not lost by assuming that $\psi(\boldsymbol{u}_2) = 0$ and $\psi(\boldsymbol{v}_2) \in \{\mathrm{E}, \mathrm{F}, \mathrm{EF}\}$. Now extend ψ to the substitution Ψ that maps every $z \in \mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$ to $\psi(z)$ and every $z \in \{x_1, x_2, \ldots, x_m, x\}$ to E. Then $\Psi(\boldsymbol{u}) \neq \Psi(\boldsymbol{v})$ because

$$\begin{aligned} \Psi(\boldsymbol{u}) &= \Psi(x_1) \cdot \Psi(x_2) \cdots \Psi(x_m) \cdot \Psi(x) \cdot \psi(\boldsymbol{u}_2) \cdot \Psi(x)^* = \mathrm{E} \cdot 0 \cdot \mathrm{F} = 0\\ \text{and}\quad \Psi(\boldsymbol{v}) &= \Psi(x_1) \cdot \Psi(x_2) \cdots \Psi(x_m) \cdot \Psi(x) \cdot \psi(\boldsymbol{v}_2) \cdot \Psi(x)^* = \mathrm{E} \cdot \psi(\boldsymbol{v}_2) \cdot \mathrm{F} = \mathrm{EF}; \end{aligned}$$

but this is impossible given that $\boldsymbol{u} \approx \boldsymbol{v}$ is satisfied by $(A_0, {}^*)$. □

Since the words $\boldsymbol{u}_2$ and $\boldsymbol{v}_2$ are bipartite, it follows from Lemmas 5.5 and 5.18 that $\boldsymbol{u}_2 \approx \boldsymbol{v}_2$ is deducible from {(1-1), (5-1)}. Since

$$\boldsymbol{u} = x_1 x_2 \cdots x_m \cdot x \cdot \boldsymbol{u}_2 \cdot x^* \quad \text{and} \quad \boldsymbol{v} = x_1 x_2 \cdots x_m \cdot x \cdot \boldsymbol{v}_2 \cdot x^*,$$

the identity $\boldsymbol{u} \approx \boldsymbol{v}$ is also deducible from {(1-1), (5-1)}. The proof of Proposition 5.1 is thus complete.

6. The involution semigroup $(S_4,^*)$

The involution semigroup $(S_4,^*)$ is isomorphic to the semigroup

$$B_0 = \langle \mathrm{A}, \mathrm{E}, \mathrm{F} \mid \mathrm{AF} = \mathrm{EA} = \mathrm{A},\ \mathrm{E}^2 = \mathrm{E},\ \mathrm{F}^2 = \mathrm{F},\ \mathrm{EF} = \mathrm{FE} = 0 \rangle = \{0, \mathrm{A}, \mathrm{E}, \mathrm{F}\}$$

with the operation * that interchanges E and F and fixes every other element.

B_0	0	A	E	F
0	0	0	0	0
A	0	0	0	A
E	0	A	E	0
F	0	0	0	F
x	0	A	E	F
x^*	0	A	F	E

The involution semigroup $(B_0,^*)$ is isomorphic to the involution subsemigroup of $(B_2^1,^{\mathsf{S}})$ that consists of the elements

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The involution semigroup $(B_0,^*)$ belongs to the variety $\mathsf{Var}(A_0,^*)$ generated by $(A_0,^*)$ [20, Proposition 3.1] and so satisfies the identities (5-1) of $(A_0,^*)$. In this section, it is shown that the identities of $(B_0,^*)$ are axiomatized by (5-1) and one additional identity.

Proposition 6.1. *The identities* (5-1) *and*

$$x^2y^2 \approx y^2x^2 \tag{6-1}$$

constitute an identity basis for $(B_0,^*)$.

It is easily checked that $(B_0,^*)$ satisfies the identities {(5-1), (6-1)}. In Section 6.1, some information on identities of $(B_0,^*)$ are given. In Section 6.2, it is shown that the identities of $(B_0,^*)$ can be used to convert every mixed word into one of two specific forms. Based on these results, it is shown in Section 6.3 that every identity of $(B_0,^*)$ is deducible from {(1-1), (5-1), (6-1)}. This completes the proof of Proposition 6.1.

Corollary 6.2. *The identities*

$$\begin{gathered} x^3 \approx x^2, \quad xyx \approx x^2y^2, \quad x^2x^* \approx xx^*, \quad x^2yx^* \approx xyx^*, \\ xy^*x^* \approx xyx^*, \quad xx^* \approx yy^*, \quad xyx^*z \approx z^*xyx^* \end{gathered} \tag{6-2}$$

constitute an identity basis for $(B_0,^*)$.

Proof. It is routine to check, say with Prover9, that the identities {(1-1), (5-1), (6-1)} and {(1-1), (6-2)} are deducible from one another. □

Remarks 6.3. (i) The variety $\mathsf{Var}\, B_0$ is defined within $\mathsf{Var}\, A_0$ by the identity (6-1), and $\mathsf{Var}\, B_0$ is the unique maximal subvariety of $\mathsf{Var}\, A_0$ [14, Lemma 4.2]; in other words, the interval $[\mathsf{Var}\, B_0,\ \mathsf{Var}\, A_0]$ is a chain of length two.

(ii) In contrast, although the variety $\mathsf{Var}(B_0,{}^*)$ is also defined within $\mathsf{Var}(A_0,{}^*)$ by the identity (6-1) (Proposition 6.1), the interval $[\mathsf{Var}(B_0,{}^*), \mathsf{Var}(A_0,{}^*)]$ contains an infinite descending chain [20, Theorem 1.5].

6.1. *Some identities of* $(B_0,{}^*)$.

Lemma 6.4. *The identities* {(5-1h), (5-1i), (6-1)} *constitute an identity basis for the semigroup* B_0.

Proof. The identities of A_0, together with (6-1), form an identity basis for B_0 [14, Section 4]. The present lemma then follows from Lemma 5.4. □

Lemma 6.5. *Let* $\boldsymbol{u} \approx \boldsymbol{v}$ *be any identity of* $(B_0,{}^*)$ *such that either* $\boldsymbol{u}$ *or* $\boldsymbol{v}$ *is bipartite. Then* $\boldsymbol{u} \approx \boldsymbol{v}$ *is deducible from* {(1-1), (5-1), (6-1)}.

Proof. Since $(S\ell_3,{}^{\mathsf{S}})$ is isomorphic to the involution subsemigroup $(\{0, \mathrm{E}, \mathrm{F}\},{}^*)$ of $(B_0,{}^*)$, the identity $\boldsymbol{u} \approx \boldsymbol{v}$ is satisfied by $(S\ell_3,{}^{\mathsf{S}})$. Since either $\boldsymbol{u}$ or $\boldsymbol{v}$ is bipartite, by Lemma 2.4, both $\boldsymbol{u}$ and $\boldsymbol{v}$ are bipartite with $\mathsf{con}(\boldsymbol{u}) = \mathsf{con}(\boldsymbol{v})$. It follows from Lemma 2.2 that $(B_0,{}^*)$ satisfies the plain identity $\bar{\boldsymbol{u}} \approx \bar{\boldsymbol{v}}$. By Lemma 6.4, the identities {(5-1h), (5-1i), (6-1)} constitute an identity basis for B_0, so that $\bar{\boldsymbol{u}} \approx \bar{\boldsymbol{v}}$ is deducible from {(5-1h), (5-1i), (6-1)}. It then follows from Lemma 2.2 that $\boldsymbol{u} \approx \boldsymbol{v}$ is deducible from {(1-1), (5-1), (6-1)}. □

An *ordered* B_0*-block* is a word of the form

$$\boldsymbol{c} = y_1^2 y_2^2 \cdots y_k^2,$$

where $y_1, y_2, \ldots, y_k \in \mathcal{X} \cup \mathcal{X}^*$ are such that $\bar{y}_1 \prec \bar{y}_2 \prec \cdots \prec \bar{y}_k$ in $\mathcal{X}$ and $k \geq 1$. Note that every ordered B_0-block is bipartite.

Lemma 6.6. *Let* $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m \in F_{\mathsf{inv}}(\mathcal{X})$ *be any pairwise disjoint bipartite connected words such that* $\mathsf{con}(\boldsymbol{w}_1\boldsymbol{w}_2 \cdots \boldsymbol{w}_m) = \{y_1, y_2, \ldots, y_k\}$ *and* $\bar{y}_1 \prec \bar{y}_2 \prec \cdots \prec \bar{y}_k$ *in* $\mathcal{X}$. *Then the identities* {(5-1h), (5-1i), (6-1)} *can be used to convert the product* $\boldsymbol{w}_1\boldsymbol{w}_2 \cdots \boldsymbol{w}_m$ *into the ordered* B_0*-block* $\boldsymbol{c} = y_1^2 y_2^2 \cdots y_k^2$.

Proof. By Lemma 5.7, the identities {(5-1h), (5-1i)} can be used to convert each $\boldsymbol{w}_i$ into some ordered A_0-block $\boldsymbol{c}_i$ with $\mathsf{con}(\boldsymbol{w}_i) = \mathsf{con}(\boldsymbol{c}_i)$. Since $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_m$ are ordered A_0-blocks, we have

$$\boldsymbol{c}_i \overset{\text{(5-1h)}}{\approx} \boldsymbol{c}_i^2 \quad \text{and} \quad \boldsymbol{c}_i\boldsymbol{c}_j \overset{\text{(6-1)}}{\approx} \boldsymbol{c}_j\boldsymbol{c}_i.$$

Hence

$$\boldsymbol{w}_1\boldsymbol{w}_2 \cdots \boldsymbol{w}_m \overset{\text{(5-1h),(5-1i)}}{\approx} \boldsymbol{c}_1\boldsymbol{c}_2 \cdots \boldsymbol{c}_m \overset{\text{(5-1h)}}{\approx} \boldsymbol{c}_1^2\boldsymbol{c}_2^2 \cdots \boldsymbol{c}_m^2 \overset{\text{(6-1)}}{\approx} (\boldsymbol{c}_1\boldsymbol{c}_2 \cdots \boldsymbol{c}_m)^2.$$

Since $(\boldsymbol{c}_1\boldsymbol{c}_2\cdots\boldsymbol{c}_m)^2$ is a bipartite connected word with content $\{y_1, y_2, \ldots, y_k\}$, by Lemma 5.7, the identities {(5-1h), (5-1i)} can be used to convert it into the ordered A_0-block $(y_1y_2\cdots y_k)^2$. Hence

$$(y_1y_2\cdots y_k)^2 \overset{(5\text{-}1\text{h})}{\approx} (y_1^2y_2^2\cdots y_k^2)^2 \overset{(6\text{-}1)}{\approx} y_1^4y_2^4\cdots y_k^4 \overset{(5\text{-}1\text{h})}{\approx} y_1^2y_2^2\cdots y_k^2. \qquad \square$$

6.2. ***Some special forms of words.*** It is easily checked that for any substitution $\varphi : \mathcal{X} \to B_0$ and any variable $z \in \mathcal{X}$, we have $\varphi(zz^*) = 0$ in B_0. Therefore, in the $\mathsf{Var}(B_0,{}^*)$-free algebra over $\mathcal{X}$, the class $[zz^*]$ containing zz^* is its zero element. This phenomenon is equivalent to the following result, whose justification is routine.

Lemma 6.7. *The identities*

$$xx^*y \approx xx^*, \quad yxx^* \approx xx^*, \quad xx^* \approx yy^* \tag{6-3}$$

are deducible from {(1-1), (5-1), (6-1)}.

Words of other possible forms in the class $[zz^*]$ are listed in the following result.

Lemma 6.8. *Let* $\boldsymbol{w} \in F_{\mathsf{inv}}(\mathcal{X})$. *Suppose that one of the following conditions holds*:

(a) $xx^*x \hookrightarrow \boldsymbol{w}$ *for some* $x \in \mathcal{X} \cup \mathcal{X}^*$;

(b) $xx^*yy^* \hookrightarrow \boldsymbol{w}$ *for some* $x, y \in \mathcal{X} \cup \mathcal{X}^*$;

(c) $\boldsymbol{w} = \boldsymbol{a}x\boldsymbol{b}x^*\boldsymbol{e}$ *for some* $x \in \mathcal{X} \cup \mathcal{X}^*$ *and* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e} \in F^1_{\mathsf{inv}}(\mathcal{X})$ *such that for each* $y \in \mathsf{con}(\boldsymbol{b})$, *we have* $\mathsf{occ}(y, \boldsymbol{w}) \geq 2$.

Then the identities {(5-1), (6-1)} *can be used to convert* $\boldsymbol{w}$ *into the word* zz^* *for any* $z \in \mathcal{X} \cup \mathcal{X}^*$.

Proof. By Lemma 6.7, it suffices to convert $\boldsymbol{w}$ into the word zz^*, using the identities {(5-1), (6-1), (6-3)}. If either (a) or (b) holds, then by Lemma 5.8,

$$\boldsymbol{w} \overset{(5\text{-}1)}{\approx} zz^*z \overset{(6\text{-}3)}{\approx} zz^*.$$

Thus suppose (c) holds. By assumption, $\boldsymbol{b} = y_1y_2\cdots y_m$ for some $y_1, y_2, \ldots, y_m \in \mathcal{X} \cup \mathcal{X}^*$ with $m \geq 0$ such that $\mathsf{occ}(y_i, \boldsymbol{w}) \geq 2$ for all i. Then by Lemma 6.7,

$$\boldsymbol{w} \overset{(5\text{-}1\text{c})}{\approx} \boldsymbol{a}x^2\boldsymbol{b}x^*\boldsymbol{e} \overset{(5\text{-}1\text{h})}{\approx} \boldsymbol{a}x^2y_1^2y_2^2\cdots y_m^2x^*\boldsymbol{e} \overset{(6\text{-}1)}{\approx} \boldsymbol{a}y_1^2y_2^2\cdots y_m^2x^2x^*\boldsymbol{e} \overset{(6\text{-}3)}{\approx} zz^*. \quad \square$$

A word $\boldsymbol{w} \in F_{\mathsf{inv}}(\mathcal{X})$ is in B_0-*standard form* if

$$\boldsymbol{w} = \boldsymbol{w}_1x\boldsymbol{w}_2x^*, \tag{6-4}$$

where $x \in \mathcal{X} \cup \mathcal{X}^*$, $\boldsymbol{w}_1 = x_1x_2\cdots x_m$, and $\boldsymbol{w}_2 = \boldsymbol{s}_0\prod_{i=1}^{p}(\boldsymbol{c}_i\boldsymbol{s}_i)$ for some $m, p \geq 0$ such that the following conditions hold:

(B1) $x_1, x_2, \ldots, x_m \in \mathcal{X} \cup \mathcal{X}^*$ are such that $\bar{x}_1 \prec \bar{x}_2 \prec \cdots \prec \bar{x}_m \prec \bar{x}$;

(B2) $\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p \in F_{\mathsf{inv}}(\mathcal{X})$ are simple and $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_p \in F_{\mathsf{inv}}(\mathcal{X})$ are ordered B_0-blocks;

(B3) $x_1, x_2, \ldots, x_m, x, \boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p, \boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_p$ are pairwise disjoint;

(B4) either

(a) $p = 0$ with $\boldsymbol{w}_2 = \boldsymbol{s}_0$ and $\boldsymbol{s}_0 \in \mathcal{X}$; or

(b) $\overline{\mathsf{h}(\boldsymbol{w}_2)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$.

Remark 6.9. The following holds for the word $\boldsymbol{w}$ in (6-4) in B_0-standard form:

(i) If $m = 0$, then $\boldsymbol{w}_1 = 1$.

(ii) If $p = 0$, then $\boldsymbol{w}_2 = \boldsymbol{s}_0 \in F_{\mathsf{inv}}(\mathcal{X})$; in particular, $\boldsymbol{w}_2$ always contains some simple variable and so is nonempty.

(iii) $\{x, x^*\}$ is the only mixed pair of $\boldsymbol{w}$ and $x, x^* \notin \mathsf{con}(\boldsymbol{w}_1\boldsymbol{w}_2)$.

(iv) $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are bipartite words such that $\mathsf{con}(\overline{\boldsymbol{w}}_1) \cap \mathsf{con}(\overline{\boldsymbol{w}}_2) = \varnothing$.

(v) Each variable in $\mathcal{X}$ occurs at most twice in $\overline{\boldsymbol{w}}$.

Lemma 6.10. *Let $\boldsymbol{w} = \boldsymbol{w}_1 x \boldsymbol{w}_2 x^*$ be the word in* (6-4) *in B_0-standard form and $z \in \mathcal{X} \cup \mathcal{X}^*$ be any simple variable in $\boldsymbol{w}$, so that $z \in \mathsf{con}(\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p)$ and $\boldsymbol{w}_2 = \boldsymbol{a}z\boldsymbol{b}$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$. Then there exists a substitution $\gamma^z_{\boldsymbol{w}} : \mathcal{X} \to B_0$ such that*

(i) $\gamma^z_{\boldsymbol{w}}(\boldsymbol{w}_1 x \boldsymbol{a}) = \mathrm{E}$, $\gamma^z_{\boldsymbol{w}}(z) = \mathrm{A}$, *and* $\gamma^z_{\boldsymbol{w}}(\boldsymbol{b}x^*) = \mathrm{F}$, *so that* $\gamma^z_{\boldsymbol{w}}(\boldsymbol{w}) = \mathrm{A}$;

(ii) $\gamma^z_{\boldsymbol{w}}(s) = 0$ *for all $s \in \mathcal{X}$ such that $s \notin \mathsf{con}(\overline{\boldsymbol{w}})$.*

Proof. It follows from Remark 6.9(iii),(iv) that $\mathsf{con}(\boldsymbol{w}_1) = \mathcal{H}_1 \cup \mathcal{K}^*_1$, $\mathsf{con}(\boldsymbol{a}) = \mathcal{H}_2 \cup \mathcal{K}^*_2$, and $\mathsf{con}(\boldsymbol{b}) = \mathcal{H}_3 \cup \mathcal{K}^*_3$ for some $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 \subseteq \mathcal{X}$ such that $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \{x, x^*\}, \{z\}$ are pairwise disjoint sets. By symmetry, it suffices to assume that $x \in \mathcal{X}$, so that $\mathsf{con}(\overline{\boldsymbol{w}}) = \mathcal{H}_1 \cup \mathcal{K}_1 \cup \mathcal{H}_2 \cup \mathcal{K}_2 \cup \mathcal{H}_3 \cup \mathcal{K}_3 \cup \{x, \bar{z}\}$. Define

$$\gamma^z_{\boldsymbol{w}}(s) = \begin{cases} \mathrm{E} & \text{if } s \in \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{K}_3 \cup \{x\}, \\ \mathrm{A} & \text{if } s = \bar{z}, \\ \mathrm{F} & \text{if } s \in \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{H}_3, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is routinely checked that the substitution $\gamma^z_{\boldsymbol{w}}$ satisfies (i) and (ii). □

Corollary 6.11. *For any word $\boldsymbol{w}$ in B_0-standard form and any $z \in \mathcal{X} \cup \mathcal{X}^*$, the identity $\boldsymbol{w} \approx zz^*$ is not satisfied by $(B_0, {}^*)$.*

Proof. Let $\boldsymbol{w} = \boldsymbol{w}_1 x \boldsymbol{w}_2 x^*$ be the word in (6-4) in B_0-standard form. Then by Remark 6.9(ii), the word $\boldsymbol{w}_2$ contains some simple variable $s \in \mathcal{X} \cup \mathcal{X}^*$, so that $\boldsymbol{w}_2 = \boldsymbol{a}s\boldsymbol{b}$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$. Under the substitution $\gamma^s_{\boldsymbol{w}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have $\gamma^s_{\boldsymbol{w}}(\boldsymbol{w}) = \mathrm{A}$ and $\gamma^s_{\boldsymbol{w}}(zz^*) = 0$. □

Lemma 6.12. *Let $\boldsymbol{w}$ be any mixed word. Then the identities* {(1-1), (5-1), (6-1)} *can be used to convert $\boldsymbol{w}$ into exactly one of the following*:

(i) *the word zz^* for any $z \in \mathcal{X} \cup \mathcal{X}^*$*;

(ii) *some word in B_0-standard form.*

Proof. By Lemma 6.7, it suffices to convert $\boldsymbol{w}$ into (i) or (ii), using the identities {(1-1), (5-1), (6-1), (6-3)}. By Lemma 5.12, the identities {(1-1), (5-1)} can first be used to convert $\boldsymbol{w}$ into either zz^*z or some word in A_0-standard form. In the former case, the first identity in (6-3) can be used to convert zz^*z into zz^*. Therefore, it remains to assume that $\boldsymbol{w} = \boldsymbol{w}_1 x \boldsymbol{w}_2 x^*$, where $\boldsymbol{w}_1 = x_1 x_2 \cdots x_m$ and $\boldsymbol{w}_2 = \boldsymbol{s}_0 \prod_{i=1}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i)$, satisfies conditions (A1)–(A4). Then conditions (B1) and (B3) hold because they coincide with conditions (A1) and (A3).

By (A2), $\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p \in F^1_{\mathsf{inv}}(\mathcal{X})$ are simple and $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_p \in F_{\mathsf{inv}}(\mathcal{X})$ are ordered A_0-blocks. By Lemma 6.6, each $\boldsymbol{c}_i$ can be converted by {(5-1), (6-1)} into some ordered B_0-block $y_{i,1}^2 y_{i,2}^2 \cdots y_{i,h_i}^2$. If $\boldsymbol{s}_0 = \boldsymbol{s}_1 = \cdots = \boldsymbol{s}_p = 1$, then by Lemma 6.8, the identities {(1-1), (5-1)} can be used to convert $\boldsymbol{w}$ into zz^*. Therefore, assume that $\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p$ are not all empty. If $\boldsymbol{s}_i = 1$ for some $i \in \{1, 2, \ldots, p-1\}$, so that the ordered B_0-blocks $\boldsymbol{c}_i$ and $\boldsymbol{c}_{i+1}$ are adjacent, then (6-1) can be used to arrange the squares $y_{i,1}^2, y_{i,2}^2, \ldots, y_{i,h_i}^2, y_{i+1,1}^2, y_{i+1,2}^2, \ldots, y_{i+1,h_{i+1}}^2$ in the product $\boldsymbol{c}_i \boldsymbol{c}_{i+1}$ in order, resulting in a single ordered B_0-block. Hence we may assume that for each $i \in \{1, 2, \ldots, p-1\}$, the words $\boldsymbol{c}_i$ and $\boldsymbol{c}_{i+1}$ are separated due to $\boldsymbol{s}_i \neq 1$. If $\boldsymbol{s}_0 = 1$, so that x is adjacent to the ordered B_0-block $\boldsymbol{c}_1$, then the identities {(5-1), (6-1)} can be used to move $\boldsymbol{c}_1$ to the left of x and turn it into a simple word:

$$\begin{aligned}
\boldsymbol{w} &\overset{\text{(5-1c)}}{\approx} x_1 x_2 \cdots x_m \cdot x^2 \cdot \overbrace{y_{1,1}^2 y_{1,2}^2 \cdots y_{1,h_1}^2}^{\boldsymbol{c}_1} \cdot \boldsymbol{s}_1 \Big(\prod_{i=2}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i) \Big) x^* \\
&\overset{\text{(6-1)}}{\approx} x_1 x_2 \cdots x_m \cdot y_{1,1}^2 y_{1,2}^2 \cdots y_{1,h_1}^2 \cdot x^2 \cdot \boldsymbol{s}_1 \Big(\prod_{i=2}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i) \Big) x^* \\
&\overset{\text{(5-1f)}}{\approx} x_1 x_2 \cdots x_m \cdot y_{1,1} y_{1,2} \cdots y_{1,h_1} \cdot x^2 \cdot \boldsymbol{s}_1 \Big(\prod_{i=2}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i) \Big) x^* \\
&\overset{\text{(5-1c)}}{\approx} x_1 x_2 \cdots x_m \cdot y_{1,1} y_{1,2} \cdots y_{1,h_1} \cdot x \cdot \boldsymbol{s}_1 \Big(\prod_{i=2}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i) \Big) x^*;
\end{aligned}$$

by the arguments in the proof of Lemma 5.12, we may assume that

$$\bar{x}_1 \prec \bar{x}_2 \prec \cdots \prec \bar{x}_m \prec \overline{y_{1,1}} \prec \overline{y_{1,2}} \prec \cdots \prec \overline{y_{1,h_1}} \prec \bar{x}.$$

If $\boldsymbol{s}_p = 1$, so that x^* is adjacent to the ordered B_0-block $\boldsymbol{c}_p$, then the identities {(5-1), (6-1)} can be used to move $\boldsymbol{c}_p$ to the left of x and turn it into a simple word:

$$\boldsymbol{w} \overset{\text{(5-1c)}}{\approx} x_1 x_2 \cdots x_m \cdot x \cdot \boldsymbol{s}_0 \Big(\prod_{i=1}^{p-1} (\boldsymbol{c}_i \boldsymbol{s}_i) \Big) \overbrace{y_{p,1}^2 y_{p,2}^2 \cdots y_{p,h_p}^2}^{\boldsymbol{c}_p} \cdot (x^*)^2$$

$$\overset{(6\text{-}1)}{\approx} x_1x_2\cdots x_m\cdot x\cdot \boldsymbol{s}_0\Big(\prod_{i=1}^{p-1}(\boldsymbol{c}_i\boldsymbol{s}_i)\Big)(x^*)^2y_{p,1}^2y_{p,2}^2\cdots y_{p,h_p}^2$$
$$\overset{(5\text{-}1e)}{\approx} x_1x_2\cdots x_m\cdot (y_{p,1}^*)^2(y_{p,2}^*)^2\cdots(y_{p,h_p}^*)^2\cdot x\cdot \boldsymbol{s}_0\Big(\prod_{i=1}^{p-1}(\boldsymbol{c}_i\boldsymbol{s}_i)\Big)(x^*)^2$$
$$\overset{(5\text{-}1f)}{\approx} x_1x_2\cdots x_m\cdot y_{p,1}^*y_{p,2}^*\cdots y_{p,h_p}^*\cdot x\cdot \boldsymbol{s}_0\Big(\prod_{i=1}^{p-1}(\boldsymbol{c}_i\boldsymbol{s}_i)\Big)(x^*)^2$$
$$\overset{(5\text{-}1c)}{\approx} x_1x_2\cdots x_m\cdot y_{p,1}^*y_{p,2}^*\cdots y_{p,h_p}^*\cdot x\cdot \boldsymbol{s}_0\Big(\prod_{i=1}^{p-1}(\boldsymbol{c}_i\boldsymbol{s}_i)\Big)x^*;$$

by repeating the arguments in the proof of Lemma 5.12, we may assume that

$$\bar{x}_1 \prec \bar{x}_2 \prec \cdots \prec \bar{x}_m \prec \overline{y_{p,1}} \prec \overline{y_{p,2}} \prec \cdots \prec \overline{y_{p,h_p}} \prec \bar{x}.$$

Therefore, we may assume that $\boldsymbol{s}_0, \boldsymbol{s}_p \neq 1$. It follows that $\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p \in F_{\mathsf{inv}}(\mathcal{X})$, so that condition (B2) is satisfied.

It remains to address condition (B4). If $\boldsymbol{w}_2$ is a single variable, so that $p = 0$ with $\boldsymbol{w}_2 = \boldsymbol{s}_0 \in \mathcal{X} \cup \mathcal{X}^*$, then the identity (5-1b) can be used to convert $\boldsymbol{s}_0$ into a variable in $\mathcal{X}$, whence condition (B4)(a) is satisfied. Hence, assume that $\boldsymbol{w}_2$ is not a single variable, so that $\overline{\mathsf{h}(\boldsymbol{w}_2)} \neq \overline{\mathsf{t}(\boldsymbol{w}_2)}$ by conditions (B2) and (B3). In this case, since each $\boldsymbol{s}_i$ is a nonempty simple word, we have $\boldsymbol{s}_i = s_{i,1}s_{i,2}\cdots s_{i,k_i}$ for some $s_{i,1}, s_{i,2}, \ldots, s_{i,k_i} \in \mathcal{X}\cup\mathcal{X}^*$ such that $\overline{s_{i,1}}, \overline{s_{i,2}}, \ldots, \overline{s_{i,k_i}}$ are distinct. If $\overline{\mathsf{h}(\boldsymbol{w}_2)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, then condition (B4)(b) is satisfied. If $\overline{\mathsf{h}(\boldsymbol{w}_2)} \not\prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, so that $\overline{s_{p,k_p}} \prec \overline{s_{0,1}}$, then

$$\boldsymbol{w} \overset{(5\text{-}1b)}{\approx} \boldsymbol{w}_1\cdot x\cdot\left(\boldsymbol{s}_0\prod_{i=1}^{p}(\boldsymbol{c}_i\boldsymbol{s}_i)\right)^* x^* \overset{(1\text{-}1)}{\approx} \boldsymbol{w}_1\cdot x\cdot\left(\prod_{i=p}^{1}(\boldsymbol{s}_i^*\boldsymbol{c}_i^*)\right)\boldsymbol{s}_0^*\cdot x^*,$$

where the identities {(1-1), (6-1)} can be used to convert $\boldsymbol{s}_i^*$ and $\boldsymbol{c}_i^*$ into the simple word $s_{i,k_i}^*s_{i,k_i-1}^*\cdots s_{i,1}^*$ and the ordered B_0-block $(y_{i,1}^*)^2(y_{i,2}^*)^2\cdots(y_{i,h_i}^*)^2$, respectively; thus, condition (B4)(b) is satisfied.

Thus the identities {(1-1), (5-1), (6-1)} can be used to convert $\boldsymbol{w}$ into either zz^* or some word $\widetilde{\boldsymbol{w}}$ in B_0-standard form. But if the identities {(1-1), (5-1), (6-1)} can be used to convert $\boldsymbol{w}$ into both zz^* and $\widetilde{\boldsymbol{w}}$, then that would imply that $(B_0,{}^*)$ satisfies the identity $\widetilde{\boldsymbol{w}} \approx zz^*$, which is impossible by Corollary 6.11. □

6.3. ***Proof of Proposition 6.1.*** Consider any identity

$$\boldsymbol{u} \approx \boldsymbol{v}$$

satisfied by $(B_0,{}^*)$. If we show that $\boldsymbol{u} \approx \boldsymbol{v}$ is deducible from {(1-1), (5-1), (6-1)}, the proposition will follow. By Lemma 6.5, this result holds if either $\boldsymbol{u}$ or $\boldsymbol{v}$ is bipartite. Therefore, suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are both mixed. By Corollary 6.11 and Lemma 6.12, the identities {(1-1), (5-1), (6-1)} can be used to convert $\boldsymbol{u}$ and $\boldsymbol{v}$ simultaneously to either zz^* or words in B_0-standard form. In the former case, $\boldsymbol{u} \approx \boldsymbol{v}$ is deducible from {(1-1), (5-1), (6-1)}, whence the proof is complete. Therefore, it remains to consider the latter case, whence we may assume that $\boldsymbol{u}$ and $\boldsymbol{v}$ are in B_0-standard form, say

$$\boldsymbol{u} = \boldsymbol{u}_1 x \boldsymbol{u}_2 x^* \quad \text{and} \quad \boldsymbol{v} = \boldsymbol{v}_1 y \boldsymbol{v}_2 y^*,$$

where $x, y \in \mathcal{X} \cup \mathcal{X}^*$, $\boldsymbol{u}_1 = x_1 x_2 \cdots x_m$, $\boldsymbol{u}_2 = \boldsymbol{s}_0 \prod_{i=1}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i)$, $\boldsymbol{v}_1 = y_1 y_2 \cdots y_n$, and $\boldsymbol{v}_2 = \boldsymbol{t}_0 \prod_{i=1}^{q} (\boldsymbol{d}_i \boldsymbol{t}_i)$ satisfy conditions (B1)–(B4).

Lemma 6.13. *The following holds for the words $\boldsymbol{u}$ and $\boldsymbol{v}$*:

(i) $\mathsf{con}(\bar{\boldsymbol{u}}) = \mathsf{con}(\bar{\boldsymbol{v}})$;

(ii) $\boldsymbol{u}_1 x = \boldsymbol{v}_1 y$;

(iii) $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$.

Proof. (i) Suppose that $\mathsf{con}(\bar{\boldsymbol{u}}) \neq \mathsf{con}(\bar{\boldsymbol{v}})$, say there exists a variable $t \in \mathsf{con}(\bar{\boldsymbol{v}})$ such that $t \notin \mathsf{con}(\bar{\boldsymbol{u}})$. Then by Remark 6.9(ii), the word $\boldsymbol{u}_2$ contains some simple variable $z \in \mathcal{X} \cup \mathcal{X}^*$, so that $\boldsymbol{u}_2 = \boldsymbol{a} z \boldsymbol{b}$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$. Under the substitution $\gamma^z_{\boldsymbol{u}} : \mathcal{X} \to B_0$ in Lemma 6.10, the contradiction $\gamma^z_{\boldsymbol{u}}(\boldsymbol{u}) = \mathrm{A} \neq 0 = \gamma^z_{\boldsymbol{u}}(\boldsymbol{v})$ is deduced.

(ii) Due to condition (B1), the equality $\boldsymbol{u}_1 x = \boldsymbol{v}_1 y$ follows from $\mathsf{con}(\boldsymbol{u}_1 x) = \mathsf{con}(\boldsymbol{v}_1 y)$; to establish the latter, by symmetry, it suffices to verify the inclusion $\mathsf{con}(\boldsymbol{u}_1 x) \subseteq \mathsf{con}(\boldsymbol{v}_1 y)$. To this end, we need to first show that $y \in \mathsf{con}(\boldsymbol{u}_1 x)$. Since $\bar{y} \in \mathsf{con}(\bar{\boldsymbol{v}}) = \mathsf{con}(\bar{\boldsymbol{u}})$ by part (i),

(a) either $y \in \mathsf{con}(\boldsymbol{u})$ or $y^* \in \mathsf{con}(\boldsymbol{u})$.

Now since $\boldsymbol{s}_p \neq 1$, the variable $z = \mathsf{t}(\boldsymbol{s}_p) = \mathsf{t}(\boldsymbol{u}_2)$ is simple in $\boldsymbol{u}$, so that $\boldsymbol{u}_2 = \boldsymbol{a} z$ for some $\boldsymbol{a} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\bar{z} \notin \mathsf{con}(\bar{\boldsymbol{a}})$. Then under the substitution $\gamma^z_{\boldsymbol{u}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^z_{\boldsymbol{u}}(\boldsymbol{u}) = \gamma^z_{\boldsymbol{u}}(\boldsymbol{u}_1 x \boldsymbol{a}) \cdot \gamma^z_{\boldsymbol{u}}(z) \cdot \gamma^z_{\boldsymbol{u}}(x^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{E}^* = \mathrm{A}.$$

If $y^* \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$, then

$$\begin{aligned} \gamma^z_{\boldsymbol{u}}(\boldsymbol{v}) &= \gamma^z_{\boldsymbol{u}}(\boldsymbol{v}_1) \cdot \gamma^z_{\boldsymbol{u}}(y) \cdot \gamma^z_{\boldsymbol{u}}(\boldsymbol{v}_2) \cdot \gamma^z_{\boldsymbol{u}}(y^*) \\ &= \begin{cases} \gamma^z_{\boldsymbol{u}}(\boldsymbol{v}_1) \cdot \mathrm{A} \cdot \gamma^z_{\boldsymbol{u}}(\boldsymbol{v}_2) \cdot \mathrm{A} & \text{if } y^* = z \\ \gamma^z_{\boldsymbol{u}}(\boldsymbol{v}_1) \cdot \mathrm{F} \cdot \gamma^z_{\boldsymbol{u}}(\boldsymbol{v}_2) \cdot \mathrm{E} & \text{if } y^* \neq z \end{cases} \\ &= 0, \end{aligned}$$

which is impossible. Therefore,

(b) $y^* \notin \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$, which implies that $y \neq x^*$.

If $y \neq x$, then together with (b), we have $y^* \notin \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2 x^*) = \mathsf{con}(\boldsymbol{u})$, so that $y \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$ by (a) and (b). On the other hand, if $y = x$, then clearly $y \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$. Therefore, $y \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$ either way.

Seeking a contradiction, suppose that $y \in \mathsf{con}(\boldsymbol{u}_2)$. Then $\mathsf{occ}(y, \boldsymbol{u}_2) \in \{1, 2\}$ by condition (B2). If $\mathsf{occ}(y, \boldsymbol{u}_2) = 1$, then under the substitution $\gamma_{\boldsymbol{u}}^{y} : \mathcal{X} \to B_0$ in Lemma 6.10, we have $\gamma_{\boldsymbol{u}}^{y}(\boldsymbol{u}) = \mathrm{A}$ and

$$\gamma_{\boldsymbol{u}}^{y}(\boldsymbol{v}) = \gamma_{\boldsymbol{u}}^{y}(\boldsymbol{v}_1) \cdot \gamma_{\boldsymbol{u}}^{y}(y) \cdot \gamma_{\boldsymbol{u}}^{y}(\boldsymbol{v}_2) \cdot \gamma_{\boldsymbol{u}}^{y}(y^*) = \gamma_{\boldsymbol{u}}^{y}(\boldsymbol{v}_1) \cdot \mathrm{A} \cdot \gamma_{\boldsymbol{u}}^{y}(\boldsymbol{v}_2) \cdot \mathrm{A} = 0,$$

which is impossible. If $\mathsf{occ}(y, \boldsymbol{u}_2) = 2$, so that $\boldsymbol{u}_2 = h\boldsymbol{b}$ for some $\boldsymbol{b} \in F_{\mathsf{inv}}(\mathcal{X})$ with $y \in \mathsf{con}(\boldsymbol{b})$ and $h = \mathsf{h}(\boldsymbol{s}_0)$ being simple in $\boldsymbol{u}_2$, then under the substitution $\gamma_{\boldsymbol{u}}^{h} : \mathcal{X} \to B_0$ in Lemma 6.10, we have $\gamma_{\boldsymbol{u}}^{h}(\boldsymbol{u}) = \mathrm{A}$ and

$$\gamma_{\boldsymbol{u}}^{h}(\boldsymbol{v}) = \gamma_{\boldsymbol{u}}^{h}(\boldsymbol{v}_1) \cdot \gamma_{\boldsymbol{u}}^{h}(y) \cdot \gamma_{\boldsymbol{u}}^{h}(\boldsymbol{v}_2) \cdot \gamma_{\boldsymbol{u}}^{h}(y^*) = \gamma_{\boldsymbol{u}}^{h}(\boldsymbol{v}_1) \cdot \mathrm{F} \cdot \gamma_{\boldsymbol{u}}^{h}(\boldsymbol{v}_2) \cdot \mathrm{E} = 0,$$

which again is impossible. Thus, $y \notin \mathsf{con}(\boldsymbol{u}_2)$; but since $y \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$, together with (b), we have

(c) $y \in \mathsf{con}(\boldsymbol{u}_1 x)$ and $y, y^* \notin \mathsf{con}(\boldsymbol{u}_2)$.

By a symmetrical argument,

(d) $x \in \mathsf{con}(\boldsymbol{v}_1 y)$ and $x, x^* \notin \mathsf{con}(\boldsymbol{v}_2)$.

Now we are ready to establish the inclusion $\mathsf{con}(\boldsymbol{u}_1 x) \subseteq \mathsf{con}(\boldsymbol{v}_1 y)$. Suppose there exists some variable $z \in \mathsf{con}(\boldsymbol{u}_1 x)$ such that $z \notin \mathsf{con}(\boldsymbol{v}_1 y)$. Then clearly $z \neq y$. But if $z = y^*$, then it follows from (c) that $y, y^* \in \mathsf{con}(\boldsymbol{u}_1 x)$, whence condition (B1) is contradicted. Hence

(e) $z \notin \{y, y^*\}$.

Since $\bar{z} \in \mathsf{con}(\bar{\boldsymbol{u}}) = \mathsf{con}(\bar{\boldsymbol{v}})$ by part (i), it follows from (e) that either $z \in \mathsf{con}(\boldsymbol{v}_1 \boldsymbol{v}_2)$ or $z^* \in \mathsf{con}(\boldsymbol{v}_1 \boldsymbol{v}_2)$. But since $z \notin \mathsf{con}(\boldsymbol{v}_1)$ by assumption, we have $z \in \mathsf{con}(\boldsymbol{v}_2)$ or $z^* \in \mathsf{con}(\boldsymbol{v}_1)$ or $z^* \in \mathsf{con}(\boldsymbol{v}_2)$. These three cases are shown in the following to be impossible. Therefore, the variable z does not exist, whence the required inclusion $\mathsf{con}(\boldsymbol{u}_1 x) \subseteq \mathsf{con}(\boldsymbol{v}_1 y)$ is established.

<u>Case 1</u>: $z \in \mathsf{con}(\boldsymbol{v}_2)$. Then $z \notin \{x, x^*\}$ by (d). But since $z \in \mathsf{con}(\boldsymbol{u}_1 x)$ by assumption, we have

(f) $z \in \mathsf{con}(\boldsymbol{u}_1)$.

By condition (B2), we have $\mathsf{occ}(z, \boldsymbol{v}_2) \in \{1, 2\}$, so there are two subcases.

SUBCASE 1.1: $\mathsf{occ}(z, \boldsymbol{v}_2) = 1$. Then $\boldsymbol{v}_2 = \boldsymbol{a} z \boldsymbol{b}$ for some $\boldsymbol{a}, \boldsymbol{b} \in F_{\mathsf{inv}}^1(\mathcal{X})$ such that $\bar{z} \notin \mathsf{con}(\overline{\boldsymbol{ab}})$. Hence under the substitution $\gamma_{\boldsymbol{v}}^{z} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma_{\boldsymbol{v}}^{z}(\boldsymbol{v}) = \gamma_{\boldsymbol{v}}^{z}(\boldsymbol{v}_1 y \boldsymbol{a}) \cdot \gamma_{\boldsymbol{v}}^{z}(z) \cdot \gamma_{\boldsymbol{v}}^{z}(\boldsymbol{b} y^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

It follows that $\gamma_{\boldsymbol{v}}^{z}(x) = \mathrm{E}$ because $x \in \mathsf{con}(\boldsymbol{v}_1 y)$ by (d), and $\gamma_{\boldsymbol{v}}^{z}(\boldsymbol{u}_1) = \cdots \mathrm{A} \cdots$ because $z \in \mathsf{con}(\boldsymbol{u}_1)$ by (f). Therefore,

$$\gamma_{\boldsymbol{v}}^{z}(\boldsymbol{u}) = \gamma_{\boldsymbol{v}}^{z}(\boldsymbol{u}_1) \cdot \gamma_{\boldsymbol{v}}^{z}(x) \cdot \gamma_{\boldsymbol{v}}^{z}(\boldsymbol{u}_2 x^*) = \cdots \mathrm{A} \cdots \mathrm{E} \cdot \gamma_{\boldsymbol{v}}^{z}(\boldsymbol{u}_2 x^*) = 0,$$

which implies a contradiction.

SUBCASE 1.2: $\mathsf{occ}(z, \boldsymbol{v}_2) = 2$. Since $\boldsymbol{t}_0 \neq 1$, the variable $h = \mathsf{h}(\boldsymbol{t}_0) = \mathsf{h}(\boldsymbol{v}_2)$ is simple in $\boldsymbol{v}$, so that $\boldsymbol{v}_2 = h\boldsymbol{a}z^2\boldsymbol{b}$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\boldsymbol{a}, \boldsymbol{b}, h, z$ are pairwise disjoint. Then under the substitution $\gamma^h_{\boldsymbol{v}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^h_{\boldsymbol{v}}(\boldsymbol{v}) = \gamma^h_{\boldsymbol{v}}(\boldsymbol{v}_1 y) \cdot \gamma^h_{\boldsymbol{v}}(h) \cdot \gamma^h_{\boldsymbol{v}}(\boldsymbol{a}z^2\boldsymbol{b}y^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

It follows that $\gamma^h_{\boldsymbol{v}}(x) = \mathrm{E}$ because $x \in \mathsf{con}(\boldsymbol{v}_1 y)$ by (d), and $\gamma^h_{\boldsymbol{v}}(z) = \mathrm{F}$. Further, since $z \in \mathsf{con}(\boldsymbol{u}_1)$ by (f), we have $\gamma^h_{\boldsymbol{v}}(\boldsymbol{u}_1) = \cdots \mathrm{F} \cdots$. Therefore,

$$\gamma^h_{\boldsymbol{v}}(\boldsymbol{u}) = \gamma^h_{\boldsymbol{v}}(\boldsymbol{u}_1) \cdot \gamma^h_{\boldsymbol{v}}(x) \cdot \gamma^h_{\boldsymbol{v}}(\boldsymbol{u}_2 x^*) = \cdots \mathrm{F} \cdots \mathrm{E} \cdot \gamma^h_{\boldsymbol{v}}(\boldsymbol{u}_2 x^*) = 0,$$

which implies a contradiction.

Case 2: $z^* \in \mathsf{con}(\boldsymbol{v}_1)$. Since $\boldsymbol{s}_0 \neq 1$, the variable $h = \mathsf{h}(\boldsymbol{s}_0) = \mathsf{h}(\boldsymbol{u}_2)$ is simple in $\boldsymbol{u}$, so that $\boldsymbol{u}_2 = h\boldsymbol{b}$ for some $\boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\bar{h} \notin \mathsf{con}(\bar{\boldsymbol{b}})$. Then under the substitution $\gamma^h_{\boldsymbol{u}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^h_{\boldsymbol{u}}(\boldsymbol{u}) = \gamma^h_{\boldsymbol{u}}(\boldsymbol{u}_1 x) \cdot \gamma^h_{\boldsymbol{u}}(h) \cdot \gamma^h_{\boldsymbol{u}}(\boldsymbol{b}x^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

It follows that $\gamma^h_{\boldsymbol{u}}(y) = \mathrm{E}$ because $y \in \mathsf{con}(\boldsymbol{u}_1 x)$ by (c), and that $\gamma^h_{\boldsymbol{u}}(z) = \mathrm{E}$ because $z \in \mathsf{con}(\boldsymbol{u}_1 x)$ by assumption. Further, since $z^* \in \mathsf{con}(\boldsymbol{v}_1)$, we have $\gamma^h_{\boldsymbol{u}}(\boldsymbol{v}_1) = \cdots \mathrm{F} \cdots$. Therefore,

$$\gamma^h_{\boldsymbol{u}}(\boldsymbol{v}) = \gamma^h_{\boldsymbol{u}}(\boldsymbol{v}_1) \cdot \gamma^h_{\boldsymbol{u}}(y) \cdot \gamma^h_{\boldsymbol{u}}(\boldsymbol{v}_2 y^*) = \cdots \mathrm{F} \cdots \mathrm{E} \cdot \gamma^h_{\boldsymbol{u}}(\boldsymbol{v}_2 y^*) = 0,$$

which implies a contradiction.

Case 3: $z^* \in \mathsf{con}(\boldsymbol{v}_2)$. Then $z \notin \{x, x^*\}$ by (d). But since $z \in \mathsf{con}(\boldsymbol{u}_1 x)$ by assumption, we have

(g) $z \in \mathsf{con}(\boldsymbol{u}_1)$.

By condition (B2), we have $\mathsf{occ}(z^*, \boldsymbol{v}_2) \in \{1, 2\}$, so there are two subcases.

SUBCASE 3.1: $\mathsf{occ}(z^*, \boldsymbol{v}_2) = 1$. Then $\boldsymbol{v}_2 = \boldsymbol{a}z^*\boldsymbol{b}$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\bar{z} \notin \mathsf{con}(\overline{\boldsymbol{a}\boldsymbol{b}})$. Hence under the substitution $\gamma^{z^*}_{\boldsymbol{v}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{v}) = \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{v}_1 y\boldsymbol{a}) \cdot \gamma^{z^*}_{\boldsymbol{v}}(z^*) \cdot \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{b}y^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

It follows that $\gamma^{z^*}_{\boldsymbol{v}}(x) = \mathrm{E}$ because $x \in \mathsf{con}(\boldsymbol{v}_1 y)$ by (d), and $\gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{u}_1) = \cdots \mathrm{A} \cdots$ because $z \in \mathsf{con}(\boldsymbol{u}_1)$ by (g). Therefore,

$$\gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{u}) = \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{u}_1) \cdot \gamma^{z^*}_{\boldsymbol{v}}(x) \cdot \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{u}_2 x^*) = \cdots \mathrm{A} \cdots \mathrm{E} \cdot \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{u}_2 x^*) = 0,$$

which implies a contradiction.

SUBCASE 3.2: $\mathsf{occ}(z^*, \boldsymbol{v}_2) = 2$. Since $\boldsymbol{t}_q \neq 1$, the variable $t = \mathsf{t}(\boldsymbol{t}_q) = \mathsf{t}(\boldsymbol{v}_2)$ is simple in $\boldsymbol{v}$, so that $\boldsymbol{v}_2 = \boldsymbol{a}(z^*)^2\boldsymbol{b}t$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\boldsymbol{a}, \boldsymbol{b}, z, t$ are pairwise disjoint. Then under the substitution $\gamma_{\boldsymbol{v}}^t : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma_{\boldsymbol{v}}^t(\boldsymbol{v}) = \gamma_{\boldsymbol{v}}^t\big(\boldsymbol{v}_1 y \boldsymbol{a}(z^*)^2\boldsymbol{b}\big) \cdot \gamma_{\boldsymbol{v}}^t(t) \cdot \gamma_{\boldsymbol{v}}^t(y^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

It follows that $\gamma_{\boldsymbol{v}}^t(x) = \mathrm{E}$ because $x \in \mathsf{con}(\boldsymbol{v}_1 y)$ by (d), and $\gamma_{\boldsymbol{v}}^t(z) = \mathrm{F}$. Further, since $z \in \mathsf{con}(\boldsymbol{u}_1)$ by (g), we have $\gamma_{\boldsymbol{v}}^t(\boldsymbol{u}_1) = \cdots \mathrm{F} \cdots$. Therefore,

$$\gamma_{\boldsymbol{v}}^t(\boldsymbol{u}) = \gamma_{\boldsymbol{v}}^t(\boldsymbol{u}_1) \cdot \gamma_{\boldsymbol{v}}^t(x) \cdot \gamma_{\boldsymbol{v}}^t(\boldsymbol{u}_2 x^*) = \cdots \mathrm{F} \cdots \mathrm{E} \cdot \gamma_{\boldsymbol{v}}^h(\boldsymbol{u}_2 x^*) = 0,$$

which implies a contradiction.

(iii) This is a consequence of parts (i) and (ii). □

Therefore, by Lemma 6.13, we now have

$$\boldsymbol{u} = \underbrace{x_1 x_2 \cdots x_m}_{\boldsymbol{u}_1} \cdot x \cdot \underbrace{\boldsymbol{s}_0 \prod_{i=1}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i)}_{\boldsymbol{u}_2} \cdot x^* \quad \text{and} \quad \boldsymbol{v} = \underbrace{x_1 x_2 \cdots x_m}_{\boldsymbol{u}_1} \cdot x \cdot \underbrace{\boldsymbol{t}_0 \prod_{i=1}^{q} (\boldsymbol{d}_i \boldsymbol{t}_i)}_{\boldsymbol{v}_2} \cdot x^*,$$

where conditions (B1)–(B4) are satisfied. In the remainder of this section, it is shown that $\boldsymbol{u}_2 = \boldsymbol{v}_2$ (Lemma 6.19), so that $\boldsymbol{u} = \boldsymbol{v}$. The identity $\boldsymbol{u} \approx \boldsymbol{v}$ is thus vacuously deducible from {(1-1), (5-1), (6-1)}, whence the proof of Proposition 6.1 is complete.

Lemma 6.14. (i) $\mathsf{con}(\overline{\boldsymbol{s}_0 \boldsymbol{s}_1 \cdots \boldsymbol{s}_p}) = \mathsf{con}(\overline{\boldsymbol{t}_0 \boldsymbol{t}_1 \cdots \boldsymbol{t}_q})$.

(ii) $\mathsf{con}(\overline{\boldsymbol{c}_1 \boldsymbol{c}_2 \cdots \boldsymbol{c}_p}) = \mathsf{con}(\overline{\boldsymbol{d}_1 \boldsymbol{d}_2 \cdots \boldsymbol{d}_q})$.

Proof. (i) Let $\boldsymbol{s} = \boldsymbol{s}_0 \boldsymbol{s}_1 \cdots \boldsymbol{s}_p$ and $\boldsymbol{t} = \boldsymbol{t}_0 \boldsymbol{t}_1 \cdots \boldsymbol{t}_q$. Suppose that $\mathsf{con}(\bar{\boldsymbol{s}}) \nsubseteq \mathsf{con}(\bar{\boldsymbol{t}})$. Then there exists some $z \in \mathsf{con}(\boldsymbol{s})$ such that $\bar{z} \in \mathsf{con}(\bar{\boldsymbol{s}})$ and $\bar{z} \notin \mathsf{con}(\bar{\boldsymbol{t}})$. Therefore, under the substitution $\gamma_{\boldsymbol{u}}^z : \mathcal{X} \to B_0$ in Lemma 6.10, we have $\gamma_{\boldsymbol{u}}^z(\boldsymbol{u}) = \mathrm{A}$. On the other hand, since $\bar{z} \in \mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$ by Lemma 6.13(iii) but $\bar{z} \notin \mathsf{con}(\bar{\boldsymbol{t}})$ by assumption, we have $\bar{z} \in \mathsf{con}(\bar{\boldsymbol{d}}_i)$ for some $i \in \{1, 2, \ldots, q\}$. Since $\boldsymbol{d}_i$ is an ordered B_0-block, we have $\gamma_{\boldsymbol{u}}^z(\boldsymbol{d}_i) = \cdots \mathrm{A}^2 \cdots = 0$, whence the contradiction $\gamma_{\boldsymbol{u}}^z(\boldsymbol{v}) = 0$ is deduced. Hence the variable z does not exist, so that the inclusion $\mathsf{con}(\bar{\boldsymbol{s}}) \subseteq \mathsf{con}(\bar{\boldsymbol{t}})$ holds. The reverse inclusion $\mathsf{con}(\bar{\boldsymbol{s}}) \supseteq \mathsf{con}(\bar{\boldsymbol{t}})$ holds by a symmetrical argument.

(ii) This follows from part (i) since $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$ by Lemma 6.13(iii). □

Lemma 6.15. (i) *Suppose that* $yz \hookrightarrow \boldsymbol{u}_2$ *for some* $y, z \in \mathsf{con}(\boldsymbol{s}_0 \boldsymbol{s}_1 \cdots \boldsymbol{s}_p)$. *Then the longest* $\{\bar{y}, \bar{z}\}$*-subsequence of* $\boldsymbol{v}_2$ *can only be* yz *or* $z^* y^*$.

(ii) *Suppose that* $yz \hookrightarrow \boldsymbol{v}_2$ *for some* $y, z \in \mathsf{con}(\boldsymbol{t}_0 \boldsymbol{t}_1 \cdots \boldsymbol{t}_q)$. *Then the longest* $\{\bar{y}, \bar{z}\}$*-subsequence of* $\boldsymbol{u}_2$ *can only be* yz *or* $z^* y^*$.

Proof. (i) By assumption,

$$\boldsymbol{u} = \boldsymbol{u}_1 \cdot x \cdot \underbrace{\boldsymbol{a} y \boldsymbol{b} z \boldsymbol{e}}_{\boldsymbol{u}_2} \cdot x^*$$

for some $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\boldsymbol{u}_1, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}, x, y, z$ are pairwise disjoint. Since $\bar{y}, \bar{z} \in \mathsf{con}(\overline{\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p}) = \mathsf{con}(\overline{\boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q})$ by Lemma 6.14(i), we have $\mathsf{occ}(\bar{y}, \bar{\boldsymbol{v}}) = \mathsf{occ}(\bar{z}, \bar{\boldsymbol{v}}) = 1$ by conditions (B2) and (B3). Therefore, the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ can only be one of

$$yz, \quad yz^*, \quad y^*z, \quad y^*z^*, \quad zy, \quad zy^*, \quad z^*y, \quad z^*y^*.$$

There are four cases to consider.

Case 1: $yz^* \hookrightarrow \boldsymbol{v}_2$. Then

$$\boldsymbol{v} = \boldsymbol{u}_1 \cdot x \cdot \underbrace{\boldsymbol{f} y \boldsymbol{g} z^* \boldsymbol{h}}_{\boldsymbol{v}_2} \cdot x^*$$

for some $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\boldsymbol{u}_1, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, x, y, z$ are pairwise disjoint. Under the substitution $\gamma^y_{\boldsymbol{v}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^y_{\boldsymbol{v}}(\boldsymbol{v}) = \gamma^y_{\boldsymbol{v}}(\boldsymbol{u}_1 x \boldsymbol{f}) \cdot \gamma^y_{\boldsymbol{v}}(y) \cdot \gamma^y_{\boldsymbol{v}}(\boldsymbol{g} z^* \boldsymbol{h} x^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

But since $\gamma^y_{\boldsymbol{v}}(y) = \mathrm{A}$ and $\gamma^y_{\boldsymbol{v}}(z) = \mathrm{E}$, we deduce the contradiction

$$\gamma^y_{\boldsymbol{v}}(\boldsymbol{u}) = \gamma^y_{\boldsymbol{v}}(\boldsymbol{u}_1 x \boldsymbol{a}) \cdot \mathrm{A} \cdot \gamma^y_{\boldsymbol{v}}(\boldsymbol{b}) \cdot \mathrm{E} \cdot \gamma^y_{\boldsymbol{v}}(\boldsymbol{e} x^*) = 0.$$

Case 2: $zy^* \hookrightarrow \boldsymbol{v}_2$. Then

$$\boldsymbol{v} = \boldsymbol{u}_1 \cdot x \cdot \underbrace{\boldsymbol{f} z \boldsymbol{g} y^* \boldsymbol{h}}_{\boldsymbol{v}_2} \cdot x^*$$

for some $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\boldsymbol{u}_1, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, x, y, z$ are pairwise disjoint. Under the substitution $\gamma^{y^*}_{\boldsymbol{v}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^{y^*}_{\boldsymbol{v}}(\boldsymbol{v}) = \gamma^{y^*}_{\boldsymbol{v}}(\boldsymbol{u}_1 x \boldsymbol{f} z \boldsymbol{g}) \cdot \gamma^{y^*}_{\boldsymbol{v}}(y^*) \cdot \gamma^{y^*}_{\boldsymbol{v}}(\boldsymbol{h} x^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

But since $\gamma^{y^*}_{\boldsymbol{v}}(y) = \mathrm{A}$ and $\gamma^{y^*}_{\boldsymbol{v}}(z) = \mathrm{E}$, we deduce the contradiction

$$\gamma^{y^*}_{\boldsymbol{v}}(\boldsymbol{u}) = \gamma^{y^*}_{\boldsymbol{v}}(\boldsymbol{u}_1 x \boldsymbol{a}) \cdot \mathrm{A} \cdot \gamma^{y^*}_{\boldsymbol{v}}(\boldsymbol{b}) \cdot \mathrm{E} \cdot \gamma^{y^*}_{\boldsymbol{v}}(\boldsymbol{e} x^*) = 0.$$

Case 3: $y^*z \hookrightarrow \boldsymbol{v}_2$ or $zy \hookrightarrow \boldsymbol{v}_2$. Under the substitution $\gamma^z_{\boldsymbol{u}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^z_{\boldsymbol{u}}(\boldsymbol{u}) = \gamma^z_{\boldsymbol{u}}(\boldsymbol{u}_1 x \boldsymbol{a} y \boldsymbol{b}) \cdot \gamma^z_{\boldsymbol{u}}(z) \cdot \gamma^z_{\boldsymbol{u}}(\boldsymbol{e} x^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

But since $\gamma^z_{\boldsymbol{u}}(y) = \mathrm{E}$ and $\gamma^z_{\boldsymbol{u}}(z) = \mathrm{A}$, we deduce the contradiction

$$\gamma^z_{\boldsymbol{u}}(\boldsymbol{v}) = \begin{cases} \cdots \mathrm{E}^* \cdots \mathrm{A} \cdots & \text{if } y^*z \hookrightarrow \boldsymbol{v}_2 \\ \cdots \mathrm{A} \cdots \mathrm{E} \cdots & \text{if } zy \hookrightarrow \boldsymbol{v}_2 \end{cases}$$
$$= 0.$$

Case 4: $z^*y \hookrightarrow \boldsymbol{v}_2$ or $y^*z^* \hookrightarrow \boldsymbol{v}_2$. Then

$$\boldsymbol{v} = \boldsymbol{u}_1 \cdot x \cdot \underbrace{\boldsymbol{f} z^* \boldsymbol{g} y \boldsymbol{h}}_{\boldsymbol{v}_2} \cdot x^* \quad \text{or} \quad \boldsymbol{v} = \boldsymbol{u}_1 \cdot x \cdot \underbrace{\boldsymbol{f} y^* \boldsymbol{g} z^* \boldsymbol{h}}_{\boldsymbol{v}_2} \cdot x^*$$

for some $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\boldsymbol{u}_1, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, x, y, z$ are pairwise disjoint. Under the substitution $\gamma^{z^*}_{\boldsymbol{v}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{v}) = \begin{cases} \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{u}_1 x \boldsymbol{f}) \cdot \gamma^{z^*}_{\boldsymbol{v}}(z^*) \cdot \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{g} y \boldsymbol{h} x^*) & \text{if } z^*y \hookrightarrow \boldsymbol{v}_2 \\ \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{u}_1 x \boldsymbol{f} y^* \boldsymbol{g}) \cdot \gamma^{z^*}_{\boldsymbol{v}}(z^*) \cdot \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{h} x^*) & \text{if } y^*z^* \hookrightarrow \boldsymbol{v}_2 \end{cases}$$
$$= \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

But since $\gamma^{z^*}_{\boldsymbol{v}}(y) = \mathrm{F}$ and $\gamma^{z^*}_{\boldsymbol{v}}(z) = \mathrm{A}$, we deduce the contradiction

$$\gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{u}) = \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{u}_1 x \boldsymbol{a}) \cdot \mathrm{F} \cdot \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{b}) \cdot \mathrm{A} \cdot \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{e} x^*) = 0.$$

Since none of the four cases is possible, the longest $\{\bar{y}, \bar{z}\}$-subsequences of $\boldsymbol{v}_2$ can only be yz or z^*y^*.

(ii) This is symmetrical to part (i). □

Lemma 6.16. (i) *Suppose that* $y^2z \hookrightarrow \boldsymbol{u}_2$ *for some* $y \in \mathsf{con}(\boldsymbol{c}_1\boldsymbol{c}_2 \cdots \boldsymbol{c}_p)$ *and* $z \in \mathsf{con}(\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p)$. *Then the longest* $\{\bar{y}, \bar{z}\}$*-subsequence of* $\boldsymbol{v}_2$ *can only be* y^2z *or* $z^*(y^*)^2$.

(ii) *Suppose that* $y^2z \hookrightarrow \boldsymbol{v}_2$ *for some* $y \in \mathsf{con}(\boldsymbol{d}_1\boldsymbol{d}_2 \cdots \boldsymbol{d}_q)$ *and* $z \in \mathsf{con}(\boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q)$. *Then the longest* $\{\bar{y}, \bar{z}\}$*-subsequence of* $\boldsymbol{u}_2$ *can only be* y^2z *or* $z^*(y^*)^2$.

Proof. (i) Since the prefix $\boldsymbol{s}_0$ of $\boldsymbol{u}_2$ consists of simple variables of $\boldsymbol{u}$, it follows from the assumption that

$$\boldsymbol{u} = \boldsymbol{u}_1 \cdot x \cdot \underbrace{\boldsymbol{a} y^2 \boldsymbol{b} z \boldsymbol{e}}_{\boldsymbol{u}_2} \cdot x^*$$

for some $\boldsymbol{a} \in F_{\mathsf{inv}}(\mathcal{X})$ and $\boldsymbol{b}, \boldsymbol{e} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\boldsymbol{u}_1, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}, x, y, z$ are pairwise disjoint. Since $\bar{z} \in \mathsf{con}(\overline{\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p}) = \mathsf{con}(\overline{\boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q})$ and $\bar{y} \in \mathsf{con}(\overline{\boldsymbol{c}_1\boldsymbol{c}_2 \cdots \boldsymbol{c}_p}) = \mathsf{con}(\overline{\boldsymbol{d}_1\boldsymbol{d}_2 \cdots \boldsymbol{d}_q})$ by Lemma 6.14, we have $\mathsf{occ}(\bar{y}, \bar{\boldsymbol{v}}_2) = 2$ and $\mathsf{occ}(\bar{z}, \bar{\boldsymbol{v}}_2) = 1$. Hence the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ can only be one of

$$y^2z, \quad y^2z^*, \quad (y^*)^2z, \quad (y^*)^2z^*, \quad zy^2, \quad z(y^*)^2, \quad z^*y^2, \quad z^*(y^*)^2.$$

There are two cases to consider.

Case 1: $y^2z^* \hookrightarrow \boldsymbol{v}_2$ or $z(y^*)^2 \hookrightarrow \boldsymbol{v}_2$. The variable $h = \mathsf{h}(\boldsymbol{a})$ is simple in $\boldsymbol{u}$, so that $\boldsymbol{a} = h\boldsymbol{f}$ for some $\boldsymbol{f} \in F^1_{\mathsf{inv}}(\mathcal{X})$. Then under the substitution $\gamma^h_{\boldsymbol{u}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^h_{\boldsymbol{u}}(\boldsymbol{u}) = \gamma^h_{\boldsymbol{u}}(\boldsymbol{u}_1 x) \cdot \gamma^h_{\boldsymbol{u}}(h) \cdot \gamma^h_{\boldsymbol{u}}(\boldsymbol{f} y^2 \boldsymbol{b} z \boldsymbol{e} x^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

But since $\gamma_{\boldsymbol{u}}^h(y) = \gamma_{\boldsymbol{u}}^h(z) = \text{F}$, we deduce the contradiction

$$\gamma_{\boldsymbol{u}}^h(\boldsymbol{v}) = \begin{cases} \cdots \text{F}^2 \cdots \text{F}^* \cdots & \text{if } y^2z^* \hookrightarrow \boldsymbol{v}_2 \\ \cdots \text{F} \cdots (\text{F}^*)^2 \cdots & \text{if } z(y^*)^2 \hookrightarrow \boldsymbol{v}_2 \end{cases} \\ = 0.$$

Case 2: $(y^*)^2z \hookrightarrow \boldsymbol{v}_2$ or $(y^*)^2z^* \hookrightarrow \boldsymbol{v}_2$ or $zy^2 \hookrightarrow \boldsymbol{v}_2$ or $z^*y^2 \hookrightarrow \boldsymbol{v}_2$. Then under the substitution $\gamma_{\boldsymbol{u}}^z : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma_{\boldsymbol{u}}^z(\boldsymbol{u}) = \gamma_{\boldsymbol{u}}^z(\boldsymbol{u}_1 x \boldsymbol{a} y^2 \boldsymbol{b}) \cdot \gamma_{\boldsymbol{u}}^z(z) \cdot \gamma_{\boldsymbol{u}}^z(\boldsymbol{e}x^*) = \text{E} \cdot \text{A} \cdot \text{F} = \text{A}.$$

But since $\gamma_{\boldsymbol{u}}^z(y) = \text{E}$ and $\gamma_{\boldsymbol{u}}^z(z) = \text{A}$, we deduce the contradiction

$$\gamma_{\boldsymbol{u}}^z(\boldsymbol{v}) = \begin{cases} \cdots (\text{E}^*)^2 \cdots \text{A} \cdots & \text{if } (y^*)^2z \hookrightarrow \boldsymbol{v}_2 \text{ or } (y^*)^2z^* \hookrightarrow \boldsymbol{v}_2 \\ \cdots \text{A} \cdots \text{E}^2 \cdots & \text{if } zy^2 \hookrightarrow \boldsymbol{v}_2 \text{ or } z^*y^2 \hookrightarrow \boldsymbol{v}_2 \end{cases} \\ = 0.$$

Since none of the two cases is possible, the longest $\{\bar{y}, \bar{z}\}$-subsequences of $\boldsymbol{v}_2$ can only be y^2z or $z^*(y^*)^2$.

(ii) This is symmetrical to part (i). □

Lemma 6.17. (i) *Suppose that* $yz^2 \hookrightarrow \boldsymbol{u}_2$ *for some* $y \in \mathsf{con}(\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p)$ *and* $z \in \mathsf{con}(\boldsymbol{c}_1\boldsymbol{c}_2 \cdots \boldsymbol{c}_p)$. *Then the longest* $\{\bar{y}, \bar{z}\}$*-subsequence of* $\boldsymbol{v}_2$ *can only be* yz^2 *or* $(z^*)^2y^*$.

(ii) *Suppose that* $yz^2 \hookrightarrow \boldsymbol{v}_2$ *for some* $y \in \mathsf{con}(\boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q)$ *and* $z \in \mathsf{con}(\boldsymbol{d}_1\boldsymbol{d}_2 \cdots \boldsymbol{d}_q)$. *Then the longest* $\{\bar{y}, \bar{z}\}$*-subsequence of* $\boldsymbol{u}_2$ *can only be* yz^2 *or* $(z^*)^2y^*$.

Proof. This is very similar to the proof of Lemma 6.16, but details are given for the sake of completeness.

(i) Since the suffix $\boldsymbol{s}_p$ of $\boldsymbol{u}_2$ consists of simple variables of $\boldsymbol{u}$, it follows from the assumption that

$$\boldsymbol{u} = \boldsymbol{u}_1 \cdot x \cdot \underbrace{\boldsymbol{a} y \boldsymbol{b} z^2 \boldsymbol{e}}_{\boldsymbol{u}_2} \cdot x^*$$

for some $\boldsymbol{a}, \boldsymbol{b} \in F_{\mathsf{inv}}^1(\mathcal{X})$ and $\boldsymbol{e} \in F_{\mathsf{inv}}(\mathcal{X})$ such that $\boldsymbol{u}_1, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}, x, y, z$ are pairwise disjoint. Since $\bar{y} \in \mathsf{con}(\overline{\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p}) = \mathsf{con}(\overline{\boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q})$ and $\bar{z} \in \mathsf{con}(\overline{\boldsymbol{c}_1\boldsymbol{c}_2 \cdots \boldsymbol{c}_p}) = \mathsf{con}(\overline{\boldsymbol{d}_1\boldsymbol{d}_2 \cdots \boldsymbol{d}_q})$ by Lemma 6.14, we have $\mathsf{occ}(\bar{y}, \bar{\boldsymbol{v}}_2) = 1$ and $\mathsf{occ}(\bar{z}, \bar{\boldsymbol{v}}_2) = 2$. Hence the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ can only be one of

$$yz^2, \quad y(z^*)^2, \quad y^*z^2, \quad y^*(z^*)^2, \quad z^2y, \quad z^2y^*, \quad (z^*)^2y, \quad (z^*)^2y^*.$$

There are two cases to consider.

Case 1: $y^*z^2 \hookrightarrow \boldsymbol{v}_2$ or $(z^*)^2y \hookrightarrow \boldsymbol{v}_2$. The variable $t = \mathsf{t}(\boldsymbol{e})$ is simple in $\boldsymbol{u}$, so that $\boldsymbol{e} = \boldsymbol{f}t$ for some $\boldsymbol{f} \in F^1_{\mathsf{inv}}(\mathcal{X})$. Then under the substitution $\gamma^t_{\boldsymbol{u}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^t_{\boldsymbol{u}}(\boldsymbol{u}) = \gamma^t_{\boldsymbol{u}}(\boldsymbol{u}_1 x\boldsymbol{a}y\boldsymbol{b}z^2\boldsymbol{f}) \cdot \gamma^t_{\boldsymbol{u}}(t) \cdot \gamma^t_{\boldsymbol{u}}(x^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

But since $\gamma^t_{\boldsymbol{u}}(y) = \gamma^t_{\boldsymbol{u}}(z) = \mathrm{E}$, we deduce the contradiction

$$\gamma^t_{\boldsymbol{u}}(\boldsymbol{v}) = \begin{cases} \cdots \mathrm{E}^* \cdots \mathrm{E}^2 \cdots & \text{if } y^*z^2 \hookrightarrow \boldsymbol{v}_2 \\ \cdots (\mathrm{E}^*)^2 \cdots \mathrm{E} \cdots & \text{if } (z^*)^2y \hookrightarrow \boldsymbol{v}_2 \end{cases}$$
$$= 0.$$

Case 2: $y(z^*)^2 \hookrightarrow \boldsymbol{v}_2$ or $y^*(z^*)^2 \hookrightarrow \boldsymbol{v}_2$ or $z^2y \hookrightarrow \boldsymbol{v}_2$ or $z^2y^* \hookrightarrow \boldsymbol{v}_2$. Then under the substitution $\gamma^y_{\boldsymbol{u}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^y_{\boldsymbol{u}}(\boldsymbol{u}) = \gamma^y_{\boldsymbol{u}}(\boldsymbol{u}_1 x\boldsymbol{a}) \cdot \gamma^y_{\boldsymbol{u}}(y) \cdot \gamma^y_{\boldsymbol{u}}(\boldsymbol{b}z^2\boldsymbol{e}x^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

But since $\gamma^y_{\boldsymbol{u}}(y) = \mathrm{A}$ and $\gamma^y_{\boldsymbol{u}}(z) = \mathrm{F}$, we deduce the contradiction

$$\gamma^y_{\boldsymbol{u}}(\boldsymbol{v}) = \begin{cases} \cdots \mathrm{A} \cdots (\mathrm{F}^*)^2 \cdots & \text{if } y(z^*)^2 \hookrightarrow \boldsymbol{v}_2 \text{ or } y^*(z^*)^2 \hookrightarrow \boldsymbol{v}_2 \\ \cdots \mathrm{F}^2 \cdots \mathrm{A} \cdots & \text{if } z^2y \hookrightarrow \boldsymbol{v}_2 \text{ or } z^2y^* \hookrightarrow \boldsymbol{v}_2 \end{cases}$$
$$= 0.$$

Since none of the two cases is possible, the longest $\{\bar{y}, \bar{z}\}$-subsequences of $\boldsymbol{v}_2$ can only be yz^2 or $(z^*)^2y^*$.

(ii) This is symmetrical to part (i). □

Lemma 6.18. $\mathsf{h}(\boldsymbol{u}_2) = \mathsf{h}(\boldsymbol{v}_2)$ *and* $\mathsf{t}(\boldsymbol{u}_2) = \mathsf{t}(\boldsymbol{v}_2)$.

Proof. Recall that $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$ by Lemma 6.13(iii). First, suppose that $|\mathsf{con}(\bar{\boldsymbol{u}}_2)| = |\mathsf{con}(\bar{\boldsymbol{v}}_2)| = 1$, say $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2) = \{z\}$ for some $z \in \mathcal{X}$. Then it follows from condition (B4)(a) that $\boldsymbol{u}_2 = \boldsymbol{v}_2 = z$, whence $\mathsf{h}(\boldsymbol{u}_2) = z = \mathsf{h}(\boldsymbol{v}_2)$ and $\mathsf{t}(\boldsymbol{u}_2) = z = \mathsf{t}(\boldsymbol{v}_2)$.

Hence, it remains to assume $|\mathsf{con}(\bar{\boldsymbol{u}}_2)| = |\mathsf{con}(\bar{\boldsymbol{v}}_2)| \geq 2$. Let $h = \mathsf{h}(\boldsymbol{u}_2) = \mathsf{h}(\boldsymbol{s}_0)$, $t = \mathsf{t}(\boldsymbol{u}_2) = \mathsf{h}(\boldsymbol{s}_p)$, $H = \mathsf{h}(\boldsymbol{v}_2) = \mathsf{h}(\boldsymbol{t}_0)$, and $T = \mathsf{t}(\boldsymbol{v}_2) = \mathsf{h}(\boldsymbol{t}_q)$, so that

$$\bar{h} \prec \bar{t} \quad \text{and} \quad \overline{H} \prec \overline{T} \tag{6-5}$$

by condition (B4)(b). Then

$$\boldsymbol{u} = \boldsymbol{u}_1 \cdot x \cdot \underbrace{h\boldsymbol{a}t}_{\boldsymbol{u}_2} \cdot x^* \quad \text{and} \quad \boldsymbol{v} = \boldsymbol{u}_1 \cdot x \cdot \underbrace{H\boldsymbol{b}T}_{\boldsymbol{v}_2} \cdot x^* \tag{6-6}$$

for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\overline{ht} \notin \mathsf{con}(\overline{\boldsymbol{u}_1 x\boldsymbol{a}})$ and $\overline{HT} \notin \mathsf{con}(\overline{\boldsymbol{u}_1 x\boldsymbol{b}})$. Since $ht \hookrightarrow \boldsymbol{u}_2$ with $h, t \in \mathsf{con}(\boldsymbol{s}_0\boldsymbol{s}_p)$, it follows from Lemma 6.15(i) that the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$ is ht or t^*h^*.

Seeking a contradiction, suppose that t^*h^* is the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$. If $H \neq t^*$, so that $Ht^* \hookrightarrow \boldsymbol{v}_2$, then by Lemma 6.15(ii), either $Ht^* \hookrightarrow \boldsymbol{u}_2$ or $tH^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (6-6). Hence $H = t^*$. By a symmetrical argument, we deduce $T = h^*$. Since $\bar{h} \prec \bar{t}$ by (6-5), we have $\overline{T} = \bar{h} \prec \bar{t} = \overline{H}$; but this contradicts $\overline{H} \prec \overline{T}$ from (6-5).

Therefore, ht is the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$. If $H \neq h$, so that $Hh \hookrightarrow \boldsymbol{v}_2$, then by Lemma 6.15(ii), either $Hh \hookrightarrow \boldsymbol{u}_2$ or $h^*H^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (6-6). Hence $H = h$. By a symmetrical argument, we deduce $T = t$. □

Lemma 6.19. $\boldsymbol{u}_2 = \boldsymbol{v}_2$.

Proof. Recall that

$$\boldsymbol{u}_2 = \boldsymbol{s}_0 \prod_{i=1}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i) \quad \text{and} \quad \boldsymbol{v}_2 = \boldsymbol{t}_0 \prod_{i=1}^{q} (\boldsymbol{d}_i \boldsymbol{t}_i),$$

and $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$ by Lemma 6.13(iii). If $|\mathsf{con}(\bar{\boldsymbol{u}}_2)| = |\mathsf{con}(\bar{\boldsymbol{v}}_2)| = 1$, then as shown in the proof of Lemma 6.18, we have $\boldsymbol{u}_2 = \boldsymbol{v}_2$. Therefore, it suffices to assume that $|\mathsf{con}(\bar{\boldsymbol{u}}_2)| = |\mathsf{con}(\bar{\boldsymbol{v}}_2)| \geq 2$. By Lemma 6.18, we have

(a) $\mathsf{h}(\boldsymbol{s}_0) = \mathsf{h}(\boldsymbol{u}_2) = \mathsf{h}(\boldsymbol{v}_2) = \mathsf{h}(\boldsymbol{t}_0)$ and $\mathsf{t}(\boldsymbol{s}_p) = \mathsf{t}(\boldsymbol{u}_2) = \mathsf{t}(\boldsymbol{v}_2) = \mathsf{t}(\boldsymbol{t}_q)$.

Suppose that $z \in \mathsf{con}(\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p)$ with $z \neq \mathsf{h}(\boldsymbol{s}_0), \mathsf{t}(\boldsymbol{s}_p)$, so that $\mathsf{h}(\boldsymbol{s}_0)z \hookrightarrow \boldsymbol{u}_2$. Then by Lemma 6.15(i), the longest $\{\overline{\mathsf{h}(\boldsymbol{s}_0)}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ can only be $\mathsf{h}(\boldsymbol{s}_0)z$ or $z^*(\mathsf{h}(\boldsymbol{s}_0))^*$. But since $\boldsymbol{v}_2$ is a bipartite word (see Remark 6.9(iv)) that contains the variable $\mathsf{h}(\boldsymbol{t}_0) = \mathsf{h}(\boldsymbol{s}_0)$, it cannot contain the variable $(\mathsf{h}(\boldsymbol{s}_0))^*$. Hence the longest $\{\overline{\mathsf{h}(\boldsymbol{s}_0)}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ must be $\mathsf{h}(\boldsymbol{s}_0)z$, so that $z \in \mathsf{con}(\boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q)$. Therefore, the inclusion $\mathsf{con}(\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p) \subseteq \mathsf{con}(\boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q)$ holds. The reverse inclusion $\mathsf{con}(\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p) \supseteq \mathsf{con}(\boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q)$ is established by a symmetrical argument, so that $\mathsf{con}(\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p) = \mathsf{con}(\boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q)$. Further, since $\mathsf{h}(\boldsymbol{s}_0) = \mathsf{h}(\boldsymbol{t}_0)$ and $\mathsf{t}(\boldsymbol{s}_p) = \mathsf{t}(\boldsymbol{t}_q)$ by (a), it is easy to show by Lemma 6.15 that $\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p = \boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q$. Hence

(b) $\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p = \boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q = z_1 z_2 \cdots z_r$ for some distinct $z_1, z_2, \ldots, z_r \in \mathcal{X} \cup \mathcal{X}^*$,

where $z_1 = \mathsf{h}(\boldsymbol{s}_0) = \mathsf{h}(\boldsymbol{t}_0)$ and $z_r = \mathsf{t}(\boldsymbol{s}_p) = \mathsf{t}(\boldsymbol{t}_q)$.

Now it follows from Lemma 6.14 that $p = 0$ if and only if $q = 0$, so there are two cases: $p = q = 0$ and $p, q \geq 1$. If $p = q = 0$, then $\boldsymbol{u}_2 = \boldsymbol{s}_0 = \boldsymbol{t}_0 = \boldsymbol{v}_2$, so the proof is complete. Hence it remains to assume $p, q \geq 1$.

Seeking a contradiction, suppose that $\boldsymbol{s}_0 \neq \boldsymbol{t}_0$. Then by (b), either $\boldsymbol{s}_0$ is a proper prefix of $\boldsymbol{t}_0$ or $\boldsymbol{t}_0$ is a proper prefix of $\boldsymbol{s}_0$. By symmetry, suppose that $\boldsymbol{s}_0$ is a proper prefix of $\boldsymbol{t}_0$, so that $\boldsymbol{s}_0 = z_1 z_2 \cdots z_k$ and $\boldsymbol{t}_0 = z_1 z_2 \cdots z_\ell$ with $k < \ell \leq r$. Then

$$\boldsymbol{u}_2 = z_1 z_2 \cdots z_k \cdot \boldsymbol{c}_1 \cdot z_{k+1} z_{k+2} \cdots$$

and

$$\boldsymbol{v}_2 = z_1 z_2 \cdots z_k \cdot z_{k+1} z_{k+2} \cdots z_\ell \cdot \boldsymbol{d}_1 \cdot z_{\ell+1} z_{\ell+2} \cdots .$$

Since $\boldsymbol{c}_1$ is an ordered B_0-block, it begins with y^2 for some $y \in \mathcal{X} \cup \mathcal{X}^*$. Then $y^2 z_{k+1} \hookrightarrow \boldsymbol{u}_2$ but $y^2 z_{k+1} \not\hookrightarrow \boldsymbol{v}_2$ and $z_{k+1}^*(y^*)^2 \not\hookrightarrow \boldsymbol{v}_2$, which is impossible in view of Lemma 6.16(i). Therefore, $\boldsymbol{s}_0 = z_1 z_2 \cdots z_k = \boldsymbol{t}_0$, so that

$$\boldsymbol{u}_2 = z_1 z_2 \cdots z_k \cdot \boldsymbol{c}_1 \cdot z_{k+1} z_{k+2} \cdots$$

and

$$\boldsymbol{v}_2 = z_1 z_2 \cdots z_k \cdot \boldsymbol{d}_1 \cdot z_{k+1} z_{k+2} \cdots .$$

Seeking a contradiction, suppose that $\mathsf{con}(\boldsymbol{c}_1) \neq \mathsf{con}(\boldsymbol{d}_1)$, say $y \in \mathsf{con}(\boldsymbol{c}_1) \setminus \mathsf{con}(\boldsymbol{d}_1)$. Then since $\boldsymbol{c}_1$ is an ordered B_0-block, $\boldsymbol{c}_1 = \boldsymbol{a} y^2 \boldsymbol{b}$ for some $\boldsymbol{a}, \boldsymbol{b} \in F_{\mathsf{inv}}^1(\mathcal{X})$. Hence $y^2 z_{k+1} \hookrightarrow \boldsymbol{u}_2$ but $y^2 z_{k+1} \not\hookrightarrow \boldsymbol{v}_2$ and $z_{k+1}^*(y^*)^2 \not\hookrightarrow \boldsymbol{v}_2$, which is impossible in view of Lemma 6.16(i). Therefore, $\mathsf{con}(\boldsymbol{c}_1) = \mathsf{con}(\boldsymbol{d}_1)$. Since $\boldsymbol{c}_1$ and $\boldsymbol{d}_1$ are ordered B_0-blocks, we have $\boldsymbol{c}_1 = \boldsymbol{d}_1$.

Without loss of generality, assume that $p \leq q$. The arguments in the previous two paragraphs can be repeated to show that $\boldsymbol{c}_i = \boldsymbol{d}_i$ and $\boldsymbol{s}_i = \boldsymbol{t}_i$ and for all $i = 1, 2, \ldots, p-1$. Hence

$$\boldsymbol{u}_2 = \boldsymbol{s}_0 \Big(\prod_{i=1}^{p-1} (\boldsymbol{c}_i \boldsymbol{s}_i) \Big) \boldsymbol{c}_p \boldsymbol{s}_p \quad \text{and} \quad \boldsymbol{v}_2 = \boldsymbol{s}_0 \Big(\prod_{i=1}^{p-1} (\boldsymbol{c}_i \boldsymbol{s}_i) \Big) \boldsymbol{d}_p \boldsymbol{t}_p \boldsymbol{d}_{p+1} \boldsymbol{t}_{p+1} \cdots \boldsymbol{d}_q \boldsymbol{t}_q .$$

Arguments that are dual to those from the previous two paragraphs (with the use of Lemma 6.17 instead of Lemma 6.16) can be repeated to show that $\boldsymbol{s}_p = \boldsymbol{t}_q$ and then $\boldsymbol{c}_p = \boldsymbol{d}_q$. It follows from (b) that $p = q$ and $\boldsymbol{u}_2 = \boldsymbol{v}_2$. □

7. Involution semigroups of order up to four

Multiplication tables of involution semigroups of order up to four are given in this section. For a more compact presentation, the column/row headers are omitted from each multiplication table. For instance, the involution semigroup $S = \{1, 2, 3\}$ given by the multiplication table on the left is abbreviated to the array on the right:

S	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2
x	1	2	3
x^*	1	3	2

$\rightarrow$

1	2	3
2	3	1
3	1	2
1	3	2

Up to isomorphism, there are three involution semigroups of order two and 15 involution semigroups of order three, all of which are commutative [20, Section 4]; see Table 2.

Up to isomorphism, there are 83 involution semigroups of order four; see Table 3.

1 1 1 1 ――― 1 2	1 1 1 2 ――― 1 2	1 2 2 1 ――― 1 2

1 1 1 1 1 1 1 1 1 ――― 1 2 3	1 1 1 1 1 1 1 1 1 ――― 1 3 2	1 1 1 1 1 1 1 1 2 ――― 1 2 3	1 1 1 1 1 1 1 1 3 ――― 1 2 3	1 1 1 1 1 2 1 2 3 ――― 1 2 3
1 1 1 1 2 1 1 1 3 ――― 1 2 3	1 1 1 1 2 1 1 1 3 ――― 1 3 2	1 1 1 1 2 2 1 2 2 ――― 1 2 3	1 1 1 1 2 2 1 2 3 ――― 1 2 3	1 1 1 1 2 3 1 3 2 ――― 1 2 3
1 1 3 1 1 3 3 3 1 ――― 1 2 3	1 1 3 1 2 3 3 3 1 ――― 1 2 3	1 2 2 2 1 1 2 1 1 ――― 1 2 3	1 2 3 2 3 1 3 1 2 ――― 1 2 3	1 2 3 2 3 1 3 1 2 ――― 1 3 2

Table 2. The three involution semigroups of order two and the 15 of order three.

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 ――― 1 2 3 4	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 ――― 1 2 4 3	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 ――― 1 2 3 4	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 4 ――― 1 2 3 4	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 4 ――― 1 3 2 4	1 1 1 1 1 1 1 1 1 1 1 1 1 1 2 1 ――― 1 2 4 3	1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 1 ――― 1 2 3 4	1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 1 ――― 1 2 4 3	1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 2 ――― 1 2 3 4	1 1 1 1 1 1 1 1 1 1 1 2 1 1 2 3 ――― 1 2 3 4	1 1 1 1 1 1 1 1 1 1 1 3 1 1 3 4 ――― 1 2 3 4
1 1 1 1 1 1 1 1 1 1 1 3 1 2 1 4 ――― 1 3 2 4	1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 2 ――― 1 2 3 4	1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 2 ――― 1 2 4 3	1 1 1 1 1 1 1 1 1 1 2 1 1 1 1 4 ――― 1 2 3 4	1 1 1 1 1 1 1 1 1 1 2 1 1 1 2 2 ――― 1 2 4 3	1 1 1 1 1 1 1 1 1 1 2 2 1 1 2 2 ――― 1 2 3 4	1 1 1 1 1 1 1 1 1 1 2 2 1 1 2 2 ――― 1 2 4 3	1 1 1 1 1 1 1 1 1 1 3 1 1 1 1 4 ――― 1 2 3 4	1 1 1 1 1 1 1 1 1 1 3 1 1 1 1 4 ――― 1 2 4 3	1 1 1 1 1 1 1 1 1 1 3 3 1 1 3 3 ――― 1 2 3 4	1 1 1 1 1 1 1 1 1 1 3 3 1 1 3 4 ――― 1 2 3 4
1 1 1 1 1 1 1 1 1 1 3 4 1 1 4 3 ――― 1 2 3 4	1 1 1 1 1 1 1 2 1 1 1 2 1 2 2 4 ――― 1 2 3 4	1 1 1 1 1 1 1 2 1 1 1 3 1 2 3 4 ――― 1 2 3 4	1 1 1 1 1 1 1 2 1 1 1 3 1 2 3 4 ――― 1 3 2 4	1 1 1 1 1 1 1 2 1 1 2 3 1 2 3 4 ――― 1 2 3 4	1 1 1 1 1 1 1 2 1 1 3 1 1 2 1 4 ――― 1 2 3 4	1 1 1 1 1 1 1 2 1 1 3 3 1 2 3 4 ――― 1 2 3 4	1 1 1 1 1 1 1 2 1 2 3 1 1 1 1 4 ――― 1 2 4 3	1 1 1 1 1 1 1 2 1 2 3 2 1 1 1 4 ――― 1 2 4 3	1 1 1 1 1 1 2 2 1 2 3 3 1 2 3 3 ――― 1 2 3 4	1 1 1 1 1 1 2 2 1 2 3 3 1 2 3 4 ――― 1 2 3 4
1 1 1 1 1 1 2 2 1 2 3 4 1 2 4 3 ――― 1 2 3 4	1 1 1 1 1 2 1 1 1 1 3 1 1 1 1 4 ――― 1 2 3 4	1 1 1 1 1 2 1 1 1 1 3 1 1 1 1 4 ――― 1 2 4 3	1 1 1 1 1 2 1 1 1 1 3 3 1 1 3 3 ――― 1 2 3 4	1 1 1 1 1 2 1 1 1 1 3 3 1 1 3 4 ――― 1 2 3 4	1 1 1 1 1 2 1 1 1 1 3 4 1 1 4 3 ――― 1 2 3 4	1 1 1 1 1 2 1 2 1 1 3 3 1 2 3 4 ――― 1 2 3 4	1 1 1 1 1 2 1 2 1 1 3 3 1 2 3 4 ――― 1 3 2 4	1 1 1 1 1 2 2 2 1 2 2 2 1 2 2 2 ――― 1 2 3 4	1 1 1 1 1 2 2 2 1 2 2 2 1 2 2 2 ――― 1 2 4 3	1 1 1 1 1 2 2 2 1 2 2 2 1 2 2 3 ――― 1 2 3 4
1 1 1 1 1 2 2 2 1 2 2 2 1 2 2 4 ――― 1 2 3 4	1 1 1 1 1 2 2 2 1 2 2 3 1 2 3 4 ――― 1 2 3 4	1 1 1 1 1 2 2 2 1 2 3 2 1 2 2 4 ――― 1 2 3 4	1 1 1 1 1 2 2 2 1 2 3 2 1 2 2 4 ――― 1 2 4 3	1 1 1 1 1 2 2 2 1 2 3 3 1 2 3 3 ――― 1 2 3 4	1 1 1 1 1 2 2 2 1 2 3 3 1 2 3 4 ――― 1 2 3 4	1 1 1 1 1 2 2 2 1 2 3 4 1 2 4 3 ――― 1 2 3 4	1 1 1 1 1 2 2 4 1 2 2 4 1 4 4 2 ――― 1 2 3 4	1 1 1 1 1 2 2 4 1 2 3 4 1 4 4 2 ――― 1 2 3 4	1 1 1 1 1 2 3 3 1 3 2 2 1 3 2 2 ――― 1 2 3 4	1 1 1 1 1 2 3 4 1 3 4 2 1 4 2 3 ――― 1 2 3 4
1 1 1 1 1 2 3 4 1 3 4 2 1 4 2 3 ――― 1 2 4 3	1 1 1 4 1 1 1 4 1 1 1 4 4 4 4 1 ――― 1 2 3 4	1 1 1 4 1 1 1 4 1 1 1 4 4 4 4 1 ――― 1 3 2 4	1 1 1 4 1 1 1 4 1 1 2 4 4 4 4 1 ――― 1 2 3 4	1 1 1 4 1 1 1 4 1 1 3 4 4 4 4 1 ――― 1 2 3 4	1 1 1 4 1 1 2 4 1 2 3 4 4 4 4 1 ――― 1 2 3 4	1 1 1 4 1 2 1 4 1 1 3 4 4 4 4 1 ――― 1 2 3 4	1 1 1 4 1 2 1 4 1 1 3 4 4 4 4 1 ――― 1 3 2 4	1 1 1 4 1 2 2 4 1 2 2 4 4 4 4 1 ――― 1 2 3 4	1 1 1 4 1 2 2 4 1 2 3 4 4 4 4 1 ――― 1 2 3 4	1 1 1 4 1 2 3 4 1 3 2 4 4 4 4 1 ――― 1 2 3 4
1 1 3 3 1 1 3 3 3 3 1 1 3 3 1 1 ――― 1 2 3 4	1 1 3 3 1 1 3 3 3 3 1 1 3 3 1 2 ――― 1 2 3 4	1 1 3 3 1 2 3 3 3 3 1 1 3 3 1 1 ――― 1 2 3 4	1 1 3 3 1 2 3 4 3 3 1 1 3 4 1 1 ――― 1 2 3 4	1 1 3 3 1 2 3 4 3 3 1 1 3 4 1 2 ――― 1 2 3 4	1 1 3 3 2 2 4 4 1 1 3 3 2 2 4 4 ――― 1 3 2 4	1 1 3 4 1 1 3 4 3 3 4 1 4 4 1 3 ――― 1 2 3 4	1 1 3 4 1 1 3 4 3 3 4 1 4 4 1 3 ――― 1 2 4 3	1 1 3 4 1 2 3 4 3 3 4 1 4 4 1 3 ――― 1 2 3 4	1 1 3 4 1 2 3 4 3 3 4 1 4 4 1 3 ――― 1 2 4 3	1 2 2 2 2 1 1 1 2 1 1 1 2 1 1 1 ――― 1 2 3 4
1 2 2 2 2 1 1 1 2 1 1 1 2 1 1 1 ――― 1 2 4 3	1 2 2 4 2 4 4 1 2 4 4 1 4 1 1 2 ――― 1 2 3 4	1 2 3 4 2 1 4 3 3 4 1 2 4 3 2 1 ――― 1 2 3 4	1 2 3 4 2 1 4 3 3 4 1 2 4 3 2 1 ――― 1 2 4 3	1 2 3 4 2 1 4 3 3 4 2 1 4 3 1 2 ――― 1 2 3 4	1 2 3 4 2 1 4 3 3 4 2 1 4 3 1 2 ――― 1 2 4 3					

Table 3. The 83 involution semigroups of order four.

Acknowledgements

The authors are grateful to the reviewers for their insightful reports and suggestions.

References

[1] J. Araújo, J. P. Araújo, P. J. Cameron, E. W. H. Lee, and J. Raminhos, "A survey on varieties generated by small semigroups and a companion website", *J. Algebra* **635** (2023), 698–735, supplementary material. MR

[2] K. Auinger, I. Dolinka, and M. V. Volkov, "Matrix identities involving multiplication and transposition", *J. Eur. Math. Soc.* **14**:3 (2012), 937–969. MR

[3] J. A. Bahturin and A. Y. Olshanskii, "Identical relations in finite Lie rings", *Mat. Sb.* (*N.S.*) **96**:4 (1975), 543–559. In Russian; translated in *Math. USSR-Sb.* **25**:4 (1975), 507–523. MR

[4] S. Burris and H. P. Sankappanavar, *A course in universal algebra*, Graduate Texts in Mathematics **78**, Springer, 1981. MR

[5] A. Distler and T. Kelsey, "The semigroups of order 9 and their automorphism groups", *Semigroup Forum* **88**:1 (2014), 93–112. MR

[6] C. C. Edmunds, "Varieties generated by semigroups of order four", *Semigroup Forum* **21**:1 (1980), 67–81. MR

[7] S. Fajtlowicz, "Equationally complete semigroups with involution", *Algebra Universalis* **1** (1971/72), 355–358. MR

[8] M. Gao, W. T. Zhang, and Y. F. Luo, "A non-finitely based involution semigroup of order five", *Algebra Universalis* **81**:3 (2020), art. id. 31, 14 pp. MR

[9] B. B. Han, W. T. Zhang, and Y. F. Luo, "Finite basis problem for involution monoids of order five", *Bull. Aust. Math. Soc.* **109**:2 (2024), 350–364. MR

[10] J. M. Howie, *Fundamentals of semigroup theory*, LMS Monographs (N.S.) **12**, Oxford University Press, 1995. MR

[11] M. Jackson and M. Volkov, "The algebra of adjacency patterns: Rees matrix semigroups with reversion", pp. 414–443 in *Fields of logic and computation*, Lecture Notes in Comput. Sci. **6300**, Springer, 2010. MR

[12] E. I. Kleiman, "Bases of identities of varieties of inverse semigroups", *Sibirsk. Mat. Zh.* **20**:4 (1979), 760–777, 926. In Russian; translated in *Siberian Math. J.* **20**:4 (1979), 530–543. MR

[13] R. L. Kruse, "Identities satisfied by a finite ring", *J. Algebra* **26** (1973), 298–318. MR

[14] E. W. H. Lee, "Identity bases for some non-exact varieties", *Semigroup Forum* **68**:3 (2004), 445–457. MR

[15] E. W. H. Lee, "Finite basis problem for semigroups of order five or less: generalization and revisitation", *Studia Logica* **101**:1 (2013), 95–115. MR

[16] E. W. H. Lee, "Finitely based finite involution semigroups with non-finitely based reducts", *Quaest. Math.* **39**:2 (2016), 217–243. MR

[17] E. W. H. Lee, "Equational theories of unstable involution semigroups", *Electron. Res. Announc. Math. Sci.* **24** (2017), 10–20. MR

[18] E. W. H. Lee, "On a class of completely join prime J-trivial semigroups with unique involution", *Algebra Universalis* **78**:2 (2017), 131–145. MR

[19] E. W. H. Lee, "Non-finitely based finite involution semigroups with finitely based semigroup reducts", *Korean J. Math.* **27**:1 (2019), 53–62. MR

[20] E. W. H. Lee, "Intervals of varieties of involution semigroups with contrasting reduct intervals", *Boll. Unione Mat. Ital.* **15**:4 (2022), 527–540. MR

[21] E. W. H. Lee, *Advances in the theory of varieties of semigroups*, Birkhäuser, 2023. MR

[22] E. W. H. Lee, "Embedding finite involution semigroups in matrices with transposition", *Discrete Appl. Math.* **340** (2023), 327–330. MR

[23] E. W. H. Lee and J. R. Li, "Minimal non-finitely based monoids", *Dissertationes Math.* **475** (2011), 65. MR

[24] E. W. H. Lee and M. V. Volkov, "On the structure of the lattice of combinatorial Rees–Sushkevich varieties", pp. 164–187 in *Semigroups and formal languages*, World Scientific, Hackensack, NJ, 2007. MR

[25] E. W. H. Lee and W. T. Zhang, "Finite basis problem for semigroups of order six", *LMS J. Comput. Math.* **18**:1 (2015), 1–129. MR

[26] E. W. H. Lee, J. R. Li, and W. T. Zhang, "Minimal non-finitely based semigroups", *Semigroup Forum* **85**:3 (2012), 577–580. MR

[27] I. V. Lvov, "Varieties of associative rings, I", *Algebra i Logika* **12** (1973), 269–297. In Russian; translated in *Algebra Logic* **12**:3 (1973), 150–167. MR

[28] W. McCune, "Prover9 and Mace4", 2005–2010, available at http://www.cs.unm.edu/∼mccune/prover9.

[29] R. McKenzie, "Equational bases for lattice theories", *Math. Scand.* **27** (1970), 24–38. MR

[30] R. McKenzie, "Tarski's finite basis problem is undecidable", *Internat. J. Algebra Comput.* **6**:1 (1996), 49–104. MR

[31] S. Oates and M. B. Powell, "Identical relations in finite groups", *J. Algebra* **1** (1964), 11–39. MR

[32] P. Perkins, "Bases for equational theories of semigroups", *J. Algebra* **11** (1969), 298–314. MR

[33] A. N. Trahtman, "The finite basis question for semigroups of order less than six", *Semigroup Forum* **27**:1-4 (1983), 387–389. MR

[34] A. N. Trahtman, "Finiteness of a basis of identities of five-element semigroups", pp. 76–97 in Полугруппы и их гомоморфизмы, Ross. Gos. Ped. Univ., Leningrad (aka Herzen Univeristy, Saint Petersburg), 1991. In Russian. MR

[35] M. V. Volkov, "The finite basis problem for finite semigroups", *Sci. Math. Jpn.* **53**:1 (2001), 171–199. MR

Received March 11, 2025. Revised January 13, 2026.

MENG GAO
SCHOOL OF MATHEMATICS AND STATISTICS
LANZHOU UNIVERSITY
LANZHOU, GANSU
CHINA
gaom2015@lzu.edu.cn

YAN FENG LUO
SCHOOL OF MATHEMATICS AND STATISTICS
LANZHOU UNIVERSITY
LANZHOU, GANSU
CHINA
luoyf@lzu.edu.cn

EDMOND W. H. LEE
DEPARTMENT OF MATHEMATICS
NOVA SOUTHEASTERN UNIVERSITY
FORT LAUDERDALE, FL
UNITED STATES
edmond.lee@nova.edu

WEN TING ZHANG
SCHOOL OF MATHEMATICS AND STATISTICS
LANZHOU UNIVERSITY
LANZHOU, GANSU
CHINA
zhangwt@lzu.edu.cn

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2026 is US $710/year for the electronic version, and $965/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 342 No. 1 May 2026