Pacific Journal of Mathematics

Volume 342 No. 1 May 2026

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 342, No. 1, 2026
https://doi.org/10.2140/pjm.2026.342.1

THE $\bar\partial$-PROBLEM ON $Z(q)$-DOMAINS

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Given a complex manifold containing a relatively compact $Z(q)$ domain, we give sufficient geometric conditions on the domain so that its L^2-cohomology in degree (p, q) (known to be finite-dimensional) vanishes. The condition consists in the existence of a smooth weight function in a neighborhood of the closure of the domain, where the complex Hessian of the weight has a prescribed number of eigenvalues of a particular sign, along with good interaction at the boundary of the Levi form with the complex Hessian, encoded in a subbundle of common positive directions for the two Hermitian forms.

1. Introduction

1.1. ***Solvability of the $\bar\partial$-problem.*** We give a sufficient geometric condition for a relatively compact $Z(q)$ domain Ω in a complex manifold to have vanishing Dolbeault cohomology at level (p, q). Among other things, our condition requires a smooth weight function defined in a neighborhood of $\bar\Omega$ whose complex Hessian has a given number of positive and negative eigenvalues. Our technique is to build a metric that turns a condition about *numbers* of eigenvalues into one about *sums* of eigenvalues. Typically, the former conditions are invariant under biholomorphisms while L^2 methods require the latter conditions. The heart of our argument is an investigation of an invariant condition in which the Levi form and the complex Hessian of the weight share positive directions. This is novel and represents a new approach to rectify the dichotomy between invariant conditions and sufficient conditions to use L^2 methods.

The solvability of the $\bar\partial$-problem on a domain in $\mathbb{C}^n$ or on a complex manifold depends on certain convexity conditions, the most natural of which is being Stein, i.e., the existence of a strictly plurisubharmonic exhaustion function. Under this

Chakrabarti was partially supported by US National Science Foundation grant number DMS-2153907 and by a grant from the Simons Foundation (706445, DC). Raich was partially supported by a grant from the Simons Foundation (707123, ASR) .

MSC2020: 32F10, 32F32, 32W05.

Keywords: Hermitian manifolds, $Z(q)$, q-complete, q-convex, q-plurisubharmonic, $\bar\partial$-problem, L^2-theory.

condition we get the vanishing of the Dolbeault cohomology in degree (p, q) for $q \geq 1$, a special case of H. Cartan's celebrated *Theorem B* (see [GR09]). It was realized early that one can generalize this substantially: by well-known results of Andreotti and Grauert (see [AG62]), if there is a smooth exhaustion function on the n-dimensional complex manifold M whose complex Hessian has $n-q+1$ positive eigenvalues, then $H^{p,q}(M) = 0$ for each p. A smooth function on an n-dimensional complex manifold whose complex Hessian has $n-q+1$ positive eigenvalues at each point is usually called a (*strictly*) *q-convex function*, but there are other competing conventions for this and other related definitions, so we mostly avoid the use of the "q-terminology" in this paper. As is common in differential geometry, the convexity condition is encoded in a Hermitian form (cf. the second fundamental form in the classical theory of surfaces in Euclidean space). For more on the q-convexity conditions, see [OP22; ES80].

In the study of complex function theory on a domain (open connected subset) Ω in a complex manifold M, methods based on L^2-estimates are usually easier to use than the classical sheaf-theoretic arguments (see [CS01; Dem12]). Establishing L^2 estimates requires the choice of a Hermitian metric on the manifold. The complex convexity of the boundary $\partial\Omega$ is encoded in its Levi form. The "interior complex convexity" of the domain is encoded in a smooth "weight function" φ in a neighborhood of the closure $\overline{\Omega}$, where the complex Hessian of φ has certain positivity conditions imposed. One can interpret φ as giving rise to a Hermitian metric $e^{-\varphi}$ on the trivial line bundle in a neighborhood of $\overline{\Omega}$. By a well-known result of Hörmander (see [Hör65]), if the domain Ω is pseudoconvex, and the weight φ is strictly plurisubharmonic, then we have the vanishing of the L^2-cohomology $H^{p,q}_{L^2}(\Omega)$ for $q \geq 1$. This is of course equivalent to the solvability, with estimates in the L^2-norm, of the $\bar{\partial}$-equation $\bar{\partial}u = g$ for a $\bar{\partial}$-closed square-integrable (p, q)-form g.

In analogy with convexity conditions on manifolds given by exhaustion or weight functions with complex Hessians of specified signature, one can also consider partial convexity conditions for the boundary in order to study the $\bar{\partial}$-problem in a fixed degree. For $1 \leq q \leq n-1$, a smoothly bounded domain Ω in an n-dimensional complex manifold is said to satisfy *condition $Z(q)$*, if at each point of the boundary $\partial\Omega$ the Levi form has at least $n-q$ positive eigenvalues, or has at least $q+1$ negative eigenvalues. This condition is fundamental in the theory of the $\bar{\partial}$-Neumann problem, since it is necessary and sufficient for $\frac{1}{2}$-subelliptic estimates on (p, q)-forms (see [FK72]).

Another type of partial convexity condition arises from a consideration of the right-hand side of the Bochner–Kohn–Morrey–Hörmander identity for higher degree forms, and the condition needed for positivity of the boundary integral (involving the Levi form) and the interior integral (involving the complex Hessian of the weight) (see [Ho91]). Following [MV15], let us introduce a notion from linear algebra.

Definition 1.1. Let H be a Hermitian form on a complex inner product space (E, g). We say that H is *strictly q-positive* with respect to g if the sum of each collection of q eigenvalues of H with respect to g is positive. If the sum of each collection of q-eigenvalues is nonnegative, we say that H is *q-positive* with respect to g.

Section 2.1 contains a more detailed discussion of the eigenstructure of Hermitian forms with respect to an inner product. Generalizing the result of Hörmander stated above, it follows from [MV15, Theorem 2.14] that given a smoothly bounded relatively compact domain Ω in a Kähler manifold, and a smooth weight φ in a neighborhood of $\overline{\Omega}$, simultaneous strict q-positivity of the Levi form of $\partial\Omega$ and of the complex Hessian of φ on $\overline{\Omega}$ constitutes a sufficient condition for the vanishing of the L^2-cohomology in degree (p, q). We will prove stronger versions of this result without the Kähler hypothesis below in Theorems 3.3 and 3.1. Related results were studied in [Ho91; Wu81] etc.

The hypotheses and conclusions of Theorems 3.3 and 3.1 have a dissatisfying incongruity about them and this is one of our motivations for starting this project. The L^2-cohomology of a bounded domain in a complex manifold is defined independently of the choice of the Hermitian metric. On the other hand, the hypotheses on the Levi form of the boundary and the complex Hessian of the weight involve the metric (since strict q-positivity of the two Hermitian forms is defined with respect to this metric). Therefore, Theorems 3.3 and 3.1 draw a metric-independent conclusion from a hypothesis that depends very much on the choice of a metric. This paper is an attempt to understand what purely complex-geometric conditions on a domain in a complex manifold suffice to ensure that the L^2-cohomology is zero.

1.2. ***Results.*** Andreotti and Vesentini in [AV65, Section 5] gave a proof of the vanishing theorem of Andreotti and Grauert stated above by constructing a metric in which the complex Hessian of the exhaustion φ is strictly q-positive and applying L^2-methods. We will use the same approach, but our metric construction is fundamentally different in that their metric is complete while we focus extensively on the interaction of the metric with $\partial\Omega$. The subtle and delicate aspect of our work is ensuring that the Levi form and the complex Hessian of the weight function are simultaneously strictly q-positivity in the metric we construct. This allows us to apply L^2-results like Theorems 3.3 and 3.1. Let us say that on a relatively compact domain Ω in a complex manifold, the $\bar{\partial}$-operator satisfies the *Morrey–Kohn–Hörmander basic estimate* in degree (p, q), if there exists a constant $C > 0$ so that for all (p, q)-forms $f \in \mathrm{Dom}(\bar{\partial}) \cap \mathrm{Dom}(\bar{\partial}^*)$ we have

$$\|f\|^2_{L^2(\partial\Omega)} \leq C\big(\|\bar{\partial} f\|^2_{L^2(\Omega)} + \|\bar{\partial}^* f\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)}\big). \tag{1-1}$$

This estimate has numerous important consequences for the function theory of Ω.

Combined with the ellipticity of $\bar\partial \oplus \bar\partial^*$ in the interior of Ω (where there are no boundary conditions), it follows that the L^2*-cohomology space* $H^{p,q}_{L^2}(\Omega)$ *is finite-dimensional.* Additionally, $\frac{1}{2}$-subelliptic estimates hold for the $\bar\partial$-Neumann problem (see [FK72; CS01; Str10]).

Our techniques provide a new proof of the following well-known result.

Theorem 1.2. *Let M be a complex manifold and let $\Omega \subset M$ be a smoothly bounded relatively compact domain, and let $1 \le q \le n-1$. Suppose that the domain Ω satisfies condition $Z(q)$. Then the $\bar\partial$-operator on (p,q)-forms on Ω satisfies the Morrey–Kohn–Hörmander basic estimate (1-1).*

Our main tool is the following proposition, which is closely related to [AV65, Lemma 18], though our proof is quite different from that offered in [AV65]. It has the added flexibility of prescribing the metric on a closed subset of the manifold.

Proposition 1.3. *Let E be a smooth complex vector bundle of rank d over a smooth manifold M, and let S be a smooth Hermitian form on E such that for some $1 \le \tilde q \le d$, at each $p \in M$, S_p has at least $d - \tilde q + 1$ strictly positive eigenvalues. Suppose that there is a (possibly empty) closed subset $F \subset M$ and a Hermitian metric g_0 on M such that on F, the Hermitian form S is strictly $\tilde q$-positive with respect to g_0. Then there is a Hermitian metric g on E such that S is strictly $\tilde q$-positive with respect to g, and in a neighborhood of F we have $g = g_0$.*

As already noted, the Morrey–Kohn–Hörmander basic estimate (1-1) proven in Theorem 1.2 suffices to prove that the L^2 cohomology $H^{p,q}_{L^2}(\Omega)$ of Ω is finite-dimensional. The main goal of this paper is to provide sufficient conditions for the *vanishing* of this cohomology, in terms of global information about the embedding of Ω in the ambient manifold M. This global structure is provided by a weight function which is compatible with the Levi form of Ω in a sense which is made precise by the following theorem. We use $T^{1,0}(\partial\Omega)$ to denote the $(1,0)$-tangent bundle of the boundary of Ω.

Theorem 1.4. *Let M be an n-dimensional complex manifold, let $\Omega \subseteq M$ be a smoothly bounded relatively compact domain, and let $1 \le q \le n-1$. Suppose that there is a smooth function φ defined in a neighborhood of $\overline{\Omega}$ such that:*

Either

(1) *there is a continuous subbundle of rank $(n-q)$ of $T^{1,0}(\partial\Omega)$ on which both the Levi form of $\partial\Omega$ and the complex Hessian of φ are positive, and*
(2) *the complex Hessian of φ has at least $(n-q+1)$ positive eigenvalues at each point of $\overline{\Omega}$.*

Or

(1) *there is a continuous subbundle of rank $(q+1)$ of $T^{1,0}(\partial\Omega)$ on which both the Levi form of $\partial\Omega$ and the complex Hessian of φ are negative*,

(2) *the complex Hessian of φ has at least $(q+1)$ negative eigenvalues at each point of $\overline{\Omega}$, and*

(3) *the restriction of the complex Hessian of φ to $T^{1,0}(\partial\Omega)$ is nondegenerate at each point of $\partial\Omega$.*

Then the L^2-cohomology $H^{p,q}_{L^2}(\Omega)$ of Ω in degree (p,q) vanishes for $0 \leq p \leq n$.

Notice that the hypotheses of Theorem 1.4 imply not only that the domain Ω satisfies condition $Z(q)$, but also that of the two mutually exclusive conditions constituting $Z(q)$ (that the Levi form has $n-q$ positive or $q+1$ negative eigenvalues) there is exactly one which is satisfied at each point of the boundary. It would be interesting to understand how to extend to general $Z(q)$-domains the techniques traditionally applied to annuli in $\mathbb{C}^n$ or Stein manifolds (see [Sha10; Sha11; LS13; CH21]).

In Theorem 1.4, the continuity of the subbundle of common positive directions of the two Hermitian forms may be a strong hypothesis. Indeed, in Corollary 8.2, we will see that there exists an example of a smoothly parameterized family of Hermitian forms such that each form admits a positive eigenvalue but there does not exist a continuously parameterized vector field on which each Hermitian form is positive. Fortunately, we can prove our result when $q = n-1$ without requiring continuity of the subbundle. Indeed, we have the following.

Theorem 1.5. *Let M be an n-dimensional complex manifold and let $\Omega \subseteq M$ be a smoothly bounded relatively compact domain. Suppose that there exists a smooth function φ defined in a neighborhood of $\overline{\Omega}$ such that*

(1) *for every point $p \in \partial\Omega$, there exists a vector $L \in T^{1,0}_p(\partial\Omega)$ on which both the Levi form of $\partial\Omega$ and the complex Hessian of φ are positive*, *and*

(2) *the complex Hessian of φ has at least 2 positive eigenvalues at each point of $\overline{\Omega}$.*

Then the L^2 cohomology $H^{p,n-1}_{L^2}(\Omega)$ of Ω in degree $(p,n-1)$ vanishes for $0 \leq p \leq n$.

Notice that in Theorems 1.2, 1.4 and 1.5, the hypotheses are independent of a choice of metric, i.e., they are determined solely by the complex structure. In particular, the L^2 spaces of forms on Ω are defined with respect to *any* Hermitian metric on M. Although the inner product and norm of $L^2_{p,q}(\Omega)$ depend on the metric chosen on M, the space $L^2_{p,q}(\Omega)$ itself is determined independently of the choice of the Hermitian metric on M. It follows that the "maximally realized" $\bar{\partial}$-operator $\bar{\partial} : L^2_{p,q-1}(\Omega) \to L^2_{p,q}(\Omega)$ is also defined independently of the choice of the metric, as is the corresponding L^2-cohomology $H^{p,q}_{L^2}(\Omega)$ of the domain Ω.

2. Definitions and preliminaries

2.1. ***Hermitian forms and eigenvalues.*** Recall that a *Hermitian form* on a complex vector space E is a map $H : E \times E \to \mathbb{C}$ such that for $u, v, w \in E$ and $a \in \mathbb{C}$ we have $H(au + v, w) = aH(u, w) + H(v, w)$, and $H(u, v) = \overline{H(v, u)}$. The Hermitian form H is a *Hermitian metric* or *inner product* if it is *positive definite*: $H(v, v) > 0$. According to *Sylvester's law of inertia*, we can write

$$H(z, z) = \sum_{j=1}^{d} \epsilon_j |\ell_j(z)|^2,$$

where $\epsilon_j \in \{1, -1, 0\}$, $\ell_j : E \to \mathbb{C}$ are linearly independent linear forms, and d is the dimension of E. The 3-tuple of integers counting respectively the number of positive, negative and zero coefficients among the ϵ_j's is known as the *inertia* or *signature* of the form and is an invariant completely classifying Hermitian forms on E up to linear automorphisms. By a standard abuse of language we will refer to the integers constituting the inertia as the number of positive, negative and zero *eigenvalues* of H.

Given a Hermitian metric g on E, and a Hermitian form H on E, recall that $v \in E$ is an *eigenvector of H with eigenvalue $\lambda \in \mathbb{R}$ with respect to g* if $H(v, u) = \lambda g(v, u)$ for all $u \in E$. Equivalently, we may define an operator $H^g : E \to E$ by

$$g(H^g u, v) = H(u, v) \qquad \text{for all } u, v \in E,$$

and the eigenpairs of the Hermitian form H with respect to g will correspond to the eigenpairs of the operator H^g. It is clear that $H^g : E \to E$ is a Hermitian operator (with respect to the metric g), and consequently, its eigenvalues are real, and the eigenvectors can be taken to be orthogonal with respect to the metric. We will denote by $\lambda_j^g(H)$ the j-th smallest eigenvalue of H^g, where eigenvalues are counted with multiplicity. Therefore

$$\lambda_1^g(H) \le \lambda_2^g(H) \le \cdots \le \lambda_d^g(H). \tag{2-1}$$

From the spectral theorem for each nonzero vector $z \in E$ we have

$$\lambda_1^g(H) \le \frac{H(z, z)}{g(z, z)} \le \lambda_d^g(H), \tag{2-2}$$

with equality if z is an eigenvector of the corresponding eigenvalues. This follows by writing $H(z, z) = g(H^g z, z)$ in terms of a basis of orthonormal eigenvectors of H^g. Notice that a Hermitian form H is strictly q-positive with respect to the metric g if and only if

$$\sum_{k=1}^{q} \lambda_k^g(H) > 0, \tag{2-3}$$

i.e., the sum of the *smallest* q eigenvalues is positive.

A Hermitian form (resp., metric) on a complex vector bundle on a smooth manifold is the assignment of a Hermitian form (resp., metric) to each fiber. Given a Hermitian form H and a Hermitian metric g on a vector bundle, we say that H is (strictly) q-positive with respect to g if it is so on each fiber with respect to the metric on that fiber.

Given a Hermitian form H and a metric g on a vector space V, the trace of H with respect to g is the sum of all the eigenvalues of H with respect to g:

$$\mathrm{tr}_g(H) = \sum_{k=1}^{\dim V} \lambda_k^g(H).$$

It is well-known that

$$\mathrm{tr}_g(H) = \sum_{k=1}^{\dim V} H(t_k, t_k)$$

for each orthonormal basis $\{t_k\}$ of V. The following characterization of strict q-positivity is classical and is a consequence of the Schur majorization theorem [HJ13, Theorem 4.3.45; IIM87] (see also [Str10, Lemma 4.7])).

Theorem 2.1. *A Hermitian form on a finite-dimensional inner-product space is strictly q-positive if and only if its restriction to each q-dimensional linear subspace has positive trace.*

2.2. ***The Levi form.*** It is possible to define the Levi form of a hypersurface fully intrinsically, with values in the line bundle of bad directions (see [BHLN20]). For our purposes, it will suffice to use a definition of the Levi form in terms of a defining function.

Let Ω be a smoothly bounded and relatively compact domain in a complex manifold M, and let ρ be a defining function of Ω, i.e., ρ is a smooth function in a neighborhood U of $\partial\Omega$ such that $\{\rho < 0\} = U \cap \Omega$ and $d\rho \neq 0$ on $\partial\Omega$. One then defines the Levi form of $\partial\Omega$ as the Hermitian form on $T^{1,0}(\partial\Omega)$ given by

$$\mathcal{L}_\rho(X, Y) = \partial\bar\partial\rho(X, \bar{Y}), \quad X, Y \in T_p^{1,0}(\partial\Omega). \tag{2-4}$$

The defining function of the domain Ω is not unique, but if r is another defining function, then there is a smooth function $f > 0$ defined near $\partial\Omega$ such that $\rho = f \cdot r$. It then easily follows that

$$\mathcal{L}_\rho = f\mathcal{L}_r,$$

so that at each $p \in \partial\Omega$, for $X \in T_p^{1,0}(\partial\Omega)$ the real number $\mathcal{L}_\rho(X, X)$ is positive (resp., negative, resp., zero) if and only if $\mathcal{L}_r(X, X)$ is positive (resp., negative, resp., zero). We see therefore that the conditions on the Levi form in the hypothesis of Theorem 1.4 are invariantly defined independently of the choice of the defining function.

In Theorem 3.3 below, we are given in addition to the domain Ω, a Hermitian metric g on the manifold M. Since $\mathcal{L}_\rho = f\mathcal{L}_r$, $\mathcal{L}_\rho$ and $\mathcal{L}_r$ have the same collection of eigenvectors in $T_p^{1,0}(\partial\Omega)$ with respect to g, and if $\{\lambda_j(p)\}$ are the eigenvalues of $\mathcal{L}_\rho(p)$, then the eigenvalues of $\mathcal{L}_r(p)$ are clearly $\{f(p)\cdot\lambda_j(p)\}$. Since $f>0$, it follows that the condition of strict q-positivity in the hypotheses of Theorems 3.3 and 3.1 are invariantly defined independently of the choice of defining function.

3. Some results from L^2-theory

3.1. ***Unweighted L^2 result.*** The following will be needed in the proof of Theorem 1.2 and is proven in Section 3.4.

Theorem 3.1. *Let M be a complex manifold, $\Omega \subseteq M$ a relatively compact C^3 domain with defining function ρ, $0 \le p \le n$, and $1 \le q \le n-1$. Let h be a Hermitian metric on M such that on each connected component of $\partial\Omega$, either the Levi form, $\mathcal{L}_\rho^h$ is strictly q-positive or the negative of the Levi form $-\mathcal{L}_\rho^h$ is strictly $(n-q-1)$-positive. Then the $\bar\partial$ operator on (p,q)-forms on Ω satisfies the Morrey–Kohn–Hörmander basic estimate (1-1).*

Remark 3.2. It is well-known that $C^2_{p,q}(\bar\Omega)\cap \mathrm{Dom}(\bar\partial^*)$ is dense in $\mathrm{Dom}(\bar\partial)\cap \mathrm{Dom}(\bar\partial^*)$ (see, e.g., [Hör65, p. 121]) so it suffices to assume that our forms have coefficients that are at least C^2 smooth on $\bar\Omega$. Further, by the comments at the end of Section 2.2, the fact that $\mathcal{L}_\rho^h$ is strictly q-positive is independent of the choice of the defining function ρ, so the hypotheses do not depend on the choice of ρ.

The function theory in [CS01, Section 5.3] is largely applicable to our setup, but we have to adapt their work to domains that are not necessarily pseudoconvex. Our analysis combines elements of [CS01] with ideas from [Ho95] and [HR15]. The latter papers, however, were concerned with domains in $\mathbb{C}^n$ and Stein manifolds, respectively. See also [Zam08, §1.9] for a related discussion but for domains in $\mathbb{C}^n$, and [MV15, Theorem 2.14] for a similar result in Kähler manifolds.

3.2. ***Weighted L^2 result.*** The following is the key result that will be used to prove Theorem 1.4 and its proof appears in Section 3.5.

Theorem 3.3. *Let (M,h) be a Hermitian manifold of complex dimension n, let $\Omega \subseteq M$ be a relatively compact domain with C^3 boundary $\partial\Omega$ and defining function ρ, let φ be a smooth function in a neighborhood of $\overline{\Omega}$ and let $1 \le q \le n-1$. Suppose that, with respect to h:*

Either

(1) *$\mathcal{L}_\rho^h$ is q-positive on $\partial\Omega$, and*

(2) *on $\overline{\Omega}$, the complex Hessian of φ is strictly q-positive.*

Or

(1) $-\mathcal{L}^h_\rho$ *is* $(n-q-1)$*-positive on* $\partial\Omega$,

(2) *on* $\overline{\Omega}$, *the complex Hessian of* $-\varphi$ *is strictly* $(n-q)$*-positive*, *and*

(3) *if* $L_n \in T^{1,0}(M)$ *is orthogonal to* $T^{1,0}(\partial\Omega)$ *and unit length on* $\partial\Omega$ *with respect to* h, *then the sum of any* $(n-q)$ *eigenvalues of the complex Hessian of* $-\varphi$ *is strictly larger than* $-\partial\bar\partial\varphi(L_n, \bar L_n)$ *on* $\partial\Omega$.

Then, *for each* $0 \le p \le n$ *and* t *sufficiently large*, *there exists a constant* $C > 0$ *so that for all* (p, q)*-forms* $f \in \mathrm{Dom}(\bar\partial) \cap \mathrm{Dom}(\bar\partial^*)$

$$\|f\|^2_{L^2(\Omega,e^{-t\varphi})} \le C\big(\|\bar\partial f\|^2_{L^2(\Omega,e^{-t\varphi})} + \|\bar\partial^*_{t\varphi} f\|^2_{L^2(\Omega,e^{-t\varphi})}\big). \tag{3-1}$$

3.3. ***Preliminaries for L^2 methods.*** Suppose that φ is a smooth function defined on a neighborhood of $\overline{\Omega}$. Let $L^2_{p,q}(\Omega, t\varphi)$ denote the L^2-space of (p, q)-forms on Ω with norm

$$\|f\|^2_{L^2(\Omega,t\varphi)} = \int_\Omega |f|^2 e^{-t\varphi}\, dV$$

and corresponding inner product. Denote the L^2-adjoint of $\bar\partial : L^2_{p,q}(\Omega, t\varphi) \to L^2_{p,q+1}(\Omega, t\varphi)$ by $\bar\partial^*_{t\varphi} : L^2_{p,q+1}(\Omega, t\varphi) \to L^2_{p,q}(\Omega, t\varphi)$. It is well-known that $\mathrm{Dom}(\bar\partial^*_{t\varphi})$ does not depend on $t\varphi$.

We assume that ρ is a defining function for Ω so that $|d\rho| = 1$ on $\partial\Omega$. Let U be a open set that intersects $\bar\Omega$ and admits a basis of orthonormal (0, 1)-forms $\omega_1, \dots, \omega_n$ so that $\omega_n = \partial\rho$, and

$$\partial\bar\partial\rho = \overline{\bar\partial\omega_n} = \sum_{j,k=1}^{n} \rho_{jk}\, \omega_j \wedge \bar\omega_k \tag{3-2}$$

where (ρ_{jk}) is the Levi matrix. We denote by $\bar L_1, \dots, \bar L_n$ the basis for $T^{0,1}(U)$ that is dual to $\bar\omega_1, \dots, \bar\omega_n$. For each $1 \le j \le n$, set $\delta^{t\varphi}_j = e^{t\varphi} L_j e^{-t\varphi}$, so that the adjoint of $\bar L_j$ with respect to $L^2(\Omega, e^{-t\varphi})$ is given by $-\delta^t_j$ up to a zero-order term that is independent of t and φ.

Let $\mathcal{I}_q = \{J = (j_1, \dots, j_q) \in \mathbb{N}^q : 1 \le j_1 < \cdots < j_q \le n\}$ denote the set of increasing q-tuples. We can then express a $(0, q)$ form f on U by

$$f = \sum_{J\in\mathcal{I}_q} f_J\, \bar\omega^J.$$

Given $I \in \mathcal{I}_{q-1}$ and $J \in \mathcal{I}_q$, we define ϵ^{jI}_J to be 0 if $\{j\} \cup I \ne J$ as sets and otherwise is the sign of the permutation that reorders jI as J. We also employ the standard notation

$$f_{jI} = \sum_{J\in\mathcal{I}_q} \epsilon^{jI}_J f_J$$

and use the shorthand $\|\bar{L}f\|_{L^2(\Omega,e^{-\varphi})}$ for

$$\|\bar{L}f\|^2_{L^2(\Omega,e^{-\varphi})} = \sum_{J\in\mathcal{I}_q}\sum_{k=1}^{n}\|\bar{L}_k f_J\|^2_{L^2(\Omega,e^{-\varphi})}.$$

The key to proving Theorem 3.1 (and Theorem 3.3) is to have a good *basic identity*, and this is provided by [CS01, equation (5.3.20)], which we now state. For forms $f\in\operatorname{Dom}(\bar\partial^*_{t\varphi})\cap C^2_{0,q}(\bar\Omega)$ supported in U, where U is an open set that admits good local coordinates,

$$\begin{aligned}\text{(3-3)}\quad &\|\bar\partial f\|^2_{L^2(\Omega,e^{-t\varphi})} + \|\bar\partial^*_{t\varphi} f\|^2_{L^2(\Omega,e^{-t\varphi})} \\ &= \|\bar{L}f\|^2_{L^2(\Omega,e^{-t\varphi})} + t\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n}(\varphi_{jk}f_{jI}, f_{kI})_\varphi \\ &\quad + \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n}\int_{\partial\Omega\cap U}\rho_{jk}f_{jI}\bar{4}0 f_{kI}e^{-t\varphi}\,d\sigma + R(f) + E(f),\end{aligned}$$

where $E(f)$ satisfies both

$$\text{(3-4)}\qquad |E(f)| \le C\|\bar{L}f\|_{L^2(\Omega,e^{-t\varphi})}\|f\|_{L^2(\Omega,e^{-t\varphi})}$$

and via integration by parts of the tangential terms,

$$\begin{aligned}\text{(3-5)}\quad &|E(f)| \le \\ &C\Big(\sum_{J\in\mathcal{I}_q}\|\bar{L}_n f_J\|_{L^2(\Omega,e^{-t\varphi})} + \sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\delta^{t\varphi}_j f_J\|_{L^2(\Omega,e^{-t\varphi})}\Big)\|f\|_{L^2(\Omega,e^{-t\varphi})}\end{aligned}$$

and for any $\epsilon>0$, there exists $C_\epsilon>0$ such that

$$\text{(3-6)}\qquad |R(f)| \le \epsilon\big(\|\bar\partial f\|^2_{L^2(\Omega,e^{-t\varphi})} + \|\bar\partial^*_\varphi f\|^2_{L^2(\Omega,e^{-t\varphi})}\big) + C_\epsilon\|f\|^2_{L^2(\Omega,e^{-t\varphi})}.$$

Additionally, the constants C and C_ϵ are independent of t and φ. The basic identity (3-3) is precisely the result we need to prove Theorems 3.3 and 3.1 for the case when $\mathcal{L}^h_\rho$ is q-positive on $U\cap\partial\Omega$.

We now establish a basic identity for the components of $\partial\Omega$ for which $-\mathcal{L}^h_\rho$ is $n-1-q$-positive. Following [CS01], we have

$$[\delta^{t\varphi}_k, \bar{L}_k]u = \phi_{kk}u + \sum_{\ell=1}^{n}c^\ell_{kk}\delta^{t\varphi}_\ell(u) - \sum_{\ell=1}^{n}\bar{c}^\ell_{kk}\bar{L}_\ell(u)$$

where $\rho_{kk}=c^n_{kk}=\bar{c}^n_{kk}$, and c^ℓ_{kk} are C^1 functions on U. Integration by parts yields

$$\begin{aligned}\text{(3-7)}\quad \|\bar{L}_k f_J\|^2_{L^2(\Omega,e^{-t\varphi})} &= \|\delta^{t\varphi}_k f_J\|^2_{L^2(\Omega,e^{-t\varphi})} - t(\varphi_{kk}f_J, f_J)_{t\varphi} \\ &\quad - \Big(\sum_{\ell=1}^{n}c^\ell_{kk}\delta^{t\varphi}_\ell f_J, f_J\Big)_{t\varphi} + O(\|\bar{L}f\|_{L^2(\Omega,e^{-t\varphi})}\|f\|_{L^2(\Omega,e^{-t\varphi})}).\end{aligned}$$

Integration by parts shows that if $\ell < n$, then

$$\left|(c^\ell_{kk}\delta^{t\varphi}_\ell f_J, f_J)_{t\varphi}\right| \le C\|\bar L f\|_{L^2(\Omega,e^{-t\varphi})}\|f\|_{L^2(\Omega,e^{-t\varphi})}.$$

However, if $\ell = n$, then

$$(c^n_{kk}\delta^{t\varphi}_n f_J, f_J)_{t\varphi} = \int_{\partial\Omega\cap U}\rho_{jk}|f_J|^2 e^{-t\varphi}\,d\sigma + O(\|\bar L f\|_{L^2(\Omega,e^{-t\varphi})}\|f\|_{L^2(\Omega,e^{-t\varphi})}).$$

Thus, we have now established [Sha85, (3.20)] but for forms $f \in \mathrm{Dom}(\bar\partial)\cap\mathrm{Dom}(\bar\partial^*)$ supported in a suitably small neighborhood in a complex manifold (vs. in $\mathbb{C}^n$), namely,

$$\text{(3-8)}\quad \|\bar\partial f\|^2_{L^2(\Omega,e^{-t\varphi})} + \|\bar\partial^*_{t\varphi} f\|^2_{L^2(\Omega,e^{-t\varphi})}
= \sum_{J\in\mathcal{I}_q}\|\bar L_n f_J\|^2_{L^2(\Omega,e^{-t\varphi})} + \sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\delta^{t\varphi}_j f_J\|^2_{L^2(\Omega,e^{-t\varphi})}
+ t\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n}(\varphi_{jk}f_{jI}, f_{kI})_{t\varphi} - t\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}(\varphi_{jj}f_J, f_J)_{t\varphi}
+ \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n}\int_{\partial\Omega}\rho_{jk}f_{jI}\overline{f_{kI}}e^{-t\varphi}\,d\sigma - \sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\int_{\partial\Omega}\rho_{jj}|f_J|^2e^{-t\varphi}\,d\sigma
+ E(f) + R(f).$$

Notice that our coordinates are such that $\sum_{j=1}^{n-1}\rho_{jj} = \mathrm{Tr}(\mathcal{L}^h_\rho)$.

3.4. ***The proof of Theorem 3.1.*** We use the basic identities (3-3) and (3-8) with $\varphi = 0$. In the case that $\mathcal{L}^h_\rho$ is strictly q-positive on $U\cap\partial\Omega$, it follows from standard multilinear algebra (see, e.g., [Str10, Lemma 4.7]) that there exists $c > 0$ so that for $f \in C^2_{p,q}(\bar\Omega)\cap\mathrm{Dom}(\bar\partial^*)$ supported in U, (3-3) becomes

$$\|\bar\partial f\|^2_{L^2(\Omega)} + \|\bar\partial^* f\|^2_{L^2(\Omega)} \ge \|\bar L f\|^2_{L^2(\Omega)} + c\|f\|^2_{L^2(\partial\Omega)} + R(f) + E(f).$$

A small constant/large constant argument and an absorption of terms establishes the Morrey–Kohn–Hörmander basic estimate for f supported on U when $\mathcal{L}^h_\rho$ is strictly q-positive on $U\cap\partial\Omega$.

Now suppose that f is supported in a neighborhood U so that $-\mathcal{L}^h_\rho$ is $(n-1-q)$-positive on $\partial\Omega\cap U$. In this case, (3-8) with $\varphi = 0$ becomes

$$\text{(3-9)}\quad \|\bar\partial f\|^2_{L^2(\Omega)} + \|\bar\partial^* f\|^2_{L^2(\Omega)}
= \sum_{J\in\mathcal{I}_q}\|\bar L_n f_J\|^2_{L^2(\Omega)} + \sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|L_j f_J\|^2_{L^2(\Omega)} + \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n}\int_{\partial\Omega}\rho_{jk}f_{jI}\overline{f_{kI}}\,d\sigma
- \sum_{J\in\mathcal{I}_q}\int_{\partial\Omega}\mathrm{Tr}(\mathcal{L})|f_J|^2\,d\sigma + E(f) + R(f)$$

For the boundary integral, we note that

$$\lambda_1^h + \cdots + \lambda_q^h - (\lambda_1^h + \cdots \lambda_{n-1}^h) = -(\lambda_{q+1}^h + \cdots + \lambda_{n-1}^h) > 0 \tag{3-10}$$

since $-\mathcal{L}_\rho^h$ is $(n-1-q)$-positive. This means that a small constant/large constant argument and an absorption of the $E(f)$ and $R(f)$ terms shows that the Morrey–Kohn–Hörmander basic estimate holds in this regime as well.

A partition of unity argument shows that the Morrey–Kohn–Hörmander basic estimate holds globally and Theorem 3.1 is proved.

3.5. ***The proof of Theorem 3.3.*** We first consider the case in which the Levi form of the boundary is q-positive and the complex Hessian of φ is strictly q-positive. We cover $\bar{\Omega}$ with good neighborhoods $\{U_k\}$ and use a partition of unity subordinate to $\{U_k\}$ such that (3-3) applies on each U_k. As in the proof of Theorem 3.1, we may use, e.g., [Str10, Lemma 4.7] to show that the boundary integral term in (3-3) is nonnegative. We then use a small constant/large constant argument with (3-4) to absorb $\|\bar{L}f\|_{L^2(\Omega,e^{-t\varphi})}$ and take ϵ small enough to absorb the $\bar{\partial}$ and $\bar{\partial}^*_{t\varphi}$ terms in (3-6). Finally, we use the strict q-positivity of the complex Hessian of φ and take t large enough to absorb the remaining error terms. Consequently, there exists $C > 0$ such that (3-1) follows.

Suppose, on the other hand, that $-\mathcal{L}_\rho^h$ is $(n-q-1)$-positive and the complex Hessian of $-\varphi$ is strictly $(n-q)$-positive. Choose a neighborhood U_k on which (3-8) holds. Using a small constant/large constant argument with (3-5) to absorb terms with derivatives in (3-8) and similar estimates to absorb $R(f)$ gives us

$$\begin{aligned} \|\bar{\partial}f\|^2_{L^2(\Omega,e^{-t\varphi})} &+ \|\bar{\partial}^*_{t\varphi}f\|^2_{L^2(\Omega,e^{-t\varphi})} \\ &\geq Ct \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n} (\varphi_{jk} f_{jI}, f_{kI})_{t\varphi} - Ct \sum_{J\in\mathcal{I}_q} \sum_{j=1}^{n-1} (\varphi_{jj} f_J, f_J)_{t\varphi} \\ &\quad + C \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n} \int_{\partial\Omega} \rho_{jk} f_{jI} \overline{f_{kI}} e^{-t\varphi}\, d\sigma \\ &\quad - C \sum_{J\in\mathcal{I}_q} \sum_{j=1}^{n-1} \int_{\partial\Omega} \rho_{jj} |f_J|^2 e^{-t\varphi}\, d\sigma - C\|f\|^2_{L^2(\Omega,e^{-t\varphi})}, \end{aligned} \tag{3-11}$$

where $C > 0$ is independent of f and t. As before, (3-10) implies that the boundary integral is nonnegative. Denote the eigenvalues of the complex Hessian of φ with respect to h in nondecreasing order by $\{\mu_1^h, \ldots, \mu_n^h\}$. By the usual multilinear algebra,

$$\sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n} (\varphi_{jk} f_{jI}, f_{kI})_{t\varphi} - \sum_{J\in\mathcal{I}_q} \sum_{j=1}^{n-1} (\varphi_{jj} f_J, f_J)_{t\varphi} \geq \epsilon \|f\|^2_{L^2(\Omega,e^{-t\varphi})}$$

for some $\epsilon > 0$ if

$$\sum_{j=1}^{q} \mu_j^h - \sum_{j=1}^{n} \mu_j^h + \varphi_{nn} = -\sum_{j=q+1}^{n} \mu_j^h + \varphi_{nn}$$

is strictly positive on some interior neighborhood of the boundary. Our hypotheses guarantee that this holds on $\partial\Omega$, so it must hold on some neighborhood of $\partial\Omega$, and hence we may choose t sufficiently large in (3-11) to obtain (3-1) for forms supported on some interior neighborhood of the boundary.

On any neighborhood U_k which does not intersect $\partial\Omega$, we may integrate by parts in $\{L_j\}_{1\le j\le n}$ for any orthonormal basis for $T^{1,0}(\partial\Omega)$. Following the same procedure used to obtain (3-11), we obtain

$$\|\bar{\partial} f\|^2_{L^2(\Omega,e^{-t\varphi})} + \|\bar{\partial}^*_{t\varphi} f\|^2_{L^2(\Omega,e^{-t\varphi})} \\ \ge Ct \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n} (\varphi_{jk} f_{jI}, f_{kI})_{t\varphi} - Ct \sum_{J\in\mathcal{I}_q} \sum_{j=1}^{n} (\varphi_{jj} f_J, f_J)_{t\varphi} - C\|f\|^2_{L^2(\Omega,e^{-t\varphi})},$$

for any form supported in U_k. Here, it suffices to observe that the complex Hessian of $-\varphi$ is $(n-q)$-positive to obtain (3-1). Increasing the size of t as needed to accommodate error terms arising from our partition of unity, we obtain (3-1) globally.

4. Constructing a metric for a single Hermitian form

4.1. ***Analyticity of spectral projections.*** Let n be a positive integer and let $\mathscr{H}_n$ denote the real vector space of $n \times n$ Hermitian matrices. For a matrix $T \in \mathscr{H}_n$, let $\sigma(T) \subset \mathbb{R}$ be its spectrum, i.e., the set of eigenvalues, and for a subset $G \subset \mathbb{C}$ denote by $\pi_G(T)$ the matrix of the orthogonal projection from $\mathbb{C}^n$ onto the direct sum of the eigenspaces of T corresponding to eigenvalues in the set G. We have

$$(\pi_G(T))^2 = \pi_G(T), \quad (\pi_G(T))^* = \pi_G(T)$$

and

$$\operatorname{range}(\pi_G(T)) = \bigoplus_{\lambda\in G\cap\sigma(T)} \{x \in \mathbb{C}^n : Tx = \lambda x\}.$$

If $G \cap \sigma(T) = \varnothing$ we let $\pi_G(T) = 0$, consistent with the convention that an empty internal direct sum of vector subspaces is the zero subspace.

Proposition 4.1. *Let $G \subset \mathbb{C}$ be a smoothly bounded open subset of the complex plane, and let $\mathscr{W}_G \subset \mathscr{H}_n$ be the set of $n \times n$ Hermitian matrices which have no eigenvalues on the boundary ∂G:*

$$\mathscr{W}_G = \{T \in \mathscr{H}_n : \sigma(T) \cap \partial G = \varnothing\}.$$

Then the mapping

$$\pi_G : \mathscr{W}_G \to \mathrm{Mat}_{n\times n}(\mathbb{C}) \tag{4-1}$$

is real analytic.

Proof. This is a consequence of the following classical representation of the matrix $\pi_G(T)$ as a contour integral (see [Kat76]):

$$\pi_G(T) = \frac{1}{2\pi i}\int_{\partial G}(\zeta I - T)^{-1} d\zeta, \tag{4-2}$$

interpreted thus:

(a) $\zeta \mapsto (\zeta I - T)^{-1} \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is a matrix valued holomorphic function defined in a neighborhood of ∂G. What we mean by this is that each entry of the output matrix is holomorphic in ζ, something that is obvious from Cramer's rule and the fact that for each $\zeta \in \partial G$, the matrix $\zeta I - T$ is invertible (by the hypothesis that $T \in \mathscr{W}_G$).

(b) The integral is a line integral, taken entrywise. The contour ∂G is given the standard orientation coming from the orientation of G as an open subset of $\mathbb{C}$.

It then follows that the mapping (4-1) is real analytic. Indeed, if we allow T to have complex entries, differentiation under the integral sign in formula (4-2) shows that the mapping (4-1) extends to a holomorphic mapping of matrices.

To complete the proof, we need to establish the formula (4-2). Fix $T \in \mathscr{W}_G \subset \mathscr{H}_n$, and note that $\zeta I - T$ is an invertible matrix provided $\zeta \notin \sigma(T)$. This means that if G does not contain any eigenvalue of T, then the function $\zeta \mapsto (\zeta I - T)^{-1}$ is holomorphic in a neighborhood of $\overline{G}$, i.e., each entry of this matrix-valued function is holomorphic in ζ. Therefore by Cauchy's theorem, in this case we have $\pi_G(T) = 0$, as needed.

Now consider the situation where $\sigma(T) = \{\lambda_1, \dots, \lambda_n\}$ and for some $1 \le k \le n$ we have $\lambda_1, \dots \lambda_k \in G$ and the rest of the eigenvalues $\lambda_{k+1}, \dots, \lambda_n$ lie outside $\overline{G}$. Let $\{v_1, \dots, v_n\}$ be an orthonormal set of eigenvectors corresponding to these eigenvalues, i.e., $Tv_j = \lambda_j v_j$, $\|v_j\| = 1$ for each j and $\langle v_j, v_k\rangle = \delta_{jk}$ for $1 \le j, k \le n$. Let U be the unitary automorphism of $\mathbb{C}^n$ defined by $Ue_j = v_j$, where $\{e_j, 1 \le j \le n\}$ is the standard basis of $\mathbb{C}^n$. Then it is clear that to prove the result, it suffices to prove that if P is the matrix defined by the integral on the right hand side of (4-2) then $U^{-1}PU$ coincides with the matrix of the orthogonal projection onto the subspace spanned by the vectors $\{e_1, \dots, e_k\}$, i.e.,

$$U^{-1}PU = \begin{pmatrix} I_{k\times k} & 0 \\ 0 & 0_{n-k\times n-k} \end{pmatrix}.$$

Observe that

$$U^{-1}(\zeta I - T)^{-1}U = \left[U^{-1}(\zeta I - T)U\right]^{-1}$$
$$= (\zeta I - U^{-1}TU)^{-1} = \operatorname{diag}\Big(\frac{1}{\zeta - \lambda_1}, \dots, \frac{1}{\zeta - \lambda_n}\Big),$$

where we use the fact that $U^{-1}TU = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. By the linearity of the integral we have

$$\begin{aligned} U^{-1}PU &= \frac{1}{2\pi i}\int_{\partial G} U^{-1}(\zeta I - T)^{-1}U d\zeta \\ &= \frac{1}{2\pi i}\int_{\partial G} \operatorname{diag}\Big(\frac{1}{\zeta - \lambda_1}, \dots, \frac{1}{\zeta - \lambda_n}\Big) d\zeta \\ &= \operatorname{diag}\Big(\frac{1}{2\pi i}\int_{\partial G}\frac{d\zeta}{\zeta - \lambda_1}, \dots, \frac{1}{2\pi i}\int_{\partial G}\frac{d\zeta}{\zeta - \lambda_n}\Big) \\ &= \begin{pmatrix} I_{k\times k} & 0 \\ 0 & 0_{n-k\times n-k}\end{pmatrix}, \end{aligned}$$

since $\lambda_1, \dots, \lambda_k \in G$ and $\lambda_{k+1}, \dots \lambda_n \in \mathbb{C} \setminus \overline{G}$. □

4.2. ***Proof of Proposition 1.3.*** Let h be a Hermitian metric on E, and for $1 \le j \le d$, let $\lambda_j^h(p)$ denote the j-th smallest eigenvalue of the Hermitian form S_p with respect to the metric h (counting with multiplicity) so that

$$\lambda_1^h(p) \le \lambda_2^h(p) \le \dots \le \lambda_d^h(p).$$

Since the j-th smallest eigenvalue of a matrix depends continuously on its entries (see [Kat76]), it follows that the real-valued function $p \mapsto \lambda_j^h(p)$ is continuous on M for each j. Denote by $\nu_+(p)$ (resp., $\nu_-(p)$) the number of positive (resp., negative) eigenvalues of S_p, which is well-known not to depend on the metric h. By hypothesis $\nu_+(p) \ge d - \tilde{q} + 1$, so it follows that $\nu_-(p) \le \tilde{q} - 1$ and therefore for any metric h we have

$$\lambda_q^h(p) > 0, \quad p \in M.$$

For $0 \le r \le \tilde{q} - 1$ introduce the subsets

$$V_r = \{p \in M : \nu_-(p) \le r\} \cup F,$$

so that $V_{\tilde{q}-1} = M$, and each V_r is closed in M, since with respect to any metric h on E, we have $V_r = \{\lambda_{r+1}^h \ge 0\} \cup F$. We also let

$$U_r = V_r \setminus V_{r-1} = \{p \in M : \nu_-(p) = r\}$$

so that $V_r = \bigcup_{j=0}^r U_j$. We will construct the metric whose existence is claimed using an inductive process. Starting with a Hermitian metric g_{r-1} on E in the r-th stage (where $1 \le r \le \tilde{q}-1$), we modify g_{r-1} so that we obtain a metric g_r for which the form S is strictly $\tilde{q}$-positive with respect to g_r on the set V_r.

To start the inductive process (i.e., $r=0$), we take g_0 to be the metric given in the hypotheses on the bundle E. Then on V_0, since either all the eigenvalues are nonnegative and $\lambda_{\tilde{q}}^{g_0} > 0$, or we are in the set F where S is strictly $\tilde{q}$-positive, it follows that the sum of the smallest $\tilde{q}$ Levi eigenvalues of S with respect to g_0 is strictly positive, and therefore S is strictly $\tilde{q}$-positive with respect to g_0 on V_0.

We now come to the r-th stage, where $1 \le r \le \tilde{q}-1$. Therefore, there is already a metric g_{r-1} on E for which S is strictly $\tilde{q}$-positive on V_{r-1}, and on U_r, the eigenvalues with respect to g_{r-1} satisfy $\lambda_r^{g_{r-1}} < 0$, $\lambda_{r+1}^{g_{r-1}} \ge 0$ and $\lambda_{\tilde{q}}^{g_{r-1}} > 0$. We claim that there is a smooth real-valued function $f \in \mathcal{C}^\infty(V_r)$ such that $f \ge 0$, f vanishes in a neighborhood of V_{r-1}, and we have

$$\sum_{j=1}^{r} \lambda_j^{g_{r-1}} + (1+f)\cdot\Big(\sum_{j=r+1}^{\tilde{q}} \lambda_j^{g_{r-1}}\Big) > 0 \tag{4-3}$$

at each point of V_r.

By the induction hypothesis, on V_{r-1}, we have $\sum_{j=1}^{\tilde{q}} \lambda_j^{g_{r-1}} > 0$, and therefore by continuity there is a neighborhood $\widetilde{V}_{r-1}$ of V_{r-1} on V_r where we continue to have $\sum_{j=1}^{\tilde{q}} \lambda_j^{g_{r-1}} > 0$. Now consider the continuous function on $U_r = V_r \setminus V_{r-1}$ given by

$$\varphi = -\frac{\sum_{j=1}^{\tilde{q}} \lambda_j^{g_{r-1}}}{\sum_{j=r+1}^{\tilde{q}} \lambda_j^{g_{r-1}}}.$$

It is well-defined since each of the summands in the denominator is nonnegative and $\lambda_{\tilde{q}}^{g_{r-1}} > 0$ at each point of M. It follows further that $\varphi < 0$ on $\widetilde{V}_{r-1}$, since both the numerator and denominator are strictly positive there. It follows that there is a $\mathcal{C}^\infty$ function f on V_r such that $f \equiv 0$ in a neighborhood of V_{r-1} and $f > \varphi$ on U_r. Such an f clearly satisfies (4-3).

Recall that $S_p^{g_{r-1}}$ denotes the Hermitian operator on E_p which is Hermitian with respect to g_{r-1} and satisfies $g_{r_1}(S_p^{g_{r-1}}u, v) = S_p(u, v)$ for $u, v \in E_p$. For $p \in U_r$, let P_p^r denote the projection from the fiber E_p of E onto the direct sum of the eigenspaces of $S_p^{g_{r-1}}$ of the negative eigenvalues $\lambda_1^{g_{r-1}}(p), \ldots \lambda_r^{g_{r-1}}(p)$, where P_p^r is orthogonal with respect to the metric g_{r-1}. We claim that $p \mapsto P_p^r$ is smooth on U_r. More precisely, this map is a smooth section of the bundle $\operatorname{Hom}_{\mathbb{C}}(E, E)$. For any compact subset $K \subset U_r$, there are *negative* numbers α, β such that $\alpha < \lambda_1^{g_{r-1}}(p)$ and $\lambda_r^{g_{r-1}}(p) < \beta$ for each $p \in K$. Let $G \subset \mathbb{C}$ denote the open disc with center at the point $\frac{1}{2}(\alpha+\beta)$ on the negative real axis and passing through the points α and β. Then, if $p \in K$, the negative eigenvalues $\lambda_1^{g_{r-1}}, \ldots, \lambda_r^{g_{r-1}}$ of S_p with respect to

g_{r-1} lie in the open set G and the other eigenvalues lie in $\mathbb{C}\setminus\overline{G}$, so $S_p^{g_{r-1}}\in\mathscr{W}_G$, where $\mathscr{W}_G$ is as in the statement of Proposition 4.1. Continuing to use the notation of Proposition 4.1, we have the representation

$$P_p^r=\pi_G(S_p^{g_{r-1}})\quad\text{ for } p\in K.$$

Since the map $p\mapsto S_p^{g_{r-1}}$ is smooth and the map $T\mapsto\pi_G(T)$ is analytic, it follows that $p\mapsto P_p^r$ is smooth on K and therefore on all of U_r.

Now define a Hermitian metric g_r on $E|_{V_r}$ by setting, for $p\in V_r$ and $X,Y\in E_p$,

$$g_r(X,Y)=\begin{cases}g_{r-1}(X,Y)+f(p)\cdot g_{r-1}(P_p^rX,P_p^rY) & \text{if } p\in U_r,\\ g_{r-1}(X,Y) & \text{elsewhere on } V_r,\end{cases}\tag{4-4}$$

where $f\in\mathcal{C}^\infty(U_r)$ is as in (4-3). We can extend the smooth metric g_r from $E|_{V_r}$ from the closed set V_r to all of M to obtain a Hermitian metric on E, which we continue to denote by g_r.

To complete the induction, we will show that the sum of the smallest $\tilde{q}$ eigenvalues of S^{g_r} is strictly positive on V_r. Since on V_{r-1} we have $g_r=g_{r-1}$ we only need to show this on U_r. Let $p\in U_r$. We have by the definitions of the operators $S^{g_{r-1}}$ and S^{g_r} that

$$S_p(X,Y)=g_{r-1}(S_p^{g_{r-1}}(X),Y)=g_r(S_p^{g_r}(X),Y).$$

Let $\{L_j(p)\}_{j=1}^d$be a g_{r-1}-orthonormal set of eigenvectors of the operator $S_p^{g_{r-1}}$ acting on E_p, corresponding to the eigenvalues $\{\lambda_j^{g_{r-1}}(p)\}$,i.e., $S_p^{g_{r-1}}L_j(p)=\lambda_j^{g_{r-1}}(p)L_j(p)$. In general one cannot choose the sections $p\mapsto L_j(p)$ of E over U_r to be even continuous. The definition of the new metric g_r in (4-4) shows that

$$g_r(L_j(p),L_k(p))=\begin{cases}(1+f(p))\delta_{jk} & \text{if both } j,k\le r,\\ \delta_{jk} & \text{if not.}\end{cases}$$

Therefore for $p\in U_r$, the vectors

$$Z_j(p)=\begin{cases}\dfrac{1}{\sqrt{1+f(p)}}\cdot L_j(p) & \text{if } 1\le j\le r,\\ L_j(p) & \text{if } r+1\le j\le d,\end{cases}$$

form an orthonormal basis of E_p with respect to the new Hermitian metric g_r. Now

$$g_r(S_p^{g_r}(Z_j(p)),Z_k(p))=\begin{cases}0 & \text{if } j\neq k,\\ \dfrac{\lambda_j^{g_{r-1}}(p)}{1+f(p)} & \text{if } j=k \text{ and } 1\le j\le r,\\ \lambda_j^{g_{r-1}}(p) & \text{if } j=k \text{ and } r+1\le j\le d,\end{cases}$$

so that the basis $\{Z_j(p)\}$ diagonalizes $S_p^{g_r}$, and consequently, its eigenvalues with respect to the new metric g_r at the point p are (in ascending order)

$$\frac{\lambda_1^{g_{r-1}}(p)}{1+f(p)}, \ldots, \frac{\lambda_r^{g_{r-1}}(p)}{1+f(p)}, \lambda_{r+1}^{g_{r-1}}(p), \ldots, \lambda_d^{g_{r-1}}(p),$$

the first r of which are negative and the remaining are nonnegative, and the $\tilde{q}$-th one is known to be strictly positive. By the choice of the function f in (4-3), it follows that the sum of the first $\tilde{q}$ of these eigenvalues is strictly positive, and this completes the proof.

5. Proof of Theorem 1.2

By hypothesis, $\partial\Omega$ satisfies condition $Z(q)$, so at each point of $\partial\Omega$ one of two mutually exclusive conditions hold: either that (i) the Levi form has at least $n-q$ positive eigenvalues, or that (ii) the Levi form has at least $q+1$ negative eigenvalues. Noticing that the Levi form $\mathcal{L}_\rho$ taken with respect to a smooth defining function ρ is continuous on $\partial\Omega$, we see each of the conditions (i) and (ii) holds on an open set, so it follows that on each point of a connected component of $\partial\Omega$, exactly one of the conditions (i) or (ii) holds.

Fix a smooth defining function ρ for Ω. For a boundary component M_0 where $\mathcal{L}_\rho$ has at least $n-q$ positive eigenvalues, we apply Proposition 1.3 with

$$d = n-1, \quad E = T^{1,0}M_0, \quad S = \mathcal{L}, \quad \tilde{q} = q, \quad F = \text{the empty set.}$$

We obtain a metric on this boundary component, with respect to which the Levi form is strictly q-positive. For a boundary component M_0 where $\mathcal{L}_\rho$ has at least $q+1$ negative eigenvalues, $-\mathcal{L}_\rho$ has at least $q+1$ positive eigenvalues. We apply Proposition 1.3 with

$$d = n-1, \quad E = T^{1,0}M_0, \quad S = -\mathcal{L}_\rho, \quad \tilde{q} = n-q-1, \quad F = \text{the empty set.}$$

Therefore, we obtain a metric on this boundary component with respect to which $-\mathcal{L}_\rho$ is $(n-1-q)$-positive. Putting these together, we obtain a Hermitian metric on $T^{1,0}(\partial\Omega)$ and then extend it arbitrarily to the whole manifold M. Theorem 3.1 applies and the proof of Theorem 1.2 is complete.

6. Constructing a metric for multiple Hermitian forms

6.1. *A result from linear algebra.*

Proposition 6.1. *Let E be a finite-dimensional inner product space, let V and W be subspaces of E, and let π_V denote the orthogonal projection from E to V. Then* *for each orthonormal basis* $\{t_j\}$ *of the space* W *we have*

$$\sum_{k=1}^{\dim W} \|\pi_V t_k\|^2 = \dim(V \cap W).$$

Proof. Consider the Hermitian operator P on the inner product space W (with the induced inner product) characterized by the condition that

$$\langle Pz, w\rangle = \langle \pi_V z, w\rangle\,, \quad z, w \in W.$$

We claim that P coincides with the orthogonal projection from W onto $V \cap W$, where as before the inner product in the subspace W is that induced from E. To see this, if $z \in V \cap W$, then for each $w \in W$ we have $\langle Pz, w\rangle = \langle z, w\rangle$, so $Pz = z$. On the other hand if $\zeta \in V^\perp \cap W$ then we have $\langle P\zeta, w\rangle = \langle \pi_V \zeta, w\rangle = 0$ for each $w \in W$, so $P\zeta = 0$, establishing the claim.

We now compute the trace of P (as an operator on the inner product space W) in two ways. Since P is a projection from W onto $V \cap W$, in an appropriate basis of W the matrix of P is diagonal, with $\dim(V \cap W)$ diagonal entries each equal to 1 and the rest equal to zero. Therefore $\operatorname{tr}(P) = \dim(V \cap W)$. On the other hand, with respect to the orthonormal basis $\{t_1, t_2, \ldots, t_q\}$ of the space W, the matrix of the operator P is $(a_{jk})_{j,k=1}^q$, where

$$a_{jk} = \langle Pt_j, t_k\rangle = \langle \pi_V t_j, t_k\rangle = \langle \pi_V^2 t_j, t_k\rangle = \langle \pi_V t_j, \pi_V t_k\rangle,$$

where we have used the fact that π_V is idempotent and self-adjoint. Therefore

$$\operatorname{tr}(P) = \sum_{k=1}^{q} a_{kk} = \sum_{k=1}^{q} \langle \pi_V t_k, \pi_V t_k\rangle = \sum_{k=1}^{q} \|\pi_V t_k\|^2. \tag{6-1}$$

The result follows. □

Corollary 6.2. *Let E be an n-dimensional inner-product space, let $1 \le q \le n$ and let V be an $(n-q+1)$-dimensional subspace of E. Denote by π_V the orthogonal projection from E onto V, and let $t_1, \ldots, t_q$ be a collection of orthonormal vectors in E. Then we have*

$$\sum_{k=1}^{q} \|\pi_V t_k\|^2 \ge 1.$$

Proof. In the preceding proposition, take W to be the span of the vectors $t_1, \ldots, t_q$, so that $\dim W = q$. Notice now that since $\dim V = n-q+1$, $\dim W = q$ and $\dim E = n$, it follows that $\dim(V \cap W) \ge 1$. The result follows from the proposition. □

6.2. *The construction for a continuous subbundle.* We begin with the result that we will need in the proof of Theorem 1.4, in the presence of a continuous subbundle.

Theorem 6.3. *Let S be a smooth compact manifold, $E \to S$ a smooth complex vector bundle of rank d, and $V \to S$ a continuous subbundle of E of rank $d-q+1$, where $1 \le q \le d$. Let $\{Q_j\}_{j=1}^N$ be a finite collection of continuous Hermitian forms on the bundle E such that the restriction of each Q_j to the subbundle V is positive definite. Then there is a smooth Hermitian metric g on E such that for each $1 \le j \le N$, the form Q_j is strictly q-positive with respect to g.*

Proof. It is sufficient to show the following:

Claim. *If H is a continuous Hermitian form on E that restricts to a positive definite form on V and γ is a continuous Hermitian metric on E, there is a constant $C(H,\gamma) > 0$ such that whenever $\kappa \ge C(H,\gamma)$, the form $H^h : E \to E$ is strictly q-positive, where the metric h is given by*

$$h(z,w) = \gamma(z,w) + \kappa \cdot \gamma(P_{V^\perp}(z), P_{V^\perp}(w)), \tag{6-2}$$

where $P_{V^\perp} : E \to V^\perp$ is the orthogonal projection with respect to the metric γ on the subbundle $V^\perp$ of E whose fibers are orthogonal to those of V with respect to γ. (Notice that this h is a continuous Hermitian metric on E.)

Indeed, granted the claim, we can choose an arbitrary continuous Hermitian metric γ on $E \to S$, and let h be given by (6-2), where we take

$$\kappa = \max\left\{C(Q_j,\gamma), 1 \le j \le N\right\}.$$

Then each Q_j is strictly q-positive with respect to h. Let g be a smooth Hermitian metric on S uniformly close to h. Such an approximation clearly exists locally, and using a partition of unity we can glue such local approximations on the compact manifold S. Such a gluing preserves uniform closeness of the local approximations. Notice that if h is sufficiently uniformly close to g then Q_j^g is also strictly q-positive for each j, and this is what we want.

To prove the claim, let

$$A_1 = \inf\{H_p(z,z) : p \in S, z \in V_p, \gamma_p(z,z) = 1\}. \tag{6-3}$$

Then $A_1 > 0$, and the infimum in (6-3) is a minimum, for the following reason. For $p \in S$, by (2-2) we have

$$\min\{H_p(z,z) : z \in V_p, \gamma_p(z,z) = 1\} = \lambda_1^\gamma(H_p|_{V_p}),$$

the smallest eigenvalue with respect to γ of the restriction of the form H_p to the subspace V_p. Since the form H, the metric γ and the subbundle $V \subset E$ are continuous, and eigenvalues depend continuously on a matrix, we easily conclude that the function $p \mapsto \lambda_1^\gamma(H_p|_{V_p})$ is continuous on S. Further, since for each p, the restriction of H_p to V_p is positive definite, this function is strictly positive at each point of S. Consequently, by the compactness of S, the number A_1, which is the infimum of this function is strictly positive and the infimum is a minimum.

Now consider

$$A_2 = \sup\left\{|H_p(z,z)| : p \in S, z \in V_p^{\perp}, \gamma_p(z,z) = 1\right\}. \tag{6-4}$$

We claim that $0 \le A_2 < \infty$. Let $p \in S$. Then clearly we have

$$\sup\left\{|H_p(z,z)| : z \in V_p^{\perp}, \gamma_p(z,z) = 1\right\} = \max\left\{|\lambda_1^{\gamma}(H_p|_{V^{\perp}})|, |\lambda_{\max}^{\gamma}(H_p|_{V^{\perp}})|\right\},$$

where $\lambda_{\max}^{\gamma}$ stands for the largest eigenvalue of the restricted Hermitian form with respect to the metric γ. An argument similar to above shows that as a function of p, the quantity above is continuous, and therefore bounded, and its supremum A_2 is finite. Lastly, introduce the quantity

$$A_3 = \sup\left\{|H_p(z,w)| : p \in S, z \in V_p, w \in V_p^{\perp}, \gamma_p(z,z) = \gamma_p(w,w) = 1\right\}. \tag{6-5}$$

We claim that $0 \le A_3 < \infty$. Notice first that if $p \in S$, then

$$\begin{aligned} &\sup\left\{|H_p(z,w)| : z \in V_p, w \in V_p^{\perp}, \gamma_p(z,z) = \gamma_p(w,w) = 1\right\} \\ &\qquad \le \sup\left\{|H_p(z,w)| : z \in E, w \in E, \gamma_p(z,z) = \gamma_p(w,w) = 1\right\} \\ &\qquad = \max\left\{|\lambda_1^{\gamma}(H_p)|, |\lambda_{\max}^{\gamma}(H_p)|\right\}. \end{aligned} \tag{6-6}$$

An argument similar to the above two cases shows that the quantity in (6-6) is a continuous function of p. It follows that $A_3 < \infty$.

It now follows that we can choose the number $C = C(H,\gamma)$ so large that

$$A_1 - \frac{q \cdot A_2}{1+C} - 2\frac{q \cdot A_3}{\sqrt{1+C}} > 0. \tag{6-7}$$

We show that this value of $C = C(H,\gamma)$ works in the claim of the previous page. To justify this, let $\kappa \ge C$ and let $p \in S$; we need to show that H_p is a strictly q-positive operator on E_p with respect to h, where h is as in (6-2). Notice that h depends on κ.

Let $t \in E_p$ be a vector of unit length with respect to the metric h_p. Set $u = P_V t$, $v = P_{V^{\perp}} t$, where $P_V : E \to V$ is the bundle map whose section over $p \in S$ is the orthogonal projection with respect to γ_p from E_p to V_p. Then

$$\begin{aligned} 1 = h_p(t,t) &= \gamma_p(t,t) + \kappa \cdot \gamma_p(P_{V^{\perp}}t, P_{V^{\perp}}t) \\ &= \gamma_p(P_V t, P_V t) + (1+\kappa)\gamma_p(P_{V^{\perp}}t, P_{V^{\perp}}t) \\ &= \gamma_p(u,u) + (1+\kappa)\gamma_p(v,v), \end{aligned}$$

from which we see that

$$\gamma_p(u,u)\le 1,\quad \gamma_p(v,v)\le \frac{1}{1+\kappa}\le\frac{1}{1+C}. \tag{6-8}$$

Therefore, since $t=u+v$,

$$\begin{aligned} H(t,t) &= H(u,u)+H(v,v)+2\operatorname{Re}H(u,v)\\ &\ge H(u,u)-|H(v,v)|-2|H(u,v)|\\ &\ge A_1\gamma_p(u,u)-A_2\gamma_p(v,v)-2A_3\sqrt{\gamma_p(u,u)}\sqrt{\gamma_p(v,v)}\\ &\ge A_1\gamma_p(P_Vt,P_Vt)-\frac{A_2}{1+C}-2\frac{A_3}{\sqrt{1+C}}, \end{aligned} \tag{6-9}$$

where we have used the definitions of the A_j as well as the bounds (6-8).

To show that H_p is strictly q-positive with respect to h_p, it suffices by Theorem 2.1 above to show that the restriction of the form H_p to each q-dimensional subspace of (E_p,γ_p) has positive trace with respect to h_p. Let W be a q-dimensional subspace of E_p, and choose an h_p-orthonormal basis $t_1,\dots,t_q$ of W. Then

$$\begin{aligned} \operatorname{tr}_h(H_p) &= \sum_{k=1}^{q} H_p(t_k,t_k)\\ &\ge \sum_{k=1}^{q}\left(A_1\gamma(P_Vt_k,P_Vt_k)-\frac{A_2}{1+C}-2\frac{A_3}{\sqrt{1+C}}\right) \quad \text{(by (6-9))}\\ &= A_1\sum_{k=1}^{q}\gamma(P_Vt_k,P_Vt_k)-\frac{q\cdot A_2}{1+C}-2\frac{q\cdot A_3}{\sqrt{1+C}}\\ &\ge A_1-\frac{q\cdot A_2}{1+C}-2\frac{q\cdot A_3}{\sqrt{1+C}} \quad \text{(by Corollary 6.2)}\\ &>0. \end{aligned}$$

□

6.3. ***The construction in the top degree.*** We next consider the construction that we will use in the $q=n-1$ case considered in Theorem 1.5, in which we still assume the existence of a direction in which both the Levi form and the Hessian of the weight function is positive, though we no longer assume that this direction varies continuously on the boundary. We will prove the following.

Theorem 6.4. *Let S be a smooth compact manifold and $E\to S$ a smooth complex vector bundle of rank d. Let Q_1 and Q_2 be continuous Hermitian forms on the bundle E such that at each point $p\in S$, there is a vector v in the fiber of E over the point p, for which $Q_1(v,v)>0$ and $Q_2(v,v)>0$. Then there is a smooth Hermitian metric g on E such that* $\operatorname{Tr}_g Q_j>0$ *for each $j\in\{1,2\}$.*

To prove this key result, we will need the following special case of Jacobi's formula for the derivative of the determinant (see [MN88, p. 169] or [Kli90, p. 798]), obtained when the matrix under consideration is Hermitian. Since this result is standard, we omit the proof.

Lemma 6.5. *For integers $d, m \geq 1$, let $\mathcal{O} \subset \mathbb{R}^m$ be an open set and let $\{M_x\}_{x\in\mathcal{O}}$ be a family of positive definite $d \times d$ Hermitian matrices with entries in $C^1(\mathcal{O})$. For any $1 \leq j \leq m$, we have, on $\mathcal{O}$,*

$$\frac{\partial}{\partial x_j} \log\det M_x = \operatorname{Tr}\left((M_x)^{-1}\frac{\partial}{\partial x_j} M_x\right) \tag{6-10}$$

The next lemma contains the main part of the proof of Theorem 6.4.

Lemma 6.6. *Let E be a finite-dimensional inner product space with $\dim E \geq 2$, and denote by $\mathcal{Q}$ the collection of ordered pairs of Hermitian forms on E sharing a common positive direction, i.e., $\mathcal{Q}$ consists of pairs (Q_1, Q_2) where each Q_j is a Hermitian form on E, and*

$$\{v \in E : Q_1(v, v) > 0\} \cap \{v \in E : Q_2(v, v) > 0\} \neq \varnothing.$$

Denoting by $\mathcal{H}$ the cone of Hermitian metrics on E, there is a continuous map

$$h : \mathcal{Q} \to \mathcal{H}$$

such that the traces $\operatorname{tr}_{h(Q_1,Q_2)} Q_1$ and $\operatorname{tr}_{h(Q_1,Q_2)} Q_2$ computed with respect to the metric $h(Q_1, Q_2)$ are both positive.

Remark 6.7. The space of Hermitian forms on a finite-dimensional vector space has a natural linear topology which can be represented by a variety of norms (see [HJ85, Section 5.6])). This induces the topologies given on the spaces $\mathcal{Q}$ and $\mathcal{H}$.

Proof. Let $d = \dim E$ and denote by $\langle\cdot,\cdot\rangle$ its metric. For $x \in \mathbb{R}^2$, let $\langle\cdot,\cdot\rangle_x$ denote the Hermitian form on E given by

$$\langle\cdot,\cdot\rangle_x = \langle\cdot,\cdot\rangle - x_1 Q_1(\cdot,\cdot) - x_2 Q_2(\cdot,\cdot).$$

For notational simplicity, when $\langle\cdot,\cdot\rangle_x$ is a metric, for a Hermitian form Q on E, we will also use the shorthand $\operatorname{Tr}_x Q$ for the trace of the form Q with respect to the metric $\langle\cdot,\cdot\rangle_x$, instead of the more correct notation $\operatorname{Tr}_{\langle\cdot,\cdot\rangle_x} Q$. We will prove the lemma by showing that there exists a point $\gamma(Q_1, Q_2) \in \mathbb{R}^2$ such that

- $\gamma_1(Q_1, Q_2) > 0$ and $\gamma_2(Q_1, Q_2) > 0$,
- the form $\langle\cdot,\cdot\rangle_{\gamma(Q_1,Q_2)}$ is a Hermitian metric on $\mathbb{C}^d$, and
- the traces $\operatorname{Tr}_{\gamma(Q_1,Q_2)} Q_1$ and $\operatorname{Tr}_{\gamma(Q_1,Q_2)} Q_2$ of Q_1 and Q_2 computed with respect to this new metric are both positive.

Furthermore, $\gamma(Q_1, Q_2)$ can be chosen to depend continuously on Q_1 and Q_2, so in the conclusion of the lemma we may take $h(Q_1, Q_2) = \langle\cdot,\cdot\rangle_{\gamma(Q_1,Q_2)}$.

Let $\mathcal{O} \subset \mathbb{R}^2$ denote the set of all $x \in \mathbb{R}^2$ such that $\langle\cdot,\cdot\rangle_x$ is positive definite. Clearly $\mathcal{O}$ is an open set and $(0,0) \in \mathcal{O}$. For any $x \in \mathbb{R}^2\backslash\{(0,0)\}$, let λ_x denote the largest eigenvalue of $x_1 Q_1 + x_2 Q_2$. If $\lambda_x \leq 0$, then $tx \in \mathcal{O}$ for all $t \geq 0$. If $\lambda_x > 0$, then $tx \in \mathcal{O}$ for $t \geq 0$ if and only if $0 \leq t < \lambda_x^{-1}$. Hence, $\mathcal{O}$ is a starlike domain centered at $(0,0)$. Let $\mathcal{O}^+$ denote the set of all $x \in \mathcal{O}$ such that $x_1 > 0$ and $x_2 > 0$. When $x \in \mathcal{O}^+$, then $x_1 Q_1(v,v) + x_2 Q_2(v,v) > 0$, so $\lambda_x > 0$ and hence

$$\mathcal{O}^+ \text{ is a bounded subset of } \mathcal{O}. \tag{6-11}$$

Fix an orthonormal basis $\{e_j\}_{1\leq j\leq d}$ for E with respect to the base metric $\langle\cdot,\cdot\rangle$. For $x \in \mathcal{O}$, define a $d \times d$ positive definite Hermitian matrix $g(x)$ by $g(x) = (g_{j\bar{k}}(x))_{1\leq j,k\leq d}$, where $g_{j\bar{k}}(x) = \langle e_j, e_k\rangle_x$. Denote the inverse of this matrix by

$$g^{-1}(x) = (g^{\bar{k}j}(x))_{1\leq j,k\leq d}.$$

Since $g(x)g^{-1}(x) = I$ for all $x \in \mathcal{O}$, we may differentiate by x_r for any $r \in \{1,2\}$ to obtain

$$\left(\frac{\partial}{\partial x_r} g(x)\right) g^{-1}(x) + g(x)\frac{\partial}{\partial x_r} g^{-1}(x) = 0,$$

or

$$\frac{\partial}{\partial x_r} g^{-1}(x) = -g^{-1}(x)\left(\frac{\partial}{\partial x_r} g(x)\right) g^{-1}(x). \tag{6-12}$$

Let $\{u_j(x)\}_{1\leq j\leq d}$ be an orthonormal basis for $\mathbb{C}^d$ with respect to $\langle\cdot,\cdot\rangle_x$. Then for every $1 \leq j \leq d$ we have $e_j = \sum_{k=1}^d A_j^k(x) u_k(x)$ for some nonsingular matrix $A(x) = (A_j^k(x))_{1\leq j,k\leq d}$. For $1 \leq j,k \leq d$,

$$g_{j\bar{k}}(x) = \langle e_j, e_k\rangle_x = \sum_{\ell=1}^d A_j^\ell(x)\overline{A_k^\ell(x)},$$

so $g(x) = A(x)A^*(x)$. If we right-multiply by $g^{-1}(x)A(x)$, we obtain

$$A(x) = A(x)A^*(x)g^{-1}(x)A(x).$$

Since $A(x)$ is nonsingular, this necessarily implies that

$$A^*(x)g^{-1}(x)A(x) = I \qquad \text{for all } x \in \mathcal{O}. \tag{6-13}$$

We define a function $\xi \in C^\infty(\mathcal{O})$ by $\xi(x) = -\log\det g(x)$. This function ξ is independent of the choice of basis $\{e_j\}$ used to define the matrix g, since if U is a unitary (with respect to $\langle\cdot,\cdot\rangle$) change of coordinates the matrix g transforms to U^*gU which has the same determinant. Observe that $g(0,0)$ is the identity matrix, so $\xi(0,0) = 0$. As $x \to b\mathcal{O}$, $\xi(x) \to \infty$, so (6-11) implies that

$$(0,\infty) \subset \operatorname{Range}(\xi|_{\text{positive } x_1\text{-axis}}) \quad\text{and}\quad (0,\infty) \subset \operatorname{Range}(\xi|_{\text{positive } x_2\text{-axis}}). \tag{6-14}$$

Fix $r \in \{1, 2\}$. Using (6-10), we have

$$\frac{\partial \xi}{\partial x_r}(x) = -\operatorname{Tr}\left(g^{-1}(x)\frac{\partial}{\partial x_r} g(x)\right),$$

where the trace is computed with respect to our base metric. Since $g(x)$ is a linear function, we may easily compute the derivatives to obtain

$$\frac{\partial \xi}{\partial x_r}(x) = \sum_{j,k=1}^{n} g^{\bar{k}j}(x) Q_r(e_j, e_k). \tag{6-15}$$

Using (6-13) to simplify after changing coordinates, we have

$$\frac{\partial \xi}{\partial x_r}(x) = \sum_{j,k,\ell,m=1}^{d} \overline{A_k^m(x)} g^{\bar{k}j}(x) A_j^\ell(x) Q_r(u_\ell(x), u_m(x)) = \sum_{j=1}^{d} Q_r(u_j(x), u_j(x)).$$

Since we denote the trace with respect to $\langle \cdot , \cdot \rangle_x$ by Tr_x, we have

$$\frac{\partial \xi}{\partial x_r}(x) = \operatorname{Tr}_x Q_r \quad \text{for all } r \in \{1, 2\}. \tag{6-16}$$

For $r, s \in \{1, 2\}$, we may use (6-12) to differentiate (6-15) and obtain

$$\frac{\partial^2 \xi}{\partial x_s \partial x_r}(x) = -\sum_{j,k,\ell,m=1}^{d} g^{\bar{k}m}(x) \left(\frac{\partial}{\partial x_s} g_{m\bar{\ell}}(x)\right) g^{\bar{\ell}j}(x) Q_r(e_j, e_k).$$

Once again, it is easy to compute derivatives of the linear functions comprising $g(x)$, so we have

$$\frac{\partial^2 \xi}{\partial x_s \partial x_r}(x) = \sum_{j,k,\ell,m=1}^{d} g^{\bar{k}m}(x) Q_s(e_m, e_\ell) g^{\bar{\ell}j}(x) Q_r(e_j, e_k).$$

As before, we may use (6-13) to simplify after changing coordinates and obtain

$$\begin{aligned}\frac{\partial^2 \xi}{\partial x_s \partial x_r}(x) &= \sum_{j,k=1}^{d} Q_s(u_k(x), u_j(x)) Q_r(u_j(x), u_k(x)) \\ &= \sum_{j,k=1}^{d} Q_r(u_j(x), u_k(x)) \overline{Q_s(u_j(x), u_k(x))}.\end{aligned}$$

In other words, $\frac{\partial^2 \xi}{\partial x_s \partial x_r}(x)$ is simply the inner product of Q_r and Q_s using the standard inner product for matrices with respect to the metric $\langle \cdot , \cdot \rangle_x$ (see Section 5.6 in [HJ85] for further discussion of the associated ℓ^2 norm). Since $Q_1(v, v) > 0$ and $Q_2(v, v) > 0$ by hypothesis, neither Hermitian form vanishes, so we must have $\frac{\partial^2 \xi}{\partial x_1^2}(x) > 0$and $\frac{\partial^2 \xi}{\partial x_2^2}(x) > 0$.By the Cauchy–Schwarz inequality, we have

$$\frac{\partial^2 \xi}{\partial x_1^2}(x) \frac{\partial^2 \xi}{\partial x_2^2}(x) \geq \left(\frac{\partial^2 \xi}{\partial x_1 \partial x_2}(x)\right)^2,$$

so the Hessian of ξ is positive semidefinite, and hence ξ is convex. Furthermore, ξ is strictly convex unless $Q_1 = \mu Q_2$ for some $\mu \in \mathbb{R}$. Since $Q_1(v, v) > 0$ and $Q_2(v, v) > 0$, we must have $\mu > 0$ in this case. To summarize:

(6-17) ξ is a convex exhaustion function on $\mathcal{O}$ and

(6-18) either ξ is strictly convex on $\mathcal{O}$ or $Q_1 = \mu Q_2$ for some $\mu > 0$.

We observe that (6-17) implies that $\mathcal{O}$ is a convex set.

If $Q_1 = \mu Q_2$ for some $\mu > 0$, then $\xi(x) = f(\mu x_1 + x_2)$ for some real-valued function f defined on some interval in $\mathbb{R}$. By (6-17), f must be convex (we can easily check that f is strictly convex, but we will not need this). By (6-11), there must exist a finite real number $b > 0$ such that $\lim_{t\to b^-} f(t) = \infty$. Since f is convex, f' is nondecreasing, and hence there exists $a \ge 0$ such that $f'(t) > 0$ for all $a < t < b$, and, if $a > 0$, $f'(t) \le 0$ for all $0 \le t < a$. By (6-16), $\operatorname{Tr}_x Q_1 = \mu f'(\mu x_1 + x_2)$ and $\operatorname{Tr}_x Q_2 = f'(\mu x_1 + x_2)$, so $\operatorname{Tr}_x Q_1 > 0$ and $\operatorname{Tr}_x Q_2 > 0$ for $x \in \mathcal{O}$ if and only if $a < \mu x_1 + x_2 < b$. By (6-14), $f(a) \le 0$, so there exists $a < c < b$ such that $f(c) = 1$. Let $\gamma(Q_1, Q_2) = \left(\frac{c}{2\mu}, \frac{c}{2}\right)$, which is the midpoint of the line segment connecting $(0, c)$ to $\left(\frac{c}{\mu}, 0\right)$, a subset of the level curve $\xi^{-1}(\{c\})$. Henceforth, (6-18) allows us to assume that ξ is strictly convex.

Consider the map $\psi : \mathcal{O} \to \mathbb{R}^2$ defined by $\psi(x) = \nabla\xi(x)$ for all $x \in \mathcal{O}$. Suppose that $\psi(y) = \psi(z)$ for some $y, z \in \mathcal{O}$. Since $\mathcal{O}$, the line segment $ty + (1-t)z \in \mathcal{O}$ for all $0 \le t \le 1$. By Rolle's Theorem, there must exist $0 < s < 1$ such that $\frac{d}{dt}\big((y - z) \cdot \nabla\xi(ty + (1-t)z)\big)\big|_{t=s} = 0$. Equivalently,

$$\sum_{j,k=1}^{2} (y_j - z_j)\frac{\partial^2\xi}{\partial x_j \partial x_k}(sy + (1-s)z)(y_k - z_k) = 0.$$

Since ξ is strictly convex, $\nabla^2\xi$ is positive definite, so this means that $y = z$. Hence, ψ is injective. The Jacobian of ψ is the Hessian of ξ, which is nonsingular since ξ is strictly convex. Hence, the Inverse Function Theorem guarantees that ψ admits a smooth local inverse. Taken together, we see that ψ is a smooth diffeomorphism from $\mathcal{O}$ onto its range.

Let $\Gamma = \xi^{-1}(\{1\})$ be a level curve of ξ, and let $\tilde{\Gamma}$ denote the set of all $x \in \Gamma \cap \mathcal{O}^+$ such that $\psi_1(x) > 0$ and $\psi_2(x) > 0$. Since ξ is strictly convex, Γ is the boundary of the strictly convex domain $\{x \in \mathcal{O} : \xi(x) < 1\}$. Hence, $\psi(\Gamma)$ is a smooth curve which crosses each radial line segment at most once, which implies that $\tilde{\Gamma}$ is a connected curve (if it is nonempty). Now (6-14) guarantees that there must exist $a > 0$ and $b > 0$ such that $\xi(a, 0) = 1$ and $\xi(0, b) = 1$. If there are two such values for either parameter, we choose the largest value, so that $\frac{\partial\xi}{\partial x_1}(a, 0) > 0$ and $\frac{\partial\xi}{\partial x_2}(0, b) > 0$. Hence $(a, 0)$ and $(0, b)$ are endpoints of a smooth curve in $\mathcal{O}^+$. By the extended mean value theorem, there must exist a point $y \in \Gamma \cap \mathcal{O}^+$ such that

$(a, -b)$ is tangential to Γ at y, and hence $\nabla\xi(y)$ is a positive multiple of (b, a). This means that $\frac{\partial\xi}{\partial x_1}(y) > 0$and $\frac{\partial\xi}{\partial x_2}(y) > 0$,so (6-16) implies that $\mathrm{Tr}_y\, Q_1 > 0$ and $\mathrm{Tr}_y\, Q_2 > 0$. As a result, $y \in \tilde\Gamma$ and so $\tilde\Gamma$ is not empty. Let $\gamma(Q_1, Q_2)$ denote the midpoint of $\tilde\Gamma$ with respect to the standard Euclidean arclength (a level curve of a convex function in a bounded set $\mathcal{O}^+$ must be of finite length).

To see that $\gamma(Q_1, Q_2)$ depends continuously on Q_1 and Q_2, it suffices to note that $g(x)$ is a linear function of the entries of Q_1 and Q_2 with respect to the coordinates $\{e_j\}_{1\le j\le d}$, and hence $g(x)$ depends continuously on Q_1 and Q_2. This means that $\xi(x)$ depends continuously on Q_1 and Q_2, and by (6-16), $\nabla\xi(x)$ also depends continuously on Q_1 and Q_2. Since 1 is never a critical value of ξ, $\Gamma \cap \mathcal{O}^+$ will depend continuously on Q_1 and Q_2 with respect to Hausdorff distance. Since $\nabla\xi(x)$ depends continuously on Q_1 and Q_2, $\tilde\Gamma$ must also depend continuously on Q_1 and Q_2. Since $\gamma(Q_1, Q_2)$ is defined to be the midpoint of $\tilde\Gamma$ (even in the case in which $Q_1 = \mu Q_2$), γ is a continuous function. □

Finally, we are ready to prove the analog to Theorem 6.3 in the top degree when the subbundle is not continuous.

Proof of Theorem 6.4. Fix a Hermitian metric on the bundle E. For each $p \in S$, now we have an inner product on E_p, and the two given Hermitian forms $Q_1(p)$, $Q_2(p)$ for which by hypothesis there is a common positive direction. Therefore, by Lemma 6.6 we have a Hermitian metric on E_p given by $h(Q_1(p), h(Q_2(p)))$. If we define a continuous metric on g_1 on E by $g_1(p) = h(Q_1(p), h(Q_2(p)))$, then $\mathrm{tr}_{g_1(p)}\, Q_j(p)$ is positive for each p and $j \in \{1, 2\}$. After a standard regularization of g_1, we obtain a smooth metric g with the required properties (positive traces on compact sets are stable under perturbations). □

7. Proof of Theorems 1.4 and 1.5

7.1. ***Proof of Theorem 1.4.*** Let ρ be a defining function for the smoothly bounded domain $\Omega \subset M$. This gives rise to a Levi form $\mathcal{L}_\rho$ which is a Hermitian form on the complex vector bundle $T^{1,0}(\partial\Omega)$ (see (2-4)). Notice that $T^{1,0}(\partial\Omega)$ has rank $(n-1)$ on the smooth manifold $\partial\Omega$ and $\mathcal{L}_\rho$ is a Hermitian form on it with at least $n-q$ positive eigenvalues or at least $q+1$ negative eigenvalues, depending on the case.

Denote by $\mathcal{H}_\varphi$ the restriction of the complex Hessian of φ to the boundary, i.e., for $p \in \partial\Omega$, $X, Y \in T^{1,0}_p(\partial\Omega)$, we have

$$\mathcal{H}_\varphi(X, Y) = \partial\bar\partial\varphi(X, \bar Y).$$

Then $\mathcal{H}_\varphi$ is a Hermitian form on the bundle $T^{1,0}(\partial\Omega)$. Since the boundary and the weight φ are smooth, these Hermitian forms $\mathcal{H}_\varphi$ and $\mathcal{L}_\rho$ are smooth.

We first consider the case in which $\mathcal{L}_\rho$ and $\mathcal{H}_\varphi$ are positive definite on a subbundle of rank $n-q$. We apply Theorem 6.3 with $S = \partial\Omega$, $E = T^{1,0}(\partial\Omega)$ (so that $d = n-1$)

and let V be the subbundle of E of rank $d-q+1 = n-q$ on which both the Hermitian forms L_ρ and $\mathcal{H}_\varphi$ are positive. Then there is a smooth metric h on the bundle $T^{1,0}(\partial\Omega)$ of rank $(n-1)$ such that both L_ρ and $\mathcal{H}_\varphi|_{\partial\Omega}$ are strictly q-positive with respect to h.

By the continuity of eigenvalues as a function of the matrix, and the compactness of $\partial\Omega$, there is a $\delta_0 > 0$ such that $\mathcal{H}_{\varphi+\delta\rho} = \mathcal{H}_\varphi + \delta\mathcal{H}_\rho$ has $(n-q+1)$ positive eigenvalues as a Hermitian form on $T^{1,0}M|_{\partial\Omega}$ whenever $0 \le \delta \le \delta_0$. It follows that there is a neighborhood $\mathcal{U}$ of $\partial\Omega$ in M where $\mathcal{H}_{\varphi+\delta\rho}$ continues to have $(n-q+1)$ positive eigenvalues whenever $0 \le \delta \le \delta_0$. Since the signature of the complex Hessian is independent of the choice of metric, δ_0 is also independent of the choice of metric. Choose $\epsilon_0 > 0$ so small that $\{-\epsilon_0 \le \rho \le 0\} \subset \overline{\Omega} \cap \mathcal{U}$. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth, convex, nondecreasing function such that $\chi(t) = 0$ whenever $t \le -1$ and $\chi'(0) = 1$. For $0 < \epsilon < \epsilon_0$, set

$$\varphi_\epsilon = \varphi + \delta_0 \epsilon \chi(\epsilon^{-1}\rho).$$

Let g_0 be a Hermitian metric on $T^{1,0}M|_{\partial\Omega}$ extending the metric h constructed above on $T^{1,0}(\partial\Omega)$. We claim the following:

(1) $\mathcal{H}_{\varphi_\epsilon}$ has $(n-q+1)$ positive eigenvalues in $\overline{\Omega}$ as a Hermitian form on $T^{1,0}M$.

(2) $\mathcal{H}_{\varphi_\epsilon}$ is strictly q-positive as a Hermitian form on $T^{1,0}(\partial\Omega)$ with respect to g_0.

(3) For sufficiently small ϵ, the form $\mathcal{H}_{\varphi_\epsilon}$ is strictly q-positive as a Hermitian form on $T^{1,0}M|_{\partial\Omega}$ with respect to g_0.

To see (1), notice first that $\varphi_\epsilon = \varphi$ on the set $\{\rho(z) \le -\epsilon\}$ which is contained in $\overline{\Omega} \setminus \mathcal{U}$, provided $\epsilon < \epsilon_0$. By a computation:

$$\mathcal{H}_{\varphi_\epsilon} = \mathcal{H}_\varphi + \delta_0 \chi'(\epsilon^{-1}\rho)\mathcal{H}_\rho + \delta_0 \epsilon^{-1} \chi''(\epsilon^{-1}\rho)\partial\rho \wedge \bar{\partial}\rho. \tag{7-1}$$

Since the third term of this sum will introduce a positive semidefinite term to $\mathcal{H}_{\varphi_\epsilon}$ and since $0 \le \delta_0 \chi'(\epsilon^{-1}\rho) \le \delta_0$ on $\mathcal{U} \cap \overline{\Omega}$, we see that $\mathcal{H}_{\varphi_\epsilon}$ must have at least $n-q+1$ positive eigenvalues on $\mathcal{U}$.

To see (2), note first that

$$\mathcal{H}_{\varphi_\epsilon}|_{T^{1,0}(\partial\Omega)} = \mathcal{H}_\varphi|_{T^{1,0}(\partial\Omega)} + \delta_0 \mathcal{L}_\rho.$$

Now since the metric g_0 restricts to the metric h constructed above on $T^{1,0}(\partial\Omega)$ and each of the Hermitian forms $\mathcal{H}_\varphi$ and $\mathcal{L}_\rho$ is strictly q-positive with respect to h, it follows that $\mathcal{H}_{\varphi_\epsilon}$ is also strictly q-positive with respect to h (and therefore g_0) on $T^{1,0}(\partial\Omega)$. Notice that the fact that the sum of two strictly q-positive Hermitian forms is again strictly q-positive is an immediate consequence of Theorem 2.1.

To see (3), note that (7-1) becomes in this case

$$\mathcal{H}_{\varphi_\epsilon}|_{T^{1,0}M|_{\partial\Omega}} = \mathcal{H}_\varphi|_{T^{1,0}M|_{\partial\Omega}} + \delta_0 \mathcal{H}_\rho|_{T^{1,0}M|_{\partial\Omega}} + \delta_0 \epsilon^{-1} \chi''(0)\partial\rho \wedge \bar{\partial}\rho|_{T^{1,0}M|_{\partial\Omega}}.$$

We will use the characterization of Theorem 2.1 to show that $\mathcal{H}_{\varphi_\epsilon}$ is strictly q-positive on $T^{1,0}M|_{\partial\Omega}$. Let

$$B_1 = \max\{\mathrm{tr}_{g_0}(\mathcal{H}_\varphi(z)|_U) : z \in \partial\Omega,\, U \subset T_z^{1,0}M \text{ is } q\text{-dimensional}\},$$
$$B_2 = \max\{\mathrm{tr}_{g_0}(\mathcal{H}_\rho(z)|_U) : z \in \partial\Omega,\, U \subset T_z^{1,0}M \text{ is } q\text{-dimensional}\}.$$

The existence of these finite maxima follows by standard continuity and compactness arguments. Let $p \in \partial\Omega$ and W be a q-dimensional subspace of $T_p^{1,0}M$, and let $t_1, \dots, t_q$ be a g_0-orthonormal basis of W. Then

$$\text{(7-2)}\quad \mathrm{tr}_{g_0}(\mathcal{H}_{\varphi_\epsilon}(p)|_W) = \sum_{j=1}^{q} \mathcal{H}_{\varphi_\epsilon}(t_j, t_j) = \mathrm{tr}_{g_0}(\mathcal{H}_\varphi(p)|_W) + \delta_0\, \mathrm{tr}_{g_0}(\mathcal{H}_\rho(p)|_W) + \frac{\delta_0}{\epsilon}\chi''(0) \sum_{j=1}^{q} |\partial\rho(t_j)|^2.$$

In the case $\sum_{j=1}^{q} |\partial\rho(t_j)|^2 = 0$, we have $W \subset T_p^{1,0}(\partial\Omega)$ and

$$\mathrm{tr}_{g_0}(\mathcal{H}_{\varphi_\epsilon}(p)|_W) = \mathrm{tr}_h(\mathcal{H}_\varphi(p)|_W) + \delta_0\, \mathrm{tr}_h(\mathcal{L}_\rho(p)|_W) > 0,$$

since both $\mathcal{H}_\varphi$ and the Levi form $\mathcal{L}_\rho$ are strictly q-positive on $T^{1,0}(\partial\Omega)$ with respect to h, and g_0 extends h. By continuity there exists an $\eta > 0$ independent of $\epsilon > 0$ such that

$$\mathrm{tr}_{g_0}(\mathcal{H}_\varphi(p)|_W) + \delta_0\, \mathrm{tr}_{g_0}(\mathcal{H}_\rho(p)|_W) > 0$$

whenever $\sum_{j=1}^{q} |\partial\rho(t_j)|^2 < \eta$. Since the third term in (7-2) is nonnegative, we have $\mathrm{tr}_{g_0}(\mathcal{H}_{\varphi_\epsilon}(p)|_W) > 0$ if $\sum_{j=1}^{q} |\partial\rho(t_j)|^2 < \eta$. In the case $\sum_{j=1}^{q} |\partial\rho(t_j)|^2 \geq \eta$ (and $W \subset T^{1,0}(M)|_{\partial\Omega}$ is no longer a subset of $T^{1,0}(\partial\Omega)$), we take

$$\epsilon < \min\left(\frac{\delta_0 \chi''(0)\eta}{B_1 + \delta_0 B_2}, \epsilon_0\right)$$

and observe that

$$\begin{aligned}\mathrm{tr}_{g_0}(\mathcal{H}_{\varphi_\epsilon}(p)|_W) &\geq -B_1 - \delta_0 B_2 + \frac{\delta_0}{\epsilon}\chi''(0) \sum_{j=1}^{q} |\partial\rho(t_j)|^2 \\ &\geq -B_1 - \delta_0 B_2 + \frac{\delta_0}{\epsilon}\chi''(0)\eta \\ &> 0.\end{aligned}$$

In Proposition 1.3, we take the bundle E to be $T^{1,0}M$ (so $d = n$) and for the closed set $F \subset M$ we take $F = \partial\Omega$. We take g_0 to be an extension of h as above, and ϵ small enough that the function φ_ϵ satisfies the three conditions above, and let this be the function ϕ of Proposition 1.3. Extend g_0 smoothly to a neighborhood of $\partial\Omega$. Then the hypotheses of Proposition 1.3 are satisfied for $\tilde{q} = q$, and consequently there is a metric g on a neighborhood of $\overline{\Omega}$ which coincides with g_0 near $\partial\Omega$, and with respect to which $\mathcal{H}_{\varphi_\epsilon}$ is strictly q-positive. Then Theorem 3.3 yields $H^{p,q}_{L^2}(\Omega) = 0$.

Now we consider the case in which $\mathcal{L}_\rho$ and $\mathcal{H}_\varphi$ are negative definite on a subbundle of rank $q+1$. We again apply Theorem 6.3 with $S=\partial\Omega$ and $E=T^{1,0}(\partial\Omega)$, but this time V is the subbundle of E of rank $d-(n-q-1)+1=q+1$ on which both the Hermitian forms $-L_\rho$ and $-\mathcal{H}_\varphi$ are positive. Then there is a smooth metric h on the bundle $T^{1,0}(\partial\Omega)$ of rank $(n-1)$ such that both $-L_\rho$ and $-\mathcal{H}_\varphi|_{\partial\Omega}$ are strictly $(n-q-1)$-positive with respect to h.

Let $\tilde{L}_n\in T^{1,0}(M)$ be a smooth vector field such that on $\partial\Omega$, $\tilde{L}_n$ is nontrivial and transverse to $T^{1,0}(\partial\Omega)$. At any $p\in\partial\Omega$, let $\{L_j\}_{1\le j\le n-1}$ be an orthonormal basis for $T^{1,0}_p(\partial\Omega)$ with respect to h. Since the restriction of the complex Hessian of φ to $T^{1,0}(\partial\Omega)$ is nondegenerate, the matrix $(\partial\bar\partial\varphi(L_j,\bar{L}_k))_{1\le j,k\le n-1}$ is invertible, and hence there exists a unique vector $v\in\mathbb{C}^{n-1}$ such that

$$\sum_{j=1}^{n-1} v^j\partial\bar\partial\varphi(L_j,\bar{L}_k)=-\partial\bar\partial\varphi(\tilde{L}_n,\bar{L}_k)$$

for all $1\le k\le n-1$. By the implicit function theorem, the dependence of v on p is smooth. For any $\epsilon>0$, we set $L_n^\epsilon=\epsilon\bigl(\tilde{L}_n+\sum_{j=1}^{n-1}v^jL_j\bigr)$, so that L_n^ϵ is smooth on $\partial\Omega$ and $\partial\bar\partial\varphi(L_n^\epsilon,\bar{L})=0$ for all $L\in T^{1,0}(\partial\Omega)$. There is a unique Hermitian metric g_0^ϵ on $T^{1,0}M|_{\partial\Omega}$ such that $\{L_j\}_{1\le j\le n-1}\cup\{L_n^\epsilon\}$ is orthonormal on $\partial\Omega$. Observe that g_0^ϵ so defined must equal h on $T^{1,0}(\partial\Omega)$.

Let $\{\mu_j\}_{1\le j\le n-1}$ denote the eigenvalues with respect to h of the complex Hessian of φ when restricted to $T^{1,0}(\partial\Omega)$, arranged in nondecreasing order. By the construction of g_0^ϵ, the μ_j are eigenvalues of the full complex Hessian of φ with respect to g_0^ϵ; and L_n^ϵ is an eigenvector of the complex Hessian of φ with respect to g_0^ϵ, with eigenvalue

$$\mu_n^\epsilon=\partial\bar\partial\varphi(L_n^\epsilon,\bar{L}_n^\epsilon)=\epsilon^2\partial\bar\partial\varphi\Bigl(\tilde{L}_n+\sum_{j=1}^{n-1}v^jL_j,\overline{\tilde{L}_n}+\sum_{j=1}^{n-1}\bar{v}^j\bar{L}_j\Bigr).$$

Observe that $\mu_n^\epsilon\to0$ as $\epsilon\to0^+$. Since the complex Hessian of $-\varphi$ is strictly $(n-q-1)$-positive on $T^{1,0}(\partial\Omega)$ with respect to h, we have

$$\sum_{j=q+1}^{n-1}(-\mu_j)>0\quad\text{on }\partial\Omega. \tag{7-3}$$

Since the eigenvalues are arranged in nondecreasing order, this is only possible if $\mu_{q+1}<0$, and hence we must also have $\mu_q<0$. Fix $\epsilon>0$ sufficiently small that

$$\sum_{j=q+1}^{n-1}(-\mu_j)>\mu_n^\epsilon>\mu_q\quad\text{on }\partial\Omega. \tag{7-4}$$

Since ϵ is now fixed, we may write $g_0=g_0^\epsilon$, $L_n=L_n^\epsilon$, and $\mu_n=\mu_n^\epsilon$. Since $\mu_n>\mu_q$ by (7-4), the sum of the $(n-q)$ smallest eigenvalues of the complex Hessian of $-\varphi$ is given by $\sum_{j=q+1}^{n}(-\mu_j)$, and this is strictly positive by (7-4). Hence, the complex Hessian of $-\varphi$ is $(n-q)$-positive on $\partial\Omega$. Furthermore, (7-3) implies that

$$\sum_{j=q+1}^{n}(-\mu_j) > -\mu_n = -\partial\bar\partial\varphi(L_n, \bar L_n),$$

so we have condition (3) in the second set of hypotheses for Theorem 3.3.

As before, in Proposition 1.3, we take the bundle E to be $T^{1,0}M$ (so $d = n$) and for the closed set $F \subset M$ we take $F = \partial\Omega$. We take g_0 to be the extension of h constructed above, and let φ be the function ϕ of Proposition 1.3. Extend g_0 smoothly to a neighborhood of $\partial\Omega$. Then the hypotheses of Proposition 1.3 are satisfied for $\tilde q = n-q$, and consequently there is a metric g on a neighborhood of $\overline\Omega$ which coincides with g_0 near $\partial\Omega$, and with respect to which $\mathcal{H}_{-\varphi}$ is strictly $(n-q)$-positive. Therefore, by Theorem 3.3, we have $H^{p,q}_{L^2}(\Omega) = 0$.

7.2. ***Proof of Theorem 1.5.*** The proof is the same as that of Theorem 1.4, except that we replace the use of Theorem 6.3 with that of Theorem 6.4.

8. Examples

8.1. ***Discontinuous subbundles.*** In Theorem 1.4, we require the subbundle of shared positive directions to vary continuously on $\partial\Omega$, but it is possible for such a subbundle to exist pointwise without a continuous representative, as the following example illustrates. This motivates the need for Theorem 1.5, although at present we only have a proof of this result when $q = n-1$.

Proposition 8.1. *For each $x \in \mathbb{R}^3$, define a Hermitian form on $\mathbb{C}^2\times\mathbb{C}^2$ by $H_0(u, v) = \langle u, v\rangle$ and when $x \neq 0$*

$$H_x(u,v) = (\bar v_1 \;\; \bar v_2)\begin{pmatrix} 1-\exp(-|x|^{-2})(|x|-x_3) & -\exp(-|x|^{-2})(x_1+ix_2) \\ -\exp(-|x|^{-2})(x_1-ix_2) & 1-\exp(-|x|^{-2})(|x|+x_3)\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}$$

for all $u, v \in \mathbb{C}^2$. Then:

(1) *For every $x \in \mathbb{R}^3$, there exists a nontrivial vector $v \in \mathbb{C}^2$ such that*

$$H_x(v,v) = |v|^2. \tag{8-1}$$

(2) *If $\mathcal{O} \subset \mathbb{R}^3$ is an open set containing $\overline{B(0,R)}$ for some $R > 0$ and $v : \mathcal{O} \to \mathbb{C}^2$ is a nowhere vanishing continuous vector field, then there exists $x \in \mathcal{O}$ at which*

$$H_x(v(x), v(x)) = |v(x)|^2(1-2\exp(-R^{-2})R). \tag{8-2}$$

Proof. To prove (8-1), given $x \in \mathbb{R}^3\backslash\{(0,0,-R) : R \geq 0\}$, let $v = \binom{|x|+x_3}{-x_1+ix_2}$. Then v is an eigenvector of H_x (with respect to the Euclidean metric) with eigenvalue 1, so (8-1) follows. When $x = (0,0,-R)$ for some $R > 0$, (8-1) follows for $v = \binom{0}{1}$. When $x = (0,0,0)$, (8-1) holds for any nonvanishing vector v.

Now, let $\mathcal{O} \subset \mathbb{R}^3$ be an open set containing $\overline{B(0,R)}$ for some $R > 0$ and let $v : \mathcal{O} \to \mathbb{C}^2$ be a nowhere vanishing continuous vector field. Define a homeomorphism $\varphi : \mathbb{CP}^1 \to S^2 \subset \mathbb{R}^3$ by

$$\varphi([z_1 : z_2]) = \left(\frac{2\operatorname{Re}(z_1\bar{z}_2)}{|z_1|^2 + |z_2|^2}, \frac{2\operatorname{Im}(z_1\bar{z}_2)}{|z_1|^2 + |z_2|^2}, \frac{|z_2|^2 - |z_1|^2}{|z_1|^2 + |z_2|^2} \right).$$

One can check that the inverse $\varphi^{-1} : S^2 \to \mathbb{CP}^1$ is given by

$$\varphi^{-1}(x) = \begin{cases} \left[\frac{x_1 + i x_2}{1 + x_3} : 1\right] & \text{if } x \neq (0,0,-1), \\ [1:0] & \text{if } x = (0,0,-1). \end{cases}$$

For $t \in [0, R]$, define a continuous map $f_t : \mathbb{CP}^1 \to \mathbb{CP}^1$ by

$$f_t([z_1 : z_2]) = [v_1(t\varphi([z_1 : z_2])) : v_2(t\varphi([z_1 : z_2]))].$$

Since f_t depends continuously on t, f_t is a homotopy between f_0 and f_R. Since f_0 is a constant, the degree of f_0 is zero, and hence the degree of f_R is also zero. This means that the induced map on the degree 2 singular cohomology of $\mathbb{CP}^1 \cong S^2$ is trivial. Since the only other nontrivial singular cohomology group of S^2 is in degree 0 and the induced map must be the identity on this group (S^2 has exactly one connected component), the Lefschetz fixed-point theorem guarantees that f_R has a fixed-point.

Let $[z_1 : z_2] \in \mathbb{CP}^1$ be a fixed-point to f_R. Set $x = R\varphi([z_1 : z_2])$, so that $[v_1(x) : v_2(x)] = \varphi^{-1}(x/R)$. If $x = (0,0,-R)$, we have $v_2(x) = 0$, so

$$H_x(v(x), v(x)) = |v_1(x)|^2\big(1 - \exp(-R^{-2})(R - (-R))\big),$$

from which (8-2) follows. If $x \neq (0,0,-R)$, then there exists $\lambda \in \mathbb{C}$ such that $v_1(x) = \lambda \frac{x_1 + i x_2}{R + x_3}$ and $v_2(x) = \lambda$, so $v(x) = \frac{\lambda}{R + x_3}\binom{x_1 + i x_2}{R + x_3}$. When $|x| = R$, the vector $\binom{x_1 + i x_2}{R + x_3}$ is an eigenvector of H_x with eigenvalue $(1 - 2\exp(-R^{-2})R)$, and hence so is its scalar multiple v. Equation (8-2) follows. □

Corollary 8.2. *There exists a convex, simply connected domain $\mathcal{O} \subset \mathbb{R}^3$ and a family of Hermitian forms H_x on $\mathbb{C}^2 \times \mathbb{C}^2$ parameterized smoothly by $x \in \mathcal{O}$ such that*:

(1) *For every $x \in \mathbb{R}^3$, there exists a vector $v \in \mathbb{C}^2$ such that $H_x(v, v) > 0$.*

(2) *If $v : \mathcal{O} \to \mathbb{C}^2$ is a nowhere vanishing continuous vector field, then there exists $x \in \mathcal{O}$ at which $H_x(v(x), v(x)) < 0$.*

Proof. It suffices to note that

$$\lim_{R \to +\infty} 1 - 2\exp(-R^{-2})R = -\infty,$$

so for R sufficiently large the right-hand side of (8-2) is strictly negative if we let $\mathcal{O} = B(0, 2R)$. □

8.2. *Domains.* For $n \geq 1$ and $1 \leq q \leq n$, set

$$\tilde{M}_q^n = \mathbb{C}^{n-q+1} \times \mathbb{CP}^{q-1}.$$

If we equip $\mathbb{C}^{n-q+1}$ with holomorphic coordinates $\{z_j\}_1^{n-q+1}$, then $\varphi(z) = |z_1|^2 + \cdots + |z_{n-q+1}|^2$ is an exhaustion function for $\tilde{M}_q^n$ such that the complex Hessian has precisely $n-q+1$ positive eigenvalues at every point of $\tilde{M}_q^n$. Furthermore, the complex Hessian of φ is strictly q-positive with respect to any Hermitian metric on $\tilde{M}_q^n$. However, the complex Hessian of φ has a kernel of dimension 2 when $q \geq 3$, so we should not expect the second case in Theorem 1.4 to hold.

Suppose $q \geq 2$. Any C^2 function on $\mathbb{CP}^{q-1}$ must have at least one point at which the complex Hessian has no positive eigenvalues, e.g., the point at which the function achieves its maximum on $\mathbb{CP}^{q-1}$. This means that any C^2 function on $\tilde{M}_q^n$ must have at least one point at which the complex Hessian has at most $n-q+1$ positive eigenvalues, so it is impossible to build an exhaustion function for which the complex Hessian has at least $n-(q-2) = n-q+2$ positive eigenvalues at every point of $\tilde{M}_q^n$.

If $\Omega' \subset \mathbb{C}^{n-q+1}$ is a bounded domain with smooth, strictly pseudoconvex boundary, then $\Omega = \Omega' \times \mathbb{CP}^{q-1}$ will be a domain in $\tilde{M}_q^n$ satisfying the hypotheses of Theorems 1.4 and 1.2.

For $n \geq 2$ and $1 \leq q \leq n$, set

$$|w|_+^2 = \sum_{j=1}^{n-q+1} |w_j|^2 \text{ and } |w|_-^2 = \sum_{j=n-q+2}^{n+1} |w_j|^2.$$

Equipping $\mathbb{CP}^n$ with homogeneous coordinates $[w_1 : \ldots : w_{n+1}]$, define a non-compact submanifold $M_q^n \subset \mathbb{CP}^n$ by

$$M_q^n = \left\{ [w_1 : \cdots : w_{n+1}] \in \mathbb{CP}^n : |w|_+^2 < |w|_-^2 \right\}.$$

The function

$$\phi(w) = -\log\left(1 - \frac{|w|_+^2}{|w|_-^2}\right)$$

induces a well-defined exhaustion function on M_q^n. One can check that the complex Hessian of φ has exactly $n-q+1$ positive eigenvalues. Observe that $S = \{[w_1 : \cdots : w_n] \in \mathbb{CP}^n : |w|_+ = 0\}$ is a $(q-1)$-dimensional complex submanifold of M_q^n, and hence, as argued above, any C^2 function on M_q^n must have at least one point at which the complex Hessian has at most $n-q+1$ positive eigenvalues. Indeed, the complex Hessian of ϕ has precisely $q-1$ negative eigenvalues on $M_q^n \backslash S$, but these $q-1$ eigenvalues vanish on S.

Let $\{\mu_j\}_{1\leq j\leq n+1}$ be a sequence of real numbers such that $\mu_j \neq 0$ for any $1 \leq j \leq n+1$, $\mu_j < 0$ for at least one $1 \leq j \leq n+1$, $\mu_j > 1$ for all $1 \leq j \leq n-q+1$, and $\mu_j > -1$ for all $n-q+2 \leq j \leq n+1$. Then $\sum_{j=1}^{n+1} \mu_j |w_j|^2 > |w|_+^2 - |w|_-^2$ for all $w \in \mathbb{C}^{n+1}$, so

$$\Omega = \left\{[w_1 : \cdots : w_{n+1}] \in \mathbb{CP}^n : \sum_{j=1}^{n+1} \mu_j |w_j|^2 < 0\right\}$$

is a subset of M_q^n satisfying the hypotheses of Theorems 1.2 and 1.4.

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Received March 14, 2025. Revised September 19, 2025.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 342, No. 1, 2026
https://doi.org/10.2140/pjm.2026.342.37

POLYNOMIAL IDENTITIES AND AZUMAYA LOCI FOR RATIONAL QUANTUM SPHERES

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We prove a number of structure and isomorphism results concerning the noncommutative Natsume–Olsen spheres $\mathbb{S}_\theta^{2n-1}$ deformed along a skew-symmetric matrix $\theta \in \mathbb{R}$. These include (a) the fact that two C^*-algebras of the form $\mathbb{S}_\theta^3 \otimes M_n$ are isomorphic precisely in the obvious cases; (b) the fact that m and n are recoverable from the isomorphism class of $C(\mathbb{S}_\theta^{2m-1}) \otimes M_n$; (c) the PI character, PI degree and Azumaya loci of $C(\mathbb{S}_\theta^{2m-1})$ for rational θ, along with a realization of their centers as (function algebras of) branched cover of $\mathbb{S}^{2n-1}$; and (d) for rational θ again, the topological finite generation of $C(\mathbb{S}_\theta^{2m-1})$ over their centers, with algebraic finite generation equivalent to being classical (equivalently, Azumaya).

Introduction

Consider a skew-symmetric real matrix $\theta \in M_n(\mathbb{R})$. We will be working extensively with the following noncommutative-geometric constructs.

- The *noncommutative* (or *quantum*) *tori* $\mathbb{T}_\theta^n$ [57, §1; 32, §12.2; 39, §1.1.5] defined as objects dual to the generator-and-relation C^*-algebras

$$A_\theta^n = C(\mathbb{T}_\theta^n) := \left\langle \text{unitaries } u_j,\ j \in [n] := \{1..n\} \;\middle|\; u_k u_j = e(\theta_{jk}) u_j u_k \right\rangle$$

for $e(-) := \exp(2\pi i -)$.

- The *noncommutative spheres* [48, Definition 2.1] analogously defined by

$$C(\mathbb{S}_\theta^{2n-1}) := \left\langle \text{normal } t_j,\ j \in [n] \;\middle|\; t_k t_j = e(\theta_{jk}) t_j t_k,\ \sum t_j^* t_j = 1 \right\rangle.$$

The former "glue" to produce the latter in ways reminiscent of classical topology (e.g., the *standard genus*-1 (*Heegaard*) *splitting* $\mathbb{S}^3 = (\mathbb{T}^2 \times [0,1]) \cup_{\mathbb{T}^2} (\mathbb{T}^2 \times [0,1])$ of [61, Proposition 3.3]): per [48, Theorem 2.5], we have

$$C(\mathbb{S}_\theta^{2n-1}) \cong \mathrm{Cont}_\partial\big(\mathbb{S}_+^{n-1} \to C(\mathbb{T}_\theta^n)\big), \tag{0-1}$$

MSC2020: 46L52, 16H05, 46M20, 55R25, 55R37, 46L85, 16R10, 55R40.
Keywords: Azumaya algebra, Chern class, PI algebra, bundle, classifying space, noncommutative sphere, noncommutative torus, projectively flat.

where

$$\begin{aligned} \mathbb{S}^{n-1}_{+} &:= \{(s_1,\ldots,s_n)\in\mathbb{S}^{n-1}\subseteq\mathbb{R}^n\ :\ s_i\ge 0\},\\ \mathbb{S}^{n-1}_{+,F\subseteq[n]} &:= \{(s_i)\in\mathbb{S}^{n-1}_{+}\ :\ s_i=0,\ \forall i\notin F\}, \end{aligned} \tag{0-2a}$$

and '∂' means

$$f(\mathbb{S}^{n-1}_{+,F})\subseteq C^*(u_i,\ i\in F)\subseteq A^n_\theta,\quad \forall F\subseteq[n]. \tag{0-2b}$$

Section 1 is concerned with reconstruction/isomorphism problems, i.e., the extent to which initial data (the deformation parameter $\theta\in M_n(\mathbb{R})$, or perhaps its size n) are determined by the isomorphism or stable isomorphism classes of the quantum-sphere algebras. A paraphrased aggregate of Theorems 1.1 and 1.2 reads as follows.

Theorem A. (1) *For $n,n'\in\mathbb{Z}_{>0}$ and $\theta,\theta'\in\mathbb{R}$ identified with skew-symmetric 2×2 matrices we have*

$$\begin{aligned} C(\mathbb{S}^3_\theta)\otimes M_n\cong C(\mathbb{S}^3_{\theta'})\otimes M_{n'} &\iff C(\mathbb{T}^2_\theta)\otimes M_n\cong C(\mathbb{T}^2_{\theta'})\otimes M_{n'}\\ &\iff n=n'\ \text{ and }\ \theta\in\pm\theta'+\mathbb{Z}. \end{aligned}$$

(2) *For positive integers m,n with $m\ge 2$ and skew-symmetric $\theta\in M_m(\mathbb{R})$ the isomorphism class of either $C(\mathbb{T}^m_\theta)\otimes M_n$ or $C(\mathbb{S}^{2m-1}_\theta)\otimes M_n$ determines m and n.*

In Section 2 and onward we assume the deformation parameter $\theta\in M_n(\mathbb{Q})$ rational and examine consequent *polynomial-identity* (*PI*) phenomena (on which background the main text offers a brief reminder). The focus is on

- *Azumaya algebras* [16, §4.4]: one way to formalize a purely algebraic analogue of the section-space $\Gamma(\mathcal{E}\otimes\mathcal{E}^*)$ for a vector bundle $\mathcal{E}$ over a compact Hausdorff space; and
- the extent to which algebras fail to qualify, as measured by the *Azumaya locus* [11, §III.1.7].

Again summarizing for brevity, one rendition of Theorems 2.8 and 2.13 below is this:

Theorem B. *Let $\theta\in M_n(\mathbb{Q})$ be a skew-symmetric matrix for $n\in\mathbb{Z}_{\ge 2}$.*

(1) *The center $Z_\theta\le C_\theta:=C(\mathbb{S}^{2n-1}_\theta)$ is the function algebra of a branched cover of $\mathbb{S}^{2n-1}$ which is not, in general, a topological manifold.*

(2) *The smallest n^2 admitting a Z_θ-algebra embedding $C_\theta\le M_n(Z_\theta)$ is*

$$h_\theta:=[(\mathbb{Z}^n+\operatorname{im}\theta)\ :\ \mathbb{Z}^n],$$

*so in particular C_θ is a PI algebra of **PI-degree*** (see [29, §1]) $\sqrt{h_\theta}$.

(3) *The Azumaya locus of* C_θ *consists precisely of those maximal ideals* $p \in \mathrm{Max}(Z_\theta)$ *whose restriction to* $\mathbb{S}_+^{n-1}$ *along equation (0-1) belongs to some face supported by* $F \subseteq [n]$ *with*

$h_\theta > h_{\theta,F} :=$ *h attached to the submatrix of* θ *supported on F-indexed rows/columns.*

In Section 3 we turn to the intimate connection between polynomial identities and the condition that an algebra be finitely generated as a module over its center (see e.g. [54, Chapter 6] and the motivating discussion on [3, p. 532]) in the context of studying the quantum spheres $\mathbb{S}_\theta^{2n-1}$. It will turn out that said finite generation (mostly) fails, but its weaker topological analogue (requiring that $C(\mathbb{S}_\theta^{2n-1})$ contain a dense finitely generated module over its center) always holds; per Theorem 3.3:

Theorem C. *For* $n \in \mathbb{Z}_{\ge 2}$ *and rational skew-symmetric* $\theta \in M_n(\mathbb{Q})$ *the algebra* $C(\mathbb{S}_\theta^{2n-1})$ *is*

- *always a topologically finitely generated module over its center* Z_θ*, but*
- ***algebraically** finitely generated as such precisely when* θ *is integral.*

Theorems B and C both link naturally to the theory of *Banach*, *Hilbert* and C^* *bundles* [27, §II.13; 21, pp. 7–9; 20, §1] over compact Hausdorff spaces and satellite topics: the theory of *noncommutative branched covers* initiated in [51] and phrased in the language of *finite-index conditional expectations* [28, Definition 2] is germane to the discussion below, which relies directly or indirectly on material from [7; 13; 51].

1. Isomorphisms of rationally deformed quantum 3-spheres

Throughout, unqualified (typically vector or algebra) *bundles* are assumed *locally trivial* [36, Definition 1.1.8]; they are to be distinguished from more general constructs termed (*F*) *Banach bundles* on [21, pp. 7–8], which will also make an appearance in Section 3.

The isomorphism problem for noncommutative 3-spheres is not difficult to resolve, given its 2-torus analogue. The following is very much in the spirit of [55, Theorem 3] for *irrational* tori. The *rational* torus version [42, Theorem 1.1] (recovered also as [56, Theorem 3.12]) typically does not involve matrix tensorands.

Theorem 1.1. *Consider* $n, n' \in \mathbb{Z}_{>0}$, $\theta, \theta' \in \mathbb{R}$, *and set*

$$C_{\theta,n} := C(\mathbb{S}_\theta^3) \otimes M_n, \quad A_{\theta,n} := C(\mathbb{T}_\theta^2) \otimes M_n,$$

and similarly for the primed parameters. The following conditions are equivalent.

(a) *We have an isomorphism*

$$C_{\theta,n} \cong C_{\theta',n'}. \tag{1-1}$$

(b) *We have an isomorphism*

$$A_{\theta,n} \cong A_{\theta',n'}. \tag{1-2}$$

(c) *The parameters coincide save for trivial modifications, in the sense that*

$$n = n' \quad \textit{and} \quad \theta \in \pm\theta' + \mathbb{Z}.$$

It might be worthwhile to first observe separately that the size n of the tensorand M_n can be recovered from the isomorphism class of either $C_{\theta,n}$ or $A_{\theta,n}$ and in fact, more generally, for noncommutative spheres/tori of arbitrary dimension.

Theorem 1.2. *Let $\theta \in M_m(\mathbb{R})$ be a skew-symmetric matrix, $m \ge 2$ and $n \in \mathbb{Z}_{>0}$.*

(1) *The isomorphism class of the C^*-algebra $C(\mathbb{T}^m_\theta) \otimes M_n$ determines m and n.*

(2) *The isomorphism class of the C^*-algebra $C(\mathbb{S}^{2m-1}_\theta) \otimes M_n$ determines m and n.*

Proof. We structure the argument conforming to the statement.

Statement (1) is a simple matter of unwinding some of the well-known K-theoretic results on noncommutative tori.

According to [22, Theorem 2.2], $K_0(C(\mathbb{T}^m_\theta))$ is free abelian of rank 2^{m-1} with the class of the identity as a generator. The usual isomorphism

$$K_0(A) \cong K_0(A \otimes M_n)$$

induced by the upper-corner embedding [67, Lemma 6.2.10] then makes it clear that in

$$K_0(C(\mathbb{T}^m_\theta) \otimes M_n) \cong \mathbb{Z}^{2^{m-1}}$$

the class of the identity is divisible precisely by n, hence the conclusion.

We turn to (2).

<u>Recovering n</u>: Equation (0-1) realizes $C(\mathbb{S}^{2m-1}_\theta)$ as a $C(X)$-*algebra* (i.e., a C^*-algebra receiving a central morphism from $C(X)$: see [38, Definition 1.5]) for $X = \mathbb{S}^{m-1}_+$. The *fibers*

$$C(\mathbb{S}^{2m-1}_\theta)|_p := C(\mathbb{S}^{2m-1}_\theta)/\big(C(\mathbb{S}^{2m-1}_\theta)\cdot\{f \in C(X) : f(p) = 0\}\big), \quad p \in \mathbb{S}^{n-1}_+,$$

are noncommutative-torus algebras A in general, and specifically to $C(\mathbb{S}^1)$ at the m vertices of the spherical simplex X. It follows that n can be recovered from the abstract C^*-algebra $C := C(\mathbb{S}^{2m-1}_\theta) \otimes M_n$ as the smallest dimension of an irreducible representation.

<u>Recovering m</u>: A small detour will first discern what can be recovered from the center

$$Z := Z(C(\mathbb{S}^{2m-1}_\theta) \otimes M_n) = Z(C(\mathbb{S}^{2m-1}_\theta)) = Z(C) \tag{1-3}$$

alone. The isomorphism (0-1) specializes to $Z \cong \mathrm{Cont}_\partial(\mathbb{S}^{m-1}_{+} \to Z(C(\mathbb{T}^n_\theta)))$, with ∂ indicating the central analogue of (0-2b):

$$f(\mathbb{S}^{m-1}_{+,F}) \subseteq Z(C(\mathbb{T}^n_\theta)) \cap C^*(\{u_j \mid j \in F\}). \tag{1-4}$$

Recall [23, §2.2] that $C(\mathbb{T}^n_\theta)$ is a cocycle twist $C^*(\mathbb{Z}^n, \sigma)$ of the group algebra $C^*(\mathbb{Z}^n)$ (with the generators of $\mathbb{Z}^n$ mapping to the u_i). The proof of [22, Lemma 2.3] then describes the center $Z(C(\mathbb{T}^n_\theta))$ as

$$\begin{aligned} &C^*(\Gamma) \subset C^*(\mathbb{Z}^n),\\ &\qquad \Gamma := \text{kernel of the bicharacter } \mathbb{Z}^n \wedge \mathbb{Z}^n \xrightarrow{e(\theta)=\exp(2\pi i\theta)} \mathbb{S}^1, \end{aligned} \tag{1-5}$$

with the matrix θ regarded, as usual, as a bilinear form. The boundary condition (1-4) then reads

$$f(\mathbb{S}^{m-1}_{+,F}) \subseteq C^*(\Gamma_F) \cong C(\mathbb{T}_F),\ \mathbb{T}_F := \widehat{\Gamma_F},$$

where

$$\Gamma_F := \Gamma \cap \mathbb{Z}^F, \quad \mathbb{Z}^F := \text{sum of the } F\text{-indexed summands in } \mathbb{Z}^n. \tag{1-6}$$

We write $\mathrm{Max}(A)$ for the *maximal spectrum* [5, Exercise 1.26] of a commutative ring, typically applying the notion to commutative C^*-algebras (and occasionally omitting the qualifier “maximal”). $X := \mathrm{Max}(Z)$ can thus be described as follows:

- To each finite-set inclusion

  $$F \subseteq F' \subseteq [m]$$

  associate

  $$\begin{aligned} &\text{an inclusion} \quad \Delta_F \hookrightarrow \Delta_{F'}, \quad \Delta_F := \mathbb{S}^{m-1}_{+,F},\\ &\text{and a quotient} \quad \mathbb{T}_F \twoheadrightarrow \mathbb{T}_{F'} \quad \text{dual to} \quad \Gamma_F \le \Gamma_{F'}. \end{aligned}$$

- And then recover X as the *coend* [43, §IX.6]

$$X = \mathrm{Max}(Z(C(\mathbb{S}^{2m-1}_\theta))) \cong \int^F \Delta_F \times \mathbb{T}_F. \tag{1-7}$$

This is a particular instance of the *geometric realization* [58, Definition 3.8.1] of a *simplicial object* [43, §VII.5]: more precisely, $F \mapsto \mathbb{T}_F$ is a simplicial topological space (a simplicial object in the category of topological spaces), and (1-7) is its simplicial realization.

The space (1-7) does have *dimension*

$$\dim X = m - 1 + \dim \mathrm{Max}(Z(C(\mathbb{T}^m_\theta))) = m - 1 + \operatorname{rank}\ker\theta, \tag{1-8}$$

where

– “dimension” is used in any of three senses: the *small inductive*, *large inductive* and *covering* dimensions [24, Definitions 1.1.1, 1.6.1, 1.6.7], all coincident for separable metric spaces [24, Theorem 1.7.7];

– and the *rank* of an abelian group is [34, Definition A1.59]

$$\operatorname{rank} \Gamma := \dim_{\mathbb{Q}} \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma.$$

By contrast to m (Example 1.3), this latter number (1-8) *is*, then, recoverable from the isomorphism class of $Z = Z(C)$ alone (so C itself is not needed for this).

The fibers of C at the various points of X are of the form $C^*(\mathbb{Z}^m/\Gamma_F, \sigma) \otimes M_n$ for subsets $F \subseteq [m]$ with the twist induced by the original cocycle σ on $\mathbb{Z}^n/\Gamma_F$ (possible, given that $\Gamma = \Gamma_{[m]}$ is by definition the kernel of θ).

The abelian groups $\mathbb{Z}^m/\Gamma_F$ have ranks

$$m - \operatorname{rank} \Gamma_F \geq m - \operatorname{rank} \Gamma_{[m]} = m - \operatorname{rank} \ker \theta, \tag{1-9}$$

with equality achieved. Since the sum between (1-8) and this minimum is $2m-1$, it will be enough to observe that the ranks on the left-hand side of (1-9) can be recovered from the corresponding fibers $C^*(\mathbb{Z}^m/\Gamma_F, \sigma) \otimes M_n$.

To see this,

- write

$$\mathbb{Z}^m/\Gamma_F \cong (\text{finite abelian group } H) \times \mathbb{Z}^r, \quad r := \operatorname{rank} \mathbb{Z}^m/\Gamma_F$$

  [10, p. VII.19, Theorem 2];

- so that $C^*(\mathbb{Z}^m/\Gamma_F, \sigma)$ can be expressed as an iterated, r-fold crossed product

$$C^*(H) \rtimes \mathbb{Z} \rtimes \cdots \rtimes \mathbb{Z}$$

  as in [32, Proposition 12.8];

- whence

$$K_0(C^*(\mathbb{Z}^m/\Gamma_F, \sigma) \otimes M_n) \cong K_0(C^*(\mathbb{Z}^m/\Gamma_F, \sigma)) \cong \mathbb{Z}^{2^{r-1}}$$

  as for noncommutative tori, via the *Pimsner–Voiculescu sequence* [6, Theorem 10.2.1].

This completes the proof. □

Note that the isomorphism class of the center (1-3) alone does *not*, in general, determine m.

Example 1.3. We exhibit noncommutative-sphere algebras $C(\mathbb{S}^{2m-1}_{\theta})$ with isomorphic centers and distinct m (so also, by necessity, distinct θ):

(I) Take first $m = 2$ and $\theta = 0$, so that

$$C(\mathbb{S}_\theta^{2m-1}) \cong C(\mathbb{S}^3)$$

and the spectrum of the center is the 3-sphere.

(II) On the other hand, take $m = 3$ and some θ for which the kernel Γ of (1-5) is of rank 1 (i.e., isomorphic to $\mathbb{Z}$) and not contained in any of the subgroups

$$\mathbb{Z}^F, \quad F \subset [3] \text{ properly}$$

of (1-6). The coend (1-7) is then

$$\mathbb{S}_+^2 \times \mathbb{S}^1 / ((x,t) = (x,t'),\ \forall x \in \partial S_+^2,\ t \in \mathbb{S}^1)$$

with ∂ denoting the boundary. Alternatively, in words: consider the product $\mathbb{D}^2 \times \mathbb{S}^1$ (where $\mathbb{D}^2$ is the closed 2-disk) and identify all copies

$$\mathbb{S}^1 \times \{t\} \subset \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{D}^2 \times \mathbb{S}^1$$

to each other in the obvious fashion. It is a simple exercise to see that this space is again (homeomorphic to) the 3-sphere.

Also in reference to Theorem 1.2, note that an isomorphism

$$A \otimes M_n \cong A' \otimes M_n \tag{1-10}$$

of unital C^*-algebras does not, in general, imply that A and A' themselves are isomorphic:

Example 1.4. Recall [45, §2] the unital C^*-algebras $A = U_{m,n}^{nc}$ generated by the mn elements of a unitary (possibly nonsquare) $m \times n$ matrix. As explained in loc.cit., we may as well assume that $m \le n$.

This is (by definition) the universal unital C^*-algebra A whose Hilbert modules A^m and A^n are isomorphic, so their endomorphism algebras (as Hilbert modules) will of course also be isomorphic:

$$A \otimes M_m \cong M_m(A) \cong M_n(A) \cong A \otimes M_n.$$

Now fix any $m \ge 2$, and consider the case $A = U_{m,m^2}^{nc}$. We then have

$$A \otimes M_m \cong A \otimes M_{m^2} \cong (A \otimes M_m) \otimes M_m,$$

which is (1-10) with $A' = A \otimes M_m$ and $n = m$. An isomorphism

$$A \cong A' \cong A \otimes M_m \tag{1-11}$$

does not exist though: according to the discussion immediately preceding [2, §6],

$$K_0(A) \cong \mathbb{Z}/(n-m) = \mathbb{Z}/(m^2 - m) \text{ generated by the class of } 1 \in A,$$

so that class cannot be a multiple of m, as it would be, given an isomorphism (1-11).

The ensuing discussion freely references *Azumaya algebras* (in their various incarnations, such as those in [1, Definition 5.1.4; 46, §13.7.6; 11, §III.1.3]) and ancillary notions, including the following.

- *Polynomial identities* (or *PIs* for short), for (always unital) rings A, are noncommutative polynomials vanishing when substituting arbitrary elements of A for the variables. (See [1, Definition 2.2.1].)

- *PI algebras* are those algebras A over commutative rings (usually fields) $\Bbbk$ satisfying at least one nonzero PI that is *stable* [1, §2.2.3] in the sense that all $A\otimes_{\Bbbk}\Bbbk'$ for commutative-ring extensions $\Bbbk\subseteq\Bbbk'$ satisfy a common nonzero identity. (See [1, Definition 2.2.41].)

- An algebra A over a commutative ring R is *Azumaya* (*of rank* n^2) as such [1, Definition 5.1.4] if for some faithfully flat extension $R\to R'$ we have

$$A\otimes_R R'\cong M_n(R').$$

- Equivalently [1, Theorem 10.3.2], A is rank-n^2 Azumaya (unqualified, i.e., over its center) precisely when it satisfies the polynomial identities of $n\times n$ matrices but has no quotient satisfying the identities of $(n-1)\times(n-1)$ matrices.

We refer the reader to the sources just cited, with more specific citations accompanying the text where appropriate.

Remark 1.5. We will repeatedly (and henceforth tacitly) take it for granted that being Azumaya over a commutative ring is stable under scalar extension: if A is Azumaya over R then $A_S := A\otimes_R S$ is Azumaya over S for any commutative-ring morphism $R\to S$ [1, Proposition 5.4.28].

Proof of Theorem 1.1. That (c) implies the other conditions is clear enough: for tori transitioning between $\pm\theta$ is effected by interchanging the two unitary generators, and the realization (0-1) settles the matter for spheres. We thus focus on the converse implications, seeking to deduce (c) from each of the other conditions.

(I) Rational vs. irrational parameters. Given an isomorphism (1-1), the deformation parameters θ and θ' are simultaneously either rational or irrational.

Indeed, if one of them (θ, say) is irrational, then $C_{\theta,n}$ has infinite-dimensional *simple* quotients isomorphic to $A_{\theta,n}$ (simple, being a minimal tensor product of simple C^*-algebras: [32, Corollary 12.12] and [64, Corollary IV.4.21]). Furthermore, (0-1) makes it clear that *all* infinite-dimensional simple quotients are of this form.

On the other hand, if $\theta'\in\mathbb{Q}$ then the simple quotients of $C_{\theta',n'}$ are plainly finite-dimensional, again by (0-1).

The same argument also (more easily) handles (1-2).

(II) <u>The irrational case.</u> This is the simpler branch of the argument: we have already observed that if θ and θ' are both irrational then (1-1) implies (1-2), $A_{\theta,n}$ and $A_{\theta',n'}$ being the only infinite-dimensional simple quotients of $C_{\theta,n}$ and $C_{\theta',n'}$ respectively. We can then simply appeal to [56, Theorem 3.12].

(III) <u>The rational case.</u> We are now assuming that

$$\theta = \frac{p}{q} \quad \text{and} \quad \theta' = \frac{p'}{q'}$$

are rational (in lowest terms, as depicted).

We focus mostly on the implication (a) $\Longrightarrow$ (c), as it will turn out to be more elaborate, indicating along the way where and how the argument changes so as to also deliver (b) $\Longrightarrow$ (c). There is no harm in assuming $n = n'$ throughout, as Theorem 1.2 allows.

An isomorphism (1-1) induces isomorphisms between

- the centers $C(\mathbb{S}^3)$ of the two C^*-algebras ([15, Proposition 4.5(1)]),
- and hence also between the $q \times q$ matrix bundles over the Azumaya loci

$$\mathbb{S}^3 \setminus (\sqcup \text{ two circles}) \cong \mathbb{T}^2 \times (0, 1)$$

of $C_{\theta,n}$ and $C_{\theta',n'}$ [15, Proposition 4.5(3)].

Over those Azumaya loci, $C_{\theta,n}$ and $C_{\theta',n'}$ are isomorphic $qn \times qn$ and $q'n' \times q'n'$ matrix bundles over $X = \mathbb{T}^2 \times (0, 1)$. We thus have $qn = q'n'$, and because

- matrix bundles are classifiable homotopically [36, §18.2.4],
- X deformation-retracts onto its slice

$$\mathbb{T}^2 \cong \mathbb{T}^2 \times \left\{\tfrac{1}{2}\right\} \subset X,$$

- and over that slice the two algebras are $A_{\theta,n}$ and $A_{\theta',n'}$, respectively,

the implication (b) $\Longrightarrow$ (c) reduces to (a) $\Longrightarrow$ (c), on which we henceforth focus. We have been assuming that $n = n'$, so that $q = q'$ because $qn = q'n'$. The rest is simple conceptually, but requires a bit of unwinding of the attendant bundle-classification theory.

The $qn \times qn$-matrix bundles on $X = \mathbb{T}^2 \times (0, 1)$ are classified by homotopy classes of maps $X \to BPU(qn)$ into the *classifying space* [36, §7.2, Definition 2.7] of the projective $qn \times qn$ unitary group ([36, §18.3, Assertion 3.2 and Remark 3.3]). Writing $[-, -]$ for homotopy classes of maps [36, §6.4, Definition 4.2], we have

$$[X, -] \cong [\mathbb{T}^2, -],$$

so we may as well assume we are working with the 2-torus.

We have an exact sequence

$$[\mathbb{T}^2,\ BSU(qn)] \to [\mathbb{T}^2,\ BPU(qn)] \to [\mathbb{T}^2,\ B^2\mathbb{Z}/qn] \cong H^2(\mathbb{T}^2,\ \mathbb{Z}/qn) \cong \mathbb{Z}/qn \tag{1-12}$$

[41, Corollary 3.2.7] attached to the top row of [36, §18.3.6], where B^2 denotes the iterated classifying-space construction discussed in [36, §7.4.6] for *abelian* groups (such as $\mathbb{Z}/qn$). The second arrow in (1-12) is an *embedding*, given that the leftmost term $[\mathbb{T}^2,\ BSU(qn)]$ is trivial: it classifies degree-0 rank-qn vector bundles on the 2-torus, and those are trivial [65, p. 2, Proposition]. In other words, $qn \times qn$-matrix-algebra bundles over the 2-torus are classified by the characteristic class denoted by β_{qn} in [36, §18.3.7] (our qn being that source's n).

Denote by $\widetilde{(-)}_q$ inverses modulo q and similarly for q'. In our case, focusing on the un-primed side, the $qn \times qn$-matrix bundle in question is of the form $\mathcal{E} \otimes \mathcal{E}^*$ for a vector bundle $\mathcal{E} \cong \mathcal{F} \otimes \mathbb{C}^n$ with rank-q degree-$\widetilde{p}_q$ $\mathcal{F}$, and the class β_{qn} is the image of the Chern class

$$n\widetilde{p}_q \in [\mathbb{T}^2,\ B^2\mathbb{Z}] \cong H^2(\mathbb{T}^2,\ \mathbb{Z}) \cong \mathbb{Z}$$

through the natural map $\mathbb{Z} \to \mathbb{Z}/nq$ (arising from the vertical maps in [36, §18.3.4]) so all in all, the isomorphism of matrix bundles simply amounts to

$$\widetilde{p}_q n = \pm \widetilde{p}\,'_q n \mod qn$$

(the $\pm$ depending on whether the homeomorphism between the two spheres preserves or reverses orientations). This is in turn equivalent to

$$qn \mid \widetilde{p}_q n \pm \widetilde{p}\,'_q n \iff q \mid \widetilde{p}_q \pm \widetilde{p}'_q \iff q \mid p \pm p' \iff \theta + \mathbb{Z} = \pm\theta' + \mathbb{Z}.$$

This concludes the proof. □

Remarks 1.6. (1) The obvious analogue of [55, Theorem 3] does in fact go through for rational deformation parameters, giving an extension of [42, Theorem 1.1].

(2) The literature on the classification of noncommutative tori up to isomorphism or Morita equivalence is extensive: the reader can consult [57; 12; 23], for instance.

(3) The proof of Theorem 1.1 takes for granted the following discussion on Chern classes and their relation to matrix-algebra-bundle characteristic classes.

The fibration sequence associated (as in [36, §18.3.6]) to the exact sequence

$$1 \to SU(n) \to U(n) \xrightarrow{\det} \mathbb{S}^1 \to 1$$

and (part of) the top portion of the diagram in [36, §18.3.6] fit together into a commutative diagram

(1-13)
$$\begin{array}{ccccccc} & & BU(n) & \xrightarrow{B\det} & B\mathbb{S}^1 & \xrightarrow{\cong} & K(\mathbb{Z},2) \\ & \nearrow & \downarrow & & \downarrow & & \downarrow \\ BSU(n) & & & & & & \\ & \searrow & BPU(n) & \longrightarrow & B^2\mathbb{Z}/n & \xrightarrow{\cong} & K(\mathbb{Z}/n,2) \end{array}$$

where

- $K(G,m)$ denotes, as usual [36, Definition 9.6.1], the m-th *Eilenberg–Mac Lane space* with homotopy group $\pi_n \cong G$ and vanishing homotopy in other degrees;
- the left-hand vertical map is the obvious one, resulting from the quotient $U(n) \to PU(n)$ that mods out the center;
- the middle vertical arrow is obtained by applying the classifying-space (homotopy) functor $B(-)$ [60, §3.7; 47, Proposition 8.1] to the map

$$\mathbb{S}^1 \to B\mathbb{Z}/n$$

classifying the $\mathbb{Z}/n$-bundle on the circle obtained via the n-fold cover

$$\mathbb{S}^1 \xhookrightarrow{\text{central embedding}} U(n) \xrightarrow{\det} \mathbb{S}^1;$$

- and the right-hand vertical map induced by $\mathbb{Z} \to \mathbb{Z}/n$, well-defined only upon choosing a generator for $\mathbb{Z}/n$ (so a choice is involved).

For a space X the upper composition

$$(n\text{-bundles on } X) \xrightarrow[\cong]{\text{[36, Assertion 18.3.2]}} [X,\ BU(n)] \to [X,\ K(\mathbb{Z},2)] \cong H^2(X,\mathbb{Z})$$

simply associates the first Chern class $c_1(\mathcal{E}) \in H^2(X,\mathbb{Z})$ to a vector bundle $\mathcal{E}$ on X, whereas the bottom composition

$$(n^2\text{-matrix bundles on } X) \xrightarrow{\cong} [X,\ BPU(n)] \to [X,\ K(\mathbb{Z}/n,2)] \cong H^2(X,\mathbb{Z}/n)$$

(where again the first isomorphism comes from [36, Assertion 18.3.2]) similarly associates the class $\beta_n(\mathcal{A})$ of [36, §18.3.7] to a matrix-algebra bundle $\mathcal{A}$.

In conclusion, the Chern class $c_1(\mathcal{E}) \in H^2(X,\mathbb{Z})$ of a vector bundle gets mapped to the $H^2(X,\mathbb{Z}/n)$-characteristic class of the corresponding matrix-algebra bundle $\mathcal{E} \otimes \mathcal{E}^*$.

(4) We will refer to the class $\beta_n \in H^2(X,\mathbb{Z}/n)$ associated in [36, §18.3.7] to an $n \times n$ matrix-algebra bundle on X as *the characteristic* 2*-class* of the matrix bundle, to distinguish it from the *Dixmier–Douady* (3-)class $\alpha \in H^3(X,\mathbb{Z})$ also discussed in [36, §18.3.7].

2. Azumaya theory for higher quantum spheres

Much can be said about higher-dimensional rational tori $C(\mathbb{T}^n_\theta) = A^n_\theta$ [32, §12.2] as well. These algebras are also Azumaya, (essentially) by [16, Proposition 7.2]. The setup there is more algebraically oriented so the discussion requires some translation, but nothing particularly problematic. The deformation-parameter data in [16, §7] consists of

- a primitive ℓ-th root of unity q in the ground field (which for us is $\mathbb{C}$); and
- a skew-symmetric $n \times n$ matrix $H = (h_{ij}) \in M_n(\mathbb{Z})$, which will turn out to be an integer multiple of our θ.

The n invertible generators x_i of loc. cit. are required to skew-commute in the sense that

$$x_j x_k = q^{h_{jk}} x_k x_j; \tag{2-1}$$

compare this with the usual skew-commutation relation [57, p. 193]

$$u_k u_j = e^{2\pi i \theta_{jk}} u_j u_k \tag{2-2}$$

for the unitary generators of A^n_θ. Now, if ℓ is a positive integer divisible by the denominators of all θ_{jk}, then

$$x_\bullet = u_\bullet, \quad q = e^{-\frac{2\pi i}{\ell}}, \quad H = \ell\theta \in M_n(\mathbb{Z})$$

will effect the transition between equations (2-1) and (2-2).

Proposition 7.2 of [16] (or rather the appropriately translated version) then shows that A^n_θ is Azumaya of rank

$$h = h_\theta := \text{cardinality of range}\big(\mathbb{Z}^n \xrightarrow{H=\ell\theta} (\mathbb{Z}/\ell)^n\big), \tag{2-3}$$

where a matrix is regarded as an operator in the usual way.

Note that although there was a choice in selecting ℓ, h itself will not depend on that choice: scaling ℓ to, say, $d\ell$ will also scale H by d, thus preserving the size of its image. In fact, the number (2-3) has the following alternative, direct description in terms of the matrix θ alone:

Lemma 2.1. *For a rational skew-symmetric $\theta \in M_n(\mathbb{Q})$ the number (2-3) can be recovered as the index*

$$h_\theta = [(\mathbb{Z}^n + \operatorname{im}\theta) : \mathbb{Z}^n]. \tag{2-4}$$

Proof. We can assume [9, §IX.5.1, Théorème 1] that θ is block-diagonal, consisting of a zero block and one of the form

$$\left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right) \tag{2-5}$$

for a diagonal matrix

$$D = \mathrm{diag}(p_1/q_1,\ p_2/q_2,\ \dots)$$

with lowest-terms nonzero entries. The quantities in (2-3) and (2-4) are now easily seen to both be equal to $\prod_i q_i^2$. ☐

Remark 2.2. The proof of Lemma 2.1 also makes it plain that (2-3) is indeed a square, as implicit in the discussion: it is the common dimension of the matrix-algebra fibers of an Azumaya algebra.

Still assuming θ rational, we have:

Notation 2.3. Consider a skew-symmetric matrix $\theta \in M_n(\mathbb{R})$ (usually rational).

(1) For a tuple $\boldsymbol{u} := (u_i)_{i=1}^n$ of generators of A_θ^n and an integer tuple $\boldsymbol{m} = (m_i)_{i=1}^n$ we write

$$\boldsymbol{u}^{\boldsymbol{m}} := \prod_{i=1}^{n} u_i^{m_i}.$$

The product is to be understood as ordering the indices increasingly unless otherwise specified, but at no point will the ordering in fact matter: we will mostly be interested in C^*-subalgebras of A_θ^n generated by such products, and a reordering simply scales by a modulus-1 complex number.

(2) The *integral kernel* of θ is

$$\theta^\perp := \{\boldsymbol{m} \in \mathbb{Z}^n \ :\ \theta \boldsymbol{m} \in \mathbb{Z}^n\},$$

regarding θ as a linear endomorphism on $\mathbb{R}^n$.

(3) For $F \subseteq [n]$ write

- $\theta|_F$ for the $|F| \times |F|$ matrix consisting of θ-entries in F-labeled rows and columns;
- $h_{\theta,F} := h_{\theta|_F}$, with $h_\bullet$ as in (2-4).

The realization of A_θ^n as a cocycle twist of the group algebra of $\mathbb{Z}^n$ [23, §2.2], coupled with the proof of [22, Lemma 2.3], describing the center of such a twisted group algebra, gives

$$Z(A_\theta^n) = C^*(\boldsymbol{u}^{\theta^\perp}) := C^*(\boldsymbol{u}^{\boldsymbol{m}} \ :\ \boldsymbol{m} \in \theta^\perp) \overset{\theta \in M_n(\mathbb{Q})}{\cong} C^*(\mathbb{Z}^n) \cong C(\mathbb{T}^n) \tag{2-6}$$

By the Azumaya claim we can once more realize A_θ^n as sections of a bundle of dimension-h_θ matrix algebras over $\mathbb{T}^n$ for the h_θ of (2-4).

Remark 2.4. In dealing with vector bundles over higher-dimensional tori, one additional complication not visible for $n=2$ (or even $n=3$) is that the cohomological classification recalled in [15, Remark 4.4(1)] no longer goes through: as observed in the introductory remarks (p. 247) of [49] and elaborated in § 3 of the same work, there are nonflat rank-2 vector bundles on complex 2-dimensional tori (*four*-dimensional when regarded as real tori) with vanishing Chern classes.

Such pathologies thus occur as soon as one can reasonably expect them, namely whenever the torus and bundle are large enough for both first Chern classes c_1 and c_2 to come into play, meaning the bundle has rank ≥ 2 and the torus dimension ≥ 4.

Incidentally, the proof of Lemma 2.1 also shows that rational noncommutative tori are decomposable into small-dimensional pieces:

Lemma 2.5. *For a skew-symmetric matrix $\theta \in M_n(\mathbb{Q})$ the corresponding noncommutative torus algebra A^n_θ decomposes as*

$$A^n_\theta \cong C(\mathbb{T}^k) \otimes \bigotimes_{i=1}^{t} A^2_{\theta_i}$$

for rationals $\theta_i \in \mathbb{Q}$ and $n = k + 2t$.

Proof. Immediate from the observation made in passing in the proof of Lemma 2.1 that after a change of (integer) basis θ is a direct sum of a ($k \times k$, say) zero matrix and a ($2t \times 2t$) matrix of the form (2-5). $\square$

It also follows that, notwithstanding Remark 2.4 (and [15, Remark 4.4(1)]), the bundles relevant to the Azumaya structure of A^n_θ are relatively well-behaved. To make sense of the statement, recall [40, §§I.2 and I.4] the notion of a *projectively flat* vector bundle over a space X (either plain or *Hermitian*, i.e., with structure group $G = \mathrm{GL}(r)$ [40, §I.4] or $G = U(r)$), respectively. Then:

- Consider the *principal* G-bundle $P \to X$ associated to the vector bundle E in question, via the usual correspondence [36, §18.2, Assertion 2.3].
- The quotient $\overline{P} := P/Z(G)$ by the center of G (scalar matrices, so either $\mathbb{C}^\times$ when $G = \mathrm{GL}(r)$ or $\mathbb{S}^1$ for $G = U(r)$) is then a principal bundle over the *projective* group
$$PG := G/Z(G) = \mathrm{PGL}(r) \text{ or } PU(r);$$
- The original bundle is projectively flat if $\overline{P}$ is flat in the usual sense [40, Proposition I.2.6].

Proposition 2.6. *For skew-symmetric $\theta \in M_n(\mathbb{Q})$ we have*

$$A^n_\theta \cong \mathrm{End}(\mathcal{E}) = \Gamma(\mathcal{E} \otimes \mathcal{E}^*) := (\textit{continuous sections of the bundle } \mathcal{E} \otimes \mathcal{E}^*)$$

for a projectively flat bundle $\mathcal{E}$ on $\mathbb{T}^n \cong \mathrm{Max}(Z(A^n_\theta))$ of rank $\sqrt{h_\theta}$ for $h_\theta =$ (2-4).

In particular, $\mathcal{E} \otimes \mathcal{E}^*$ itself is a flat matrix-algebra bundle.

Proof. That the rank is as claimed has already been noted above, as a consequence of [16, Proposition 7.2]. Lemma 2.5 reduces the problem to noncommutative 2-tori. Indeed, it shows that $\mathcal{E}$ decomposes as an *external* (*or exterior*) *tensor product* [4, §2.6], and external tensor products preserve projective flatness: the latter is definable [40, Proposition I.2.8] as the existence of a connection whose curvature takes scalar values, and there is a simple formula for the curvature of the tensor product of the natural two connections [40, §I.5, (5.15)].

The conclusion now follows from the fact that *all* complex vector bundles on a 2-torus (and indeed a compact orientable surface) are projectively flat: being completely classified by rank and Chern class ([65, Proposition, p. 2] again) a rank-q bundle splits as the sum between a line bundle and a trivial rank-$(q-1)$ bundle, etc.

As for the last claim on flatness, it follows from the general remark that $\mathcal{E}\otimes\mathcal{E}^*$ is flat whenever $\mathcal{E}$ is projectively flat [40, Propositions I.2.9 and/or I.4.23]. □

Remark 2.7. The flatness of the relevant $q\times q$ matrix bundle for $n=2$ and $\theta=p/q$ is plain from its direct construction in the proof of [32, Proposition 12.2] (or [42, §2], on which the latter account is based): the bundle is obtained as the (total space of the) quotient

$$(\mathbb{T}^2\times M_q)/(\mathbb{Z}/q)^2$$

through the diagonal action, where

$$(\mathbb{Z}/q)^2\subset\mathbb{T}^2$$

acts on the torus via translation by the q-torsion subgroup and on M_q as described in loc. cit. (In particular, the two generators act by conjugation by two order-q unitary matrices that commute up to scalars.)

A few reminders will help make sense of the statement of Theorem 2.8, a higher-dimensional generalization of [15, Proposition 4.5].

- The *PI-degree* [19, Definitions A.7.1.8 and B.4.15; 29, §1] of a PI algebra is the largest n for which the polynomial identities of A are among those of M_n. Rank-n^2 Azumaya algebras, for instance, have PI-degree n (as observed just after Definition B.4.15 of [19]).

The slight definition variations one typically encounters in the literature will not make a difference here, so the above will do.

- The *Azumaya locus* of an algebra A consists of those maximal ideals

$$\mathfrak{m}\in\mathrm{Max}(Z:=Z(A)):=\text{maximal spectrum of the center } Z$$

for which the *localization* [5, Example (1) post Corollary 3.2] $A_{\mathfrak{m}}$ is Azumaya over $Z_{\mathfrak{m}}$.

There are again minor departures from this setup in the literature (one might, for instance, consider all *prime* ideals with the requisite property [66, §1], impose additional finiteness/primality constraints on the algebra A [11, §III.1.7]).

Set

$$I_Y := \{f|_Y \equiv 0\} \trianglelefteq \mathrm{Cont}_\partial(\mathbb{S}_+^{n-1} \to C(\mathbb{T}_\theta^n)) \overset{(0\text{-}1)}{\cong} C_\theta, \quad Y = \overline{Y} \subseteq \mathbb{S}_+^{n-1} \tag{2-7}$$

and

$$C_\theta|_Y := C_\theta / I_Y, \quad Z_\theta|_Y := Z_\theta/(I_Y \cap Z_\theta). \tag{2-8}$$

For subsets $S \subseteq \mathbb{Z}^n$ and $F \subseteq [n]$ we write

$$S_{\downarrow F} := \{\boldsymbol{s} = (s_i)_{i=1}^n \in S \ : \ s_i = 0, \ \forall i \notin F\};$$

the portion of S *supported on* F, in other words. This applies to the symbol $\theta^\perp_{\downarrow F}$ employed below.

Theorem 2.8. *Let $\theta \in M_n(\mathbb{Q})$ be an $n \times n$ skew-symmetric rational matrix for some fixed $n \geq 2$.*

(1) *The center of the noncommutative sphere algebra $C_\theta := C(\mathbb{S}_\theta^{2n-1})$ is*

$$\begin{aligned} Z_\theta &:= Z(C(\mathbb{S}_\theta^{2n-1})) \\ &\cong \mathrm{Cont}_\partial(\mathbb{S}_+^{n-1} \to Z(C(\mathbb{T}_\theta^n))) \\ &= \{\mathbb{S}_+^{n-1} \xrightarrow[\text{cont.}]{f} C^*(\boldsymbol{u}^{\theta^\perp}) \ : \ f(\mathbb{S}_{+,F}^{n-1}) \in C^*(\boldsymbol{u}^{\theta^\perp_{\downarrow F}}), \ \forall F \subseteq [n]\}. \end{aligned} \tag{2-9}$$

(2) *The spectrum $X_\theta := \mathrm{Max}(Z_\theta)$ branch-covers the classical sphere*

$$\mathbb{S}^{2n-1} \cong \mathrm{Max}\{\mathbb{S}_+^{n-1} \xrightarrow[\text{cont.}]{f} C^*(u_i^{q_i}) : f(\mathbb{S}_{+,F}^{n-1}) \in C^*(u_i^{q_i}, \ i \in F), \forall F \subseteq [n]\} \tag{2-10}$$

for

$$q_i := \mathrm{lcm}(\textit{lowest-term denominators of } \theta_{ij}, \ j \in [n]).$$

(3) *C_θ embeds into $M_{\sqrt{h_\theta}}(Z_\theta)$ with h_θ as in (2-4) and is not a Z_θ-subalgebra of $M_n(Z_\theta)$ for any smaller n. In particular, C_θ is a PI algebra of PI-degree $\sqrt{h_\theta}$.*

Proof. (1) We have already recalled (2-6) that the centers $Z(C(\mathbb{T}_\theta^n))$ of the noncommutative torus algebras are as claimed, so the conclusion follows from the identification (0-1).

(2) That the right-hand side of (2-10) is indeed a sphere is a simple topology exercise (the classical counterpart to (0-1)), given that the $u_i^{q_i} \in A_\theta^n$ are central and hence the generators of a $C(\mathbb{T}^n)$. The claim is that the map

$$X_\theta \xrightarrow[\text{onto}]{\pi} \mathbb{S}^{2n-1}$$

dualizing the embedding

$$C(\mathbb{S}^{2n-1}) \cong \mathrm{Cont}_\partial\big(\mathbb{S}^{n-1}_+ \to Z_\downarrow(A^n_\theta) := C^*(u_i^{q_i})\big) \tag{2-11}$$
$$:= \big\{\mathbb{S}^{n-1}_+ \xrightarrow[\text{cont.}]{f} Z_\downarrow(A^n_\theta) : f(\mathbb{S}^{n-1}_{+,F}) \in C^*(u_i^{q_i}, i \in F), \forall F \subseteq [n]\big\}$$
$$\hookrightarrow (2\text{-}9) =: C(X_\theta)$$

is a *branched cover* in the sense of [51, §1]: an open surjection of compact Hausdorff spaces, with a finite upper bound on the cardinalities of the fibers $\pi^{-1}(p)$, $p \in \mathbb{S}^{2n-1}$. Surjectivity is not at issue, so it is the two other requirements that require attention. Rather than attack that classical statement directly, we appeal to the theory of *noncommutative branched covers* developed in [51; 7], revolving around the notion of a C^* *conditional expectation* [6, Definition II.6.10.1].

We have embeddings

$$Z_\downarrow(A^n_\theta) \hookrightarrow Z(A^n_\theta) \hookrightarrow C^*(\mathbb{Z}^n) \cong C^*(t_i, i \in [n]), \quad t_i^{q_i} = u_i^{q_i}.$$

The usual [53, Proposition 8.5] (plainly of *finite index* [28, Definition 2]) expectation

$$C^*(t_i, i \in [n]) \xrightarrow{E} Z_\downarrow(A^n_\theta)$$

restricts to another such (also E) on the intermediate $Z(A^n_\theta)$, compatible with the inclusions of the subgroups generated by t_i, $i \in F$ for $F \subseteq [n]$.

We then have a finite-index

$$\mathrm{Cont}(\mathbb{S}^{n-1}_+ \to Z(A^n_\theta)) \xrightarrow{E\circ} \mathrm{Cont}(\mathbb{S}^{n-1}_+ \to Z_\downarrow(A^n_\theta)),$$

hence also (because of the noted F-compatibility) an analogous expectation between Cont_∂ function algebras. These, however, are exactly the two extremes of (2-11), whence the conclusion by [7, Theorem 1.4].

(3) The isomorphism (0-1) and Proposition 2.6 ensure that the irreducible C_θ-representations are all at most $\sqrt{h_\theta}$-dimensional. That estimate is the best possible: in the language of [6, Definition IV.1.4.1], C_θ is sharply $\sqrt{h_\theta}$*-subhomogeneous*. To verify this last sharpness assertion, quotient out the ideal $I_p := I_{\{p\}}$ of (2-7) in (0-1) to obtain a surjection

$$C_\theta \twoheadrightarrow A^n_\theta \xrightarrow{\text{Proposition 2.6}} M_{\sqrt{h_\theta}}.$$

C_θ thus embeds [18, §2.7.3] into a product of matrix algebras M_n, $n \le \sqrt{h_\theta}$ with at least one $M_{\sqrt{h_\theta}}$ quotient, and the conclusion follows. ☐

Corollary 2.9. *A noncommutative sphere algebra C_θ is Azumaya if and only if it is commutative, i.e., $\theta \in M_n(\mathbb{Z})$.*

Proof. This is immediate from Theorem 2.8 and its proof: $\theta \notin M_n(\mathbb{Z})$ is equivalent to $h_\theta > 1$, and it remains to observe that if that condition holds then C_θ cannot be Azumaya, for on the one hand it has an h_θ-dimensional matrix algebra quotient, while on the other hand $C_\theta|_p$ is abelian for any of the n vertices $p \in \mathbb{S}^{n-1}_+$. □

The branch-covering qualification in Theorem 2.8(2) is not redundant: as soon as $n \geq 3$ the center Z_θ of (2-9) need not be (the function algebra of) a sphere.

Example 2.10. Set

$$\theta := \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \in M_3(\mathbb{Q}).$$

Denoting by u_1, u_2, u_3 the generators of A^3_θ, the sphere $\mathbb{S}^{2n-1} = \mathbb{S}^5$ of Theorem 2.8(2) is

$$\mathbb{S}^5 \cong \operatorname{Max}\operatorname{Cont}_\partial(\mathbb{S}^2_+ \to C^*(u_i^2,\ i \in [3])),$$

while

$$X_\theta \cong \operatorname{Max}\operatorname{Cont}_\partial(\mathbb{S}^2_+ \to Z(A^3_\theta) = C^*(u_i^2,\ i \in [3],\ u_1u_2u_3)).$$

The latter space thus consists of two 5-spheres glued along the 3-dimensional complex

$$\operatorname{Max}\operatorname{Cont}_\partial(\partial\mathbb{S}^2_+ \to Z(A^3_\theta)) \tag{2-12}$$

(one copy of which is embedded in each of the two 5-spheres). Said complex can be described as follows:

- Consider three 3-spheres $\mathbb{S}^3_i$ indexed by $i \in \mathbb{Z}/3$, regarded as total spaces of the *Hopf fibration* [17, §14.1.9] $\mathbb{S}^3 \xrightarrow{\pi} \mathbb{S}^2$.
- In each $\mathbb{S}^3_i$ set $\mathbb{S}^1_{i,\pm} := \pi^{-1}(p_\pm)$ for antipodes $p_\pm \in \mathbb{S}^2$.
- Glue $\mathbb{S}^3_i$ to $\mathbb{S}^3_{i+1}$ (indices modulo 3) by identifying $\mathbb{S}^1_{i,+} \cong \mathbb{S}^1_{i+1,-}$ to obtain (2-12).

X_θ is now easily seen not to be homeomorphic to $\mathbb{S}^5$, and indeed, not a topological manifold: points in $\mathbb{S}^3_0$ but off the exceptional circles $\mathbb{S}^1_{0,\pm}$ have neighborhoods homeomorphic to two copies of $\mathbb{R}^5$ glued along a closed $\mathbb{R}^3$. The removal of that $\mathbb{R}^3$ disconnects such a neighborhood, so it cannot [25, Theorem 1.8.13] be homeomorphic to a Euclidean space.

Remark 2.11. Example 2.10 illustrates a qualitative distinction between quantum tori and spheres: while *equivalent* $\theta, \theta' \in M_n(\mathbb{Q})$, in the sense [9, §IX.1.6] (appropriate for bilinear forms) that

$$\exists (T \in \mathrm{GL}_n(\mathbb{Z})) \quad : \quad \theta' = T\theta T^t,$$

will produce isomorphic torus algebras $A^n_\theta \cong A^n_{\theta'}$, it may well be that $C_\theta \not\cong C_{\theta'}$. In fact, not even the centers of the latter two algebras need be isomorphic.

Indeed, Z_θ will be (the function algebra of) a $(2n-1)$-sphere whenever θ is block-diagonal with blocks (2-5), for in that case, in the notation of Theorem 2.8,

$$Z(A^n_\theta) = C^*(u_i^{q_i},\ i \in [n]).$$

Per Example 2.10, then, with that choice of θ and

$$\theta' := \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = T\theta T^t, \quad T := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

The respective centers $Z_{\theta,\theta'} = Z(C_{\theta,\theta'})$ are nonisomorphic.

To follow up on Corollary 2.9, determining the Azumaya locus of C_θ is now also. We again need some vocabulary and notation preliminaries.

Definition 2.12. Let $\theta \in M_n(\mathbb{Q})$ be a rational skew-symmetric matrix and recall Notation 2.3.

The *jump (sub)complex* JMP_θ of θ (just plain JMP when θ is understood) is

$$\mathrm{JMP} = \mathrm{JMP}_\theta := \{F \in 2^{[n]}\ :\ h_{\theta,F} < h_\theta\}.$$

Identifying *simplicial complexes* and their respective *geometric realizations* [17, §8.1], JMP_θ is a *subcomplex* ([17, § 8.1], [62, §3.1]) of the simplex $\Delta^{n-1} \cong \mathbb{S}^{n-1}_+$ of (0-2a) with vertex set $[n]$; this justifies the term.

Theorem 2.13. *The Azumaya locus of C_θ is the complement*

$$(X_\theta = \mathrm{Max}(Z_\theta)) \setminus \mathrm{Max}(Z_\theta|_{\mathrm{JMP}}) \tag{2-13}$$

in the notation of (2-8).

Proof. (2-13) contains the Azumaya locus: Suppose $p \in \mathrm{Max}(Z_\theta)$ belongs precisely to the subspaces

$$X_{\theta,\hat{\imath}} := \mathrm{Max}\big(Z_\theta|_{\mathbb{S}^{n-1}_{+,\hat{\imath}}}\big) \subseteq X_\theta, \quad i \in F \in 2^{[n]}.$$

The intersection

$$p \cap C(\mathbb{S}^{n-1}_+) \in \mathrm{Max}\, C(\mathbb{S}^{n-1}_+) \cong \mathbb{S}^{n-1}_+$$

will then belong to the interior of the face $\mathbb{S}^{n-1}_{+,\widehat{F}}$, and (0-1) specializes to

$$C_\theta/\big(C_\theta \cdot (p \cap C(\mathbb{S}^{n-1}_+))\big) \cong C^*(u_i,\ i \in \widehat{F}) \cong A^{|\widehat{F}|}_{\theta|_F} \subseteq A^n_\theta.$$

If $\widehat{F} \in \mathrm{JMP}$ then (by the very definition of the jump complex) this further specializes, at every maximal ideal of the center $Z(A^n_\theta) \cong C(\mathbb{T}^n)$, to a *proper* inclusion

$$C_\theta/(C_\theta \cdot p) \ \cong\ \big(M_{\sqrt{h_{\theta,\widehat{F}}}}\big)^{[(\theta|_{\widehat{F}})^\perp : \theta^\perp_{\downarrow \widehat{F}}]} \ \precneqq\ M_{\sqrt{h_\theta}}$$

of a nonmatrix algebra, with $(\theta|_{\widehat{F}})^{\perp} \le \mathbb{Z}^{|\widehat{F}|}$ regarded as a subgroup of $\mathbb{Z}^n$ by padding vectors with zero entries. C_θ, then, cannot be Azumaya at p, for its localization $(C_\theta)_p$ does not satisfy the polynomial identities of any M_n, $n < \sqrt{h_\theta}$.

(2-13) is contained in the Azumaya locus: Every maximal ideal $p \in$ (2-13) is contained in the interior of some

$$\mathrm{Max}(Z_\theta|_Y) \subset \mathrm{Max}(Z_\theta), \quad Y = \overline{Y} \subseteq \mathbb{S}_+^{n-1} \setminus \mathrm{JMP},$$

because the intersection $p \cap C(\mathbb{S}_+^{n-1})$ belongs to the latter open set, and it suffices to take for Y a closed neighborhood of that point still contained in that open. The conclusion now follows from the fact that for such Y the algebras $C_\theta|_Y$ are Azumaya:

$$C_\theta|Y \cong \mathrm{End}(\pi^*\mathcal{E}), \quad Y \times \mathbb{T}^n \xrightarrow{\pi:=\text{2nd projection}} \mathbb{T}^n \cong \mathrm{Max}\big(Z(A_\theta^n)\big). \qquad \square$$

3. On and around center-finiteness

Finiteness (i.e., being a finitely generated module) over the center is well-known not to be automatic for PI algebras: [54, Example post Proposition 5.1.3], say, is of a finitely generated algebra satisfying the identities of M_4 but not embeddable in a ring finite over its center. The noncommutative sphere algebras C_θ themselves, as will become apparent, are another case in point.

Subhomogeneous C^*-algebras A (such as C_θ) admit (unital) embeddings

$$A \le M_n^I \cong M_n(C(\beta I)), \qquad \beta(-) := \textit{Stone-Čech compactification } [30, §6.5] \text{ of } I, \tag{3-1}$$

as noted in the proof of Theorem 2.8 (essentially by [18, §2.7.3]). But this will also not (generally) suffice to ensure center-finiteness, per Example 3.1: the latter will indeed not even be *topologically* center-finite, in the sense of containing a (norm-topology-)dense module over its center. Naturally, rings sandwiched as $R \le A \le M_n(R)$ for *Noetherian* commutative R are center-finite, so Example 3.1 is in a sense a manifestation of the non-Noetherianness of the continuous-function algebras $C(X)$ involved.

Example 3.1. [52, Example 3.5] (also [7, Example 3.6]) provides a C^*-algebra A equipped with a central morphism $C(X) \to A$ (a $C(X)$-algebra) for

$$X := \Big(X_0 := \bigsqcup_{n\ge 1} \mathbb{CP}^n\Big)^+ := \textit{one-point compactification } [68, \text{Problem 19A}] \text{ of } X_0$$

with fiber $\mathbb{C}$ at the distinguished point $\infty \in X$ and fibers $M_2(\mathbb{C})$ over X_0. Because the associated M_2-bundle over X is by construction not *of finite type* (that is, [35, Definition 3.5.7] trivializable over a finite open cover of X_0), A cannot be topologically finitely generated as a $C(X)$-module by [31, Theorem 1.1] (or [14, Theorem A]) and [7, Proposition 3.7].

Remarks 3.2. (1) Example 3.1 is one instance of the following general setup.

- Consider compact Hausdorff spaces Y_n respectively equipped with (complex) vector d-bundles $\mathcal{E}_n$; we abuse notation and conflate these [36, Remark 15.1.2] with their corresponding principal $U(d)$-bundles.
- Assume the attached set

$$\{\mathrm{ind}_{U(d)}(\mathcal{E}_n)\}_n \subseteq \mathbb{Z}_{\geq 0}$$

of $U(d)$-*indices* [44, Definition 6.2.3] is unbounded: there is no finite upper bound on the cardinality of an open cover of Y_n that will trivialize $\mathcal{E}_n$.

- The $\mathcal{E}_n$ glue to a vector d-bundle over $X_0 := \bigsqcup_n Y_n$.
- Form the bundle $\mathcal{F} := \mathcal{E} \oplus (X_0 \times \mathbb{C})$ (i.e., add a trivial 1-dimensional summand to $\mathcal{E} \twoheadrightarrow X_0$).
- Construct the corresponding endomorphism bundle $\mathcal{F} \otimes \mathcal{F}^*$ (a $(d+1) \times (d+1)$-matrix bundle over X_0).
- And finally, take for the C^*-algebra A (supposed to play the same role as in Example 3.1) the *unitization* [6, §II.1.2.1] $\Gamma_0(\mathcal{F} \otimes \mathcal{F}^*)^+$, the subscript 0 indicating sections on X_0 vanishing at ∞.

(2) For path-connected (compact Hausdorff) Y_n in (1) above the unboundedness

$$\sup_n \dim Y_n, \quad \dim := \textit{covering dimension } [25, \text{Definition } 1.6.7]$$

is an essential feature of this family of examples: per [37, Proposition 2.1] (or as an immediate consequence of [50, Lemma 2.4], say) for any *paracompact* [26, §5.1] path-connected space Y there is an open cover

$$Y = \bigcup_{i=0}^{\dim Y} U_i, \quad U_i \text{ contractible in } Y.$$

The restriction of a bundle on Y will be trivializable [17, Theorem 14.3.3] over every U_i, rendering (1) inoperative.

(3) In reference to (3-1), observe that endomorphism algebras $\mathrm{End}(\mathcal{E}) = \Gamma(\mathcal{E} \otimes \mathcal{E}^*)$ for vector-bundles $\mathcal{E} \twoheadrightarrow X$ over compact Hausdorff X can always be embedded unitally in some $M_n(C(X))$ (same space X: one need not involve the typically nonmetrizable $\beta(\text{discrete } X)$).

To see this, consider a decomposition $\mathcal{E} \oplus \mathcal{F} \cong \mathbf{1}^m$ of a trivial bundle over X, with $\mathbf{1}$ denoting the trivial rank-1 bundle (one such exists by [63, Theorems 1 and

2]). We then have

$$\mathcal{E}^d \oplus \mathcal{F}^d \cong \mathbf{1}^{md}, \quad d := \operatorname{rank} \mathcal{E},$$

and obtain an embedding

$$\operatorname{End}(\mathcal{E}) \xhookrightarrow{\iota := \mathrm{id}^d \oplus (\mathrm{id} \otimes \mathrm{ev}_x)} \operatorname{End}(\mathcal{E}^d) \times \operatorname{End}(\mathcal{F}^d) \le \operatorname{End}(\mathbf{1}^{md}) \cong M_{md}(C(X)),$$

where the second component, $\mathrm{id} \otimes \mathrm{ev}_x$, means

- identifying $\mathcal{F}^d$ with $\mathcal{F} \otimes \mathbb{C}^d$ (focusing on the complex case to fix ideas), and
- mapping a global endomorphism $s \in \operatorname{End}(\mathcal{E})$ to the endomorphism of $\mathcal{F} \otimes \mathbb{C}^d$ operating trivially on the $\mathcal{F}$ tensorand and as the restriction $s_x \in \operatorname{End}(\mathbb{C}^d) \cong \operatorname{End}(\mathcal{E}_x)$ of x to the fiber at some fixed $x \in X$.

Returning to the C_θ: while (mostly) not center-finite, they nevertheless exhibit less pathology in that regard than Example 3.1 and the like.

Theorem 3.3. *Let $\theta \in M_n(\mathbb{Q})$ be a skew-symmetric matrix for some $n \in \mathbb{Z}_{\ge 2}$.*

(1) *The quantum sphere C^*-algebra C_θ is center-finite precisely when it is commutative, i.e., $\theta \in M_n(\mathbb{Z})$.*

(2) *On the other hand, C_θ is always topologically center-finite.*

Proof. (1) Theorem 2.8 makes C_θ into a subhomogeneous Z_θ-algebra, which can thus [13, Theorem A] be regarded as a Z_θ*-Hilbert module* [67, Definition 15.1.5]. Finite generation would imply [67, Corollary 15.4.8] that C_θ is also *projective* [1, Definition 5.3.1].

The Hilbert-module-to-Hilbert-bundle correspondence of [33, Scholium 6.7] and Swan's celebrated [59, Theorem 1.6.3] (originally [63, Theorems 1 and 2]) then imply that the (F) *Hilbert bundle* [21, p. 9] over $X := \operatorname{Max}(Z_\theta)$ with fibers

$$X \ni p \mapsto C_\theta|_p$$

is locally trivial, so (the sphere being connected) of constant rank. This, in turn, is equivalent (Corollary 2.9) to the commutativity of C_θ.

(2) Once more regarding C_θ, via Theorem 2.8(1), as the section-space $\Gamma(\mathcal{E})$ of a subhomogeneous (F) Banach bundle $\mathcal{E} \twoheadrightarrow \mathbb{S}^{2n-1} \cong \operatorname{Max}(Z_\theta)$, observe that the *strata*

$$X_d := \{p \in X \ : \ \dim \text{fiber } \mathcal{E}_p = \dim C_\theta / C_\theta \cdot p = d\}$$

are all members of the *set ring* [8, I, Definition 1.2.13] generated by the spaces

$$\operatorname{Max}(Z_\theta|_{\mathbb{S}^{n-1}_{+,F}}) \subseteq \operatorname{Max}(Z_\theta) = X, \quad F \in 2^{[n]}.$$

Thus they are all finite unions of path-connected paracompact spaces of finite (covering) dimension, so by [37, Proposition 2.1] they admit respective covers

$$X_d = \bigcup_{j=0}^{N} U_j, \quad U_j = \mathring{U}_j \text{ contractible in } \mathbb{S}_d^{2n-1} \quad (\text{some } N \in \mathbb{Z}_{\ge 0}).$$

The restrictions $\mathcal{E}|_{X_d}$ are thus all trivializable by finite covers, hence topological finite generation via [31, Theorem 1.1]. □

Acknowledgments

The paper has benefited from fruitful exchanges with B. Badzioch, M. Khalkhali, B. Passer and I. Thompson.

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Received August 6, 2025.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 342, No. 1, 2026
https://doi.org/10.2140/pjm.2026.342.63

FROM i-BOXES TO SIGNED WORDS

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The combinatorics of i-boxes has recently been introduced by Kashiwara, Kim, Oh and Park in the study of cluster algebras arising from the representation theory of quantum affine algebras. In this article, we associate to each chain of i-boxes a signed word, which canonically determines a cluster seed, following Berenstein, Fomin and Zelevinsky. By bridging these two different languages, we are able to provide a quick solution to the problem of explicitly determining the exchange matrices associated with chains of i-boxes.

1. Introduction

1.1. ***Background.*** Cluster algebras were introduced by Fomin and Zelevinsky [FZ02]. These algebras possess distinguished elements called cluster monomials. One of the main reasons for the interest in the theory of cluster algebras is their unexpected emergence across diverse areas of mathematics. An illustrative case is the representation theory of quantum affine algebras, where cluster algebra structures are studied through the framework of monoidal categorification, starting with the inspiring work of Hernandez and Leclerc [HL10]. A monoidal category $\mathscr{C}$ is a monoidal categorification of a cluster algebra $\mathcal{A}$ if there is an isomorphism between the Grothendieck ring of $\mathscr{C}$ and $\mathcal{A}$, such that the cluster monomials of $\mathcal{A}$ correspond to certain simple objects of $\mathscr{C}$.

Let $\mathfrak{g}$ be a simple complex finite-dimensional Lie algebra. In 2023, Kashiwara, Kim, Oh and Park [KKOP24] defined certain monoidal Serre subcategories $\mathscr{C}^{[a,b],\mathcal{D}_\mathcal{Q},\underline{w}_0}$ of the module category of the quantum affine algebra of $\mathfrak{g}$, where $[a,b]$ denotes a possibly unbounded integer interval (for details; see [KKOP24, §4, §6]). To show that these categories are instances of monoidal categorification,

MSC2020: primary 13F60; secondary 17B37.

Keywords: cluster algebra, quantum affine algebra, monoidal categorification.

they introduced the combinatorics of *chains of* $\boldsymbol{i}$ *-boxes* [KKOP24, §4, §5]. The definition of $\boldsymbol{i}$-boxes (Section 2) is based on the choice of an infinite sequence of indexes $\boldsymbol{i}$, and a chain of $\boldsymbol{i}$-boxes is a sequence of integer intervals satisfying certain technical conditions.

Let us fix a category $\mathscr{C}^{[a,b],\mathcal{D}_{\mathcal{Q}},\underline{w}_0}$. In this setting, the sequence $\boldsymbol{i}$ is of the form $\widehat{\underline{w}}_0$, a particular sequence whose elements belong to the index set of a simply laced Dynkin diagram canonically associated to $\mathfrak{g}$. Kashiwara, Kim, Oh and Park associate an affine determinant module $M(\mathfrak{c})$ (a generalization of the Kirillov–Reshetikhin modules) to each $\boldsymbol{i}$-box $\mathfrak{c}$. Moreover, for each chain of $\boldsymbol{i}$-boxes $\mathfrak{C} = (\mathfrak{c}_i)$, they show the existence of a skew-symmetric exchange matrix $B(\mathfrak{C})$ such that, together with the representatives $([M(\mathfrak{c})_i)])_i$ of the modules associated to the $\boldsymbol{i}$-boxes of the chain, they give a seed for a cluster algebra structure of the Grothendieck ring of $\mathscr{C}^{[a,b],\mathcal{D}_{\mathcal{Q}},\underline{w}_0}$ [KKOP24, Theorem 8.1].

More precisely, when b is an integer, Kashiwara, Kim, Oh and Park start by explicitly providing the exchange matrix $B(\mathfrak{C}_{-}^{[a,b]})$ associated to a specific chain of $\boldsymbol{i}$-boxes, denoted $\mathfrak{C}_{-}^{[a,b]}$ (see Definition 2.5), generalizing a construction of Hernandez and Leclerc [HL11], who showed that the matrix $B(\mathfrak{C})$ can be obtained from the matrix $B(\mathfrak{C}_{-}^{[a,b]})$ through a sequence of mutations. The case $b = \infty$ is treated through a limit procedure. At the end of [KKOP24], they state the problem of finding an explicit formula for all the matrices $B(\mathfrak{C})$. See Remark 2.11 for more details of this problem in terms of monoidal categories.

This problem is natural and fundamental for understanding the cluster structures appearing in the representation theory of quantum affine algebras. Recently, two solutions have been proposed:

- In [Con26], the first author proposed a solution translating the problem in a framework of additive categorification. Each exchange matrix $B(\mathfrak{C})$ is obtained as a submatrix of a square matrix $\overline{B}(\mathfrak{C})$, which, starting from the explicitly given matrix $\overline{B}(\mathfrak{C}_{-}^{[a,b]})$, can be expressed via a matrix multiplication, known as Palu's generalized mutation rule [Pal]:

$$\overline{B}(\mathfrak{C}) = P\,\overline{B}(\mathfrak{C}_{-}^{[a,b]})\,P^{-t},$$

  where P is an invertible matrix obtained through the computation of indices of cluster-tilting objects of a suitable cluster category.

- In [KK24], Kashiwara and Kim work with sequences $\boldsymbol{i}$ taking values in the index set of a generalized Cartan matrix C and with *maximal commuting families* of $\boldsymbol{i}$-boxes. Each chain of $\boldsymbol{i}$-boxes forms a maximal commuting family. For any such family $\mathcal{F}$, they define an $\mathcal{F}\times\mathcal{F}$-skew-symmetrizable matrix $\widetilde{B}^{\mathrm{KK}}(\mathcal{F})$ (see Section 4.1). When C is a symmetric Cartan matrix of finite type and $\boldsymbol{i}$ is of the form $\underline{\widehat{w}}_0$, each exchange matrix $B(\mathfrak{C})$ can be obtained as a submatrix $B^{\mathrm{KK}}(\mathfrak{C})$ of the matrix $\widetilde{B}^{\mathrm{KK}}(\mathfrak{C})$.

The solution to the Kashiwara–Kim–Oh–Park problem in [Con26], although interesting for bridging monoidal and additive categorification of cluster algebras, is not as explicit, since it requires a multiplication of matrices. Kashiwara and Kim provide a direct formula relying on monoidal categorification and elaborate combinatorial machinery.

1.2. ***Main results.*** In this work, we propose an alternative and straightforward solution to Kashiwara, Kim, Oh and Park's problem. The starting point is the combinatorics and the formalism of signed words. Recall that a signed word on an index set I is a sequence whose elements are of the form εh, where ε is in $\{\pm 1\}$ and h is in I. Assume that I is the index set of a generalized Cartan matrix. To each signed word $\underline{\boldsymbol{h}}$, one can associate a seed $\mathbf{t}(\underline{\boldsymbol{h}})$ following [BFZ05], which plays an important role in studying the cluster structures on double Bott–Samelson cells [SW21; Qin24b].

Let $B(\underline{\boldsymbol{h}})$ be the corresponding exchange matrix. Assume that $\boldsymbol{i}$ takes value in I and that b is in $\mathbb{Z}$. For each chain of $\boldsymbol{i}$-boxes $\mathfrak{C}$ associated to I, we define algorithmically a skew-symmetrizable matrix $B(\mathfrak{C})$ in a similar fashion to [KKOP24]. Using the indices of the $\boldsymbol{i}$-boxes of the chain, we define a signed word $\underline{\boldsymbol{h}}(\mathfrak{C})$. Our main result states that the desired matrix $B(\mathfrak{C})$ is given by the well-known matrix $B(\underline{\boldsymbol{h}}(\mathfrak{C}))$.

Theorem 1.1. *The matrix $B(\mathfrak{C})$ equals $B(\underline{\boldsymbol{h}}(\mathfrak{C}))$.*

Therefore, by applying our main result to the setting where the Cartan matrix is symmetric of finite type and $\boldsymbol{i}$ is of the form $\underline{\widehat{w}}_0$, we obtain a solution to the problem of Kashiwara Kim Oh Park, via a translation from the combinatorics of signed words to that of $\boldsymbol{i}$-boxes.

The outline of the proof of Theorem 1.1 is the following:

(1) By direct comparison, we verify that the matrices $B(\mathfrak{C}_{-}^{[a,b]})$ and $B(\underline{\boldsymbol{h}}(\mathfrak{C}_{-}^{[a,b]}))$ are equal.

(2) Let $\mathfrak{C}'$ and $\mathfrak{C}$ be any chains related by a box move. We show that, if $B(\mathfrak{C})$ equals $B(\underline{\boldsymbol{h}}(\mathfrak{C}))$, then $B(\mathfrak{C}')$ also equals $B(\underline{\boldsymbol{h}}(\mathfrak{C}'))$.

Additionally, when I is the set of indices of a generalized Cartan matrix, we directly verify that our matrix $B(\underline{\boldsymbol{h}}(\mathfrak{C}))$ corresponds to Kashiwara–Kim's matrix $B^{\mathrm{KK}}(\mathfrak{C})$ (Proposition 4.5).

Theorem 1.1 implies that the matrix $B(\mathfrak{C})$ is independent of the choice of box moves from $\mathfrak{C}_{-}^{[a,b]}$ to $\mathfrak{C}$ (Corollary 4.3). It also determines matrices in the case $b = +\infty$, as colimits of the matrices in the cases $b \in \mathbb{Z}$ (Corollary 4.4; see also [Qin24b]).

1.3. ***Notations and conventions.*** Choose any finite subset $K^{\mathrm{ex}} \subset K$. For any $(K \times K^{\mathrm{ex}})$-matrix $Z = (Z_{ij})$ and permutation σ on K, we define the $(K \times \sigma(K^{\mathrm{ex}}))$-matrix σZ such that $(\sigma Z)_{\sigma i, \sigma j} := Z_{i,j}$. Let $\sigma_{j,j+1}$ denote the transposition $(j, j+1)$.

The matrix Z is called an exchange matrix if $Z_{ik} \in \mathbb{Z}$ for $(i,k) \in K \times K^{\mathrm{ex}}$ and, moreover, it is skew-symmetrizable, i.e., there exists a diagonal matrix $D = (D_{kk})_{k \in K^{\mathrm{ex}}}$ with diagonal entries $D_{kk} \in \mathbb{N}_{>0}$, called a skew-symmetrizer, such that $D_{ii} Z_{ik} = -D_{kk} Z_{ki}$, $\forall i, k \in K^{\mathrm{ex}}$.

Let $[\ ]_+$ denote $\max(\ ,0)_+$. Let Z denote an exchange matrix. Following [FZ02], for any $k \in K^{\mathrm{ex}}$, the mutation μ_k gives us a new exchange matrix $\mu_k Z$ such that

$$(\mu_k Z)_{ij} = \begin{cases} Z_{ij} + [Z_{ik}]_+[Z_{kj}]_+ - [-Z_{ik}]_+[-Z_{kj}]_+ & \text{if } i, j \neq k, \\ -Z_{ij} & \text{if } i = k \text{ or } j = k. \end{cases}$$

2. Combinatorics of $\boldsymbol{i}$-boxes

In this section, following [KKOP24; KK24], we recall the definition and the properties of $\boldsymbol{i}$-boxes.

For $a, b \in \mathbb{Z} \sqcup \{\pm\infty\}$, we write $[a,b]$ for the integer interval

$$[a,b] = \{k \in \mathbb{Z} \mid a \leq k \leq b\}.$$

The *length* l of an integer interval $[a,b]$ is defined as $l = \max(b - a + 1, 0)$.

Let I be a finite set of indices and Z be an integer interval. We write $\boldsymbol{i} = (i_Z)_{k \in Z}$ for a sequence of elements of I indexed by the elements of Z. For $s \in Z$ and $j \in I$, we introduce the symbols

$$\begin{aligned} s^+ &= \min(\{t \in Z \mid s < t,\ i_t = i_s\} \cup \{\infty\}), \\ s^- &= \max(\{t \in Z \mid s > t,\ i_t = i_s\} \cup \{-\infty\}), \\ s(j)^{\oplus} &= \min(\{t \in Z \mid s \leq t,\ i_t = j\} \cup \{\infty\}), \\ s(j)^{\ominus} &= \max(\{t \in Z \mid s \geq t,\ i_t = j\} \cup \{-\infty\}). \end{aligned} \tag{1}$$

Definition 2.1 [KK24, §2.1; Qin24a, §6.1]. A finite integer interval $[a,b]$ in Z is an $\boldsymbol{i}$*-box* if $i_a = i_b$. For an $\boldsymbol{i}$-box $[a,b]$, we define its *color* $i([a,b])$ as $i([a,b]) = i_a = i_b$ and its i*-cardinality* or *order* as the number of times that the index $i([a,b])$ appears in the subinterval of $\boldsymbol{i}$ corresponding to $[a,b]$.

Remark 2.2. The intervals in Definition 2.1 are closely related to Kirillov–Reshetikhin modules of quantum affine algebras. They were called $\boldsymbol{i}$-boxes in [KKOP24], which focused on the case $I = I_{\mathrm{g}}$, $\boldsymbol{i} = \widehat{\underline{w}}_0$ and $Z = \mathbb{Z}$, where I_{g} is the set of Dynkin indices of a simply laced Lie algebra g and $\widehat{\underline{w}}_0 = (i_k)_{k \in \mathbb{Z}}$ is an infinite sequence obtained from a reduced expression $\underline{w}_0 = s_{i_1} \ldots s_{i_l}$ of the longest element of the

Weyl group of g by extending the sequence $i_1, \dots, i_l$ via the rule

$$i_{k+l} = i_k^*,$$

where $(-)^*$ is the involution on the index set I_{g} induced by $w_0(\alpha_i) = -\alpha_{i^*}$, for any simple root α_i, $i \in I_{\mathrm{g}}$.

For a finite interval $[a, b]$ in Z, we define the $\boldsymbol{i}$-boxes

$$[a, b\} = [a, b(i_a)^{\ominus}] \text{ and } \{a, b] = [a(i_b)^{\oplus}, b].$$

In other terms, $[a, b\}$ and $\{a, b]$ are the largest $\boldsymbol{i}$-boxes contained in $[a, b]$ with colors i_a and i_b respectively. When we want to emphasize that an $\boldsymbol{i}$-box is of color j, we use the notation $[a, b]_j$.

Definition 2.3 [KKOP24, Definition 5.1]. Let l be in $\mathbb{N} \cup \{\infty\}$. A *chain* of $\boldsymbol{i}$-boxes of *length* l is a sequence of $\boldsymbol{i}$-boxes $\mathfrak{C} = (\mathfrak{c}_k)_{1 \le k < l+1}$ satisfying the following conditions for any $1 \le s < l + 1$:

(i) The union $[\tilde{a}_s, \tilde{b}_s] := \bigcup_{1 \le k \le s} \mathfrak{c}_k$ is an interval of length s.

(ii) The $\boldsymbol{i}$-box $\mathfrak{c}_s$ is the largest $\boldsymbol{i}$-box of color $i(\mathfrak{c}_s)$ contained in the interval $\bigcup_{1 \le k \le s} \mathfrak{c}_k$.

Condition (ii) implies $b_s \le \tilde{b}_s < b_s^-$ and $a_s \ge \tilde{a}_s > a_s^-$. In addition, we have $[\tilde{a}_s, \tilde{b}_s] \subset [\tilde{a}_t, \tilde{b}_t]$ whenever $s < t$. We call the interval $\bigcup_{1 \le k < l+1} \mathfrak{c}_k$ the *range* of the chain. For any $1 \le s < l + 1$, the sequence $(\mathfrak{c}_k)_{1 \le k \le s}$ is a chain of $\boldsymbol{i}$-boxes, called a *subchain* of $\mathfrak{C}$.

Remark 2.4 [KKOP24, §5]. To each chain of $\boldsymbol{i}$-boxes $\mathfrak{C} = (\mathfrak{c}_k)_{1 \le k < l+1}$, we can bijectively associate a pair $(c, (E_k)_{1 \le k < l})$, where $c \in Z$ and each E_k is a symbol in $\{L, R\}$, in the following way: let c be the integer such that $\{c\} = \mathfrak{c}_1$ and, for any $1 \le k < l$, set

$$E_k = \begin{cases} L & \text{if } [\tilde{a}_{k+1}, \tilde{b}_{k+1}] = [\tilde{a}_k - 1, \tilde{b}_k], \\ R & \text{otherwise.} \end{cases}$$

In fact, given such a pair $(c, (E_k)_{1 \le k < l})$, the associated chain of $\boldsymbol{i}$-boxes $\mathfrak{C}$ can be recursively recovered as follows:

- $\mathfrak{c}_1 = \{c\}$;
- for any $2 \le k < l + 1$, we have

$$\mathfrak{c}_k = \begin{cases} [\tilde{a}_{k-1} - 1, \tilde{b}_{k-1}\} & \text{if } E_{k-1} = L, \\ \{\tilde{a}_{k-1}, \tilde{b}_{k-1} + 1] & \text{if } E_{k-1} = R. \end{cases}$$

We refer to $T = L$ (resp. $= R$) as a *left* (resp. *right*) *expansion operator* and to a sequence $(c, (E_k)_{1 \le k < l})$ as a *rooted sequence of expansion operators*; see [Con26].

Definition 2.5. Let $[a, b]$ be an integer interval with $a \leq b$, $a \in \mathbb{Z} \sqcup \{-\infty\}$ and $b \in \mathbb{Z}$. Denote its length by $l = b - a + 1$. Following [KKOP24], we define $\underline{\mathfrak{C}}^{[a,b]}$ as the chain of $\boldsymbol{i}$-boxes associated to the rooted sequence of expansion operators $(b, (E_k)_{1 \leq k < l})$, where $E_k = L$ for any k. Explicitly, the k-th $\boldsymbol{i}$-boxes are $\mathfrak{c}_k = [b-k+1, b\}$, $\forall k \in [1, l]$.

Definition 2.6 [KKOP24, §5]. Let $\mathfrak{C} = (\mathfrak{c}_k)$ be a chain of $\boldsymbol{i}$-boxes of length $l \leq \infty$ corresponding to a pair $(c, (E_k)_{1 \leq k < l})$.

(i) For $1 \leq s < l$, the $\boldsymbol{i}$-box $\mathfrak{c}_s$ is defined to be *movable* if $s = 1$ or $s \geq 2$ and $E_{s-1} \neq E_s$.

(ii) For a movable $\boldsymbol{i}$-box $\mathfrak{c}_s$, the *box move* at s, denoted by ν_s, is the operation sending $\mathfrak{C}$ to the chain $\nu_s \mathfrak{C}$, whose associated pair (c', E') is defined as follows:

$$c' = \begin{cases} c+1 & \text{if } s = 1, E_1 = R, \\ c-1 & \text{if } s = 1, E_1 = L, \\ c & \text{if } s > 1, \end{cases} \quad \text{and} \quad E'_k = \begin{cases} R & \text{if } E_k = L, \ k \in \{s-1, s\}, \\ L & \text{if } E_k = R, \ k \in \{s-1, s\}, \\ E_k & \text{if } k \notin \{s-1, s\}. \end{cases}$$

(iii) We call a finite composition of box moves a *chain transformation*.

Example 2.7. Let $I = \{1, 2, 3\}$ be the set of Dynkin indices of a simple Lie algebra of type A_3. Let $\underline{w}_0 = s_1 s_2 s_3 s_1 s_2 s_1$ be a reduced expression of the longest element Weyl group of type A and let $\boldsymbol{i}$ be the sequence $\widehat{\underline{w}}_0$:

$$\widehat{\underline{w}}_0 = \ldots \underbrace{1,\ 3,\ 2,\ 3,\ 1,\ 2,\ 3,\ 1}_{[-3,4]} \ldots .$$

The chain of $\boldsymbol{i}$-boxes $\mathfrak{C} = (\mathfrak{c}_k)_{1 \leq k \leq 8}$ of range $[-3, 4]$ associated to the rooted sequence of expansion operators $(4, (L, L, L, L, L, L, L))$ is given by

$$\begin{aligned} &\mathfrak{c}_1 = [4]_1, && \mathfrak{c}_2 = [3]_3, && \mathfrak{c}_3 = [2]_2, && \mathfrak{c}_4 = [1, 4]_1, \\ &\mathfrak{c}_5 = [0, 3]_3, && \mathfrak{c}_6 = [-1, 2]_2, && \mathfrak{c}_7 = [-2, 3]_3 && \mathfrak{c}_8 = [-3, 4]_1. \end{aligned}$$

Notice that the only movable $\boldsymbol{i}$-box in the chain $\mathfrak{C}$ is $\mathfrak{c}_1$. The box move at 1 sends $\mathfrak{C}$ to the chain $\nu_1 \mathfrak{C} = (\mathfrak{c}'_k)_{1 \leq k \leq 6}$ associated to the sequence $(3, (R, L, L, L, L, L, L))$:

$$\begin{aligned} &\mathfrak{c}'_1 = [3]_3, && \mathfrak{c}'_2 = [4]_1, && \mathfrak{c}'_3 = [2]_2, && \mathfrak{c}'_4 = [1, 4]_1, \\ &\mathfrak{c}'_5 = [0, 3]_3, && \mathfrak{c}'_6 = [-1, 2]_2, && \mathfrak{c}'_7 = [-2, 3]_3 && \mathfrak{c}'_8 = [-3, 4]_1. \end{aligned}$$

Notice that the $\boldsymbol{i}$-box $\mathbf{c}'_2$ is movable and that the associated box move sends $\nu_1 \mathfrak{C}$ to the chain associated to the sequence $(3, (L, R, L, L, L, L, L)))$. Iterating this process, we see that, through a composition of box moves, the chain $\mathfrak{C}$ is sent to the chain $\widetilde{\mathfrak{C}} = (\tilde{\mathfrak{c}}_k)_{1\le k\le 8}$ associated to the sequence $(3, (L, L, L, L, L, L, R))$:

$$\begin{array}{llll} \tilde{\mathfrak{c}}_1 = [3]_3, & \tilde{\mathfrak{c}}_2 = [2]_2, & \tilde{\mathfrak{c}}_3 = [1]_1, & \tilde{\mathfrak{c}}_4 = [0, 3]_3, \\ \tilde{\mathfrak{c}}_5 = [-1, 2]_2, & \tilde{\mathfrak{c}}_6 = [-2, 3]_3, & \tilde{\mathfrak{c}}_7 = [-3, 1]_1 & \tilde{\mathfrak{c}}_8 = [-3, 4]_1. \end{array}$$

Remark 2.8 [KKOP24, Lemma 5.10; Con26, Remark 2.10]. Let $[a, b]$ be an integer interval with $b \in \mathbb{Z}$ and $a \in \mathbb{Z} \sqcup \{-\infty\}$. Then any two chains of $\boldsymbol{i}$-boxes of range $[a, b]$ are related by a chain transformation.

Remark 2.9. Let $\mathfrak{C}$ and $\mathfrak{C}'$ be two chain of $\boldsymbol{i}$-boxes. Assume that $\mathfrak{C}$ and $\mathfrak{C}'$ are related by a box move at $s \ge 1$.

- If $s \ge 2$, we have

$$i(\mathfrak{c}_{s+1}) = i(\mathfrak{c}'_s), \quad i(\mathfrak{c}_s) = i(\mathfrak{c}'_{s+1}) \quad \text{and} \quad i(\mathfrak{c}_k) = i(\mathfrak{c}'_k) \ \text{ for any } k \notin \{s, s+1\}.$$

- If $s = 1$, we have

$$i(\mathfrak{c}_2) = i(\mathfrak{c}'_1), \quad i(\mathfrak{c}'_2) = i(\mathfrak{c}_1) \quad \text{and} \quad i(\mathfrak{c}_k) = i(\mathfrak{c}'_k) \ \text{ for any } k \ge 3.$$

Definition 2.10 [KK24, Definition 2.13]. Let $\mathfrak{C} = (\mathfrak{c}_k)_k$ be a chain of $\boldsymbol{i}$-boxes with associated sequence of extension operators $(E_k)_k$. For any k, the *effective end* z of the $\boldsymbol{i}$-box $\mathfrak{c}_k = [x, y]$ is defined as

$$z = \begin{cases} y & \text{if } k = 1 \text{ or } E_{k-1} = R, \\ x & \text{if } k = 1 \text{ or } E_{k-1} = L. \end{cases}$$

2.1. ***The matrix associated to a chain of i-boxes.*** From now on, let I be the set of indices of a generalized Cartan matrix $C = (c_{ij})_{i,j\in I}$. Let $[a, b]$ be an integer interval and let l be its length. Let $\mathfrak{C} = (\mathfrak{c}_k)_{k\in[1,l]}$ be a chain of $\boldsymbol{i}$-boxes of range $[a, b]$. In this subsection, following [KKOP24], under the assumption that b is an integer, we recursively associate to the chain $\mathfrak{C}$ an exchange matrix $B(\mathfrak{C})$.

We introduce the following sets

$$K(\mathfrak{C}) = \begin{cases} \{1, \dots, l\} & \text{if } l < \infty, \\ \mathbb{N}_{\ge 1} & \text{if } l = \infty; \end{cases}$$

$$K(\mathfrak{C})^{\mathrm{fr}} = \{s \in K(\mathfrak{C}) \mid \mathfrak{c}_s = [a(i)^{\oplus}, b(i)^{\ominus}] \text{ for some } i \in I\};$$

$$K(\mathfrak{C})^{\mathrm{ex}} = K(\mathfrak{C}) \backslash K(\mathfrak{C})^{\mathrm{fr}}.$$

For any $k \in K(\mathfrak{C})$, we define $k[1]$ as

$$k[1] = \min(\{k' \in [k+1, l] \mid \mathfrak{c}_{k'} \text{ has the same color of } \mathfrak{c}_k\} \sqcup \{+\infty\}.$$

Assume that b is in $\mathbb{Z}$. Following [KKOP24, §7.5], let

$$B(\mathfrak{C}_-^{[a,b]}) = (b_{jk})_{j\in K(\mathfrak{C}), k\in K(\mathfrak{C})^{\mathrm{ex}}}$$

be the exchange matrix given by

$$b_{jk} = \begin{cases} 1 & \text{if } k = j[1], \\ -1 & \text{if } j = k[1], \\ c_{i_j,i_k} & \text{if } j < k < j[1] < k[1], \\ -c_{i_j,i_k} & \text{if } k < j < k[1] < j[1], \\ 0 & \text{otherwise.} \end{cases}$$

Although this construction was given in [KKOP24] only for Cartan matrices of ADE type (see also Remark 2.12), we apply the same formula in the more general setting considered here.

Next, for any chain of $\boldsymbol{i}$-boxes $\mathfrak{C} = (\mathfrak{c}_k)_{k\in[1,l]}$ of range $[a,b]$, fix a sequence of box moves $\nu_1, \dots, \nu_N$ whose composition sends $\mathfrak{C}_{-}^{[a,b]}$ to $\mathfrak{C}$. For $0 \le s \le N$, we write $\mathfrak{C}_s = (\mathfrak{c}_k^s)_{k\in[1,l]}$ for the chain of $\boldsymbol{i}$-boxes

$$\mathfrak{C}_s = \begin{cases} \mathfrak{C}_{-}^{[a,b]} & \text{if } s = 0, \\ \nu_s \circ \cdots \circ \nu_1 \mathfrak{C}_{-}^{[a,b]} & \text{otherwise .} \end{cases}$$

Let B_0 be the matrix $B(\mathfrak{C}_{-}^{[a,b]})$ and, for any $1 \le s \le N$, recursively define the exchange matrix B_s as follows:

(i) If the $\boldsymbol{i}$-box $\mathfrak{c}_{s+1}^{s-1}$ has the same color as $\mathfrak{c}_s^{s-1}$, then

$$B_s = \mu_s(B_{s-1}).$$

(ii) If the $\boldsymbol{i}$-boxes $\mathfrak{c}_{s+1}^{s-1}$ and $\mathfrak{c}_s^{s-1}$ have different colors, then

$$B_s = \sigma_{s,s+1}(B_{s-1}).$$

Finally, we set $B(\mathfrak{C}) = B_N$. In the next section, we will show that the matrix $B(\mathfrak{C})$ does not depend on the choice of the composition of box-moves sending $\mathfrak{C}_{-}^{[a,b]}$ to $\mathfrak{C}$.

Remark 2.11. Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra and let g be the Lie algebra of simply laced type associated to the unfolding of the Dynkin diagram of $\mathfrak{g}$ [FO21]. Assume that b is in $\mathbb{Z} \cup \{\infty\}$ and that $\boldsymbol{i} = (i_k)_{k\in[a,b]}$ is a subsequence of the sequence $\widehat{\underline{w}}_0 = (i_k)_{k\in\mathbb{Z}}$, for a certain reduced expression $\underline{w}_0$ of the longest element of the Weyl group of g. In [KKOP24], Kashiwara, Kim, Oh and Park associate a *monoidal seed* (see [KKOP24, Definition 7.2] for the terminology) to each chain of $\boldsymbol{i}$-boxes $\mathfrak{C}$, defining a family of *commuting real prime* modules $M(\mathfrak{C})$ and showing the existence of a companion exchange matrix $B(\mathfrak{C})$. To do this, when b is an integer they use the above procedure [KKOP24, Proposition 8.11], and when $b = +\infty$ they define $B(\mathfrak{C})$ as a colimit of the matrices associated to the subchain of $\mathfrak{C}$. By [KKOP24, Proposition 7.14, Lemma 7.16], this operation is well defined. It follows from this last proposition that the exchange matrix $B(\mathfrak{C})$ does not depend on the choice of the sequence of box moves sending $\mathfrak{C}_{-}^{[a,b]}$ to $\mathfrak{C}$.

Nevertheless, [KKOP24] does not provide an explicit description of the coefficients of $B(\mathfrak{C})$, whose determination is stated as an open problem.

3. Signed words

Recall that $C = (c_{ij})_{i,j\in I}$ is a generalized Cartan matrix. Let l be in $\mathbb{N} \sqcup \{+\infty\}$.

Definition 3.1. A signed word on the index set I is a sequence $\underline{\boldsymbol{h}} = (\varepsilon_k h_k)_{1\le k<l+1}$ such that, for any k, $\varepsilon_k \in \{\pm 1\}$ and $h_k \in I$. Denote $\boldsymbol{h}_k = \varepsilon_k h_k$ and $|\boldsymbol{h}_k| = h_k$.

We introduce the following sets

$$\begin{aligned} K(\underline{\boldsymbol{h}}) &= [1, l]; \\ K(\underline{\boldsymbol{h}})^{\mathrm{ex}} &= \{s \in K(\underline{\boldsymbol{h}}) \mid \exists t \in [s+1, l],\ h_t = h_s\}; \\ K(\underline{\boldsymbol{h}})^{\mathrm{fr}} &= K(\mathfrak{C}) \backslash K(\underline{\boldsymbol{h}})^{\mathrm{ex}}. \end{aligned}$$

For any $k \in K(\underline{\boldsymbol{h}})$, we define $k[1]$ as

$$k[1] = \min\big(\{k' \in [k+1, l] \mid |\boldsymbol{h}_{k'}| = |\boldsymbol{h}_k|\} \sqcup \{+\infty\}\big). \tag{2}$$

Note that $k[1]$ in (2) and s^+ in (1) should not be confused.

Following [BFZ05; SW21; Qin24a, (6.1)], for any signed word $\underline{\boldsymbol{h}}$, we define $\widetilde{B}(\underline{\boldsymbol{h}}) = (\tilde{b}_{jk})_{j,k\in K(\underline{\boldsymbol{h}})}$ by

$$\tilde{b}_{jk} = \begin{cases} \varepsilon_k & \text{if } k = j[1], \\ -\varepsilon_j & \text{if } j = k[1], \\ \varepsilon_k c_{|\boldsymbol{h}_j|,|\boldsymbol{h}_k|} & \text{if } \varepsilon_{j[1]} = \varepsilon_k,\ j < k < j[1] < k[1], \\ \varepsilon_k c_{|\boldsymbol{h}_j|,|\boldsymbol{h}_k|} & \text{if } \varepsilon_k = -\varepsilon_{k[1]},\ j < k < k[1] < j[1], \\ -\varepsilon_j c_{|\boldsymbol{h}_j|,|\boldsymbol{h}_k|} & \text{if } \varepsilon_{k[1]} = \varepsilon_j,\ k < j < k[1] < j[1], \\ -\varepsilon_j c_{|\boldsymbol{h}_j|,|\boldsymbol{h}_k|} & \text{if } \varepsilon_j = -\varepsilon_{j[1]},\ k < j < j[1] < k[1], \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

It has a skew-symmetrizer $\widetilde{D}(\underline{\boldsymbol{h}})$, which is a diagonal matrix with diagonal entries $\widetilde{D}_{ii} = h_i$. Let $B(\underline{\boldsymbol{h}})$ denote its $K(\underline{\boldsymbol{h}}) \times K(\underline{\boldsymbol{h}})^{\mathrm{ex}}$-submatrix.

From now on, we will always assume that $B(\underline{\boldsymbol{h}})$ is a locally finite matrix, i.e., for any j, only finitely many b_{jk} are nonzero and, for any k, only finitely many b_{jk} are nonzero. This assumption allows us to extend results in [SW21; Qin24a] for $l \in \mathbb{N}$ to the case $l = +\infty$ as in [Qin24b].

Let $\underline{\boldsymbol{h}}_{[j,k]}$ denote the sequence $(\boldsymbol{h}_s)_{s\in[j,k]}$.

Definition 3.2 [SW21, Section 2.3; Qin24b, Section 3.2]. Let $\underline{\boldsymbol{h}} = (\underline{\boldsymbol{h}}_k)_{k\in[1,l]}$ be a signed word.

- The *left reflection* of $\underline{\boldsymbol{h}}$ is the operation sending $\underline{\boldsymbol{h}}$ to the signed word $\underline{\boldsymbol{h}}' = (-\boldsymbol{h}_1, \underline{\boldsymbol{h}}_{[2,l]})$.

- Let $j \in [1, l-1]$ be such that $\underline{\boldsymbol{h}}_j$ and $\underline{\boldsymbol{h}}_{j+1}$ have different signs. Then the *flip* of $\underline{\boldsymbol{h}}$ at j is the operation sending $\underline{\boldsymbol{h}}$ to the signed word $\underline{\boldsymbol{h}}' = (\underline{\boldsymbol{h}}_{[1,j-1]}, \boldsymbol{h}_{j+1}, \boldsymbol{h}_j, \underline{\boldsymbol{h}}_{[j+2,l]})$.

Proposition 3.3 [SW21, Proposition 3.7; Qin24b, Section 3.2]. *Let* $\underline{\boldsymbol{h}} = (\underline{\boldsymbol{h}}_k)_{k\in[1,l]}$ *and* $\underline{\boldsymbol{h}}' = (\underline{\boldsymbol{h}}'_k)_{k\in[1,l]}$ *be two signed words.*

(1) *If* $\underline{\boldsymbol{h}}'$ *is obtained from* $\underline{\boldsymbol{h}}$ *via a left reflection, then* $B(\underline{\boldsymbol{h}}') = B(\underline{\boldsymbol{h}})$.

(2) *If* $\underline{\boldsymbol{h}}'$ *is obtained from* $\underline{\boldsymbol{h}}$ *via a flip at* j, *then*

$$B(\underline{\boldsymbol{h}}') = \begin{cases} \sigma_{j,j+1} B(\underline{\boldsymbol{h}}) & \text{if } h_j \neq h'_j, \\ \mu_j(B(\underline{\boldsymbol{h}})) & \text{if } h_j = h'_j. \end{cases}$$

4. Comparison of matrices

Recall that I denotes the index set of a generalized Cartan matrix C. Let $[a,b]$ be an interval and $\boldsymbol{i} = (i_k)_{k\in[a,b]}$ a sequence of elements of I. In the following, we will consider chains of $\boldsymbol{i}$-boxes defined with respect to the sequence $\boldsymbol{i}$. Let $\mathfrak{C} = (\mathfrak{c}_k)_{k\in[1,l]}$ be a chain of $\boldsymbol{i}$-boxes on $[a,b]$ and let $(E_k)_{1\leq k<l}$ be the associated sequence of expansion operators. For any $1 \leq k < l+1$, we write $\mathfrak{c}_k = [a_k, b_k]$. We associate to $\mathfrak{C}$ a signed word $\underline{\boldsymbol{h}}(\mathfrak{C}) = (\boldsymbol{h}_k)_{1\leq k<l+1}$ as follows:

- We set $\boldsymbol{h}_1$ equal to the color h_1 of the $\boldsymbol{i}$-box $\mathfrak{c}_1$.
- For any $2 \leq k < l+1$, we set $\boldsymbol{h}_k = \varepsilon_k h_k$ where h_k is the color of the $\boldsymbol{i}$-box $\mathfrak{c}_k$ and the sign ε_k is defined by

$$\begin{cases} 1 & \text{if } E_{k-1} = L, \\ -1 & \text{if } E_{k-1} = R. \end{cases}$$

It follows from our construction that $K(\mathfrak{C}) = K(\underline{\boldsymbol{h}}(\mathfrak{C}))$, $K(\mathfrak{C})^{\text{ex}} = K(\underline{\boldsymbol{h}}(\mathfrak{C}))^{\text{ex}}$ and $K(\mathfrak{C})^{\text{fr}} = K(\underline{\boldsymbol{h}}(\mathfrak{C}))^{\text{fr}}$, and the definition of $k[1]$ becomes identical. We will simply denote $K(\mathfrak{C})$ by K below. Note that different chains of $\boldsymbol{i}$-boxes correspond to different signed words, but not every signed word comes from a chain of $\boldsymbol{i}$-boxes.

Remark 4.1. Assume $k'[1] = k$ for some $k', k \in K = [1, \ell]$, and set $k' = k[-1]$. By our constructions, the following properties (1) to (6) are equivalent in each case:

(1) $\varepsilon_k = 1$	(1) $\varepsilon_k = -1$
(2) $E_{k-1} = L$	(2) $E_{k-1} = R$
(3) a_k is the effective end of $[a_k, b_k]$	(3) b_k is the effective end of $[a_k, b_k]$
(4) $\tilde{a}_k = a_k$	(4) $\tilde{b}_k = b_k$
(5) $a_k = a^-_{k[-1]}$	(5) $b_k = b^+_{k[-1]}$
(6) $b_k = b_{k[-1]}$	(6) $a_k = a_{k[-1]}$

Assume that b is an integer. In the following, we want to show that the matrices $B(\mathfrak{C})$ and $B(\underline{\boldsymbol{h}}(\mathfrak{C}))$ are equal, thus providing a solution to the Kashiwara–Kim–Oh–Park problem.

To start with, consider the chain of $\boldsymbol{i}$-boxes $\mathfrak{C}_{-}^{[a,b]}$. Then the signed word $\underline{\boldsymbol{h}}(\mathfrak{C}_{-}^{[a,b]}) = (\boldsymbol{h}_k)_{k\in[1,l]}$ is given by

$$\boldsymbol{h}_k = i_{b-k+1}.$$

Then the matrix $B(\underline{\boldsymbol{h}}(\mathfrak{C}_{-}^{[a,b]})) = (\tilde{b}_{jk})_{j\in K, k\in K^{\mathrm{ex}}}$ simplifies to

$$\tilde{b}_{jk} = \begin{cases} 1 & \text{if } k = j[1], \\ -1 & \text{if } j = k[1], \\ c_{|\boldsymbol{h}_j|,|\boldsymbol{h}_k|} & \text{if } j < k < j[1] < k[1], \\ -c_{|\boldsymbol{h}_j|,|\boldsymbol{h}_k|} & \text{if } k < j < k[1] < j[1]. \end{cases}$$

Therefore, $B(\underline{\boldsymbol{h}}(\mathfrak{C}_{-}^{[a,b]}))$ coincides with the matrix $B(\mathfrak{C}_{-}^{[a,b]})$.

Proposition 4.2. *Let $\mathfrak{C} = (\mathfrak{c}_k)_k$ be any chain of $\boldsymbol{i}$-boxes of range $[a,b]$. Suppose that the matrix $B(\mathfrak{C})$ equals $B(\underline{\boldsymbol{h}}(\mathfrak{C}))$. Then for any movable $\boldsymbol{i}$-box $\mathfrak{c}_s$ of $\mathfrak{C}$, we also have $B(\nu_s(\mathfrak{C})) = B(\underline{\boldsymbol{h}}(\nu_s(\mathfrak{C})))$.*

Proof. Recall that we write $\underline{\boldsymbol{h}}(\mathfrak{C}) = (\boldsymbol{h}_k)_k = (\varepsilon_k h_k)_k$. Denote the sequence of expansion operators associated to $\mathfrak{C}$ by $(E_k)_k$.

Suppose that s is greater than 1. Since $\mathfrak{c}_s$ is movable, we have $E_{s-1} \neq E_s$. Then, at the level of the associated signed word, we have $\varepsilon_s \neq \varepsilon_{s+1}$. Moreover, by Remark 2.9, the signed word associated to $\nu_s(\mathfrak{C})$ is $\underline{\boldsymbol{h}}(\nu_s(\mathfrak{C})) = (\boldsymbol{h}'_k)_k$, where

$$\boldsymbol{h}'_k = \begin{cases} \boldsymbol{h}_k & \text{if } k \notin \{s, s+1\}, \\ \boldsymbol{h}_{s+1} & \text{if } k = s, \\ \boldsymbol{h}_s & \text{if } k = s+1. \end{cases}$$

In particular, the signed word $\underline{\boldsymbol{h}}(\nu_s(\mathfrak{C}))$ is obtained from $\underline{\boldsymbol{h}}(\mathfrak{C})$ through a flip move at s. We have two cases:

(1) If the $\boldsymbol{i}$-boxes $\mathfrak{c}_s$ and $\mathfrak{c}_{s+1}$ have the same color, then $|\boldsymbol{h}_s| = |\boldsymbol{h}_{s+1}|$. Therefore, the matrices $B(\underline{\boldsymbol{h}}(\nu_s(\mathfrak{C})))$ and $B(\nu_s(\mathfrak{C}))$ are equal, since they are the mutation of the matrix $B(\underline{\boldsymbol{h}}(\mathfrak{C})) = B(\mathfrak{C})$ at s.

(2) If the $\boldsymbol{i}$-boxes $\mathfrak{c}_s$ and $\mathfrak{c}_{s+1}$ have different color, then $|\boldsymbol{h}_s| \neq |\boldsymbol{h}_{s+1}|$. Therefore, the matrices $B(\underline{\boldsymbol{h}}(\nu_s(\mathfrak{C})))$ and $B(\nu_s(\mathfrak{C}))$ are equal, since they are obtained from the matrix $B(\underline{\boldsymbol{h}}(\mathfrak{C})) = B(\mathfrak{C})$ by applying σ_s.

If $s = 1$, by Remark 2.9, the signed word $\underline{\boldsymbol{h}}(\nu_s(\mathfrak{C})) = (\boldsymbol{h}'_k)_k$ associated to $\nu_s(\mathfrak{C})$ is given by

$$\boldsymbol{h}'_k = \begin{cases} \boldsymbol{h}_k & \text{if } k \geq 3, \\ |\boldsymbol{h}_2| & \text{if } k = 1, \\ -\varepsilon_2 \boldsymbol{h}_1 & \text{if } k = 2. \end{cases}$$

Therefore, up to a left reflection which does not change the B-matrix, the signed word $\underline{\boldsymbol{h}}(\nu_s(\mathfrak{C}))$ is obtained from $\underline{\boldsymbol{h}}(\mathfrak{C})$ through a flip move, and we can conclude as above that $B(\nu_s(\mathfrak{C})) = B(\underline{\boldsymbol{h}}(\nu_s(\mathfrak{C})))$. □

Proof of Theorem 1.1. Recall that $\mathfrak{C}$ could be obtained from $\mathfrak{C}^{[a,b]}_{-}$ by finite many box moves, and $B(\mathfrak{C}^{[a,b]}_{-}) = B(\underline{\boldsymbol{h}}(\mathfrak{C}^{[a,b]}_{-}))$. Applying Proposition 4.2 repeatedly, we obtain $B(\mathfrak{C}) = B(\underline{\boldsymbol{h}}(\mathfrak{C}))$. □

Corollary 4.3. *The matrix $B(\mathfrak{C})$ does not depend on the choice of the chain transformation sending $\mathfrak{C}^{[a,b]}_{-}$ to $\mathfrak{C}$.*

Finally, assume that $b = +\infty$. Consider, for $s \geq 1$, the subchains $\mathfrak{C}_s = (\mathfrak{c}_k)_{1 \leq k \leq s}$ of $\mathfrak{C}$. As a corollary of Proposition 4.2, the coefficients of the exchange matrices $B(\mathfrak{C}_s)$ stabilize:

Corollary 4.4. *For any $s \geq 1$, if $(i, j) \in K(\mathfrak{C}_s) \times K^{\mathrm{ex}}(\mathfrak{C}_s)$, then $B(\mathfrak{C}_t)_{ij} = B(\mathfrak{C}_s)_{ij}$ for any $t \geq s$.*

Therefore, we can define the exchange matrix $B(\mathfrak{C})$ as the colimit of the matrices $B(\mathfrak{C}_s)$. In other words, we have

$$B(\mathfrak{C})_{|K(\mathfrak{C}_s) \times K^{\mathrm{ex}}(\mathfrak{C}_s)} = B(\mathfrak{C}_s) \text{for any} \quad s \geq 1.$$

4.1. *Comparison of $B(\underline{\boldsymbol{h}}(\mathfrak{C}))$ with Kashiwara–Kim's matrix.* We still write $[a, b]$ for an interval and $\mathfrak{C} = (\mathfrak{c}_k)_{k \in [1,l]}$ for a chain of $\boldsymbol{i}$-boxes $\mathfrak{c}_k = [a_k, b_k]$ of range $[a, b]$. For $1 \leq k \leq l$, h_k is the color of the $\boldsymbol{i}$-box $\mathfrak{c}_k$. Let $\widetilde{D}$ be the diagonal $l \times l$-matrix with diagonal entries $d_1 = d_{h_1}, \ldots, d_l = d_{h_l}$, where the $(d_i)_{i \in I}$ are the diagonal entries of the minimal symmetrizer of the generalized Cartan matrix C. Following [KK24], we define $\widetilde{B}^{\mathrm{KK}}(\mathfrak{C}) = (b^{\mathrm{KK}}_{kk'})_{k,k' \in K(\mathfrak{C})}$ as the skew-symmetrizable $l \times l$-matrix with skew-symmetrizer $\widetilde{D}$ and whose positive entries are given as follows:

$$b^{\mathrm{KK}}_{jk} = \begin{cases} 1 & \text{if } (a_j = a_k \text{ and } b_k = b_j^-) \text{ or } (b_j = b_k \text{ and } a_k = a_j^-), \\ -c_{h_j,h_k} & \text{if } c_{h_j,h_k} < 0 \text{ and one of the following conditions holds:} \end{cases} \tag{4}$$

(a) $[a_j, b_j^+] \in \mathfrak{C}$, a_j is the effective end of $[a_j, b_j]$, $a_k^- < a_j < a_k < b_k < b_j^+ < b_k^+$,

(b) $[a_j, b_j^+] \in \mathfrak{C}$, b_k is the effective end of $[a_k, b_k]$, $a_k^- < a_j < b_j < b_k < b_j^+ < b_k^+$,

(c) $[a_k^-, b_k] \in \mathfrak{C}$, b_k is the effective end of $[a_k, b_k]$, $a_j^- < a_k^- < a_j < b_j < b_k < b_j^+$,

(d) $[a_k^-, b_k] \in \mathfrak{C}$, a_j is the effective end of $[a_j, b_j]$, $a_j^- < a_k^- < a_j < a_k < b_k < b_j^+$.

Let $B^{\mathrm{KK}}(\mathfrak{C})$ denote the restriction of the matrix $\widetilde{B}(\mathfrak{C})$ to the indexes $K(\mathfrak{C}) \times K^{\mathrm{ex}}(\mathfrak{C})$. When the Cartan matrix is symmetric of finite type and $\underline{\boldsymbol{i}}$ is of the form $\underline{\widehat{w}}_0$, it is proved in [KK24] that $B^{\mathrm{KK}}(\mathfrak{C}) = B(\mathfrak{C})$. Then Theorem 1.1 implies that $B^{\mathrm{KK}}(\mathfrak{C}) = B(\underline{\boldsymbol{h}}(\mathfrak{C}))$. In the next proposition, in the general case where C is a generalized Cartan matrix, we compare them directly without using Theorem 1.1.

Proposition 4.5. *The matrix* $B^{\mathrm{KK}}(\mathfrak{C})$ *is equal to the matrix* $B(\underline{\boldsymbol{h}}(\mathfrak{C}))$.

Proof. $\widetilde{B}^{\mathrm{KK}}(\mathfrak{C})$ and $\widetilde{B}(\underline{\boldsymbol{h}}(\mathfrak{C}))$ have the same skew-symmetrizer. It suffices to prove that the positive (j,k)-entries are the same, where at least one of j,k belongs to K^{ex}.

Let $j,k \in [1,l]$ be such that $c_{h_j,h_k} < 0$. Recall from (4) the conditions (a), (b), (c) and (d) which characterize such entries. Define another four conditions:

(i) $\varepsilon_{j[1]} = \varepsilon_k = -1,\ j < k < j[1] < k[1]$;

(ii) $\varepsilon_k = -\varepsilon_{k[1]} = -1,\ j < k < k[1] < j[1]$;

(iii) $\varepsilon_{k[1]} = \varepsilon_j = 1,\ k < j < k[1] < j[1]$;

(iv) $\varepsilon_j = -\varepsilon_{j[1]} = 1,\ k < j < j[1] < k[1]$.

If any of these conditions is true, then $b_{jk} = -c_{h_j,h_k}$. Moreover, conditions (i), (ii), (iii) and (iv) are mutually exclusive, while (a) (resp. (c)) and (b) (resp. (d)) can hold at the same time.

<u>Step 1.</u> First, assume that (i) holds. Using Remark 4.1, we have:

- $\varepsilon_k = -1$ is equivalent to b_k being the effective end of $[a_k, b_k]$.
- $\varepsilon_{j[1]} = -1$ is equivalent to $[a_j, b_j^+]$ belonging to $\mathfrak{C}$, i.e., $\mathfrak{c}_{j[1]} = [a_j, b_j^+]$.
- $j < k$ implies $[a_j, b_j] \subset [\tilde{a}_k, \tilde{a}_k]$. Then we deduce $a_k^- < \tilde{a}_k < a_j$.
- $j < k$ and $\varepsilon_k = -1$ imply that $b_j < \tilde{b}_k = b_k$.
- $k < j[1]$ and $\varepsilon_{j[1]} = -1$ imply that $b_k < \tilde{b}_{j[1]} = b_j^+$.
- $j[1] < k[1]$ and $\varepsilon_{j[1]} = -1$ implies $b_j^+ < b_k^+$. This claim follows from the fact that $b_j^+ = b_{j[1]} \le \tilde{b}_{k[1]}$ and from the construction of $\mathfrak{c}_{k[1]}$, which implies that $\tilde{b}_{k[1]} \le b_k^+$.

Therefore, (i) implies (b). Now, assume that (b) holds. Then we have the effective ends $b_k = \tilde{b}_k$ and $b_{j[1]} = \tilde{b}_{j[1]} = b_j^+$. Moreover, $\mathfrak{c}_{j[1]} = [a_j, b_j^+]$. We claim that (b) implies one of (i)–(iv).

First, we cannot have $j[1] < k$ since $b_j^+ = \tilde{b}_{j[1]} > \tilde{b}_k = b_k$. In addition, we cannot have $k[1] < j$. Otherwise, it would follow that $\mathfrak{c}_{k[1]} \subset [\tilde{a}_j, \tilde{b}_j]$, so that $b_{k[1]} \le \tilde{b}_j < b_j^+$ and $a_{k[1]} \ge \tilde{a}_j > a_j^-$. But since we have either $b_{k[1]} = b_k^+$ or $a_{k[1]} = a_k^-$, both cases lead to contradiction with (b).

If $j < k < j[1] < k[1]$, we are in case (i).

If $j<k<k[1]<j[1]$, we claim that $\varepsilon_{k[1]}=1$, i.e, we are in case (ii). To see this, assume $\varepsilon_{k[1]}=-1$. Then $\tilde{b}_{k[1]}=b_k^+>b_j^+=\tilde{b}_{j[1]}$, which contradicts $k[1]<j[1]$.

If $k<j<k[1]<j[1]$, as before, $k[1]<j[1]$ and (b) imply $\varepsilon_{k[1]}=1$. We claim that $\varepsilon_j=1$, i.e, we are in case (iii). To see this, note that $b_j<b_k$ but $j>k$, so b_j is not the effective end of $\mathfrak{c}_j$, i.e, $\varepsilon_j=1$. The claim follows.

If $k<j<j[1]<k[1]$, as before, $j>k$ and $b_j<b_k$ imply $\varepsilon_j=1$. So we are in case (iv).

<u>Step 2.</u> Similarly, assume that (ii) holds. We have:

- $\varepsilon_k=-1$ is equivalent to b_k being the effective end of $[a_k,b_k]$.
- $\varepsilon_{k[1]}=1$ is equivalent to $[a_k^-,b_k]$ belonging to $\mathfrak{C}$, i.e., $\mathfrak{c}_{k[1]}=[a_k^-,b_k]$.
- $j<k$ implies $a_k^-<\tilde{a}_k<a_j$.
- $j<k$ and $\varepsilon_k=-1$ imply that $b_j<\tilde{b}_k=b_k$.
- $k[1]<j[1]$ and $\varepsilon_{k[1]}=1$ imply $b_k<b_j^+$ and $a_j^-<a_k^-$. This follows from the fact that $a_k^-=a_{k[1]}\geq\tilde{a}_{j[1]}\geq a_j^-$ and $b_k=b_{k[1]}\leq\tilde{b}_{j[1]}\leq b_j^+$.

Therefore, (ii) implies (c).

Now assume that (c) holds. Then we have the effective ends $b_k=\tilde{b}_k$ and $a_{k[1]}=\tilde{a}_{k[1]}=a_k^-$. Moreover, $\mathfrak{c}_{k[1]}=[a_k^-,b_k]$. We claim that (c) implies one of (i)–(iv).

First, we cannot have $k[1]<j$. In fact, since $\mathfrak{c}_j=[a_j,b_j]\subset[a_k^-,b_k]=\mathfrak{c}_{k[1]}$, if $j>k[1]$, none of a_j,b_j could be an effective end. In addition, we cannot have $j[1]<k$. Otherwise, it would follow that $\mathfrak{c}_{j[1]}\subset[\tilde{a}_k,\tilde{b}_k]$, thus $b_{j[1]}\leq\tilde{b}_k=b_k$ and $a_{j[1]}\geq\tilde{a}_k>a_k^-$. Since either $b_{j[1]}=b_j^+$ or $a_{j[1]}=a_j^-$ holds, and both contradict (c), we are led to a contradiction.

If $j<k<k[1]<j[1]$, we are in case (ii).

If $j<k<j[1]<k[1]$, we claim that $\varepsilon_{j[1]}=-1$, i.e, we are in case (i). To see this, assume $\varepsilon_{j[1]}=1$. Then $\tilde{a}_{j[1]}=a_j^-<a_k^-=\tilde{a}_{k[1]}$, which contradicts $j[1]<k[1]$.

If $k<j<j[1]<k[1]$, as before, $j[1]<k[1]$ and (c) imply $\varepsilon_{j[1]}=-1$. Moreover, $b_j<b_k$ and $j>k$ imply that b_j is not the effective end of $\mathfrak{c}_j$, i.e., $\varepsilon_j=1$. So we are in case (iv).

If $k<j<k[1]<j[1]$, as before, $j>k$ and $b_j<b_k$ imply $\varepsilon_j=1$. So we are in case (iii).

<u>Step 3.</u> Let us deduce the remaining cases from the previous steps. Let σ denote the order reversing automorphism on $\mathbb{Z}$ such that $\sigma(x)=-x$. Consider the word $\boldsymbol{i}'=(i'_k):=(i_{\sigma k})$ and the chain of $\boldsymbol{i}'$-boxes $\mathfrak{C}'=(\mathfrak{c}'_s)_{s\in[1,l]}$ such that $\mathfrak{c}'_s:=[a'_s,b'_s]:=[\sigma b_s,\sigma a_s]$. Then, for any j,k in K, the signed word $\underline{\boldsymbol{h}}(\mathfrak{C})$ satisfies conditions (i), (ii), (iii), or (iv) if and only if, respectively, the signed word $\underline{\boldsymbol{h}}(\mathfrak{C}')=(\varepsilon'_i h_i)$ satisfies conditions (iii), (iv), (i), or (ii) with j and k swapped:

(iii) $\varepsilon_{j[1]} = \varepsilon_k = 1,\ j < k < j[1] < k[1]$,

(iv) $\varepsilon_k = -\varepsilon_{k[1]} = 1,\ j < k < k[1] < j[1]$,

(i) $\varepsilon_{k[1]} = \varepsilon_j = -1,\ k < j < k[1] < j[1]$,

(ii) $\varepsilon_j = -\varepsilon_{j[1]} = -1,\ k < j < j[1] < k[1]$.

Similarly, for any j, k in K, the chain of $\boldsymbol{i}$-boxes $\mathfrak{C}$ satisfies the conditions (a), (b), (c), or (d) if and only if, respectively, the chain of $\boldsymbol{i}$-boxes $\mathfrak{C}'$ satisfies the conditions (c), (d), (a), or (b) with j and k swapped:

(c) $[(a')_j^-, (b')_j] \in \mathfrak{C}'$, $(b')_j$ is the effective end of $[(a')_j, (b')_j]$, $(b')_k^+ > (b')_j > (b')_k > (a')_k > (a')_j^- > (a')_k^-$;

(d) $[(a')_j^-, (b')_j] \in \mathfrak{C}'$, $(a')_k$ is the effective end of $[(a')_k, (b')_k]$, $(b')_k^+ > (b')_j > (a')_j > (a')_k > (a')_j^- > (a')_k^-$;

(a) $[(a')_k, b_k^+] \in \mathfrak{C}'$, $(a')_k$ is the effective end of $[(a')_k, (b')_k]$, $(b')_j^+ > (b')_k^+ > (b')_j > (a')_j > (a')_k > (a')_j^-$;

(b) $[(a')_k, (b')_k^+] \in \mathfrak{C}'$, $(b')_j$ is the effective end of $[(a')_j, (b')_j]$, $(b')_j^+ > (b')_k^+ > (b')_j > (b')_k > (a')_k > (a')_j^-$.

Combining with the results in Steps 1 and 2, we obtain that (iii) implies (d), that (iv) implies (a) and that, if any among (a), and (d) holds, then we are in one of the cases (i)–(iv).

Finally, we obtain the desired claim by comparing the explicit formula for the entries of the matrices, using the results of the previous steps. □

Funding. This work was supported by the National Natural Science Foundation of China [grant number 12422102 to Fan Qin].

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Received August 17, 2025.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 342, No. 1, 2026
https://doi.org/10.2140/pjm.2026.342.79

NEAR COINCIDENCES AND NILPOTENT DIVISION FIELDS

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Let $E/\mathbb{Q}$ be an elliptic curve. We say that E has a near coincidence of level (n, m) if $m \mid n$ and $\mathbb{Q}(E[n]) = \mathbb{Q}(E[m], \zeta_n)$. We classify near coincidences of prime power level and use this result to give a classification of values of n for which $\mathrm{Gal}(\mathbb{Q}(E[n])/\mathbb{Q})$ is a nilpotent group. Along the way we prove a Gauss–Wantzel analog for the elliptic curve $E : y^2 = x^3 - x$, showing that $\mathbb{Q}(E[n])/\mathbb{Q}$ is constructible if and only if $\varphi(n)$ is a power of 2. Assuming that there are no non-CM rational points on the modular curves $X_{ns}^{+}(p)$ for primes $p > 11$, we show that $\mathrm{Gal}(\mathbb{Q}(E[n])/\mathbb{Q})$ nilpotent implies that n is a power of 2 or $n \in \{3, 5, 6, 7, 15, 21\}$.

1. Introduction

For much of mathematical history, compass and straightedge constructions have served as exemplars of mathematical reasoning. Central to the study of such constructions is the question of which regular polygons are constructible. First considered by the Greeks, an answer to this question long eluded mathematicians, with no progress made for hundreds of years. This changed when Gauss proved the following theorem:

Theorem 1.1 [15]. *Suppose that n is of the form*

$$n = 2^k \cdot p_1 \cdots p_j,$$

where p_i is a Fermat prime. Then it is possible to construct a regular n-gon using only a straightedge and compass.

Here we remind the reader that a Fermat prime is a Fermat number (i.e., a number of the form $2^{2^m} + 1$ with $m \geq 0$) which is prime.

In fact, in [15] Gauss states that he has a proof of the converse, but he says, "the limits of the present work exclude this demonstration here." The final part of the question was formally answered in 1837 when Pierre Wantzel proved the converse.

Theorem 1.2 [43]. *Suppose n is a positive integer such that a regular n-gon is constructible. Then n is of the form $n = 2^k \cdot p_1 \cdots p_j$, where each p_i is a Fermat prime.*

MSC2020: primary 11G05; secondary 11R32, 14H52.
Keywords: elliptic curve, division field, Galois group, modular curve.

Together, these two results together form the Gauss–Wantzel theorem. If we let φ be Euler's totient function, then we can summarize the Gauss–Wantzel theorem as saying a regular n-gon is constructible if and only if there exists a k such that $\varphi(n) = 2^k$.

A modern reformulation of this is to take the standard plane and turn it into the complex plane. A point in the complex plane is said to be *constructible* if there is a way to define that point using just a straightedge and compass. Interpreting the Gauss–Wantzel theorem through this more modern lens gives us a new interpretation of what it means for a complex number to be constructible.

Theorem 1.3 [32, Theorem 4.53]. *Let $\alpha \in \mathbb{C}$. Then, α is constructible if and only if there exists an ascending chain of fields*

$$\mathbb{Q} = K_0 \subseteq K_1 \subseteq K_2 \subset \cdots \subseteq K_j$$

such that $\alpha \in K_j$ and $[K_i : K_{i-1}] = 2$.

A subfield of $\mathbb{C}$ is said to be *constructible* if all of the elements in the field are constructible. Using Galois theory from here we can update Theorem 1.3 to say that α is constructible if and only if there is a Galois extension $K/\mathbb{Q}$ that contains α such that $\mathrm{Gal}(K/\mathbb{Q})$ is a 2-group. Given that the nth roots of unity are equally spaced around the unit circle, from this updated perspective, the question of about the constructibility of a regular n-gon is exactly the same as the question about the constructibility of $\mathbb{Q}(\zeta_n)$. Here ζ_n is a primitive nth root of unity. This modern perspective allows us to examine the constructibility of more complicated fields. The primary objects of study in this paper are the division fields of elliptic curves.

First we fix $\overline{\mathbb{Q}}$, an algebraic closure of $\mathbb{Q}$ and let $E/\mathbb{Q}$ be an elliptic curve. A classical result is that the points on E can be given the structure of an abelian group. We let $E[n]$ be the n-torsion on E defined over $\overline{\mathbb{Q}}$, that is $E[n] = \{P \in E(\overline{Q}) : n \cdot P = \mathcal{O}\}$. Then we can define $\mathbb{Q}(E[n])$ to be the field of definition of the n-torsion points. With this we present the following theorem.

Theorem 1.4 (a Gauss–Wantzel analog). *Let E be the elliptic curve given by $y^2 = x^3 - x$ and let $n \geq 2$. Then, $\mathbb{Q}(E[n])$ is a constructible field if and only if $\varphi(n) = 2^k$ for some integer k.*

Proof. Suppose $n \in \mathbb{Z}$ such that $\mathbb{Q}(E[n])/\mathbb{Q}$ is a constructible extension. By properties of the Weil pairing, $\mathbb{Q}(\zeta_n) \subseteq \mathbb{Q}(E[n])$, and so by the Gauss–Wantzel theorem $\varphi(n)$ is a power of 2.

In the opposite direction, suppose that $n \in \mathbb{Z}$ such that $\varphi(n)$ is a power of 2. Given that constructibility is closed under compositum, it is enough to understand when $\mathbb{Q}(E[p^k])$ is constructible for some prime power p^k.

Suppose $p = 2$. In this case, $\mathbb{Q}(E[2]) = \mathbb{Q}$ and $[\mathbb{Q}(E[2^{k+1}]) : \mathbb{Q}(E[2^k])] = 2^m$ for some $m \in \mathbb{Z}$ depending on k. Thus, for any k, $\mathrm{Gal}(\mathbb{Q}(E[2^k])/\mathbb{Q})$ is a 2-group and hence constructible.

Next suppose that $p = 2^{2^n} + 1$ is a Fermat prime and $E : y^2 = x^3 - x$. We start this case by showing that $[\mathbb{Q}(E[p], i) : \mathbb{Q}(i)]$ is a power of 2, which implies the desired result. The endomorphism ring of E over $\mathbb{Q}(i)$ is isomorphic to $\mathbb{Z}[i]$. Moreover, since $p \equiv 1 \pmod 4$ splits we can factor $p = \pi\bar{\pi}$ for $\pi \in \mathbb{Z}[i]$. (Explicitly, we have $\pi = 2^{2^{n-1}} + i$ and $\bar{\pi} = 2^{2^{n-1}} - i$.) We therefore have two isogenies $\phi_1 : E \to E$ and $\phi_2 : E \to E$ given by $\phi_1(P) = [\pi]P$ and $\phi_2(P) = [\bar{\pi}]P$ which are defined over $\mathbb{Q}(i)$. Let $E[\pi]$ and $E[\bar{\pi}]$ be the kernels of these two isogenies. It is straightforward to see that $E[p]$ is the direct product of $E[\pi]$ and $E[\bar{\pi}]$, that each of $E[\pi]$ and $E[\bar{\pi}]$ has order p, and both $E[\pi]$ and $E[\bar{\pi}]$ are Galois stable over $\mathbb{Q}(i)$. This shows that the image of the mod p Galois representation $\rho_{E,p} : \mathrm{Gal}(\mathbb{Q}(E[p], i)/\mathbb{Q}(i)) \to \mathrm{GL}_2(\mathbb{F}_p)$ is conjugate to a subgroup of the diagonal matrices. The set of diagonal matrices has order $(p-1)^2 = 2^{2^{n+1}}$. Since ρ is injective, Lagrange's theorem implies that $[\mathbb{Q}(E[p], i) : \mathbb{Q}(i)]$ is a power of 2. □

One might think that the more general question of when $\mathrm{Gal}(\mathbb{Q}(E[n])/\mathbb{Q})$ can be a p-group would add some interest, but it turns out the only interesting case here is when $p = 2$. This is easy to see since $[\mathbb{Q}(E[n]) : \mathbb{Q}]$ is even for all $n \geq 3$. If we want to study a condition on $\mathrm{Gal}(\mathbb{Q}(E[n])/\mathbb{Q})$ that is more general than being a p-group, but is restrictive enough to yield meaningful results, a naive thing to do would be to study when $\mathrm{Gal}(\mathbb{Q}(E[n])/\mathbb{Q})$ is the direct product of a finite number of p-groups. This allows some additional flexibility, but still imposes some structure on the situation.

While at first this seems arbitrary and naive, it is a very natural thing to do.

Proposition 1.5 [18, Proposition II.7.5]. *Let G be a finite group, then G is nilpotent if and only if it is the direct product of its p-Sylow subgroups.*

Thus, our naive consideration is actually logically equivalent to the condition that $\mathrm{Gal}(\mathbb{Q}(E[n])/\mathbb{Q})$ is nilpotent. Looking at the literature, we find the following result about when $\mathrm{Gal}(\mathbb{Q}(E[n])/\mathbb{Q})$ is abelian.

Theorem 1.6 [16, Theorem 1.1]. *Let $E/\mathbb{Q}$ be an elliptic curve and $n \geq 2$. If $\mathbb{Q}(E[n])/\mathbb{Q}$ is an abelian extension, then $n = 2, 3, 4, 5, 6,$ or 8.*

With this added context, our new goal is to understand when $\mathrm{Gal}(\mathbb{Q}(E[n])/\mathbb{Q})$ is nilpotent for $E/\mathbb{Q}$. To that end, we give the following definition:

Definition 1.7. Let K/k be a Galois extension of fields. We say that K/k is a *nilpotent extension* if $\mathrm{Gal}(K/k)$ is a nilpotent group. When the base field k is clear from context, we will just say that K is a *nilpotent field* for brevity.

We will be able to give a complete answer to the question of when $\mathbb{Q}(E[n])/\mathbb{Q}$ is nilpotent that is conditional on a positive answer to Serre's uniformity question.

Conjecture 1.8 [46, Conjecture 1.1; 34, Conjecture 1.1.5]. *If $p > 11$, then there is no non-CM elliptic curve $E/\mathbb{Q}$ for which the image of the mod p Galois representation is contained in the normalizer of the nonsplit Cartan subgroup.*

Before stating our next theorem we remind the reader that a Mersenne prime is a prime which is one less than a power of 2. Such a prime always has the form $2^p - 1$ for p a prime.

Theorem 1.9. *Let $E/\mathbb{Q}$ be an elliptic curve.*

(1) *If E does not have complex multiplication, Conjecture 1.8 holds, and $\mathbb{Q}(E[n])/\mathbb{Q}$ is nilpotent then $n \in \{3, 5, 6, 7, 15, 21\} \cup \{2^k : k \in \mathbb{Z}^+\}$. Each of these cases occurs for infinitely many different rational j-invariants.*

(2) *If E does not have complex multiplication, $n \notin \{3, 5, 6, 7, 15, 21\} \cup \{2^k : k \in \mathbb{Z}^+\}$, then $\mathbb{Q}(E[n])/\mathbb{Q}$ is nilpotent if and only if one of the following holds*: (i) *n is a product of distinct Mersenne primes with the property that the mod p image of Galois is contained in the normalizer of the nonsplit Cartan for all primes $p \mid n$, or* (ii) *n is twice a product of distinct Mersenne primes with the property that the mod p image of Galois is contained in the normalizer of the nonsplit Cartan, and the mod 2 image has RSZB label* 2.2.0.1.

(3) *If E has complex multiplication by the order of discriminant $D \in \{-4, -7, -8, -12, -16, -28\}$, then $\mathbb{Q}(E[n])/\mathbb{Q}$ is nilpotent if and only if n is a power of two times a product of distinct Mersenne and Fermat primes, where the Mersenne primes are inert in the CM field and the Fermat primes are split in the CM field.*

(4) *If E has complex multiplication by the order of discriminant $D \in \{-11, -19, -43, -67, -163\}$, then $\mathbb{Q}(E[n])/\mathbb{Q}$ is nilpotent if and only if n is a product of distinct Mersenne and Fermat primes, where the Mersenne primes are inert in the CM field and the Fermat primes are split in the CM field.*

(5) *If E has complex multiplication by the order of discriminant $D = -27$, then $\mathbb{Q}(E[n])/\mathbb{Q}$ is never nilpotent.*

(6) *If $j(E) = 0$, then E is isomorphic over $\mathbb{Q}$ to an elliptic curve of the form $E_d : y^2 = x^3 + d$. Then, $\mathbb{Q}(E_d[n])/\mathbb{Q}$ is nilpotent if and only if $n = p$ is a prime and*

$$\begin{cases} d \equiv 1 \pmod{(\mathbb{Q}^\times)^3} \text{ if } p = 2, \\ d \equiv 2 \pmod{(\mathbb{Q}^\times)^3} \text{ if } p = 3, \\ d \equiv 2 \cdot p^{\frac{1}{3}(p-1)} \pmod{(\mathbb{Q}^\times)^3} \text{ if } p = 3 \cdot 2^k + 1 \text{ for some } k \geq 1, \\ d \equiv 2 \cdot p^{\frac{1}{3}(p+1)} \pmod{(\mathbb{Q}^\times)^3} \text{ if } p = 3 \cdot 2^k - 1 \text{ for some } k \geq 1. \end{cases}$$

Corollary 1.10. *There is no elliptic curve $E/\mathbb{Q}$ for which $\mathbb{Q}(E[19])/\mathbb{Q}$ is a nilpotent extension. Further*, 19 *is the smallest prime with this property.*

Along the way to proving Theorem 1.9, a phenomenon appears that we point out here. When studying the question of when $\mathbb{Q}(E[p^n])/\mathbb{Q}$ nilpotent given that $\mathbb{Q}(E[p^{n-1}])/\mathbb{Q}$ is, we realized that we would need to classify exactly when $\mathbb{Q}(E[p^n]) = \mathbb{Q}(E[p^{n-1}], \zeta_{p^n})$.

Definition 1.11. Let $E/\mathbb{Q}$ be an elliptic curve and let $m, n \in \mathbb{Z}^+$ such that $m \mid n$. We say that there is a *near coincidence* between the m- and n-division fields of E if

$$\mathbb{Q}(E[n]) = \mathbb{Q}(E[m], \zeta_n).$$

The terminology here is motivated by the definition given in [7] where an elliptic curve $E/\mathbb{Q}$ is said to have a *coincidence* between its division fields if $\mathbb{Q}(E[m]) = \mathbb{Q}(E[n])$ for distinct integers m and n. In that paper is the following theorem:

Theorem 1.12 [7, Theorem 1.4]. *Let $E/\mathbb{Q}$ be an elliptic curve*, *p be a prime*, *and let $n \in \mathbb{Z}^+$.*

(1) *Suppose $\mathbb{Q}(E[p^{n+1}]) = \mathbb{Q}(E[p^n])$. Then $p = 2$ and $n = 1$.*

(2) *If $\mathbb{Q}(E[p^n]) \cap \mathbb{Q}(\zeta_{p^{n+1}}) = \mathbb{Q}(\zeta_{p^{n+1}})$, then $p = 2$.*

The problem of classifying such coincidences for elliptic curves defined over number fields was also recently investigated by Yvon in [44].

In the end, we are able to give a complete classification of elliptic curves over $\mathbb{Q}$ with a near coincidence of their p^{n+1} and p^n division fields; i.e., when $\mathbb{Q}(E[p^{n+1}]) = \mathbb{Q}(E[p^n], \zeta_p^{n+1})$.

Theorem 1.13. *Let $E/\mathbb{Q}$ be an elliptic curve*, *$p \in \mathbb{Z}$ prime*, *and $n \in \mathbb{Z}^+$. Suppose that*

$$\mathbb{Q}(E[p^{n+1}]) = \mathbb{Q}(E[p^n], \zeta_{p^{n+1}}).$$

Then $p \in \{2, 3\}$ and $n = 1$. Further, *if $p = 2$*, *then E must correspond to a rational point on the one of the modular curves with RSZB label* 4.48.0.3 *or* 4.16.0.2, *while if $p = 3$*, *then E must come from a rational point on* 9.27.0.1.

Remark 1.14. Before moving on, we point out that elliptic curves corresponding to rational points on 4.16.0.2 have the property that $\mathbb{Q}(E[4]) = \mathbb{Q}(E[2]) = \mathbb{Q}(E[2], i)$. Thus, these curves actually have a coincidence of division fields. In contrast, the elliptic curves corresponding to rational points on 4.48.0.3 have the property that $\mathbb{Q}(E[2]) = \mathbb{Q}$ and $\mathbb{Q}(E[4]) = \mathbb{Q}(i)$. So these curves have a near coincidence without having a coincidence of division fields. It is interesting to note that in order to have a near coincidence between the 2- and 4-division fields, without having an actual coincidence between the 2- and 4-division fields, the 2-division field has to be trivial.

Similarly, the rational points on 9.27.0.1 yield elliptic curves with a near coincidence between their 3- and 9-division fields. Unlike the case with $p = 2$, a rational point on 9.27.0.1 corresponds to an elliptic curve E with either $j(E) = 0$ or a surjective $\bar{\rho}_{E,3}$. In particular, if E is a non-CM elliptic curve with $\mathbb{Q}(E[9]) = \mathbb{Q}(E[3], \zeta_9)$, we must have $[\mathbb{Q}(E[3]) : \mathbb{Q}] = 48$.

Remark 1.15. We choose to begin this paper with the Gauss–Wantzel theorem not only because of its classical significance, but because it serves as a natural entry point for a broad audience into the kinds of questions that we hope to address in this paper. The main objective here is to classify when the division field $\mathbb{Q}(E[n])/\mathbb{Q}$ of an elliptic curve E over $\mathbb{Q}$ is nilpotent. Along the way, we uncover a surprising analog of the Gauss–Wantzel theorem as well as a result about near coincidences of division fields. We hope that the Gauss–Wantzel analog, while not the main thrust of the paper, illustrates the value of considering questions like these.

Outline of the paper. In Section 2 we recall basic facts related to elliptic and modular curves that will be necessary for the proof of the main results. The proof of Theorem 1.13 will be handled in Section 3 by finding the smallest power n such that $\mathbb{Q}(E[p^{n+1}])$ cannot be $\mathbb{Q}(E[p^n], \zeta_{p^{n+1}})$ and then we prove that if $\mathbb{Q}(E[p^{n+1}]) \neq \mathbb{Q}(E[p^n], \zeta_{p^{n+1}})$, then $\mathbb{Q}(E[p^{n+2}]) \neq \mathbb{Q}(E[p^{n+1}], \zeta_{p^{n+2}})$. The proof will have to be broken down into cases depending on if $p = 2$, 3, 5, or $p \geq 7$.

The proof of Theorem 1.9 starts in Section 4 by using group theory to classify the nilpotent subgroups of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ according to their image in $\mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$. We then apply known results about the corresponding modular curves to determine when the extension $\mathbb{Q}(E[p])/\mathbb{Q}$ can be nilpotent. This is where we will employ Conjecture 1.8.

In Section 5 we use the information from the previous section to prove that if p is odd, then the p^2-division field is never nilpotent. Lastly, in Section 6 we study which combinations of mod p images can occur simultaneously, and prove Theorem 1.9. Throughout the paper we address the case of elliptic curves with complex multiplication separately from those without complex multiplication because of their unique properties (which are outlined in Section 2.1.1).

All of the computations in this paper were performed using Magma [4] and the code can be found at [8].

2. Background

The goal of this section is to review some of the background information necessary for the proofs of the main theorems. In each subsection readers should find some additional resources to supplement what is written here.

2.1. *Elliptic curves.* For background about elliptic curves, see [39]. Given an elliptic curve $E/\mathbb{Q}$ and a natural number n, the points of order dividing n defined over $\overline{\mathbb{Q}}$ form a group. Considering $E(\mathbb{C})$ as the quotient of $\mathbb{C}$ by a lattice shows that

$$E[n] := \{P \in E(\overline{\mathbb{Q}}) : nP = \mathcal{O}\} \simeq \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}.$$

This isomorphism is noncanonical, but it only requires a choice of basis for $E[n]$.

Because the group law on an elliptic curve is given by rational functions, $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on $E[n]$ component-wise. That is, if $P = (x, y) \in E[n]$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, then $P^\sigma = (\sigma(x), \sigma(y)) \in E[n]$. This component-wise action induces a representation

$$\bar{\rho}_{E,n} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

with the property that $\mathrm{Im}\,\bar{\rho}_{E,n} \simeq \mathrm{Gal}(\mathbb{Q}(E[n])/\mathbb{Q})$. We remark here that because the isomorphism $E[n] \simeq \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ is noncanonical, $\mathrm{Im}\,\bar{\rho}_{E,n}$ is really only defined up to conjugation.

A guiding principle in this paper is that oftentimes things can be broken down into cases depending on the shape of $\mathrm{Im}\,\bar{\rho}_{E,p}$. We are able to do this thanks to the following proposition.

Proposition 2.1 [42, Lemma 2]. *Let p be a prime and let G be a subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$. If p divides $|G|$ then, either $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}) \subseteq G$, or G is contained inside a Borel subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$. If p does not divide $|G|$, let H be the image of G in $\mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$; then*

1. *H is cyclic and G is contained inside a Cartan subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$, or*
2. *H is dihedral and G is contained in the normalizer of a Cartan subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$, but not the Cartan itself, or*
3. *H is isomorphic to A_4, S_4 and A_5.*

In case (2), p must be odd. In case (3), p must be relatively prime to 6, 6, and 30 respectively.

We will say more about the Cartan subgroups when we discuss elliptic curves with complex multiplication in Section 2.1.1.

Before moving on from Galois representations attached to elliptic curves, we draw attention to the fact that we can combine mod p^k representations using inverse limits to define the p-adic Galois representations

$$\rho_{E,p^\infty} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{Z}_p).$$

An important point, for our purposes, is that if $p \geq 5$, then the group $\mathrm{SL}_2(\mathbb{Z}_p)$ has no proper closed subgroups whose image is $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ under the standard reduction map. As Serre explains, this leads to the following proposition.

Proposition 2.2. [35, Section IV] *Let $E/\mathbb{Q}$ be an elliptic curve and let $p \geq 5$ be a prime. If $\bar{\rho}_{E,p}$ is surjective, then ρ_{E,p^∞} is also surjective.*

To see how this breaks down when $p = 2$ or 3, the reader is encouraged to see [11] and [13].

Lastly, we note that given an elliptic curve over $\mathbb{Q}$, the group $\operatorname{Im}\bar{\rho}_{E,n}$ must have a few special properties.

Definition 2.3. Let $n \geq 2$ be a positive integer and let G be a subgroup of $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$. We say that G is an *admissible* group if

- $\det(G) = (\mathbb{Z}/n\mathbb{Z})^\times$, and
- G contains an element of determinant -1 and trace 0 that fixes a point of order n inside of $(\mathbb{Z}/n\mathbb{Z})^2$.

Proposition 2.4 [45, Proposition 2.2]. *Let $n \geq 2$ be an integer and let $E/\mathbb{Q}$ be an elliptic curve. Then, $\operatorname{Im}\bar{\rho}_{E,n}$ is an admissible subgroup of $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$.*

2.1.1. *Elliptic curves with complex multiplication.* Elliptic curves come in two distinct types depending on their endomorphism rings. Given an elliptic curve E, defined over a field of characteristic 0, the endomorphism ring of E over $\overline{\mathbb{Q}}$ is either isomorphic to $\mathbb{Z}$ or an order of an imaginary quadratic field, usually denoted by $\mathcal{O}$. When the endomorphism ring is larger than $\mathbb{Z}$ we say that E has *complex multiplication* by $\mathcal{O}$. Throughout this section we follow the work done in [34, Section 12]. Many of the results we use were first proven in [28]. A reader looking for an introduction to elliptic curves with complex multiplication should see [38, Chapter II].

One way to think about elliptic curves with complex multiplication is as elliptic curves with additional symmetries. These added symmetries manifest themselves in many ways. They endow elliptic curves with complex multiplication with many interesting properties that make them unique among elliptic curves in general. Of particular interest to us is that the Galois representations attached to elliptic curves with complex multiplication behave very differently than those without complex multiplication.

We introduce some notation to state results about the Galois representations attached to elliptic curves with complex multiplication.

Given $\mathcal{O}$, an order of a quadratic imaginary field K. We define the *adelic Cartan subgroup associated to* $\mathcal{O}$ to be

$$\mathcal{C}_\mathcal{O} = \varprojlim (\mathcal{O}/N\mathcal{O})^\times$$

where N is a positive integer and the inverse limit is taken with respect to divisibility.

Next, we let $\mathcal{O}_K$ be the maximal order inside of K and we let $f = [\mathcal{O}_K : \mathcal{O}]$ be the conductor of $\mathcal{O}$. Continuing, we let $D = \operatorname{disc}(\mathcal{O}) = f^2 \operatorname{disc}\mathcal{O}$, and

$$\phi = \begin{cases} f & D \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Then, we let

$$\omega = \frac{\phi + \sqrt{D}}{2} \quad \text{and} \quad \delta = \frac{D - \phi^2}{4},$$

so that $\mathcal{O} = \operatorname{Span}_{\mathbb{Z}}\{1, \omega\}$ with $\omega^2 - \phi\omega - \delta = 0$. We can now define the level N Cartan subgroup associated to $\mathcal{O}$ as

$$\mathcal{C}_{\mathcal{O}}(N) = \left\{ \begin{pmatrix} a + b\phi & b \\ \delta b & a \end{pmatrix} : a, b \in \mathbb{Z}/N\mathbb{Z} \quad \text{and} \quad a^2 + ab\phi - \delta b^2 \in (\mathbb{Z}/N\mathbb{Z})^\times \right\}$$
$$\subseteq \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}).$$

Taking inverse limits as N runs over the positive integers ordered by divisibility of the level N, we can define

$$\mathcal{C}_{\mathcal{O}}(\widehat{\mathbb{Z}}) = \varprojlim \mathcal{C}_{\mathcal{O}}(N) \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}}).$$

The group $\mathcal{C}_{\mathcal{O}}(\widehat{\mathbb{Z}})$ is a closed subgroup of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ that is isomorphic to $\mathcal{C}_{\mathcal{O}}$ under the isomorphism

$$a + b\omega \mapsto \begin{pmatrix} a + b\phi & b \\ \delta b & a \end{pmatrix}.$$

Next, we define

$$\mathcal{N}_{\mathcal{O}}(N) = \left\langle \mathcal{C}_{\mathcal{O}}(N), \left(\begin{smallmatrix} -1 & 0 \\ \phi & 1 \end{smallmatrix} \right) \right\rangle$$

and let $\mathcal{N}_{\mathcal{O}}(\widehat{\mathbb{Z}}) \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}})$ and $\mathcal{N}_{\mathcal{O}}(\mathbb{Z}_p) \subseteq \mathrm{GL}_2(\widehat{\mathbb{Z}}_p)$ be the usual inverse limits of the $\mathcal{N}_{\mathcal{O}}(N)$.

Remark 2.5. Frequently the group $\mathcal{N}_{\mathcal{O}}$ is called the normalizer of $\mathcal{C}_{\mathcal{O}}$. It turns out that the group $\mathcal{N}_{\mathcal{O}}(\mathbb{Z}_p)$ is the normalizer of $\mathcal{C}_{\mathcal{O}}(\mathbb{Z}_p)$ in $\mathrm{GL}_2(\mathbb{Z}_p)$ (by [28, Proposition 5.6(2)]), but $\mathcal{N}_{\mathcal{O}}$ is not the normalizer of $\mathcal{C}_{\mathcal{O}}$ in $\mathrm{GL}_2(\widehat{\mathbb{Z}})$. See [34, Remark 12.1.2].

Before moving on we make a few more observations about these groups that will be useful later on. First we note that by construction each of the groups $\mathcal{C}_{\mathcal{O}}(N)$ is an abelian group since $\mathcal{O}/N\mathcal{O}$ is abelian. In contrast, the groups $\mathcal{N}_{\mathcal{O}}$ are not abelian unless $p = 2$. In order to compute the center of $\mathcal{N}_{\mathcal{O}}(N)$, it would be a simple matter of determining which matrices in $\mathcal{C}_{\mathcal{O}}(N)$ commute with $M = \left(\begin{smallmatrix} -1 & 0 \\ \phi & 1 \end{smallmatrix} \right)$. Let

$$A = \begin{pmatrix} a + b\phi & b \\ \delta b & a \end{pmatrix} \in \mathcal{C}_{\mathcal{O}}(N).$$

Computing the entry in the first row, second column of MA and AM we see that A commutes with M if and only if $b = -b$. Thus, if $p \neq 2$, then $A \in Z(\mathcal{N}_{\mathcal{O}}(N))$ if and only if $A = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right) = aI \in Z(\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}))$. From this we get:

Lemma 2.6. *Let $N > 2$ be an integer. Let G be a subgroup of $\mathcal{N}_{\mathcal{O}}(N)$ such that*

$$\left(\begin{smallmatrix} -1 & 0 \\ \phi & 1 \end{smallmatrix}\right) \in G.$$

Then the center of G, $Z(G)$, is exactly the set of scalar matrices in G. In other words,

$$Z(G) = Z(\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})) \cap G.$$

The last few theorems we need will help us pin down the exact image of the Galois representations attached to elliptic curves with complex multiplication. We will state the following theorems for elliptic curves with complex multiplication defined over $\mathbb{Q}$, but we note that both [34, Section 12] and [28] handle the case where E is defined over a number field.

Proposition 2.7 [34, Proposition 12.1.4]. *Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication by $\mathcal{O}$, let p be a prime, and let*

$$e = \begin{cases} 4 & \text{if } p = 2, \\ 3 & \text{if } p = 3, \\ 1 & \text{otherwise.} \end{cases}$$

Then $\operatorname{Im} \rho_{E,p^\infty}$ is the inverse image of $\operatorname{Im} \bar{\rho}_{E,p^e}$ under the reduction map $\mathcal{N}_{\mathcal{O}}(\mathbb{Z}_p) \to \mathcal{N}_{\mathcal{O}}(\mathbb{Z}/p^e\mathbb{Z})$.

Proposition 2.7 will be exactly what we need in order to understand how the division fields change as we go up the p-adic tower. This will be useful in Section 5.

Proposition 2.8. [28, Theorem 1.2(4)] *Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication by $\mathcal{O} \neq \mathbb{Z}[\zeta_3]$. If p does not divide $2\operatorname{disc}(\mathcal{O})$, then there is a choice of basis such that $\operatorname{Im} \rho_{E,p^\infty} = \mathcal{N}_{\mathcal{O}}(\mathbb{Z}_p)$.*

2.2. ***Modular curves.*** Modular curves are the main objects that we will use to classify the elliptic curves over $\mathbb{Q}$ with a given admissible group as the image of their mod n representation. Given a natural number $n \geq 2$ and an admissible subgroup $G \subseteq \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ there is a smooth, projective, and geometrically irreducible curve defined over $\mathbb{Q}$ denoted X_G whose $\mathbb{Q}$-rational points classify elliptic curves with the property that $\operatorname{Im} \bar{\rho}_{E,n}$ is conjugate to a *subgroup* of G. Here we emphasize that the image of $\operatorname{Im} \bar{\rho}_{E,n}$ need not be all of G in order to have a corresponding point on X_G. Indeed, since subgroups of nilpotent groups are nilpotent, this will allow us to focus on finding the maximal admissible nilpotent groups of a given level.

The nature of this correspondence depends on whether $-I \in G$ or not, but if G were a nilpotent group that did not contain $-I$, then adding $-I$ would preserve nilpotency. For this reason, we can assume that $-I \in G$. Then the curve X_G always comes with a natural morphism

$$\pi_G : X_G \to \mathbb{P}^1_{\mathbb{Q}}$$

such that an elliptic curve $E/\mathbb{Q}$ with j-invariant $j_E \notin \{0, 1728\}$, has $\operatorname{Im} \bar{\rho}_{E,n}$ conjugate to a subgroup of G if and only if $j_E = \pi_G(P)$ for some $P \in X_G(\mathbb{Q})$. The interested reader should see [34, Subsection 2.3] to see more details when $-I \notin G$.

We end this section by emphasizing that a complete classification of the points on these curves would give a corresponding classification of $\operatorname{im} \bar{\rho}_{E,n}$ for every elliptic curve $E/\mathbb{Q}$. That is, if we can determine all of the maximal nilpotent subgroups H of $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ and classify all the rational points on the corresponding X_H's, then we would have classified all the elliptic curves with nilpotent n-division fields.

3. Near coincidences

In order to classify near coincidences, it will be useful to be able to detect them using the image of the corresponding Galois representation. With this in mind, we give the following definition.

Definition 3.1. Let $G \subseteq \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$ with surjective determinant and suppose that $m \mid n$. We say that G *represents a near coincidence* of level (n, m) if

$$(G \cap \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})) \cap \operatorname{Ker}(\pi) = \{I\},$$

where $\pi : \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$ is the standard componentwise reduction map.

Remark 3.2. The idea behind this definition is that classically we know that $(G \cap \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z}))$ fixes $\mathbb{Q}(\zeta_n) \subseteq \mathbb{Q}(E[n])$ and $\operatorname{Ker}(\pi)$ fixes $\mathbb{Q}(E[m]) \subseteq \mathbb{Q}(E[n])$. Therefore, $(G \cap \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})) \cap \operatorname{Ker}(\pi)$ should fix $\mathbb{Q}(E[m], \zeta_n)$. Thus, the only way that $\mathbb{Q}(E[n]) = \mathbb{Q}(E[m], \zeta_n)$ is if $(G \cap \mathrm{SL}_2(\mathbb{Z}/n\mathbb{Z})) \cap \operatorname{Ker}(\pi) = \{I\}$.

Of course this definition requires that $m \mid n$, but that lines up with the original definition of near coincidence.

The proof of Theorem 1.13 will be done by first considering the case of prime level and then moving on to the case of prime power level. We will have to break the prime level case down into 3 smaller cases. These cases consist of when $p = 2$, $p = 3$, or $p \geq 5$.

3.1. *Proof of Theorem 1.13 for prime levels.*

We will start the case when $n = 1$ of the theorem by dealing with primes $p \geq 5$ and break the argument into cases depending on $\operatorname{Im} \bar{\rho}_{E,p}$ based on Proposition 2.1. First we handle the case that $\bar{\rho}_{E,p}$ is surjective.

Proposition 3.3. *Assume that $E/\mathbb{Q}$ is an elliptic curve, $p \geq 5$ is prime, and* $\operatorname{Im}\bar{\rho}_{E,p} = \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$. *Then* $\mathbb{Q}(E[p^2]) \neq \mathbb{Q}(E[p], \zeta_{p^2})$.

Proof. By Proposition 2.2, we have $\operatorname{Im}\rho_{E,p^\infty} = \mathrm{GL}_2(\mathbb{Z}_p)$ and hence $\operatorname{Im}\bar{\rho}_{E,p^2} = \mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$. Hence

$$[\mathbb{Q}(E[p^2]) : \mathbb{Q}] = |\mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})| = p^5(p-1)^2(p+1),$$

while

$$\begin{aligned}[\mathbb{Q}(E[p], \zeta_{p^2})] &\leq [\mathbb{Q}(E[p], \zeta_{p^2}) : \mathbb{Q}(E[p])][\mathbb{Q}(E[p]) : \mathbb{Q}] \\ &\leq p(p-1) \cdot |\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})| \\ &= p^2(p-1)^3(p+1) < p^5(p-1)^2(p+1).\end{aligned}$$

This proves the claim. □

Next we handle the case that $\operatorname{Im}\bar{\rho}_{E,p}$ is a proper subgroup whose order is a multiple of p.

Proposition 3.4. *Assume that $E/\mathbb{Q}$ is an elliptic curve, $p \geq 5$ is prime, and* $\operatorname{Im}\bar{\rho}_{E,p}$ *is a proper subgroup of* $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ *whose order is a multiple of p. Then* $\mathbb{Q}(E[p^2]) \neq \mathbb{Q}(E[p], \zeta_{p^2})$.

Proof. In this case, the image of $\operatorname{Im}\bar{\rho}_{E,p}$ is contained in a Borel subgroup by Proposition 2.1 and hence E has a cyclic p-isogeny defined over $\mathbb{Q}$. By the classification of Mazur, we have $p \in \{5, 7, 11, 17, 19, 43, 67, 163\}$. Let $H = \operatorname{im}\rho_{E,p^\infty}$.

If E has complex multiplication, choose a basis so that $H \subseteq \mathcal{N}_{\mathcal{O}}(\mathbb{Z}_p)$. By Proposition 2.7, H is the full preimage in $\mathcal{N}_{\mathcal{O}}(\mathbb{Z}_p)$ of $\operatorname{Im}\bar{\rho}_{E,p}$. Since $\mathcal{N}_{\mathcal{O}}(\mathbb{Z}_p)$ contains all matrices of the form $\left(\begin{smallmatrix} 1+kp & 0 \\ 0 & 1-kp \end{smallmatrix}\right)$ with $k \in \mathbb{Z}/p\mathbb{Z}$ and this implies that $\mathbb{Q}(E[p^2])/\mathbb{Q}(E[p], \zeta_{p^2})$ is a nontrivial extension.

If E does not have complex multiplication, then Theorem 1.1.6 of [34] implies that X_H is isomorphic to $\mathbb{P}^1$ or a positive rank elliptic curve, or that X_H is listed in Table 1 of [34] as the other cases are ruled out by the assumption that $p \mid |\operatorname{Im}\bar{\rho}_{E,p}|$. If we assume that $\mathbb{Q}(E[p^2]) = \mathbb{Q}(E[p], \zeta_{p^2})$, then $p^2 \nmid |\operatorname{Im}\bar{\rho}_{E,p^2}|$ and this implies that the index of H in $\mathrm{GL}_2(\mathbb{Z}_p)$ is a multiple of p^3. Since $p \geq 5$, this implies that the index is ≥ 125 and this implies that X_H has genus ≥ 2. There are groups H contained in a Borel subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ listed in Table 1 of [34] for $p = 11$, 17 and 37, but in no case is the index of H a multiple of p^3. This concludes the proof. □

Next we handle the case that $\operatorname{Im}\bar{\rho}_{E,p}$ is contained in the normalizer of a split Cartan subgroup. We divide the argument into the CM case and the non-CM case.

Proposition 3.5. *Let $E/\mathbb{Q}$ be an elliptic curve and $p \geq 5$ a prime such that* $\operatorname{Im}\bar{\rho}_{E,p}$ *is contained in the normalizer of a split Cartan subgroup of* $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$. *Then* $\mathbb{Q}(E[p^2]) \neq \mathbb{Q}(E[p], \zeta_{p^2})$.

Proof. We first will show that E has complex multiplication. If E does not have complex multiplication, then the results of [3] rule out $p = 11$ and $p \geq 17$, and the results of [1] rule out $p = 13$. Hence $p \in \{5, 7\}$. If $H = \operatorname{Im} \bar{\rho}_{E,p^2}$ and $\mathbb{Q}(E[p^2]) = \mathbb{Q}(E[p], \zeta_{p^2})$, then H is an index p^3 subgroup of the full preimage in $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ of $\operatorname{Im} \bar{\rho}_{E,p}$ and hence

$$(*) \qquad |H| \text{ divides } \frac{|\mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})|}{p^4(p+1)/2},$$

since $\frac{p(p+1)}{2}$ is the index of the normalizer of the split Cartan in $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$. In addition, by Proposition 2.4, H must also be an admissible group. Finally, H must be nonabelian, since by [16] $\mathbb{Q}(E[n])/\mathbb{Q}$ is nonabelian if $n > 8$.

For these two primes p, we use Magma to search for admissible subgroups H of $\mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$ such that $(*)$ is satisfied and H is nonabelian. For $p = 5$ there are two conjugacy classes of such subgroups and for $p = 7$ there are four. In both cases, all such subgroups are contained in the normalizer of a split Cartan subgroup of $\mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$. (The code for these calculations can be found in the file `prop35.m` at [8].) The rational points on the modular curve corresponding to the normalizer of the split Cartan subgroup mod 25 are determined by Momose and Shimura in [30], and the mod 49 case is handled by Momose in [29]. In both cases, all such elliptic curves have CM. It follows that E has complex multiplication.

Now we show that if E has complex multiplication, then $\mathbb{Q}(E[p^2]) \neq \mathbb{Q}(E[p], \zeta_{p^2})$. The argument here is identical to that in the proof of Proposition 3.4. Choose a basis so that $\operatorname{Im} \bar{\rho}_{E,p^2} \subseteq \mathcal{N}_{\mathcal{O}}(\mathbb{Z}_p)$. By Proposition 2.7, $\operatorname{Im} \bar{\rho}_{E,p^2}$ is the full preimage in $\mathcal{N}_{\mathcal{O}}(\mathbb{Z}_p)$ of $\operatorname{Im} \bar{\rho}_{E,p}$. Therefore, $\operatorname{Im} \bar{\rho}_{E,p^2}$ contains all matrices of the form $\left(\begin{smallmatrix} 1+kp & 0 \\ 0 & 1-kp \end{smallmatrix}\right)$ with $k \in \mathbb{Z}/p\mathbb{Z}$ and this implies that $\mathbb{Q}(E[p^2])/\mathbb{Q}(E[p], \zeta_{p^2})$ is a nontrivial extension. □

The last case to be dealt with is when $\operatorname{Im} \bar{\rho}_{E,p}$ is contained in the normalizer of a nonsplit Cartan subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ for $p \geq 5$. Importantly, we know from [23, Appendix B] or [45, Proposition 1.13] that in this case any such elliptic curve has to have potentially supersingular reduction at p.

Proposition 3.6 [36, p. 312]. *Let $E/\mathbb{Q}$ be an elliptic curve with potential good reduction at $p \geq 5$ and discriminant Δ. Then, E acquires good reduction over $\mathbb{Q}(\sqrt[12]{\Delta})$ at all primes over p.*

In [loc. cit.], Serre explains that not only does E gain good reduction over $\mathbb{Q}(\sqrt[12]{\Delta})$, but also that $\operatorname{ord}_p(\Delta) \in \{0, 2, 3, 4, 6, 8, 9, 10\}$. Thus, if $\mathfrak{p}$ is a prime above p in $\mathbb{Q}(\sqrt[12]{\Delta})$, then $e(\mathfrak{p}/p) \in \{1, 2, 3, 4, 6\}$.

Even with this in hand, for $p = 5$ we will have to rely on the classification of rational points provided in [34].

Proposition 3.7. *Let $E/\mathbb{Q}$ be such that* $\operatorname{im}\bar{\rho}_{E,5}$ *is contained in the normalizer of the nonsplit Cartan subgroup mod* 5*. Then* $\mathbb{Q}(E[25]) \neq \mathbb{Q}(E[5], \zeta_{25})$.

Proof. We search $\mathrm{GL}_2(\mathbb{Z}/25\mathbb{Z})$ for groups that represent (25, 5) near coincidences. We then take the ones that are maximal with respect to containment (up to conjugation) and check if they have points. These groups are exactly the groups with RSZB labels

25.625.36.1, 25.1250.76.1, 25.2500.156.{2,3}, 25.3750.236.{1,2}

(where we have used braces to combine almost identical labels). Using the data in [34], we see that there are no noncuspidal $\mathbb{Q}$-rational points on any of these curves and so there are no elliptic curves over $\mathbb{Q}$ with a (25, 5) near coincidence. □

We now turn to the case that $p \geq 7$.

Proposition 3.8. *Let $E/\mathbb{Q}$ be an elliptic curve and let $p \geq 7$ be a prime such that* $\operatorname{Im}\bar{\rho}_{E,p}$ *is a subgroup of the normalizer of a nonsplit Cartan subgroup* $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$*. Let Δ be the discriminant of E and suppose that $\mathbb{Q}(E[p^2]) = \mathbb{Q}(E[p], \zeta_{p^2})$. Let $\mathfrak{p}$ be a prime over p in $\mathbb{Q}(\sqrt[12]{\Delta})$. The extension $\mathbb{Q}(\sqrt[12]{\Delta}, E[p^2])/\mathbb{Q}(\sqrt[12]{\Delta})$ is a degree $2p(p^2-1)$ extension that is totally ramified at $\mathfrak{p}$.*

Proof. Suppose towards a contradiction that there is a prime $\mathfrak{P}$ above $\mathfrak{p}$ such that $e(\mathfrak{P}/\mathfrak{p})$ is a proper divisor of $2p(p^2-1)$. Since $\operatorname{Im}\bar{\rho}_{E,p}$ is a subgroup of the normalizer of a nonsplit Cartan subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$, we know that $[\mathbb{Q}(E[p]) : \mathbb{Q}]$ is a divisor of $2(p^2-1)$. Further, because we assumed that $\mathbb{Q}(E[p^2]) = \mathbb{Q}(E[p], \zeta_{p^2})$, we know that $[\mathbb{Q}(E[p^2]) : \mathbb{Q}]$ is a divisor of $2p(p^2-1)$. Thus $\mathbb{Q}(\sqrt[12]{\Delta}, E[p^2])/\mathbb{Q}(\sqrt[12]{\Delta})$ has degree dividing $2p(p^2-1)$. Now, $e(\mathfrak{P}/\mathfrak{p})$ is a proper divisor of $\left|\bar{\rho}_{E,p^2}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt[12]{\Delta})))\right|$, which itself divides $2p(p^2-1)$. Thus

$$e(\mathfrak{P}/\mathfrak{p}) \leq p(p^2-1) \quad \text{and} \quad e(\mathfrak{P}/p) = e(\mathfrak{P}/\mathfrak{p})e(\mathfrak{p}/p) \leq p(p^2-1)\cdot 6 < p^2(p^2-1).$$

But a result of Hanson Smith [40, Theorem 1.1] says that $e(\mathfrak{P}/p) \geq p^4 - p^2 = p^2(p^2-1)$, giving us our contradiction. □

We now explain how the previous proposition rules out the possibility of a (p^2, p) near coincidence.

Proposition 3.9. *Let $E/\mathbb{Q}$ be an elliptic curve and let $p \geq 7$ be a prime such that* $\operatorname{Im}\bar{\rho}_{E,p}$ *is a subgroup of the normalizer of a nonsplit Cartan subgroup* $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$*. Then $\mathbb{Q}(E[p^2]) \neq \mathbb{Q}(E[p], \zeta_{p^2})$.*

Proof. Suppose towards a contradiction that $\mathbb{Q}(E[p^2]) = \mathbb{Q}(E[p], \zeta_{p^2})$. By the previous proposition, we know that $\mathbb{Q}(\sqrt[12]{\Delta}, E[p^2])/\mathbb{Q}(\sqrt[12]{\Delta})$ is totally ramified at

$\mathfrak{p}$. The assumption that $\mathbb{Q}(E[p^2]) = \mathbb{Q}(E[p], \zeta_{p^2})$ implies that $\operatorname{Im} \bar{\rho}_{E,p^2}$ has order dividing $2p(p^2-1)$. On the other hand, Proposition 3.8 implies that

$$|\bar{\rho}_{E,p^2}(\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt[12]{\Delta})))| = 2p(p^2-1).$$

It follows from this that

$$\operatorname{Im} \bar{\rho}_{E,p^2} = \bar{\rho}_{E,p^2}(\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\sqrt[12]{\Delta})))$$

and hence

$$\operatorname{Gal}(\mathbb{Q}(\sqrt[12]{\Delta}, E[p^2])/\mathbb{Q}(\sqrt[12]{\Delta})) \simeq \operatorname{Im} \bar{\rho}_{E,p^2}.$$

However, a Galois extension that is totally ramified at a prime over p must have a Galois group which is an extension of a finite p-group (the wild inertia group) by a finite cyclic group of order coprime to p (the tame inertia group). This means that for any prime $q \neq p$, the q-Sylow subgroup of $\operatorname{Im} \bar{\rho}_{E,p^2}$ must be cyclic. In particular, the Sylow 2-subgroup of $\operatorname{Im} \bar{\rho}_{E,p^2}$ must be cyclic. However, by [16], $\mathbb{Q}(E[p])/\mathbb{Q}$ must be nonabelian, and so the projectivization of $\operatorname{Im} \bar{\rho}_{E,p}$ must have a dihedral Sylow 2-subgroup. This is a contradiction. □

The last remaining case is that $\operatorname{Im} \bar{\rho}_{E,p}$ is contained in an exceptional group — that is, the image in $\operatorname{PGL}_2(\mathbb{F}_p)$ falls into case 3 of Proposition 2.1.

Proposition 3.10. *Let $E/\mathbb{Q}$ be an elliptic curve and let $p \geq 5$ be a prime such that $\operatorname{Im} \bar{\rho}_{E,p}$ is contained in an exceptional subgroup. Then $\mathbb{Q}(E[p^2]) \neq \mathbb{Q}(E[p], \zeta_{p^2})$.*

Proof. Serre showed [37, §8.4, lemme 18] that $\operatorname{Im} \bar{\rho}_{E,p}$ can only be an exceptional subgroup if image in $\operatorname{PGL}_2(\mathbb{F}_p)$ is isomorphic to S_4, $p \leq 13$ and $p \equiv 3$ or $5 \pmod 8$.

The elliptic curves with $\operatorname{Im} \bar{\rho}_{E,13}$ contained in an exceptional subgroup were determined in [2] and in [34] it was shown that for each such elliptic curve, $\operatorname{Im} \bar{\rho}_{E,13^2}$ contains all matrices $\equiv I \pmod{13}$, which implies that $\mathbb{Q}(E[p^2]) \neq \mathbb{Q}(E[p], \zeta_{p^2})$.

For $p = 11$, the elliptic curves with $\operatorname{Im} \bar{\rho}_{E,11}$ contained an exceptional subgroup were determined by Ligozat [26, Proposition II.4.4.8.1] and the only possibility is $j(E) = 0$. For such an elliptic curve, we cannot have $\mathbb{Q}(E[11^2]) = \mathbb{Q}(E[11], \zeta_{11^2})$ by Proposition 2.7.

For $p = 5$, if $\operatorname{Im} \bar{\rho}_{E,5}$ is contained in an exceptional subgroup, then it either equals an exceptional subgroup, or is contained in a Borel or the normalizer of a split Cartan. The latter two cases are impossible by Proposition 3.4 and Proposition 3.5. If the image is the exceptional subgroup mod 5, then a group theory computation with Magma shows that the mod 25 image of Galois has RSZB label 25.625.36.1, which is shown to be impossible in [34, Subsection 8.6]. □

In the case when $p = 2$ and 3, we search $\operatorname{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$ for subgroups that represent a near coincidence of level (p^2, p) and then compute the maximal groups ordered by containment up to conjugation. Doing this yields curves with labels

4.16.0.1, 4.16.0.2 and 4.48.0.3 when $p = 2$, and 9.27.0.1, 9.162.4.1 and 9.324.10.1 when $p = 3$. Thanks to [27; 34; 33] we know that 4.16.0.2, 9.162.4.1, and 9.324.10.1 do not have any rational points and so can be omitted. To summarize:

Proposition 3.11. *Let $E/\mathbb{Q}$ be an elliptic curve and let $p \in \mathbb{Z}$ be a prime such that $\mathbb{Q}(E[p^2]) = \mathbb{Q}(E[p], \zeta_{p^2})$. Then, $p = 2$ or $p = 3$ and E corresponds to a rational point on one of the modular curves with RSZB labels* 4.48.0.3, 4.16.0.2, *or* 9.27.0.1.

3.2. ***Proof of Theorem 1.13 for prime power levels.*** To start this section, we push a little further in the cases where $p = 2$ and 3. These searches were carried out using a Magma script, NearCoin.m, which can be found in [8]. In both of these cases, we can have near coincidence between the p^2- and p-division fields, but what about the p^3- and p^2-division fields?

So we search for groups that represent $(8, 4)$ and $(27, 9)$ coincidences. In the first case, we find that the maximal groups that represent an $(8, 4)$ coincidence all have genus 1 or higher. Using the data in [33], we know that this means that there is no elliptic curve without complex multiplication that has these images, and we have already completely dealt with the CM case.

When considering $(27, 9)$ coincidences, the maximal groups are the ones with RSZB labels

27.729.43.1, 27.4374.280.{1,2,3,4}, 27.8748.568.{1,2,3,5}.

Again, [34] says that the corresponding modular curves have no noncuspidal $\mathbb{Q}$-rational points and so there are no elliptic curves over $\mathbb{Q}$ with a $(27, 9)$ near coincidence.

Remark 3.12. In [34], it was shown that 27.729.43.1 cannot occur as the image of $\rho_{E,27}$ for any elliptic curve over $\mathbb{Q}$ by writing down the canonical model of this modular curve in $\mathbb{P}^{42}$ and showing it has no mod 9 points. The argument given above in Proposition 3.8 and Proposition 3.9 can be modified to give a simpler proof that this modular curve has no rational points. In particular, one can show that if $E/\mathbb{Q}$ has mod 9 image contained in 9.27.0.1 (a supergroup of 27.729.43.1), then $\mathrm{ord}_3(j(E)) \geq 7$. Since any elliptic curve with $j(E) \equiv 0 \pmod{3}$ has potentially supersingular reduction at 3, the argument (using Theorem 1.1 of [40]) can proceed along similar lines.

Next we prove by induction the case of Theorem 1.13 when $n \geq 2$.

Proposition 3.13. *Suppose that $E/\mathbb{Q}$ is an elliptic curve and $p > 2$ is a prime such that p^k divides $[\mathbb{Q}(E[p^{n+1}]) : \mathbb{Q}(E[p^n])]$ for some $k \in \{1, 2, 3, 4\}$ and $n \geq 1$. Then, p^k divides $[\mathbb{Q}(E[p^{n+2}]) : \mathbb{Q}(E[p^{n+1}])]$*

Proof. Assume that $n \geq 1$ and p^k divides $[\mathbb{Q}(E[p^{n+1}]) : \mathbb{Q}(E[p^n])]$ for some $k \in \{1, 2, 3, 4\}$ This means that the set

$$S = \{A \in \operatorname{Im} \bar{\rho}_{E,p^{n+1}} : A \equiv I \mod p^n\}$$

must have size at least p^k. Next, we let

$$\tilde{S} = \{A \in \operatorname{Im} \bar{\rho}_{E,p^{n+2}} : A \equiv I \mod p^{n+1}\}.$$

Our goal now is to show that there is an injective homomorphism from S to $\tilde{S}$. Doing this would allow us to conclude that $|S|$ divides $|\tilde{S}|$.

We can represent an element of S in the form $I + p^n X$ with X in the additive group $M_2(\mathbb{Z}/p\mathbb{Z})$ of 2×2 matrices. Define $\phi : S \to \tilde{S}$ by $\phi(I + p^n X) = I + p^{n+1} X$. It is straightforward to see that this formula defines an injective homomorphism, but it is not immediately clear that if $I + p^n X \in S$, then $I + p^{n+1} X \in \tilde{S}$. We now justify this.

If $I + p^n X \in S$ for some $X \in M_2(\mathbb{Z}/p\mathbb{Z})$, then there is a $\sigma_0 \in \operatorname{Gal}(\mathbb{Q}(E[p^{n+1}]/\mathbb{Q}))$ such that $\bar{\rho}_{E,p^{n+1}}(\sigma_0) = I + p^n X$. Further, there must be a $\sigma \in \operatorname{Gal}(\mathbb{Q}(E[p^{n+2}])/\mathbb{Q})$ such that $\sigma\big|_{\mathbb{Q}(E[p^{n+1}])} = \sigma_0$. In this case, since $\bar{\rho}_{E,p^{n+1}}(\sigma_0) \equiv I + p^n X$ we have

$$\bar{\rho}_{E,p^{n+2}}(\sigma) = I + p^n \tilde{X}$$

for some $\tilde{X} \in M_2(\mathbb{Z}/p^2\mathbb{Z})$ such that $\tilde{X} \equiv X \mod p$. Then

$$\begin{aligned}
\bar{\rho}_{E,p^{n+2}}(\sigma^p) &\equiv (I + p^n \tilde{X})^p \mod p^{n+2} \\
&\equiv I + p \cdot p^n \tilde{X} + \tfrac{1}{2} p(p-1) p^{2n} \tilde{X}^2 + \cdots \mod p^{n+2} \\
&\equiv I + p^{n+1} \tilde{X} \mod p^{n+2} \equiv I + p^{n+1} X \mod p^{n+2}.
\end{aligned}$$

Thus $I + p^{n+1} X \in \tilde{S}$.

Since $|S|$ divides $|\tilde{S}|$, this forces $\mathbb{Q}(E[p^{n+2}])/\mathbb{Q}(E[p^{n+1}])$ to have degree at least $|S|$ and so

$$p^k \mid [\mathbb{Q}(E[p^{n+2}]) : \mathbb{Q}(E[p^{n+1}])]. \qquad \square$$

Remark 3.14. If $p \geq 3$, then the statement $p^2 \mid [\mathbb{Q}(E[p^{n+1}]) : \mathbb{Q}(E[p^n])]$ is equivalent to $\mathbb{Q}(E[p^{n+1}]) \neq \mathbb{Q}(E[p^n], \zeta_{p^{n+1}})$. This is because the field extension $\mathbb{Q}(E[p^{n+1}])/\mathbb{Q}(E[p^n])$ is a Galois extension whose Galois group is isomorphic to a subgroup of the additive group $M_2(\mathbb{Z}/p\mathbb{Z})$ of 2×2 matrices with entries in $\mathbb{Z}/p\mathbb{Z}$. The group $M_2(\mathbb{Z}/p\mathbb{Z})$ has order p^4 and so a priori $[\mathbb{Q}(E[p^{n+1}]) : \mathbb{Q}(E[p^n])] = p^k$ for some $k \in \{0, 1, 2, 3, 4\}$. We omit the case when $k = 0$ in Proposition 3.13 since it is uninteresting.

Next, we notice that

$$[\mathbb{Q}(E[p^{n+1}]):\mathbb{Q}(E[p^n])] = [\mathbb{Q}(E[p^{n+1}]):\mathbb{Q}(E[p^n],\zeta_{p^{n+1}})][\mathbb{Q}(E[p^n],\zeta_{p^{n+1}}):\mathbb{Q}(E[p^n])].$$

But, by Theorem 1.12, in this case $\zeta_{p^{n+1}} \notin \mathbb{Q}(E[p^n])$ and so $[\mathbb{Q}(E[p^n],\zeta_{p^{n+1}}) : \mathbb{Q}(E[p^n])] = p$. Bringing it all together we see that

$$p^2 \mid [\mathbb{Q}(E[p^n]):\mathbb{Q}(E[p^{n-1}])] \iff p \mid [\mathbb{Q}(E[p^n]):\mathbb{Q}(E[p^{n-1}],\zeta_{p^n})] \iff [\mathbb{Q}(E[p^n]):\mathbb{Q}(E[p^{n-1}],\zeta_{p^n})] \neq 1.$$

Examining the proof of Proposition 3.13, the term $\frac{1}{2}p(p-1)p^{2n}\tilde{X}^2$ is $\equiv 0 \pmod{p^{n+2}}$ if $(p,n) \neq (2,1)$ but could fail if $p=2$ and $n=1$. With this in mind, we immediately get the following corollary.

Corollary 3.15. *Suppose that $p=2$ and $E/\mathbb{Q}$ is an elliptic curve such that p^k divides $[\mathbb{Q}(E[p^{n+1}]) : \mathbb{Q}(E[p^n])]$ for some $k \in \{1,2,3,4\}$ and $n \geq 2$. Then, p^k divides $[\mathbb{Q}(E[p^{n+2}]) : \mathbb{Q}(E[p^{n+1}])]$*

Thus, we find ourselves at the end. The work of 3.1 together with Proposition 3.13 and Corollary 3.15 completes the proof of Theorem 1.13.

4. Nilpotent division fields of prime level

We are now ready to start classifying when the division fields of elliptic curves can give us nilpotent extensions of $\mathbb{Q}$. Before starting the classification in earnest, we will quickly remind the reader of some basic facts about nilpotent groups.

4.1. ***Nilpotent groups.*** This subsection will only cover the very basics of subgroups series and nilpotent groups. For more context the reader can consult [5; 6; 12; 19].

Definition 4.1. Let G be a group. An ascending series

$$\{e\} = G_0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G$$

is called a *central series* if for all i, $G_i \lhd G$ and $G_{i+1}/G_i \subseteq Z(G/G_i)$. Here $Z(G)$ is the center of G. A descending series

$$G = G_0 \supseteq G_2 \supseteq G_2 \supseteq \cdots \supseteq \{e\}$$

is called a *central series* if $G_i \lhd G$ and $G_i/G_{i+1} \subseteq Z(G/G_{i+1})$.

Definition 4.2. A group G is called *nilpotent* if it has a central series.

Theorem 4.3. [19, Theorem 1.26] *Let G be a finite nontrivial group. The following are equivalent*:

(1) *G is a nilpotent group.*

(2) *Every Sylow subgroup of G is normal.*

(3) *G is the direct product of its Sylow subgroups.*

(4) *If d divides $|G|$, then G has a normal subgroup of order d.*

An immediate consequence of this result is that every abelian group and every finite p-group is nilpotent.

Proposition 4.4. [5, Theorem 5.7] *Nilpotency is closed under subgroups, quotients, and direct products.*

In general, given a group G and a nilpotent normal subgroup N, it is not true that G/N nilpotent implies that G is nilpotent. However if $N \leq Z(G)$, this follows from Theorem 5.13 of [5].

Proposition 4.5. *If G is a finite nontrivial group such that $Z(G) = \{e\}$, then G is not nilpotent.*

Proof. Suppose G is a finite nontrivial group that is nilpotent. Then, by Theorem 4.3 part 3, G is the direct products of its p-Sylow subgroups. A classical result in group theory is that p-group have nontrivial centers and so G must have a nontrivial center. □

Example 4.6. Let D_n be the dihedral group of order $2n$. More specifically, let

$$D_n = \langle r, s \mid r^n = s^2 = e, srs^{-1} = r^{-1} \rangle.$$

A classical result is that D_n is nilpotent exactly when $n = 2^k$ for some $k \geq 2$. In order to keep the statement of Proposition 2.1 as clean as possible, we will need to adopt the convention that $(\mathbb{Z}/2\mathbb{Z})^2$ is a dihedral group.

Example 4.7. Let p be a prime. The goal of this example is to show that $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ is not nilpotent. A simple computation shows that $Z(\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})) = \langle -I \rangle$ and by definition $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})/Z(\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}))$ is $\mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})$. A classical result [20] is that $\mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})$ is simple for all $p \geq 5$. Since the center of a group is always normal and $\mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})$ is clearly nonabelian, $Z(\mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z}))$ is trivial. Thus, $\mathrm{PSL}_2(\mathbb{Z}/p\mathbb{Z})$ is not nilpotent. This together with Proposition 4.4 shows that $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ is not nilpotent when $p \geq 5$. The cases when $p = 2$ and $p = 3$ can be easily checked by hand.

4.2. ***Classification of nilpotent division fields of prime level.*** Step 1 in the process of determining when an elliptic curve $E/\mathbb{Q}$ can have a nilpotent n-division field, is determining when the p-division fields can be nilpotent extensions of $\mathbb{Q}$. Proposition 4.4 tells us that if $\mathbb{Q}(E[n])/\mathbb{Q}$ is nilpotent, then $\mathbb{Q}(E[d])/\mathbb{Q}$ is nilpotent for all $d \mid n$. Moreover, if $n = p_1^{a_1} \cdots p_k^{a_k}$ is the prime factorization of n, then $\mathbb{Q}(E[n])/\mathbb{Q}$ is nilpotent if and only if $\mathbb{Q}(E[p_i^{a_i}])/\mathbb{Q}$ is nilpotent for all i. To see why this is true, one only needs to recall the Galois correspondence as well as the fact that nilpotency is preserved under subgroups, quotients, and direct products.

For this reason, we start by studying $\mathbb{Q}(E[p])/\mathbb{Q}$ and use that information to understand what happens at level p^2 and further up the p-adic tower.

To that end, we need a way to divide up the subgroups of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ so that we can study them. Fortunately, Proposition 2.1 gives us exactly what we need.

Remark 4.8. If G is a subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$, then G is nilpotent if and only if its image in $\mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$ is nilpotent. The reason is as follows. If G is nilpotent, then its image in $\mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$ is a quotient of G and is hence nilpotent. Conversely, if $G \leq \mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ and the image of G in $\mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$ is nilpotent, then the image of G in $\mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$ is the quotient G/N, where N is the set of scalar multiples of the identity in G. This subgroup $N \leq Z(G)$ and by Theorem 5.13 of [5], it follows that G is nilpotent, since N can be extended into a central series.

The following result gives a classification of when an admissible subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ is nilpotent.

Proposition 4.9. *Let p be a prime and G an admissible subgroup of* $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$. *Then G is nilpotent if and only if G is abelian, or the image of G in* $\mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$ *is isomorphic to D_{2^k} for some $k \geq 2$. Further, if p is odd, then G is either contained in the normalizer of a split or nonsplit Cartan subgroup of* $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$.

Proof. If G is abelian, then it must be nilpotent. Remark 4.8 shows that if the image of G in $\mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$ is isomorphic to D_{2^k} (a 2-group) then G is nilpotent.

Now we assume that G is nilpotent and consider the cases based on Proposition 2.1. First, if G contains $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$, then G cannot be nilpotent since $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ is not nilpotent.

Next, suppose that G is an admissible nilpotent group such that $p \mid |G|$, and G is contained in a Borel subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$. Conjugating G if necessary, we may assume G is contained in the set of upper triangular matrices. The set of upper triangular matrices contains a unique subgroup of order p, namely the cyclic group generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Since we assumed that G was nilpotent, we have from part (3) of Theorem 4.3 every element in G must commute with A. Let

$$B = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

be an arbitrary element of G. Next we compute

$$AB = \begin{pmatrix} a & b+d \\ 0 & d \end{pmatrix} \quad \text{and} \quad BA = \begin{pmatrix} a & a+b \\ 0 & d \end{pmatrix}.$$

From this we get that the only way that $AB = BA$ is if $a = d$. This implies that every element of G must have square determinant, and $\det(G) = (\mathbb{Z}/p\mathbb{Z})^\times$ now forces $p = 2$ and

$$G = \left\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \right\rangle \subseteq \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}),$$

which is abelian.

We now consider case (1) of Proposition 2.1. In this case, G is contained in a Cartan subgroup and is hence abelian.

In case (2) of Proposition 2.1, the projective image is dihedral and by Example 4.6, a dihedral group is nilpotent if and only if its order is a power of 2.

In case (3) of Proposition 2.1 the projective image is isomorphic to A_4, S_4 or A_5. If G were nilpotent this would imply that one of A_4, S_4 or A_5 is nilpotent. Proposition 4.5 shows this cannot happen, since all three have trivial center. □

4.3. ***Modular curves associated to split Cartan subgroups.*** We now survey, and will soon apply, what is known about the modular curves associated to split Cartan subgroups of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$. We let $C_s^+(p)$ be the normalizer of a split Cartan subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ and $X_s^+(p)$ the corresponding modular curve.

The work of Bilu, Parent, and Rebolledo in [3] gives an almost complete picture of rational points on $X_s^+(p)$. This work together with the work of Balakrishnan, Dogra, Müller, Tuitman and Vonk in [1] gives, among other things, the following theorem.

Theorem 4.10 [1; 3]. *If $p \geq 11$ is a prime, then the $\mathbb{Q}$-rational points on $X_s^+(p)$ are cusps or correspond to elliptic curves with complex multiplication.*

Using this result we prove the following.

Proposition 4.11. *Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication and let p be a prime such that $\operatorname{Im}\bar{\rho}_{E,p}$ is contained in $C_s^+(p)$. If $\mathbb{Q}(E[p])/\mathbb{Q}$ is a nilpotent extension, then $p \in \{2, 3, 5\}$.*

Proof. By Theorem 4.10, it suffices to rule out the case that $p = 7$. When $p = 7$, the projective image of $C_s^+(p)$ has size $2(7-1) = 12$ and so is not nilpotent. As shown in [34], there are three maximal admissible subgroups of $C_s^+(7)$, and for each of these, the corresponding modular curve has genus 1. For two of these, there are no non-CM points, while for the third of these, there is a non-CM point with $j = 3^3 \cdot 5 \cdot 7^5/2^7$. An elliptic curve with this j-invariant has mod 7 image isomorphic to either $\mathbb{Z}/6\mathbb{Z} \times S_3$ or $\mathbb{Z}/3\mathbb{Z} \times S_3$ and neither of these groups is nilpotent. □

4.4. ***Modular curves associated to nonsplit Cartan subgroups.*** In this section, we survey what is known about the modular curves associated to nonsplit Cartan subgroups of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$. To start, we will let p be an odd prime and we define the nonsplit Cartan subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ to be

$$C_{ns}(p) := \left\{ \left(\begin{smallmatrix} a & \epsilon b \\ b & a \end{smallmatrix}\right) : a, b \in \mathbb{Z}/p\mathbb{Z} \quad \text{and} \quad (a,b) \neq (0,0) \right\},$$

where ϵ is a generator of $(\mathbb{Z}/p\mathbb{Z})^\times$. The normalizer of this group in $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ is

$$C_{ns}^+(p) := \left\langle C_{ns}(p), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \right\rangle.$$

We will denote the modular curves corresponding to these groups by $X_{ns}(p)$ and $X_{ns}^+(p)$ respectively.

Much less is known about the rational points on the modular curves associated with the normalizers of the nonsplit Cartan subgroups compared to what is know about the split Cartan cases. These modular curves have some arithmetic properties that make analysis of their rational points particularly challenging. In particular, the Jacobians of these modular curves always have analytic rank at least as big as the genus of the curve. This rules out the traditional method of Chabauty and Coleman and requires more advanced techniques (which have been successful in two cases: see [1] and [2]). Fortunately for us, enough is known that we will be able to say quite a bit about the situation unconditionally, and the remainder of what we need is covered by Conjecture 1.8.

We state some of the relevant theorems for these modular curves.

Proposition 4.12. *Let $E/\mathbb{Q}$ be an elliptic curve that does not have complex multiplication and let $p \geq 7$ be a prime such that* $\operatorname{Im} \bar{\rho}_{E,p}$ *is contained in $C_{ns}^+(p)$. Then* $\operatorname{Im} \bar{\rho}_{E,p} = C_{ns}^+(p)$.

Proof. In [45] the stated result is proven for $p = 7$ (Theorem 1.5) and $p = 11$ (Theorem 1.6). For $p = 13$, [1] shows that there are no non-CM elliptic curves for which $\operatorname{Im} \bar{\rho}_{E,p}$ is contained in $C_{ns}^+(p)$. The cases that $p \geq 17$ are handled by combining Proposition 1.13 of [45] with Theorem 1.6 of [14]. □

From this we can see that if $E/\mathbb{Q}$ is an elliptic curve and $p \geq 7$ is a prime such that $\operatorname{Im} \bar{\rho}_{E,p}$ is conjugate to a subgroup of $C_{ns}^+(p)$, then $\operatorname{Im} \bar{\rho}_{E,p} = C_{ns}^+(p)$ and the image of $\bar{\rho}_{E,p}$ in $\mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$ is a dihedral group with order

$$\frac{|C_{ns}^+(p)|}{p-1} = \frac{2(p^2-1)}{p-1} = 2(p+1).$$

Combining this with Example 4.6 and Remark 4.8, we get that in this case $\operatorname{Im} \bar{\rho}_{E,p}$ is nilpotent exactly when $2(p+1)$ is a power of 2 which can happen only when $p+1$ is a power of 2 or p is a Mersenne prime.

We summarize the discussion up to this point in the following proposition.

Proposition 4.13. *Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication and let p be a prime such that* $\operatorname{Im} \bar{\rho}_{E,p}$ *is conjugate to a subgroup of $C_{ns}^+(p)$ and $\mathbb{Q}(E[p])/\mathbb{Q}$ is a nilpotent extension. Then p is a Mersenne prime.*

This will be as much as we can say unconditionally. What we really need here is something like Theorem 4.10, but for $X_{ns}^{+}(p)$. This is exactly why we need Conjecture 1.8.

Proposition 4.14. *Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication and let p be a Mersenne prime. Assuming Conjecture 1.8, if $\operatorname{Im}\bar{\rho}_{E,p}$ is conjugate to a subgroup of $C_{ns}^{+}(p)$, then $p = 3$ or $p = 7$.*

Before moving on to the case where $E/\mathbb{Q}$ has complex multiplication, we state a proposition summarizing this section.

Proposition 4.15. *Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication and let p be a prime such that $\mathbb{Q}(E[p])/\mathbb{Q}$ is a nilpotent extension. Then Conjecture 1.8 implies that $p \in \{2, 3, 5, 7\}$.*

In Table 2 we give models for all modular curves of prime level $p \in \{2, 3, 5, 7\}$ for which $\mathbb{Q}(E[p])/\mathbb{Q}$ is nilpotent. These modular curves are isomorphic to $\mathbb{P}^1$ and hence there are infinitely many rational j-invariants of elliptic curves $E/\mathbb{Q}$ for which $\mathbb{Q}(E[p])/\mathbb{Q}$ is nilpotent.

4.5. ***The case of complex multiplication.*** One of the interesting properties of elliptic curves with complex multiplication is that their mod p representations almost always have their images in the normalizer of a Cartan subgroup. Proposition 1.14(i) and (ii) of [45] state the following.

Proposition 4.16. *Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication by an order $\mathcal{O} \neq \mathbb{Z}[\zeta_3]$ of a quadratic imaginary field K. Next, let $p \geq 3$ be a prime such that $p \nmid \operatorname{disc}(\mathcal{O})$. Then, $\operatorname{Im}\bar{\rho}_{E,p}$ is conjugate to*

$$\begin{cases} C_s^{+}(p) & \text{if } p\mathcal{O}_K \text{ splits in } \mathcal{O}_K, \\ C_{ns}^{+}(p) & \text{if } p\mathcal{O}_K \text{ is inert in } \mathcal{O}_K. \end{cases}$$

The point of this proposition is that in these cases the mod p images is as large as possible. This is useful because we know that in these cases the image of $\operatorname{Im}\bar{\rho}_{E,p}$ of in $\mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$ is dihedral by Proposition 2.1 and has easily computable size. Remark 4.8 and Example 4.6 now imply that $\operatorname{Im}\bar{\rho}_{E,p}$ is nilpotent exactly when its image in $\mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$ is a 2-group.

Since $|C_s^{+}(p)| = 2(p-1)^2$ while $|C_{ns}^{+}(p)| = 2(p^2-1)$, we have that $\operatorname{Im}\bar{\rho}_{E,p}$ is nilpotent if it equals $C_s^{+}(p)$ and p is a Fermat prime, or if it equals $C_{ns}^{+}(p)$ and p is a Mersenne prime.

The next result summarizes the situation and handles the cases that $p \mid \operatorname{disc}(\mathcal{O})$.

Proposition 4.17. *Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication by $\mathcal{O} \neq \mathbb{Z}[\zeta_3]$ and p an odd prime. Then, $\mathbb{Q}(E[p])/\mathbb{Q}$ is nilpotent if and only if either p splits in $\mathcal{O}$ and p is a Fermat prime or p is inert in $\mathcal{O}$ and p is a Mersenne prime.*

Proof. The discussion preceding this theorem covers the case where $p \nmid \operatorname{disc}(\mathcal{O})$. To handle the case where $p \mid \operatorname{disc}(\mathcal{O})$, we refer to [45, Theorem 1.14] which shows that in this case $\operatorname{Im} \bar{\rho}_{E,p}$ is isomorphic to one of the following groups:

$$\begin{aligned} G &:= \left\{ \left(\begin{smallmatrix} a & b \\ 0 & \pm a \end{smallmatrix} \right) : a \in (\mathbb{Z}/p\mathbb{Z})^\times, b \in \mathbb{Z}/p\mathbb{Z} \right\}, \\ H_1 &:= \left\{ \left(\begin{smallmatrix} a & b \\ 0 & \pm a \end{smallmatrix} \right) : a \in ((\mathbb{Z}/p\mathbb{Z})^\times)^2, b \in \mathbb{Z}/p\mathbb{Z} \right\}, \text{ or} \\ H_2 &:= \left\{ \left(\begin{smallmatrix} \pm a & b \\ 0 & a \end{smallmatrix} \right) : a \in ((\mathbb{Z}/p\mathbb{Z})^\times)^2, b \in \mathbb{Z}/p\mathbb{Z} \right\}. \end{aligned}$$

In all three cases, the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$ is in $\operatorname{Im} \bar{\rho}_{E,p}$ and so $\operatorname{Im} \bar{\rho}_{E,p}$ is not nilpotent by Proposition 4.9. □

4.5.1. *The case when E has complex multiplication by $\mathbb{Z}[\zeta_3]$.* If $E/\mathbb{Q}$ has complex multiplication by $\mathcal{O} = \mathbb{Z}[\zeta_3]$ we know that $j(E) = 0$. Given such an elliptic curve, we know that there is always a $d \in \mathbb{Q}^\times$ such that E is isomorphic to the curve

$$E_d : y^2 = x^3 + d.$$

The images of the mod p representations of E depend on the value of d modulo 6th powers. This relationship is explicitly classified in [45, Propositions 1.15 and 1.16]. We summarize the relevant parts of those propositions here for the convenience of the reader.

Theorem 4.18 [45, Propositions 1.15 and 1.16]. *Let $E/\mathbb{Q}$ be an elliptic curve with complex multiplication by $\mathbb{Z}[\zeta_3]$. Then the curve E can be given by a Weierstrass equation of the form*

$$y^2 = x^3 + d$$

for some $d \in \mathbb{Q}^\times$.

(1) *If d is a cube, then* $\operatorname{Im} \bar{\rho}_{E,2} = \left\langle \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \right\rangle$. *Otherwise,* $\operatorname{Im} \bar{\rho}_{E,2} = \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$.

(2) *If $4d$ is not a cube, then* $\operatorname{Im} \bar{\rho}_{E,3}$ *is conjugate to*

$$\begin{array}{ll} \left\{ \left(\begin{smallmatrix} \pm 1 & a \\ 0 & b \end{smallmatrix} \right) : a \in \mathbb{Z}/3\mathbb{Z} \quad \text{and} \quad b \in (\mathbb{Z}/3\mathbb{Z})^\times \right\} & \text{if neither } d \text{ nor } -3d \text{ is a square}, \\ \left\{ \left(\begin{smallmatrix} 1 & a \\ 0 & b \end{smallmatrix} \right) : a \in (\mathbb{Z}/3\mathbb{Z})^\times \quad \text{and} \quad b \in \mathbb{Z}/3\mathbb{Z} \right\} & \text{if } d \text{ is a square}, \\ \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix} \right) : a \in (\mathbb{Z}/3\mathbb{Z})^\times \quad \text{and} \quad b \in \mathbb{Z}/3\mathbb{Z} \right\} & \text{if } 3d \text{ is a square.} \end{array}$$

On the other hand, if $4d$ is a cube, then $\operatorname{Im} \bar{\rho}_{E,3}$ *is conjugate to*

$$\begin{array}{ll} \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix} \right) : a, b \in (\mathbb{Z}/3\mathbb{Z})^\times \right\} & \text{if neither } d \text{ nor } -3d \text{ is a square}, \\ \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & b \end{smallmatrix} \right) : b \in (\mathbb{Z}/3\mathbb{Z})^\times \right\} & \text{if either } d \text{ or } -3d \text{ is a square.} \end{array}$$

(3) *If $p \equiv 1 \bmod 9$, then* $\operatorname{Im} \bar{\rho}_{E,p}$ *is conjugate to $C_s^+(p)$.*

(4) *If $p \equiv 8 \bmod 9$, then* $\operatorname{Im} \bar{\rho}_{E,p}$ *is conjugate to $C_{ns}^+(p)$.*

(5) *Suppose that* $p \equiv 4$ *or* $7 \bmod 9$ *and* $e \in \{1, 2\}$ *such that* $e \equiv \frac{1}{3}(p-1) \bmod 3$. *If* $d \not\equiv 16p^e \bmod (\mathbb{Q}^\times)^3$, *then* $\operatorname{Im} \bar{\rho}_{E,p}$ *is conjugate to* $C_s^+(p)$. *If* $d \equiv 16p^e \bmod (\mathbb{Q}^\times)^3$, *then* $\operatorname{Im} \bar{\rho}_{E,p}$ *is conjugate in* $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ *to the subgroup of* $C_s^+(p)$ *consisting of matrices of the form* $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ *and* $\left(\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right)$, *with* $a, b \in (\mathbb{Z}/p\mathbb{Z})^\times$ *such that* a/b *is a cube.*

(6) *Suppose that* $p \equiv 2$ *or* $5 \bmod 9$ *and let* $e \in \{1, 2\}$ *such that* $-e \equiv \frac{1}{3}(p+1) \bmod 3$. *If* $d \not\equiv 16p^e \bmod (\mathbb{Q}^\times)^3$, *then* $\operatorname{Im} \bar{\rho}_{E,p}$ *is conjugate to* $C_{ns}^+(p)$. *If* $d \equiv 16p^e \bmod (\mathbb{Q}^\times)^3$, *then* $\operatorname{Im} \bar{\rho}_{E,p}$ *is conjugate in* $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$ *to the subgroup generated by the unique index* 3 *subgroup of* $C_{ns}(p)$ *and* $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.

The take-away from this theorem is that for $E_d : y^2 = x^3 + d$, whether or not $\operatorname{Im} \bar{\rho}_{E,p}$ is nilpotent is completely controlled by $d \bmod (\mathbb{Q}^\times)^3$.

For example, condition (1) tells us that $\mathbb{Q}(E_d[2])/\mathbb{Q}$ is a nilpotent extension exactly when d is a cube. Similarly, condition (2) says that $\mathbb{Q}(E_d[3])/\mathbb{Q}$ is nilpotent when $4d$ is a cube.

We note that there are no Fermat primes $\equiv 1 \pmod 9$. By Remark 4.8, $\operatorname{Im} \bar{\rho}_{E,p}$ is nilpotent if and only if its image in $\mathrm{PGL}_2(\mathbb{F}_p)$, namely a dihedral group of order $2(p-1)$, is nilpotent. By Example 4.6 this cannot occur since $2(p-1)$ is not a power of 2.

Thus if $p \equiv 1 \pmod 9$ the image of $\bar{\rho}_{E,p}$ in $\mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$ is nilpotent if and only if $2(p-1)$ is a power of two and so condition (3) never yields a nilpotent $\mathbb{Q}(E[p])/\mathbb{Q}$. Likewise, there are no Mersenne primes $p \equiv 8 \pmod 9$.

The last cases that we have to deal with are the special cases that arise in cases (5) and (6) of Theorem 4.18. In cases (5) and (6) respectively, the image of $\bar{\rho}_{E,p}$ is contained in an index 3 subgroup of $C_s^+(p)$ and $C_{ns}^+(p)$ respectively. If we are in condition (5), then image of $\operatorname{Im} \bar{\rho}_{E,p}$ inside of $\mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$ is a dihedral group of size $\frac{2}{3}(p-1)$, while in condition (6) the image of $\operatorname{Im} \bar{\rho}_{E,p}$ inside of $\mathrm{PGL}_2(\mathbb{Z}/p\mathbb{Z})$ is a dihedral group of size $\frac{2}{3}(p+1)$. This along with our previous analysis gives the following proposition.

Proposition 4.19. *Let* $E_d : y^2 = x^3 + d$ *and let* p *be a prime. Then* $\mathbb{Q}(E_d[p])/\mathbb{Q}$ *is nilpotent if and only if*

$$
\begin{aligned}
&d \equiv 1 \pmod{(\mathbb{Q}^\times)^3} \text{ if } p = 2,\\
&d \equiv 2 \pmod{(\mathbb{Q}^\times)^3} \text{ if } p = 3,\\
&d \equiv 2 \cdot p^{\frac{1}{3}(p-1)} \pmod{(\mathbb{Q}^\times)^3} \text{ if } p = 3 \cdot 2^k + 1 \text{ for some } k \geq 1,\\
&d \equiv 2 \cdot p^{\frac{1}{3}(p+1)} \pmod{(\mathbb{Q}^\times)^3} \text{ if } p = 3 \cdot 2^k - 1 \text{ for some } k \geq 1.
\end{aligned}
$$

Example 4.20. Let E be the elliptic curve given by

$$y^2 = x^3 + 16 \cdot 97^2.$$

We check in the LMFDB that the image of $\bar{\rho}_{E,97}$ is conjugate to the group with RSZB label 97.14259.1103.1. One can check directly that this group is nilpotent and so $\mathbb{Q}(E[97])/\mathbb{Q}$ is nilpotent.

Remark 4.21. If E is an elliptic curve with complex multiplication by $\mathcal{O} = \mathbb{Z}[\zeta_3]$, then Proposition 4.19 shows that $\mathbb{Q}(E[p])/\mathbb{Q}$ is nilpotent for at most one prime p.

Proof of Corollary 1.10. If $E/\mathbb{Q}$ is an elliptic curve for which $\mathbb{Q}(E[19])/\mathbb{Q}$ is nilpotent then by Proposition 4.9, $\operatorname{Im} \bar{\rho}_{E,p}$ is contained in the normalizer of a split or nonsplit Cartan subgroup. If E is non-CM, the split Cartan case cannot occur by Proposition 4.11 and the nonsplit Cartan case cannot occur by Proposition 4.13. If E has CM by an order $\mathcal{O}$, then Proposition 4.17 forces $\mathcal{O} = \mathbb{Z}[\zeta_3]$. However, by Proposition 4.19 only cases where $p = 3 \cdot 2^k \pm 1$ can occur and 19 does not have this form.

Proposition 4.19 allows us to construct elliptic curves $E_d : y^2 = x^3 + d$ for which $\mathbb{Q}(E_d[p])/\mathbb{Q}$ is nilpotent for $p = 2$, $p = 3$, $p = 5 = 3 \cdot 2 - 1$, $p = 7 = 3 \cdot 2 + 1$, $p = 11 = 3 \cdot 2^2 - 1$ and $p = 13 = 3 \cdot 2^2 + 1$. The prime $p = 17$ is a Fermat prime which splits in $\mathbb{Z}[i]$ and so if $E : y^2 = x^3 - x$, then $\mathbb{Q}(E[17])/\mathbb{Q}$ is nilpotent by Proposition 4.17. □

5. Nilpotent groups of prime-power level

Suppose that p is a prime and that $E/\mathbb{Q}$ is an elliptic curve such that $\mathbb{Q}(E[p^k])/\mathbb{Q}$ is a nilpotent extension for some $k \geq 2$. The first observation we make is that since nilpotency is closed under quotients, Proposition 4.4, we know that $\mathbb{Q}(E[p^i])/\mathbb{Q}$ is a nilpotent extension for all $1 \leq i \leq k$, in particular, and this would mean $\mathbb{Q}(E[p])/\mathbb{Q}$ is a nilpotent extension. As usual, we will have to handle the case when $p = 2$ separately, but thanks to Proposition 4.9, when p is odd, we only have to deal with the case that $\operatorname{Im} \bar{\rho}_{E,p}$ is contained in either the normalizer of a split or nonsplit Cartan subgroup of $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$.

5.1. ***The case when $p = 2$.*** In this case there are exactly two ways that $\mathbb{Q}(E[2])/\mathbb{Q}$ can be nilpotent. In order for $\mathbb{Q}(E[2])/\mathbb{Q}$ to be nilpotent, either E can have a square discriminant or E can have a point of order 2 defined over $\mathbb{Q}$.

We start this case by considering what the image of $\bar{\rho}_{E,4}$ could be if we know that E has square discriminant and

$$\operatorname{Im} \bar{\rho}_{E,2} \subseteq \left\langle \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle.$$

Let $\pi_2 : \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$ be the standard component-wise reduction map and let

$$G_2 = \left\langle \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right) \right\rangle \quad \text{and} \quad G_4 := \pi_2^{-1}(G_2).$$

The next step is to search for admissible nilpotent subgroups of G_4 up to conjugation with the additional property that their image mod 2 is exactly equal to G_2. Such a subgroup must have order which is a multiple of 6 (a factor of 3 coming from the image in G_2 and a factor of 2 coming from the determinant being surjective). There are two such groups up to conjugacy: one with order 6 and one with order 12, with the former contained in the latter. Neither of these groups are admissible. Computing conjugacy classes of the order 12 subgroup shows that there are three conjugacy classes of elements of order 2 and none of these elements fix an element of $(\mathbb{Z}/4\mathbb{Z})^2$ of order 4, which the image of complex conjugation under $\bar{\rho}_{E,4}$ must. (Code for this calculation can be found in the file `subsec51.m` at [8].) Thus in the case when $\operatorname{Im}\bar{\rho}_{E,2}$ is conjugate to G_2, there is no way that $\mathbb{Q}(E[4])/\mathbb{Q}$ can be a nilpotent extension.

The next case is when $E/\mathbb{Q}$ has a point of order 2 defined over $\mathbb{Q}$. In this case, $\mathbb{Q}(E[2])/\mathbb{Q}$ is either a quadratic extension or trivial. Letting $\pi_2 : \mathrm{GL}_2(\mathbb{Z}/2^k\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$, we have $|\mathrm{Ker}(\pi_2)| = 2^{4(k-1)}$. From this we have

$$|\pi_2^{-1}(G_2)| = 2^{4(k-1)}|G_2|$$

and in particular $\pi_2^{-1}(G_2)$ is a 2-group. From Theorem 4.3, we know that $\pi_2^{-1}(G_2)$ is *always* nilpotent. The upshot of this is that if $E/\mathbb{Q}$ is an elliptic curve with a point of order two defined over $\mathbb{Q}$, then for every $k \geq 1$, $\mathbb{Q}(E[2^k])/\mathbb{Q}$ is a nilpotent extension.

Proposition 5.1. *Let $E/\mathbb{Q}$ be an elliptic curve such that $\mathbb{Q}(E[2])/\mathbb{Q}$ is a nilpotent extension. Then, either the discriminant of E is a square, in which case $\mathbb{Q}(E[2^k])/\mathbb{Q}$ is not nilpotent for any $k \geq 2$, or E has a rational point of order 2, in which case $\mathbb{Q}(E[2^k])/\mathbb{Q}$ is nilpotent for all $k \geq 1$.*

5.2. *The case when p is odd.*

Proposition 5.2. *Suppose that G is a nilpotent subgroup of $\mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$ and let $\pi : G \to \mathrm{GL}_2(\mathbb{F}_p)$ be the reduction mod p map. Assume that $p \nmid |\pi(G)|$. Then at least one of the following is true:*

(1) *G is abelian.*

(2) $\mathrm{Ker}(\pi) \subseteq \{\alpha I : \alpha \in (\mathbb{Z}/p^2\mathbb{Z})^\times$ *with* $\alpha \equiv 1 \pmod p\}$.

Proof. Let P be a Sylow p-subgroup of G. Since $|\pi(G)|$ has order coprime to p, we have $\pi(P) = \{1\}$. In particular, P is contained in the set of matrices $\equiv I \pmod p$. The set of matrices $\equiv I \pmod p$ is an abelian subgroup of $\mathrm{GL}_2(\mathbb{Z}/p^2\mathbb{Z})$ order p^4. In particular P is abelian and $\mathrm{Ker}\,\pi = P$. Let

$$H = \prod_{\substack{Q \in \mathrm{Syl}_q(G) \\ q \neq p}} Q$$

be a complement of P in G. Note that $\pi(G) = \pi(PH) = \pi(P)\pi(H) = \{1\} \cdot \pi(H) = \pi(H)$ and also $H \cap \operatorname{Ker} \pi \subseteq H \cap P = \{1\}$. Thus $\pi : H \to \pi(G)$ is an isomorphism.

Case I: There exists an element of P that is not a scalar multiple of the identity.

This implies that there is some $X \in M_2(\mathbb{F}_p)$ so that $I + pX \in P$ and X is not a scalar multiple of the identity. If $Y \in \pi(G)$, there is some $\tilde{Y} \in H$ so that $\pi(\tilde{Y}) = Y$. The assumption that G is nilpotent implies that $(I + pX)$ must commute with $\tilde{Y}$, and this implies that $XY = YX$ in $M_2(\mathbb{F}_p)$. The assumption on X implies that X is a *cyclic matrix*. This is a matrix X whose minimal polynomial and characteristic polynomial are the same.

Corollary 4.4.18 of [17] implies that for every cyclic matrix X, its centralizer in $M_2(\mathbb{F}_p)$ is equal to $\mathbb{F}_p[X]$, the set of all polynomials in X with coefficients in $\mathbb{F}_p$. This is a commutative subring of $M_2(\mathbb{F}_p)$, and this implies that $\pi(G) \subseteq \mathbb{F}_p[X]^\times$ is abelian. Since $H \simeq \pi(G)$, it follows that H is abelian. Since $G \simeq P \times H$, it follows that G is abelian.

Case II: Every element of P is a scalar multiple of the identity.

Since $P = \operatorname{Ker} \pi$, in this case, condition (2) is clearly true. □

As a consequence of this result, we can establish the following result.

Proposition 5.3. *Let $E/\mathbb{Q}$ be an elliptic curve and let p be an odd prime. Then $\mathbb{Q}(E[p^2])/\mathbb{Q}$ is not a nilpotent extension.*

Proof. Assume that $E/\mathbb{Q}$ is an elliptic curve, p is an odd prime, and $\mathbb{Q}(E[p^2])/\mathbb{Q}$ is nilpotent. Let $G = \operatorname{Im} \bar{\rho}_{E,p^2}$. If we are in case (1) of Proposition 5.2, then $\mathbb{Q}(E[p^2])/\mathbb{Q}$ is abelian, which contradicts the main result of [16]. If we are in case (2) of Proposition 5.2 and $\pi : G \to \mathrm{GL}_2(\mathbb{F}_p)$ is the reduction mod p map, then $\operatorname{Ker}(\pi) \cap (G \cap \mathrm{SL}_2(\mathbb{Z}/p^2\mathbb{Z})) = 1$ and this implies that we have a near coincidence of level (p^2, p). By Proposition 3.11, we must have that $p = 3$ and E corresponds to a rational point on the curve with RSZB label 9.27.0.1. However, the main result of [34] implies that for such an elliptic curve, the mod 9 image of Galois must equal 9.27.0.1, which is not nilpotent. □

As a consequence, if $E/\mathbb{Q}$ is an elliptic curve, p is an odd prime, and $n \geq 2$, $\mathbb{Q}(E[p^n])/\mathbb{Q}$ is not a nilpotent extension.

6. Nilpotent groups of composite level

In this section, we will complete the proof of Theorem 1.9. Since $\mathbb{Q}(E[n])$ is the composite of $\mathbb{Q}(E[p^k])$ for every prime power factor p^k of n, the extension $\mathbb{Q}(E[n])/\mathbb{Q}$ is nilpotent if and only if every $\mathbb{Q}(E[p^k])/\mathbb{Q}$ is nilpotent. From Section 5, this only occurs if $k = 1$, or $p = 2$ and E has a rational point of order 2.

First, we consider the case that $E/\mathbb{Q}$ is an elliptic curve with complex multiplication. From Section 4, we have a classification of when $\mathbb{Q}(E[p])/\mathbb{Q}$ is nilpotent based on the mod p image of Galois for E, which is determined by whether p splits, is inert, or ramifies in the CM field. From Section 5, $\mathbb{Q}(E[p^k])/\mathbb{Q}$ is nilpotent for $k \geq 2$ if and only if $p = 2$ and E has a rational point of order 2. From these results, the complex multiplication cases of Theorem 1.9 follow.

Next we consider the case that $E/\mathbb{Q}$ is an elliptic curve without complex multiplication under the assumption of Conjecture 1.8. In this case, Section 4 implies that if $\mathbb{Q}(E[p])/\mathbb{Q}$ is nilpotent then either $p \in \{2, 5\}$ or p is a Mersenne prime and $\operatorname{Im} \bar{\rho}_{E,p}$ is contained in the normalizer of the nonsplit Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$, which entails that $p \in \{2, 3, 5, 7\}$. All that remains is for us to determine which combinations of the possible nilpotent mod p images can occur simultaneously. Given possible mod p and mod q images of Galois G and H, we construct the fiber product $X_G \times_{X_0(1)} X_H$. This is the curve given by $\pi_G(x) = \pi_H(y)$. The results of this computation are listed in Table 1.

The curves that are genus 1 or 2 with rank zero can be handled using standard techniques. The rank 0 genus 3 curve was shown to have no noncuspidal rational points corresponding to elliptic curves without complex multiplication in [31]. This leaves us with one remaining curve, with label 35.315.19.1. Theorem A.7 of Appendix A to [34] shows that if X_H is a modular curve of level N, every simple factor of the Jacobian of X_H is isogenous to a simple factor of $J_1(N^2)$, and Section 6 of [34] explains how this information can be used to determine the decomposition of the Jacobian of X_H. This decomposition is recorded in the beta version of the LMFDB at the link here. In particular, X_H factors up to isogeny as the product of nine $\mathbb{Q}$-simple abelian varieties (of dimensions 2, 3 and 4).

(p, q)	$\operatorname{Im} \bar{\rho}_{E,p}$	$\operatorname{Im} \bar{\rho}_{E,q}$	$\operatorname{Im} \bar{\rho}_{E,pq}$	has noncuspidal rational points?
(2,3)	2.2.0.1	3.3.0.1	6.6.1.1	no – genus 1, rank 0
	2.3.0.1	3.3.0.1	6.9.0.1	yes
(2,5)	2.2.0.1	5.15.0.1	10.30.2.2	no – genus 2, rank 0
	2.3.0.1	5.15.0.1	10.45.1.1	no - genus 1, rank 0
(2,7)	2.2.0.1	7.21.0.1	14.42.3.1	no – genus 3, rank 0
	2.3.0.1	7.21.0.1	14.63.2.1	no - genus 2, rank 0
(3,5)	3.3.0.1	5.15.0.1	15.45.1.1	yes
(3,7)	3.3.0.1	7.21.0.1	21.63.1.1	yes
(5,7)	5.15.0.1	7.21.0.1	35.315.19.1	no – genus 19, analytic rank 15

Table 1. Potential composite level nilpotent images.

Work of Kolyvagin and Logachev [21] shows that if a simple abelian variety A of GL_2-type has analytic rank 0 it must have rank 0, while if it has analytic rank r equal to its dimension, then it must also have algebraic rank r. (This latter result is not stated by Kolyvagin and Logachev, but their ideas suffice to prove it. For more detail about how this follows, see [10, Section 7].) This hypothesis is easy to verify since $L(A, s)$ factors as a product of $\dim A$ modular L-functions. From this, it follows that the analytic and algebraic rank of the Jacobian of 35.315.19.1 are both 15. In theory, this curve could be attacked using the method of Chabauty and Coleman since the genus higher than the rank, but computing on a curve of genus 19 is rather unwieldy. However, the techniques of Lemos [25] apply to this situation:

Theorem 6.1 (Lemos ([25, Theorem 1.4])). *Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication. Suppose that there exists a prime q for which $\operatorname{Im} \bar{\rho}_{E,q}$ is contained in a subgroup of $C_s^+(q)$. Then $\bar{\rho}_{E,p}$ is surjective for all $p > 37$.*

The work in [25] does not, however, require full strength of the assumption that $p > 37$. We wish to explain why that work implies the following result.

Theorem 6.2. *Let $E/\mathbb{Q}$ be an elliptic curve without complex multiplication. Suppose that there exist a prime q for which $\operatorname{Im} \bar{\rho}_{E,q}$ is contained in a subgroup of $C_s^+(q)$. Then $\bar{\rho}_{E,p}$ is not contained in $C_{ns}^+(p)$ for any $p > 3$ with $p \neq q$.*

Setting $p = 7$ and $q = 5$ above implies that the only rational points on the modular curve 35.315.19.1 are cusps or CM points.

Proof of Theorem 6.2. As this result is really contained in [25] we will give an overview of the steps in Lemos's argument highlighting the necessary hypotheses on p and q. First, Theorem 4.10 implies that $q \in \{2, 3, 5, 7\}$. As a consequence, $X_0(q)$ has genus zero.

The modular curve $X_s(q)$ parametrizes elliptic curves with two independent cyclic q-isogenies. In Lemma 3.4 of [25], Lemos shows that there is an isomorphism

$$\theta : X_s(q) \times_{X_0(1)} X_{ns}^+(p) \to X_0(q^2) \times_{X_0(1)} X_{ns}^+(p)$$

and that θ commutes with natural involutions on the source and the target. (The involution on the source interchanges the kernels of the two isogenies, and the involution on the target is the Atkin–Lehner involution w_{q^2}.) Lemos then defines a map $g : X_0(q^2) \times_{X_0(1)} X_{ns}^+(p) \to J(X_0(q) \times_{X_0(1)} X_{ns}^+(p))$ by taking a point P and mapping it to the difference of its images under the two different degeneracy maps (coming from the two covers $X_0(q^2) \to X_0(q)$).

Work of Darmon and Merel [9, Proposition 7.1] shows that there is a projection $\pi : J(X_0(q) \times_{X_0(1)} X_{ns}^+(p)) \to A$ to a positive-dimensional abelian variety with rank 0 for which the kernel of π is connected and is stable under the action of

Hecke operators. (For their application, Darmon and Merel only need this for $q = 2$ or $q = 3$, but as they indicate, this part of the argument applies to any q with $p \nmid q$.)

Define $h = \pi \circ g$ and let $\mathcal{O}$ be the ring of integers in $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$ and $R = \mathcal{O}[1/(2qp)]$. Roughly speaking, Proposition 3.6 of [25] shows that h is a formal immersion at ∞ for every prime ideal $\mathfrak{p}$ of R. The proof of this proposition is standard and the key assumption is that $X_0(q)$ has genus 0. It follows from this that $\theta \circ h : X_s(q) \times_{X_0(1)} X_{ns}^+(p) \to A$ is a formal immersion at ∞ for every prime ideal $\mathfrak{p}$ of R.

Now, if P is a rational point on $X_s(q) \times_{X_0(1)} X_{ns}^+(p)$ corresponding to an elliptic curve E which has potential multiplicative reduction at some prime $\ell \notin \{2, p, q\}$, then it meets one of the cusps at the fiber at ℓ. Without loss of generality this cusp can be chosen to be infinity and this implies that $\theta \circ h(P) = Q \in A(\mathbb{Q})$ reduces to zero in $\tilde{A}(\mathbb{F}_\ell)$. (Here $\tilde{A}$ is the special fiber of the Néron model of A over $\mathbb{Z}_\ell$.) This implies that $Q = 0$ and the fact that $\theta \circ h$ is a formal immersion and $\ell > 2$ implies that $P = \infty$. (One way to make this conclusion is using Proposition 2.4 of [22].)

It follows from this that if $E/\mathbb{Q}$ is an elliptic curve with mod p image contained in $C_{ns}^+(p)$ and mod q image contained in $C_s^+(q)$, then E cannot have potentially multiplicative reduction at any prime $\ell \notin \{2, p, q\}$. The primes of potentially multiplicative reduction are precisely those that divide the denominator of $j(E)$. As a consequence $j(E) \in \mathbb{Z}[1/(2pq)]$. More is true however. Lemos notes in [25, Proposition 3.3] that the assumption that the mod p image of Galois is contained in $C_{ns}^+(p)$ implies that E has potentially supersingular reduction at p, and that if E has potentially multiplicative reduction at $\ell \neq p$, then $\ell \equiv \pm 1 \pmod{p}$. (These same observations were made earlier by Zywina.) It follows that the only primes that can divide the denominator of $j(E)$ are those that are $\equiv \pm 1 \pmod{p}$. Since $p > 3$, none of 2, 3, 5 or 7 (the only options for q) can be $\equiv \pm 1 \pmod{p}$. It follows that $j(E) \in \mathbb{Z}$.

Lemos proceeds to show using an explicit isomorphism $X_s^+(q) \simeq \mathbb{P}^1$ that there are only finite integral j-invariants of elliptic curves $E/\mathbb{Q}$ with mod q image of Galois contained in $C_s^+(q)$. For $q \in \{3, 5, 7\}$ these are explicitly listed on [25, p. 749]. They all have CM except for $j = -5000$ ($q = 5$) and $j = -1728$ ($q = 3$). For elliptic curves with these j-invariants, the image of $\rho_{E,p}$ is not contained in the normalizer of a nonsplit Cartan subgroup for any $p > 3$. The case of $q = 2$ was previously handled in [24, p. 142]. Here there are 25 integral j-invariants of elliptic curves E with image in $C_s^+(2)$ (which is a Borel subgroup of $\mathrm{GL}_2(\mathbb{F}_2)$). Of these 25, there are 18 are non-CM j-invariants. For elliptic curves with these j-invariants, the LMFDB indicates that the image of $\rho_{E,p}$ is contained in $C_{ns}^+(p)$ for some p only for $p = 3$ and $j \in \{-64, 4913, 238328, 16974593\}$. □

In Table 2, we give models for the modular curves from Table 1 that have noncuspidal rational points.

Working without the assumption of Conjecture 1.8, we must consider the

G	X_G	$\pi_G : X_G \to \mathbb{P}^1_{\mathbb{Q}}$
2.2.0.1	$\mathbb{P}^1$	$f_2(t) = t^2 + 1728$
2.3.0.1	$\mathbb{P}^1$	$h_2(t) = \frac{(256-t)^3}{(256-t)^3}$
3.3.0.1	$\mathbb{P}^1$	$f_3(t) = t^3$
5.15.0.1	$\mathbb{P}^1$	$f_5(t) = \frac{(t+5)^3(t^2-5)^3(t^2+5t+10)^3}{(t+5)^3(t^2-5)^3(t^2+5t+10)^3}$
6.9.0.1	$\mathbb{P}^1$	$f_6(t) = \frac{(t^3+3t^2+3t-15)^3}{(t^3+3t^2+3t-15)^3}$
7.21.0.1	$\mathbb{P}^1$	$f_7(t) = \frac{(2t-1)^3(t^2-t+2)^3(2t^2+5t+4)^3(5t^2+2t-4)^3}{(2t-1)^3(t^2-t+2)^3(2t^2+5t+4)^3(5t^2+2t-4)^3}$
15.45.1.1	$y^2+y=x^3+1$	$f_{15}(x,y) = \frac{(y+3)^3(y^2-4y-1)^3(y^2+y+4)^3}{(y+3)^3(y^2-4y-1)^3(y^2+y+4)^3}$
21.63.1.1	$y^2+y=x^3+12$	$f_{21}(x,y) = f_7\left(\frac{x^2+5x-14}{x^2+5x-14}\right)$

Table 2. Models of the modular curves relevant for Theorem 1.9. The models for curves of prime level come from [41]. The remaining genus 0 curves can be computed as fiber products of curves of prime level. The models for the genus 1 modular curves of composite level are computed as fiber products of the prime level modular curves also computed in [41].

possibility that there is an elliptic curve $E/\mathbb{Q}$ for which $\operatorname{Im} \bar{\rho}_{E,5}$ is contained in the normalizer of the split Cartan mod 5 and for which $\operatorname{Im} \bar{\rho}_{E,p}$ is contained in the normalizer of a nonsplit Cartan modulo p for some Mersenne prime p. Theorem 6.2 above shows this is only possible for $p = 3$, and the elliptic curves for which this occurs are parametrized by the modular curve with label 15.45.1.1.

In addition, we must consider the possibility that there is an elliptic curve $E/\mathbb{Q}$ for which $\operatorname{im} \bar{\rho}_{E,p}$ is contained in the normalizer of a nonsplit Cartan modulo p for some Mersenne prime p, and which also has a rational point of order 2. This implies the mod 2 image of Galois is contained in a Borel subgroup, which for $p = 2$ is equal to $C_s^+(2)$. The desired result again follows from Theorem 6.2. This concludes the proof of Theorem 1.9.

Acknowledgements

We thank David Zureick-Brown and Álvaro Lozano-Robledo for insightful conversations during the writing of this paper, and we thank Pedro Lemos for helpful communications about the results in [25]. We also thank the referees whose careful reading has helped improve the clarity and accuracy of the paper.

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Received December 22, 2024. Revised December 9, 2025.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 342, No. 1, 2026
https://doi.org/10.2140/pjm.2026.342.115

THE CAUCHY PROBLEM FOR 1D NONLINEAR SCHRÖDINGER EQUATIONS WITH REPULSIVE DELTA POTENTIAL FOR DATA IN L^p-BASED SPACES

QINGQUAN DENG, PING LI AND XIUHONG LONG

Let $\lambda \in \mathbb{R}$ and let $H = -\frac{1}{2}\partial_x^2 + q\delta_0$ be the one-dimensional Schrödinger equation with a repulsive delta potential. We study the Cauchy problem for the nonlinear equation

$$\begin{cases} i\,\partial_t u(t,x) = Hu(t,x) + \lambda |u(t,x)|^2 u(t,x), & (t,x) \in \mathbb{R}\times\mathbb{R}, \\ u(0,x) = u_0(x), \end{cases}$$

in L^p-based spaces. Using the boundedness of wave operators, a characterization of Besov space adapted to H, and the cancellation property of the trilinear form $\mathcal{T}(v_1(\tau), v_2(\tau), v_3(\tau)) = \mathcal{U}(-\tau)\big(\mathcal{U}(-\tau)v_1(\tau)\mathcal{U}(\tau)v_2(\tau)\mathcal{U}(\tau)v_3(\tau)\big)$ with $\mathcal{U}(\tau) = e^{-itH}$, we demonstrate that under the linear transformation $v(t) = \mathcal{U}(-t)u(t)$, the problem is locally well-posed in $L^p(\mathbb{R})$ for $1 < p < 2$ and in the homogeneous Besov space $\dot{B}^s_{p,1}(\mathbb{R})$ with $1 < p < 2$ and $s = 1 - \frac{1}{p}$.

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MSC2020: 35Q55, 35A01, 42A38, 42B35.

Keywords: Cauchy problem, nonlinear Schrödinger equation, wave operator, locally well-posedness, delta potential.

1. Introduction

This paper is devoted to the Cauchy problem for the cubic nonlinear Schrödinger equation (NLSE) with a repulsive delta potential in dimension one. The equation is introduced as

(1-1)
$$\begin{cases} i\partial_t u(t,x) = Hu(t,x) + \lambda |u(t,x)|^2 u(t,x), & (t,x) \in \mathbb{R}\times\mathbb{R}, \quad \lambda \in \mathbb{R}, \\ u(0,x) = u_0(x). \end{cases}$$

Here H is the delta perturbation of the self-adjoint operator $H_0 = -\frac{1}{2}\partial_x^2$, that is,

$$H = -\tfrac{1}{2}\partial_x^2 + q\delta_0(x),$$

where δ_0 is the Dirac delta measure supported at the origin and $q \in \mathbb{R}$ is the strength of the perturbation. A more detailed introduction to the operator H will be presented in Section 2. For convenience, we will refer to the case $q > 0$ as repulsive, and $q < 0$ as attractive throughout the paper. We will speak of the focusing nonlinear Schrödinger equation (1-1) when $\lambda < 0$ and the defocusing case when $\lambda > 0$.

Equation (1-1) is of great interest in both mathematics and physics. It is frequently used to model the interaction between a quantum mechanical wave or particle and an impurity or a localized defect. See Adami, Golse and Teta [2] for the interaction of a one dimensional Bose condensate with an impurity, as well as Bergé [7] for the propagation of an optical wave pulse in an optical fiber in the presence of defects or junctions. A different occurrence of (1-1) is found in the study of soliton-soliton collisions within the framework of the coupled NLSEs, when considering the interaction between a narrow soliton and a wider one, which are governed by different equations, a suitable limiting process reduces the system of two coupled NLSEs to a single equation, in which the narrow soliton in the mate mode is effectively represented by a delta function; see Cao and Malomed [13].

In mathematical physics, Hilbert spaces provide a rigorous and versatile framework for describing the behavior of physical systems. For (1-1), Hilbert spaces such as L^2-spaces, L^2-based Sobolev spaces $W^{s,2}$ and weighted L^2-spaces are used to study local and global well-posedness, as well as the long-time behavior of the solutions. A briefly review of the relevant details will be provided later.

Much less is known about the properties of the solution to (1-1), when the initial data u_0 does not belong to L^2-based spaces. Mathematically, the L^p-based spaces ($p \neq 2$) would be popular and interesting to work with. In other words, from the viewpoint of mathematics, it is of great interest to consider the local and global well-posedness for evolution equations like (1-1) in L^p-based spaces, with the initial data u_0 also residing in these spaces. But this is not the whole story since the study of NLSE in L^p-based spaces is not merely a mathematical generalization. If we go back to the works of NLSEs with the initial data that is periodic, or localized perturbation of a periodic function, or quasi-periodic, we find that the initial data is usually not in L^2, but rather in L^∞ in many cases, see [9; 10; 12; 17; 26; 54; 64] and so on. For example, Dodson, Soffer and Spencer [26] and Oh [64] considered the local well-posedness of NLSE in dimension one with quasi-periodic initial data, they allow the initial data to be of the form

$$u_0(x) = \cos(x) + \cos(\sqrt{2}x),$$

which is obviously in $L^\infty \setminus L^2$. We note that the above models have their origins in mathematical physics. For instance, the periodic case appears naturally in the context of periodic signals propagating through fibers, while localized perturbations may be related to noise. Moreover, as mentioned by Vargas and Vega [77] that the LIA model for the vortex filament can be reduced to equation (1-1), where the L^2 theory no longer works, it leads to seek solution of (1-1) with infinite L^2 data. Further studies on NLSE in L^p-based spaces ($p \neq 2$) will be introduced in next section.

1.1. ***Background.*** This paper is concerned with the well-posedness of equation (1-1) with the initial data residing in either $L^p(\mathbb{R})$ or $\dot{B}^s_{p,r}(\mathbb{R})$. In this section, we will briefly review the known results for (1-1), and our attention is primarily restricted to the one-dimensional case.

***The case q* = 0.** Let $H_0 = -\frac{1}{2}\partial_x^2$. We are concerned with the following cubic NLSE in dimension one:

$$i\partial_t u(t,x) = H_0 u(t,x) + \lambda |u(t,x)|^2 u(t,x), \quad u(0,x) = u_0(x). \tag{1-2}$$

It is well known that the Cauchy problem (1-2) is globally well-posed in L^2 and $W^{1,2}$, see e.g. Kato [55] and Tsutsumi [75]. An elementary problem that needs to be considered is the long-time behavior of global solutions. Notice that the cubic nonlinearity in dimension one is the borderline for short range and long range problems. The main reason lies in the fact that the free wave $e^{-itH_0}u_0$ decays at the rate $t^{-\frac{1}{2}}$, which causes the cubic nonlinearity $|u|^2u$ to behave like $t^{-1}u$, a term that is not integrable in t. In fact, it was proved by Barab [6] and Tsutsumi and Yajima [76] that the scattering in L^2 only occurs for the trivial zero solution. Consequently, a phase correction is required, which is known as modified scattering. When $\lambda > 0$, Deift and Zhou [23] obtained the long-time asymptotic behavior for all solutions based on the fact that equation (1-2) is completely integrable. The sign of λ does not make any difference if one considers the Cauchy problem (1-2) with initial data being small in L^2-based weighted Sobolev spaces. One can see Ozawa [67] for the existence of modified wave operators, as well as Hayashi and Naumkin [37], Lindblad and Soffer [59], Kato and Pusateri [57] and Ifrim and Tataru [52] the results for modified scattering.

The theory of equation (1-2) in L^p-based spaces has been investigated in the literature. In the following, we will omit specifying the spatial dimension in this circumstance, as some of the known results hold in the higher-dimensional setting. By using the Duhamel formula, the solution to (1-2) can be expressed as

$$u(t) = \mathcal{U}_0(t)u_0 - i\lambda \int_0^t \mathcal{U}_0(t-s)|u(s)|^2 u(s)\, ds, \tag{1-3}$$

where $\mathcal{U}_0(t)$ denotes the linear propagator e^{-itH_0}, and we will use the simplified notation $u(t)$ to denote $u(t,x)$ from time to time, whenever the spatial variable x is clear from the context. It is well-known from Hörmander [44] that if $u_0 \in L^p$, $\mathcal{U}_0(t)u_0$ is not necessarily in L^p unless $p = 2$, which is a basic obstacle to study equation the in L^p-based spaces. The linear propagator $\mathcal{U}_0(t)$ has the factorization

$$\mathcal{U}_0(t) = M(t)D(t)\mathcal{F}M(t), \tag{1-4}$$

where $M(t)f = e^{i\frac{|x|^2}{2t}} f(x)$ is multiplication, $D(t)f = (2\pi i t)^{-\frac{1}{2}} f\left(\frac{x}{2\pi i t}\right)$ is dilation and $\mathcal{F}$ is the Fourier transform. This implies that the boundedness properties of $\mathcal{U}_0(t)$ are analogous to those of the Fourier transform $\mathcal{F}$, which is bounded from L^p to $L^{p'}$ with $p \in [1,2]$ by Young's inequality, where p' is the conjugate of p. This indicates that one cannot expect (1-3) to be well-posed in L^p-based spaces with initial data belonging to the same spaces.

Zhou [82] found a way around this by considering the well-posedness of the integral equation

$$v(t) = u_0 - i\lambda \int_0^t \mathcal{U}_0(-s)|\mathcal{U}_0(s)v(s)|^2 \mathcal{U}_0(s)v(s)\, ds \tag{1-5}$$

in L^p and Besov space $\dot{B}^s_{p,r}$ with $1 < p < 2$, where $v(t) = \mathcal{U}_0(-t)u(t)$. A key observation lies in the cancellation of the multilinear form

$$\mathcal{T}(v_1, v_2, v_3; s) = \mathcal{U}_0(-s)\big(\mathcal{U}_0(s)v_1(s)\mathcal{U}_0(-s)\bar{v}_2(s)\mathcal{U}_0(s)v_3(s)\big),$$

which is arisen in the Duhamel term of (1-5). The cancellation for $\mathcal{T}$ leads to the following $L^1(\mathbb{R}^d)$ estimate

$$\|\mathcal{T}(v_1, v_2, v_3; s)\|_{L^1(\mathbb{R}^d)} \lesssim s^{-d} \|v_1\|_{L^1(\mathbb{R}^d)} \|v_2\|_{L^1(\mathbb{R}^d)} \|v_3\|_{L^1(\mathbb{R}^d)} \tag{1-6}$$

which combined with a frequency localized L^2 estimate implies the multilinear estimates for $\mathcal{T}$ in both L^p and Besov spaces, based on these estimates, the author established the local well-posedness for $v(t)$, the solution of (1-5) in L^p and Besov spaces. The work [82] gave an efficient way to solve the Cauchy problem in L^p-based spaces by considering the integral equation for $\mathcal{U}_0(-t)u(t)$, even if only local theory is understood. Such an idea has been extended to study other NLSEs in non L^2-based spaces. Hoshino and Hyakuna [45] studied the local well-posedness of

NLSE in Sobolev spaces $W^{s,p}$ and Besov spaces $\dot{B}^s_{p,r}$ with Hartree type nonlinearity $(|x|^{-\gamma} * |u|^2)u$, the authors considered the integral equation similarly to (1-5) for $\mathcal{U}_0(-t)u(t)$ and the key point in their work is the new multilinear estimates for

$$\mathcal{U}_0(-s)\big[\big(|\cdot|^{-\gamma} * \mathcal{U}_0(s)v_1(s)\mathcal{U}_0(-s)\bar{v}_2(s)\big)\mathcal{U}_0(s)v_3(s)\big]$$

in Sobolev spaces, which compared to [82], this result is obtained by using the factorization for $\mathcal{U}_0(s)$ (see (1-4)) instead. Hyakuna [47] obtained the local and global well-posedness for Hartree type NLSE in $L^2 \cap L^p$ by using a similar approach, along with the blowup alternative argument. The same author [49] studied the well-posedness of the cubic NLSE in L^p for $p > 2$, establishing the multilinear estimates for $\mathcal{T}$ defined by (1-6) in terms of the factorization for $\mathcal{U}_0(s)$. Finally, we mention that in Hyakuna [46] and [48], the solvability of NLSEs for $\mathcal{U}_0(-t)u(t)$ was considered, with general nonlinearity $N(u)$ and Hartree type instead, respectively. Nevertheless, the author did not invoke the integral equation (1-5) nor exploit the cancellation property inherent to the mult-linear operator.

Notice that the work of [82] concerns the integral equation (1-5) for $v(t) = \mathcal{U}_0(-t)u(t)$ in L^p-based spaces. One can also study the solution $u(t)$ itself straightforwardly by using Strichartz estimates for initial data in non L^2-based spaces. The existing results in this direction are primarily based on the homogeneous Strichartz-type estimates

$$\|\mathcal{U}_0(t)f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^p} \quad \text{and} \quad \|\mathcal{U}_0(t)f\|_{L^q_t L^r_x} \lesssim \|f\|_{\widehat{L}^p},$$

where $\hat{f}$ is the Fourier transform of f and $\widehat{L}^p$ is defined by

$$\widehat{L}^p := \{f : \hat{f} \in L^{p'}\},$$

it follows from the Hausdorff–Young inequality that $L^p \subset \widehat{L}^p$ if $p \le 2$ and $\widehat{L}^p \subset L^p$ if $p \ge 2$. The triplet (q, r, p) in above estimates, called a p-admissible pair in dimension d, from the scaling point of view satisfies

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{p}$$

and some further restrictions. The generalized and variant Strichartz estimates for the Duhamel term

$$\int_0^t \mathcal{U}_0(t-s)N(u)(s)\,ds$$

are also needed; see [36; 45; 48; 56; 78]. The Strichartz estimates have been extensively applied in the study of well-posedness and long-time behavior of NLSE in L^2-based spaces, we refer to the books by Cazenave [15] and Tao [74] to see the basic philosophy. For the local and global well-posedness of (1-2) in non-L^2-based spaces, Vargas and Vega [77] established local well-posedness in the space $L^3_{t,\mathrm{loc}}L^6_x$

in one spatial dimension, for initial data belonging to certain function spaces such that $\|\mathcal{U}_0(t)u_0\|_{L_t^3 L_x^6(I\times\mathbb{R})} < \infty$. They also obtained global solutions by employing a splitting argument, originally developed by Bourgain [11], which was used to establish global existence for large data in H^s-critical NLSEs with s near one. Such an idea has been generalized to the study of NLSEs with different nonlinearities $N(u)$ and function spaces X for the initial data. Here $N(u)$ could be one of

$$|u|^\alpha u, \quad (|\cdot|^{-\gamma} * |u|^2)u, \quad \partial_x(|u|^2 u),$$

and the function space X can be taken as any of

$$L^{p,\infty},\ \widehat{L}^p,\ \widehat{W}^{s,p},\ M^s_{p,\sigma},\ \dot{W}^{1,2}\cap L^p \text{ and the Wiener algebra } W.$$

($\widehat{W}^{s,p} = \{f : (1+|\xi|^2)^{\frac{s}{2}}\hat{f} \in L^{p'}\}$ and $M^s_{p,\sigma}$ are modulation spaces.) One can see related results in [14; 16; 17; 19; 25; 36; 46; 50; 51; 60; 70]. We note that one advantage of employing Strichartz estimates in combination with the splitting method is that it allows for the development of a global theory for NLSEs in L^p-based spaces, whereas the approach in [82] is primarily suited for establishing local results. Finally, there are different approaches to study the global well-posedness for NLSE (1-2) in L^p-based spaces; see Dodson, Soffer and Spencer [26; 27] for results in L^p with $2 < p \le \infty$, where use is made of Strichartz estimates and the energy method, as well as Ru and Chen [68] and Wang and Hudzik [79] for results in modulation spaces, where the authors applied the blow-up criterion and the smallness condition on the initial data, respectively.

The case $q \neq 0$. Returning to equation (1-1), the well-posedness of the solution in L^2-based spaces was studied by Adami and Noja [1] and Fukuizumu, Ohta and Ozawa [32]. The asymptotic behavior of solutions, the scattering theory in H^1, and the blow-up phenomenon have been studied by Banica and Visciglia [5] and Tang and Xu [73]. One can also see Segata [71] for the construction of the modified wave operator for (1-2) with $q > 0$. Conversely, the modified scattering was given by Masaki, Murphy and Segata [61] and Chen and Pusateri [18]. Results of nonlinear dynamics around solitons can be found in [20; 31; 32; 34; 42; 43; 41; 53; 58; 62; 63; 65; 72].

As for the results in L^p-based spaces, to the best of our knowledge, Angulo Pava and Ferreira [4] considered the local well-posedness of NLSE with double-well potential and a general nonlinearity of the form $|u|^{\rho-1}u$ in the spaces $L^{p,\infty}$. Obviously, the result in [4] can be applied to (1-1) by selecting the locations of two wells at zero and setting $\rho = 3$.

1.2. ***The main results.*** Before presenting our main results, concerning the local and global well-posedness of (1-1) in $L^p(\mathbb{R})$ and $\dot{B}^s_{p,r}(\mathbb{R})$ spaces, we recall relevant definitions and notation.

Denote by $\mathcal{U}(t)=e^{-itH}$ the linear propagator of $H=H_0+q\delta_0$ with $H_0=-\frac{1}{2}\partial_x^2$. It follows from the Duhamel formula that the solution u of (1-1) can be written as

$$u(t)=\mathcal{U}(t)u_0-i\lambda\int_0^t \mathcal{U}(t-s)|u(s)|^2u(s)\,ds. \tag{1-7}$$

We introduce the linear transformation

$$v(t)=\mathcal{U}(-t)u(t),\qquad \text{or equivalently,}\qquad u(t)=\mathcal{U}(t)v(t).$$

By combining these two identities with (1-7), we have

$$v(t)=u_0-i\lambda\int_0^t \mathcal{U}(-s)\big(\mathcal{U}(s)v(s)\,\overline{\mathcal{U}(s)v(s)}\,\mathcal{U}(s)v(s)\big)\,ds, \tag{1-8}$$

where we use the fact that $\overline{\mathcal{U}(t)}=\mathcal{U}(-t)$. Let us start with the local well-posedness of the integral equation (1-8) in the Besov space $\dot{B}^s_{p,r}(\mathbb{R})$.

Theorem 1.1. *Assume that $u_0\in\dot{B}^s_{p,1}(\mathbb{R})$ with $1<p<2$ and $s=1-\frac{1}{p}$. There exists a time T depending only on $\|u_0\|_{\dot{B}^s_{p,1}(\mathbb{R})}$ and such that the integral equation (1-8) has a unique solution*

$$v\in C\big([0,T),\dot{B}^s_{p,1}(\mathbb{R})\big)$$

satisfying, for all $t\in[0,T)$,

$$\|v(t)\|_{\dot{B}^s_{p,1}(\mathbb{R})}\le 2\|u_0\|_{\dot{B}^s_{p,1}(\mathbb{R})}.$$

If v_1 and v_2 are two solutions for (1-8) with initial data u_{01} and u_{02}, then

$$\|v_1(t)-v_2(t)\|_{\dot{B}^s_{p,1}(\mathbb{R})}\le 2\,\|u_{01}-u_{02}\|_{\dot{B}^s_{p,1}(\mathbb{R})}\,.$$

As mentioned previously, by factorization (1-4), $\mathcal{U}_0(t)$ is bounded from $L^p(\mathbb{R})$ to $L^{p'}(\mathbb{R})$ with $p\in[1,2]$ by Young's inequality, which combined with the identity

$$u(t)=\mathcal{U}_0(t)\mathcal{U}_0(-t)u(t)=\mathcal{U}_0(t)v(t),$$

implies that the solution $u(t)$ to the original equation (1-2) with $q=0$ stays in $L^{p'}(\mathbb{R})$ when $t\neq 0$, see Hyakuna [47] the results for Hartree type nonlinearity. In what follows, we establish a similar result for (1-1) within the framework of Besov spaces.

Corollary 1.2. *Let $v(t)=\mathcal{U}(-t)u(t)$ given in Theorem 1.1 be the solution to the integral equation (1-8). Then*

$$u\in C\big((0,T);\dot{B}^s_{p',1}(\mathbb{R})\big).$$

Next, in the case where the initial data lies in $L^p(\mathbb{R})$, an appropriate space-time norm is needed to control the evolution of the solution $v(t)$. The following result provides such an estimate.

Theorem 1.3. *Suppose that $u_0 \in L^p(\mathbb{R})$ for $1 < p < 2$. Then there exists T depending only on $\|u_0\|_{L^p(\mathbb{R})}$ such that the integral equation (1-8) has a unique solution $v \in C\big([0,T), L^p(\mathbb{R})\big)$ satisfying*

$$\|v(t)\|_{L^p(\mathbb{R})} \le C\|u_0\|_{L^p(\mathbb{R})}, \quad \text{and} \quad \|t^{\frac{2}{p}-1}\partial_t v(t)\|_{L_t^{p'}L_x^p([0,T)\times\mathbb{R})} \le C\|u_0\|^3_{L^p(\mathbb{R})}$$

for all $t \in [0,T)$. If $v_1(t)$ and $v_2(t)$ are solutions for (1-8) with initial data u_{01} and u_{02}, then, for all $t \in [0,T)$,

$$\|v_1(t) - v_2(t)\|_{L^p(\mathbb{R})} \le C\|u_{01} - u_{02}\|_{L^p(\mathbb{R})},$$

where C is a absolute positive constant and $\frac{1}{p} + \frac{1}{p'} = 1$.

Corollary 1.4. *Let $v(t) = \mathcal{U}(-t)u(t)$ given in Theorem 1.3 be the solution to the integral equation (1-8). Then*

$$u \in C\big((0,T); L^{p'}(\mathbb{R})\big).$$

Outline of proofs. The starting point in the proofs of Theorems 1.1 and 1.3 is the integral equation (1-8), where the central task is to analyze the associated trilinear form

$$\mathcal{T}(v_1(\tau), v_2(\tau), v_3(\tau)) = \mathcal{U}(-\tau)\big(\mathcal{U}(-\tau)v_1(\tau)\mathcal{U}(\tau)v_2(\tau)\mathcal{U}(\tau)v_3(\tau)\big)$$

in the Besov space $\dot{B}^s_{p,1}(\mathbb{R})$, if the well-posedness of equation (1-8) in $\dot{B}^s_{p,1}(\mathbb{R})$ is considered. If $\mathcal{U}(t)$ is replaced by $\mathcal{U}_0(t)$ in this form, by exploiting a remarkable cancellation property, one obtains an estimate of $\mathcal{T}$ in L^1; see (1-6). The case of general $\mathcal{U}$ presents two obstacles: the cancellation for the trilinear form and the noncommutativity of $\mathcal{U}(t)$ with the classic Littlewood–Paley projection $\varphi_j(\sqrt{2H_0})$ (see Section 2.1 for the definition).

The cancellation for the trilinear form $\mathcal{T}(v_1(\tau), v_2(\tau), v_3(\tau))$ is based on the explicit formula for the propagator $\mathcal{U}(t)$. In fact, in Proposition 2.2, we obtain the formula for the general multiplier $m(\sqrt{2H})$ in terms of the distorted Fourier transform $\mathcal{F}_q$ (see (2-10) for the definition), which can be applied to $\mathcal{U}(t)$ to obtain

$$e^{-itH}f(x) = \chi_+(x)K_t * \mathcal{L}_+(f)(x) + \chi_-(x)K_t * \mathcal{L}_-(f)(x)$$

where $K_t(x) = e^{-\frac{i\pi}{4}}(2\pi t)^{-1/2}e^{\frac{i|x|^2}{2t}}$. Notice that $\chi_\pm(x)$ are nonsmooth cutoff functions, the Hilbert transform shows up in the expansion of $\mathcal{T}(v_1(\tau), v_2(\tau), v_3(\tau))$. Thus we can't obtain the favorable L^1 estimate just like (1-6). To proceed, we will work in the Hardy space $H^1(\mathbb{R})$ instead of $L^1(\mathbb{R})$ and then estimate $\|\mathcal{T}\|_{L^1}$ via the H^1-norm. The details for cancellation have been established in Section 4.

The noncommutativity between $\mathcal{U}(t)$ and $\varphi_j(\sqrt{2H_0})$ can be circumvented by constructing the Besov spaces $\dot{B}^{s,H}_{p,r}(\mathbb{R})$ associated to the Schrödinger operator H.

The space $\dot{B}^{s,H}_{p,r}(\mathbb{R})$ is defined as the completion of the Schwartz space under

$$\|f\|_{\dot{B}^{s,H}_{p,r}(\mathbb{R})} = \Big(\sum_{j\in\mathbb{Z}} \big(2^{js}\|\varphi_j(\sqrt{2H})f\|_{L^p(\mathbb{R})}\big)^r\Big)^{\frac{1}{r}},$$

where $\varphi_j(\sqrt{2H})$ is the Littlewood–Paley projection associated with H and can be defined in terms of the distorted Fourier transform $\mathcal{F}_q$ (see Proposition 2.2). The advantage of using $\dot{B}^{s,H}_{p,r}(\mathbb{R})$ is that one has the commutation relation

$$\mathcal{U}(t)\varphi_j(\sqrt{2H}) = \varphi_j(\sqrt{2H})\mathcal{U}(t),$$

which is crucial for the estimate of the trilinear form $\mathcal{T}$ in $\dot{B}^{s,H}_{p,r}(\mathbb{R})$. The next step is to establish the equivalence between the two types Besov spaces under consideration. That is, we show that

$$\dot{B}^{s,H}_{p,r}(\mathbb{R}) = \dot{B}^{s}_{p,r}(\mathbb{R})$$

for some s, p and r. Similar results have been obtained by Georgieva and Giammetta [33] in the case $r = 2$, and by Cuccagna, Visciglia and Georgiev [21], where Sobolev spaces were considered instead of Besov spaces, for Schrödinger operators $-\Delta + V$ under different assumptions on the potential V. To prove the equivalence, in Section 3.1, we introduce wave operators $W_\pm$, which are defined by

$$W_\pm f = s - \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}f,$$

and investigate the boundedness on various function spaces. And then we use the results (see Proposition 3.2, Proposition 3.4 and Corollary 3.5) that the wave operator and its conjugate are bounded in $L^p(\mathbb{R})$ and $\dot{B}^{s}_{p,r}(\mathbb{R})$ to get the embedding

$$\dot{B}^{s}_{p,r}(\mathbb{R}) \subset \dot{B}^{s,H}_{p,r}(\mathbb{R}).$$

To establish inverse inclusion, we make use of an estimate of the following type,

$$\|\varphi_j(\sqrt{2H_0})\varphi_k(\sqrt{2H})f\|_{L^p(\mathbb{R})} \lesssim 2^{-|j-k|s}\|f\|_{L^p(\mathbb{R})}$$

for some $s > 0$ and certain restrictions on j, k. We note that since the potential in our setting is singular, we can't apply the perturbation argument used in [21; 33].

Outline. Section 2 introduces some notions, including the distorted Fourier transform associated with H, the explicit formulas for the multipliers $m(\sqrt{2H})$ and some linear estimates for the propagator e^{-itH}. Section 3 is devoted to the equivalence between Besov spaces associated to H and the classical Besov spaces, where the boundedness of wave operators will be involved. In Section 4, we explore the cancellation property for the trilinear form $\mathcal{T}$ and prove Theorems 1.1 and 1.3.

2. Preliminaries

2.1. *Notation.* In this section, we introduce notation for several function spaces. We use $A \lesssim B$ to mean that $A \le CB$ for some $C > 0$ that changes from line to line, independent of the main parameters. For $r \in [1, \infty]$, we let $r' \in [1, \infty]$ denote the Hölder dual of r, given by $\frac{1}{r} + \frac{1}{r'} = 1$. By χ_+ and χ_- we mean the characteristic functions of the intervals $[0, +\infty)$ and $(-\infty, 0)$, respectively.

The Hilbert transform of a function $f \in \mathcal{S}(\mathbb{R})$ is defined by

$$\mathcal{H}f(x) = p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy, \quad x \in \mathbb{R},$$

where $p.v.$ denotes the principal value, which is equivalent to the definition in terms of a or else "multipliers" in the plural Fourier multiplier,

$$\mathcal{H}f(x) = \mathcal{F}^{-1}\big(-i \operatorname{sgn}\xi \ \hat{f}(\xi)\big)(x), \quad f \in \mathcal{S}(\mathbb{R}).$$

As usual, $L^p(\mathbb{R})$ denotes the space of measurable functions $f : \mathbb{R} \to \mathbb{C}$ such that

$$\|f\|_{L^p(\mathbb{R})} = \Big(\int_{\mathbb{R}} |f(x)|^p dx\Big)^{\frac{1}{p}} < \infty, \quad 1 \le p < \infty,$$

or, for $p = \infty$,

$$\|f\|_{L^\infty(\mathbb{R})} := \text{ess sup}_{x\in\mathbb{R}}|f(x)| < \infty.$$

Let $I \subset \mathbb{R}$ be an interval and $1 \le p, r \le \infty$. The space $L_t^p L_x^r(I \times \mathbb{R})$ contains all measurable functions u on $I \times \mathbb{R}$ with $\|u\|_{L_t^p L_x^r(I\times\mathbb{R})} = \big\| \|u(t)\|_{L_x^r(\mathbb{R})} \big\|_{L_t^p(I)} < \infty$. We use $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ to denote the Schwartz space and its dual.

Denote by $\mathcal{F}$ (resp. $\hat{}$) and $\mathcal{F}^{-1}$ (resp. $\check{}$) the standard Fourier transform and its inverse. That is,

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx,$$
$$\mathcal{F}^{-1}(f)(x) = \check{f}(x) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi} f(\xi)\,d\xi.$$

For a given real-valued measurable function m on the real line, we define the Fourier multiplier operator $m(i\nabla) = \mathcal{F}^{-1}m(\xi)\mathcal{F}$. Let φ be a smooth even function on $\mathbb{R}$ such that

$$\operatorname{supp}\varphi \subset \big\{\xi \ \big|\tfrac{1}{2} \le |\xi| \le 2\big\} \quad \text{and} \quad \sum_{j\in\mathbb{Z}} \varphi_j(\xi) := \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad (\xi \neq 0).$$

The classic Littlewood–Paley projections $\varphi_j(\sqrt{2H_0})$ $(j \in \mathbb{Z})$ associated to $H_0 = -\frac{1}{2}\partial_x^2$ are defined in terms of Fourier multipliers by

$$\varphi_j(\sqrt{2H_0})f(x) = \mathcal{F}^{-1}\big(\varphi_j(\xi)\hat{f}(\xi)\big)(x). \tag{2-1}$$

The classic homogeneous Besov spaces $\dot{B}^s_{p,r}(\mathbb{R})$ for $1 \le p, r \le \infty$ can be defined as the closure of $\mathcal{S}(\mathbb{R})$ functions f with respect to the norm

$$\|f\|_{\dot{B}^s_{p,r}(\mathbb{R})} = \Big(\sum_{j\in\mathbb{Z}} 2^{jsr} \|\varphi_j(\sqrt{2H_0}) f\|^r_{L^p(\mathbb{R})}\Big)^{1/r}.$$

We use $H^1(\mathbb{R})$ to denote the Hardy space. The space $H^1(\mathbb{R})$ is a proper subspace of $L^1(\mathbb{R})$, which is usually used as a substitution of $L^1(\mathbb{R})$ when one considers the boundedness of operators at the endpoint. One way to define the Hardy space $H^1(\mathbb{R})$ is as

$$H^1(\mathbb{R}) = \{f \in L^1(\mathbb{R}) \mid \mathcal{H} f \in L^1(\mathbb{R})\}.$$

2.2. ***The Schrödinger operator with delta potential.*** The Hamiltonian

$$H = -\tfrac{1}{2}\partial_x^2 + q\delta_0(x) \tag{2-2}$$

associated with the linear Schrödinger equation with a delta potential describes a δ-interaction of strength q centered at $x = 0$. This kind of interaction, also known as Fermi pseudopotential, gives rise to a variety of models that are widely used in contemporary physics. Throughout this paper, we will restrict our attention to the case of a repulsive delta potential, that is, $q > 0$ in (2-2).

The domain of H is given by

$$\mathcal{D}(H) = \{f \in H^1(\mathbb{R}) \cap H^2(\mathbb{R}\backslash\{0\}) : \partial_x f(0+) - \partial_x f(0-) = 2qf(0)\},$$

where $\pm$ denote limits from the right or left, and $H = -\frac{1}{2}\partial_x^2$ on its domain. Then H is a self-adjoint operator on $L^2(\mathbb{R})$, the Stone theorem yields that it generates a strongly continuous L^2-unitary group e^{-itH} for $t \in \mathbb{R}$. The spectrum of H is well understood, it is known that the essential spectrum and the absolutely continuous spectrum are identical, and $\sigma_{ess}(H) = \sigma_{ac}(H) = [0, +\infty), \sigma_{sc}(H) = \varnothing$. The eigenvalue of H depends on the sign of q. In the repulsive case $q > 0$, the operator H has no eigenvalues, whereas in the attractive case $q < 0$, H has only one simple negative eigenvalue $-\frac{1}{2}q^2$, see for example Albeverio et al. [3] for more details.

2.3. ***The distorted Fourier transform.*** The *Jost functions* associated with our problem are the solutions $f_\pm = f_\pm(x, \xi)$ to the equation

$$Hf = \tfrac{1}{2}\xi^2 f \tag{2-3}$$

with boundary conditions

$$f_\pm(x, \xi) - e^{\pm ix\xi} \to 0 \quad \text{as } x \to \pm\infty.$$

Their explicit expressions are

$$f_+(x,\xi)=\begin{cases} e^{ix\xi} & \text{if } x\ge 0,\\ \frac{1}{t_q(\xi)}e^{ix\xi}+\frac{r_q(\xi)}{t_q(\xi)}e^{-ix\xi} & \text{if } x<0,\end{cases} \tag{2-4}$$

$$f_-(x,\xi)=\begin{cases} \frac{1}{t_q(\xi)}e^{-ix\xi}+\frac{r_q(\xi)}{t_q(\xi)}e^{ix\xi} & \text{if } x\ge 0,\\ e^{-ix\xi} & \text{if } x<0,\end{cases} \tag{2-5}$$

where the so-called transmission and reflection coefficients $t_q(\xi)$ and $r_q(\xi)$, $\xi\in\mathbb{R}$, are given by

$$t_q(\xi)=\frac{i\xi}{i\xi-q} \quad\text{and}\quad r_q(\xi)=\frac{q}{i\xi-q}.$$

The transmission and reflection coefficients enjoy the identities

$$\overline{t_q(\xi)}=t_q(-\xi),\quad \overline{r_q(\xi)}=r_q(-\xi),\quad t_q(\xi)=r_q(\xi)+1, \tag{2-6}$$

$$|t_q(\xi)|^2+|r_q(\xi)|^2=1,\quad t_q(\xi)\overline{r_q(\xi)}+\overline{t_q(\xi)}r_q(\xi)=0. \tag{2-7}$$

(Jost functions also arise when H is associated with more general potentials than the delta of our study. In this generality they may not be easy to write out explicitly, but equation (2-7) still holds.)

The distorted Fourier transform associated with H can be constructed via Jost functions, which serve as generalized eigenfunctions of H. To do so, we define the distorted plane wave by Jost functions:

$$e_+(x,\xi)=t_q(\xi)f_+(x,\xi),\quad e_-(x,\xi)=t_q(\xi)f_-(x,\xi). \tag{2-8}$$

Define

$$\Psi(x,\xi)=\begin{cases} (2\pi)^{-\frac{1}{2}}e_+(x,\xi) & \text{if } \xi\ge 0,\\ (2\pi)^{-\frac{1}{2}}e_-(x,-\xi) & \text{if } \xi<0.\end{cases} \tag{2-9}$$

Note that $\Psi(x,0)=0$ and $\Psi(x,\cdot)$ is continuous at $k=0$ provided $q>0$. The distorted Fourier transform $\mathcal{F}_q$ and its inverse $\mathcal{F}_q^{-1}$ associated with H are defined by

$$\mathcal{F}_q(f)(\xi)=\int_{\mathbb{R}}\overline{\Psi(x,\xi)}f(x)\,dx,\quad \mathcal{F}_q^{-1}(f)(x)=\int_{\mathbb{R}}\Psi(x,\xi)f(\xi)\,d\xi. \tag{2-10}$$

Just as the classic Fourier transform can diagonalize H_0 by $H_0=\mathcal{F}^{-1}\big(\frac{1}{2}\xi^2\big)\mathcal{F}$, the distorted Fourier can be used to diagonalize the Schrödinger operator H by $H=\mathcal{F}_q^{-1}\big(\frac{1}{2}\xi^2\big)\mathcal{F}_q$. Consequently, the multipliers $m(2H)=\mathcal{F}_q^{-1}m(\xi^2)\mathcal{F}_q$ are well defined for some bounded measurable functions m. Below, we will derive the explicit expression for $m(\sqrt{2H})$.

The distorted Fourier transform and its inverse maintain a close structural resemblance to the classical Fourier transform. In fact, it follows from the results in Segata [71] that

(2-11) $$\mathcal{F}_q(\phi)(\xi)=\begin{cases}\mathcal{F}(\phi)(\xi)+r_q(\xi)\mathcal{F}(\chi_+\phi)(-\xi)+r_q(\xi)\mathcal{F}(\chi_-\phi)(\xi) & \text{if } \xi\ge 0,\\ \mathcal{F}(\phi)(\xi)+\overline{r_q(\xi)}\mathcal{F}(\chi_-\phi)(-\xi)+\overline{r_q(\xi)}\mathcal{F}(\chi_+\phi)(\xi) & \text{if } \xi<0,\end{cases}$$

(2-12)
$$\mathcal{F}_q^{-1}(\phi)(x)=\begin{cases}\mathcal{F}^{-1}(\phi)(x)+\mathcal{F}^{-1}(\chi_+\bar{r}_q\phi)(-x)+\mathcal{F}^{-1}(\chi_-r_q\phi)(x) & \text{if } x\ge 0,\\ \mathcal{F}^{-1}(\phi)(x)+\mathcal{F}^{-1}(\chi_-r_q\phi)(-x)+\mathcal{F}^{-1}(\chi_+\bar{r}_q\phi)(x) & \text{if } x<0.\end{cases}$$

For self-containedness, we give a brief proof of (2-11) when $\xi\ge 0$; the other cases can be dealt with in a similar manner. It follows from the definition (2-10) and the identities (2-6)–(2-7) that

$$\begin{aligned}\mathcal{F}_q(\phi)(\xi)&=\int_{\mathbb{R}}\overline{\Psi(x,\xi)}\phi(x)\,dx\\ &=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{t_q(\xi)f_+(x,\xi)}\phi(x)\,dx\\ &=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\overline{t_q(\xi)}e^{-ix\xi}\phi(x)\,dx+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^0\big(e^{-ix\xi}+\overline{r_q(\xi)}e^{ix\xi}\big)\phi(x)\,dx\\ &=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\big(1+\overline{r_q(\xi)}\big)e^{-ix\xi}\phi(x)\,dx+\frac{1}{\sqrt{2\pi}}\int_{-\infty}^0\big(e^{-ix\xi}+\overline{r_q(\xi)}e^{ix\xi}\big)\phi(x)\,dx\\ &=\mathcal{F}(\chi_+\phi)(\xi)+\overline{r_q(\xi)}\mathcal{F}(\chi_+\phi)(\xi)+\mathcal{F}(\chi_-\phi)(\xi)+\overline{r_q(\xi)}\mathcal{F}(\chi_-\phi)(-\xi)\\ &=\mathcal{F}(\phi)(\xi)+r_q(\xi)\mathcal{F}(\chi_+\phi)(-\xi)+r_q(\xi)\mathcal{F}(\chi_-\phi)(\xi),\end{aligned}$$

which gives (2-11) for $\xi\ge 0$.

We collect some basic properties of $\mathcal{F}_q$ and $\mathcal{F}_q^{-1}$. For more details, please refer to Segata [71] and Masaki, Murphy and Segata [61].

Lemma 2.1. *Assume that H is the Schrödinger operator given by (2-2) with $q>0$. Let $\mathcal{F}_q$ and $\mathcal{F}_q^{-1}$ be defined by (2-10).*

(i) *$\mathcal{F}_q$ and $\mathcal{F}_q^{-1}$ are unitary on $L^2(\mathbb{R})$, and*

$$\mathcal{F}_q^{-1}\mathcal{F}_q=\mathcal{F}_q\mathcal{F}_q^{-1}=I\quad\text{on}\quad L^2(\mathbb{R}).$$

(ii) $\|\mathcal{F}_q(f)\|_{L^\infty(\mathbb{R})}\lesssim\|f\|_{L^1(\mathbb{R})}$. *The same estimate is true for $\mathcal{F}_q^{-1}$ as well.*

(iii) $\mathcal{F}_q(f)(0)=0$ *whenever* $\langle x\rangle f\in L^2(\mathbb{R})$.

Proof. Statements (i) and (iii) were established in [71] and [61]. Statement (ii) follows from (2-11), (2-12) and the fact that $r_q\in L^\infty(\mathbb{R})$. □

Next we give an explicit formula for the multipliers associated with H, which will be used to investigate the linear estimates for e^{-itH}, as well as the formula for the Littlewood–Paley projection $\varphi_j(\sqrt{2H})$.

Proposition 2.2. *If m is a bounded radial measurable function, $m(\sqrt{2H})f(x)$ is given by*

$$\begin{cases} \mathcal{F}^{-1}\big(m(\xi)(\mathcal{F}(f)(\xi)+r_q(\xi)\mathcal{F}(\chi_+ f)(-\xi)+r_q(\xi)\mathcal{F}(\chi_- f)(\xi))\big)(x) & \text{if } x\ge 0,\\ \mathcal{F}^{-1}\big(m(\xi)(\mathcal{F}(f)(\xi)+\bar{r}_q(\xi)\mathcal{F}(\chi_- f)(-\xi)+\bar{r}_q(\xi)\mathcal{F}(\chi_+ f)(\xi))\big)(x) & \text{if } x< 0.\end{cases}$$

Proof. The explicit representation for the spectral projection implies

$$m(\sqrt{2H})f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Big(\int_0^{\infty} m(\xi)|t_q(\xi)|^2(f_+(x,\xi)\overline{f_+(y,\xi)}+f_-(x,\xi)\overline{f_-(y,\xi)})\,d\xi\Big)f(y)\,dy,$$

where $f_\pm$ are Jost functions and t_q is the transmission coefficient. We first consider the case $x\ge 0$. We write

$$m(\sqrt{2H})f(x)=F_1(x)+F_2(x)+F_3(x)+F_4(x),$$

with

$$\begin{aligned}
F_1(x)&:=\frac{1}{2\pi}\int_0^\infty m(\xi)|t_q(\xi)|^2 f_+(x,\xi)\Big(\int_0^\infty \overline{f_+(y,\xi)}f(y)\,dy\Big)\,d\xi,\\
F_2(x)&:=\frac{1}{2\pi}\int_0^\infty m(\xi)|t_q(\xi)|^2 f_+(x,\xi)\Big(\int_{-\infty}^0 \overline{f_+(y,\xi)}f(y)\,dy\Big)\,d\xi,\\
F_3(x)&:=\frac{1}{2\pi}\int_0^\infty m(\xi)|t_q(\xi)|^2 f_-(x,\xi)\Big(\int_0^\infty \overline{f_-(y,\xi)}f(y)\,dy\Big)\,d\xi,\\
F_4(x)&:=\frac{1}{2\pi}\int_0^\infty m(\xi)|t_q(\xi)|^2 f_-(x,\xi)\Big(\int_{-\infty}^0 \overline{f_-(y,\xi)}f(y)\,dy\Big)\,d\xi.
\end{aligned}$$

It follows from the expressions (2-4)–(2-5) for $f_\pm$ that

$$\begin{aligned}
F_1(x)=&\frac{1}{2\pi}\int_0^\infty m(\xi)|t_q(\xi)|^2e^{ix\xi}\Big(\int_0^\infty e^{-iy\xi}f(y)\,dy\Big)\,d\xi\\
=&\frac{1}{\sqrt{2\pi}}\int_0^\infty m(\xi)e^{ix\xi}|t_q(\xi)|^2\mathcal{F}(\chi_+ f)(\xi)\,d\xi
\end{aligned}$$

and

$$F_2(x)=F_{21}(x)+F_{22}(x),$$

with

$$\begin{aligned}
F_{21}(x)&:=\frac{1}{\sqrt{2\pi}}\int_0^\infty m(\xi)t_q(\xi)e^{ix\xi}\mathcal{F}(\chi_- f)(\xi)\,d\xi,\\
F_{22}(x)&:=+\frac{1}{\sqrt{2\pi}}\int_0^\infty m(\xi)t_q(\xi)e^{ix\xi}\overline{r_q(\xi)}\mathcal{F}(\chi_- f)(-\xi)\,d\xi.
\end{aligned}$$

Notice that $\overline{t_q(\xi)}=t_q(-\xi)$, we further have

$$F_3(x)=F_{31}(x)+F_{32}(x)+F_{33}(x)+F_{34}(x),$$

with

$$F_{31}(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} m(\xi) e^{ix\xi} \mathcal{F}(\chi_+ f)(\xi)\, d\xi,$$
$$F_{32}(x) := \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} m(\xi) r_q(\xi) e^{ix\xi} \mathcal{F}(\chi_+ f)(-\xi)\, d\xi,$$
$$F_{33}(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} m(\xi) e^{ix\xi} r_q(\xi) \mathcal{F}(\chi_+ f)(-\xi)\, d\xi,$$
$$F_{34}(x) := +\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} m(\xi) e^{ix\xi} |r_q(\xi)|^2 \mathcal{F}(\chi_+ f)(\xi)\, d\xi.$$

Similarly and

$$F_4(x) = F_{41}(x) + F_{42}(x),$$

with

$$F_{41}(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} m(\xi) e^{ix\xi} t_q(\xi) \mathcal{F}(\chi_- f)(\xi)\, d\xi,$$
$$F_{42}(x) := \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} m(\xi) e^{ix\xi} \overline{t_q(\xi)} r_q(\xi) \mathcal{F}(\chi_- f)(-\xi)\, d\xi.$$

Now by using identities (2-6)–(2-7), we have

$$F_1(x)+F_{31}(x)+F_{34}(x) = \mathcal{F}^{-1}\big(m(\xi)\mathcal{F}(\chi_+ f)(\xi)\big)(x),$$
$$F_{21}(x)+F_{41}(x) = \mathcal{F}^{-1}\big(m(\xi)\mathcal{F}(\chi_- f)(\xi)\big)(x)+\mathcal{F}^{-1}\big(m(\xi)r_q(\xi)\mathcal{F}(\chi_- f)(\xi)\big)(x),$$
$$F_{22}(x)+F_{42}(x) = 0,$$
$$F_{32}(x)+F_{33}(x) = \mathcal{F}^{-1}\big(m(\xi)r_q(\xi)\mathcal{F}(\chi_+ f)(-\xi)\big)(x),$$

which imply that for $x \ge 0$,

$$m(\sqrt{2H}) f(x) = \mathcal{F}^{-1}\big(m(\xi)(\mathcal{F}(f)(\xi) + r_q(\xi)\mathcal{F}(\chi_+ f)(-\xi) + r_q(\xi)\mathcal{F}(\chi_- f)(\xi))\big)(x).$$

As for the case $x < 0$, noticing that $f_+(x,\xi) = f_-(-x,\xi)$, it follows that

$$\begin{aligned} m(\sqrt{2H}) f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big(\int_0^{\infty} m(\xi) |t_q(\xi)|^2 (f_-(-x,\xi)\, \overline{f_-(-y,\xi)} + f_+(-x,\xi) d\xi \Big)\, \overline{f_+(-y,\xi)}) \\ &\qquad \times f(y)\, dy \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \Big(\int_0^{\infty} m(\xi) |t_q(\xi)|^2 (f_-(-x,\xi)\, \overline{f_-(y,\xi)} + f_+(-x,\xi)\, \overline{f_+(y,\xi)})\, d\xi \Big) \\ &\qquad \times f(-y)\, dy \\ &= m(\sqrt{2H}) f(-\cdot)(-x), \end{aligned}$$

concluding the proof. □

We now present a set of corollaries that provide a representation of the linear propagator e^{-itH} and its associated Strichartz estimates. These are essential tools in the analysis of the Cauchy problem (1-1) with initial data in $L^p(\mathbb{R})$.

First, choosing $m(\lambda) = e^{-\frac{1}{2}it\lambda^2}$ in Proposition 2.2, we obtain:

Corollary 2.3. *Let H be defined by (2-2). Then*

$$e^{-itH} f(x) = \chi_+(x) K_t * \mathcal{L}_+(f)(x) + \chi_-(x) K_t * \mathcal{L}_-(f)(x)$$

where $K_t(x) = e^{-\frac{i\pi}{4}}(2\pi t)^{-1/2} e^{\frac{i|x|^2}{2t}}$ and

$$\mathcal{L}_+(f) = f + \mathcal{F}^{-1}\big(r_q \mathcal{F}(\chi_+ f)(-\,\cdot)\big) + \mathcal{F}^{-1}\big(r_q \mathcal{F}(\chi_- f)\big),$$
$$\mathcal{L}_-(f) = f + \mathcal{F}^{-1}\big(\overline{r_q} \mathcal{F}(\chi_- f)(-\,\cdot)\big) + \mathcal{F}^{-1}\big(\overline{r_q} \mathcal{F}(\chi_+ f)\big).$$

The next result appears in [42] and [71], but we include the proof for convenience.

Corollary 2.4. *Let H be defined by (2-2) and*

$$\tfrac{2}{r_j} + \tfrac{1}{p_j} = \tfrac{1}{2}, \quad 4 \le r_j \le \infty, \quad j = 1, 2.$$

Then for any interval I and $s \in \bar{I}$, we have $\|e^{-itH} f\|_{L_t^{r_1} L_x^{q_1}(I\times\mathbb{R})} \lesssim \|f\|_{L_x^2(\mathbb{R})}$ and

$$\Big\| \int_s^t e^{-i(t-\tau)H} F(\tau)\, d\tau \Big\|_{L_t^{r_1} L_x^{q_1}(I\times\mathbb{R})} \lesssim \|F\|_{L_t^{r_2'} L_x^{q_2'}(I\times\mathbb{R})}.$$

Proof. The operators $\mathcal{L}_\pm$ are bounded on $L^p(\mathbb{R})$ for all $1 \le p \le \infty$, since $r_q \in L^\infty$. It then follows from Corollary 2.3 that

$$\|e^{-itH} f\|_{L^\infty(\mathbb{R})} \lesssim |t|^{-\frac{1}{2}} \|f\|_{L^1(\mathbb{R})};$$

combined with

$$\|e^{-itH} f\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$$

and the TT^* argument, this implies the desired estimates. □

Remark 2.5. The formula for e^{-itH} in Corollary 2.3 can be expressed in terms of e^{-itH_0}:

$$e^{-itH} f = \chi_+ e^{-itH_0} \mathcal{L}_+(f) + \chi_- e^{-itH_0} \mathcal{L}_-(f).$$

The dispersive estimates and Strichartz estimates for e^{-itH} can also be derived from those for e^{-itH_0} and the L^p-boundedness of $\mathcal{L}_\pm$, where $1 \le p \le \infty$.

3. Homogeneous Besov spaces associated to H

The function space theory associated to operator H is an important topic in harmonic analysis and has been extensively studied in recent years, one can see for example [24; 29; 30; 38; 40; 39], the theories of Hardy spaces, BMO spaces and Sobolev spaces associated with operators. On the one hand, they generalize the classic theories of corresponding function spaces. On the other hand, they are used to investigate the boundedness of singular integrals such as the Riesz transform, square function and area integral associated to operators at some endpoint. Such theories primarily rely on the point-wise estimates or off-diagonal estimates for the heat semigroup e^{-tH}, and can be applied to Schrödinger operator $H=-\Delta+V$, where V is an unbounded positive potential or V has small negative part.

However, if the potential V has large negative component, the Schrödinger operator $H=-\Delta+V$ may admit eigenvalue and resonance at zero energy. Under such circumstance, the development of function spaces adapted to $H=-\Delta+V$ is primarily grounded in spectral analysis and the theory of distorted Fourier transforms. Ólafsson and Zheng [66] studied the Triebel–Lizorkin spaces and Besov spaces associated to H, where V is taken to be the Pöschl–Teller potential. Cuccagna, Visciglia and Georgiev [21] considered the Sobolev spaces associated to H under the assumptions that $V\in\mathcal{S}(\mathbb{R})$ being both generic and exceptional. Moreover, Georgiev and Giammetta [33] studied the homogeneous Besov spaces associated to H with short range potential V. It is noteworthy that in [21] and [33], the authors proved the equivalence between the classical function spaces and the corresponding spaces associated to H.

The homogeneous Besov space $\dot{B}^{s,H}_{p,r}(\mathbb{R})$ associated with H for some $1<p<\infty$, $1\le r\le\infty$ and $s\in\mathbb{R}$ is defined as follows. Let $\varphi_j(\sqrt{2H})$ be the Littlewood–Paley associated with H, which can be defined by Proposition 2.2 with $m(\lambda)=\varphi_j(\lambda)$, where φ_j is defined as in (2-1). Let $1<p<\infty$, $1\le r\le\infty$ and $s\in\mathbb{R}$, the homogeneous Besov space $\dot{B}^{s,H}_{p,r}(\mathbb{R})$ associated with the perturbed Hamiltonian H is defined as the closure of $\mathcal{S}(\mathbb{R})$ function f with respect to the norm

$$\|f\|_{\dot{B}^{s,H}_{p,r}(\mathbb{R})}=\Big(\sum_{j\in\mathbb{Z}}2^{jrs}\|\varphi_j(\sqrt{2H})f\|^r_{L^p(\mathbb{R})}\Big)^{1/r}. \tag{3-1}$$

The space $\dot{B}^{s,H}_{p,r}(\mathbb{R})$ is independent of the choice of the Littlewood–Paley function φ. In fact, if ψ is of the same type as φ in Section 2.1, it follows that $\varphi_j\psi_k=0$ if $|j-k|>3$. Then by using the distorted Fourier transform and the uniform boundedness of $\varphi_j(\sqrt{2H})$ on $L^p(\mathbb{R})$ for $1<p<\infty$ (see Remark 3.3(ii)), we have

$$\begin{aligned}\|\varphi_j(\sqrt{2H})f\|_{L^p(\mathbb{R})}&\le\sum_{k\in\mathbb{Z}}\|\varphi_j(\sqrt{2H})\psi_k(\sqrt{2H})f\|_{L^p(\mathbb{R})}\\&=\sum_{k\in\mathbb{Z}}\|\mathcal{F}_q^{-1}(\varphi_j(\xi)\psi_k(\xi)\mathcal{F}_qf(\xi))\|_{L^p(\mathbb{R})}\\&\lesssim\sum_{k=j-3}^{j+3}\|\psi_k(\sqrt{2H})f\|_{L^p(\mathbb{R})},\end{aligned}$$

which means that $\dot{B}^{s,H}_{p,r}(\mathbb{R})$ is well defined.

We are now in a position to state the main theorem of this section.

Theorem 3.1. *Assume that* $1 < p < \infty$, $1 \le r \le \infty$ *and* $-\frac{1}{p'} < s < \frac{1}{p}$. *We have*

$$\dot{B}^{s}_{p,r}(\mathbb{R}) = \dot{B}^{s,H}_{p,r}(\mathbb{R})$$

with equivalent norms.

The proof, given in Section 3.2, depends on the boundedness of wave operators on Besov spaces and the cancellation property of the operator $\varphi_j(\sqrt{2H_0})\varphi_k(\sqrt{2H})$.

3.1. ***Wave operators.*** Wave operator methods are fundamental in the study of the evolution flow generated by the Hamiltonian H, typically considered as perturbations of the free Hamiltonian H_0. They are often used to deal with the behavior of particles interacting with each other or external potentials.

Namely, we have the free Hamiltonian $H_0 = -\frac{1}{2}\partial_x^2$ and the perturbed Hamiltonian $H = H_0 + q\delta_0$ with $q > 0$. The corresponding wave operators are defined by

$$W_\pm f = s - \lim_{t\to\pm\infty} e^{itH} e^{-itH_0} f,$$

and their conjugates

$$W_\pm^* f = s - \lim_{t\to\pm\infty} e^{itH_0} e^{-itH} f.$$

If the potential $q\delta_0$ is replaced by a general potential, $W_\pm^*$ is well-defined for $P_c(H)f$, where $P_c(H)$ denotes the projection onto the continuous spectrum. The wave operators enjoy the splitting property

$$HW_\pm = W_\pm H_0,$$

and can lead to the functional calculus for H,

$$g(H) = W_+ g(H_0) W_+^* = W_- g(H_0) W_-^* \tag{3-2}$$

for any function $g \in L^\infty_{\mathrm{loc}}(\mathbb{R})$.

There are several ways to represent the wave operators $W_\pm$. It follows from Schechter [69] and (2-12) that

$$W_+ f(x) = \mathcal{F}_q^{-1}\mathcal{F}(f)(x) = I_d f(x) + \sum_{\ell=1}^{4} T_\ell f(x), \tag{3-3}$$

where I_d is the identity operator and

$$\begin{aligned}
T_1 f(x) &= \chi_+(x)\mathcal{F}^{-1}\big(\chi_+\bar{r}_q\mathcal{F}(f)\big)(-x),\\
T_2 f(x) &= \chi_+(x)\mathcal{F}^{-1}\big(\chi_- r_q\mathcal{F}(f)\big)(x),\\
T_3 f(x) &= \chi_-(x)\mathcal{F}^{-1}\big(\chi_- r_q\mathcal{F}(f)\big)(-x),\\
T_4 f(x) &= \chi_-(x)\mathcal{F}^{-1}\big(\chi_+\bar{r}_q\mathcal{F}(f)\big)(x).
\end{aligned}$$

We will prove the boundedness of wave operators on various function spaces, which can be used to prove the equivalence between $\dot{B}^{s,H}_{p,r}(\mathbb{R})$ and $\dot{B}^{s}_{p,r}(\mathbb{R})$. The boundedness of wave operators under different assumptions regarding the potential V in Lebesgue spaces and Sobolev spaces has been extensively studied, so we will not go further on this topic; for more information, one can see for example D'Ancona and Fanelli [22], Weder [80] and Yajima [81].

Denote by $\mathbb{B}(X, Y)$ the space of bounded linear operators from X to Y.

Proposition 3.2. *The wave operators $W_\pm$ lie in $\mathbb{B}(X, Y)$, where*

(i) $X = Y = L^p(\mathbb{R})$ *with* $1 < p < \infty$, *or*

(ii) $X = L^1(\mathbb{R})$ *and* $Y = L^{1,\infty}(\mathbb{R})$, *or*

(iii) $X = H^1(\mathbb{R})$ *and* $Y = L^1(\mathbb{R})$.

Proof. By using the identity (3-3), we only prove $T_2 \in \mathbb{B}(X, Y)$. Write

$$T_2 f(x) = \chi_+(x)\mathcal{F}^{-1}\big(m_2(\xi)\mathcal{F}(f)(\xi)\big)(x)$$

with

$$m_2(\xi) = \chi_-(\xi) r_q(\xi) = \frac{\chi_-(\xi) q}{i\xi - q}.$$

Notice that m_2 is smooth away from the origin and

$$\big|\partial_\xi^k m_2(\xi)\big| \le C\langle\xi\rangle^{-k-1}, \quad \xi \in \mathbb{R}\backslash\{0\} \quad \text{and} \quad k \in \mathbb{N}_+,$$

where the constant $C > 0$ depends on k, q. Then it follows from a standard multiplier theorem (see Grafakos [35]) that W_+ is bounded from X to Y with X, Y satisfying any of the sets of conditions above. □

Remark 3.3. (1) By duality, the operators $W_\pm^*$ belong to $\mathbb{B}(L^p(\mathbb{R}), L^p(\mathbb{R}))$ with $1 < p < \infty$ and $\mathbb{B}(L^\infty(\mathbb{R}), BMO(\mathbb{R}))$.

(2) For $1 < p < \infty$, the functional calculus (3-2) and Proposition 3.2 show that if the classic multiplier $m(H_0)$ is bounded on $L^p(\mathbb{R})$, then so is $m(H)$.

(3) Duchêne, Marzuola and Weinstein [28] considered the boundedness of wave operators $W_\pm$ on Sobolev spaces $W^{1,p}(\mathbb{R})$ for singular potentials in dimension one. Their results also apply to the Schrödinger operator with delta potential.

(4) The wave operators $W_\pm$ may not map the Hardy space $H^1(\mathbb{R})$ to itself. This is because the projection $\chi_\pm$ will break the cancellation property for the atoms of $H^1(\mathbb{R})$.

Next we investigate the boundedness of the wave operators on homogeneous Besov spaces.

Proposition 3.4. *Assume that* $1 < p < \infty$, $1 \le r \le \infty$ *and* $-\frac{1}{p'} < s < \frac{1}{p}$. *Then*

$$W_\pm \in \mathbb{B}\big(\dot{B}^s_{p,r}(\mathbb{R}), \dot{B}^s_{p,r}(\mathbb{R})\big).$$

Proof. We prove that W_+ is bounded on $\dot{B}^s_{p,r}(\mathbb{R})$; the same argument can be applied for the proof on the boundedness of W_-. We will establish the inequality

$$\|W_+ f\|_{\dot{B}^s_{p,r}(\mathbb{R})} = \Big(\sum_{j\in\mathbb{Z}} 2^{jrs} \|\varphi_j(\sqrt{2H_0})W_+ f\|^r_{L^p(\mathbb{R})}\Big)^{1/r} \lesssim \|f\|_{\dot{B}^s_{p,r}(\mathbb{R})}, \tag{3-4}$$

where φ is defined in Section 2.1 and $\varphi_j(\xi) = \varphi(2^{-j}\xi)$. Let $\psi \in C_0^\infty(\mathbb{R})$ be an even function such that $\psi = 1$ on the support of φ and $0 \le \psi \le 1$. We write

$$f = \sum_{k\in\mathbb{Z}} \varphi_k(\sqrt{2H_0})\psi_k(\sqrt{2H_0}) f = \sum_{k\in\mathbb{Z}} \varphi_k(\sqrt{2H_0}) f_k$$

with $f_k = \psi_k(\sqrt{2H_0}) f$. Then

$$\begin{aligned} \varphi_j(\sqrt{2H_0})W_+ f &= \sum_{k\in\mathbb{Z}} \varphi_j(\sqrt{2H_0})W_+\big(\varphi_k(\sqrt{2H_0}) f_k\big) \\ &= \sum_{k\in\mathbb{Z}} \varphi_j(\sqrt{2H_0})\varphi_k(\sqrt{2H_0}) f_k + \sum_{\ell=1}^{4}\sum_{k\in\mathbb{Z}} \varphi_j(\sqrt{2H_0}) T_\ell\big(\varphi_k(\sqrt{2H_0}) f_k\big) \\ &=: \sum_{\ell=0}^{4}\sum_{k\in\mathbb{Z}} R_{\ell,j,k}. \end{aligned} \tag{3-5}$$

Notice that for $j \in \mathbb{Z}$, $\varphi_j\varphi_k$ is nonzero when $|j-k| \le 3$ and otherwise it is zero. Thus

$$\sum_{k\in\mathbb{Z}} \|R_{0,j,k}\|_{L^p(\mathbb{R})} \lesssim \sum_{k=j-3}^{j+3} \|f_k\|_{L^p(\mathbb{R})}. \tag{3-6}$$

As for $R_{\ell,j,k}$ ($\ell = 1, \dots, 4$) in (3-5), it follows from the proof of Proposition 3.2 that T_ℓ ($\ell = 1, \dots, 4$) are bounded on L^p for all $1 < p < \infty$. Then for fixed $j \in \mathbb{Z}$ and $|j-k| \le 3$,

$$\sum_{\ell=1}^{4}\sum_{k=j-3}^{j+3} \|R_{\ell,j,k}\|_{L^p(\mathbb{R})} \lesssim \sum_{k=j-3}^{j+3} \|f_k\|_{L^p(\mathbb{R})}. \tag{3-7}$$

It remains to consider the estimates for $R_{\ell,j,k}$ ($\ell = 1, \dots, 4$) in (3-5) with $|j-k| > 3$. We deal with the case $\ell = 2$. Notice that

$$\begin{aligned} &\mathcal{F}(T_2\varphi_k(\sqrt{2H_0}) f_k)(\xi) \\ &\quad= \mathcal{F}\big(\chi_+\mathcal{F}^{-1}(\chi_- r_q \mathcal{F}(\varphi_k(\sqrt{2H_0}) f_k))\big)(\xi) \\ &\quad= \mathcal{F}\big(\tfrac{1}{2}(1+\operatorname{sgn}\cdot)\mathcal{F}^{-1}(\chi_- r_q \varphi_k \mathcal{F}(f_k))\big)(\xi) \end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{2}\chi_-(\xi) r_q(\xi)\varphi_k(\xi)\mathcal{F}(f_k)(\xi) + \tfrac{1}{2}(2\pi)^{-\frac{1}{2}}\mathcal{F}(\operatorname{sgn}\cdot) * (r_q\chi_-\varphi_k\mathcal{F}(f_k))(\xi) \\
&= \tfrac{1}{2}\chi_-(\xi) r_q(\xi)\varphi_k(\xi)\mathcal{F}(f_k)(\xi) \\
&\qquad\qquad + i(2\pi)^{-\frac{3}{2}}\int \frac{1}{y-\xi}\chi_-(y) r_q(y)\varphi_k(y)\int e^{-izy} f_k(z)\,dz\,dy.
\end{aligned}
$$

Thus we can represent each $R_{2,j,k}$ as

(3-8)

$$
\begin{aligned}
R_{2,j,k} &= \mathcal{F}^{-1}\big(\varphi_j(\xi)\mathcal{F}(T_2\varphi_k(\sqrt{2H_0})f_k)\big)(x) \\
&= \tfrac{1}{2}\mathcal{F}^{-1}\big(\varphi_j(\xi)\varphi_k(\xi)\chi_-(\xi) r_q(\xi)\mathcal{F}(f_k)(\xi)\big)(x) \\
&\quad + i(2\pi)^{-2}\iiint e^{ix\xi}\varphi_j(\xi)\frac{1}{y-\xi}\chi_-(y) r_q(y)\varphi_k(y) e^{-izy} f_k(z)\,dz\,dy\,d\xi \\
&= \frac{1}{2}\mathcal{F}^{-1}\big(\varphi_j(\xi)\varphi_k(\xi)\chi_-(\xi) r_q(\xi)\mathcal{F}(f_k)(\xi)\big)(x) + \int \mathcal{K}_{j,k}(x,z) f_k(z)\,dz \\
&=: R_{21,j,k} + R_{22,j,k},
\end{aligned}
$$

where the kernel $\mathcal{K}_{j,k}(x,z)$ is given by

$$
\mathcal{K}_{j,k}(x,z) = i(2\pi)^{-2}\iint e^{ix\xi}e^{-izy}\varphi_j(\xi)\varphi_k(y)\frac{1}{y-\xi}\chi_-(y) r_q(y)\,dy\,d\xi.
$$

By using the fact that $\varphi_j\varphi_k = 0$ when $|j-k|>3$, the term $R_{21,j,k}$ in (3-8) vanishes, which means that

(3-9) $$\|R_{2,j,k}\|_{L^p(\mathbb{R})} \le \|R_{22,j,k}\|_{L^p(\mathbb{R})} \lesssim \Big\| \int \mathcal{K}_{j,k}(x,z) f_k(z)\,dz \Big\|_{L^p(\mathbb{R})}.$$

By a change of variables, we have

$$
\begin{aligned}
&\mathcal{K}_{j,k}(x,z) = i(2\pi)^{-2}2^j 2^k \\
&\qquad\qquad \iint e^{i2^j x\xi}e^{-i2^k zy}\varphi(\xi)\varphi(y)\frac{1}{2^k y - 2^j\xi}\chi_-(2^k y) r_q(2^k y)\,dy\,d\xi.
\end{aligned}
$$

By integration by parts in y and ξ, it follows

$$
\begin{aligned}
\big|\mathcal{K}_{j,k}(x,z)\big| &\lesssim \frac{2^j 2^k}{\langle 2^j x\rangle^2\langle 2^k z\rangle^2}\iint \Big|\partial_y^2\partial_\xi^2 \frac{\varphi(\xi)\varphi(y)\chi_-(2^k y) r_q(2^k y)}{2^k y - 2^j\xi}\Big|\,dy\,d\xi \\
&\lesssim \frac{2^j 2^k}{\langle 2^j x\rangle^2\langle 2^k z\rangle^2}\frac{1}{\max(2^k, 2^j)}.
\end{aligned}
$$

Now for $k < j-3$, it follows from Hölder's inequality that

(3-10) $$
\begin{aligned}
\Big\| \int \mathcal{K}_{j,k}(x,z) f_k(z)\,dz \Big\|_{L^p(\mathbb{R})} &\lesssim \frac{2^k}{(2^j)^{\frac{1}{p}}(2^k)^{1-\frac{1}{p}}}\|f_k\|_{L^p(\mathbb{R})} \\
&\lesssim 2^{-|j-k|\frac{1}{p}}\|f_k\|_{L^p(\mathbb{R})}.
\end{aligned}
$$

As for $k > j+3$, by Hölder's inequality again, we have

$$\Big\| \int \mathcal{K}_{j,k}(x,z) f_k(z)\, dz \Big\|_{L^p(\mathbb{R})} \lesssim \frac{2^j}{(2^j)^{\frac{1}{p}}(2^k)^{(1-\frac{1}{p})}} \| f_k\|_{L^p(\mathbb{R})} \lesssim 2^{-|j-k|(1-\frac{1}{p})} \| f_k\|_{L^p(\mathbb{R})}. \tag{3-11}$$

For fixed $j \in \mathbb{Z}$, combining (3-7), (3-9), (3-10) and (3-11), we have

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \|R_{2,j,k}\|_{L^p(\mathbb{R})} &\lesssim \sum_{k\in\mathbb{Z}} \big(\chi_{\le 3}(|k-j|) + \chi_{>3}(k-j) + \chi_{<-3}(k-j)\big) \|R_{2,j,k}\|_{L^p(\mathbb{R})} \\ &\lesssim \sum_{k=j-3}^{j+3} \|f_k\|_{L^p(\mathbb{R})} \\ &\quad + \sum_{k\in\mathbb{Z}} \big(\chi_{>3}(k-j) 2^{-|j-k|(1-\frac{1}{p})} + \chi_{<-3}(k-j) 2^{-|j-k|\frac{1}{p}}\big) \|f_k\|_{L^p(\mathbb{R})}, \end{aligned} \tag{3-12}$$

where $\chi_{\le 3}(n)$ is the characteristic function of the set $(-\infty, 3]$ on $\mathbb{Z}$, and $\chi_{>3}$ and $\chi_{<-3}$ are defined similarly. The same estimate (3-12) is also true if the operator $R_{2,j,k}$ is replaced by $R_{\ell,j,k}$ with $(\ell \neq 2)$. Writing $\|a_n\|^r_{\ell^r_n(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} |a_n|^r$, notice that for

$$\begin{aligned} \big\| \chi_{>3}(n) 2^{-n(1-\frac{1}{p})-ns} \big\|_{\ell^1_n(\mathbb{Z})} &\lesssim 1 \quad \text{for } s > -\big(1-\tfrac{1}{p}\big) = -\tfrac{1}{p'}, \\ \big\| \chi_{<-3}(n) 2^{-\frac{|n|}{p}-ns} \big\|_{\ell^1_n(\mathbb{Z})} &\lesssim 1 \quad \text{for } s < \tfrac{1}{p}. \end{aligned}$$

Then it follows from (3-5), (3-6), (3-12) and Young's inequality that for $-\frac{1}{p'} < s < \frac{1}{p}$,

$$\begin{aligned} \Big\| 2^{js} \|\varphi_j(\sqrt{2H_0}) W_+ f\|_{L^p(\mathbb{R})} \Big\|_{\ell^r_j(\mathbb{Z})} &\lesssim \Big\| 2^{js} \sum_{\ell=0}^{4} \sum_{k\in\mathbb{Z}} \|R_{\ell,j,k}\|_{L^p(\mathbb{R})} \Big\|_{\ell^r_j(\mathbb{Z})} \\ &\lesssim \Big\| 2^{js} \sum_{|k-j|\le 3} \|f_k\|_{L^p(\mathbb{R})} \Big\|_{\ell^r_j(\mathbb{Z})} \\ &\qquad + \Big\| \sum_{k\in\mathbb{Z}} \chi_{>3}(k-j) 2^{-|j-k|(1-\frac{1}{p})+(j-k)s} 2^{ks} \|f_k\|_{L^p(\mathbb{R})} \Big\|_{\ell^r_j(\mathbb{Z})} \\ &\qquad + \Big\| \sum_{k\in\mathbb{Z}} \chi_{<-3}(k-j) 2^{-|j-k|\frac{1}{p}+(j-k)s} 2^{ks} \|f_k\|_{L^p(\mathbb{R})} \Big\|_{\ell^r_j(\mathbb{Z})} \\ &\lesssim \Big\| 2^{ks} \|f_k\|_{L^p(\mathbb{R})} \Big\|_{\ell^r_k(\mathbb{Z})}, \end{aligned} \tag{3-13}$$

which implies (3-4). □

Corollary 3.5. *Assume that* $1 < p < \infty$, $1 \le r < \infty$ *and* $-\frac{1}{p'} < s < \frac{1}{p}$. *Then*

$$W_\pm^* \in \mathbb{B}(\dot{B}^s_{p,r}(\mathbb{R}), \dot{B}^s_{p,r}(\mathbb{R})).$$

Proof. For given $1 < p < \infty$, $1 \le r < \infty$ and $-\frac{1}{p'} < s < \frac{1}{p}$, it is easy to see that $1 < p' < \infty$, $1 < r' \le \infty$ and $-s \in \left(-\frac{1}{p}, \frac{1}{p'}\right) = \left(-\frac{1}{(p')'}, \frac{1}{p'}\right)$. Then for any $f \in \dot{B}^s_{p,r}(\mathbb{R})$ and $g \in \dot{B}^{-s}_{p',r'}(\mathbb{R})$, by Proposition 3.4, we have

$$\left|\langle W_\pm^* f, g\rangle\right| \lesssim \|f\|_{\dot{B}^s_{p,r}(\mathbb{R})} \|W_\pm g\|_{\dot{B}^{-s}_{p',r'}(\mathbb{R})} \lesssim \|f\|_{\dot{B}^s_{p,r}(\mathbb{R})} \|g\|_{\dot{B}^{-s}_{p',r'}(\mathbb{R})}, \tag{3-14}$$

which implies the desired result. □

Remark 3.6. We didn't prove the boundedness of $W_\pm^*$ at the endpoint $r = \infty$. This doesn't mean the result is not true for $r = \infty$. In fact, we can represent $W_\pm^*$ similarly to (3-3) in terms of the Fourier transform, and use the same argument used in the proof of Proposition 3.4 to obtain the boundedness at $r = \infty$. However, Corollary 3.5 is enough for further applications.

3.2. ***Proof of Theorem 3.1.*** In this section, we prove Theorem 3.1, the equivalence between $\dot{B}^s_{p,r}(\mathbb{R})$ and $\dot{B}^{s,H}_{p,r}(\mathbb{R})$ for $1 < p < \infty$, $1 \le r < \infty$ and $-\frac{1}{p'} < s < \frac{1}{p}$.

Step I. We prove the embedding $\dot{B}^s_{p,r}(\mathbb{R}) \hookrightarrow \dot{B}^{s,H}_{p,r}(\mathbb{R})$, where the boundedness of the wave operators on Lebesgue spaces and Besov spaces obtained in Section 3.1 will be involved. In fact, using (3-1), the identity (3-2), Proposition 3.2 and Corollary 3.5, we have

$$\begin{aligned}\|f\|^r_{\dot{B}^{s,H}_{p,r}(\mathbb{R})} &= \sum_{j\in\mathbb{Z}} 2^{jrs} \|\varphi_j(\sqrt{2H}) f\|^r_{L^p(\mathbb{R})} \\ &= \sum_{j\in\mathbb{Z}} 2^{jrs} \|W_+ \varphi_j(\sqrt{2H_0}) W_+^* f\|^r_{L^p(\mathbb{R})} \\ &\le \sum_{j\in\mathbb{Z}} 2^{jrs} \|\varphi_j(\sqrt{2H_0}) W_+^* f\|^r_{L^p(\mathbb{R})} = \|W_+^* f\|^r_{\dot{B}^s_{p,r}(\mathbb{R})} \lesssim \|f\|^r_{\dot{B}^s_{p,r}(\mathbb{R})},\end{aligned}$$

which implies the desired conclusion.

Step II. We prove the inverse embedding, $\dot{B}^{s,H}_{p,r}(\mathbb{R}) \hookrightarrow \dot{B}^s_{p,r}(\mathbb{R})$. Let φ be as defined in Section 2.1, let $\varphi_j(\xi) = \varphi(2^{-j}\xi)$, and let $\psi \in C_0^\infty(\mathbb{R})$ be an even function such that $\psi = 1$ on the support of φ and $0 \le \psi \le 1$. We write

$$f = \sum_{k\in\mathbb{Z}} \varphi_k(\sqrt{2H}) f = \sum_{k\in\mathbb{Z}} \tilde{f}_k,$$

and then if we set $f_k = \psi_k(\sqrt{2H}) f$, we further have

$$\begin{aligned}\varphi_j(\sqrt{2H_0}) f &= \sum_{k\in\mathbb{Z}} \varphi_j(\sqrt{2H_0}) \tilde{f}_k = \sum_{k\in\mathbb{Z}} \varphi_j(\sqrt{2H_0}) \psi_k(\sqrt{2H}) \tilde{f}_k \\ &= \sum_{k\in\mathbb{Z}} \varphi_j(\sqrt{2H_0}) \varphi_k(\sqrt{2H}) f_k.\end{aligned}$$

We next estimate $\varphi_j(\sqrt{2H_0})\varphi_k(\sqrt{2H})$ in $L^p(\mathbb{R})$. For given $j\in\mathbb{Z}$ and any k with $|j-k|\le 3$, using the boundedness of the Littlewood–Paley projection and Remark 3.3(ii), we have

$$\|\varphi_j(\sqrt{2H_0})\varphi_k(\sqrt{2H})f_k\|_{L^p(\mathbb{R})}\lesssim\|f_k\|_{L^p(\mathbb{R})}. \tag{3-15}$$

It suffices to consider the estimates for $|j-k|>3$. By applying Proposition 2.2 to $m(\lambda)=\varphi_k(\lambda)$, we have

$$\varphi_j(\sqrt{2H_0})\varphi_k(\sqrt{2H})f_k(x)=\mathcal{F}^{-1}\big(\varphi_j\mathcal{F}(\varphi_k(\sqrt{2H})f_k)\big)(x)=\sum_{\ell=1}^{5}J_{\ell,j,k}, \tag{3-16}$$

where

$$\begin{aligned}
J_{1,j,k}&:=\mathcal{F}^{-1}(\varphi_j\varphi_k\mathcal{F}(f_k))(x),\\
J_{2,j,k}&:=\mathcal{F}^{-1}\big[\varphi_j\mathcal{F}\big(\chi_+\mathcal{F}^{-1}(\varphi_k r_q\mathcal{F}(\chi_-f_k))\big)\big](x),\\
J_{3,j,k}&:=\mathcal{F}^{-1}\big[\varphi_j\mathcal{F}\big(\chi_+\mathcal{F}^{-1}(\varphi_k r_q\mathcal{F}(\chi_+f_k)(-\cdot))\big)\big](x),\\
J_{4,j,k}&:=\mathcal{F}^{-1}\big[\varphi_j\mathcal{F}\big(\chi_-\mathcal{F}^{-1}(\varphi_k \bar r_q\mathcal{F}(\chi_+f_k))\big)\big](x),\\
J_{5,j,k}&:=\mathcal{F}^{-1}\big[\varphi_j\mathcal{F}\big(\chi_-\mathcal{F}^{-1}(\varphi_k \bar r_q\mathcal{F}(\chi_-f_k)(-\cdot))\big)\big](x).
\end{aligned}$$

Notice that $J_{1,j,k}$ vanishes, since $\varphi_j(\xi)\varphi_k(\xi)=0$ when $|k-j|>3$, and the other four summands $J_{\ell,j,k}$ ($\ell\neq 1$) are of the same type, so we only estimate $J_{2,j,k}$. We write

$$\begin{aligned}
&\mathcal{F}\big(\chi_+\mathcal{F}^{-1}(\varphi_k r_q\mathcal{F}(\chi_-f_k))\big)(\xi)\\
&=\mathcal{F}\big(\tfrac{1}{2}(1+\operatorname{sgn}\cdot)\mathcal{F}^{-1}(r_q\varphi_k\mathcal{F}(\chi_-f_k))\big)(\xi)\\
&=\tfrac12 r_q(\xi)\varphi_k(\xi)\mathcal{F}(\chi_-f_k)(\xi)+\frac{1}{2\sqrt{2\pi}}\mathcal{F}(\operatorname{sgn}\cdot)*(r_q\varphi_k\mathcal{F}(\chi_-f_k))(\xi)\\
&=\tfrac12 r_q(\xi)\varphi_k(\xi)\mathcal{F}(\chi_-f_k)(\xi)+\frac{1}{2\pi}\frac{1}{i\cdot}*(r_q\varphi_k\mathcal{F}(\chi_-f_k))(\xi)\\
&=\tfrac12 r_q(\xi)\varphi_k(\xi)\mathcal{F}(\chi_-f_k)(\xi)+i(2\pi)^{-\frac32}\\
&\qquad\qquad\times\int\frac{1}{y-\xi}r_q(y)\varphi_k(y)\int e^{-izy}\chi_-(z)f_k(z)\,dz\,dy,
\end{aligned} \tag{3-17}$$

which implies that

$$\begin{aligned}
J_{2,j,k}&=\tfrac12\mathcal{F}^{-1}\big(\varphi_j(\xi)\varphi_k(\xi)r_q(\xi)\mathcal{F}(\chi_-f_k)(\xi)\big)(x)+\int\widetilde{\mathcal{K}}_{j,k}(x,z)f_k(z)\,dz\\
&=:J_{21,j,k}+J_{22,j,k},
\end{aligned}$$

with the kernel $\widetilde{\mathcal{K}}_{j,k}(x,z)$ given by

$$\widetilde{\mathcal{K}}_{j,k}(x,z)=i(2\pi)^{-2}\chi_-(z)\iint e^{ix\xi}e^{-izy}\varphi_j(\xi)\varphi_k(y)\frac{1}{y-\xi}r_q(y)\,dy\,d\xi.$$

Similarly to the proof of Proposition 3.4, by changing variables and integrating by parts, we have

$$\begin{aligned}\big|\widetilde{\mathcal{K}}_{j,k}(x,z)\big| &\lesssim \frac{2^j 2^k}{\langle 2^j x\rangle^2 \langle 2^k z\rangle^2} \iint \left|\partial_y^2 \partial_\xi^2 \frac{\varphi(\xi)\varphi(y)}{2^k y - 2^j \xi}\right| dy\, d\xi \\ &\lesssim \frac{2^j 2^k}{\langle 2^j x\rangle^2 \langle 2^k z\rangle^2} \frac{1}{\max(2^k, 2^j)},\end{aligned}$$

which leads to the estimates

$$\|J_{22,j,k}\|_{L^p(\mathbb{R})} \le \left\| \int \widetilde{\mathcal{K}}_{j,k}(x,z) f_k(z)\, dz \right\|_{L^p(\mathbb{R})} \lesssim 2^{-|j-k|\frac{1}{p}} \|f_k\|_{L^p(\mathbb{R})} \quad (k<j-3),$$

$$\|J_{22,j,k}\|_{L^p(\mathbb{R})} \le \left\| \int \widetilde{\mathcal{K}}_{j,k}(x,z) f_k(z)\, dz \right\|_{L^p(\mathbb{R})} \lesssim 2^{-|k-j|(1-\frac{1}{p})} \|f_k\|_{L^p(\mathbb{R})} \quad (k>j-3).$$

Notice that $J_{21,j,k}$ vanishes, since $\varphi_j(\xi)\varphi_k(\xi) = 0$ when $|k-j| > 3$. With the same meaning for the χ's as in (3-12), we have

$$\|J_{2,j,k}\|_{L^p(\mathbb{R})} \lesssim \big(\chi_{<-3}(k-j) 2^{-|j-k|\frac{1}{p}} + \chi_{>3}(k-j) 2^{-|k-j|(1-\frac{1}{p})}\big) \|f_k\|_{L^p(\mathbb{R})}. \tag{3-18}$$

The same estimate holds for $J_{\ell,j,k}$ with $\ell = 3, 4, 5$. Now combining (3-15)–(3-18) with the inequalities

$$\begin{aligned}\big\|\chi_{>3}(n) 2^{-n(1-\frac{1}{p})-ns}\big\|_{\ell^1_n(\mathbb{Z})} &\lesssim 1, \quad s > -(1-\tfrac{1}{p}) = -\tfrac{1}{p'}, \\ \big\|\chi_{<-3}(n) 2^{-\frac{|n|}{p}-ns}\big\|_{\ell^1_n(\mathbb{Z})} &\lesssim 1, \quad s < \tfrac{1}{p},\end{aligned}$$

and Young's inequality, we obtain

$$\begin{aligned}\|f\|_{\dot{B}^s_{p,r}(\mathbb{R})} &= \big\|2^{js} \|\varphi_j(\sqrt{2H_0}) f\|_{L^p(\mathbb{R})}\big\|_{\ell^r_j(\mathbb{Z})} \\ &\lesssim \Big\|2^{js} \sum_{k\in\mathbb{Z}} \|\varphi_j(\sqrt{2H_0})\varphi_k(\sqrt{2H}) f_k\|_{L^p(\mathbb{R})}\Big\|_{\ell^r_j(\mathbb{Z})} \\ &\lesssim \Big\|2^{js} \sum_{|k-j|\le 3} \|f_k\|_{L^p}\Big\|_{\ell^r_j(\mathbb{Z})} \\ &\quad + \Big\|\sum_{k\in\mathbb{Z}} \chi_{>3}(k-j) 2^{-|j-k|(1-\frac{1}{p})+(j-k)s} 2^{ks} \|f_k\|_{L^p(\mathbb{R})}\Big\|_{\ell^r_j(\mathbb{Z})} \\ &\quad + \Big\|\sum_{k\in\mathbb{Z}} \chi_{<-3}(k-j) 2^{-|j-k|\frac{1}{p}+(j-k)s} 2^{ks} \|f_k\|_{L^p(\mathbb{R})}\Big\|_{\ell^r_j(\mathbb{Z})} \\ &\lesssim \big\|2^{ks} \|f_k\|_{L^p(\mathbb{R})}\big\|_{\ell^r_k(\mathbb{Z})} = \|f\|_{\dot{B}^{s,H}_{p,r}(\mathbb{R})}.\end{aligned}$$

This concludes the proof of Theorem 3.1.

4. Proofs of Theorems of 1.1 and 1.3

We recall the integral equation

$$v(t) = u_0 - i\lambda \int_0^t \mathcal{U}(-s)\big(\mathcal{U}(s)v(s)\,\overline{\mathcal{U}(s)v(s)}\,\mathcal{U}(s)v(s)\big)\,ds,$$

where $v(t) = \mathcal{U}(-t)u(t)$, $u(t)$ is the solution to the original nonlinear Schrödinger equation (1-1) and $\mathcal{U}(t) = e^{-itH}$ is the linear propagator with $H = -\frac{1}{2}\partial_x^2 + q\delta_0$. In this section, we will prove Theorems 1.1 and 1.3, the local well-posedness of $v(t)$ in L^p-based spaces.

We start with the estimate for trilinear form

$$\mathcal{T}(v_1(\tau), v_2(\tau), v_3(\tau)) = \mathcal{U}(-\tau)\big(\mathcal{U}(-\tau)v_1(\tau)\mathcal{U}(\tau)v_2(\tau)\mathcal{U}(\tau)v_3(\tau)\big). \tag{4-1}$$

We will exploit the cancellation of $\mathcal{T}$, which is analogous to (1-6), with $\mathcal{U}(t)$ replaced by $e^{it\Delta}$. Set

$$M(t)f(x) = e^{i\frac{|x|^2}{2t}}f(x) \quad \text{and} \quad Jf(x) = f(-x).$$

Lemma 4.1. *Let the trilinear form $\mathcal{T}$ be defined by (4-1). We have, up to a constant,*

$$\begin{aligned}\mathcal{T}(v_1(t), v_2(t), v_3(t)) = t^{-1}\textstyle\sum_{\ell=1}^2\big(&\mathcal{Q}_\ell^{(2)}(v_1(t), v_2(t), v_3(t))\\ &+ \mathcal{Q}_{\ell+2}^{(1)}(v_1(t), v_2(t), v_3(t))\big)\\ &+ t^{-1}\chi_+\mathcal{F}^{-1}(r_q) * \textstyle\sum_{\ell=1}^4 \mathcal{Q}_\ell^{(1)}(v_1(t), v_2(t), v_3(t))\\ &+ t^{-1}\chi_-\mathcal{F}^{-1}(\bar{r}_q) * \textstyle\sum_{\ell=1}^4 \mathcal{Q}_\ell^{(2)}(v_1(t), v_2(t), v_3(t)),\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{Q}_1^{(1)}(v_1(t), v_2(t), v_3(t))\\
&\quad = M(t)\big\{(\overline{M(t)}\mathcal{L}_+(v_1(t))) * (JM(t)\mathcal{L}_+\big(v_2(t))) * (JM(t)\mathcal{L}_+(v_3(t)))\big\},\\
&\mathcal{Q}_2^{(1)}(v_1(t), v_2(t), v_3(t))\\
&\quad = M(t)\big\{(\mathcal{H}\overline{M(t)}\mathcal{L}_+(v_1(t))) * (JM(t)\mathcal{L}_+\big(v_2(t))) * (JM(t)\mathcal{L}_+(v_3(t)))\big\},\\
&\mathcal{Q}_3^{(1)}(v_1(t), v_2(t), v_3(t))\\
&\quad = M(t)\big\{(J\overline{M(t)}\mathcal{L}_-(v_1(t))) * (M(t)\mathcal{L}_-\big(v_2(t))) * (M(t)\mathcal{L}_-(v_3(t)))\big\},\\
&\mathcal{Q}_4^{(1)}(v_1(t), v_2(t), v_3(t))\\
&\quad = M(t)\big\{(J\mathcal{H}\overline{M(t)}\mathcal{L}_-(v_1(t))) * (M(t)\mathcal{L}_-\big(v_2(t))) * (M(t)\mathcal{L}_-(v_3(t)))\big\},\\
&\mathcal{Q}_1^{(2)}(v_1(t), v_2(t), v_3(t))\\
&\quad = M(t)\big\{(J\overline{M(t)}\mathcal{L}_+(v_1(t))) * (M(t)\mathcal{L}_+\big(v_2(t))) * (M(t)\mathcal{L}_+(v_3(t)))\big\},\\
&\mathcal{Q}_2^{(2)}(v_1(t), v_2(t), v_3(t))\\
&\quad = M(t)\big\{(J\mathcal{H}\overline{M(t)}\mathcal{L}_+(v_1(t))) * (M(t)\mathcal{L}_+\big(v_2(t))) * (M(t)\mathcal{L}_+(v_3(t)))\big\},
\end{aligned}$$

$$\begin{aligned}
&\mathcal{Q}_3^{(2)}(v_1(t), v_2(t), v_3(t)) \\
&\qquad = M(t)\big\{(\overline{M(t)}\mathcal{L}_-(v_1(t))) * (JM(t)\mathcal{L}_-\big(v_2(t))) * (JM(t)\mathcal{L}_-(v_3(t)))\big\}, \\
&\mathcal{Q}_4^{(2)}(v_1(t), v_2(t), v_3(t)) \\
&\qquad = M(t)\big\{(\mathcal{H}\overline{M(t)}\mathcal{L}_-(v_1(t))) * (JM(t)\mathcal{L}_-\big(v_2(t))) * (JM(t)\mathcal{L}_-(v_3(t)))\big\}.
\end{aligned}$$

Proof. Set

$$v(t) = \mathcal{U}(-t)v_1(t)\mathcal{U}(t)v_2(t)\mathcal{U}(t)v_3(t).$$

It follows from Corollary 2.3 that

$$\begin{aligned}
\text{(4-2)}\quad &\mathcal{T}\big((v_1)(t), v_2(t), v_3(t)\big)(x) \\
&= \big(\mathcal{U}(-t)v(t)\big)(x) \\
&= \chi_+(x)\overline{K}_t * \mathcal{L}_+(v(t))(x) + \chi_-(x)\overline{K}_t * \mathcal{L}_-(v(t))(x) \\
&=: \chi_+(x)\mathcal{T}^+(t)(x) + \chi_-(x)\mathcal{T}^-(t)(x),
\end{aligned}$$

where $K_t(x) = e^{-\frac{i\pi}{4}}(2\pi t)^{-1/2}e^{\frac{i|x|^2}{2t}}$ and

$$\begin{aligned}
\mathcal{L}_+v(t) &= v(t) + \mathcal{F}^{-1}\big(r_q J\mathcal{F}\big(\chi_+v(t)\big)\big) + \mathcal{F}^{-1}\big(r_q\mathcal{F}\big(\chi_-v(t)\big)\big), \\
\mathcal{L}_-v(t) &= v(t) + \mathcal{F}^{-1}\big(\overline{r_q} J\mathcal{F}\big(\chi_-v(t)\big)\big) + \mathcal{F}^{-1}\big(\overline{r_q}\mathcal{F}\big(\chi_+v(t)\big)\big).
\end{aligned}$$

In the following, we will omit absolute constants in some identities.

Let us turn to the expansion of $v(t)$. By using Corollary 2.3 again, we have

$$\begin{aligned}
\text{(4-3)}\quad &v(t)(y) \\
&= \big(\mathcal{U}(-t)v_1(t)\big)(y)\big(\mathcal{U}(t)v_2(t)\big)(y)\big(\mathcal{U}(t)v_3(t)\big)(y) \\
&= \chi_+(y)\overline{K}_t * \mathcal{L}_+(v_1(t))(y)K_t * \mathcal{L}_+(v_2(t))(y)K_t * \mathcal{L}_+(v_3(t))(y) \\
&\qquad + \chi_-(y)\overline{K}_t * \mathcal{L}_-(v_1(t))(y)K_t * \mathcal{L}_-(v_2(t))(y)K_t * \mathcal{L}_-(v_3(t))(y) \\
&=: \chi_+(y)v_+(t)(y) + \chi_-(y)v_-(t)(y).
\end{aligned}$$

Then it follows that

$$\begin{aligned}
\text{(4-4)}\quad v_+(t)(y) &= \iiint \overline{K}_t(y-\alpha)\mathcal{L}_+(v_1(t))(\alpha)K_t(y-\beta)\mathcal{L}_+(v_2(t))(\beta) \\
&\qquad\qquad \times K_t(y-\gamma)\mathcal{L}_+(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma \\
&= t^{-\frac{3}{2}}\iiint e^{\frac{-i(|y-\alpha|^2-|y-\beta|^2-|y-\gamma|^2)}{2t}} \\
&\qquad \times \mathcal{L}_+(v_1(t))(\alpha)\mathcal{L}_+(v_2(t))(\beta)\mathcal{L}_+(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma,
\end{aligned}$$

$$\begin{aligned}
\text{(4-5)}\quad v_-(t)(y) &= t^{-\frac{3}{2}}\iiint e^{\frac{-i(|y-\alpha|^2-|y-\beta|^2-|y-\gamma|^2)}{2t}} \\
&\qquad \times \mathcal{L}_-(v_1(t))(\alpha)\mathcal{L}_-(v_2(t))(\beta)\mathcal{L}_-(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma.
\end{aligned}$$

Now by (4-2), we have

$$\begin{aligned}\mathcal{T}^{+}(t)(x) &= \overline{K}_t * \big(v(t) + \mathcal{F}^{-1}\big(r_q J\mathcal{F}(\chi_+ v(t))\big) + \mathcal{F}^{-1}\big(r_q \mathcal{F}(\chi_- v(t))\big)\big)(x)\\ &=: \mathcal{T}_1^{+}(t)(x) + \mathcal{T}_2^{+}(t)(x) + \mathcal{T}_3^{+}(t)(x).\end{aligned}$$

Set $\tilde{e}_1(t,x,y,\alpha,\beta,\gamma) = e^{\frac{-i(|x-y|^2+|y-\alpha|^2-|y-\beta|^2-|y-\gamma|^2)}{2t}}$. It is easy to see that

$$|x-y|^2+|y-\alpha|^2-|y-\beta|^2-|y-\gamma|^2 = (x^2+\alpha^2-\beta^2-\gamma^2)+2y(\beta+\gamma-\alpha-x),$$

$$\int \tilde{e}_1(t,x,y,\alpha,\beta,\gamma)\chi_+(y)\,dy = te^{-i\frac{(x^2+\alpha^2-\gamma^2-\beta^2)}{2t}}\Big(\sqrt{\tfrac{\pi}{2}}\delta_0(\beta+\gamma-\alpha-x) + \tfrac{1}{i\sqrt{2\pi}}\tfrac{1}{\beta+\gamma-\alpha-x}\Big),$$

$$\int \tilde{e}_1(t,x,y,\alpha,\beta,\gamma)\chi_-(y)\,dy = te^{-i\frac{(x^2+\alpha^2-\gamma^2-\beta^2)}{2t}}\Big(\sqrt{\tfrac{\pi}{2}}\delta_0(\beta+\gamma-\alpha-x) - \tfrac{1}{i\sqrt{2\pi}}\tfrac{1}{\beta+\gamma-\alpha-x}\Big),$$

which combined with (4-4) and (4-5) imply that

$$\begin{aligned}\text{(4-6)}\quad &\overline{K}_t * \big(\chi_+ v_+(t)\big)(x)\\ &= t^{-2}\iiiint \tilde{e}_1(t,x,y,\alpha,\beta,\gamma)\\ &\qquad\times\chi_+(y)\mathcal{L}_+(v_1(t))(\alpha)\mathcal{L}_+(v_2(t))(\beta)\mathcal{L}_+(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma\,dy\\ &= t^{-1}e^{-i\frac{x^2}{2t}}\iiint\Big(\delta_0(\beta+\gamma-\alpha-x)+\tfrac{1}{\beta+\gamma-\alpha-x}\Big)\overline{M(t)}\mathcal{L}_+(v_1(t))(\alpha)\\ &\qquad\times M(t)\mathcal{L}_+(v_2(t))(\beta)M(t)\mathcal{L}_+(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma\\ &= t^{-1}M(t)\big\{\big(J\overline{M(t)}\mathcal{L}_+(v_1(t))\big)*\big(M(t)\mathcal{L}_+\big(v_2(t)\big)\big)*\big(M(t)\mathcal{L}_+(v_3(t))\big)\big\}(x)\\ &\quad+t^{-1}M(t)\big\{\big(J\mathcal{H}\overline{M(t)}\mathcal{L}_+(v_1(t))\big)*\big(M(t)\mathcal{L}_+(v_2(t))\big)*\big(M(t)\mathcal{L}_+(v_3(t))\big)\big\}(x)\end{aligned}$$

and

$$\begin{aligned}&\overline{K}_t * \big(\chi_- v_-(t)\big)(x)\\ &= t^{-2}\iiiint \tilde{e}_1(t,x,y,\alpha,\beta,\gamma)\\ &\qquad\times\chi_-(y)\mathcal{L}_-(v_1(t))(\alpha)\mathcal{L}_-(v_2(t))(\beta)\mathcal{L}_-(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma\,dy\\ &= t^{-1}e^{-i\frac{x^2}{2t}}\iiint\Big(\delta_0(\beta+\gamma-\alpha-x)-\tfrac{1}{\beta+\gamma-\alpha-x}\Big)\overline{M(t)}\mathcal{L}_-(v_1(t))(\alpha)\\ &\qquad\times M(t)\mathcal{L}_-(v_2(t))(\beta)M(t)\mathcal{L}_-(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma\\ &= t^{-1}M(t)\big\{\big(J\overline{M(t)}\mathcal{L}_-(v_1(t))\big)*\big(M(t)\mathcal{L}_-\big(v_2(t)\big)\big)*\big(M(t)\mathcal{L}_-(v_3(t))\big)\big\}(x)\\ &\quad-t^{-1}M(t)\big\{\big(J\mathcal{H}\overline{M(t)}\mathcal{L}_-(v_1(t))\big)*\big(M(t)\mathcal{L}_-(v_2(t))\big)*\big(M(t)\mathcal{L}_-(v_3(t))\big)\big\}(x).\end{aligned}$$

Combining these formulas with (4-3), we further have

$$(4\text{-}7)\quad \begin{aligned}\mathcal{T}_1^+(t)(x) &= \overline{K}_t * (\chi_+ v_+(t))(x) + \overline{K}_t * (\chi_- v_-(t))(x)\\ &=: t^{-1} \sum_{\ell=1}^{2} \mathcal{Q}_\ell^{(2)}(v_1(t), v_2(t), v_3(t)) + t^{-1} \sum_{\ell=3}^{4} \mathcal{Q}_\ell^{(1)}(v_1(t), v_2(t), v_3(t)).\end{aligned}$$

Let us turn to $T_2^+(t)$. Set $\tilde{e}_2(t,z,y,\alpha,\beta,\gamma) = e^{\frac{-i(|z+y|^2+|y-\alpha|^2-|y-\beta|^2-|y-\gamma|^2)}{2t}}$. Similarly as above, we have

$$|z+y|^2+|y-\alpha|^2-|y-\beta|^2-|y-\gamma|^2 = (z^2+\alpha^2-\beta^2-\gamma^2)+2y(\beta+\gamma+z-\alpha),$$

$$\int \tilde{e}_2(t,z,y,\alpha,\beta,\gamma)\chi_+(y)\,dy = te^{-i\frac{(z^2+\alpha^2-\gamma^2-\beta^2)}{2t}}\Big(\sqrt{\tfrac{\pi}{2}}\delta_0(\beta+\gamma+z-\alpha) + \frac{1}{i\sqrt{2\pi}}\frac{1}{\beta+\gamma+z-\alpha}\Big),$$

which combined with (4-4) and (4-5) imply that

$$\begin{aligned}&\overline{K}_t * \big(J\chi_+ v_+(t)\big)(z)\\ &= t^{-2}\iiiint \tilde{e}_2(t,z,y,\alpha,\beta,\gamma)\\ &\qquad\qquad \times\chi_+(y)\mathcal{L}_+(v_1(t))(\alpha)\mathcal{L}_+(v_2(t))(\beta)\mathcal{L}_+(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma\,dy\\ &= t^{-1}e^{-i\frac{z^2}{2t}}\iiint \Big(\delta_0(\beta+\gamma+z-\alpha)+\frac{1}{\beta+\gamma+z-\alpha}\Big)\overline{M(t)}\mathcal{L}_+(v_1(t))(\alpha)\\ &\qquad\qquad \times M(t)\mathcal{L}_+(v_2(t))(\beta)M(t)\mathcal{L}_+(v_3(t))(\gamma)\,d\alpha\,d\beta\,d\gamma\\ &= t^{-1}M(t)\big\{\big(\overline{M(t)}\mathcal{L}_+(v_1(t))\big)*\big(JM(t)\mathcal{L}_+(v_2(t))\big)*\big(JM(t)\mathcal{L}_+(v_3(t))\big)\big\}(z)\\ &\quad + t^{-1}M(t)\big\{\big(\mathcal{H}\overline{M(t)}\mathcal{L}_+(v_1(t))\big)*\big(JM(t)\mathcal{L}_+(v_2(t))\big)*\big(JM(t)\mathcal{L}_+(v_3(t))\big)\big\}(z).\end{aligned}$$

Combining this with (4-3), we have

$$(4\text{-}8)\quad \begin{aligned}\mathcal{T}_2^+(t)(x) &= \mathcal{F}^{-1}(r_q) * \overline{K}_t * \big(J\chi_+ v_+(t)\big)(x)\\ &= t^{-1}\mathcal{F}^{-1}(r_q) * \sum_{\ell=1}^{2} \mathcal{Q}_\ell^{(1)}(v_1(t), v_2(t), v_3(t)).\end{aligned}$$

The expansion for $\mathcal{T}_3^+(t)$ follows from the expansion of $\mathcal{T}_1^+(t)$. Indeed, we have

$$(4\text{-}9)\quad \begin{aligned}\mathcal{T}_3^+(t)(x) &= \mathcal{F}^{-1}(r_q) * \overline{K}_t * \big(\chi_- v_-(t)\big)(x)\\ &= t^{-1}\mathcal{F}^{-1}(r_q) * \sum_{\ell=3}^{4} \mathcal{Q}_\ell^{(1)}(v_1(t), v_2(t), v_3(t)).\end{aligned}$$

Turning to $\mathcal{T}^-(t)$, notice that

$$\begin{aligned}\mathcal{T}^-(t)(x) &= \overline{K}_t * \big(v(t) + \mathcal{F}^{-1}\big(\bar{r}_q \mathcal{F}(\chi_+ v(t))\big) + \mathcal{F}^{-1}\big(\bar{r}_q \mathcal{F}(J\chi_- v(t))\big)\big)(x)\\ &=: \mathcal{T}_1^-(t)(x) + \mathcal{T}_2^-(t)(x) + \mathcal{T}_3^-(t)(x).\end{aligned}$$

Obviously, $\mathcal{T}_1^-(t)(x) = \mathcal{T}_1^+(t)(x)$. From (4-6) and the identity

$$\begin{aligned}&\overline{K}_t * \big(J\chi_- v_-(t)\big)(z)\\ &= t^{-2} \iiiint \tilde{e}_2(t, z, y, \alpha, \beta, \gamma)\\ &\qquad\qquad \times \chi_-(y)\mathcal{L}_-(v_1(t))(\alpha)\mathcal{L}_-(v_2(t))(\beta)\mathcal{L}_-(v_3(t))(\gamma)\, d\alpha\, d\beta\, d\gamma\, dy\\ &= t^{-1} e^{-i\frac{z^2}{2t}} \iiint \Big(\delta_0(\beta+\gamma+z-\alpha) - \tfrac{1}{\beta+\gamma+z-\alpha}\Big)\overline{M(t)}\mathcal{L}_-(v_1(t))(\alpha)\\ &\qquad\qquad \times M(t)\mathcal{L}_-(v_2(t))(\beta)M(t)\mathcal{L}_-(v_3(t))(\gamma)\, d\alpha\, d\beta\, d\gamma\\ &= t^{-1} M(t)\big\{\big(\overline{M(t)}\mathcal{L}_-(v_1(t))\big) * \big(JM(t)\mathcal{L}_-(v_2(t))\big) * \big(JM(t)\mathcal{L}_-(v_3(t))\big)\big\}(z)\\ &\quad - t^{-1} M(t)\big\{\big(\mathcal{H}\overline{M(t)}\mathcal{L}_-(v_1(t))\big) * \big(JM(t)\mathcal{L}_-(v_2(t))\big) * \big(JM(t)\mathcal{L}_-(v_3(t))\big)\big\}(z)\end{aligned}$$

it follows that

$$\begin{aligned}\mathcal{T}_2^-(t)(x) &= \mathcal{F}^{-1}(\bar{r}_q) * \overline{K}_t * \big(\chi_+ v_+(t)\big)(x)\\ &= t^{-1}\mathcal{F}^{-1}(\bar{r}_q) * \textstyle\sum_{\ell=1}^2 \mathcal{Q}_\ell^{(2)}(v_1(t), v_2(t), v_3(t)),\end{aligned}\tag{4-10}$$

$$\begin{aligned}\mathcal{T}_3^-(t)(x) &= \mathcal{F}^{-1}(\bar{r}_q) * \overline{K}_t * \big(J\chi_- v_-(t)\big)(x)\\ &= t^{-1}\mathcal{F}^{-1}(\bar{r}_q) * \textstyle\sum_{\ell=3}^4 \mathcal{Q}_\ell^{(2)}(v_1(t), v_2(t), v_3(t)).\end{aligned}\tag{4-11}$$

Hence, by (4-2) and (4-8)–(4-11), we have

$$\begin{aligned}&\mathcal{T}(v_1(t), v_2(t), v_3(t))\\ &\quad =: \chi_+\mathcal{T}^+(t) + \chi_-\mathcal{T}^-(t)\\ &\quad = \mathcal{T}_1^+(t) + \chi_+\big(\mathcal{T}_2^+(t) + \mathcal{T}_3^+(t)\big) + \chi_-\big(\mathcal{T}_2^-(t) + \mathcal{T}_3^-(t)\big)\\ &\quad = t^{-1}\textstyle\sum_{\ell=1}^2\big(\mathcal{Q}_\ell^{(2)}(v_1(t), v_2(t), v_3(t)) + \mathcal{Q}_{\ell+2}^{(1)}(v_1(t), v_2(t), v_3(t))\big)\\ &\qquad\qquad + t^{-1}\chi_+\mathcal{F}^{-1}(r_q) * \textstyle\sum_{\ell=1}^4 \mathcal{Q}_\ell^{(1)}(v_1(t), v_2(t), v_3(t))\\ &\qquad\qquad + t^{-1}\chi_-\mathcal{F}^{-1}(\bar{r}_q) * \textstyle\sum_{\ell=1}^4 \mathcal{Q}_\ell^{(2)}(v_1(t), v_2(t), v_3(t)),\end{aligned}$$

which finishes the proof. □

4.1. ***Proof of Theorem 1.1.*** We next prove Theorem 1.1 and Corollary 1.2, the well-posedness of the integral equation (1-8) in homogeneous Besov spaces $\dot{B}^s_{p,1}(\mathbb{R})$. The following lemma will be used to deal with the nonlinear term in (1-8).

Lemma 4.2. *Let $1 < p \le 2$ and $\mathcal{T}$ be the trilinear form defined by (4-1). Assume that $v_2(t) = \varphi_k(\sqrt{2H})v_2(t)$ and $v_3(t) = \varphi_j(\sqrt{2H})v_3(t)$. Then we have*

$$\begin{aligned}&\big\|\mathcal{T}(v_1(t), v_2(t), v_3(t))\big\|_{L^p(\mathbb{R})}\\ &\qquad\qquad \lesssim t^{1-\frac{2}{p}} 2^{(1-\frac{1}{p})\frac{j+k}{2}} \|v_1(t)\|_{L^p(\mathbb{R})} \|v_2(t)\|_{L^p(\mathbb{R})} \|v_3(t)\|_{L^p(\mathbb{R})}.\end{aligned}$$

Proof. We use interpolation. Consider the estimate of $\mathcal{T}$ for $p = 1$. Notice that $\mathcal{F}^{-1}(r_q)(x) = -\sqrt{2\pi} q \chi_-(x) e^{qx} \in L^1(\mathbb{R})$, $\mathcal{L}_\pm$ are bounded on $L^p(\mathbb{R})$ for all $1 \le p \le \infty$ and $H^1(\mathbb{R}) \hookrightarrow L^1(\mathbb{R})$. It follows from Lemma 4.1 and Young's inequality that

$$\begin{aligned}&\big\|\mathcal{T}(v_1(t), v_2(t), v_3(t))\big\|_{L^1(\mathbb{R})}\\ &\lesssim t^{-1} \sum_{\ell=1}^{4} \Big(\big\|\mathcal{Q}_\ell^{(1)}(v_1(t), v_2(t), v_3(t))\big\|_{L^1(\mathbb{R})} + \big\|\mathcal{Q}_\ell^{(2)}(v_1(t), v_2(t), v_3(t))\big\|_{L^1(\mathbb{R})} \Big)\\ &\lesssim t^{-1} \big(\|\mathcal{L} v_1(t)\|_{L^1(\mathbb{R})} + \|\mathcal{H} M(t) \mathcal{L} v_1(t)\|_{L^1(\mathbb{R})} \big) \|v_2(t)\|_{L^1(\mathbb{R})} \|v_3(t)\|_{L^1(\mathbb{R})}\\ &\lesssim t^{-1} \|M(t)\mathcal{L} v_1(t)\|_{H^1(\mathbb{R})} \|v_2(t)\|_{L^1(\mathbb{R})} \|v_3(t)\|_{L^1(\mathbb{R})},\end{aligned}$$

where we write $\mathcal{L} = (\mathcal{L}_-, \mathcal{L}_+)$ and $\|\mathcal{L} f\|_{L^p(\mathbb{R})} = \|\mathcal{L}_+ f\|_{L^p(\mathbb{R})} + \|\mathcal{L}_- f\|_{L^p(\mathbb{R})}$.

On the other hand, it follows from Corollary 2.3 that

$$\begin{aligned}\|\mathcal{U}(-t) v_1(t)\|_{L^2(\mathbb{R})} &= \|\chi_+ \overline{K}_t * \mathcal{L}_+(v_1(t)) + \chi_- \overline{K}_t * \mathcal{L}_-(v_1(t))\|_{L^2(\mathbb{R})}\\ &\le \|M(t)\mathcal{L} v_1(t)\|_{L^2(\mathbb{R})}.\end{aligned} \tag{4-12}$$

By Lemma 2.1 and Hölder's inequality, we have

$$\|\mathcal{U}(t) v_2(t)\|_{L^\infty(\mathbb{R})} \lesssim \|\mathcal{F}_q(\mathcal{U}(t) v_2(t))\|_{L^1(\mathbb{R})} \lesssim 2^{\frac{k}{2}} \|v_2(t)\|_{L^2(\mathbb{R})},$$

which combined with (4-12) leads to

$$\begin{aligned}&\|\mathcal{T}(v_1(t), v_2(t), v_3(t))\|_{L^2(\mathbb{R})}\\ &\qquad = \|\mathcal{U}(-t)\big(\mathcal{U}(-t) v_1(t) \mathcal{U}(t) v_2(t) \mathcal{U}(t) v_3(t)\big)\|_{L^2(\mathbb{R})}\\ &\qquad \lesssim \|\mathcal{U}(-t) v_1(t)\|_{L^2(\mathbb{R})} \|\mathcal{U}(t) v_2(t)\|_{L^\infty(\mathbb{R})} \|\mathcal{U}(t) v_3(t)\|_{L^\infty(\mathbb{R})}\\ &\qquad \lesssim 2^{\frac{j+k}{2}} \|M(t)\mathcal{L} v_1(t)\|_{L^2(\mathbb{R})} \|v_2(t)\|_{L^2(\mathbb{R})} \|v_3(t)\|_{L^2(\mathbb{R})}.\end{aligned}$$

Now we achieve the result by using multilinear interpolation and the fact that both $M(t)$ and $\mathcal{L}$ are bounded on $L^p(\mathbb{R})$ for all $1 \le p \le \infty$. □

Proof of Theorem 1.1. Let us recall the integral equation

$$v(t) = u_0 - i\lambda \int_0^t \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}(s)(\mathcal{U}(s)v(s))^2\big)\, ds.$$

For fixed $1 < p < 2$, we define the space

$$X_T = \Big\{ v \in C\big([0, T], \dot{B}^\sigma_{p,1}(\mathbb{R})\big) \;\Big|\; \sup_{0 \le t \le T} \|v(t)\|_{\dot{B}^{s,H}_{p,1}(\mathbb{R})} \le 2\|u_0\|_{\dot{B}^s_{p,1}(\mathbb{R})} \Big\},$$

with $\sigma = 1 - \frac{1}{p}$, and the map

$$Tv(t) = u_0 - i\lambda \int_0^t \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}(s)(\mathcal{U}(s)v(s))^2\big)\, ds.$$

We show that T maps X_T to itself and is a contraction.

• Given $v \in X_T$, we aim to prove that $Tv \in X_T$. By Theorem 3.1, we know that $\dot{B}^{s,H}_{p,1}(\mathbb{R}) = \dot{B}^{s}_{p,1}(\mathbb{R})$ for $1 < p < \infty$ and $-\frac{1}{p'} < s < \frac{1}{p}$. In the proof of Theorem 1.1, for $1 < p < 2$ and $\sigma = 1 - \frac{1}{p}$, one has $0 < \sigma < \frac{1}{p}$ and thus $\dot{B}^{\sigma,H}_{p,1}(\mathbb{R}) = \dot{B}^{\sigma}_{p,1}(\mathbb{R})$, which means that we can use $\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})$ instead of $\dot{B}^{\sigma}_{p,1}(\mathbb{R})$ when dealing with the nonlinear term.

We make the decomposition

$$\sum_{j\in\mathbb{Z}} \varphi_j(\sqrt{2H})v(s) = \sum_{j\in\mathbb{Z}} v_j(s) = v(s),$$

and estimate

$$\begin{aligned}
F(s) &= \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}(s)\mathcal{U}(s)v(s)\mathcal{U}(s)v(s)\big)\\
&= \sum_{j,k,l\in\mathbb{Z}} \mathcal{U}(-s)\big(\mathcal{U}(-s)\varphi_j(\sqrt{2H})\bar{v}(s)\mathcal{U}(s)\varphi_k(\sqrt{2H})v(s)\mathcal{U}(s)\varphi_l(\sqrt{2H})v(s)\big)\\
&= \sum_{j,k,l\in\mathbb{Z}} \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}_j(s)\mathcal{U}(s)v_k(s)\mathcal{U}(s)v_l(s)\big)
\end{aligned}$$

in $\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})$. By symmetry, it suffices to estimate

$$F_1(s) = \sum_{j\geq k\geq l} \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}_j(s)\,\mathcal{U}(s)v_k(s)\,\mathcal{U}(s)v_l(s)\big).$$

By Remark 3.3(ii), $\varphi_m(\sqrt{2H})$ are uniformly bounded on $L^p(\mathbb{R})$ for all $1 \leq p \leq \infty$; thus it follows from the definition of $\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})$ that, setting $\sigma = 1 - \frac{1}{p}$,

$$\begin{aligned}
&\|F_1(s)\|_{\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})}\\
&\leq \sum_{j\in\mathbb{Z}} \sum_{m=-\infty}^{j+4} 2^{m\sigma} \Big\| \varphi_m(\sqrt{2H})\\
&\qquad\qquad \times \sum_{k,l=-\infty}^{j} \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}_j(s)\,\mathcal{U}(s)v_k(s)\,\mathcal{U}(s)v_l(s)\big)\Big\|_{L^p(\mathbb{R})}\\
&\leq \sum_{j\in\mathbb{Z}} \sum_{m=-\infty}^{j+4} 2^{m\sigma} \Big\| \sum_{k,l=-\infty}^{j} \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}_j(s)\,\mathcal{U}(s)v_k(s)\big)\Big\|_{L^p(\mathbb{R})}\\
&\lesssim \sum_{j\in\mathbb{Z}} \sum_{m=-\infty}^{j+k} 2^{j\sigma}\\
&\qquad \times \sum_{k\geq l\geq -\infty}^{j} s^{-(\frac{2}{p}-1)} 2^{(1-\frac{1}{p})(k+l)} \|v_j(s)\|_{L^p(\mathbb{R})} \|v_k(s)\|_{L^p(\mathbb{R})} \|v_l(s)\|_{L^p(\mathbb{R})}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim s^{-(\frac{2}{p}-1)} \sum_{j\in\mathbb{Z}} \sum_{k,l}^{j} 2^{j\sigma} 2^{\sigma(k+l)} \|v_j(s)\|_{L^p(\mathbb{R})} \|v_k(s)\|_{L^p(\mathbb{R})} \|v_l(s)\|_{L^p(\mathbb{R})} \\
&\lesssim s^{-(\frac{2}{p}-1)} \big(\sum_{j\in\mathbb{Z}} 2^{j\sigma} \|v_j(s)\|_{L^p}\big)^3 \\
&\lesssim s^{-(\frac{2}{p}-1)} \|v\|^3_{\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})},
\end{aligned}
$$

Noting that $0 < \frac{2}{p} - 1 < 1$ when $1 < p < 2$, we have

$$
\begin{aligned}
\|Tv(t)\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} &\le \|u_0\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} + \int_0^t \big\|\mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}(s)((\mathcal{U}(s)v(s))^2)\big)\big\|_{\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})} ds \\
&\le \|u_0\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} + \int_0^t s^{-(\frac{2}{p}-1)} \|v(s)\|^3_{\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})} ds \\
&\le \|u_0\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} + CT^{1-(\frac{2}{p}-1)} \big(\sup_{0\le t\le T} \|v(s)\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})}\big)^3 \\
&\le \|u_0\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} + CT^{2-\frac{2}{p}} \|u_0\|^3_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} \\
&\le 2\|u_0\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})},
\end{aligned}
$$

where T is chosen small enough that $CT^{2-\frac{2}{p}} \|u_0\|^2_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} \le 1$.

- To prove that T is a contraction on X_T, let $v_1, v_2 \in X_T$. One shows easily that

$$
\begin{aligned}
&Tv_1(t) - Tv_2(t) \\
&\quad = -i\lambda \int_0^t \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}_1(s)\mathcal{U}(s)(v_1(s)-v_2(s))\mathcal{U}(s)(v_1(s)+v_2(s))\big)\, ds \\
&\qquad\qquad - i\lambda \int_0^t \mathcal{U}(-s)\big(\mathcal{U}(-s)(\bar{v}_1(s)-\bar{v}_2(s))(\mathcal{U}(s)v_2(s))^2\big)\, ds.
\end{aligned}
$$

Set $v^*(t) = Tv_1(t) - Tv_2(t)$. Without loss of generality we choose $\lambda = \pm 1$. By using the same procedure as in the estimation of F_1, we obtain

$$
\begin{aligned}
&\|v^*(t)\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} \\
&\quad \le \int_0^t \big\|\mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}_1(s)\mathcal{U}(s)(v_1(s)-v_2(s))\mathcal{U}(s)(v_1(s)+v_2(s))\big)\big\|_{\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})} ds \\
&\qquad\qquad + \int_0^t \big\|\mathcal{U}(-s)\big(\mathcal{U}(-s)(\bar{v}_1(s)-\bar{v}_2(s))(\mathcal{U}(s)v_2(s))^2\big)\big\|_{\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})} ds \\
&\quad \le C \int_0^t s^{-\frac{2}{p}+1} \|v_1(s)-v_2(s)\|_{\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})} \big(\|v_1(s)\|_{\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})} + \|v_2(s)\|_{\dot{B}^{\sigma,H}_{p,1}(\mathbb{R})}\big)^2 ds \\
&\quad \le CT^{2-\frac{2}{p}} \sup_{0\le t\le T} \|v_1(s)-v_2(s)\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} \|u_0\|^2_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} \\
&\quad \le \tfrac{1}{2} \sup_{0\le t\le T} \|v_1(t)-v_2(t)\|_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})},
\end{aligned}
$$

where we choose T small enough that

$$
CT^{2-\frac{2}{p}} \|u_0\|^2_{\dot{B}^{\sigma}_{p,1}(\mathbb{R})} \le \tfrac{1}{2}.
$$

Thus we have proved that T is a contraction from X_T to itself, and then by a stand fixed point argument, we conclude that there exists a unique solution $v(t)$ to the integral equation (1-8).

It remains to prove the continuous dependence of the solution $v(t)$. Let $v_1(t)$ and $v_2(t)$ be two solutions of the integral equation (1-8), corresponding to the initial data u_{01} and u_{02}. Then

$$\begin{aligned} v_1(t)-v_2(t) &= (u_{01}-u_{02}) - i\lambda \int_0^t \mathcal{U}(-s)\big(\mathcal{U}(-s)(\bar v_1(s)-\bar v_2(s))(\mathcal{U}(s)v_1(s))^2\big)\,ds \\ &\quad - i\lambda \int_0^t \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar v_2(s)\mathcal{U}(s)(v_1(s)+v_2(s))\mathcal{U}(s)(v_1(s)-v_2(s))\big)\,ds. \end{aligned}$$

Using the same argument as in the proof of contraction just above, we further have

$$\begin{aligned} &\|v_1(t)-v_2(t)\|_{\dot B^{\sigma}_{p,1}(\mathbb{R})} \\ &\quad\le \|u_{01}-u_{02}\|_{\dot B^{\sigma}_{p,1}(\mathbb{R})} + CT^{2-\frac{2}{p}} \sup\nolimits_{0\le t\le T} \|v_1(t)-v_2(t)\|_{\dot B^{\sigma,H}_{p,1}} \|u_0\|^2_{\dot B^{\sigma}_{p,1}(\mathbb{R})} \\ &\quad\le \|u_{01}-u_{02}\|_{\dot B^{\sigma}_{p,1}(\mathbb{R})} + \tfrac12 \sup\nolimits_{0\le t\le T} \|v_1(t)-v_2(t)\|_{\dot B^{\sigma}_{p,1}(\mathbb{R})}, \end{aligned}$$

where T is chosen small enough that $CT^{2-\frac{2}{p}}\|u_0\|^2_{\dot B^{\sigma}_{p,1}(\mathbb{R})} \le \frac12$. The inequality above further implies

$$\sup_{0\le t\le T} \|v_1(t)-v_2(t)\|_{\dot B^{\sigma}_{p,1}(\mathbb{R})} \leqslant 2\,\|u_{01}-u_{02}\|_{\dot B^{\sigma}_{p,1}(\mathbb{R})},$$

which finishes the proof. □

Proof of Corollary 1.2. Let $v(t)$ $(t\in[0,T))$ be the solution to the integral equation (1-8) given by Theorem 1.1 and $u(t)=\mathcal{U}(t)v(t)$. We will show that $u\in C\big((0,T), \dot B^{s}_{p',1}(\mathbb{R})\big)$ with $1<p<2$ and $s=1-\frac{1}{p}$. We proceed in three steps.

Step I. For $1<p<2$, $s=1-\frac1p$ and any $\phi\in\mathcal{S}(\mathbb{R})$, we have

(4-13)
$$\begin{aligned} \big\|\big(\mathcal{U}(t_1)-\mathcal{U}(t_2)\big)\phi\big\|_{\dot B^{s}_{p',1}(\mathbb{R})} &= \sum_{j\in\mathbb{Z}} 2^{js}\big\|\varphi_j(\sqrt{2H})\big(\mathcal{U}(t_1)-\mathcal{U}(t_2)\big)\phi\big\|_{L^{p'}(\mathbb{R})} \\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{js}\big\|\mathcal{F}_q\big((\mathcal{U}(t_1)-\mathcal{U}(t_2))\varphi_j(\sqrt{2H})\phi\big)\big\|_{L^{p}(\mathbb{R})} \\ &= \sum_{j\in\mathbb{Z}} 2^{js}\big\|(e^{-\frac{i}{2}t_1\xi^2}-e^{-\frac{i}{2}t_2\xi^2})\varphi_j(\xi)\mathcal{F}_q(\phi)\big\|_{L^{p}(\mathbb{R})} \\ &\lesssim |t_1-t_2|^{\frac14} \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac12}\varphi_j(\xi)\mathcal{F}_q(\phi)\big\|_{L^{p}(\mathbb{R})}, \end{aligned}$$

where we used the estimate $\|\mathcal{F}_q^{-1}\phi\|_{L^{p'}(\mathbb{R})} \lesssim \|\phi\|_{L^p(\mathbb{R})}$, which derives from Lemma 2.1. Next we prove that the right side of (4-13) is finite:

(4-14)
$$\sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac12}\varphi_j(\xi)\mathcal{F}_q(\phi)(\xi)\big\|_{L^{p}(\mathbb{R})} < \infty.$$

By the definition of the distorted Fourier transform, we have

$$\mathcal{F}_q(\phi)(\xi) = \mathcal{F}(\phi)(\xi) + \chi_+(\xi)\big(r_q(\xi)J\mathcal{F}(\chi_+\phi)(\xi) + r_q(\xi)\mathcal{F}(\chi_-\phi)(\xi)\big) + \chi_-(\xi)\big(\overline{r_q(\xi)}J\mathcal{F}(\chi_-\phi)(\xi) + \overline{r_q(\xi)}\mathcal{F}(\chi_+\phi)(\xi)\big),$$

where $Jf(x) = f(-x)$. Then it follows that

$$\begin{aligned}(4\text{-}15)\quad \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac12}\varphi_j(\xi)\mathcal{F}_q(\phi)(\xi)\big\|_{L^p(\mathbb{R})} &\lesssim \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac12}\varphi_j(\xi)\mathcal{F}(\phi)(\xi)\big\|_{L^p(\mathbb{R})}\\ &\quad + \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac12}\varphi_j(\xi)\chi_+(\xi)r_q(\xi)J\mathcal{F}(\chi_+\phi)(\xi)\big\|_{L^p(\mathbb{R})}\\ &\quad + \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac12}\varphi_j(\xi)\chi_+(\xi)r_q(\xi)\mathcal{F}(\chi_-\phi)(\xi)\big\|_{L^p(\mathbb{R})}\\ &\quad + \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac12}\varphi_j(\xi)\chi_-(\xi)\overline{r_q(\xi)}\mathcal{F}(\chi_+\phi)(\xi)\big\|_{L^p(\mathbb{R})}\\ &\quad + \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac12}\varphi_j(\xi)\chi_-(\xi)\overline{r_q(\xi)}J\mathcal{F}(\chi_-\phi)(\xi)\big\|_{L^p(\mathbb{R})}.\end{aligned}$$

For the first term on the right side, setting $\widetilde{\varphi}_j(\xi) = |\xi/2^j|^{\frac12}\varphi_j(\xi)$, we have

$$\begin{aligned}(4\text{-}16)\quad \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac12}\varphi_j(\xi)\mathcal{F}(\phi)(\xi)\big\|_{L^p(\mathbb{R})} &= \sum_{j\in\mathbb{Z}} 2^{j(s+\frac12)}\|\widetilde{\varphi}_j(\xi)\mathcal{F}(\phi)\|_{L^p(\mathbb{R})}\\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{j(s+\frac1p)}\|\widetilde{\varphi}_j(\xi)\mathcal{F}(\phi)\|_{L^2(\mathbb{R})}\\ &= C\|\phi\|_{\dot{B}^{s+\frac1p}_{2,1}(\mathbb{R})}.\end{aligned}$$

The remaining terms on the right side of (4-15) are of the same type, so we only treat the third one. We write

$$\begin{aligned}(4\text{-}17)\quad &\sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac12}\varphi_j(\xi)\chi_+(\xi)r_q(\xi)\mathcal{F}(\chi_-\phi)(\xi)\big\|_{L^p(\mathbb{R})}\\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac12}\varphi_j(\xi)r_q(\xi)\mathcal{F}\big((1-\operatorname{sgn}\cdot)\phi\big)(\xi)\big\|_{L^p(\mathbb{R})}\\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{j(s+\frac12)}\|\widetilde{\varphi}_j(\xi)\mathcal{F}(\phi)(\xi)\|_{L^p(\mathbb{R})} + \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac12}\varphi_j(\xi)r_q(\xi)(\tfrac{1}{\cdot}*\mathcal{F}(\phi))(\xi)\big\|_{L^p(\mathbb{R})}\\ &\lesssim \sum_{j\in\mathbb{Z}} 2^{j(s+\frac1p)}\|\widetilde{\varphi}_j(\xi)\mathcal{F}(\phi)(\xi)\|_{L^2(\mathbb{R})} + \sum_{j\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac12}\varphi_j(\xi)r_q(\xi)(\tfrac{1}{\cdot}*\mathcal{F}(\phi))(\xi)\big\|_{L^p(\mathbb{R})}\\ &\lesssim \|\phi\|_{\dot{B}^{s+\frac1p}_{2,1}(\mathbb{R})} + \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{js}\big\||\xi|^{\frac12}\varphi_j(\xi)r_q(\xi)(\frac{1}{\cdot}*\varphi_k\hat{\phi}_k)(\xi)\big\|_{L^p(\mathbb{R})},\end{aligned}$$

where $\widetilde{\varphi}_j(\xi)=|\xi/2^j|^{\frac{1}{2}}\varphi_j(\xi)$, and make the decomposition

$$\phi=\sum_{k\in\mathbb{Z}}\varphi_k(\sqrt{2H_0})\phi=\sum_{k\in\mathbb{Z}}\varphi_k(\sqrt{2H_0})\widetilde{\psi}_k(\sqrt{2H_0})\phi=\sum_{k\in\mathbb{Z}}\varphi_k(\sqrt{2H_0})\phi_k$$

with

$$\phi_k=\widetilde{\psi}_k(\sqrt{2H_0})\phi.$$

Setting $R_{j,k}(\xi)=|\xi|^{\frac{1}{2}}\varphi_j(\xi)r_q(\xi)(\frac{1}{\cdot}*\varphi_k\hat{\phi}_k)(\xi)$ in (4-17), we have

$$\begin{aligned}R_{j,k}(\xi)&=|\xi|^{\frac{1}{2}}\varphi_j(\xi)r_q(\xi)\int\frac{1}{\xi-y}\varphi_k(y)\hat{\phi}_k(y)\,dy\\&=\frac{1}{\sqrt{2\pi}}|\xi|^{\frac{1}{2}}\varphi_j(\xi)r_q(\xi)\iint\frac{1}{\xi-y}\varphi_k(y)e^{-iyz}\,dy\phi_k(z)\,dz\\&=:|\xi|^{\frac{1}{2}}\varphi_j(\xi)r_q(\xi)\int\mathcal{K}(\xi,z)\phi_k(z)\,dz\end{aligned}$$

with

$$\mathcal{K}(\xi,z)=\frac{1}{\sqrt{2\pi}}\int\frac{1}{\xi-y}\varphi_k(y)e^{-iyz}\,dy.$$

When $|k-j|\le 3$, since $|\cdot|^{\frac{1}{2}}r_q\in L^\infty(\mathbb{R})$, it follows from Hölder's inequality and the $L^2(\mathbb{R})$ boundedness of the Hilbert transform $\mathcal{H}$ that

$$\begin{aligned}\|R_{j,k}\|_{L^p(\mathbb{R})}&\lesssim 2^{(\frac{1}{p}-\frac{1}{2})j}\big\|\mathcal{H}(\varphi_k\hat{\phi}_k)\big\|_{L^2(\mathbb{R})}\lesssim 2^{(\frac{1}{p}-\frac{1}{2})j}\big\|\varphi_k\hat{\phi}_k\big\|_{L^2(\mathbb{R})}\\&\lesssim 2^{(\frac{1}{p}-\frac{1}{2})j}\|\phi_k\|_{L^2(\mathbb{R})}.\end{aligned}\tag{4-18}$$

When $k>j+3$, by integration by parts, we have, for any integer $N>0$,

$$\begin{aligned}|\mathcal{K}(\xi,z)|&\lesssim 2^k\left|\int\frac{1}{\xi-2^ky}e^{-i2^kz\cdot y}\varphi(y)\,dy\right|\\&\lesssim\frac{2^k}{\langle 2^kz\rangle^N}\int\left|\partial_y^N\Big(\frac{1}{\xi-2^ky}\varphi(y)\Big)\right|dy\lesssim\frac{1}{\langle 2^kz\rangle^N}.\end{aligned}$$

Combined with the fact that $|\cdot|^{\frac{1}{2}}r_q\in L^\infty(\mathbb{R})$ and Hölder's inequality, this implies

$$\begin{aligned}\|R_{j,k}\|_{L^p(\mathbb{R})}&\le\|\varphi_j\|_{L^p(\mathbb{R})}\left\|\int\mathcal{K}(\cdot,z)\phi_k(z)\,dz\right\|_{L^\infty(\mathbb{R})}\\&\lesssim 2^{\frac{1}{p}(j-k)}\|\phi_k\|_{L^{p'}(\mathbb{R})}.\end{aligned}\tag{4-19}$$

When $k<j-3$, similarly as above, we have

$$\begin{aligned}|\mathcal{K}(\xi,z)|&\lesssim 2^k\left|\int\frac{1}{\xi-2^ky}\varphi(y)e^{-i2^kzy}\,dy\right|\\&\lesssim 2^k\langle 2^kz\rangle^{-N}\int\left|\partial_y^N\Big(\frac{\varphi(y)}{\xi-2^ky}\Big)\right|dy\lesssim\frac{2^{k-j}}{\langle 2^kz\rangle^N},\end{aligned}$$

which, combined with the inequality $\big||\xi|^{\frac12} r_q(\xi)\big| \lesssim (1+|\xi|)^{-\frac12}$, leads to

$$\|R_{j,k}\|_{L^p(\mathbb{R})} \le \big\||\cdot|^{\frac12}\varphi_j r_q\big\|_{L^p(\mathbb{R})}\Big\|\int \mathcal{K}(\cdot,z)\phi_k(z)\,dz\Big\|_{L^\infty(\mathbb{R})} \lesssim 2^{-\frac{1}{p'}(j-k)-\frac{j}{2}}\|\phi_k\|_{L^{p'}(\mathbb{R})}. \tag{4-20}$$

Combining (4-18)–(4-20), we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{js}\|R_{j,k}\|_{L^p(\mathbb{R})} &\le \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{js}\big(\chi_{\le 3}(|k-j|)+\chi_{>3}(k-j)+\chi_{<-3}(k-j)\big)\|R_{j,k}\|_{L^p(\mathbb{R})}\\ &\lesssim \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{js}\big(\chi_{\le 3}(|k-j|)2^{(\frac1p-\frac12)j}\|\phi_k\|_{L^2}\\ &\qquad\qquad + \chi_{>3}(k-j)2^{\frac1p(j-k)}\|\phi_k\|_{L^{p'}(\mathbb{R})}\\ &\qquad\qquad + \chi_{<-3}(k-j)2^{-\frac{1}{p'}(j-k)-\frac{j}{2}}\|\phi_k\|_{L^{p'}(\mathbb{R})}\big)\\ &\lesssim \|\phi\|_{\dot B^{s+\frac{2-p}{2p}}_{2,1}(\mathbb{R})} + \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{-(k-j)(s+\frac1p)}\chi_{>3}(k-j)2^{ks}\|\phi_k\|_{L^{p'}(\mathbb{R})}\\ &\qquad + \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} 2^{-(\frac12+\frac{1}{p'}-s)(j-k)}\chi_{<-3}(k-j)2^{k(s-\frac12)}\|\phi_k\|_{L^{p'}(\mathbb{R})}\\ &\lesssim \|\phi\|_{\dot B^{s+\frac{2-p}{2p}}_{2,1}(\mathbb{R})} + \|\phi\|_{\dot B^{s}_{p',1}(\mathbb{R})} + \|\phi\|_{\dot B^{s-\frac12}_{p',1}(\mathbb{R})},\end{aligned} \tag{4-21}$$

where the discrete Young's inequality is used in the last inequality. By (4-16), (4-17) and (4-21), we show that (4-15) is finite, which verifies (4-14). Hence for $1<p<2$, $s=1-\frac1p$ and any $\phi\in\mathcal{S}(\mathbb{R})$, by (4-13), we have

$$\|\big(\mathcal{U}(t_1)-\mathcal{U}(t_2)\big)\phi\|_{\dot B^s_{p',1}(\mathbb{R})} \lesssim |t_1-t_2|^{\frac14}. \tag{4-22}$$

<u>Step II.</u> For general $\phi\in\dot B^s_{p',1}(\mathbb{R})$ with $1<p<2$ and $s=1-\frac1p$, let $\tilde\phi\in\mathcal{S}(\mathbb{R})$ and $t_1,t_2\in\mathbb{R}$, by the decay estimates of $\mathcal{U}(t)$ (see the proof of Corollary 2.4), we have

$$\begin{aligned}&\big\|\big(\mathcal{U}(t_1)-\mathcal{U}(t_2)\big)\phi\big\|_{\dot B^s_{p',1}(\mathbb{R})}\\ &\lesssim \big\|\mathcal{U}(t_1)\phi-\mathcal{U}(t_1)\tilde\phi\big\|_{\dot B^s_{p',1}(\mathbb{R})}+\big\|\mathcal{U}(t_1)\tilde\phi-\mathcal{U}(t_2)\tilde\phi\big\|_{\dot B^s_{p',1}(\mathbb{R})}+\big\|\mathcal{U}(t_2)\phi-\mathcal{U}(t_2)\tilde\phi\big\|_{\dot B^s_{p',1}(\mathbb{R})}\\ &\le |t_1|^{-(\frac1p-\frac12)}\big\|\phi-\tilde\phi\big\|_{\dot B^s_{p',1}(\mathbb{R})}+\big\|\mathcal{U}(t_1)\tilde\phi-\mathcal{U}(t_2)\tilde\phi\big\|_{\dot B^s_{p',1}(\mathbb{R})}+|t_2|^{-(\frac1p-\frac12)}\big\|\phi-\tilde\phi\big\|_{\dot B^s_{p',1}(\mathbb{R})},\end{aligned}$$

which, combined with (4-22) and the standard $\frac{\varepsilon}{3}$-argument, implies that for any $\phi\in L^p(\mathbb{R})$ with $1<p<2$ and $s=1-\frac1p$, the map $t\to\mathcal{U}(t)\phi$ is continuous from $\mathbb{R}\setminus\{0\}$ to $\dot B^s_{p',1}(\mathbb{R})$.

<u>Step III.</u> Let $v(t)$ be the solution of the integral equation (1-8) given by Theorem 1.1 and let $u(t)=\mathcal{U}(t)v(t)$, for any $t,t_0\in(0,T)$. It follows from the decay estimates

of $\mathcal{U}(t)$ that

$$\begin{aligned}\|u(t)-u(t_0)\|_{\dot{B}^s_{p',1}(\mathbb{R})} &\lesssim \|\mathcal{U}(t)\big(\mathcal{U}(-t)u(t)-\mathcal{U}(-t_0)u(t_0)\big)\|_{\dot{B}^s_{p',1}(\mathbb{R})} \\ &\qquad + \|\mathcal{U}(t)\mathcal{U}(-t_0)u(t_0)-\mathcal{U}(t_0)\mathcal{U}(-t_0)u(t_0)\|_{\dot{B}^s_{p',1}(\mathbb{R})} \\ &\lesssim |t|^{-(\frac{1}{p}-\frac{1}{2})}\|v(t)-v(t_0)\|_{\dot{B}^s_{p',1}(\mathbb{R})} + \|\mathcal{U}(t)v(t_0)-\mathcal{U}(t_0)v(t_0)\|_{\dot{B}^s_{p',1}(\mathbb{R})}.\end{aligned}$$

Letting $t \to t_0$, the first term of the right side tends to 0 by the assumption, and the second term goes to 0 by Step II. This concludes the proof. □

4.2. ***Proof of Theorem 1.3.*** We now prove Theorem 1.3 and Corollary 1.4, the well-posedness of the integral equation (1-8) in $L^p(\mathbb{R})$. Set

$$v_0(t) = \mathcal{U}(-t)\big(\mathcal{U}(-t)\bar{v}_1(t)\mathcal{U}(t)v_2(t)\mathcal{U}(t)v_3(t)\big).$$

The next lemma will be used to treat the nonlinear term.

Lemma 4.3. *Let* $1 < p < 2$. *For any* $T > 0$,

$$\big\|t^{\frac{2}{p}-1}v_0(t)\big\|_{L_t^{p'}L_x^p([0,T]\times\mathbb{R})} \lesssim \prod_{j=1}^{3}\big(\|v_j(0)\|_{L^p(\mathbb{R})}+\|\partial_t v_j(t)\|_{L_t^1 L_x^p([0,T]\times\mathbb{R})}\big). \tag{4-23}$$

Proof. Notice that

$$v_j(t) = v_j(0) + \int_0^t \partial_\tau v_j(\tau)\,d\tau.$$

By the L^1 estimate for the trilinear form in the proof of Lemma 4.2, we have

$$\begin{aligned}\|tv_0(t)\|_{L_t^\infty L_x^1([0,T]\times\mathbb{R})} &\lesssim \|M(t)\mathcal{L}v_1(t)\|_{H_x^1(\mathbb{R})}\|v_2(t)\|_{L_x^1(\mathbb{R})}\|v_3(t)\|_{L_x^1(\mathbb{R})} \\ &\lesssim \Big(\|M(t)\mathcal{L}v_1(0)\|_{H_x^1(\mathbb{R})} + \Big\|M(t)\mathcal{L}\int_0^t \partial_\tau v_i(\tau)\,d\tau\Big\|_{H_x^1(\mathbb{R})}\Big) \\ &\qquad \times\Big(\|v_2(0)\|_{L_x^1(\mathbb{R})} + \Big\|\int_0^t \partial_\tau v_2(\tau)\,d\tau\Big\|_{L_x^1(\mathbb{R})}\Big) \\ &\qquad \times\Big(\|v_3(0)\|_{L_x^1(\mathbb{R})} + \Big\|\int_0^t \partial_\tau v_3(\tau)\,d\tau\Big\|_{L_x^1(\mathbb{R})}\Big).\end{aligned}$$

Next we show that (4-23) holds for $p = 2$. To see this, let

$$u_j(t) = \mathcal{U}(t)v_j(t) \quad (j = 1, 2, 3).$$

It is easy to see that

$$i\partial_t u_j(t) = Hu_j(t) + i\mathcal{U}(t)\partial_t v_j(t) \quad \text{and} \quad u_j(0) = v_j(0).$$

By Duhamel's formula, we have for $j = 1, 2, 3$,

$$u_j(t) = \mathcal{U}(t)v_j(0) - i\int_0^t \mathcal{U}(t-s)\mathcal{U}(s)\partial_s v_j(s)\,ds \tag{4-24}$$
$$= \mathcal{U}(t)\Big(v_j(0) - i\int_0^t \partial_s v_j(s)\,ds\Big),$$

which, combined with the Strichartz estimates from Corollary 2.4, implies

$$\|u_j(t)\|_{L_t^6 L_x^6([0,T]\times\mathbb{R})} \lesssim \|v_j(0)\|_{L^2(\mathbb{R})} + \Big\|\int_0^t \partial_s v_j(s)\,ds\Big\|_{L^2(\mathbb{R})}.$$

On the other hand, it follows from Remark 2.5, (4-24) and the Strichartz estimates of $\mathcal{U}_0(t)$ that

$$\begin{aligned}\|u_1(t)\|_{L_t^6 L_x^6([0,T]\times\mathbb{R})} &= \Big\|\chi_+\mathcal{U}_0(t)\mathcal{L}_+\Big(v_1(0) - i\int_0^t \partial_s v_1(s)\,ds\Big)\\ &\qquad + \chi_-\mathcal{U}_0(t)\mathcal{L}_-\Big(v_1(0) - i\int_0^t \partial_s v_1(s)\,ds\Big)\Big\|_{L_t^6 L_x^6([0,T]\times\mathbb{R})}\\ &\lesssim \|M(t)\mathcal{L}v_1(0)\|_{L_x^2(\mathbb{R})} + \Big\|M(t)\mathcal{L}\int_0^t \partial_\tau v_i(\tau)\,d\tau\Big\|_{L_x^2(\mathbb{R})}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\|v_0(t)\|_{L_t^2 L_x^2([0,T]\times\mathbb{R})}\\ &\le \prod_{i=1}^3 \|u_i(t)\|_{L_t^6 L_x^6([0,T]\times\mathbb{R})}\\ &\lesssim \Big(\|M(t)\mathcal{L}v_1(0)\|_{L_x^2(\mathbb{R})} + \|M(t)\mathcal{L}\int_0^t \partial_\tau v_i(\tau)\,d\tau\|_{L_x^2(\mathbb{R})}\Big)\\ &\quad\times\Big(\|v_2(0)\|_{L_x^2} + \Big\|\int_0^t \partial_s v_2(s)\,ds\Big\|_{L_x^2(\mathbb{R})}\Big)\Big(\|v_3(0)\|_{L_x^2} + \Big\|\int_0^t \partial_s v_3(s)\,ds\Big\|_{L_x^2(\mathbb{R})}\Big).\end{aligned}$$

By the interpolation theorem on the multilinear functionals (see Bergh and Löfström [8, Theorme 4.4.1]) and the fact that both $M(t)$ and $\mathcal{L}_\pm$ are bounded on $L^p(\mathbb{R})$ for all $1 \le p \le \infty$, we obtain the desired estimate. □

Proof of Theorem 1.3. Recall the integral equation

$$v(t) = u_0 - i\lambda\int_0^t \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}(s)(\mathcal{U}(s)v(s))^2\big)\,ds.$$

For fixed $1 < p < 2$, we define the space

$$X_T = \big\{v \mid v(0) = u_0,\ \|t^{\frac{2}{p}-1}\partial_t v(t)\|_{L_t^{p'} L_x^p([0,T]\times\mathbb{R})} \le C_1\|u_0\|_{L^p(\mathbb{R})}^3\big\},$$

where $C_1 > 0$ is a large constant independent of the initial data u_0, and a map

$$Tv(t) = u_0 - i\lambda\int_0^t \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}(s)(\mathcal{U}(s)v(s))^2\big)\,ds.$$

We show that T maps X_T to itself. For any $v \in X_T$, we have $Tv(0) = u_0$ and

$$\partial_t(Tv(t)) = -i\lambda\mathcal{U}(-t)\big(\mathcal{U}(-t)\bar{v}(t)(\mathcal{U}(t)v(t))^2\big) = -i\lambda v_0(t).$$

Then it follows from Lemma 4.3 and Hölder's inequality that

$$\begin{aligned}\big\|t^{\frac{2}{p}-1}\partial_t(Tv(t))\big\|_{L_t^{p'}L_x^p([0,T]\times\mathbb{R})} &\le C\big(\|u_0\|_{L^p(\mathbb{R})}+\|\partial_t v(t)\|_{L_t^1L_x^p([0,T]\times\mathbb{R})}\big)^3\\ &\le C\big(\|u_0\|_{L^p(\mathbb{R})}+\|t^{-\frac{2}{p}+1}\|_{L_t^p([0,T])}\,\|t^{\frac{2}{p}-1}\partial_t v(t)\|_{L_t^{p'}L_x^p([0,T]\times\mathbb{R})}\big)^3\\ &\le C\big(\|u_0\|_{L^p(\mathbb{R})}+CT^{\frac{1}{p'}}C_1\|u_0\|^3_{L^p(\mathbb{R})}\big)^3\\ &\le C_1\|u_0\|^3_{L^p(\mathbb{R})},\end{aligned}$$

where we choose T small enough that $CC_1T^{\frac{1}{p'}}\|u_0\|^2_{L^p(\mathbb{R})}\le 1$.

Now we show that T is a contraction on X_T; that is, for any $v_1, v_2\in X_T$,

$$\big\|t^{\frac{2}{p}-1}\partial_t(Tv_1(t)-Tv_2(t))\big\|_{L_t^{p'}L_x^p([0,T]\times\mathbb{R})}\le\tfrac{1}{2}\big\|t^{\frac{2}{p}-1}\partial_t(v_1(t)-v_2(t))\big\|_{L_t^{p'}L_x^p([0,T]\times\mathbb{R})}.$$

Note that

$$\begin{aligned}\partial_t\big(Tv_1(t)-Tv_2(t)\big) &= -i\lambda\mathcal{U}(-t)\big(\mathcal{U}(-t)(\bar v_1(t)-\bar v_2(t))(\mathcal{U}(t)v_1(t))^2\big)\\ &\quad -i\lambda\mathcal{U}(-t)\big(\mathcal{U}(-t)\bar v_2(t)\mathcal{U}(t)(v_1(t)+v_2(t))\mathcal{U}(t)(v_1(t)-v_2(t))\big).\end{aligned}$$

The two terms on the right side of this identity are of the same type as $v_0(t)$. We apply the same argument used in the proof of Lemma 4.3 to get (henceforth we abbreviate $L_t^1L_x^p([0,T]\times\mathbb{R})$ to $L_t^1L_x^p$)

$$\begin{aligned}\text{(4-25)}\quad \big\|t^{\frac{2}{p}-1}\partial_t(Tv_1(t)-Tv_2(t))\big\|_{L_t^{p'}L_x^p} &\le C\big(\|v_1(0)\|_{L^p(\mathbb{R})}+\|\partial_t v_1(t)\|_{L_t^1L_x^p}\big)^2\|\partial_t(v_1(t)-v_2(t))\|_{L_t^1L_x^p}\\ &\quad + C\big(\|(v_1+v_2)(0)\|_{L^p(\mathbb{R})}+\|\partial_t(v_1+v_2)(t)\|_{L_t^1L_x^p}\big)\\ &\qquad\times\big(\|v_2(0)\|_{L^p(\mathbb{R})}+\|\partial_t v_2(t)\|_{L_t^1L_x^p}\big)\|\partial_t(v_1(t)-v_2(t))\|_{L_t^1L_x^p}\\ &\le C\big(\|v_1(0)\|_{L^p(\mathbb{R})}+\|v_2(0)\|_{L^p(\mathbb{R})}+\|\partial_t v_1(t)\|_{L_t^1L_x^p}+\|\partial_t v_2(t)\|_{L_t^1L_x^p}\big)^2\\ &\qquad\times\|\partial_t(v_1(t)-v_2(t))\|_{L_t^1L_x^p}\end{aligned}$$

For $v_i\in X_T$ $(i=1,2)$, by Hölder's inequality, we have

$$\begin{aligned}\|\partial_t v_i(t)\|_{L_t^1L_x^p} &\le \|t^{-\frac{2}{p}+1}\|_{L_t^p[0,T]}\|t^{\frac{2}{p}-1}\partial_t v(t)\|_{L_t^{p'}L_x^p}\\ &\le CT^{\frac{1}{p'}}C_1\|u_0\|^3_{L^p(\mathbb{R})}.\end{aligned}$$

Therefore,

$$\begin{aligned}
&\big\| t^{\frac{2}{p}-1} \partial_t \big(T v_1(t) - T v_2(t)\big) \big\|_{L_t^{p'} L_x^p} \\
&\quad \le C T^{\frac{1}{p'}} \big(\|u_0\|_{L^p(\mathbb{R})} + C T^{\frac{1}{p'}} C_1 \|u_0\|_{L^p(\mathbb{R})}^3 \big)^2 \big\| t^{\frac{2}{p}-1} \partial_t (v_1(t) - v_2(t)) \big\|_{L_t^{p'} L_x^p} \\
&\quad \le \tfrac{1}{2} \big\| t^{\frac{2}{p}-1} \partial_t (v_1(t) - v_2(t)) \big\|_{L_t^{p'} L_x^p},
\end{aligned}$$

provided that T of order $\|u_0\|_{L^p(\mathbb{R})}^{-2}$ is small enough.

It remains to prove stability. Let $u_{01}, u_{02} \in L^p(\mathbb{R})$ and let $v_1(t), v_2(t)$ be the corresponding solutions of the integral equation (1-8). Then

$$\begin{aligned}
v_1(t) - v_2(t)(u_{01} - u_{02}) - i\lambda \int_0^t \mathcal{U}(-s)\big(\mathcal{U}(-s)(\bar{v}_1(s) - \bar{v}_2(s))(\mathcal{U}(s) v_1(s))^2\big)\, ds \\
- i\lambda \int_0^t \mathcal{U}(-s)\big(\mathcal{U}(-s)\bar{v}_2(s)\mathcal{U}(s)(v_1(s) + v_2(s))\mathcal{U}(s)(v_1(s) - v_2(s))\big)\, ds.
\end{aligned}$$

Similarly to (4-25), we have

$$\begin{aligned}
&\big\| t^{\frac{2}{p}-1} \partial_t (v_1(t) - v_2(t)) \big\|_{L_t^{p'} L_x^p} \\
&\le \big(\|u_{01}\|_{L^p(\mathbb{R})} + \|u_{02}\|_{L^p(\mathbb{R})} + C T^{\frac{1}{p'}} C_1 \|u_{01}\|_{L^p(\mathbb{R})}^3 + C T^{\frac{1}{p'}} C_1 \|u_{02}\|_{L^p(\mathbb{R})}^3 \big)^2 \\
&\qquad\qquad \times \big(\|u_{01} - u_{02}\|_{L^p(\mathbb{R})} + C T^{\frac{1}{p'}} \big\| t^{\frac{2}{p}-1} \partial_t (v_1(t) - v_2(t)) \big\|_{L_t^{p'} L_x^p} \big),
\end{aligned}$$

which implies

$$\big\| t^{\frac{2}{p}-1} \partial_t (v_1(t) - v_2(t)) \big\|_{L_t^{p'} L_x^p} \le C \big(\|u_{01}\|_{L^p(\mathbb{R})} + \|u_{02}\|_{L^p(\mathbb{R})} \big)^2 \|u_{01} - u_{02}\|_{L^p(\mathbb{R})},$$

by choosing T of order $(\|u_{01}\|_{L^p(\mathbb{R})} + |u_{02}\|_{L^p(\mathbb{R})})^{-2}$ to be sufficiently small. It follows from the last inequality that

$$\begin{aligned}
&\|v_1(t) - v_2(t)\|_{L^p(\mathbb{R})} \\
&\quad \le \|u_{01} - u_{02}\|_{L^p(\mathbb{R})} + \Big(\int_0^T t^{-(\frac{2}{p}-1)p} dt \Big)^{\frac{1}{p}} \big\| t^{\frac{2}{p}-1} \partial_t (v_1(t) - v_2(t)) \big\|_{L_t^{p'} L_x^p} \\
&\quad \le \|u_{01} - u_{02}\|_{L^p(\mathbb{R})} + C T^{\frac{1}{p'}} \big(\|u_{01}\|_{L^p(\mathbb{R})} + \|u_{02}\|_{L^p(\mathbb{R})} \big)^2 \|u_{01} - u_{02}\|_{L^p(\mathbb{R})}.
\end{aligned}$$

Provided that T of order $(\|u_{01}\|_{L^p(\mathbb{R})} + |u_{02}\|_{L^p(\mathbb{R})})^{-2}$ is sufficiently small, this gives the stability of the solution.

To conclude, the same procedure as above also implies that for the solution $v(t)$ with initial data u_0,

$$\|v(t)\|_{L^p(\mathbb{R})} \le C \|u_0\|_{L^p(\mathbb{R})}, \quad t \in [0, T),$$

where T of order $\|u_0\|_{L^p(\mathbb{R})}^{-2}$ is small enough. □

Proof of Corollary 1.4. Let $v(t)$ $(t \in (0, T))$ be the solution to the integral equation (1-8) given by Theorem 1.3 and let $u(t) = U(t)v(t)$. We will show that $u \in C\big((0, T), L^{p'}(\mathbb{R})\big)$ with $1 < p < 2$.

We claim that for any $\phi \in L^p(\mathbb{R})$ with $1 < p < 2$, the map $t \to \mathcal{U}(t)\phi$ is continuous from $\mathbb{R} \setminus \{0\}$ to $L^{p'}(\mathbb{R})$. For any $\phi \in \mathcal{S}(\mathbb{R})$, we write

$$\phi = um_{k\in\mathbb{Z}}\varphi_k(\sqrt{2H})\phi,$$

where $\varphi_k(\sqrt{2H})$ is the standard Littlewood–Paley projection. Then for any $t_1, t_2 \in (0, T)$, the same procedure used in Step I of the proof of Corollary 1.2 leads to

$$\begin{aligned} \text{(4-26)} \quad \big\|\big(\mathcal{U}(t_1) - \mathcal{U}(t_2)\big)\phi\big\|_{L^{p'}(\mathbb{R})} &\lesssim \sum_{k\in\mathbb{Z}} \big\|\big(\mathcal{U}(t_1) - \mathcal{U}(t_2)\big)\varphi_k(\sqrt{2H})\phi\big\|_{L^{p'}(\mathbb{R})} \\ &\le C|t_1 - t_2|^{\frac{1}{4}}, \end{aligned}$$

where the constant C depends on some Besov norm of ϕ. For general $\phi \in L^p(\mathbb{R})$ with $1 < p < 2$, we choose $\tilde{\phi} \in \mathcal{S}(\mathbb{R})$ such that

$$\begin{aligned} &\big\|\mathcal{U}(t_1)\phi - \mathcal{U}(t_2)\phi\big\|_{L^{p'}(\mathbb{R})} \\ &\quad\le \big\|\mathcal{U}(t_1)\phi - \mathcal{U}(t_1)\tilde{\phi}\big\|_{L^{p'}(\mathbb{R})} + \big\|\mathcal{U}(t_1)\tilde{\phi} - \mathcal{U}(t_2)\tilde{\phi}\big\|_{L^{p'}(\mathbb{R})} + \big\|\mathcal{U}(t_2)\tilde{\phi} - \mathcal{U}(t_2)\phi\big\|_{L^{p'}(\mathbb{R})} \\ &\quad\lesssim |t_1|^{-(\frac{1}{p}-\frac{1}{2})}\|\phi - \tilde{\phi}\|_{L^p(\mathbb{R})} + \big\|\mathcal{U}(t_1)\tilde{\phi} - \mathcal{U}(t_2)\tilde{\phi}\big\|_{L^{p'}(\mathbb{R})} + |t_2|^{-(\frac{1}{p}-\frac{1}{2})}\|\phi - \tilde{\phi}\|_{L^p(\mathbb{R})}, \end{aligned}$$

where we use the decay estimates of $\mathcal{U}(t)$ (see the proof of Corollary 2.4). From this estimate and (4-26), via the standard $\frac{\varepsilon}{3}$-argument, the claim follows.

For any $t_0, t \in (0, T)$, it follows from the decay estimate of $\mathcal{U}(t)$ that

$$\begin{aligned} &\|u(t) - u(t_0)\|_{L^{p'}(\mathbb{R})} \\ &\quad\le \big\|\mathcal{U}(t)(\mathcal{U}(-t)u(t) - \mathcal{U}(-t_0)u(t_0))\big\|_{L^{p'}(\mathbb{R})} \\ &\qquad\qquad + \big\|\mathcal{U}(t)\mathcal{U}(-t_0)u(t_0) - \mathcal{U}(t_0)\mathcal{U}(-t_0)u(t_0)\big\|_{L^{p'}(\mathbb{R})} \\ &\quad\lesssim |t|^{-(\frac{1}{p}-\frac{1}{2})}\|v(t) - v(t_0)\|_{L^p(\mathbb{R})} + \big\|(\mathcal{U}(t) - \mathcal{U}(t_0))\mathcal{U}(-t_0)u(t_0)\big\|_{L^{p'}(\mathbb{R})}. \end{aligned}$$

Letting $t \to t_0$, the first term on the last line tends to 0 by assumption, and the second term goes to 0 by the above claim. □

Acknowledgements

Qingquan Deng is supported by NSFC (no. 12471092 and no. 12531005). Ping Li is supported by NSFC (no. 12371136) and the Open Research Fund of Key Laboratory of Nonlinear Analysis and Applications (Central China Normal University), Ministry of Education, China.

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Received November 2, 2024. Revised June 24, 2025.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 342, No. 1, 2026
https://doi.org/10.2140/pjm.2026.342.163

FINITE BASIS PROBLEM FOR INVOLUTION SEMIGROUPS OF ORDER FOUR

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Since the 1980s, it has been known that the smallest non-finitely based semigroups are of order six. Surprisingly, for involution semigroups, a non-finitely based example of order five was recently discovered. In this article, it is confirmed that every involution semigroup of order four is finitely based. Since every involution semigroup of order three or less is already known to be finitely based, it follows that the smallest non-finitely based involution semigroups are of order five.

1. Introduction

1.1. ***Minimal non-finitely based involution semigroups.*** An *identity basis* for an algebra A is a set of identities of A that axiomatizes all the identities of A. An algebra is *finitely based* if it has some finite identity basis; otherwise, it is *non-finitely based*. A prominent research problem in universal algebra is the *finite basis problem*: determine which finite algebras are finitely based. Finite groups [31], finite associative rings [3], finite Lie rings [13; 27], and finite lattices [29] are finitely based, but in general, not all finite algebras are finitely based. For instance, the multiplicative matrix semigroup

$$B_2^1 = \left\{ \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \right\},$$

published by Perkins in 1969, is non-finitely based [32]. The discovery of this example focused much attention upon the finite basis problem for small semigroups. Decades of cumulative work that followed has shown that every semigroup of order five or less is finitely based [15; 33; 34], and among all semigroups of order six — 28,634 of them up to isomorphism [5] — only four are non-finitely based [23; 25; 26]. The four non-finitely based semigroups of order six, which include B_2^1, are thus *minimal non-finitely based*.

This research was partially supported by the National Natural Science Foundation of China (nos. 12271224, 12571018, and 12171213) and the Fundamental Research Funds for the Central Universities (no. lzujbky-2023-ey06).
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MSC2020: 20M05.
Keywords: semigroup, involution semigroup, identity basis, finitely based, finite basis problem.

The present article is concerned with *involution semigroups* $(S,{}^*)$, that is, semigroups S with a unary operation * that satisfy the identities

$$(x^*)^* \approx x, \quad (xy)^* \approx y^* x^*; \tag{1-1}$$

the unary operation * is called an *involution* of S. An *inverse semigroup* is an involution semigroup $(S,{}^*)$ that satisfies the additional identities

$$xx^*x \approx x, \quad xx^*yy^* \approx yy^*xx^*.$$

Examples of inverse semigroups include any group $(G,{}^{-1})$ with inversion $^{-1}$ and the Perkins semigroup $(B_2^1,{}^\mathsf{T})$ under the usual matrix transposition $^\mathsf{T}$. Examples of involution semigroups that are not inverse semigroups include the multiplicative $n\times n$ matrix semigroup $(M_n(\mathbb{F}),{}^\mathsf{T})$ over any field $\mathbb{F}$ with the usual transposition $^\mathsf{T}$ and the Perkins semigroup $(B_2^1,{}^\mathsf{S})$ under the *skew transposition* $^\mathsf{S}$ across the secondary diagonal, that is,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\mathsf{S} = \begin{bmatrix} d & b \\ c & a \end{bmatrix}.$$

Given how close involution semigroups are to semigroups, it seems reasonable to conjecture that a finite involution semigroup $(S,{}^*)$ and its semigroup *reduct* S are always simultaneously finitely based. For instance, the involution semigroups $(B_2^1,{}^\mathsf{T})$ and $(B_2^1,{}^\mathsf{S})$ are both non-finitely based [2; 12], while their reduct B_2^1 is also non-finitely based [32]. However, this conjecture has been refuted by several counterexamples [8; 11; 16; 19], the smallest of which is the multiplicative matrix semigroup

$$A_0^1 = \left\{ \begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 1&1&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&1\\0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix} \right\}$$

under the skew transposition $^\mathsf{S}$. As noted above, all semigroups of order five or less — which include A_0^1 — are finitely based, but the involution semigroup $(A_0^1,{}^\mathsf{S})$ is non-finitely based [8]. It follows that minimal non-finitely based involution semigroups are of order at most five; this result is quite unexpected given that minimal non-finitely based semigroups are of order six [26].

It is of fundamental importance to examine if there exists a non-finitely based involution semigroup that is smaller than $(A_0^1,{}^\mathsf{S})$. Since every involution semigroup of order three or less is finitely based [20], the answer would require addressing the finite basis problem for those of order four. Solving this problem is the objective of the present article.

Theorem 1.1. *Every involution semigroup of order four is finitely based.*

Consequently, minimal non-finitely based involution semigroups are of order five, and $(A_0^1,{}^\mathsf{S})$ is one such example. An obvious next step in the investigation is to question the uniqueness of this example.

Question 1.2. Is there a non-finitely based involution semigroup of order five that is not isomorphic to $(A_0^1, {}^{\mathsf{S}})$?

It has recently been confirmed that up to isomorphism, $(A_0^1, {}^{\mathsf{S}})$ is the unique smallest non-finitely based involution semigroup within the class of all involution semigroups with a unit element [9]. Therefore, to answer Question 1.2, it suffices to only examine involution semigroups without a unit element.

1.2. ***Finite basis problem for finite (involution) semigroups.*** Let $\mathfrak{M}_n$ denote the set of all subsemigroups of $M_n(\mathbb{R})$ consisting of binary matrices and let $\mathfrak{M}_\infty = \bigcup_{n\geq 1} \mathfrak{M}_n$. It is not a coincidence that all explicit examples of finite involution semigroups given so far are semigroups from $\mathfrak{M}_\infty$ with transpositions ${}^{\mathsf{T}}$ or ${}^{\mathsf{S}}$, given that every finite semigroup is isomorphic to some semigroup in $\mathfrak{M}_\infty$ and every finite inverse semigroup is isomorphic to some semigroup in $\mathfrak{M}_\infty$ with the usual transposition ${}^{\mathsf{T}}$; see, for instance, Howie [10, Theorems 1.1.2 and 5.1.7].

Regarding finite involution semigroups in general, it turns out that every one of them is isomorphic to some semigroup in $\mathfrak{M}_\infty$ with the skew transposition ${}^{\mathsf{S}}$ but not necessarily the usual transposition ${}^{\mathsf{T}}$ [22]. Therefore, when addressing the finite basis problem for finite semigroups (with involution) — which is currently open — it is equivalent to focus on finite semigroups in $\mathfrak{M}_\infty$ (with the skew transposition ${}^{\mathsf{S}}$). Refer to the survey by Volkov [35] for more information on the finite basis problem for finite semigroups.

On the other hand, the finite basis problem for finite algebras is undecidable in general [30].

1.3. ***Organization.*** Notation and background information are first given in Section 2. An outline of the proof of Theorem 1.1 is then given in Section 3, while the finer details of the proof are deferred to Sections 4–6. Multiplication tables of all involution semigroups of order up to four are listed in Section 7.

2. Preliminaries

Most of the notation and background results of this article are given in this section. Refer to the monograph of Burris and Sankappanavar [4] for any undefined notation and terminology of universal algebra in general.

2.1. ***Words.*** Let $\preceq$ be a total order on a countably infinite alphabet $\mathcal{X}$ that excludes the symbol 0; write $x \prec y$ to indicate that $x \preceq y$ and $x \neq y$.

Let $\mathcal{X}^* = \{x^* \mid x \in \mathcal{X}\}$ be a disjoint copy of $\mathcal{X}$. Elements of $\mathcal{X} \cup \mathcal{X}^*$ are called *variables*. The *free involution semigroup* over $\mathcal{X}$ is the free semigroup $F_{\mathsf{inv}}(\mathcal{X}) = (\mathcal{X} \cup \mathcal{X}^*)^+$ with unary operation given by $(x^*)^* = x$ for all $x \in \mathcal{X}$ and

$$(x_1 x_2 \cdots x_{n-1} x_n)^* = x_n^* x_{n-1}^* \cdots x_2^* x_1^*$$

for all $x_1, x_2, \ldots, x_{n-1}, x_n \in \mathcal{X} \cup \mathcal{X}^*$. The *free involution monoid* over $\mathcal{X}$ is $F^1_{\mathsf{inv}}(\mathcal{X}) = F_{\mathsf{inv}}(\mathcal{X}) \cup \{1\}$, where 1 is the empty word with $1^* = 1$. Elements of $F^1_{\mathsf{inv}}(\mathcal{X})$ are called *words* and words in $\mathcal{X}^+ \cup \{1\}$ are said to be *plain*.

Any word $\boldsymbol{w} \in F^1_{\mathsf{inv}}(\mathcal{X})$ can be written in the form

$$\boldsymbol{w} = x_1^{\circledast_1} x_2^{\circledast_2} \cdots x_n^{\circledast_n}$$

for some $x_1, x_2, \ldots, x_n \in \mathcal{X}$ and $\circledast_1, \circledast_2, \ldots, \circledast_n \in \{1, *\}$ with $n \geq 0$; the *plain projection* of such a word is the plain word

$$\overline{\boldsymbol{w}} = x_1 x_2 \cdots x_n.$$

For any words $\boldsymbol{u}, \boldsymbol{v} \in F^1_{\mathsf{inv}}(\mathcal{X})$, write $\boldsymbol{u} \hookrightarrow \boldsymbol{v}$ to indicate that $\boldsymbol{u}$ is a *subsequence* of $\boldsymbol{v}$, that is, $\boldsymbol{u} = x_1 x_2 \cdots x_n$ for some $x_1, x_2, \ldots, x_n \in \mathcal{X} \cup \mathcal{X}^*$ and

$$\boldsymbol{v} = \boldsymbol{v}_0 x_1 \boldsymbol{v}_1 x_2 \boldsymbol{v}_2 \cdots x_n \boldsymbol{v}_n$$

for some $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in F^1_{\mathsf{inv}}(\mathcal{X})$. Specifically, a subsequence $\boldsymbol{u}$ of $\boldsymbol{v}$ such that $\mathsf{con}(\overline{\boldsymbol{u}}) = \{x, y\}$ for some $x, y \in \mathcal{X}$ is called an $\{x, y\}$-*subsequence* of $\boldsymbol{v}$.

For any word $\boldsymbol{w} \in F_{\mathsf{inv}}(\mathcal{X})$, the *content* of $\boldsymbol{w}$, denoted by $\mathsf{con}(\boldsymbol{w})$, is the set of variables occurring in $\boldsymbol{w}$; the number of times that a variable $x \in \mathcal{X} \cup \mathcal{X}^*$ occurs in $\boldsymbol{w}$ is denoted by $\mathsf{occ}(x, \boldsymbol{w})$; the *head* of $\boldsymbol{w}$, denoted by $\mathsf{h}(\boldsymbol{w})$, is the first variable occurring in $\boldsymbol{w}$; the *tail* of $\boldsymbol{w}$, denoted by $\mathsf{t}(\boldsymbol{w})$, is the last variable occurring in $\boldsymbol{w}$; and the length of $\boldsymbol{w}$ is denoted by $|\boldsymbol{w}|$.

Example 2.1. If $\boldsymbol{w} = x^* x y^* x^2 y z^* x^* y$ for some $x, y, z \in \mathcal{X}$, then

- $\overline{\boldsymbol{w}} = x^2 y x^2 y z x y$;
- $\mathsf{con}(\boldsymbol{w}) = \{x, x^*, y, y^*, z^*\}$;
- $\mathsf{occ}(x, \boldsymbol{w}) = 3,\ \mathsf{occ}(x^*, \boldsymbol{w}) = 2,\ \mathsf{occ}(y, \boldsymbol{w}) = 2,\ \mathsf{occ}(y^*, \boldsymbol{w}) = \mathsf{occ}(z^*, \boldsymbol{w}) = 1$;
- $\mathsf{h}(\boldsymbol{w}) = x^*,\ \mathsf{t}(\boldsymbol{w}) = y$; and
- $|\boldsymbol{w}| = 9$.

For any word $\boldsymbol{w} \in F_{\mathsf{inv}}(\mathcal{X})$, a variable $x \in \mathsf{con}(\boldsymbol{w})$ is *simple* if $\mathsf{occ}(\bar{x}, \overline{\boldsymbol{w}}) = 1$. A word $\boldsymbol{w}$ is *simple* if every variable in $\boldsymbol{w}$ is simple. If $x, x^* \in \mathsf{con}(\boldsymbol{w})$ for some $x \in \mathcal{X}$, then $\{x, x^*\}$ is a *mixed pair* of $\boldsymbol{w}$. A word $\boldsymbol{w}$ is *mixed* if it has some mixed pair; otherwise, $\boldsymbol{w}$ is *bipartite*.

Two words $\boldsymbol{w}_1, \boldsymbol{w}_2 \in F^1_{\mathsf{inv}}(\mathcal{X})$ are *disjoint* if $\mathsf{con}(\overline{\boldsymbol{w}}_1) \cap \mathsf{con}(\overline{\boldsymbol{w}}_2) = \varnothing$. A non-simple word $\boldsymbol{w}$ is *connected* if it cannot be decomposed into a product of two disjoint nonempty words.

2.2. *Identities.* An *identity* is an expression $\boldsymbol{u} \approx \boldsymbol{v}$ formed by words $\boldsymbol{u}, \boldsymbol{v} \in F_{\mathsf{inv}}(\mathcal{X})$; it is *nontrivial* if $\boldsymbol{u} \neq \boldsymbol{v}$. An identity $\boldsymbol{u} \approx \boldsymbol{v}$ is *bipartite* if both $\boldsymbol{u}$ and $\boldsymbol{v}$ are bipartite words. A bipartite identity $\boldsymbol{u} \approx \boldsymbol{v}$ is *plain* if $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{X}^+$.

An involution semigroup $(S,^*)$ *satisfies* an identity $\boldsymbol{s} \approx \boldsymbol{t}$, or $\boldsymbol{s} \approx \boldsymbol{t}$ is *satisfied by* $(S,^*)$, if for any substitution $\varphi : \mathcal{X} \to S$, the elements $\varphi(\boldsymbol{s})$ and $\varphi(\boldsymbol{t})$ of S coincide; in this case, $\boldsymbol{s} \approx \boldsymbol{t}$ is also said to be an *identity of* $(S,^*)$.

Lemma 2.2 (Lee [17, Lemma 9]). *An involution semigroup satisfies a bipartite identity* $\boldsymbol{u} \approx \boldsymbol{v}$ *with* $\mathsf{con}(\boldsymbol{u}) = \mathsf{con}(\boldsymbol{v})$ *if and only if it satisfies* $\bar{\boldsymbol{u}} \approx \bar{\boldsymbol{v}}$.

Lemma 2.3 (Lee [21, Lemma 2.12]). *An involution semigroup satisfies an identity* $\boldsymbol{u} \approx \boldsymbol{v}$ *if and only if it satisfies the identity* $\overleftarrow{\boldsymbol{u}} \approx \overleftarrow{\boldsymbol{v}}$, *where* $\overleftarrow{\boldsymbol{u}}$ *and* $\overleftarrow{\boldsymbol{v}}$ *are the words* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *written in reverse order.*

Recall that a *semilattice* is a semigroup that is commutative and idempotent. Up to isomorphism, the smallest semilattice with nontrivial involution is the multiplicative matrix semigroup

$$S\ell_3 = \left\{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]\right\}$$

under the skew transposition $^{\mathsf{S}}$.

Lemma 2.4 (Lee [17, Lemma 8]). *Let* $\boldsymbol{u} \approx \boldsymbol{v}$ *be any identity of* $(S\ell_3, ^{\mathsf{S}})$. *Then* $\boldsymbol{u}$ *is bipartite if and only if* $\boldsymbol{v}$ *is bipartite*; *in this case*, $\mathsf{con}(\boldsymbol{u}) = \mathsf{con}(\boldsymbol{v})$.

A set Σ of identities of $(S,^*)$ is an *identity basis* for $(S,^*)$ if every identity of $(S,^*)$ is deducible from Σ. An involution semigroup is *finitely based* if it possesses a finite identity basis. It is unambiguous and sometimes convenient to take the involution axioms (1-1) for granted and omit them from an identity basis for an involution semigroup.

3. Proof of Theorem 1.1

Every finite commutative involution semigroup is finitely based [9, Proposition 2.2]. Since every involution semigroup of order three or less is commutative [20], they are all finitely based. Therefore, it remains to consider only noncommutative involution semigroups of order four. With the help of a computer, it is routine to check that up to isomorphism, there exist 83 involution semigroups of order four; see Section 7. Only six of these 83 involution semigroups are noncommutative; see Table 1.

Since $(S_1,^*)$ and $(S_3,^*)$ satisfy the identity $x_1x_2x_3 \approx y_1y_2y_3$, their identities can be axiomatized by those formed by words of length at most four, whence they are finitely based. The involution semigroups $(S_2,^*)$, $(S_5,^*)$, and $(S_4,^*)$ are shown to be finitely based in Sections 4, 5, and 6, respectively. The identities of $(S_6,^*)$ — a rectangular band with involution — is long known to be axiomatized by $\{x^2 \approx x,\ xyz \approx xz\}$ [7, Lemma 2]. Consequently, every involution semigroup of order four is finitely based.

S_1	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	1
4	1	1	2	1
x	1	2	3	4
x^*	1	2	4	3

S_2	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	3
4	1	2	1	4
x	1	2	3	4
x^*	1	3	2	4

S_3	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	2	1
4	1	1	2	2
x	1	2	3	4
x^*	1	2	4	3

S_4	1	2	3	4
1	1	1	1	1
2	1	1	1	2
3	1	2	3	1
4	1	1	1	4
x	1	2	3	4
x^*	1	2	4	3

S_5	1	2	3	4
1	1	1	1	1
2	1	1	1	2
3	1	2	3	2
4	1	1	1	4
x	1	2	3	4
x^*	1	2	4	3

S_6	1	2	3	4
1	1	1	3	3
2	2	2	4	4
3	1	1	3	3
4	2	2	4	4
x	1	2	3	4
x^*	1	3	2	4

Table 1. The noncommutative involution semigroups of order four.

4. The involution semigroup $(S_2, {}^*)$

For any word $\boldsymbol{w} \in F_{\mathsf{inv}}(\mathcal{X})$, the *interior* of $\boldsymbol{w}$, denoted by $\mathsf{int}(\boldsymbol{w})$, is the word obtained from $\boldsymbol{w}$ by removing its first and last variables. Specifically, if $\boldsymbol{w} = h\boldsymbol{w}_0 t$ for some $h, t \in \mathcal{X} \cup \mathcal{X}^*$ and $\boldsymbol{w}_0 \in F^1_{\mathsf{inv}}(\mathcal{X})$, then $\mathsf{int}(\boldsymbol{w}) = \boldsymbol{w}_0$. Note that if $|\boldsymbol{w}| \leq 2$, then $\mathsf{int}(\boldsymbol{w}) = 1$.

Proposition 4.1. *The identities*

(4-1a) $\quad x^3 \approx x^2, \quad xyx \approx x^2y^2, \quad xyx \approx y^2x^2, \quad xy^2z \approx xyz,$

(4-1b) $\quad x^*x \approx x^2, \quad xx^* \approx x^2, \quad x^*yx \approx xyx, \quad xyx^* \approx xyx, \quad xy^*z \approx xyz$

constitute an identity basis for the involution semigroup $(S_2, {}^*)$.

Proof. In this proof, a word $\boldsymbol{w}$ is in *canonical form* if every variable in $\mathsf{int}(\boldsymbol{w})$ and every non-simple variable in $\boldsymbol{w}$ are plain. It is easy to see that the identities (4-1b) can be used to convert any word into one in canonical form.

It is routine to check that $(S_2, {}^*)$ satisfies the identities (4-1). Hence, there exists some set Σ of identities of $(S_2, {}^*)$, formed by words in canonical form, such that $\{(4\text{-}1)\} \cup \Sigma$ is an identity basis for $(S_2, {}^*)$. Now the identities (4-1a) in fact constitute an identity basis for the semigroup S_2 [1, Variety 24], so every plain identity of $(S_2, {}^*)$ is deducible from (4-1a). Therefore, the identities in Σ can be assumed non-plain. Let $\boldsymbol{u} \approx \boldsymbol{v}$ be any identity in Σ, say with $\boldsymbol{u}$ non-plain. If $|\boldsymbol{u}| = 1$, then $\boldsymbol{u} = x^*$ for some $x \in \mathcal{X}$, whence it is easy to show that the identity $\boldsymbol{u} \approx \boldsymbol{v}$ is trivial and so is clearly deducible from (4-1).

It remains to consider the case $|\boldsymbol{u}| \geq 2$. Since $\boldsymbol{u}$ is in canonical form and contains a non-plain variable, say x^* with $x \in \mathcal{X}$, the variable x^* is simple and so is either the head or the tail of $\boldsymbol{u}$. In view of Lemma 2.3, it suffices to assume the former, so that $\boldsymbol{u} = x^* \,\mathsf{int}(\boldsymbol{u})\mathsf{t}(\boldsymbol{u})$ with $\mathsf{int}(\boldsymbol{u}) \in \mathcal{X}^+ \cup \{1\}$ and $\mathsf{t}(\boldsymbol{u}) \in \mathcal{X} \cup \mathcal{X}^*$ such that $x \notin \mathsf{con}(\mathsf{int}(\boldsymbol{u})\overline{\mathsf{t}(\boldsymbol{u})})$. If $\mathsf{h}(\boldsymbol{v}) \neq x^*$, then letting $\varphi_1 : \mathcal{X} \to S_2$ be the substitution that maps x to 2 and every other variable to 4, the contradiction $\varphi_1(\boldsymbol{u}) = 3 \neq \varphi_1(\boldsymbol{v})$ is obtained. Therefore, $\mathsf{h}(\boldsymbol{v}) = x^*$, so that $\boldsymbol{v} = x^* \,\mathsf{int}(\boldsymbol{v})\mathsf{t}(\boldsymbol{v})$ with $\mathsf{int}(\boldsymbol{v}) \in \mathcal{X}^+ \cup \{1\}$ and $\mathsf{t}(\boldsymbol{v}) \in \mathcal{X} \cup \mathcal{X}^*$ such that $x \notin \mathsf{con}(\mathsf{int}(\boldsymbol{v})\overline{\mathsf{t}(\boldsymbol{v})})$. There are two cases.

Case 1: $\mathsf{t}(\boldsymbol{u}), \mathsf{t}(\boldsymbol{v}) \in \mathcal{X}$. Then $\mathsf{int}(\boldsymbol{u})\mathsf{t}(\boldsymbol{u})$ and $\mathsf{int}(\boldsymbol{v})\mathsf{t}(\boldsymbol{v})$ are plain, so that $\bar{\boldsymbol{u}} = x\,\mathsf{int}(\boldsymbol{u})\mathsf{t}(\boldsymbol{u})$ and $\bar{\boldsymbol{v}} = x\,\mathsf{int}(\boldsymbol{v})\mathsf{t}(\boldsymbol{v})$.

Case 2: $\mathsf{t}(\boldsymbol{u}) \notin \mathcal{X}$ or $\mathsf{t}(\boldsymbol{v}) \notin \mathcal{X}$. By symmetry, suppose that $\mathsf{t}(\boldsymbol{u}) \in \mathcal{X}^*$, say $\mathsf{t}(\boldsymbol{u}) = y^*$ for some $y \in \mathcal{X}\backslash\{x\}$. Then $\boldsymbol{u} = x^* \,\mathsf{int}(\boldsymbol{u}) y^*$ with $\mathsf{int}(\boldsymbol{u}) \in \mathcal{X}^+ \cup \{1\}$ such that $x, y \notin \mathsf{con}(\mathsf{int}(\boldsymbol{u}))$. If $\mathsf{t}(\boldsymbol{v}) \neq y^*$, then letting $\varphi_2 : \mathcal{X} \to S_2$ be the substitution that maps y to 3 and every other variable to 4, the contradiction $\varphi_2(\boldsymbol{u}) = 2 \neq \varphi_2(\boldsymbol{v})$ is obtained. Therefore, $\mathsf{t}(\boldsymbol{v}) = y^*$, so that $\boldsymbol{v} = x^* \,\mathsf{int}(\boldsymbol{v}) y^*$ with $\mathsf{int}(\boldsymbol{v}) \in \mathcal{X}^+ \cup \{1\}$ such that $x, y \notin \mathsf{con}(\mathsf{int}(\boldsymbol{v}))$. It follows that $\bar{\boldsymbol{u}} = x\,\mathsf{int}(\boldsymbol{u}) y$ and $\bar{\boldsymbol{v}} = x\,\mathsf{int}(\boldsymbol{v}) y$.

It is clear that in both cases, the identities $\{$(1-1), $\boldsymbol{u} \approx \boldsymbol{v}\}$ and $\{$(1-1), $\bar{\boldsymbol{u}} \approx \bar{\boldsymbol{v}}\}$ are deducible from one another. Since $\bar{\boldsymbol{u}} \approx \bar{\boldsymbol{v}}$ is an identity of S_2, it is deducible from (4-1a). It follows that $\boldsymbol{u} \approx \boldsymbol{v}$ is deducible from $\{$(1-1), (4-1)$\}$. Consequently, every identity in Σ is deducible from $\{$(1-1), (4-1)$\}$, so that (4-1) is an identity basis for $(S_2, {}^*)$. □

5. The involution semigroup $(S_5, {}^*)$

The involution semigroup $(S_5, {}^*)$ is isomorphic to the semigroup

$$A_0 = \langle \mathrm{E}, \mathrm{F} \mid \mathrm{E}^2 = \mathrm{E},\ \mathrm{F}^2 = \mathrm{F},\ \mathrm{FE} = 0 \rangle = \{0, \mathrm{E}, \mathrm{F}, \mathrm{EF}\}$$

with the operation * that interchanges E and F and fixes every other element.

A_0	0	EF	E	F
0	0	0	0	0
EF	0	0	0	EF
E	0	EF	E	EF
F	0	0	0	F
x	0	EF	E	F
x^*	0	EF	F	E

The involution semigroup $(A_0, {}^*)$ is isomorphic to the involution subsemigroup of $(A_0^1, {}^{\mathsf{S}})$ that consists of its non-unit elements.

The semigroup A_0 has been known to be finitely based for over 40 years [6]. The finite basis problem for the involution semigroup $(A_0, {}^*)$ has not been considered and has only recently been questioned [18, Question 6.4]. The present section addresses this problem by showing that $(A_0, {}^*)$ is finitely based.

Proposition 5.1. *The identities*

(5-1a) $$xx^*xy \approx xx^*x, \quad yxx^*x \approx xx^*x, \quad xx^*x \approx yy^*y,$$

(5-1b) $$xyx^* \approx xy^*x^*,$$

(5-1c) $$x^2\mathsf{H}x^* \approx x\mathsf{H}x^*, \quad x\mathsf{H}(x^*)^2 \approx x\mathsf{H}x^*,$$

(5-1d) $$xy\mathsf{H}y^* \approx yx\mathsf{H}x^*,$$

(5-1e) $$x\mathsf{H}x^*y \approx y^*x\mathsf{H}x^*,$$

(5-1f) $$x^2\mathsf{H}y\mathsf{T}y^* \approx x\mathsf{H}y\mathsf{T}y^*,$$

(5-1g) $$xy\mathsf{H}z\mathsf{T}z^* \approx yx\mathsf{H}z\mathsf{T}z^*,$$

(5-1h) $$x^3 \approx x^2, \quad x^2yx \approx xyx, \quad xyx^2 \approx xyx,$$

(5-1i) $$xyx \approx yxy,$$

(5-1j) $$x\mathsf{H}yz\mathsf{T}x \approx x\mathsf{H}zy\mathsf{T}x,$$

where $\mathsf{H} \in \{1, h\}$ *and* $\mathsf{T} \in \{1, t\}$, *constitute an identity basis for* $(A_0, {}^*)$.

It is easily checked that $(A_0, {}^*)$ satisfies the identities (5-1). In Section 5.1, some information on identities of $(A_0, {}^*)$ are given. In Section 5.2, it is shown that the identities of $(A_0, {}^*)$ can be used to convert every mixed word into one of two specific forms. Based on these results, it is shown in Section 5.3 that every identity of $(A_0, {}^*)$ is deducible from {(1-1), (5-1)}. This completes the proof of Proposition 5.1.

Corollary 5.2. *The identities*

(5-2) $$\begin{gathered} x^3 \approx x^2, \quad xyxy \approx xyx, \quad x^2x^* \approx xx^*, \quad x^2yx^* \approx xyx^*, \\ xy^*x^* \approx xyx^*, \quad xx^*x \approx yy^*y, \quad xyx^*z \approx z^*xyx^* \end{gathered}$$

constitute an identity basis for $(A_0, {}^*)$.

Proof. It is routine to check, say with Prover9 [28], that the identities {(1-1), (5-1)} and {(1-1), (5-2)} are deducible from one another. □

Remarks 5.3. (i) Not only is the semigroup A_0 finitely based, it is *hereditarily finitely based* in the sense that every semigroup in the variety $\mathsf{Var}\, A_0$ is finitely based [14, Corollary 4.3].

(ii) In contrast, the finitely based involution semigroup $(A_0, {}^*)$ is not hereditarily finitely based because the variety $\mathsf{Var}(A_0, {}^*)$ contains a non-finitely based involution semigroup [20, Proposition 3.8].

5.1. *Some identities of* $(A_0, {}^*)$.

Lemma 5.4. *The identities* {(5-1h), (5-1i)} *constitute an identity basis for the semigroup* A_0.

Proof. The identities $\Sigma = \{x^3 \approx x^2,\ xyxy \approx xyx,\ xyxy \approx yxy\}$ constitute an identity basis for A_0 [24, Theorem 4.1]; it is routine to verify that Σ and {(5-1h), (5-1i)} are deducible from one another. ☐

Lemma 5.5. *Let $\boldsymbol{u} \approx \boldsymbol{v}$ be any identity of $(A_0, {}^*)$ such that either $\boldsymbol{u}$ or $\boldsymbol{v}$ is bipartite. Then $\boldsymbol{u} \approx \boldsymbol{v}$ is deducible from* {(1-1), (5-1)}.

Proof. Since $(S\ell_3, {}^{\mathsf{S}})$ is isomorphic to $(A_0, {}^*)$ modulo the ideal $\{0, \text{EF}\}$, the identity $\boldsymbol{u} \approx \boldsymbol{v}$ is satisfied by $(S\ell_3, {}^{\mathsf{S}})$. Then since either $\boldsymbol{u}$ or $\boldsymbol{v}$ is bipartite, by Lemma 2.4, both $\boldsymbol{u}$ and $\boldsymbol{v}$ are bipartite with $\mathsf{con}(\boldsymbol{u}) = \mathsf{con}(\boldsymbol{v})$. It follows from Lemma 2.2 that $(A_0, {}^*)$ satisfies the plain identity $\bar{\boldsymbol{u}} \approx \bar{\boldsymbol{v}}$. By Lemma 5.4, the identities {(5-1h), (5-1i)} constitute an identity basis for A_0, so that $\bar{\boldsymbol{u}} \approx \bar{\boldsymbol{v}}$ is deducible from {(5-1h), (5-1i)}. It then follows from Lemma 2.2 that $\boldsymbol{u} \approx \boldsymbol{v}$ is deducible from {(1-1), (5-1)}. ☐

An *ordered A_0-block* is a word of the form

$$\boldsymbol{c} = (y_1 y_2 \cdots y_k)^2,$$

where $y_1, y_2, \ldots, y_k \in \mathcal{X} \cup \mathcal{X}^*$ are such that $\bar{y}_1 \prec \bar{y}_2 \prec \cdots \prec \bar{y}_k$ in $\mathcal{X}$ and $k \geq 1$. Note that every ordered A_0-block is bipartite and connected.

Lemma 5.6 (Lee [21, Lemma 2.1]). *Let $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{X}^+$ be any plain connected words such that $\mathsf{con}(\boldsymbol{u}) = \mathsf{con}(\boldsymbol{v})$. Then $\boldsymbol{u} \approx \boldsymbol{v}$ is an identity of the semigroup A_0.*

Lemma 5.7. *Let $\boldsymbol{w} \in F_{\mathsf{inv}}(\mathcal{X})$ be any bipartite connected word such that $\mathsf{con}(\boldsymbol{w}) = \{y_1, y_2, \ldots, y_k\}$ and $\bar{y}_1 \prec \bar{y}_2 \prec \cdots \prec \bar{y}_k$ in $\mathcal{X}$. Then the identities* {(5-1h), (5-1i)} *can be used to convert $\boldsymbol{w}$ into the ordered A_0-block $\boldsymbol{c} = (y_1 y_2 \cdots y_k)^2$.*

Proof. Since $\boldsymbol{w}$ is bipartite, there exists a partition $I \cup J = \{1, 2, \ldots, k\}$ such that $y_i \in \mathcal{X}$ for all $i \in I$ and $y_j \in \mathcal{X}^*$ for all $j \in J$. Let $\varphi : \mathcal{X} \to \mathcal{X} \cup \mathcal{X}^*$ denote the substitution

$$\varphi(t) = \begin{cases} t^* & \text{if } t \in \{y_j, y_j^* \mid j \in J\}, \\ t & \text{otherwise.} \end{cases}$$

Then it is easy to check that for any word $\boldsymbol{v} \in \mathsf{con}(\boldsymbol{w})^+$, we have $\varphi(\boldsymbol{v}) = \bar{\boldsymbol{v}}$ and $\varphi(\bar{\boldsymbol{v}}) = \boldsymbol{v}$. Since $\overline{\boldsymbol{w}}$ and $\bar{\boldsymbol{c}}$ are plain words such that $\mathsf{con}(\overline{\boldsymbol{w}}) = \mathsf{con}(\bar{\boldsymbol{c}})$, by Lemma 5.6, the identity $\overline{\boldsymbol{w}} \approx \bar{\boldsymbol{c}}$ is satisfied by A_0. Then by Lemma 5.4, $\overline{\boldsymbol{w}} \approx \bar{\boldsymbol{c}}$ is deducible from {(5-1h), (5-1i)}. Consequently,

$$\boldsymbol{w} = \varphi(\overline{\boldsymbol{w}}) \overset{(5\text{-}1\text{h}),(5\text{-}1\text{i})}{\approx} \varphi(\bar{\boldsymbol{c}}) = \boldsymbol{c}.$$ ☐

5.2. ***Some special forms of words.*** It is easily checked that for any substitution $\varphi : \mathcal{X} \to A_0$ and any variable $z \in \mathcal{X}$, we have $\varphi(zz^*z) = 0$ in A_0. Therefore, in the $\mathsf{Var}(A_0, {}^*)$-free algebra over $\mathcal{X}$, the class $[zz^*z]$ containing zz^*z is its zero element. This phenomenon can also be seen from the identities (5-1a) of $(A_0, {}^*)$.

Words of other possible forms in the class $[zz^*z]$ are given in the following result.

Lemma 5.8. *Let $\boldsymbol{w} \in F_{\mathsf{inv}}(\mathcal{X})$. Suppose that either $xx^*x \hookrightarrow \boldsymbol{w}$ for some $x \in \mathcal{X} \cup \mathcal{X}^*$ or $xx^*yy^* \hookrightarrow \boldsymbol{w}$ for some $x, y \in \mathcal{X} \cup \mathcal{X}^*$. Then the identities (5-1) can be used to convert $\boldsymbol{w}$ into the word zz^*z for any $z \in \mathcal{X} \cup \mathcal{X}^*$.*

Proof. If $\boldsymbol{w} = \boldsymbol{w}_0 x \boldsymbol{w}_1 x^* \boldsymbol{w}_2 x \boldsymbol{w}_3$ for some $\boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3 \in F^1_{\mathsf{inv}}(\mathcal{X})$, then

$$\boldsymbol{w} \overset{\text{(5-1j)}}{\approx} \boldsymbol{w}_0 x \boldsymbol{w}_1 \boldsymbol{w}_2 x^* x \boldsymbol{w}_3 \overset{\text{(5-1g)}}{\approx} \boldsymbol{w}_0 \boldsymbol{w}_1 \boldsymbol{w}_2 x x^* x \boldsymbol{w}_3 \overset{\text{(5-1a)}}{\approx} zz^*z.$$

If $\boldsymbol{w} = \boldsymbol{w}_0 x \boldsymbol{w}_1 x^* \boldsymbol{w}_2 y \boldsymbol{w}_3 y^* \boldsymbol{w}_4$ for some $\boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3, \boldsymbol{w}_4 \in F^1_{\mathsf{inv}}(\mathcal{X})$, then

$$\boldsymbol{w} \overset{\text{(5-1c)}}{\approx} \boldsymbol{w}_0 x^2 \boldsymbol{w}_1 x^* \boldsymbol{w}_2 y \boldsymbol{w}_3 y^* \boldsymbol{w}_4 \overset{\text{(5-1g)}}{\approx} \boldsymbol{w}_0 x x^* x \boldsymbol{w}_1 \boldsymbol{w}_2 y \boldsymbol{w}_3 y^* \boldsymbol{w}_4 \overset{\text{(5-1a)}}{\approx} zz^*z. \qquad \square$$

A word $\boldsymbol{w} \in F_{\mathsf{inv}}(\mathcal{X})$ is in A_0-*standard form* if

$$\boldsymbol{w} = \boldsymbol{w}_1 x \boldsymbol{w}_2 x^*, \tag{5-3}$$

where $x \in \mathcal{X} \cup \mathcal{X}^*$, $\boldsymbol{w}_1 = x_1 x_2 \cdots x_m$, and $\boldsymbol{w}_2 = \boldsymbol{s}_0 \prod_{i=1}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i)$ for some $m, p \geq 0$ such that the following conditions hold:

(A1) $x_1, x_2, \ldots, x_m \in \mathcal{X} \cup \mathcal{X}^*$ are such that $\bar{x}_1 \prec \bar{x}_2 \prec \cdots \prec \bar{x}_m \prec \bar{x}$;

(A2) $\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p \in F^1_{\mathsf{inv}}(\mathcal{X})$ are simple and $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_p \in F_{\mathsf{inv}}(\mathcal{X})$ are ordered A_0-blocks;

(A3) $x_1, x_2, \ldots, x_m, x, \boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p, \boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_p$ are pairwise disjoint;

(A4) if $\boldsymbol{w}_2 \neq 1$, then
   - (a) $p = 0$ with $\boldsymbol{w}_2 = \boldsymbol{s}_0$ and $\boldsymbol{s}_0 \in \mathcal{X}$; or
   - (b) $p = 1$ with $\boldsymbol{w}_2 = \boldsymbol{c}_1$, $\boldsymbol{s}_0 = \boldsymbol{s}_1 = 1$, and $\mathsf{h}(\boldsymbol{c}_1) \in \mathcal{X}$; or
   - (c) $\overline{\mathsf{h}(\boldsymbol{s}_0)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$ when $\boldsymbol{s}_0 \neq 1$; or
   - (d) $\overline{\mathsf{t}(\boldsymbol{c}_1)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$ when $\boldsymbol{s}_0 = 1$.

Remark 5.9. The following holds for the word $\boldsymbol{w}$ in (5-3) in A_0-standard form:

(i) If $m = 0$, then $\boldsymbol{w}_1 = 1$.

(ii) If $p = 0$, then $\boldsymbol{w}_2 = \boldsymbol{s}_0$.

(iii) $\{x, x^*\}$ is the only mixed pair of $\boldsymbol{w}$ and $x, x^* \notin \mathsf{con}(\boldsymbol{w}_1 \boldsymbol{w}_2)$.

(iv) $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are bipartite words such that $\mathsf{con}(\overline{\boldsymbol{w}}_1) \cap \mathsf{con}(\overline{\boldsymbol{w}}_2) = \varnothing$.

(v) Each variable in $\mathcal{X}$ occurs at most twice in $\overline{\boldsymbol{w}}$.

(vi) If $\boldsymbol{w}_2 \neq 1$ and $|\mathsf{con}(\overline{\boldsymbol{w}}_2)| \geq 2$, then $\overline{\mathsf{h}(\boldsymbol{w}_2)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$ by condition (A4). This is because
   - if $\boldsymbol{s}_0 \neq 1$, then $\overline{\mathsf{h}(\boldsymbol{w}_2)} = \overline{\mathsf{h}(\boldsymbol{s}_0)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$ by condition (A4)(c);
   - if $\boldsymbol{s}_0 = 1$, then since $\overline{\mathsf{h}(\boldsymbol{w}_2)} = \overline{\mathsf{h}(\boldsymbol{c}_1)} \preceq \overline{\mathsf{t}(\boldsymbol{c}_1)}$ due to $\boldsymbol{c}_1$ being an ordered A_0-block, we have $\overline{\mathsf{h}(\boldsymbol{w}_2)} \preceq \overline{\mathsf{t}(\boldsymbol{c}_1)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$ by condition (A4)(d).

Lemma 5.10. *Let $\boldsymbol{w} = \boldsymbol{w}_1 x \boldsymbol{w}_2 x^*$ be the word in (5-3) in A_0-standard form. Then there exist substitutions $\alpha_{\boldsymbol{w}}, \beta_{\boldsymbol{w}} : \mathcal{X} \to A_0$ such that*

(i) $\alpha_{\boldsymbol{w}}(\boldsymbol{w}_1 x \boldsymbol{w}_2) = \mathrm{E}$ *and* $\alpha_{\boldsymbol{w}}(x^*) = \mathrm{F}$, *so that* $\alpha_{\boldsymbol{w}}(\boldsymbol{w}) = \mathrm{EF}$;

(ii) $\beta_{\boldsymbol{w}}(\boldsymbol{w}_1 x) = \mathrm{E}$ *and* $\beta_{\boldsymbol{w}}(\boldsymbol{w}_2 x^*) = \mathrm{F}$, *so that* $\beta_{\boldsymbol{w}}(\boldsymbol{w}) = \mathrm{EF}$;

(iii) $\alpha_{\boldsymbol{w}}(t) = \beta_{\boldsymbol{w}}(t) = 0$ *for all* $t \in \mathcal{X}$ *such that* $t \notin \mathsf{con}(\overline{\boldsymbol{w}})$.

Proof. It follows from Remark 5.9(iii),(iv) that $\mathsf{con}(\boldsymbol{w}_1) = \mathcal{H}_1 \cup \mathcal{K}_1^*$ and $\mathsf{con}(\boldsymbol{w}_2) = \mathcal{H}_2 \cup \mathcal{K}_2^*$ for some $\mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_1, \mathcal{K}_2 \subseteq \mathcal{X}$ such that $\mathcal{H}_1, \mathcal{H}_2, \mathcal{K}_1, \mathcal{K}_2, \{x, x^*\}$ are pairwise disjoint sets. By symmetry, it suffices to assume that $x \in \mathcal{X}$, so that $\mathsf{con}(\overline{\boldsymbol{w}}) = \mathcal{H}_1 \cup \mathcal{K}_1 \cup \mathcal{H}_2 \cup \mathcal{K}_2 \cup \{x\}$. Define

$$\alpha_{\boldsymbol{w}}(t) = \begin{cases} \mathrm{E} & \text{if } t \in \mathcal{H}_1 \cup \mathcal{H}_2 \cup \{x\}, \\ \mathrm{F} & \text{if } t \in \mathcal{K}_1 \cup \mathcal{K}_2, \\ 0 & \text{otherwise;} \end{cases} \qquad \beta_{\boldsymbol{w}}(t) = \begin{cases} \mathrm{E} & \text{if } t \in \mathcal{H}_1 \cup \mathcal{K}_2 \cup \{x\}, \\ \mathrm{F} & \text{if } t \in \mathcal{K}_1 \cup \mathcal{H}_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is routinely checked that the substitutions $\alpha_{\boldsymbol{w}}$ and $\beta_{\boldsymbol{w}}$ satisfy (i)–(iii). □

Corollary 5.11. *For any word $\boldsymbol{w}$ in A_0-standard form and any $z \in \mathcal{X} \cup \mathcal{X}^*$, the identity $\boldsymbol{w} \approx zz^*z$ is not satisfied by $(A_0, {}^*)$.*

Proof. Under the substitution $\alpha_{\boldsymbol{w}} : \mathcal{X} \to A_0$ in Lemma 5.10, we have $\alpha_{\boldsymbol{w}}(\boldsymbol{w}) = \mathrm{EF}$ and $\alpha_{\boldsymbol{w}}(zz^*z) = 0$. □

Lemma 5.12. *Let $\boldsymbol{w}$ be any mixed word. Then the identities* {(1-1), (5-1)} *can be used to convert $\boldsymbol{w}$ into exactly one of the following*:

(i) *the word zz^*z for any $z \in \mathcal{X} \cup \mathcal{X}^*$*;

(ii) *some word in A_0-standard form.*

Proof. In view of the identities {(1-1), (5-1c), (5-1e), (5-1g)}, we may assume that $\boldsymbol{w} = \boldsymbol{w}_1 x \boldsymbol{w}_2 x^*$, where $x \in \mathcal{X} \cup \mathcal{X}^*$ and $\boldsymbol{w}_1, \boldsymbol{w}_2 \in F^1_{\mathsf{inv}}(\mathcal{X})$ are such that $x \notin \mathsf{con}(\boldsymbol{w}_1)$, $x, x^* \notin \mathsf{con}(\boldsymbol{w}_2)$, and $\boldsymbol{w}_2$ is bipartite. If either $x^* \in \mathsf{con}(\boldsymbol{w}_1)$ or $\boldsymbol{w}_1$ contains some mixed pair, then by Lemma 5.8, the identities (5-1) can be used to convert $\boldsymbol{w}$ into the word zz^*z for any $z \in \mathcal{X} \cup \mathcal{X}^*$. Therefore, suppose that $x^* \notin \mathsf{con}(\boldsymbol{w}_1)$ and $\boldsymbol{w}_1$ is bipartite. In summary, we may assume that

(a) $x, x^* \notin \mathsf{con}(\boldsymbol{w}_1 \boldsymbol{w}_2)$ and

(b) $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are bipartite.

Suppose that $\mathsf{con}(\boldsymbol{w}_1) \cap \mathsf{con}(\boldsymbol{w}_2) \neq \varnothing$. Then the x in $\boldsymbol{w}$ is sandwiched between two occurrences of the same variable, one occurring in $\boldsymbol{w}_1$ and one in $\boldsymbol{w}_2$. Therefore, $\boldsymbol{w}_1 = \boldsymbol{a} y \boldsymbol{b}$ and $\boldsymbol{w}_2 = \boldsymbol{e} y \boldsymbol{f}$ for some $y \in \mathcal{X} \cup \mathcal{X}^*$ and $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}, \boldsymbol{f} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that

$x \notin \mathsf{con}(\boldsymbol{abef})$ and $y \notin \mathsf{con}(\boldsymbol{bf})$, whence

$$\begin{aligned} \boldsymbol{w} &= \boldsymbol{a}y\boldsymbol{b}x\boldsymbol{e}y\boldsymbol{f}x^* \overset{(5\text{-}1h)}{\approx} \boldsymbol{a}y\boldsymbol{b}x\boldsymbol{e}y^2\boldsymbol{f}x^* \overset{(5\text{-}1j)}{\approx} \boldsymbol{a}y\boldsymbol{b}\boldsymbol{e}yxy\boldsymbol{f}x^* \\ &\overset{(5\text{-}1i)}{\approx} \boldsymbol{a}y\boldsymbol{b}\boldsymbol{e}xyx\boldsymbol{f}x^* \overset{(5\text{-}1g)}{\approx} \boldsymbol{a}y\boldsymbol{b}\boldsymbol{e}yx^2\boldsymbol{f}x^* \overset{(5\text{-}1c)}{\approx} \boldsymbol{a}y\boldsymbol{b}\boldsymbol{e}yx\boldsymbol{f}x^*; \end{aligned}$$

in other words, x can be moved to the right until it is no longer sandwiched by any two occurrences of y. This process can be repeated until x is not sandwiched by any two occurrences of the same variable. Therefore, we may further assume that

(c) $\mathsf{con}(\boldsymbol{w}_1) \cap \mathsf{con}(\boldsymbol{w}_2) = \varnothing$.

Suppose that $\mathsf{con}(\overline{\boldsymbol{w}}_1) \cap \mathsf{con}(\overline{\boldsymbol{w}}_2) \neq \varnothing$. Then the x in $\boldsymbol{w}$ is sandwiched by some mixed pair $\{y, y^*\}$ with $y \in \mathsf{con}(\boldsymbol{w}_1)$ and $y^* \in \mathsf{con}(\boldsymbol{w}_2)$. Therefore, $\boldsymbol{w}_1 = \boldsymbol{a}y\boldsymbol{b}$ and $\boldsymbol{w}_2 = \boldsymbol{e}y^*\boldsymbol{f}$ for some $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}, \boldsymbol{f} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $x \notin \mathsf{con}(\boldsymbol{abef})$ and $y^* \notin \mathsf{con}(\boldsymbol{e})$. By (c), we also have $y \notin \mathsf{con}(\boldsymbol{e})$. Then

$$\begin{aligned} \boldsymbol{w} &= \boldsymbol{a}y\boldsymbol{b}x\boldsymbol{e}y^*\boldsymbol{f}x^* \overset{(5\text{-}1b)}{\approx} \boldsymbol{a}y\boldsymbol{b}x(\boldsymbol{e}y^*\boldsymbol{f})^*x^* \overset{(1\text{-}1)}{\approx} \boldsymbol{a}y\boldsymbol{b}x\boldsymbol{f}^*y\boldsymbol{e}^*x^* \\ &\overset{(5\text{-}1h)}{\approx} \boldsymbol{a}y\boldsymbol{b}x\boldsymbol{f}^*y^2\boldsymbol{e}^*x^* \overset{(5\text{-}1j)}{\approx} \boldsymbol{a}y\boldsymbol{b}\boldsymbol{f}^*yxy\boldsymbol{e}^*x^* \overset{(5\text{-}1i)}{\approx} \boldsymbol{a}y\boldsymbol{b}\boldsymbol{f}^*xyx\boldsymbol{e}^*x^* \\ &\overset{(5\text{-}1g)}{\approx} \boldsymbol{a}y\boldsymbol{b}\boldsymbol{f}^*yx^2\boldsymbol{e}^*x^* \overset{(5\text{-}1c)}{\approx} \boldsymbol{a}y\boldsymbol{b}\boldsymbol{f}^*yx\boldsymbol{e}^*x^* \overset{(5\text{-}1b)}{\approx} \boldsymbol{a}y\boldsymbol{b}\boldsymbol{f}^*yx\boldsymbol{e}x^*. \end{aligned}$$

Since $y, y^* \notin \mathsf{con}(\boldsymbol{e})$, the variable x is no longer sandwiched by the mixed pair $\{y, y^*\}$. This process can be repeated until x is not sandwiched by any mixed pair. Therefore, we may further assume that

(d) $\mathsf{con}(\overline{\boldsymbol{w}}_1) \cap \mathsf{con}(\overline{\boldsymbol{w}}_2) = \varnothing$.

Since the prefix $\boldsymbol{w}_1$ of $\boldsymbol{w}$ is bipartite by (b), the identities (5-1g) can be used to put the variables in $\boldsymbol{w}_1$ in order, and the identities (5-1f) can be used to reduce the exponent of any non-simple variable to 1. Hence, we may assume that $\boldsymbol{w}_1 = x_1x_2\cdots x_m$ for some $x_1, x_2, \ldots, x_m \in \mathcal{X} \cup \mathcal{X}^*$ with $m \geq 0$ such that $\bar{x}_1 \prec \bar{x}_2 \prec \cdots \prec \bar{x}_m$. By (a), we have $\bar{x} \notin \{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m\}$. If $\bar{x}_m \not\prec \bar{x}$, then

$$\boldsymbol{w} = x_1x_2\cdots x_{m-1}x_m x \boldsymbol{w}_2 x^* \overset{(5\text{-}1d)}{\approx} x_1x_2\cdots x_{n-1}x x_m \boldsymbol{w}_2 x_m^*,$$

and the identities (5-1g) can be used to put the variables in the prefix $x_1x_2\cdots x_{m-1}x$ in an order such that condition (A1) is satisfied.

Since $\boldsymbol{w}_2$ is bipartite by (b), it can be written in the form $\boldsymbol{w}_2 = \boldsymbol{s}_0 \prod_{i=1}^{p}(\boldsymbol{c}_i\boldsymbol{s}_i)$, where $\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p \in F^1_{\mathsf{inv}}(\mathcal{X})$ are simple words and $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_p \in F_{\mathsf{inv}}(\mathcal{X})$ are connected words with $p \geq 0$ such that

(e) $\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p, \boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_p$ are pairwise disjoint.

Then by Lemma 5.7, the identities {(5-1h), (5-1i)} can be used to convert each $\boldsymbol{c}_i$ into the ordered A_0-block word $\widehat{\boldsymbol{c}}_i$ with $\mathsf{con}(\boldsymbol{c}_i) = \mathsf{con}(\widehat{\boldsymbol{c}}_i)$. Therefore, condition (A2) is satisfied, and it follows from (d) and (e) that condition (A3) is also satisfied.

It remains to address condition (A4). Suppose that $\boldsymbol{w}_2 = \boldsymbol{s}_0 \prod_{i=1}^{p}(\boldsymbol{c}_i \boldsymbol{s}_i) \neq 1$. There are five cases.

Case 1: $p = 0$. Then $\boldsymbol{w}_2 = \boldsymbol{s}_0$.

SUBCASE 1.1: $|\mathsf{con}(\bar{\boldsymbol{s}}_0)| = 1$. Then $\boldsymbol{w}_2 = \boldsymbol{s}_0 \in \{y, y^*\}$ for some $y \in \mathcal{X}$. If $\boldsymbol{w}_2 = y$, then condition (A4)(a) is satisfied. If $\boldsymbol{w}_2 = y^*$, then

$$\boldsymbol{w} = \boldsymbol{w}_1 x y^* x^* \overset{(5\text{-}1b)}{\approx} \boldsymbol{w}_1 x (y^*)^* x^* \overset{(1\text{-}1)}{\approx} \boldsymbol{w}_1 x y x^*;$$

in other words, the identities {(1-1), (5-1b)} can be used to convert the y^* in $\boldsymbol{w}$ to $y \in \mathcal{X}$, so that condition (A4)(a) is satisfied.

SUBCASE 1.2: $|\mathsf{con}(\bar{\boldsymbol{s}}_0)| \geq 2$. Then $\boldsymbol{w}_2 = \boldsymbol{s}_0 = y_1 y_2 \cdots y_r$ for some distinct $y_1, y_2, \ldots, y_r \in \mathcal{X} \cup \mathcal{X}^*$ with $r \geq 2$. If $\overline{\mathsf{h}(\boldsymbol{s}_0)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, then condition (A4)(c) is satisfied. If $\overline{\mathsf{h}(\boldsymbol{s}_0)} \not\prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, so that $\bar{y}_1 = \overline{\mathsf{h}(\boldsymbol{s}_0)} \succ \overline{\mathsf{t}(\boldsymbol{s}_0)} = \bar{y}_r$, then

$$\boldsymbol{w} \overset{(5\text{-}1b)}{\approx} \boldsymbol{w}_1 x \boldsymbol{w}_2^* x^* = \boldsymbol{w}_1 x \underbrace{y_r^* y_{r-1}^* \cdots y_1^*}_{\boldsymbol{s}_0^* = \boldsymbol{w}_2^*} x^*,$$

where $\overline{\mathsf{h}(\boldsymbol{s}_0^*)} = \bar{y}_r \prec \bar{y}_1 = \overline{\mathsf{t}(\boldsymbol{s}_0^*)} = \overline{\mathsf{t}(\boldsymbol{w}_2^*)}$, whence condition (A4)(c) is satisfied.

Case 2: $p \geq 1$ and $\boldsymbol{s}_0 \neq 1 \neq \boldsymbol{s}_p$. Then $\boldsymbol{w}_2 = \boldsymbol{s}_0 \prod_{i=1}^{p}(\boldsymbol{c}_i \boldsymbol{s}_i)$, where $\boldsymbol{s}_0 = y_1 y_2 \cdots y_r$ and $\boldsymbol{s}_p = z_1 z_2 \cdots z_s$ for some distinct $y_1, y_2, \ldots, y_r, z_1, z_2, \ldots, z_s \in \mathcal{X} \cup \mathcal{X}^*$ with $r, s \geq 1$. If $\overline{\mathsf{h}(\boldsymbol{s}_0)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, then condition (A4)(c) is satisfied. Hence suppose that $\overline{\mathsf{h}(\boldsymbol{s}_0)} \not\prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, so that $\bar{y}_1 = \overline{\mathsf{h}(\boldsymbol{s}_0)} \succ \overline{\mathsf{t}(\boldsymbol{s}_p)} = \bar{z}_s$. Then

$$\boldsymbol{w} \overset{(5\text{-}1b)}{\approx} \boldsymbol{w}_1 x \boldsymbol{w}_2^* x^* \overset{(1\text{-}1)}{\approx} \boldsymbol{w}_1 x \cdot \boldsymbol{s}_p^* \boldsymbol{c}_p^* \boldsymbol{s}_{p-1}^* \boldsymbol{c}_{p-1}^* \cdots \boldsymbol{s}_1^* \boldsymbol{c}_1^* \boldsymbol{s}_0^* \cdot x^*,$$

where $\overline{\mathsf{h}(\boldsymbol{s}_p^*)} = \bar{z}_s \prec \bar{y}_1 = \overline{\mathsf{t}(\boldsymbol{s}_0^*)}$, and the identities {(1-1), (5-1h), (5-1i)} can be used to convert each $\boldsymbol{c}_i^*$ into an ordered A_0-block (see Lemma 5.7). Therefore, condition (A4)(c) is satisfied.

Case 3: $p \geq 1$ and $\boldsymbol{s}_0 \neq 1 = \boldsymbol{s}_p$. Then $\boldsymbol{w}_2 = \boldsymbol{s}_0 \prod_{i=1}^{p-1}(\boldsymbol{c}_i \boldsymbol{s}_i) \boldsymbol{c}_p$, where $\boldsymbol{s}_0 = z_1 z_2 \cdots z_s$ and $\boldsymbol{c}_p = (y_1 y_2 \cdots y_r)^2$ for some distinct $y_1, y_2, \ldots, y_r, z_1, z_2, \ldots, z_s \in \mathcal{X} \cup \mathcal{X}^*$ with $r, s \geq 1$ such that $\bar{y}_1 \prec \bar{y}_2 \prec \cdots \prec \bar{y}_r$. If $\overline{\mathsf{h}(\boldsymbol{s}_0)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, then condition (A4)(c) is satisfied. If $\overline{\mathsf{h}(\boldsymbol{s}_0)} \not\prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, so that $\bar{z}_1 = \overline{\mathsf{h}(\boldsymbol{s}_0)} \succ \overline{\mathsf{t}(\boldsymbol{c}_p)} = \bar{y}_r$, then

$$\boldsymbol{w} \overset{(5\text{-}1b)}{\approx} \boldsymbol{w}_1 x \boldsymbol{w}_2^* x^* \overset{(1\text{-}1)}{\approx} \boldsymbol{w}_1 x \cdot \boldsymbol{c}_p^* \boldsymbol{s}_{p-1}^* \boldsymbol{c}_{p-1}^* \cdots \boldsymbol{s}_1^* \boldsymbol{c}_1^* \boldsymbol{s}_0^* \cdot x^*,$$

and the identities {(1-1), (5-1h), (5-1i)} can be used to convert each $\boldsymbol{c}_i^*$ into an ordered A_0-block (Lemma 5.7); specifically, $\boldsymbol{c}_p^*$ is converted into $\widehat{\boldsymbol{c}}_p^* = (y_1^* y_2^* \cdots y_r^*)^2$. Since $\overline{\mathsf{t}(\boldsymbol{c}_p^*)} = \bar{y}_r \prec \bar{z}_1 = \overline{\mathsf{t}(\boldsymbol{s}_0^*)}$, condition (A4)(d) is satisfied.

Case 4: $p \geq 1$ and $\boldsymbol{s}_0 = 1 \neq \boldsymbol{s}_p$. Then $\boldsymbol{w}_2 = \prod_{i=1}^{p}(\boldsymbol{c}_i \boldsymbol{s}_i)$, where $\boldsymbol{c}_1 = (y_1 y_2 \cdots y_r)^2$ and $\boldsymbol{s}_p = z_1 z_2 \cdots z_s$ for some distinct $y_1, y_2, \ldots, y_r, z_1, z_2, \ldots, z_s \in \mathcal{X} \cup \mathcal{X}^*$ with

$r, s \geq 1$ such that $\bar{y}_1 \prec \bar{y}_2 \prec \cdots \prec \bar{y}_r$. If $\overline{\mathsf{t}(\boldsymbol{c}_1)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, then condition (A4)(d) is satisfied. If $\overline{\mathsf{t}(\boldsymbol{c}_1)} \not\prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, so that $\bar{y}_r = \overline{\mathsf{t}(\boldsymbol{c}_1)} \succ \overline{\mathsf{t}(\boldsymbol{s}_p)} = \bar{z}_s$, then

$$\boldsymbol{w} \overset{\text{(5-1b)}}{\approx} \boldsymbol{w}_1 x \boldsymbol{w}_2^* x^* \overset{\text{(1-1)}}{\approx} \boldsymbol{w}_1 x \cdot \boldsymbol{s}_p^* \boldsymbol{c}_p^* \boldsymbol{s}_{p-1}^* \boldsymbol{c}_{p-1}^* \cdots \boldsymbol{s}_1^* \boldsymbol{c}_1^* \cdot x^*,$$

and the identities {(1-1), (5-1h), (5-1i)} can be used to convert each $\boldsymbol{c}_i^*$ into an ordered A_0-block (Lemma 5.7); specifically, $\boldsymbol{c}_1^*$ is converted into $\widehat{\boldsymbol{c}}_1^* = (y_1^* y_2^* \cdots y_r^*)^2$. Since $\overline{\mathsf{h}(\boldsymbol{s}_p^*)} = \bar{z}_s \prec \bar{y}_r = \overline{\mathsf{t}(\widehat{\boldsymbol{c}}_1^*)}$, condition (A4)(c) is satisfied.

Case 5: $p \geq 1$ and $\boldsymbol{s}_0 = 1 = \boldsymbol{s}_p$.

SUBCASE 5.1: $p = 1$. Then $\boldsymbol{w}_2 = \boldsymbol{c}_1 = (y_1 y_2 \cdots y_k)^2$ for some $y_1, y_2, \ldots, y_k \in \mathcal{X} \cup \mathcal{X}^*$ with $k \geq 1$ such that $\bar{y}_1 \prec \bar{y}_2 \prec \cdots \prec \bar{y}_k$. If $y_1 \in \mathcal{X}$, then condition (A4)(b) is satisfied. Hence suppose that $y_1 \in \mathcal{X}^*$. Then

$$\boldsymbol{w} \overset{\text{(5-1b)}}{\approx} \boldsymbol{w}_1 x \boldsymbol{w}_2^* x^* = \boldsymbol{w}_1 x \cdot \boldsymbol{c}_1^* \cdot x^*,$$

and the identities {(1-1), (5-1h), (5-1i)} can be used to convert $\boldsymbol{c}_1^*$ into an ordered A_0-block $\widehat{\boldsymbol{c}}_1^* = (y_1^* y_2^* \cdots y_k^*)^2$ (see Lemma 5.7). Now since $y_1^* \in \mathcal{X}$, condition (A4)(b) is satisfied.

SUBCASE 5.2: $p \geq 2$. Then $\boldsymbol{w}_2 = \left(\prod_{i=1}^{p-1} (\boldsymbol{c}_i \boldsymbol{s}_i)\right) \boldsymbol{c}_p$, where $\boldsymbol{c}_1 = (y_1 y_2 \cdots y_r)^2$ and $\boldsymbol{c}_p = (z_1 z_2 \cdots z_s)^2$ for some distinct $y_1, y_2, \ldots, y_r, z_1, z_2, \ldots, z_s \in \mathcal{X} \cup \mathcal{X}^*$ with $r, s \geq 1$ such that $\bar{y}_1 \prec \bar{y}_2 \prec \cdots \prec \bar{y}_r$ and $\bar{z}_1 \prec \bar{z}_2 \prec \cdots \prec \bar{z}_s$. If $\overline{\mathsf{t}(\boldsymbol{c}_1)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, then condition (A4)(d) is satisfied. Hence suppose that $\overline{\mathsf{t}(\boldsymbol{c}_1)} \not\prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, so that $\bar{z}_s = \overline{\mathsf{t}(\boldsymbol{c}_p)} \succ \overline{\mathsf{t}(\boldsymbol{c}_1)} = \bar{y}_r$. Then

$$\boldsymbol{w} \overset{\text{(5-1b)}}{\approx} \boldsymbol{w}_1 x \boldsymbol{w}_2^* x^* \overset{\text{(1-1)}}{\approx} \boldsymbol{w}_1 x \cdot \boldsymbol{c}_p^* \boldsymbol{s}_{p-1}^* \boldsymbol{c}_{p-1}^* \cdots \boldsymbol{s}_1^* \boldsymbol{c}_1^* \cdot x^*,$$

and the identities {(1-1), (5-1h), (5-1i)} can be used to convert each $\boldsymbol{c}_i^*$ into an ordered A_0-block (Lemma 5.7); specifically, $\boldsymbol{c}_1^*$ is converted into $\widehat{\boldsymbol{c}}_1^* = (y_1^* y_2^* \cdots y_r^*)^2$ and $\boldsymbol{c}_p^*$ is converted into $\widehat{\boldsymbol{c}}_p^* = (z_1^* z_2^* \cdots z_s^*)^2$. Since $\overline{\mathsf{t}(\widehat{\boldsymbol{c}}_p^*)} = \bar{z}_s \prec \bar{y}_r = \overline{\mathsf{t}(\widehat{\boldsymbol{c}}_1^*)}$, condition (A4)(d) is satisfied.

Consequently, the identities {(1-1), (5-1)} can be used to convert $\boldsymbol{w}$ into either zz^*z or some word $\widetilde{\boldsymbol{w}}$ in A_0-standard form. But if the identities {(1-1), (5-1)} can be used to convert $\boldsymbol{w}$ into both zz^*z and $\widetilde{\boldsymbol{w}}$, then that would imply that $(A_0, {}^*)$ satisfies the identity $\widetilde{\boldsymbol{w}} \approx zz^*z$, which is impossible by Corollary 5.11. □

5.3. ***Proof of Proposition 5.1.*** Consider any identity

$$\boldsymbol{u} \approx \boldsymbol{v}$$

satisfied by $(A_0, {}^*)$. It suffices to show that $\boldsymbol{u} \approx \boldsymbol{v}$ is deducible from {(1-1), (5-1)}. By Lemma 5.5, this result holds if either $\boldsymbol{u}$ or $\boldsymbol{v}$ is bipartite. Therefore, suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are both mixed. By Corollary 5.11 and Lemma 5.12, the identities {(1-1), (5-1)} can be used to convert $\boldsymbol{u}$ and $\boldsymbol{v}$ simultaneously to either zz^*z or words in A_0-standard form. In the former case, $\boldsymbol{u} \approx \boldsymbol{v}$ is deducible from {(1-1), (5-1)}, whence the proof is complete. Therefore, it remains to consider the latter case, whence we may assume that $\boldsymbol{u}$ and $\boldsymbol{v}$ are in A_0-standard form, say

$$\boldsymbol{u} = \boldsymbol{u}_1 x \boldsymbol{u}_2 x^* \quad \text{and} \quad \boldsymbol{v} = \boldsymbol{v}_1 y \boldsymbol{v}_2 y^*,$$

where $x, y \in \mathcal{X} \cup \mathcal{X}^*$, $\boldsymbol{u}_1 = x_1 x_2 \cdots x_m$, $\boldsymbol{u}_2 = \boldsymbol{s}_0 \prod_{i=1}^{p}(\boldsymbol{c}_i \boldsymbol{s}_i)$, $\boldsymbol{v}_1 = y_1 y_2 \cdots y_n$, and $\boldsymbol{v}_2 = \boldsymbol{t}_0 \prod_{i=1}^{q}(\boldsymbol{d}_i \boldsymbol{t}_i)$ satisfy conditions (A1)–(A4).

Lemma 5.13. *The following holds for the words $\boldsymbol{u}$ and $\boldsymbol{v}$:*

(i) $\mathsf{con}(\bar{\boldsymbol{u}}) = \mathsf{con}(\bar{\boldsymbol{v}})$;

(ii) $\boldsymbol{u}_1 x = \boldsymbol{v}_1 y$;

(iii) $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$.

Proof. (i) Suppose that $\mathsf{con}(\bar{\boldsymbol{u}}) \neq \mathsf{con}(\bar{\boldsymbol{v}})$, say there exists a variable $z \in \mathsf{con}(\bar{\boldsymbol{u}})$ such that $z \notin \mathsf{con}(\bar{\boldsymbol{v}})$. Then under the substitution $\alpha_{\boldsymbol{v}} : \mathcal{X} \to A_0$ in Lemma 5.10, the contradiction $\alpha_{\boldsymbol{v}}(\boldsymbol{u}) = 0 \neq \mathrm{EF} = \alpha_{\boldsymbol{v}}(\boldsymbol{v})$ is deduced.

(ii) Due to condition (A1), the equality $\boldsymbol{u}_1 x = \boldsymbol{v}_1 y$ follows from $\mathsf{con}(\boldsymbol{u}_1 x) = \mathsf{con}(\boldsymbol{v}_1 y)$; to establish the latter, by symmetry, it suffices to verify the inclusion $\mathsf{con}(\boldsymbol{u}_1 x) \subseteq \mathsf{con}(\boldsymbol{v}_1 y)$. To this end, we need to first show that $y \in \mathsf{con}(\boldsymbol{u}_1 x)$. Since $\bar{y} \in \mathsf{con}(\bar{\boldsymbol{v}}) = \mathsf{con}(\bar{\boldsymbol{u}})$ by part (i),

(a) either $y \in \mathsf{con}(\boldsymbol{u})$ or $y^* \in \mathsf{con}(\boldsymbol{u})$.

If $y^* \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$, then by Lemma 5.10, we have $\alpha_{\boldsymbol{u}}(\boldsymbol{u}) = \mathrm{EF}$ and

$$\alpha_{\boldsymbol{u}}(\boldsymbol{v}) = \alpha_{\boldsymbol{u}}(\boldsymbol{v}_1) \cdot \alpha_{\boldsymbol{u}}(y) \cdot \alpha_{\boldsymbol{u}}(\boldsymbol{v}_2) \cdot \alpha_{\boldsymbol{u}}(y^*) = \alpha_{\boldsymbol{u}}(\boldsymbol{v}_1) \cdot \mathrm{F} \cdot \alpha_{\boldsymbol{u}}(\boldsymbol{v}_2) \cdot \mathrm{E} = 0,$$

which is impossible. Therefore,

(b) $y^* \notin \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$, which implies that $y \neq x^*$.

If $y \neq x$, then together with (b), we have $y^* \notin \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2 x^*) = \mathsf{con}(\boldsymbol{u})$, so that $y \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$ by (a) and (b). On the other hand, if $y = x$, then clearly $y \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$. Therefore, $y \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$ either way. Now if $y \in \mathsf{con}(\boldsymbol{u}_2)$, then by Lemma 5.10, we have $\beta_{\boldsymbol{u}}(\boldsymbol{u}) = \mathrm{EF}$ and

$$\beta_{\boldsymbol{u}}(\boldsymbol{v}) = \beta_{\boldsymbol{u}}(\boldsymbol{v}_1) \cdot \beta_{\boldsymbol{u}}(y) \cdot \beta_{\boldsymbol{u}}(\boldsymbol{v}_2) \cdot \beta_{\boldsymbol{u}}(y^*) = \beta_{\boldsymbol{u}}(\boldsymbol{v}_1) \cdot \mathrm{F} \cdot \beta_{\boldsymbol{u}}(\boldsymbol{v}_2) \cdot \mathrm{E} = 0,$$

which is impossible. Hence $y \notin \mathsf{con}(\boldsymbol{u}_2)$; but since $y \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$, we in fact have

(c) $y \in \mathsf{con}(\boldsymbol{u}_1 x)$.

Now we are ready to establish the inclusion $\mathsf{con}(\boldsymbol{u}_1 x) \subseteq \mathsf{con}(\boldsymbol{v}_1 y)$. Suppose there exists some variable $z \in \mathsf{con}(\boldsymbol{u}_1 x)$ such that $z \notin \mathsf{con}(\boldsymbol{v}_1 y)$. Then clearly $z \neq y$. If $z = y^*$, then $y, y^* \in \mathsf{con}(\boldsymbol{u}_1 x)$ by (c), so that condition (A1) is contradicted. Hence

(d) $z \notin \{y, y^*\}$.

Since $\bar{z} \in \mathsf{con}(\bar{\boldsymbol{u}}) = \mathsf{con}(\bar{\boldsymbol{v}})$ by part (i), it follows from (d) that either $z \in \mathsf{con}(\boldsymbol{v}_1\boldsymbol{v}_2)$ or $z^* \in \mathsf{con}(\boldsymbol{v}_1\boldsymbol{v}_2)$. But since $z \notin \mathsf{con}(\boldsymbol{v}_1)$ by assumption, we have $z \in \mathsf{con}(\boldsymbol{v}_2)$ or $z^* \in \mathsf{con}(\boldsymbol{v}_1)$ or $z^* \in \mathsf{con}(\boldsymbol{v}_2)$. These three cases are shown in the following to be impossible. Therefore, the variable z does not exist, whence the required inclusion $\mathsf{con}(\boldsymbol{u}_1 x) \subseteq \mathsf{con}(\boldsymbol{v}_1 y)$ is established.

Case 1: $z \in \mathsf{con}(\boldsymbol{v}_2)$. By Lemma 5.10, we have $\beta_{\boldsymbol{v}}(\boldsymbol{v}_1 y) = \mathrm{E}$ and $\beta_{\boldsymbol{v}}(\boldsymbol{v}_2 y^*) = \mathrm{F}$, so that $\beta_{\boldsymbol{v}}(\boldsymbol{v}) = \mathrm{EF}$. Since $z \in \mathsf{con}(\boldsymbol{v}_2)$, we also have $\beta_{\boldsymbol{v}}(z) = \mathrm{F}$. Note that

$$\mathrm{EF} = \beta_{\boldsymbol{v}}(\boldsymbol{v}) = \beta_{\boldsymbol{v}}(\boldsymbol{u}) = \beta_{\boldsymbol{v}}(\boldsymbol{u}_1) \cdot \beta_{\boldsymbol{v}}(x) \cdot \beta_{\boldsymbol{v}}(\boldsymbol{u}_2) \cdot \beta_{\boldsymbol{v}}(x^*),$$

so we must have $\beta_{\boldsymbol{v}}(x^*) = \mathrm{F}$ and $\beta_{\boldsymbol{v}}(x) = \mathrm{E}$, so that $z \neq x$. But since $z \in \mathsf{con}(\boldsymbol{u}_1 x)$ by assumption, it follows that $\beta_{\boldsymbol{v}}(\boldsymbol{u}_1 x) = \cdots \mathrm{F} \cdots \mathrm{E} = 0$, whence the contradiction $\beta_{\boldsymbol{v}}(\boldsymbol{u}) = 0$ is deduced.

Case 2: $z^* \in \mathsf{con}(\boldsymbol{v}_1)$. By Lemma 5.10, we have $\alpha_{\boldsymbol{u}}(\boldsymbol{u}_1 x \boldsymbol{u}_2) = \mathrm{E}$ and $\alpha_{\boldsymbol{u}}(x^*) = \mathrm{F}$, so that $\alpha_{\boldsymbol{u}}(\boldsymbol{u}) = \mathrm{EF}$. Since $z \in \mathsf{con}(\boldsymbol{u}_1 x)$ by assumption, we also have $\alpha_{\boldsymbol{u}}(z) = \mathrm{E}$. Note that

$$\mathrm{EF} = \alpha_{\boldsymbol{u}}(\boldsymbol{u}) = \alpha_{\boldsymbol{u}}(\boldsymbol{v}) = \alpha_{\boldsymbol{u}}(\boldsymbol{v}_1) \cdot \alpha_{\boldsymbol{u}}(y) \cdot \alpha_{\boldsymbol{u}}(\boldsymbol{v}_2) \cdot \alpha_{\boldsymbol{u}}(y^*),$$

so we must have $\alpha_{\boldsymbol{u}}(y^*) = \mathrm{F}$ and $\alpha_{\boldsymbol{u}}(y) = \mathrm{E}$. But since $z^* \in \mathsf{con}(\boldsymbol{v}_1)$, it follows that $\alpha_{\boldsymbol{u}}(\boldsymbol{v}_1 y) = \cdots \mathrm{F} \cdots \mathrm{E} = 0$, whence the contradiction $\alpha_{\boldsymbol{u}}(\boldsymbol{v}) = 0$ is deduced.

Case 3: $z^* \in \mathsf{con}(\boldsymbol{v}_2)$. By Lemma 5.10, we have $\alpha_{\boldsymbol{v}}(\boldsymbol{v}_1 y \boldsymbol{v}_2) = \mathrm{E}$ and $\alpha_{\boldsymbol{v}}(y^*) = \mathrm{F}$, so that $\alpha_{\boldsymbol{v}}(\boldsymbol{v}) = \mathrm{EF}$. Since $z^* \in \mathsf{con}(\boldsymbol{v}_2)$, we also have $\alpha_{\boldsymbol{v}}(z) = \mathrm{F}$. Note that

$$\mathrm{EF} = \alpha_{\boldsymbol{v}}(\boldsymbol{v}) = \alpha_{\boldsymbol{v}}(\boldsymbol{u}) = \alpha_{\boldsymbol{v}}(\boldsymbol{u}_1) \cdot \alpha_{\boldsymbol{v}}(x) \cdot \alpha_{\boldsymbol{v}}(\boldsymbol{u}_2) \cdot \alpha_{\boldsymbol{v}}(x^*),$$

so we must have $\alpha_{\boldsymbol{v}}(x^*) = \mathrm{F}$ and $\alpha_{\boldsymbol{v}}(x) = \mathrm{E}$. But since $z \in \mathsf{con}(\boldsymbol{u}_1 x)$ by assumption, it follows that $\alpha_{\boldsymbol{v}}(\boldsymbol{u}_1 x) = \cdots \mathrm{F} \cdots \mathrm{E} = 0$, whence the contradiction $\alpha_{\boldsymbol{v}}(\boldsymbol{u}) = 0$ is deduced.

(iii) This is a consequence of parts (i) and (ii). □

Therefore, by Lemma 5.13, we now have

$$\boldsymbol{u} = \underbrace{x_1 x_2 \cdots x_m}_{\boldsymbol{u}_1} \cdot x \cdot \underbrace{\boldsymbol{s}_0 \prod_{i=1}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i)}_{\boldsymbol{u}_2} \cdot x^* \quad \text{and} \quad \boldsymbol{v} = \underbrace{x_1 x_2 \cdots x_m}_{\boldsymbol{u}_1} \cdot x \cdot \underbrace{\boldsymbol{t}_0 \prod_{i=1}^{q} (\boldsymbol{d}_i \boldsymbol{t}_i)}_{\boldsymbol{v}_2} \cdot x^*,$$

where conditions (A1)–(A4) are satisfied.

Lemma 5.14. (i) $\mathsf{con}(\overline{\boldsymbol{s}_0 \boldsymbol{s}_1 \cdots \boldsymbol{s}_p}) = \mathsf{con}(\overline{\boldsymbol{t}_0 \boldsymbol{t}_1 \cdots \boldsymbol{t}_q})$.

(ii) $\mathsf{con}(\overline{\boldsymbol{c}_1 \boldsymbol{c}_2 \cdots \boldsymbol{c}_p}) = \mathsf{con}(\overline{\boldsymbol{d}_1 \boldsymbol{d}_2 \cdots \boldsymbol{d}_q})$.

Proof. (i) Let $\boldsymbol{s} = \boldsymbol{s}_0 \boldsymbol{s}_1 \cdots \boldsymbol{s}_p$ and $\boldsymbol{t} = \boldsymbol{t}_0 \boldsymbol{t}_1 \cdots \boldsymbol{t}_q$. Suppose there exists some variable $z \in \mathsf{con}(\bar{\boldsymbol{s}})$ such that $z \notin \mathsf{con}(\bar{\boldsymbol{t}})$. Then

$$\boldsymbol{u} = x_1 x_2 \cdots x_m \cdot x \cdot \boldsymbol{a} z^{\circledast} \boldsymbol{b} \cdot x^*$$

for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$ and $\circledast \in \{1, *\}$. By the definition of A_0-standard form, $\mathsf{occ}(z, \bar{\boldsymbol{u}}) = 1$ and the sets $\mathsf{con}(\overline{x_1 x_2 \cdots x_m \cdot x \cdot \boldsymbol{a}})$, $\{z\}$, $\mathsf{con}(\bar{\boldsymbol{b}})$ are pairwise disjoint. Therefore, if $\varphi : \mathcal{X} \cup \mathcal{X}^* \to A_0$ is the substitution given by

$$\varphi(t) = \begin{cases} \mathrm{E} & \text{if } t \in \mathsf{con}(x_1 x_2 \cdots x_m \cdot x \cdot \boldsymbol{a}), \\ \mathrm{EF} & \text{if } t = z, \\ \mathrm{F} & \text{otherwise}, \end{cases}$$

then $\varphi(\boldsymbol{u}) = \mathrm{E}^m \cdot \mathrm{E} \cdot \mathrm{E}^{|\boldsymbol{a}|}(\mathrm{EF})^{\circledast}\mathrm{F}^{|\boldsymbol{b}|} \cdot \mathrm{E}^* = \mathrm{EF}$. Now since $z \in \mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$ by Lemma 5.13(iii) and $z \notin \mathsf{con}(\bar{\boldsymbol{t}})$, there exists some $j \in \{1, 2, \ldots, q\}$ such that $z \in \mathsf{con}(\bar{\boldsymbol{d}}_j)$; further, since $\boldsymbol{d}_j$ is an ordered A_0-block, $\mathsf{occ}(z, \bar{\boldsymbol{d}}_j) = 2$. It follows that $\varphi(\boldsymbol{v}) = \cdots \mathrm{EF} \cdots \mathrm{EF} \cdots = 0 \neq \varphi(\boldsymbol{u})$, which is a contradiction. Consequently, the variable z does not exist, so that the inclusion $\mathsf{con}(\bar{\boldsymbol{s}}) \subseteq \mathsf{con}(\bar{\boldsymbol{t}})$ holds. The reverse inclusion $\mathsf{con}(\bar{\boldsymbol{s}}) \supseteq \mathsf{con}(\bar{\boldsymbol{t}})$ holds by a symmetrical argument.

(ii) This follows from part (i) since $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$ by Lemma 5.13(iii). □

Lemma 5.15. (i) *Suppose that $yzyz \hookrightarrow \boldsymbol{u}_2$ for some $y, z \in \mathsf{con}(\boldsymbol{c}_i)$ with $1 \le i \le p$. Then the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ can only be $yzyz$ or $y^*z^*y^*z^*$.*

(ii) *Suppose that $yzyz \hookrightarrow \boldsymbol{v}_2$ for some $y, z \in \mathsf{con}(\boldsymbol{d}_i)$ with $1 \le i \le q$. Then the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{u}_2$ can only be $yzyz$ or $y^*z^*y^*z^*$.*

Proof. (i) By Remark 5.9(v), $yzyz$ is the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{u}$. Further, $\bar{y} \prec \bar{z}$ because $\boldsymbol{c}_i$ is an ordered A_0-block. Hence $\bar{y}, \bar{z} \in \mathsf{con}(\bar{\boldsymbol{c}}_i) \subseteq \mathsf{con}(\overline{\boldsymbol{d}_1 \boldsymbol{d}_2 \cdots \boldsymbol{d}_q})$ by Lemma 5.14(ii). If $\bar{y}, \bar{z} \in \mathsf{con}(\bar{\boldsymbol{d}}_j)$ for some $j \in \{1, 2, \ldots, q\}$, then the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ is one of

$$yzyz, \quad y^*zy^*z, \quad yz^*yz^*, \quad y^*z^*y^*z^*;$$

and if $\bar{y} \in \mathsf{con}(\bar{\boldsymbol{d}}_j)$ and $\bar{z} \in \mathsf{con}(\bar{\boldsymbol{d}}_k)$ for some distinct $j, k \in \{1, 2, \ldots, q\}$, then the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ is one of

$$yyzz, \quad y^*y^*zz, \quad yyz^*z^*, \quad y^*y^*z^*z^*, \quad zzyy, \quad z^*z^*yy, \quad zzy^*y^*, \quad z^*z^*y^*y^*.$$

There are four cases to consider.

Case 1: y^*zy^*z or yz^*yz^* or y^*y^*zz or z^*z^*yy is a subsequence of $\boldsymbol{v}_2$. Then under the substitution $\alpha_{\boldsymbol{u}} : \mathcal{X} \to A_0$ in Lemma 5.10, we have $\alpha_{\boldsymbol{u}}(\boldsymbol{u}_1 x \boldsymbol{u}_2) = \mathrm{E}$ and $\alpha_{\boldsymbol{u}}(x^*) = \mathrm{F}$, so that $\alpha_{\boldsymbol{u}}(\boldsymbol{u}) = \mathrm{EF}$. Specifically, $\alpha_{\boldsymbol{u}}(y) = \alpha_{\boldsymbol{u}}(z) = \mathrm{E}$ because $y, z \in \mathsf{con}(\boldsymbol{u}_2)$. But this implies the contradiction $\alpha_{\boldsymbol{u}}(\boldsymbol{v}) = 0$.

Case 2: either $yyz^*z^* \hookrightarrow \boldsymbol{v}_2$ or $zzy^*y^* \hookrightarrow \boldsymbol{v}_2$. Then under the substitution $\beta_{\boldsymbol{u}} : \mathcal{X} \to A_0$ in Lemma 5.10, we have $\beta_{\boldsymbol{u}}(\boldsymbol{u}_1 x) = \mathrm{E}$ and $\beta_{\boldsymbol{u}}(\boldsymbol{u}_2 x^*) = \mathrm{F}$, so that $\beta_{\boldsymbol{u}}(\boldsymbol{u}) = \mathrm{EF}$. Specifically, $\beta_{\boldsymbol{u}}(y) = \beta_{\boldsymbol{u}}(z) = \mathrm{F}$ because $y, z \in \mathsf{con}(\boldsymbol{u}_2)$. But this implies the contradiction $\beta_{\boldsymbol{u}}(\boldsymbol{v}) = 0$.

Case 3: either $yyzz \hookrightarrow \boldsymbol{v}_2$ or $y^*y^*z^*z^* \hookrightarrow \boldsymbol{v}_2$. Then we can write $\boldsymbol{v}_2 = \boldsymbol{ab}$ such that $\mathsf{occ}(\bar{y}, \bar{\boldsymbol{a}}) = 2$ and $\mathsf{occ}(\bar{z}, \bar{\boldsymbol{b}}) = 2$. Let $\varphi : \mathcal{X} \cup \mathcal{X}^* \to A_0$ be any substitution that maps each variable in $\mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{a})$ to E and each variable in $\mathsf{con}(\boldsymbol{b})$ to F. Then

$$\varphi(\boldsymbol{v}) = \varphi(\boldsymbol{u}_1 x \boldsymbol{a}) \cdot \varphi(\boldsymbol{b}) \cdot \varphi(x^*) = \mathrm{E} \cdot \mathrm{E} \cdot \mathrm{E}^* = \mathrm{EF}.$$

Now depending on whether $yyzz \hookrightarrow \boldsymbol{v}_2$ or $y^*y^*z^*z^* \hookrightarrow \boldsymbol{v}_2$, the pair $(\varphi(y), \varphi(z))$ is either (E, F) or (F, E); but in either case,

$$\varphi(\boldsymbol{u}) = \cdots \varphi(y) \cdots \varphi(z) \cdots \varphi(y) \cdots \varphi(z) \cdots = 0,$$

which is a contradiction.

Case 4: either $zzyy \hookrightarrow \boldsymbol{v}_2$ or $z^*z^*y^*y^* \hookrightarrow \boldsymbol{v}_2$. This is symmetrical to the previous case and so also leads to a contradiction.

Since none of the four cases is possible, the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ can only be $yzyz$ or $y^*z^*y^*z^*$.

(ii) This is symmetrical to part (i). □

Lemma 5.16. (i) *Suppose that $yz \hookrightarrow \boldsymbol{u}_2$ for some $y, z \in \mathcal{X} \cup \mathcal{X}^*$ that are not in the same ordered A_0-block. Then the $\{\bar{y}, \bar{z}\}$-subsequences of $\boldsymbol{v}_2$ of length two can only be yz or z^*y^*.*

(ii) *Suppose that $yz \hookrightarrow \boldsymbol{v}_2$ for some $y, z \in \mathcal{X} \cup \mathcal{X}^*$ that are not in the same ordered A_0-block. Then the $\{\bar{y}, \bar{z}\}$-subsequences of $\boldsymbol{u}_2$ of length two can only be yz or z^*y^*.*

Proof. (i) By symmetry, it suffices to assume that within $\boldsymbol{u}_2$, the first y appears before the first z. Then by assumption, depending on which of y and z is simple or in an ordered A_0-block, the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{u}_2$ is $y^r z^s$ for some $r, s \in \{1, 2\}$. Since $\boldsymbol{u}$ is in A_0-standard form, $\boldsymbol{u}_2 = \boldsymbol{a} y \boldsymbol{b} z \boldsymbol{e}$ for some pairwise disjoint $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $y \notin \mathsf{con}(\boldsymbol{be})$ and $z \notin \mathsf{con}(\boldsymbol{ab})$. Since $\bar{y}, \bar{z} \in \mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$ by Lemma 5.14, the $\{\bar{y}, \bar{z}\}$-subsequences of $\boldsymbol{v}_2$ of length two can only be

$$yz, \quad y^*z, \quad yz^*, \quad y^*z^*, \quad zy, \quad z^*y, \quad zy^*, \quad z^*y^*.$$

There are three cases to consider.

Case 1: either $y^*z \hookrightarrow \boldsymbol{v}_2$ or $z^*y \hookrightarrow \boldsymbol{v}_2$. Then under the substitution $\alpha_{\boldsymbol{u}} : \mathcal{X} \to A_0$ in Lemma 5.10, we have $\alpha_{\boldsymbol{u}}(\boldsymbol{u}_1 x \boldsymbol{u}_2) = \mathrm{E}$ and $\alpha_{\boldsymbol{u}}(x^*) = \mathrm{F}$, so that $\alpha_{\boldsymbol{u}}(\boldsymbol{u}) = \mathrm{EF}$. Specifically, $\alpha_{\boldsymbol{u}}(y) = \alpha_{\boldsymbol{u}}(z) = \mathrm{E}$ because $y, z \in \mathsf{con}(\boldsymbol{u}_2)$. But this implies the contradiction $\alpha_{\boldsymbol{u}}(\boldsymbol{v}) = 0$.

Case 2: either $yz^* \hookrightarrow \boldsymbol{v}_2$ or $zy^* \hookrightarrow \boldsymbol{v}_2$. Then under the substitution $\beta_{\boldsymbol{u}} : \mathcal{X} \to A_0$ in Lemma 5.10, we have $\beta_{\boldsymbol{u}}(\boldsymbol{u}_1 x) = \mathrm{E}$ and $\beta_{\boldsymbol{u}}(\boldsymbol{u}_2 x^*) = \mathrm{F}$, so that $\beta_{\boldsymbol{u}}(\boldsymbol{u}) = \mathrm{EF}$. Specifically, $\beta_{\boldsymbol{u}}(y) = \beta_{\boldsymbol{u}}(z) = \mathrm{F}$ because $y, z \in \mathsf{con}(\boldsymbol{u}_2)$. But this implies the contradiction $\beta_{\boldsymbol{u}}(\boldsymbol{v}) = 0$.

Case 3: either $y^*z^* \hookrightarrow \boldsymbol{v}_2$ or $zy \hookrightarrow \boldsymbol{v}_2$. Then under any substitution $\varphi : \mathcal{X} \cup \mathcal{X}^* \to A_0$

that maps each variable in $\mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{a} y \boldsymbol{b})$ to E and each variable in $\mathsf{con}(z\boldsymbol{e})$ to F, we have $\varphi(\boldsymbol{u}) = \varphi(\boldsymbol{u}_1 x \boldsymbol{a} y \boldsymbol{b}) \cdot \varphi(z\boldsymbol{e}) \cdot \varphi(x^*) = \mathrm{E} \cdot \mathrm{F} \cdot \mathrm{E}^* = \mathrm{EF}$. But $\varphi(\boldsymbol{v}) = 0$ is a contradiction.

Since none of the three cases is possible, the $\{\bar{y}, \bar{z}\}$-subsequences of $\boldsymbol{v}_2$ of length two can only be yz or $z^* y^*$.

(ii) This is symmetrical to part (i). □

Lemma 5.17. *Suppose that* $\boldsymbol{u}_2, \boldsymbol{v}_2 \neq 1$. *Then* $\mathsf{h}(\boldsymbol{u}_2) = \mathsf{h}(\boldsymbol{v}_2)$ *and* $\mathsf{t}(\boldsymbol{u}_2) = \mathsf{t}(\boldsymbol{v}_2)$.

Proof. Recall from Lemma 5.13(iii) that $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$. First assume that $|\mathsf{con}(\bar{\boldsymbol{u}}_2)| = |\mathsf{con}(\bar{\boldsymbol{v}}_2)| = 1$, say $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2) = \{z\}$ for some $z \in \mathcal{X}$. Then each of $\boldsymbol{u}_2$ and $\boldsymbol{v}_2$ can only be simple or an ordered A_0-block, so that $\boldsymbol{u}_2, \boldsymbol{v}_2 \in \{z, z^2\}$ by conditions (A4)(a) and (A4)(b). Hence $\mathsf{h}(\boldsymbol{u}_2) = z = \mathsf{h}(\boldsymbol{v}_2)$ and $\mathsf{t}(\boldsymbol{u}_2) = z = \mathsf{t}(\boldsymbol{v}_2)$.

Now assume that $|\mathsf{con}(\bar{\boldsymbol{u}}_2)| = |\mathsf{con}(\bar{\boldsymbol{v}}_2)| \geq 2$. Let $h = \mathsf{h}(\boldsymbol{u}_2)$, $t = \mathsf{t}(\boldsymbol{u}_2)$, $H = \mathsf{h}(\boldsymbol{v}_2)$, and $T = \mathsf{t}(\boldsymbol{v}_2)$, so that

$$\bar{h} \prec \bar{t} \quad \text{and} \quad \bar{H} \prec \bar{T} \tag{5-4}$$

by conditions (A4)(c) and (A4)(d). Recall that $\boldsymbol{u}_2$ and $\boldsymbol{v}_2$ are bipartite words such that $\mathsf{occ}(z, \boldsymbol{u}_2), \mathsf{occ}(z, \boldsymbol{v}_2) \leq 2$ for all $z \in \mathcal{X} \cup \mathcal{X}^*$. There are five cases to consider; in each case, several intermediate results are established to eventually show that $h = H$ and $t = T$.

Case 1: $\mathsf{occ}(h, \boldsymbol{u}_2) = \mathsf{occ}(t, \boldsymbol{u}_2) = 1$. Then $h = \mathsf{h}(\boldsymbol{s}_0)$ and $t = \mathsf{t}(\boldsymbol{s}_p)$, so that $\boldsymbol{s}_0 = h\boldsymbol{a}$ and $\boldsymbol{s}_p = \boldsymbol{b}t$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$:

$$\boldsymbol{u} = x_1 x_2 \cdots x_m \cdot x \cdot \underbrace{h\boldsymbol{a} \cdot \boldsymbol{c}_1 \boldsymbol{s}_1 \cdot \boldsymbol{c}_2 \boldsymbol{s}_2 \cdots \boldsymbol{c}_{p-1} \boldsymbol{s}_{p-1} \cdot \boldsymbol{c}_p \boldsymbol{b} t}_{\boldsymbol{u}_2} \cdot x^*. \tag{5-5}$$

Result A. $\mathsf{occ}(\bar{h}, \bar{\boldsymbol{v}}_2) = \mathsf{occ}(\bar{t}, \bar{\boldsymbol{v}}_2) = 1$.

Proof. This holds because $\bar{h}, \bar{t} \in \mathsf{con}(\overline{\boldsymbol{s}_0 \boldsymbol{s}_p}) \subseteq \mathsf{con}(\overline{\boldsymbol{t}_0 \boldsymbol{t}_1 \cdots \boldsymbol{t}_q})$ by Lemma 5.14(i). □

Result B. The longest $\{\bar{h}, \bar{t}\}$*-subsequence of* $\boldsymbol{v}_2$ *is* ht.

Proof. Since $ht \hookrightarrow \boldsymbol{u}_2$, it follows from Lemma 5.16(i) and Result A that the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$ is either ht or $t^* h^*$. Seeking a contradiction, suppose that $t^* h^*$ is the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$. Note that t^* does not occur in any ordered A_0-block in $\boldsymbol{v}_2$ due to $\mathsf{occ}(\bar{t}, \bar{\boldsymbol{v}}_2) = 1$ by Result A. It follows that if $H \neq t^*$, so that $Ht^* \hookrightarrow \boldsymbol{v}_2$, then by Lemma 5.16(ii), either $Ht^* \hookrightarrow \boldsymbol{u}_2$ or $tH^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-5). By a symmetrical argument, it is impossible for $T \neq h^*$. Therefore, $H = t^*$ and $T = h^*$. It follows that $\mathsf{occ}(\bar{H}, \bar{\boldsymbol{v}}_2) = \mathsf{occ}(\bar{t}, \bar{\boldsymbol{v}}_2) = 1$ and $H = \mathsf{h}(\boldsymbol{v}_2) = \mathsf{h}(\boldsymbol{t}_0)$, so that $\boldsymbol{t}_0 \neq 1$. Given that $\boldsymbol{v}$ is in A_0-standard form, we have $\bar{H} \prec \bar{T}$ by condition (A4)(c); but this implies that $\bar{t} = \bar{H} \prec \bar{T} = \bar{h}$, which contradicts (5-4). □

Result C. $h = H$ *and* $t = T$.

Proof. Suppose that $h \neq H$. Then it follows from Result B that $Hh \hookrightarrow \boldsymbol{v}_2$, and h is not in any ordered A_0-block in $\boldsymbol{v}_2$ due to Result A. Therefore, by Lemma 5.16(ii), either $Hh \hookrightarrow \boldsymbol{u}_2$ or $h^*H^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-5). By a symmetrical argument, it is impossible for $t \neq T$. □

Case 2: $\mathsf{occ}(h, \boldsymbol{u}_2) = 2$ and $\mathsf{occ}(t, \boldsymbol{u}_2) = 1$. Then $\boldsymbol{s}_0 = 1$, $h = \mathsf{h}(\boldsymbol{c}_1)$, and $t = \mathsf{t}(\boldsymbol{s}_p)$, so that $\boldsymbol{c}_1 = h\boldsymbol{a}h\boldsymbol{a}$ and $\boldsymbol{s}_p = \boldsymbol{b}t$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$:

$$\boldsymbol{u} = x_1x_2\cdots x_m \cdot x \cdot \underbrace{h\boldsymbol{a}h\boldsymbol{a}\,\boldsymbol{s}_1 \cdot \boldsymbol{c}_2\boldsymbol{s}_2 \cdots \boldsymbol{c}_{p-1}\boldsymbol{s}_{p-1} \cdot \boldsymbol{c}_p\boldsymbol{b}t}_{\boldsymbol{u}_2} \cdot x^*. \tag{5-6}$$

Result D. $\mathsf{occ}(\bar{h}, \bar{\boldsymbol{v}}_2) = 2$ *and* $\mathsf{occ}(\bar{t}, \bar{\boldsymbol{v}}_2) = 1$.

Proof. This holds because $\bar{h} \in \mathsf{con}(\bar{\boldsymbol{c}}_1) \subseteq \mathsf{con}(\overline{\boldsymbol{d}_1\boldsymbol{d}_2\cdots\boldsymbol{d}_q})$ and $\bar{t} \in \mathsf{con}(\bar{\boldsymbol{s}}_p) \subseteq \mathsf{con}(\overline{\boldsymbol{t}_0\boldsymbol{t}_1\cdots\boldsymbol{t}_q})$ by Lemma 5.14. □

Result E. The longest $\{\bar{h}, \bar{t}\}$*-subsequence of* $\boldsymbol{v}_2$ *is* h^2t.

Proof. Since $h^2t \hookrightarrow \boldsymbol{u}_2$, it follows from Lemma 5.16(i) and Result D that the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$ is either h^2t or $t^*(h^*)^2$. Seeking a contradiction, suppose that $t^*(h^*)^2$ is the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$. Note that t^* does not occur in any ordered A_0-block in $\boldsymbol{v}_2$ due to $\mathsf{occ}(\bar{t}, \bar{\boldsymbol{v}}_2) = 1$ by Result D. It follows that if $H \neq t^*$, so that $Ht^* \hookrightarrow \boldsymbol{v}_2$, then by Lemma 5.16(ii), either $Ht^* \hookrightarrow \boldsymbol{u}_2$ or $tH^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-6). Hence $H = t^*$.

Now suppose that $T \neq h^*$, so that $h^*T \hookrightarrow \boldsymbol{v}_2$. Then since $\boldsymbol{v}_2$ is bipartite, we have $h, T^* \notin \mathsf{con}(\boldsymbol{v}_2)$. There are two cases.

(a) h^* and T are not in the same ordered A_0-block in $\boldsymbol{v}_2$. By Lemma 5.16(ii), either $h^*T \hookrightarrow \boldsymbol{u}_2$ or $T^*h \hookrightarrow \boldsymbol{u}_2$. We see from (5-6) that $h^*T \not\hookrightarrow \boldsymbol{u}_2$, so only $T^*h \hookrightarrow \boldsymbol{u}_2$ holds. Specifically, $T^*h \hookrightarrow \boldsymbol{c}_1 = h\boldsymbol{a}h\boldsymbol{a}$, so that $hT^*hT^* \hookrightarrow \boldsymbol{u}_2$. Hence, by Lemma 5.15(i), either $hT^*hT^* \hookrightarrow \boldsymbol{v}_2$ or $h^*Th^*T \hookrightarrow \boldsymbol{v}_2$; the former contradicts $h, T^* \notin \mathsf{con}(\boldsymbol{v}_2)$, while the latter contradicts the assumption of the present case.

(b) h^* and T are in the same ordered A_0-block in $\boldsymbol{v}_2$. Specifically, since $T = \mathsf{t}(\boldsymbol{v}_2)$, we have $h^*, T \in \mathsf{con}(\boldsymbol{d}_q)$, so that $h^*Th^*T \hookrightarrow \boldsymbol{v}_2$ and $\bar{h} \prec \overline{T}$. By Lemma 5.15(ii), either $h^*Th^*T \hookrightarrow \boldsymbol{u}_2$ or $hT^*hT^* \hookrightarrow \boldsymbol{u}_2$. It follows from (5-6) that $h^*Th^*T \not\hookrightarrow \boldsymbol{u}_2$, so only $hT^*hT^* \hookrightarrow \boldsymbol{u}_2$ holds, whence $T^* \in \mathsf{con}(\boldsymbol{c}_1)$. Since $\boldsymbol{s}_0 = 1$ and $\boldsymbol{u}$ is in A_0-standard form, by condition (A4)(d), we have $\overline{T} \preceq \overline{\mathsf{t}(\boldsymbol{c}_1)} \prec \overline{\mathsf{t}(\boldsymbol{u}_2)} = \bar{t}$. Now the first variable in $\boldsymbol{v}_2$ is simple because $\mathsf{h}(\boldsymbol{v}_2) = H = t^*$ and $\mathsf{occ}(\bar{t}, \bar{\boldsymbol{v}}_2) = 1$; hence $\boldsymbol{t}_0 \neq 1$. Since $\boldsymbol{v}$ is in A_0-standard form, by condition (A4)(c), we have $\bar{t} = \overline{H} = \overline{\mathsf{h}(\boldsymbol{t}_0)} \prec \overline{\mathsf{t}(\boldsymbol{v}_2)} = \overline{T}$, which is a contradiction.

Since neither (a) nor (b) is possible, we have $T = h^*$. As observed in (b), the first variable in $\boldsymbol{v}_2$ is simple because $\mathsf{h}(\boldsymbol{v}_2) = H = t^*$ and $\mathsf{occ}(\bar{t}, \bar{\boldsymbol{v}}_2) = 1$, thus $\boldsymbol{t}_0 \neq 1$. Given that $\boldsymbol{v}$ is in A_0-standard form, by condition (A4)(c), we have $\bar{t} = \overline{H} = \overline{\mathsf{h}(\boldsymbol{t}_0)} \prec \overline{\mathsf{t}(\boldsymbol{v}_2)} = \overline{T} = \bar{h}$, which contradicts (5-4). □

Result F. $h = H$ *and* $t = T$.

Proof. First, suppose that $t \neq T$. Then $tT \hookrightarrow \boldsymbol{v}_2$ by Result E, and t is not in any ordered A_0-block in $\boldsymbol{v}_2$ due to Result D. Therefore, by Lemma 5.16(ii), either $tT \hookrightarrow \boldsymbol{u}_2$ or $T^*t^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-6). Hence $t = T$.

Now suppose that $h \neq H$. Then $Hh \hookrightarrow \boldsymbol{v}_2$ by Result E. Since $\boldsymbol{v}_2$ is bipartite, we have $h^*, H^* \notin \mathsf{con}(\boldsymbol{v}_2)$. There are two cases.

(a) H and h are not in the same ordered A_0-block in $\boldsymbol{v}_2$. Then by Lemma 5.16(ii), either $Hh \hookrightarrow \boldsymbol{u}_2$ or $h^*H^* \hookrightarrow \boldsymbol{u}_2$. But it is clear from (5-6) that $h^*H^* \not\hookrightarrow \boldsymbol{u}_2$, so only $Hh \hookrightarrow \boldsymbol{u}_2$ holds. Specifically, $Hh \hookrightarrow \boldsymbol{c}_1 = h\boldsymbol{a}h\boldsymbol{a}$, so that $hHhH \hookrightarrow \boldsymbol{u}_2$. Therefore, by Lemma 5.15(i), either $hHhH \hookrightarrow \boldsymbol{v}_2$ or $h^*H^*h^*H^* \hookrightarrow \boldsymbol{v}_2$; but the former contradicts the assumption of the present case, while the latter contradicts $h^*, H^* \notin \mathsf{con}(\boldsymbol{v}_2)$.

(b) H and h are in the same ordered A_0-block in $\boldsymbol{v}_2$. Then $HhHh \hookrightarrow \boldsymbol{v}_2$. Hence by Lemma 5.15(ii), either $HhHh \hookrightarrow \boldsymbol{u}_2$ or $H^*h^*H^*h^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-6).

Since neither (a) nor (b) is possible, we have $h = H$. □

Case 3: $\mathsf{occ}(h, \boldsymbol{u}_2) = 1$ and $\mathsf{occ}(t, \boldsymbol{u}_2) = 2$. Then $h = \mathsf{h}(\boldsymbol{s}_0)$, $t = \mathsf{t}(\boldsymbol{c}_p)$, and $\boldsymbol{s}_p = 1$, so that $\boldsymbol{s}_0 = h\boldsymbol{a}$ and $\boldsymbol{c}_p = \boldsymbol{b}t\boldsymbol{b}t$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$:

$$\boldsymbol{u} = x_1x_2\cdots x_m \cdot x \cdot \underbrace{h\boldsymbol{a} \cdot \boldsymbol{c}_1\boldsymbol{s}_1 \cdot \boldsymbol{c}_2\boldsymbol{s}_2 \cdots \boldsymbol{c}_{p-1}\boldsymbol{s}_{p-1} \cdot \boldsymbol{b}t\boldsymbol{b}t}_{\boldsymbol{u}_2} \cdot x^*. \tag{5-7}$$

Result G. $\mathsf{occ}(\bar{h}, \bar{\boldsymbol{v}}_2) = 1$ *and* $\mathsf{occ}(\bar{t}, \bar{\boldsymbol{v}}_2) = 2$.

Proof. This holds because $\bar{h} \in \mathsf{con}(\bar{\boldsymbol{s}}_0) \subseteq \mathsf{con}(\overline{\boldsymbol{t}_0\boldsymbol{t}_1\cdots\boldsymbol{t}_q})$ and $\bar{t} \in \mathsf{con}(\bar{\boldsymbol{c}}_p) \subseteq \mathsf{con}(\overline{\boldsymbol{d}_1\boldsymbol{d}_2\cdots\boldsymbol{d}_q})$ by Lemma 5.14. □

Result H. The longest $\{\bar{h}, \bar{t}\}$*-subsequence of* $\boldsymbol{v}_2$ *is* ht^2.

Proof. Since $ht^2 \hookrightarrow \boldsymbol{u}_2$, it follows from Lemma 5.16(i) and Result G that the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$ is either ht^2 or $(t^*)^2h^*$. Seeking a contradiction, suppose that $(t^*)^2h^*$ is the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$. Note that h^* does not occur in any ordered A_0-block in $\boldsymbol{v}_2$ due to $\mathsf{occ}(\bar{h}, \bar{\boldsymbol{v}}_2) = 1$ by Result G. It follows that if $T \neq h^*$, so that $h^*T \hookrightarrow \boldsymbol{v}_2$, then by Lemma 5.16(ii), either $h^*T \hookrightarrow \boldsymbol{u}_2$ or $T^*h \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-7). Hence $T = h^*$.

Now suppose that $H \neq t^*$, so that $Ht^* \hookrightarrow \boldsymbol{v}_2$. Then since $\boldsymbol{v}_2$ is bipartite, $H^*, t \notin \mathsf{con}(\boldsymbol{v}_2)$. There are two cases.

(a) H and t^* are not in the same ordered A_0-block in $\boldsymbol{v}_2$. Then by Lemma 5.16(ii), either $Ht^* \hookrightarrow \boldsymbol{u}_2$ or $tH^* \hookrightarrow \boldsymbol{u}_2$. But it is clear from (5-7) that $Ht^* \not\hookrightarrow \boldsymbol{u}_2$, so only $tH^* \hookrightarrow \boldsymbol{u}_2$ holds. Specifically, $tH^* \hookrightarrow \boldsymbol{c}_p = \boldsymbol{b}t\boldsymbol{b}t$, so that $H^*tH^*t \hookrightarrow \boldsymbol{u}_2$. Therefore, by Lemma 5.15(i), either $H^*tH^*t \hookrightarrow \boldsymbol{v}_2$ or $Ht^*Ht^* \hookrightarrow \boldsymbol{v}_2$; but the former contradicts $H^*, t \notin \mathsf{con}(\boldsymbol{v}_2)$, while the latter contradicts the assumption of the present case.

(b) H and t^* are in the same ordered A_0-block in $\boldsymbol{v}_2$. Specifically, since $H = \mathsf{h}(\boldsymbol{v}_2)$, we have $\boldsymbol{t}_0 = 1$ and $H, t^* \in \mathsf{con}(\boldsymbol{d}_1)$. Given that $\boldsymbol{v}$ is in A_0-standard form, by condition (A4)(d), we have $\bar{t} \preceq \overline{\mathsf{t}(\boldsymbol{d}_1)} \prec \overline{\mathsf{t}(\boldsymbol{v}_2)} = \overline{T} = \bar{h}$, but this contradicts (5-4).

Since neither (a) nor (b) is possible, we must have $H = t^*$. Now since $\overline{H} \prec \overline{T}$ by (5-4), we have $\bar{t} = \overline{H} \prec \overline{T} = \bar{h}$; but this contradicts $\bar{h} \prec \bar{t}$ in (5-4). □

Result I. $h = H$ *and* $t = T$.

Proof. First, suppose that $h \neq H$. Then $Hh \hookrightarrow \boldsymbol{v}_2$ by Result H, and h is not in any ordered A_0-block in $\boldsymbol{v}_2$ due to Result G. Therefore, by Lemma 5.16(ii), either $Hh \hookrightarrow \boldsymbol{u}_2$ or $h^*H^* \hookrightarrow \boldsymbol{u}_2$; but this is impossible in view of (5-7). Hence $h = H$.

Now suppose that $t \neq T$. Then $tT \hookrightarrow \boldsymbol{v}_2$ by Result H, and $t^*, T^* \notin \mathsf{con}(\boldsymbol{v}_2)$ due to $\boldsymbol{v}_2$ being bipartite. If t and T are in the same ordered A_0-block in $\boldsymbol{v}_2$, so that $tTtT \hookrightarrow \boldsymbol{v}_2$, then by Lemma 5.15(ii), either $tTtT \hookrightarrow \boldsymbol{u}_2$ or $t^*T^*t^*T^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-7). Therefore, t and T are not in the same ordered A_0-block in $\boldsymbol{v}_2$. Then by Lemma 5.16(ii), either $tT \hookrightarrow \boldsymbol{u}_2$ or $T^*t^* \hookrightarrow \boldsymbol{u}_2$. But it is clear from (5-7) that $T^*t^* \not\hookrightarrow \boldsymbol{u}_2$, so only $tT \hookrightarrow \boldsymbol{u}_2$ holds. Specifically, $tT \hookrightarrow \boldsymbol{c}_p = \boldsymbol{b}t\boldsymbol{b}t$, so that $TtTt \hookrightarrow \boldsymbol{u}_2$. Hence by Lemma 5.15(i), either $TtTt \hookrightarrow \boldsymbol{v}_2$ or $T^*t^*T^*t^* \hookrightarrow \boldsymbol{v}_2$; but the former contradicts t and T not being in the same ordered A_0-block in $\boldsymbol{v}_2$, while the latter contradicts $t^*, T^* \notin \mathsf{con}(\boldsymbol{v}_2)$. □

<u>Case 4</u>: $\mathsf{occ}(h, \boldsymbol{u}_2) = \mathsf{occ}(t, \boldsymbol{u}_2) = 2$ with $p \geq 2$. Then $\boldsymbol{s}_0 = 1$, $h = \mathsf{h}(\boldsymbol{c}_1)$, $t = \mathsf{t}(\boldsymbol{c}_p)$, and $\boldsymbol{s}_p = 1$, so that $\boldsymbol{c}_1 = h\boldsymbol{a}h\boldsymbol{a}$ and $\boldsymbol{c}_p = \boldsymbol{b}t\boldsymbol{b}t$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$:

$$\boldsymbol{u} = x_1x_2\cdots x_m \cdot x \cdot \underbrace{h\boldsymbol{a}h\boldsymbol{a}\,\boldsymbol{s}_1 \cdot \boldsymbol{c}_2\boldsymbol{s}_2 \cdots \boldsymbol{c}_{p-1}\boldsymbol{s}_{p-1} \cdot \boldsymbol{b}t\boldsymbol{b}t}_{\boldsymbol{u}_2} \cdot x^*. \tag{5-8}$$

Result J. $\mathsf{occ}(\bar{h}, \bar{\boldsymbol{v}}_2) = \mathsf{occ}(\bar{t}, \bar{\boldsymbol{v}}_2) = 2$.

Proof. This holds because $\bar{h}, \bar{t} \in \mathsf{con}(\overline{\boldsymbol{c}_1\boldsymbol{c}_p}) \subseteq \mathsf{con}(\overline{\boldsymbol{d}_1\boldsymbol{d}_2\cdots\boldsymbol{d}_q})$ by Lemma 5.14(ii). □

Result K. The longest $\{\bar{h}, \bar{t}\}$*-subsequence of* $\boldsymbol{v}_2$ *is either* h^2t^2 *or* $(t^*)^2(h^*)^2$.

Proof. Since $h^2t^2 \hookrightarrow \boldsymbol{u}_2$, it follows from Lemma 5.16(i) and Result J that the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$ is one of the following six words: h^2t^2, $htht$, $thth$, $(t^*)^2(h^*)^2$, $t^*h^*t^*h^*$, and $h^*t^*h^*t^*$. If either $htht \hookrightarrow \boldsymbol{v}_2$ or $h^*t^*h^*t^* \hookrightarrow \boldsymbol{v}_2$, then it follows from Lemma 5.15(ii) that either $htht \hookrightarrow \boldsymbol{u}_2$ or $h^*t^*h^*t^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-8). If $thth \hookrightarrow \boldsymbol{v}_2$, then t and h are in the same ordered A_0-block in $\boldsymbol{v}_2$, whence $\bar{t} \prec \bar{h}$; but this contradicts (5-4). A similar contradiction is obtained if $t^*h^*t^*h^* \hookrightarrow \boldsymbol{v}_2$. □

Result L. Suppose that $(t^*)^2(h^*)^2 \hookrightarrow \boldsymbol{v}_2$ *and* $H \neq t^*$. *Then* $\bar{t} \prec \overline{T}$.

Proof. By assumption, $Ht^* \hookrightarrow \boldsymbol{v}_2$. Since $\boldsymbol{v}_2$ is bipartite, we have $H^*, t \notin \mathsf{con}(\boldsymbol{v}_2)$. Suppose that H and t^* are not in the same ordered A_0-block in $\boldsymbol{v}_2$. Then by Lemma 5.16(ii), either $Ht^* \hookrightarrow \boldsymbol{u}_2$ or $tH^* \hookrightarrow \boldsymbol{u}_2$. But it is clear from (5-8) that $Ht^* \not\hookrightarrow \boldsymbol{u}_2$, so only $tH^* \hookrightarrow \boldsymbol{u}_2$ holds. Specifically, $tH^* \hookrightarrow \boldsymbol{c}_p = \boldsymbol{b}t\boldsymbol{b}t$, so that $H^*tH^*t \hookrightarrow \boldsymbol{u}_2$. Therefore, by Lemma 5.15(i), either $H^*tH^*t \hookrightarrow \boldsymbol{v}_2$ or $Ht^*Ht^* \hookrightarrow \boldsymbol{v}_2$; but the former contradicts $H^*, t \notin \mathsf{con}(\boldsymbol{v}_2)$, while the latter contradicts H and t^* not being in the same ordered A_0-block in $\boldsymbol{v}_2$.

Therefore, H and t^* are in the same ordered A_0-block in $\boldsymbol{v}_2$. Specifically, since $H = \mathsf{h}(\boldsymbol{v}_2)$, we have $H, t^* \in \mathsf{con}(\boldsymbol{d}_1)$ and $\boldsymbol{t}_0 = 1$. Given that $\boldsymbol{v}$ is in A_0-standard form, it follows from condition (A4)(d) that $\bar{t} \preceq \overline{\mathsf{t}(\boldsymbol{d}_1)} \prec \overline{\mathsf{t}(\boldsymbol{v}_2)} = \overline{T}$. □

Result M. Suppose that $(t^*)^2(h^*)^2 \hookrightarrow \boldsymbol{v}_2$ and $T \neq h^*$. Then $\overline{T} \prec \bar{t}$.

Proof. By assumption, $h^*T \hookrightarrow \boldsymbol{v}_2$. There are two cases.

(a) T and h^* are not in the same ordered A_0-block in $\boldsymbol{v}_2$. Then by Lemma 5.16(ii), either $h^*T \hookrightarrow \boldsymbol{u}_2$ or $T^*h \hookrightarrow \boldsymbol{u}_2$. It is clear from (5-8) that $h^*T \not\hookrightarrow \boldsymbol{u}_2$, so only $T^*h \hookrightarrow \boldsymbol{u}_2$ holds. Specifically, $T^*h \hookrightarrow \boldsymbol{c}_1 = h\boldsymbol{a}h\boldsymbol{a}$.

(b) T and h^* are in the same ordered A_0-block in $\boldsymbol{v}_2$. Then $h^*Th^*T \hookrightarrow \boldsymbol{v}_2$. By Lemma 5.15(ii), either $h^*Th^*T \hookrightarrow \boldsymbol{u}_2$ or $hT^*hT^* \hookrightarrow \boldsymbol{u}_2$. It is clear from (5-8) that $h^*Th^*T \not\hookrightarrow \boldsymbol{u}_2$, so only $hT^*hT^* \hookrightarrow \boldsymbol{u}_2$ holds, whence $hT^*hT^* \hookrightarrow \boldsymbol{c}_1 = h\boldsymbol{a}h\boldsymbol{a}$.

Therefore, in any case, we have $T^* \in \mathsf{con}(\boldsymbol{c}_1)$. Since $\boldsymbol{s}_0 = 1$ and $\boldsymbol{u}$ is in A_0-standard form, by condition (A4)(d), we have $\overline{T} \preceq \overline{\mathsf{t}(\boldsymbol{c}_1)} \prec \overline{\mathsf{t}(\boldsymbol{u}_2)} = \bar{t}$. □

Result N. The longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$ is h^2t^2.

Proof. By Result K, the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$ is either h^2t^2 or $(t^*)^2(h^*)^2$. Seeking a contradiction, suppose that $(t^*)^2(h^*)^2 \hookrightarrow \boldsymbol{v}_2$. If $H \neq t^*$, then $\bar{t} \prec \overline{T}$ by Result L, whence $T = h^*$ by Result M; but this implies that $\bar{t} \prec \overline{T} = \bar{h}$, which contradicts (5-4). If $T \neq h^*$, then $\overline{T} \prec \bar{t}$ by Result M, whence $H = t^*$ by Result L; but this implies that $\overline{T} \prec \bar{t} = \overline{H}$, which contradicts (5-4) again. Therefore, we must have $H = t^*$ and $T = h^*$. Now since $\overline{H} \prec \overline{T}$ by (5-4), we have $\bar{t} = \overline{H} \prec \overline{T} = \bar{h}$; but this contradicts $\bar{h} \prec \bar{t}$ in (5-4). □

Result O. $h = H$ and $t = T$.

Proof. Seeking a contradiction, suppose that $h \neq H$. Then $Hh \hookrightarrow \boldsymbol{v}_2$ by Result N, and $H^*, h^* \notin \mathsf{con}(\boldsymbol{v}_2)$ due to $\boldsymbol{v}_2$ being bipartite. If H and h are in the same ordered A_0-block in $\boldsymbol{v}_2$, so that $HhHh \hookrightarrow \boldsymbol{v}_2$, then by Lemma 5.15(ii), either $HhHh \hookrightarrow \boldsymbol{u}_2$ or $H^*h^*H^*h^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-8). Therefore, H and h are not in the same ordered A_0-block in $\boldsymbol{v}_2$. Then by Lemma 5.16(ii), either $Hh \hookrightarrow \boldsymbol{u}_2$ or $h^*H^* \hookrightarrow \boldsymbol{u}_2$. It is clear from (5-8) that $h^*H^* \not\hookrightarrow \boldsymbol{u}_2$, so only $Hh \hookrightarrow \boldsymbol{u}_2$ holds. Specifically, $Hh \hookrightarrow \boldsymbol{c}_1 = h\boldsymbol{a}h\boldsymbol{a}$, so that $hHhH \hookrightarrow \boldsymbol{u}_2$. Therefore, by Lemma 5.15(i), either $hHhH \hookrightarrow \boldsymbol{v}_2$ or $h^*H^*h^*H^* \hookrightarrow \boldsymbol{v}_2$; but the former contradicts H and h not being in the same ordered A_0-block in $\boldsymbol{v}_2$, while the latter contradicts $H^*, h^* \notin \mathsf{con}(\boldsymbol{v}_2)$.

A symmetrical argument shows that the assumption $t \neq T$ also leads to a contradiction. □

Case 5: $\mathsf{occ}(h, \boldsymbol{u}_2) = \mathsf{occ}(t, \boldsymbol{u}_2) = 2$ with $p = 1$. Then $\boldsymbol{s}_0 = 1$, $h = \mathsf{h}(\boldsymbol{c}_1)$, $t = \mathsf{t}(\boldsymbol{c}_1)$, and $\boldsymbol{s}_1 = 1$, so that $\boldsymbol{c}_1 = hath\boldsymbol{a}t$ for some $\boldsymbol{a} \in F^1_{\mathsf{inv}}(\mathcal{X})$:

$$\boldsymbol{u} = x_1 x_2 \cdots x_m \cdot x \cdot \underbrace{h\boldsymbol{a}th\boldsymbol{a}t}_{\boldsymbol{u}_2} \cdot x^*. \tag{5-9}$$

Note that due to condition (A4)(b), we have $h \in \mathcal{X}$.

Result P. $\mathsf{occ}(\bar{y}, \bar{\boldsymbol{v}}_2) = 2$ *for all* $y \in \mathsf{con}(\boldsymbol{v}_2)$.

Proof. Since $\boldsymbol{s}_0 = \boldsymbol{s}_1 = 1$ and $\mathsf{con}(\overline{\boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q}) = \mathsf{con}(\overline{\boldsymbol{s}_0\boldsymbol{s}_1})$ by Lemma 5.14(i), we have $\boldsymbol{t}_0 = \boldsymbol{t}_1 = \cdots = \boldsymbol{t}_q = 1$. Therefore, $\boldsymbol{v}_2 = \boldsymbol{d}_1\boldsymbol{d}_2 \cdots \boldsymbol{d}_q$ and the result follows. □

Result Q. The longest $\{\bar{h}, \bar{t}\}$*-subsequence of* $\boldsymbol{v}_2$ *is* $htht$.

Proof. Since $htht \hookrightarrow \boldsymbol{u}_2$, it follows from Lemma 5.15(i) that the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$ is either $htht$ or $h^*t^*h^*t^*$. Seeking a contradiction, suppose that $h^*t^*h^*t^* \hookrightarrow \boldsymbol{v}_2$.

First consider the case when $H \neq h^*$, so that $Hh^* \hookrightarrow \boldsymbol{v}_2$. Then $H^*, h \notin \mathsf{con}(\boldsymbol{v}_2)$ due to $\boldsymbol{v}_2$ being bipartite. If H and h^* are in the same ordered A_0-block in $\boldsymbol{v}_2$, so that $Hh^*Hh^* \hookrightarrow \boldsymbol{v}_2$, then it follows from Lemma 5.15(ii) that either $Hh^*Hh^* \hookrightarrow \boldsymbol{u}_2$ or $H^*hH^*h \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-9). Therefore, H and h^* are not in the same ordered A_0-block in $\boldsymbol{v}_2$. Then by Lemma 5.16(ii), either $Hh^* \hookrightarrow \boldsymbol{u}_2$ or $hH^* \hookrightarrow \boldsymbol{u}_2$. It is clear from (5-9) that $Hh^* \not\hookrightarrow \boldsymbol{u}_2$, so only $hH^* \hookrightarrow \boldsymbol{u}_2$ holds. It follows that $hH^*hH^* \hookrightarrow \boldsymbol{u}_2$, so that by Lemma 5.15(i), either $hH^*hH^* \hookrightarrow \boldsymbol{v}_2$ or $h^*Hh^*H \hookrightarrow \boldsymbol{v}_2$; but the former contradicts $H^*, h \notin \mathsf{con}(\boldsymbol{v}_2)$, while the latter contradicts H and h^* not being in the same ordered A_0-block in $\boldsymbol{v}_2$.

Therefore, $H = h^*$. By a symmetrical argument, we have $T = t^*$. It follows that $HTHT = h^*t^*h^*t^* \hookrightarrow \boldsymbol{v}_2$, so that $\boldsymbol{v}_2 = \boldsymbol{d}_1 = h^*\boldsymbol{b}t^*h^*\boldsymbol{b}t^*$ for some $\boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$ (with $q = 1$). As deduced in the proof of Result P, we have $\boldsymbol{t}_0 = \boldsymbol{t}_1 = 1$. Since $\boldsymbol{v}$ is in A_0-standard form, by condition (A4)(b), we have $h^* = \mathsf{h}(\boldsymbol{d}_1) \in \mathcal{X}$, which contradicts the observation $h \in \mathcal{X}$ made after (5-9). □

Result R. $h = H$ *and* $t = T$.

Proof. Seeking a contradiction, suppose that $h \neq H$. Then $Hh \hookrightarrow \boldsymbol{v}_2$ by Result Q, and $H^*, h^* \notin \mathsf{con}(\boldsymbol{v}_2)$ due to $\boldsymbol{v}_2$ being bipartite. If H and h are in the same ordered A_0-block in $\boldsymbol{v}_2$, so that $HhHh \hookrightarrow \boldsymbol{v}_2$, then it follows from Lemma 5.15(ii) that either $HhHh \hookrightarrow \boldsymbol{u}_2$ or $H^*h^*H^*h^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (5-9). Therefore, H and h are not in the same ordered A_0-block in $\boldsymbol{v}_2$. Then by Lemma 5.16(ii), either $Hh \hookrightarrow \boldsymbol{u}_2$ or $h^*H^* \hookrightarrow \boldsymbol{u}_2$. It is clear from

(5-9) that $h^*H^* \not\hookrightarrow \boldsymbol{u}_2$, so only $Hh \hookrightarrow \boldsymbol{u}_2$ holds. It follows that $hHhH \hookrightarrow \boldsymbol{u}_2$, so that by Lemma 5.15(i), either $hHhH \hookrightarrow \boldsymbol{v}_2$ or $h^*H^*h^*H^* \hookrightarrow \boldsymbol{v}_2$; but the former contradicts H and h not being in the same ordered A_0-block in $\boldsymbol{v}_2$, while the latter contradicts $H^*, h^* \notin \mathsf{con}(\boldsymbol{v}_2)$.

A contradiction can be similarly deduced if $t \neq T$. □

In conclusion, we have $h = H$ and $t = T$ in all five cases (Results C, F, I, O, and R). The proof of Lemma 5.17 is thus complete. □

Lemma 5.18. *The identity $\boldsymbol{u}_2 \approx \boldsymbol{v}_2$ is satisfied by $(A_0,{}^*)$.*

Proof. Recall from Lemma 5.13(iii) that $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$. As shown in the beginning of the proof of Lemma 5.17, if $|\mathsf{con}(\bar{\boldsymbol{u}}_2)| = |\mathsf{con}(\bar{\boldsymbol{v}}_2)| = 1$, then $\boldsymbol{u}_2, \boldsymbol{v}_2 \in \{z, z^2\}$ for some $z \in \mathcal{X}$, so that $\boldsymbol{u}_2 = \boldsymbol{v}_2$ by Lemma 5.14 and conditions (A4)(a) and (A4)(b). Therefore, it suffices to assume that $|\mathsf{con}(\bar{\boldsymbol{u}}_2)| = |\mathsf{con}(\bar{\boldsymbol{v}}_2)| \geq 2$, so that by Lemma 5.17, we have $h = \mathsf{h}(\boldsymbol{u}_2) = \mathsf{h}(\boldsymbol{v}_2)$ and $t = \mathsf{t}(\boldsymbol{u}_2) = \mathsf{t}(\boldsymbol{v}_2)$ with $\bar{h} \prec \bar{t}$. Specifically, $\boldsymbol{u}_2 = h\boldsymbol{a}t$ and $\boldsymbol{v}_2 = h\boldsymbol{b}t$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$.

Seeking a contradiction, suppose that $\boldsymbol{u}_2 \approx \boldsymbol{v}_2$ is not satisfied by $(A_0,{}^*)$. Then there exists a substitution $\psi : \mathsf{con}(\bar{\boldsymbol{u}}_2) \to A_0$ such that $\psi(\boldsymbol{u}_2) \neq \psi(\boldsymbol{v}_2)$, so that

$$\psi(h) \cdot \psi(\boldsymbol{a}) \cdot \psi(t) \neq \psi(h) \cdot \psi(\boldsymbol{b}) \cdot \psi(t). \tag{5-10}$$

Clearly, $\psi(h) \neq 0 \neq \psi(t)$. Further, it is routinely checked that

$$\begin{array}{lll}
\mathrm{E} \cdot A_0^1 \cdot \mathrm{E} = \{0, \mathrm{E}\}, & \mathrm{E} \cdot A_0^1 \cdot \mathrm{F} = \{0, \mathrm{EF}\}, & \mathrm{E} \cdot A_0^1 \cdot \mathrm{EF} = \{0, \mathrm{EF}\}, \\
\mathrm{F} \cdot A_0^1 \cdot \mathrm{E} = \{0\}, & \mathrm{F} \cdot A_0^1 \cdot \mathrm{F} = \{0, \mathrm{F}\}, & \mathrm{F} \cdot A_0^1 \cdot \mathrm{EF} = \{0\}, \\
\mathrm{EF} \cdot A_0^1 \cdot \mathrm{E} = \{0\}, & \mathrm{EF} \cdot A_0^1 \cdot \mathrm{F} = \{0, \mathrm{EF}\}, & \mathrm{EF} \cdot A_0^1 \cdot \mathrm{EF} = \{0\}.
\end{array}$$

Thus for (5-10) to hold, we need $(\psi(h), \psi(t)) \in \{(\mathrm{E},\mathrm{E}), (\mathrm{E},\mathrm{F}), (\mathrm{E},\mathrm{EF}), (\mathrm{F},\mathrm{F}), (\mathrm{EF},\mathrm{F})\}$, whence $\{\psi(\boldsymbol{u}_2), \psi(\boldsymbol{v}_2)\}$ can be $\{0, \mathrm{E}\}$, $\{0, \mathrm{F}\}$, or $\{0, \mathrm{EF}\}$. Generality is not lost by assuming that $\psi(\boldsymbol{u}_2) = 0$ and $\psi(\boldsymbol{v}_2) \in \{\mathrm{E}, \mathrm{F}, \mathrm{EF}\}$. Now extend ψ to the substitution Ψ that maps every $z \in \mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$ to $\psi(z)$ and every $z \in \{x_1, x_2, \dots, x_m, x\}$ to E. Then $\Psi(\boldsymbol{u}) \neq \Psi(\boldsymbol{v})$ because

$$\Psi(\boldsymbol{u}) = \Psi(x_1) \cdot \Psi(x_2) \cdots \Psi(x_m) \cdot \Psi(x) \cdot \psi(\boldsymbol{u}_2) \cdot \Psi(x)^* = \mathrm{E} \cdot 0 \cdot \mathrm{F} = 0$$

$$\text{and } \Psi(\boldsymbol{v}) = \Psi(x_1) \cdot \Psi(x_2) \cdots \Psi(x_m) \cdot \Psi(x) \cdot \psi(\boldsymbol{v}_2) \cdot \Psi(x)^* = \mathrm{E} \cdot \psi(\boldsymbol{v}_2) \cdot \mathrm{F} = \mathrm{EF};$$

but this is impossible given that $\boldsymbol{u} \approx \boldsymbol{v}$ is satisfied by $(A_0,{}^*)$. □

Since the words $\boldsymbol{u}_2$ and $\boldsymbol{v}_2$ are bipartite, it follows from Lemmas 5.5 and 5.18 that $\boldsymbol{u}_2 \approx \boldsymbol{v}_2$ is deducible from {(1-1), (5-1)}. Since

$$\boldsymbol{u} = x_1x_2 \cdots x_m \cdot x \cdot \boldsymbol{u}_2 \cdot x^* \quad \text{and} \quad \boldsymbol{v} = x_1x_2 \cdots x_m \cdot x \cdot \boldsymbol{v}_2 \cdot x^*,$$

the identity $\boldsymbol{u} \approx \boldsymbol{v}$ is also deducible from {(1-1), (5-1)}. The proof of Proposition 5.1 is thus complete.

6. The involution semigroup $(S_4, {}^*)$

The involution semigroup $(S_4, {}^*)$ is isomorphic to the semigroup

$$B_0 = \langle \mathrm{A}, \mathrm{E}, \mathrm{F} \mid \mathrm{AF} = \mathrm{EA} = \mathrm{A},\ \mathrm{E}^2 = \mathrm{E},\ \mathrm{F}^2 = \mathrm{F},\ \mathrm{EF} = \mathrm{FE} = 0 \rangle = \{0, \mathrm{A}, \mathrm{E}, \mathrm{F}\}$$

with the operation * that interchanges E and F and fixes every other element.

B_0	0	A	E	F
0	0	0	0	0
A	0	0	0	A
E	0	A	E	0
F	0	0	0	F
x	0	A	E	F
x^*	0	A	F	E

The involution semigroup $(B_0, {}^*)$ is isomorphic to the involution subsemigroup of $(B_2^1, {}^{\mathsf{S}})$ that consists of the elements

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The involution semigroup $(B_0, {}^*)$ belongs to the variety $\mathsf{Var}(A_0, {}^*)$ generated by $(A_0, {}^*)$ [20, Proposition 3.1] and so satisfies the identities (5-1) of $(A_0, {}^*)$. In this section, it is shown that the identities of $(B_0, {}^*)$ are axiomatized by (5-1) and one additional identity.

Proposition 6.1. *The identities* (5-1) *and*

$$x^2y^2 \approx y^2x^2 \tag{6-1}$$

constitute an identity basis for $(B_0, {}^*)$.

It is easily checked that $(B_0, {}^*)$ satisfies the identities {(5-1), (6-1)}. In Section 6.1, some information on identities of $(B_0, {}^*)$ are given. In Section 6.2, it is shown that the identities of $(B_0, {}^*)$ can be used to convert every mixed word into one of two specific forms. Based on these results, it is shown in Section 6.3 that every identity of $(B_0, {}^*)$ is deducible from {(1-1), (5-1), (6-1)}. This completes the proof of Proposition 6.1.

Corollary 6.2. *The identities*

$$\begin{gathered} x^3 \approx x^2, \quad xyx \approx x^2y^2, \quad x^2x^* \approx xx^*, \quad x^2yx^* \approx xyx^*, \\ xy^*x^* \approx xyx^*, \quad xx^* \approx yy^*, \quad xyx^*z \approx z^*xyx^* \end{gathered} \tag{6-2}$$

constitute an identity basis for $(B_0, {}^*)$.

Proof. It is routine to check, say with Prover9, that the identities {(1-1), (5-1), (6-1)} and {(1-1), (6-2)} are deducible from one another. □

Remarks 6.3. (i) The variety $\mathsf{Var}\, B_0$ is defined within $\mathsf{Var}\, A_0$ by the identity (6-1), and $\mathsf{Var}\, B_0$ is the unique maximal subvariety of $\mathsf{Var}\, A_0$ [14, Lemma 4.2]; in other words, the interval $[\mathsf{Var}\, B_0,\ \mathsf{Var}\, A_0]$ is a chain of length two.

(ii) In contrast, although the variety $\mathsf{Var}(B_0,{}^*)$ is also defined within $\mathsf{Var}(A_0,{}^*)$ by the identity (6-1) (Proposition 6.1), the interval $[\mathsf{Var}(B_0,{}^*), \mathsf{Var}(A_0,{}^*)]$ contains an infinite descending chain [20, Theorem 1.5].

6.1. *Some identities of* $(B_0,{}^*)$.

Lemma 6.4. *The identities* {(5-1h), (5-1i), (6-1)} *constitute an identity basis for the semigroup* B_0.

Proof. The identities of A_0, together with (6-1), form an identity basis for B_0 [14, Section 4]. The present lemma then follows from Lemma 5.4. □

Lemma 6.5. *Let* $\boldsymbol{u} \approx \boldsymbol{v}$ *be any identity of* $(B_0,{}^*)$ *such that either* $\boldsymbol{u}$ *or* $\boldsymbol{v}$ *is bipartite. Then* $\boldsymbol{u} \approx \boldsymbol{v}$ *is deducible from* {(1-1), (5-1), (6-1)}.

Proof. Since $(S\ell_3,{}^{\mathsf{S}})$ is isomorphic to the involution subsemigroup $(\{0, \mathrm{E}, \mathrm{F}\},{}^*)$ of $(B_0,{}^*)$, the identity $\boldsymbol{u} \approx \boldsymbol{v}$ is satisfied by $(S\ell_3,{}^{\mathsf{S}})$. Since either $\boldsymbol{u}$ or $\boldsymbol{v}$ is bipartite, by Lemma 2.4, both $\boldsymbol{u}$ and $\boldsymbol{v}$ are bipartite with $\mathsf{con}(\boldsymbol{u}) = \mathsf{con}(\boldsymbol{v})$. It follows from Lemma 2.2 that $(B_0,{}^*)$ satisfies the plain identity $\bar{\boldsymbol{u}} \approx \bar{\boldsymbol{v}}$. By Lemma 6.4, the identities {(5-1h), (5-1i), (6-1)} constitute an identity basis for B_0, so that $\bar{\boldsymbol{u}} \approx \bar{\boldsymbol{v}}$ is deducible from {(5-1h), (5-1i), (6-1)}. It then follows from Lemma 2.2 that $\boldsymbol{u} \approx \boldsymbol{v}$ is deducible from {(1-1), (5-1), (6-1)}. □

An *ordered* B_0*-block* is a word of the form

$$\boldsymbol{c} = y_1^2 y_2^2 \cdots y_k^2,$$

where $y_1, y_2, \ldots, y_k \in \mathcal{X} \cup \mathcal{X}^*$ are such that $\bar{y}_1 \prec \bar{y}_2 \prec \cdots \prec \bar{y}_k$ in $\mathcal{X}$ and $k \geq 1$. Note that every ordered B_0-block is bipartite.

Lemma 6.6. *Let* $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m \in F_{\mathsf{inv}}(\mathcal{X})$ *be any pairwise disjoint bipartite connected words such that* $\mathsf{con}(\boldsymbol{w}_1 \boldsymbol{w}_2 \cdots \boldsymbol{w}_m) = \{y_1, y_2, \ldots, y_k\}$ *and* $\bar{y}_1 \prec \bar{y}_2 \prec \cdots \prec \bar{y}_k$ *in* $\mathcal{X}$. *Then the identities* {(5-1h), (5-1i), (6-1)} *can be used to convert the product* $\boldsymbol{w}_1 \boldsymbol{w}_2 \cdots \boldsymbol{w}_m$ *into the ordered* B_0*-block* $\boldsymbol{c} = y_1^2 y_2^2 \cdots y_k^2$.

Proof. By Lemma 5.7, the identities {(5-1h), (5-1i)} can be used to convert each $\boldsymbol{w}_i$ into some ordered A_0-block $\boldsymbol{c}_i$ with $\mathsf{con}(\boldsymbol{w}_i) = \mathsf{con}(\boldsymbol{c}_i)$. Since $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_m$ are ordered A_0-blocks, we have

$$\boldsymbol{c}_i \overset{\text{(5-1h)}}{\approx} \boldsymbol{c}_i^2 \quad \text{and} \quad \boldsymbol{c}_i \boldsymbol{c}_j \overset{\text{(6-1)}}{\approx} \boldsymbol{c}_j \boldsymbol{c}_i.$$

Hence

$$\boldsymbol{w}_1 \boldsymbol{w}_2 \cdots \boldsymbol{w}_m \overset{\text{(5-1h),(5-1i)}}{\approx} \boldsymbol{c}_1 \boldsymbol{c}_2 \cdots \boldsymbol{c}_m \overset{\text{(5-1h)}}{\approx} \boldsymbol{c}_1^2 \boldsymbol{c}_2^2 \cdots \boldsymbol{c}_m^2 \overset{\text{(6-1)}}{\approx} (\boldsymbol{c}_1 \boldsymbol{c}_2 \cdots \boldsymbol{c}_m)^2.$$

Since $(\boldsymbol{c}_1\boldsymbol{c}_2\cdots\boldsymbol{c}_m)^2$ is a bipartite connected word with content $\{y_1, y_2, \dots, y_k\}$, by Lemma 5.7, the identities {(5-1h), (5-1i)} can be used to convert it into the ordered A_0-block $(y_1y_2\cdots y_k)^2$. Hence

$$(y_1y_2\cdots y_k)^2 \overset{\text{(5-1h)}}{\approx} (y_1^2y_2^2\cdots y_k^2)^2 \overset{\text{(6-1)}}{\approx} y_1^4y_2^4\cdots y_k^4 \overset{\text{(5-1h)}}{\approx} y_1^2y_2^2\cdots y_k^2. \qquad \square$$

6.2. *Some special forms of words.* It is easily checked that for any substitution $\varphi : \mathcal{X} \to B_0$ and any variable $z \in \mathcal{X}$, we have $\varphi(zz^*) = 0$ in B_0. Therefore, in the $\mathsf{Var}(B_0,{}^*)$-free algebra over $\mathcal{X}$, the class $[zz^*]$ containing zz^* is its zero element. This phenomenon is equivalent to the following result, whose justification is routine.

Lemma 6.7. *The identities*

$$\text{(6-3)} \qquad xx^*y \approx xx^*, \quad yxx^* \approx xx^*, \quad xx^* \approx yy^*$$

are deducible from {(1-1), (5-1), (6-1)}.

Words of other possible forms in the class $[zz^*]$ are listed in the following result.

Lemma 6.8. *Let* $\boldsymbol{w} \in F_{\mathsf{inv}}(\mathcal{X})$. *Suppose that one of the following conditions holds*:

(a) $xx^*x \hookrightarrow \boldsymbol{w}$ *for some* $x \in \mathcal{X} \cup \mathcal{X}^*$;

(b) $xx^*yy^* \hookrightarrow \boldsymbol{w}$ *for some* $x, y \in \mathcal{X} \cup \mathcal{X}^*$;

(c) $\boldsymbol{w} = \boldsymbol{a}x\boldsymbol{b}x^*\boldsymbol{e}$ *for some* $x \in \mathcal{X} \cup \mathcal{X}^*$ *and* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e} \in F^1_{\mathsf{inv}}(\mathcal{X})$ *such that for each* $y \in \mathsf{con}(\boldsymbol{b})$, *we have* $\mathsf{occ}(y, \boldsymbol{w}) \geq 2$.

Then the identities {(5-1), (6-1)} *can be used to convert* $\boldsymbol{w}$ *into the word* zz^* *for any* $z \in \mathcal{X} \cup \mathcal{X}^*$.

Proof. By Lemma 6.7, it suffices to convert $\boldsymbol{w}$ into the word zz^*, using the identities {(5-1), (6-1), (6-3)}. If either (a) or (b) holds, then by Lemma 5.8,

$$\boldsymbol{w} \overset{\text{(5-1)}}{\approx} zz^*z \overset{\text{(6-3)}}{\approx} zz^*.$$

Thus suppose (c) holds. By assumption, $\boldsymbol{b} = y_1y_2\cdots y_m$ for some $y_1, y_2, \dots, y_m \in \mathcal{X} \cup \mathcal{X}^*$ with $m \geq 0$ such that $\mathsf{occ}(y_i, \boldsymbol{w}) \geq 2$ for all i. Then by Lemma 6.7,

$$\boldsymbol{w} \overset{\text{(5-1c)}}{\approx} \boldsymbol{a}x^2\boldsymbol{b}x^*\boldsymbol{e} \overset{\text{(5-1h)}}{\approx} \boldsymbol{a}x^2y_1^2y_2^2\cdots y_m^2x^*\boldsymbol{e} \overset{\text{(6-1)}}{\approx} \boldsymbol{a}y_1^2y_2^2\cdots y_m^2x^2x^*\boldsymbol{e} \overset{\text{(6-3)}}{\approx} zz^*. \quad \square$$

A word $\boldsymbol{w} \in F_{\mathsf{inv}}(\mathcal{X})$ is in B_0-*standard form* if

$$\text{(6-4)} \qquad \boldsymbol{w} = \boldsymbol{w}_1x\boldsymbol{w}_2x^*,$$

where $x \in \mathcal{X} \cup \mathcal{X}^*$, $\boldsymbol{w}_1 = x_1x_2\cdots x_m$, and $\boldsymbol{w}_2 = \boldsymbol{s}_0\prod_{i=1}^{p}(\boldsymbol{c}_i\boldsymbol{s}_i)$ for some $m, p \geq 0$ such that the following conditions hold:

(B1) $x_1, x_2, \dots, x_m \in \mathcal{X} \cup \mathcal{X}^*$ are such that $\bar{x}_1 \prec \bar{x}_2 \prec \cdots \prec \bar{x}_m \prec \bar{x}$;

(B2) $\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p \in F_{\mathsf{inv}}(\mathcal{X})$ are simple and $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_p \in F_{\mathsf{inv}}(\mathcal{X})$ are ordered B_0-blocks;

(B3) $x_1, x_2, \ldots, x_m, x, \boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p, \boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_p$ are pairwise disjoint;

(B4) either

(a) $p = 0$ with $\boldsymbol{w}_2 = \boldsymbol{s}_0$ and $\boldsymbol{s}_0 \in \mathcal{X}$; or

(b) $\overline{\mathsf{h}(\boldsymbol{w}_2)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$.

Remark 6.9. The following holds for the word $\boldsymbol{w}$ in (6-4) in B_0-standard form:

(i) If $m = 0$, then $\boldsymbol{w}_1 = 1$.

(ii) If $p = 0$, then $\boldsymbol{w}_2 = \boldsymbol{s}_0 \in F_{\mathsf{inv}}(\mathcal{X})$; in particular, $\boldsymbol{w}_2$ always contains some simple variable and so is nonempty.

(iii) $\{x, x^*\}$ is the only mixed pair of $\boldsymbol{w}$ and $x, x^* \notin \mathsf{con}(\boldsymbol{w}_1\boldsymbol{w}_2)$.

(iv) $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are bipartite words such that $\mathsf{con}(\overline{\boldsymbol{w}}_1) \cap \mathsf{con}(\overline{\boldsymbol{w}}_2) = \varnothing$.

(v) Each variable in $\mathcal{X}$ occurs at most twice in $\overline{\boldsymbol{w}}$.

Lemma 6.10. *Let $\boldsymbol{w} = \boldsymbol{w}_1 x \boldsymbol{w}_2 x^*$ be the word in (6-4) in B_0-standard form and $z \in \mathcal{X} \cup \mathcal{X}^*$ be any simple variable in $\boldsymbol{w}$, so that $z \in \mathsf{con}(\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p)$ and $\boldsymbol{w}_2 = \boldsymbol{a}z\boldsymbol{b}$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$. Then there exists a substitution $\gamma^z_{\boldsymbol{w}} : \mathcal{X} \to B_0$ such that*

(i) $\gamma^z_{\boldsymbol{w}}(\boldsymbol{w}_1 x \boldsymbol{a}) = \mathrm{E}$, $\gamma^z_{\boldsymbol{w}}(z) = \mathrm{A}$, *and* $\gamma^z_{\boldsymbol{w}}(\boldsymbol{b}x^*) = \mathrm{F}$, *so that* $\gamma^z_{\boldsymbol{w}}(\boldsymbol{w}) = \mathrm{A}$;

(ii) $\gamma^z_{\boldsymbol{w}}(s) = 0$ *for all* $s \in \mathcal{X}$ *such that* $s \notin \mathsf{con}(\overline{\boldsymbol{w}})$.

Proof. It follows from Remark 6.9(iii),(iv) that $\mathsf{con}(\boldsymbol{w}_1) = \mathcal{H}_1 \cup \mathcal{K}^*_1$, $\mathsf{con}(\boldsymbol{a}) = \mathcal{H}_2 \cup \mathcal{K}^*_2$, and $\mathsf{con}(\boldsymbol{b}) = \mathcal{H}_3 \cup \mathcal{K}^*_3$ for some $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 \subseteq \mathcal{X}$ such that $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \{x, x^*\}, \{z\}$ are pairwise disjoint sets. By symmetry, it suffices to assume that $x \in \mathcal{X}$, so that $\mathsf{con}(\overline{\boldsymbol{w}}) = \mathcal{H}_1 \cup \mathcal{K}_1 \cup \mathcal{H}_2 \cup \mathcal{K}_2 \cup \mathcal{H}_3 \cup \mathcal{K}_3 \cup \{x, \bar{z}\}$. Define

$$\gamma^z_{\boldsymbol{w}}(s) = \begin{cases} \mathrm{E} & \text{if } s \in \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{K}_3 \cup \{x\}, \\ \mathrm{A} & \text{if } s = \bar{z}, \\ \mathrm{F} & \text{if } s \in \mathcal{K}_1 \cup \mathcal{K}_2 \cup \mathcal{H}_3, \\ 0 & \text{otherwise.} \end{cases}$$

Then it is routinely checked that the substitution $\gamma^z_{\boldsymbol{w}}$ satisfies (i) and (ii). □

Corollary 6.11. *For any word $\boldsymbol{w}$ in B_0-standard form and any $z \in \mathcal{X} \cup \mathcal{X}^*$, the identity $\boldsymbol{w} \approx zz^*$ is not satisfied by $(B_0, {}^*)$.*

Proof. Let $\boldsymbol{w} = \boldsymbol{w}_1 x \boldsymbol{w}_2 x^*$ be the word in (6-4) in B_0-standard form. Then by Remark 6.9(ii), the word $\boldsymbol{w}_2$ contains some simple variable $s \in \mathcal{X} \cup \mathcal{X}^*$, so that $\boldsymbol{w}_2 = \boldsymbol{a}s\boldsymbol{b}$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$. Under the substitution $\gamma^s_{\boldsymbol{w}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have $\gamma^s_{\boldsymbol{w}}(\boldsymbol{w}) = \mathrm{A}$ and $\gamma^s_{\boldsymbol{w}}(zz^*) = 0$. □

Lemma 6.12. *Let $\boldsymbol{w}$ be any mixed word. Then the identities* {(1-1), (5-1), (6-1)} *can be used to convert $\boldsymbol{w}$ into exactly one of the following*:

(i) *the word zz^* for any $z \in \mathcal{X} \cup \mathcal{X}^*$*;

(ii) *some word in B_0-standard form.*

Proof. By Lemma 6.7, it suffices to convert $\boldsymbol{w}$ into (i) or (ii), using the identities {(1-1), (5-1), (6-1), (6-3)}. By Lemma 5.12, the identities {(1-1), (5-1)} can first be used to convert $\boldsymbol{w}$ into either zz^*z or some word in A_0-standard form. In the former case, the first identity in (6-3) can be used to convert zz^*z into zz^*. Therefore, it remains to assume that $\boldsymbol{w} = \boldsymbol{w}_1 x \boldsymbol{w}_2 x^*$, where $\boldsymbol{w}_1 = x_1 x_2 \cdots x_m$ and $\boldsymbol{w}_2 = \boldsymbol{s}_0 \prod_{i=1}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i)$, satisfies conditions (A1)–(A4). Then conditions (B1) and (B3) hold because they coincide with conditions (A1) and (A3).

By (A2), $\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p \in F^1_{\mathsf{inv}}(\mathcal{X})$ are simple and $\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_p \in F_{\mathsf{inv}}(\mathcal{X})$ are ordered A_0-blocks. By Lemma 6.6, each $\boldsymbol{c}_i$ can be converted by {(5-1), (6-1)} into some ordered B_0-block $y^2_{i,1} y^2_{i,2} \cdots y^2_{i,h_i}$. If $\boldsymbol{s}_0 = \boldsymbol{s}_1 = \cdots = \boldsymbol{s}_p = 1$, then by Lemma 6.8, the identities {(1-1), (5-1)} can be used to convert $\boldsymbol{w}$ into zz^*. Therefore, assume that $\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p$ are not all empty. If $\boldsymbol{s}_i = 1$ for some $i \in \{1, 2, \ldots, p-1\}$, so that the ordered B_0-blocks $\boldsymbol{c}_i$ and $\boldsymbol{c}_{i+1}$ are adjacent, then (6-1) can be used to arrange the squares $y^2_{i,1}, y^2_{i,2}, \ldots, y^2_{i,h_i}, y^2_{i+1,1}, y^2_{i+1,2}, \ldots, y^2_{i+1,h_{i+1}}$ in the product $\boldsymbol{c}_i \boldsymbol{c}_{i+1}$ in order, resulting in a single ordered B_0-block. Hence we may assume that for each $i \in \{1, 2, \ldots, p-1\}$, the words $\boldsymbol{c}_i$ and $\boldsymbol{c}_{i+1}$ are separated due to $\boldsymbol{s}_i \neq 1$. If $\boldsymbol{s}_0 = 1$, so that x is adjacent to the ordered B_0-block $\boldsymbol{c}_1$, then the identities {(5-1), (6-1)} can be used to move $\boldsymbol{c}_1$ to the left of x and turn it into a simple word:

$$\begin{aligned}
\boldsymbol{w} &\overset{(5\text{-}1c)}{\approx} x_1 x_2 \cdots x_m \cdot x^2 \cdot \overbrace{y^2_{1,1} y^2_{1,2} \cdots y^2_{1,h_1}}^{\boldsymbol{c}_1} \cdot \boldsymbol{s}_1 \Big(\prod_{i=2}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i) \Big) x^* \\
&\overset{(6\text{-}1)}{\approx} x_1 x_2 \cdots x_m \cdot y^2_{1,1} y^2_{1,2} \cdots y^2_{1,h_1} \cdot x^2 \cdot \boldsymbol{s}_1 \Big(\prod_{i=2}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i) \Big) x^* \\
&\overset{(5\text{-}1f)}{\approx} x_1 x_2 \cdots x_m \cdot y_{1,1} y_{1,2} \cdots y_{1,h_1} \cdot x^2 \cdot \boldsymbol{s}_1 \Big(\prod_{i=2}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i) \Big) x^* \\
&\overset{(5\text{-}1c)}{\approx} x_1 x_2 \cdots x_m \cdot y_{1,1} y_{1,2} \cdots y_{1,h_1} \cdot x \cdot \boldsymbol{s}_1 \Big(\prod_{i=2}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i) \Big) x^*;
\end{aligned}$$

by the arguments in the proof of Lemma 5.12, we may assume that

$$\bar{x}_1 \prec \bar{x}_2 \prec \cdots \prec \bar{x}_m \prec \overline{y_{1,1}} \prec \overline{y_{1,2}} \prec \cdots \prec \overline{y_{1,h_1}} \prec \bar{x}.$$

If $\boldsymbol{s}_p = 1$, so that x^* is adjacent to the ordered B_0-block $\boldsymbol{c}_p$, then the identities {(5-1), (6-1)} can be used to move $\boldsymbol{c}_p$ to the left of x and turn it into a simple word:

$$\boldsymbol{w} \overset{(5\text{-}1c)}{\approx} x_1 x_2 \cdots x_m \cdot x \cdot \boldsymbol{s}_0 \Big(\prod_{i=1}^{p-1} (\boldsymbol{c}_i \boldsymbol{s}_i) \Big) \overbrace{y^2_{p,1} y^2_{p,2} \cdots y^2_{p,h_p}}^{\boldsymbol{c}_p} \cdot (x^*)^2$$

$$\overset{(6\text{-}1)}{\approx} x_1x_2\cdots x_m\cdot x\cdot \boldsymbol{s}_0\Big(\prod_{i=1}^{p-1}(\boldsymbol{c}_i\boldsymbol{s}_i)\Big)(x^*)^2y_{p,1}^2y_{p,2}^2\cdots y_{p,h_p}^2$$

$$\overset{(5\text{-}1e)}{\approx} x_1x_2\cdots x_m\cdot (y_{p,1}^*)^2(y_{p,2}^*)^2\cdots (y_{p,h_p}^*)^2\cdot x\cdot \boldsymbol{s}_0\Big(\prod_{i=1}^{p-1}(\boldsymbol{c}_i\boldsymbol{s}_i)\Big)(x^*)^2$$

$$\overset{(5\text{-}1f)}{\approx} x_1x_2\cdots x_m\cdot y_{p,1}^*y_{p,2}^*\cdots y_{p,h_p}^*\cdot x\cdot \boldsymbol{s}_0\Big(\prod_{i=1}^{p-1}(\boldsymbol{c}_i\boldsymbol{s}_i)\Big)(x^*)^2$$

$$\overset{(5\text{-}1c)}{\approx} x_1x_2\cdots x_m\cdot y_{p,1}^*y_{p,2}^*\cdots y_{p,h_p}^*\cdot x\cdot \boldsymbol{s}_0\Big(\prod_{i=1}^{p-1}(\boldsymbol{c}_i\boldsymbol{s}_i)\Big)x^*;$$

by repeating the arguments in the proof of Lemma 5.12, we may assume that

$$\bar{x}_1 \prec \bar{x}_2 \prec \cdots \prec \bar{x}_m \prec \overline{y_{p,1}} \prec \overline{y_{p,2}} \prec \cdots \prec \overline{y_{p,h_p}} \prec \bar{x}.$$

Therefore, we may assume that $\boldsymbol{s}_0, \boldsymbol{s}_p \neq 1$. It follows that $\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_p \in F_{\mathsf{inv}}(\mathcal{X})$, so that condition (B2) is satisfied.

It remains to address condition (B4). If $\boldsymbol{w}_2$ is a single variable, so that $p = 0$ with $\boldsymbol{w}_2 = \boldsymbol{s}_0 \in \mathcal{X} \cup \mathcal{X}^*$, then the identity (5-1b) can be used to convert $\boldsymbol{s}_0$ into a variable in $\mathcal{X}$, whence condition (B4)(a) is satisfied. Hence, assume that $\boldsymbol{w}_2$ is not a single variable, so that $\overline{\mathsf{h}(\boldsymbol{w}_2)} \neq \overline{\mathsf{t}(\boldsymbol{w}_2)}$ by conditions (B2) and (B3). In this case, since each $\boldsymbol{s}_i$ is a nonempty simple word, we have $\boldsymbol{s}_i = s_{i,1}s_{i,2}\cdots s_{i,k_i}$ for some $s_{i,1}, s_{i,2}, \ldots, s_{i,k_i} \in \mathcal{X}\cup\mathcal{X}^*$ such that $\overline{s_{i,1}}, \overline{s_{i,2}}, \ldots, \overline{s_{i,k_i}}$ are distinct. If $\overline{\mathsf{h}(\boldsymbol{w}_2)} \prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, then condition (B4)(b) is satisfied. If $\overline{\mathsf{h}(\boldsymbol{w}_2)} \not\prec \overline{\mathsf{t}(\boldsymbol{w}_2)}$, so that $\overline{s_{p,k_p}} \prec \overline{s_{0,1}}$, then

$$\boldsymbol{w} \overset{(5\text{-}1b)}{\approx} \boldsymbol{w}_1\cdot x\cdot \left(\boldsymbol{s}_0\prod_{i=1}^{p}(\boldsymbol{c}_i\boldsymbol{s}_i)\right)^* x^* \overset{(1\text{-}1)}{\approx} \boldsymbol{w}_1\cdot x\cdot \left(\prod_{i=p}^{1}(\boldsymbol{s}_i^*\boldsymbol{c}_i^*)\right)\boldsymbol{s}_0^*\cdot x^*,$$

where the identities {(1-1), (6-1)} can be used to convert $\boldsymbol{s}_i^*$ and $\boldsymbol{c}_i^*$ into the simple word $s_{i,k_i}^*s_{i,k_i-1}^*\cdots s_{i,1}^*$ and the ordered B_0-block $(y_{i,1}^*)^2(y_{i,2}^*)^2\cdots(y_{i,h_i}^*)^2$, respectively; thus, condition (B4)(b) is satisfied.

Thus the identities {(1-1), (5-1), (6-1)} can be used to convert $\boldsymbol{w}$ into either zz^* or some word $\widetilde{\boldsymbol{w}}$ in B_0-standard form. But if the identities {(1-1), (5-1), (6-1)} can be used to convert $\boldsymbol{w}$ into both zz^* and $\widetilde{\boldsymbol{w}}$, then that would imply that $(B_0, {}^*)$ satisfies the identity $\widetilde{\boldsymbol{w}} \approx zz^*$, which is impossible by Corollary 6.11. □

6.3. ***Proof of Proposition 6.1.*** Consider any identity

$$\boldsymbol{u} \approx \boldsymbol{v}$$

satisfied by $(B_0, {}^*)$. If we show that $\boldsymbol{u} \approx \boldsymbol{v}$ is deducible from {(1-1), (5-1), (6-1)}, the proposition will follow. By Lemma 6.5, this result holds if either $\boldsymbol{u}$ or $\boldsymbol{v}$ is bipartite. Therefore, suppose that $\boldsymbol{u}$ and $\boldsymbol{v}$ are both mixed. By Corollary 6.11 and Lemma 6.12, the identities {(1-1), (5-1), (6-1)} can be used to convert $\boldsymbol{u}$ and $\boldsymbol{v}$ simultaneously to either zz^* or words in B_0-standard form. In the former case, $\boldsymbol{u} \approx \boldsymbol{v}$ is deducible from {(1-1), (5-1), (6-1)}, whence the proof is complete. Therefore, it remains to consider the latter case, whence we may assume that $\boldsymbol{u}$ and $\boldsymbol{v}$ are in B_0-standard form, say

$$\boldsymbol{u} = \boldsymbol{u}_1 x \boldsymbol{u}_2 x^* \quad \text{and} \quad \boldsymbol{v} = \boldsymbol{v}_1 y \boldsymbol{v}_2 y^*,$$

where $x, y \in \mathcal{X} \cup \mathcal{X}^*$, $\boldsymbol{u}_1 = x_1 x_2 \cdots x_m$, $\boldsymbol{u}_2 = \boldsymbol{s}_0 \prod_{i=1}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i)$, $\boldsymbol{v}_1 = y_1 y_2 \cdots y_n$, and $\boldsymbol{v}_2 = \boldsymbol{t}_0 \prod_{i=1}^{q} (\boldsymbol{d}_i \boldsymbol{t}_i)$ satisfy conditions (B1)–(B4).

Lemma 6.13. *The following holds for the words $\boldsymbol{u}$ and $\boldsymbol{v}$*:

(i) $\mathsf{con}(\bar{\boldsymbol{u}}) = \mathsf{con}(\bar{\boldsymbol{v}})$;

(ii) $\boldsymbol{u}_1 x = \boldsymbol{v}_1 y$;

(iii) $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$.

Proof. (i) Suppose that $\mathsf{con}(\bar{\boldsymbol{u}}) \neq \mathsf{con}(\bar{\boldsymbol{v}})$, say there exists a variable $t \in \mathsf{con}(\bar{\boldsymbol{v}})$ such that $t \notin \mathsf{con}(\bar{\boldsymbol{u}})$. Then by Remark 6.9(ii), the word $\boldsymbol{u}_2$ contains some simple variable $z \in \mathcal{X} \cup \mathcal{X}^*$, so that $\boldsymbol{u}_2 = \boldsymbol{a} z \boldsymbol{b}$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$. Under the substitution $\gamma^z_{\boldsymbol{u}} : \mathcal{X} \to B_0$ in Lemma 6.10, the contradiction $\gamma^z_{\boldsymbol{u}}(\boldsymbol{u}) = \mathrm{A} \neq 0 = \gamma^z_{\boldsymbol{u}}(\boldsymbol{v})$ is deduced.

(ii) Due to condition (B1), the equality $\boldsymbol{u}_1 x = \boldsymbol{v}_1 y$ follows from $\mathsf{con}(\boldsymbol{u}_1 x) = \mathsf{con}(\boldsymbol{v}_1 y)$; to establish the latter, by symmetry, it suffices to verify the inclusion $\mathsf{con}(\boldsymbol{u}_1 x) \subseteq \mathsf{con}(\boldsymbol{v}_1 y)$. To this end, we need to first show that $y \in \mathsf{con}(\boldsymbol{u}_1 x)$. Since $\bar{y} \in \mathsf{con}(\bar{\boldsymbol{v}}) = \mathsf{con}(\bar{\boldsymbol{u}})$ by part (i),

(a) either $y \in \mathsf{con}(\boldsymbol{u})$ or $y^* \in \mathsf{con}(\boldsymbol{u})$.

Now since $\boldsymbol{s}_p \neq 1$, the variable $z = \mathsf{t}(\boldsymbol{s}_p) = \mathsf{t}(\boldsymbol{u}_2)$ is simple in $\boldsymbol{u}$, so that $\boldsymbol{u}_2 = \boldsymbol{a} z$ for some $\boldsymbol{a} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\bar{z} \notin \mathsf{con}(\bar{\boldsymbol{a}})$. Then under the substitution $\gamma^z_{\boldsymbol{u}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^z_{\boldsymbol{u}}(\boldsymbol{u}) = \gamma^z_{\boldsymbol{u}}(\boldsymbol{u}_1 x \boldsymbol{a}) \cdot \gamma^z_{\boldsymbol{u}}(z) \cdot \gamma^z_{\boldsymbol{u}}(x^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{E}^* = \mathrm{A}.$$

If $y^* \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$, then

$$\begin{aligned} \gamma^z_{\boldsymbol{u}}(\boldsymbol{v}) &= \gamma^z_{\boldsymbol{u}}(\boldsymbol{v}_1) \cdot \gamma^z_{\boldsymbol{u}}(y) \cdot \gamma^z_{\boldsymbol{u}}(\boldsymbol{v}_2) \cdot \gamma^z_{\boldsymbol{u}}(y^*) \\ &= \begin{cases} \gamma^z_{\boldsymbol{u}}(\boldsymbol{v}_1) \cdot \mathrm{A} \cdot \gamma^z_{\boldsymbol{u}}(\boldsymbol{v}_2) \cdot \mathrm{A} & \text{if } y^* = z \\ \gamma^z_{\boldsymbol{u}}(\boldsymbol{v}_1) \cdot \mathrm{F} \cdot \gamma^z_{\boldsymbol{u}}(\boldsymbol{v}_2) \cdot \mathrm{E} & \text{if } y^* \neq z \end{cases} \\ &= 0, \end{aligned}$$

which is impossible. Therefore,

(b) $y^* \notin \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$, which implies that $y \neq x^*$.

If $y \neq x$, then together with (b), we have $y^* \notin \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2 x^*) = \mathsf{con}(\boldsymbol{u})$, so that $y \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$ by (a) and (b). On the other hand, if $y = x$, then clearly $y \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$. Therefore, $y \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$ either way.

Seeking a contradiction, suppose that $y \in \mathsf{con}(\boldsymbol{u}_2)$. Then $\mathsf{occ}(y, \boldsymbol{u}_2) \in \{1, 2\}$ by condition (B2). If $\mathsf{occ}(y, \boldsymbol{u}_2) = 1$, then under the substitution $\gamma_{\boldsymbol{u}}^y : \mathcal{X} \to B_0$ in Lemma 6.10, we have $\gamma_{\boldsymbol{u}}^y(\boldsymbol{u}) = \mathrm{A}$ and

$$\gamma_{\boldsymbol{u}}^y(\boldsymbol{v}) = \gamma_{\boldsymbol{u}}^y(\boldsymbol{v}_1) \cdot \gamma_{\boldsymbol{u}}^y(y) \cdot \gamma_{\boldsymbol{u}}^y(\boldsymbol{v}_2) \cdot \gamma_{\boldsymbol{u}}^y(y^*) = \gamma_{\boldsymbol{u}}^y(\boldsymbol{v}_1) \cdot \mathrm{A} \cdot \gamma_{\boldsymbol{u}}^y(\boldsymbol{v}_2) \cdot \mathrm{A} = 0,$$

which is impossible. If $\mathsf{occ}(y, \boldsymbol{u}_2) = 2$, so that $\boldsymbol{u}_2 = h\boldsymbol{b}$ for some $\boldsymbol{b} \in F_{\mathsf{inv}}(\mathcal{X})$ with $y \in \mathsf{con}(\boldsymbol{b})$ and $h = \mathsf{h}(\boldsymbol{s}_0)$ being simple in $\boldsymbol{u}_2$, then under the substitution $\gamma_{\boldsymbol{u}}^h : \mathcal{X} \to B_0$ in Lemma 6.10, we have $\gamma_{\boldsymbol{u}}^h(\boldsymbol{u}) = \mathrm{A}$ and

$$\gamma_{\boldsymbol{u}}^h(\boldsymbol{v}) = \gamma_{\boldsymbol{u}}^h(\boldsymbol{v}_1) \cdot \gamma_{\boldsymbol{u}}^h(y) \cdot \gamma_{\boldsymbol{u}}^h(\boldsymbol{v}_2) \cdot \gamma_{\boldsymbol{u}}^h(y^*) = \gamma_{\boldsymbol{u}}^h(\boldsymbol{v}_1) \cdot \mathrm{F} \cdot \gamma_{\boldsymbol{u}}^h(\boldsymbol{v}_2) \cdot \mathrm{E} = 0,$$

which again is impossible. Thus, $y \notin \mathsf{con}(\boldsymbol{u}_2)$; but since $y \in \mathsf{con}(\boldsymbol{u}_1 x \boldsymbol{u}_2)$, together with (b), we have

(c) $y \in \mathsf{con}(\boldsymbol{u}_1 x)$ and $y, y^* \notin \mathsf{con}(\boldsymbol{u}_2)$.

By a symmetrical argument,

(d) $x \in \mathsf{con}(\boldsymbol{v}_1 y)$ and $x, x^* \notin \mathsf{con}(\boldsymbol{v}_2)$.

Now we are ready to establish the inclusion $\mathsf{con}(\boldsymbol{u}_1 x) \subseteq \mathsf{con}(\boldsymbol{v}_1 y)$. Suppose there exists some variable $z \in \mathsf{con}(\boldsymbol{u}_1 x)$ such that $z \notin \mathsf{con}(\boldsymbol{v}_1 y)$. Then clearly $z \neq y$. But if $z = y^*$, then it follows from (c) that $y, y^* \in \mathsf{con}(\boldsymbol{u}_1 x)$, whence condition (B1) is contradicted. Hence

(e) $z \notin \{y, y^*\}$.

Since $\bar{z} \in \mathsf{con}(\bar{\boldsymbol{u}}) = \mathsf{con}(\bar{\boldsymbol{v}})$ by part (i), it follows from (e) that either $z \in \mathsf{con}(\boldsymbol{v}_1 \boldsymbol{v}_2)$ or $z^* \in \mathsf{con}(\boldsymbol{v}_1 \boldsymbol{v}_2)$. But since $z \notin \mathsf{con}(\boldsymbol{v}_1)$ by assumption, we have $z \in \mathsf{con}(\boldsymbol{v}_2)$ or $z^* \in \mathsf{con}(\boldsymbol{v}_1)$ or $z^* \in \mathsf{con}(\boldsymbol{v}_2)$. These three cases are shown in the following to be impossible. Therefore, the variable z does not exist, whence the required inclusion $\mathsf{con}(\boldsymbol{u}_1 x) \subseteq \mathsf{con}(\boldsymbol{v}_1 y)$ is established.

Case 1: $z \in \mathsf{con}(\boldsymbol{v}_2)$. Then $z \notin \{x, x^*\}$ by (d). But since $z \in \mathsf{con}(\boldsymbol{u}_1 x)$ by assumption, we have

(f) $z \in \mathsf{con}(\boldsymbol{u}_1)$.

By condition (B2), we have $\mathsf{occ}(z, \boldsymbol{v}_2) \in \{1, 2\}$, so there are two subcases.

SUBCASE 1.1: $\mathsf{occ}(z, \boldsymbol{v}_2) = 1$. Then $\boldsymbol{v}_2 = \boldsymbol{a} z \boldsymbol{b}$ for some $\boldsymbol{a}, \boldsymbol{b} \in F_{\mathsf{inv}}^1(\mathcal{X})$ such that $\bar{z} \notin \mathsf{con}(\overline{\boldsymbol{ab}})$. Hence under the substitution $\gamma_{\boldsymbol{v}}^z : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma_{\boldsymbol{v}}^z(\boldsymbol{v}) = \gamma_{\boldsymbol{v}}^z(\boldsymbol{v}_1 y \boldsymbol{a}) \cdot \gamma_{\boldsymbol{v}}^z(z) \cdot \gamma_{\boldsymbol{v}}^z(\boldsymbol{b} y^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

It follows that $\gamma_{\boldsymbol{v}}^z(x) = \mathrm{E}$ because $x \in \mathsf{con}(\boldsymbol{v}_1 y)$ by (d), and $\gamma_{\boldsymbol{v}}^z(\boldsymbol{u}_1) = \cdots \mathrm{A} \cdots$ because $z \in \mathsf{con}(\boldsymbol{u}_1)$ by (f). Therefore,

$$\gamma_{\boldsymbol{v}}^z(\boldsymbol{u}) = \gamma_{\boldsymbol{v}}^z(\boldsymbol{u}_1) \cdot \gamma_{\boldsymbol{v}}^z(x) \cdot \gamma_{\boldsymbol{v}}^z(\boldsymbol{u}_2 x^*) = \cdots \mathrm{A} \cdots \mathrm{E} \cdot \gamma_{\boldsymbol{v}}^z(\boldsymbol{u}_2 x^*) = 0,$$

which implies a contradiction.

SUBCASE 1.2: $\mathsf{occ}(z, \boldsymbol{v}_2) = 2$. Since $\boldsymbol{t}_0 \neq 1$, the variable $h = \mathsf{h}(\boldsymbol{t}_0) = \mathsf{h}(\boldsymbol{v}_2)$ is simple in $\boldsymbol{v}$, so that $\boldsymbol{v}_2 = h\boldsymbol{a}z^2\boldsymbol{b}$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\boldsymbol{a}, \boldsymbol{b}, h, z$ are pairwise disjoint. Then under the substitution $\gamma^h_{\boldsymbol{v}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^h_{\boldsymbol{v}}(\boldsymbol{v}) = \gamma^h_{\boldsymbol{v}}(\boldsymbol{v}_1 y) \cdot \gamma^h_{\boldsymbol{v}}(h) \cdot \gamma^h_{\boldsymbol{v}}(\boldsymbol{a}z^2\boldsymbol{b}y^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

It follows that $\gamma^h_{\boldsymbol{v}}(x) = \mathrm{E}$ because $x \in \mathsf{con}(\boldsymbol{v}_1 y)$ by (d), and $\gamma^h_{\boldsymbol{v}}(z) = \mathrm{F}$. Further, since $z \in \mathsf{con}(\boldsymbol{u}_1)$ by (f), we have $\gamma^h_{\boldsymbol{v}}(\boldsymbol{u}_1) = \cdots \mathrm{F} \cdots$. Therefore,

$$\gamma^h_{\boldsymbol{v}}(\boldsymbol{u}) = \gamma^h_{\boldsymbol{v}}(\boldsymbol{u}_1) \cdot \gamma^h_{\boldsymbol{v}}(x) \cdot \gamma^h_{\boldsymbol{v}}(\boldsymbol{u}_2 x^*) = \cdots \mathrm{F} \cdots \mathrm{E} \cdot \gamma^h_{\boldsymbol{v}}(\boldsymbol{u}_2 x^*) = 0,$$

which implies a contradiction.

<u>Case 2</u>: $z^* \in \mathsf{con}(\boldsymbol{v}_1)$. Since $\boldsymbol{s}_0 \neq 1$, the variable $h = \mathsf{h}(\boldsymbol{s}_0) = \mathsf{h}(\boldsymbol{u}_2)$ is simple in $\boldsymbol{u}$, so that $\boldsymbol{u}_2 = h\boldsymbol{b}$ for some $\boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\bar{h} \notin \mathsf{con}(\bar{\boldsymbol{b}})$. Then under the substitution $\gamma^h_{\boldsymbol{u}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^h_{\boldsymbol{u}}(\boldsymbol{u}) = \gamma^h_{\boldsymbol{u}}(\boldsymbol{u}_1 x) \cdot \gamma^h_{\boldsymbol{u}}(h) \cdot \gamma^h_{\boldsymbol{u}}(\boldsymbol{b}x^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

It follows that $\gamma^h_{\boldsymbol{u}}(y) = \mathrm{E}$ because $y \in \mathsf{con}(\boldsymbol{u}_1 x)$ by (c), and that $\gamma^h_{\boldsymbol{u}}(z) = \mathrm{E}$ because $z \in \mathsf{con}(\boldsymbol{u}_1 x)$ by assumption. Further, since $z^* \in \mathsf{con}(\boldsymbol{v}_1)$, we have $\gamma^h_{\boldsymbol{u}}(\boldsymbol{v}_1) = \cdots \mathrm{F} \cdots$. Therefore,

$$\gamma^h_{\boldsymbol{u}}(\boldsymbol{v}) = \gamma^h_{\boldsymbol{u}}(\boldsymbol{v}_1) \cdot \gamma^h_{\boldsymbol{u}}(y) \cdot \gamma^h_{\boldsymbol{u}}(\boldsymbol{v}_2 y^*) = \cdots \mathrm{F} \cdots \mathrm{E} \cdot \gamma^h_{\boldsymbol{u}}(\boldsymbol{v}_2 y^*) = 0,$$

which implies a contradiction.

<u>Case 3</u>: $z^* \in \mathsf{con}(\boldsymbol{v}_2)$. Then $z \notin \{x, x^*\}$ by (d). But since $z \in \mathsf{con}(\boldsymbol{u}_1 x)$ by assumption, we have

(g) $z \in \mathsf{con}(\boldsymbol{u}_1)$.

By condition (B2), we have $\mathsf{occ}(z^*, \boldsymbol{v}_2) \in \{1, 2\}$, so there are two subcases.

SUBCASE 3.1: $\mathsf{occ}(z^*, \boldsymbol{v}_2) = 1$. Then $\boldsymbol{v}_2 = \boldsymbol{a}z^*\boldsymbol{b}$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\bar{z} \notin \mathsf{con}(\overline{\boldsymbol{a}\boldsymbol{b}})$. Hence under the substitution $\gamma^{z^*}_{\boldsymbol{v}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{v}) = \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{v}_1 y\boldsymbol{a}) \cdot \gamma^{z^*}_{\boldsymbol{v}}(z^*) \cdot \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{b}y^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

It follows that $\gamma^{z^*}_{\boldsymbol{v}}(x) = \mathrm{E}$ because $x \in \mathsf{con}(\boldsymbol{v}_1 y)$ by (d), and $\gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{u}_1) = \cdots \mathrm{A} \cdots$ because $z \in \mathsf{con}(\boldsymbol{u}_1)$ by (g). Therefore,

$$\gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{u}) = \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{u}_1) \cdot \gamma^{z^*}_{\boldsymbol{v}}(x) \cdot \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{u}_2 x^*) = \cdots \mathrm{A} \cdots \mathrm{E} \cdot \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{u}_2 x^*) = 0,$$

which implies a contradiction.

SUBCASE 3.2: $\mathsf{occ}(z^*, \boldsymbol{v}_2) = 2$. Since $\boldsymbol{t}_q \neq 1$, the variable $t = \mathsf{t}(\boldsymbol{t}_q) = \mathsf{t}(\boldsymbol{v}_2)$ is simple in $\boldsymbol{v}$, so that $\boldsymbol{v}_2 = \boldsymbol{a}(z^*)^2\boldsymbol{b}t$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\boldsymbol{a}, \boldsymbol{b}, z, t$ are pairwise disjoint. Then under the substitution $\gamma_{\boldsymbol{v}}^t : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma_{\boldsymbol{v}}^t(\boldsymbol{v}) = \gamma_{\boldsymbol{v}}^t\big(\boldsymbol{v}_1 y \boldsymbol{a}(z^*)^2\boldsymbol{b}\big) \cdot \gamma_{\boldsymbol{v}}^t(t) \cdot \gamma_{\boldsymbol{v}}^t(y^*) = \mathrm{E}\cdot\mathrm{A}\cdot\mathrm{F} = \mathrm{A}.$$

It follows that $\gamma_{\boldsymbol{v}}^t(x) = \mathrm{E}$ because $x \in \mathsf{con}(\boldsymbol{v}_1 y)$ by (d), and $\gamma_{\boldsymbol{v}}^t(z) = \mathrm{F}$. Further, since $z \in \mathsf{con}(\boldsymbol{u}_1)$ by (g), we have $\gamma_{\boldsymbol{v}}^t(\boldsymbol{u}_1) = \cdots \mathrm{F} \cdots$. Therefore,

$$\gamma_{\boldsymbol{v}}^t(\boldsymbol{u}) = \gamma_{\boldsymbol{v}}^t(\boldsymbol{u}_1)\cdot\gamma_{\boldsymbol{v}}^t(x)\cdot\gamma_{\boldsymbol{v}}^t(\boldsymbol{u}_2 x^*) = \cdots\mathrm{F}\cdots\mathrm{E}\cdot\gamma_{\boldsymbol{v}}^h(\boldsymbol{u}_2 x^*) = 0,$$

which implies a contradiction.

(iii) This is a consequence of parts (i) and (ii). □

Therefore, by Lemma 6.13, we now have

$$\boldsymbol{u} = \underbrace{x_1x_2\cdots x_m}_{\boldsymbol{u}_1}\cdot x\cdot \underbrace{\boldsymbol{s}_0\prod_{i=1}^{p}(\boldsymbol{c}_i\boldsymbol{s}_i)}_{\boldsymbol{u}_2}\cdot x^* \quad\text{and}\quad \boldsymbol{v} = \underbrace{x_1x_2\cdots x_m}_{\boldsymbol{u}_1}\cdot x\cdot \underbrace{\boldsymbol{t}_0\prod_{i=1}^{q}(\boldsymbol{d}_i\boldsymbol{t}_i)}_{\boldsymbol{v}_2}\cdot x^*,$$

where conditions (B1)–(B4) are satisfied. In the remainder of this section, it is shown that $\boldsymbol{u}_2 = \boldsymbol{v}_2$ (Lemma 6.19), so that $\boldsymbol{u} = \boldsymbol{v}$. The identity $\boldsymbol{u} \approx \boldsymbol{v}$ is thus vacuously deducible from {(1-1), (5-1), (6-1)}, whence the proof of Proposition 6.1 is complete.

Lemma 6.14. (i) $\mathsf{con}(\overline{\boldsymbol{s}_0\boldsymbol{s}_1\cdots\boldsymbol{s}_p}) = \mathsf{con}(\overline{\boldsymbol{t}_0\boldsymbol{t}_1\cdots\boldsymbol{t}_q})$.

(ii) $\mathsf{con}(\overline{\boldsymbol{c}_1\boldsymbol{c}_2\cdots\boldsymbol{c}_p}) = \mathsf{con}(\overline{\boldsymbol{d}_1\boldsymbol{d}_2\cdots\boldsymbol{d}_q})$.

Proof. (i) Let $\boldsymbol{s} = \boldsymbol{s}_0\boldsymbol{s}_1\cdots\boldsymbol{s}_p$ and $\boldsymbol{t} = \boldsymbol{t}_0\boldsymbol{t}_1\cdots\boldsymbol{t}_q$. Suppose that $\mathsf{con}(\bar{\boldsymbol{s}}) \nsubseteq \mathsf{con}(\bar{\boldsymbol{t}})$. Then there exists some $z \in \mathsf{con}(\boldsymbol{s})$ such that $\bar{z} \in \mathsf{con}(\bar{\boldsymbol{s}})$ and $\bar{z} \notin \mathsf{con}(\bar{\boldsymbol{t}})$. Therefore, under the substitution $\gamma_{\boldsymbol{u}}^z : \mathcal{X} \to B_0$ in Lemma 6.10, we have $\gamma_{\boldsymbol{u}}^z(\boldsymbol{u}) = \mathrm{A}$. On the other hand, since $\bar{z} \in \mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$ by Lemma 6.13(iii) but $\bar{z} \notin \mathsf{con}(\bar{\boldsymbol{t}})$ by assumption, we have $\bar{z} \in \mathsf{con}(\bar{\boldsymbol{d}}_i)$ for some $i \in \{1, 2, \ldots, q\}$. Since $\boldsymbol{d}_i$ is an ordered B_0-block, we have $\gamma_{\boldsymbol{u}}^z(\boldsymbol{d}_i) = \cdots\mathrm{A}^2\cdots = 0$, whence the contradiction $\gamma_{\boldsymbol{u}}^z(\boldsymbol{v}) = 0$ is deduced. Hence the variable z does not exist, so that the inclusion $\mathsf{con}(\bar{\boldsymbol{s}}) \subseteq \mathsf{con}(\bar{\boldsymbol{t}})$ holds. The reverse inclusion $\mathsf{con}(\bar{\boldsymbol{s}}) \supseteq \mathsf{con}(\bar{\boldsymbol{t}})$ holds by a symmetrical argument.

(ii) This follows from part (i) since $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$ by Lemma 6.13(iii). □

Lemma 6.15. (i) *Suppose that $yz \hookrightarrow \boldsymbol{u}_2$ for some $y, z \in \mathsf{con}(\boldsymbol{s}_0\boldsymbol{s}_1\cdots\boldsymbol{s}_p)$. Then the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ can only be yz or z^*y^*.*

(ii) *Suppose that $yz \hookrightarrow \boldsymbol{v}_2$ for some $y, z \in \mathsf{con}(\boldsymbol{t}_0\boldsymbol{t}_1\cdots\boldsymbol{t}_q)$. Then the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{u}_2$ can only be yz or z^*y^*.*

Proof. (i) By assumption,

$$\boldsymbol{u} = \boldsymbol{u}_1\cdot x\cdot \underbrace{\boldsymbol{a}y\boldsymbol{b}z\boldsymbol{e}}_{\boldsymbol{u}_2}\cdot x^*$$

for some $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\boldsymbol{u}_1, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}, x, y, z$ are pairwise disjoint. Since $\bar{y}, \bar{z} \in \mathsf{con}(\overline{\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p}) = \mathsf{con}(\overline{\boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q})$ by Lemma 6.14(i), we have $\mathsf{occ}(\bar{y}, \bar{\boldsymbol{v}}) = \mathsf{occ}(\bar{z}, \bar{\boldsymbol{v}}) = 1$ by conditions (B2) and (B3). Therefore, the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ can only be one of

$$yz, \quad yz^*, \quad y^*z, \quad y^*z^*, \quad zy, \quad zy^*, \quad z^*y, \quad z^*y^*.$$

There are four cases to consider.

Case 1: $yz^* \hookrightarrow \boldsymbol{v}_2$. Then

$$\boldsymbol{v} = \boldsymbol{u}_1 \cdot x \cdot \underbrace{\boldsymbol{f} y \boldsymbol{g} z^* \boldsymbol{h}}_{\boldsymbol{v}_2} \cdot x^*$$

for some $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\boldsymbol{u}_1, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, x, y, z$ are pairwise disjoint. Under the substitution $\gamma^y_{\boldsymbol{v}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^y_{\boldsymbol{v}}(\boldsymbol{v}) = \gamma^y_{\boldsymbol{v}}(\boldsymbol{u}_1 x \boldsymbol{f}) \cdot \gamma^y_{\boldsymbol{v}}(y) \cdot \gamma^y_{\boldsymbol{v}}(\boldsymbol{g} z^* \boldsymbol{h} x^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

But since $\gamma^y_{\boldsymbol{v}}(y) = \mathrm{A}$ and $\gamma^y_{\boldsymbol{v}}(z) = \mathrm{E}$, we deduce the contradiction

$$\gamma^y_{\boldsymbol{v}}(\boldsymbol{u}) = \gamma^y_{\boldsymbol{v}}(\boldsymbol{u}_1 x \boldsymbol{a}) \cdot \mathrm{A} \cdot \gamma^y_{\boldsymbol{v}}(\boldsymbol{b}) \cdot \mathrm{E} \cdot \gamma^y_{\boldsymbol{v}}(\boldsymbol{e} x^*) = 0.$$

Case 2: $zy^* \hookrightarrow \boldsymbol{v}_2$. Then

$$\boldsymbol{v} = \boldsymbol{u}_1 \cdot x \cdot \underbrace{\boldsymbol{f} z \boldsymbol{g} y^* \boldsymbol{h}}_{\boldsymbol{v}_2} \cdot x^*$$

for some $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\boldsymbol{u}_1, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, x, y, z$ are pairwise disjoint. Under the substitution $\gamma^{y^*}_{\boldsymbol{v}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^{y^*}_{\boldsymbol{v}}(\boldsymbol{v}) = \gamma^{y^*}_{\boldsymbol{v}}(\boldsymbol{u}_1 x \boldsymbol{f} z \boldsymbol{g}) \cdot \gamma^{y^*}_{\boldsymbol{v}}(y^*) \cdot \gamma^{y^*}_{\boldsymbol{v}}(\boldsymbol{h} x^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

But since $\gamma^{y^*}_{\boldsymbol{v}}(y) = \mathrm{A}$ and $\gamma^{y^*}_{\boldsymbol{v}}(z) = \mathrm{E}$, we deduce the contradiction

$$\gamma^{y^*}_{\boldsymbol{v}}(\boldsymbol{u}) = \gamma^{y^*}_{\boldsymbol{v}}(\boldsymbol{u}_1 x \boldsymbol{a}) \cdot \mathrm{A} \cdot \gamma^{y^*}_{\boldsymbol{v}}(\boldsymbol{b}) \cdot \mathrm{E} \cdot \gamma^{y^*}_{\boldsymbol{v}}(\boldsymbol{e} x^*) = 0.$$

Case 3: $y^*z \hookrightarrow \boldsymbol{v}_2$ or $zy \hookrightarrow \boldsymbol{v}_2$. Under the substitution $\gamma^z_{\boldsymbol{u}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^z_{\boldsymbol{u}}(\boldsymbol{u}) = \gamma^z_{\boldsymbol{u}}(\boldsymbol{u}_1 x \boldsymbol{a} y \boldsymbol{b}) \cdot \gamma^z_{\boldsymbol{u}}(z) \cdot \gamma^z_{\boldsymbol{u}}(\boldsymbol{e} x^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

But since $\gamma^z_{\boldsymbol{u}}(y) = \mathrm{E}$ and $\gamma^z_{\boldsymbol{u}}(z) = \mathrm{A}$, we deduce the contradiction

$$\begin{aligned}\gamma^z_{\boldsymbol{u}}(\boldsymbol{v}) &= \begin{cases} \cdots \mathrm{E}^* \cdots \mathrm{A} \cdots & \text{if } y^*z \hookrightarrow \boldsymbol{v}_2 \\ \cdots \mathrm{A} \cdots \mathrm{E} \cdots & \text{if } zy \hookrightarrow \boldsymbol{v}_2 \end{cases} \\ &= 0.\end{aligned}$$

Case 4: $z^*y \hookrightarrow \boldsymbol{v}_2$ or $y^*z^* \hookrightarrow \boldsymbol{v}_2$. Then

$$\boldsymbol{v} = \boldsymbol{u}_1 \cdot x \cdot \underbrace{\boldsymbol{f} z^* \boldsymbol{g} y \boldsymbol{h}}_{\boldsymbol{v}_2} \cdot x^* \quad \text{or} \quad \boldsymbol{v} = \boldsymbol{u}_1 \cdot x \cdot \underbrace{\boldsymbol{f} y^* \boldsymbol{g} z^* \boldsymbol{h}}_{\boldsymbol{v}_2} \cdot x^*$$

for some $\boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\boldsymbol{u}_1, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}, x, y, z$ are pairwise disjoint. Under the substitution $\gamma^{z^*}_{\boldsymbol{v}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{v}) = \begin{cases} \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{u}_1 x \boldsymbol{f}) \cdot \gamma^{z^*}_{\boldsymbol{v}}(z^*) \cdot \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{g} y \boldsymbol{h} x^*) & \text{if } z^*y \hookrightarrow \boldsymbol{v}_2 \\ \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{u}_1 x \boldsymbol{f} y^* \boldsymbol{g}) \cdot \gamma^{z^*}_{\boldsymbol{v}}(z^*) \cdot \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{h} x^*) & \text{if } y^*z^* \hookrightarrow \boldsymbol{v}_2 \end{cases}$$
$$= \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

But since $\gamma^{z^*}_{\boldsymbol{v}}(y) = \mathrm{F}$ and $\gamma^{z^*}_{\boldsymbol{v}}(z) = \mathrm{A}$, we deduce the contradiction

$$\gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{u}) = \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{u}_1 x \boldsymbol{a}) \cdot \mathrm{F} \cdot \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{b}) \cdot \mathrm{A} \cdot \gamma^{z^*}_{\boldsymbol{v}}(\boldsymbol{e} x^*) = 0.$$

Since none of the four cases is possible, the longest $\{\bar{y}, \bar{z}\}$-subsequences of $\boldsymbol{v}_2$ can only be yz or z^*y^*.

(ii) This is symmetrical to part (i). □

Lemma 6.16. (i) *Suppose that $y^2z \hookrightarrow \boldsymbol{u}_2$ for some $y \in \mathsf{con}(\boldsymbol{c}_1\boldsymbol{c}_2 \cdots \boldsymbol{c}_p)$ and $z \in \mathsf{con}(\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p)$. Then the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ can only be y^2z or $z^*(y^*)^2$.*

(ii) *Suppose that $y^2z \hookrightarrow \boldsymbol{v}_2$ for some $y \in \mathsf{con}(\boldsymbol{d}_1\boldsymbol{d}_2 \cdots \boldsymbol{d}_q)$ and $z \in \mathsf{con}(\boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q)$. Then the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{u}_2$ can only be y^2z or $z^*(y^*)^2$.*

Proof. (i) Since the prefix $\boldsymbol{s}_0$ of $\boldsymbol{u}_2$ consists of simple variables of $\boldsymbol{u}$, it follows from the assumption that

$$\boldsymbol{u} = \boldsymbol{u}_1 \cdot x \cdot \underbrace{\boldsymbol{a} y^2 \boldsymbol{b} z \boldsymbol{e}}_{\boldsymbol{u}_2} \cdot x^*$$

for some $\boldsymbol{a} \in F_{\mathsf{inv}}(\mathcal{X})$ and $\boldsymbol{b}, \boldsymbol{e} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\boldsymbol{u}_1, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}, x, y, z$ are pairwise disjoint. Since $\bar{z} \in \mathsf{con}(\overline{\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p}) = \mathsf{con}(\overline{\boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q})$ and $\bar{y} \in \mathsf{con}(\overline{\boldsymbol{c}_1\boldsymbol{c}_2 \cdots \boldsymbol{c}_p}) = \mathsf{con}(\overline{\boldsymbol{d}_1\boldsymbol{d}_2 \cdots \boldsymbol{d}_q})$ by Lemma 6.14, we have $\mathsf{occ}(\bar{y}, \bar{\boldsymbol{v}}_2) = 2$ and $\mathsf{occ}(\bar{z}, \bar{\boldsymbol{v}}_2) = 1$. Hence the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ can only be one of

$$y^2z, \quad y^2z^*, \quad (y^*)^2z, \quad (y^*)^2z^*, \quad zy^2, \quad z(y^*)^2, \quad z^*y^2, \quad z^*(y^*)^2.$$

There are two cases to consider.

Case 1: $y^2z^* \hookrightarrow \boldsymbol{v}_2$ or $z(y^*)^2 \hookrightarrow \boldsymbol{v}_2$. The variable $h = \mathsf{h}(\boldsymbol{a})$ is simple in $\boldsymbol{u}$, so that $\boldsymbol{a} = h\boldsymbol{f}$ for some $\boldsymbol{f} \in F^1_{\mathsf{inv}}(\mathcal{X})$. Then under the substitution $\gamma^h_{\boldsymbol{u}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^h_{\boldsymbol{u}}(\boldsymbol{u}) = \gamma^h_{\boldsymbol{u}}(\boldsymbol{u}_1 x) \cdot \gamma^h_{\boldsymbol{u}}(h) \cdot \gamma^h_{\boldsymbol{u}}(\boldsymbol{f} y^2 \boldsymbol{b} z \boldsymbol{e} x^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

But since $\gamma_{\boldsymbol{u}}^{h}(y) = \gamma_{\boldsymbol{u}}^{h}(z) = \mathrm{F}$, we deduce the contradiction

$$\begin{aligned}\gamma_{\boldsymbol{u}}^{h}(\boldsymbol{v}) &= \begin{cases} \cdots \mathrm{F}^2 \cdots \mathrm{F}^* \cdots & \text{if } y^2 z^* \hookrightarrow \boldsymbol{v}_2 \\ \cdots \mathrm{F} \cdots (\mathrm{F}^*)^2 \cdots & \text{if } z(y^*)^2 \hookrightarrow \boldsymbol{v}_2 \end{cases} \\ &= 0.\end{aligned}$$

Case 2: $(y^*)^2 z \hookrightarrow \boldsymbol{v}_2$ or $(y^*)^2 z^* \hookrightarrow \boldsymbol{v}_2$ or $zy^2 \hookrightarrow \boldsymbol{v}_2$ or $z^* y^2 \hookrightarrow \boldsymbol{v}_2$. Then under the substitution $\gamma_{\boldsymbol{u}}^{z} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma_{\boldsymbol{u}}^{z}(\boldsymbol{u}) = \gamma_{\boldsymbol{u}}^{z}(\boldsymbol{u}_1 x \boldsymbol{a} y^2 \boldsymbol{b}) \cdot \gamma_{\boldsymbol{u}}^{z}(z) \cdot \gamma_{\boldsymbol{u}}^{z}(\boldsymbol{e} x^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

But since $\gamma_{\boldsymbol{u}}^{z}(y) = \mathrm{E}$ and $\gamma_{\boldsymbol{u}}^{z}(z) = \mathrm{A}$, we deduce the contradiction

$$\begin{aligned}\gamma_{\boldsymbol{u}}^{z}(\boldsymbol{v}) &= \begin{cases} \cdots (\mathrm{E}^*)^2 \cdots \mathrm{A} \cdots & \text{if } (y^*)^2 z \hookrightarrow \boldsymbol{v}_2 \text{ or } (y^*)^2 z^* \hookrightarrow \boldsymbol{v}_2 \\ \cdots \mathrm{A} \cdots \mathrm{E}^2 \cdots & \text{if } zy^2 \hookrightarrow \boldsymbol{v}_2 \text{ or } z^* y^2 \hookrightarrow \boldsymbol{v}_2 \end{cases} \\ &= 0.\end{aligned}$$

Since none of the two cases is possible, the longest $\{\bar{y}, \bar{z}\}$-subsequences of $\boldsymbol{v}_2$ can only be $y^2 z$ or $z^*(y^*)^2$.

(ii) This is symmetrical to part (i). □

Lemma 6.17. (i) *Suppose that $yz^2 \hookrightarrow \boldsymbol{u}_2$ for some $y \in \mathsf{con}(\boldsymbol{s}_0 \boldsymbol{s}_1 \cdots \boldsymbol{s}_p)$ and $z \in \mathsf{con}(\boldsymbol{c}_1 \boldsymbol{c}_2 \cdots \boldsymbol{c}_p)$. Then the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ can only be yz^2 or $(z^*)^2 y^*$.*

(ii) *Suppose that $yz^2 \hookrightarrow \boldsymbol{v}_2$ for some $y \in \mathsf{con}(\boldsymbol{t}_0 \boldsymbol{t}_1 \cdots \boldsymbol{t}_q)$ and $z \in \mathsf{con}(\boldsymbol{d}_1 \boldsymbol{d}_2 \cdots \boldsymbol{d}_q)$. Then the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{u}_2$ can only be yz^2 or $(z^*)^2 y^*$.*

Proof. This is very similar to the proof of Lemma 6.16, but details are given for the sake of completeness.

(i) Since the suffix $\boldsymbol{s}_p$ of $\boldsymbol{u}_2$ consists of simple variables of $\boldsymbol{u}$, it follows from the assumption that

$$\boldsymbol{u} = \boldsymbol{u}_1 \cdot x \cdot \underbrace{\boldsymbol{a} y \boldsymbol{b} z^2 \boldsymbol{e}}_{\boldsymbol{u}_2} \cdot x^*$$

for some $\boldsymbol{a}, \boldsymbol{b} \in F_{\mathsf{inv}}^1(\mathcal{X})$ and $\boldsymbol{e} \in F_{\mathsf{inv}}(\mathcal{X})$ such that $\boldsymbol{u}_1, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{e}, x, y, z$ are pairwise disjoint. Since $\bar{y} \in \mathsf{con}(\overline{\boldsymbol{s}_0 \boldsymbol{s}_1 \cdots \boldsymbol{s}_p}) = \mathsf{con}(\overline{\boldsymbol{t}_0 \boldsymbol{t}_1 \cdots \boldsymbol{t}_q})$ and $\bar{z} \in \mathsf{con}(\overline{\boldsymbol{c}_1 \boldsymbol{c}_2 \cdots \boldsymbol{c}_p}) = \mathsf{con}(\overline{\boldsymbol{d}_1 \boldsymbol{d}_2 \cdots \boldsymbol{d}_q})$ by Lemma 6.14, we have $\mathsf{occ}(\bar{y}, \bar{\boldsymbol{v}}_2) = 1$ and $\mathsf{occ}(\bar{z}, \bar{\boldsymbol{v}}_2) = 2$. Hence the longest $\{\bar{y}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ can only be one of

$$yz^2, \quad y(z^*)^2, \quad y^* z^2, \quad y^*(z^*)^2, \quad z^2 y, \quad z^2 y^*, \quad (z^*)^2 y, \quad (z^*)^2 y^*.$$

There are two cases to consider.

Case 1: $y^*z^2 \hookrightarrow \boldsymbol{v}_2$ or $(z^*)^2y \hookrightarrow \boldsymbol{v}_2$. The variable $t = \mathsf{t}(\boldsymbol{e})$ is simple in $\boldsymbol{u}$, so that $\boldsymbol{e} = \boldsymbol{f}t$ for some $\boldsymbol{f} \in F^1_{\mathsf{inv}}(\mathcal{X})$. Then under the substitution $\gamma^t_{\boldsymbol{u}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^t_{\boldsymbol{u}}(\boldsymbol{u}) = \gamma^t_{\boldsymbol{u}}(\boldsymbol{u}_1 x\boldsymbol{a}y\boldsymbol{b}z^2\boldsymbol{f}) \cdot \gamma^t_{\boldsymbol{u}}(t) \cdot \gamma^t_{\boldsymbol{u}}(x^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

But since $\gamma^t_{\boldsymbol{u}}(y) = \gamma^t_{\boldsymbol{u}}(z) = \mathrm{E}$, we deduce the contradiction

$$\begin{aligned}\gamma^t_{\boldsymbol{u}}(\boldsymbol{v}) &= \begin{cases} \cdots \mathrm{E}^* \cdots \mathrm{E}^2 \cdots & \text{if } y^*z^2 \hookrightarrow \boldsymbol{v}_2 \\ \cdots (\mathrm{E}^*)^2 \cdots \mathrm{E} \cdots & \text{if } (z^*)^2y \hookrightarrow \boldsymbol{v}_2 \end{cases} \\ &= 0.\end{aligned}$$

Case 2: $y(z^*)^2 \hookrightarrow \boldsymbol{v}_2$ or $y^*(z^*)^2 \hookrightarrow \boldsymbol{v}_2$ or $z^2y \hookrightarrow \boldsymbol{v}_2$ or $z^2y^* \hookrightarrow \boldsymbol{v}_2$. Then under the substitution $\gamma^y_{\boldsymbol{u}} : \mathcal{X} \to B_0$ in Lemma 6.10, we have

$$\gamma^y_{\boldsymbol{u}}(\boldsymbol{u}) = \gamma^y_{\boldsymbol{u}}(\boldsymbol{u}_1 x\boldsymbol{a}) \cdot \gamma^y_{\boldsymbol{u}}(y) \cdot \gamma^y_{\boldsymbol{u}}(\boldsymbol{b}z^2\boldsymbol{e}x^*) = \mathrm{E} \cdot \mathrm{A} \cdot \mathrm{F} = \mathrm{A}.$$

But since $\gamma^y_{\boldsymbol{u}}(y) = \mathrm{A}$ and $\gamma^y_{\boldsymbol{u}}(z) = \mathrm{F}$, we deduce the contradiction

$$\begin{aligned}\gamma^y_{\boldsymbol{u}}(\boldsymbol{v}) &= \begin{cases} \cdots \mathrm{A} \cdots (\mathrm{F}^*)^2 \cdots & \text{if } y(z^*)^2 \hookrightarrow \boldsymbol{v}_2 \text{ or } y^*(z^*)^2 \hookrightarrow \boldsymbol{v}_2 \\ \cdots \mathrm{F}^2 \cdots \mathrm{A} \cdots & \text{if } z^2y \hookrightarrow \boldsymbol{v}_2 \text{ or } z^2y^* \hookrightarrow \boldsymbol{v}_2 \end{cases} \\ &= 0.\end{aligned}$$

Since none of the two cases is possible, the longest $\{\bar{y}, \bar{z}\}$-subsequences of $\boldsymbol{v}_2$ can only be yz^2 or $(z^*)^2y^*$.

(ii) This is symmetrical to part (i). □

Lemma 6.18. $\mathsf{h}(\boldsymbol{u}_2) = \mathsf{h}(\boldsymbol{v}_2)$ *and* $\mathsf{t}(\boldsymbol{u}_2) = \mathsf{t}(\boldsymbol{v}_2)$.

Proof. Recall that $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$ by Lemma 6.13(iii). First, suppose that $|\mathsf{con}(\bar{\boldsymbol{u}}_2)| = |\mathsf{con}(\bar{\boldsymbol{v}}_2)| = 1$, say $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2) = \{z\}$ for some $z \in \mathcal{X}$. Then it follows from condition (B4)(a) that $\boldsymbol{u}_2 = \boldsymbol{v}_2 = z$, whence $\mathsf{h}(\boldsymbol{u}_2) = z = \mathsf{h}(\boldsymbol{v}_2)$ and $\mathsf{t}(\boldsymbol{u}_2) = z = \mathsf{t}(\boldsymbol{v}_2)$.

Hence, it remains to assume $|\mathsf{con}(\bar{\boldsymbol{u}}_2)| = |\mathsf{con}(\bar{\boldsymbol{v}}_2)| \geq 2$. Let $h = \mathsf{h}(\boldsymbol{u}_2) = \mathsf{h}(\boldsymbol{s}_0)$, $t = \mathsf{t}(\boldsymbol{u}_2) = \mathsf{h}(\boldsymbol{s}_p)$, $H = \mathsf{h}(\boldsymbol{v}_2) = \mathsf{h}(\boldsymbol{t}_0)$, and $T = \mathsf{t}(\boldsymbol{v}_2) = \mathsf{h}(\boldsymbol{t}_q)$, so that

$$\bar{h} \prec \bar{t} \quad \text{and} \quad \overline{H} \prec \overline{T} \tag{6-5}$$

by condition (B4)(b). Then

$$\boldsymbol{u} = \boldsymbol{u}_1 \cdot x \cdot \underbrace{h\boldsymbol{a}t}_{\boldsymbol{u}_2} \cdot x^* \quad \text{and} \quad \boldsymbol{v} = \boldsymbol{u}_1 \cdot x \cdot \underbrace{H\boldsymbol{b}T}_{\boldsymbol{v}_2} \cdot x^* \tag{6-6}$$

for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$ such that $\overline{ht} \notin \mathsf{con}(\overline{\boldsymbol{u}_1 x\boldsymbol{a}})$ and $\overline{HT} \notin \mathsf{con}(\overline{\boldsymbol{u}_1 x\boldsymbol{b}})$. Since $ht \hookrightarrow \boldsymbol{u}_2$ with $h, t \in \mathsf{con}(\boldsymbol{s}_0\boldsymbol{s}_p)$, it follows from Lemma 6.15(i) that the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$ is ht or t^*h^*.

Seeking a contradiction, suppose that t^*h^* is the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$. If $H \neq t^*$, so that $Ht^* \hookrightarrow \boldsymbol{v}_2$, then by Lemma 6.15(ii), either $Ht^* \hookrightarrow \boldsymbol{u}_2$ or $tH^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (6-6). Hence $H = t^*$. By a symmetrical argument, we deduce $T = h^*$. Since $\bar{h} \prec \bar{t}$ by (6-5), we have $\overline{T} = \bar{h} \prec \bar{t} = \overline{H}$; but this contradicts $\overline{H} \prec \overline{T}$ from (6-5).

Therefore, ht is the longest $\{\bar{h}, \bar{t}\}$-subsequence of $\boldsymbol{v}_2$. If $H \neq h$, so that $Hh \hookrightarrow \boldsymbol{v}_2$, then by Lemma 6.15(ii), either $Hh \hookrightarrow \boldsymbol{u}_2$ or $h^*H^* \hookrightarrow \boldsymbol{u}_2$; but neither subsequence is possible in view of (6-6). Hence $H = h$. By a symmetrical argument, we deduce $T = t$. □

Lemma 6.19. $\boldsymbol{u}_2 = \boldsymbol{v}_2$.

Proof. Recall that

$$\boldsymbol{u}_2 = \boldsymbol{s}_0 \prod_{i=1}^{p} (\boldsymbol{c}_i \boldsymbol{s}_i) \quad \text{and} \quad \boldsymbol{v}_2 = \boldsymbol{t}_0 \prod_{i=1}^{q} (\boldsymbol{d}_i \boldsymbol{t}_i),$$

and $\mathsf{con}(\bar{\boldsymbol{u}}_2) = \mathsf{con}(\bar{\boldsymbol{v}}_2)$ by Lemma 6.13(iii). If $|\mathsf{con}(\bar{\boldsymbol{u}}_2)| = |\mathsf{con}(\bar{\boldsymbol{v}}_2)| = 1$, then as shown in the proof of Lemma 6.18, we have $\boldsymbol{u}_2 = \boldsymbol{v}_2$. Therefore, it suffices to assume that $|\mathsf{con}(\bar{\boldsymbol{u}}_2)| = |\mathsf{con}(\bar{\boldsymbol{v}}_2)| \geq 2$. By Lemma 6.18, we have

(a) $\mathsf{h}(\boldsymbol{s}_0) = \mathsf{h}(\boldsymbol{u}_2) = \mathsf{h}(\boldsymbol{v}_2) = \mathsf{h}(\boldsymbol{t}_0)$ and $\mathsf{t}(\boldsymbol{s}_p) = \mathsf{t}(\boldsymbol{u}_2) = \mathsf{t}(\boldsymbol{v}_2) = \mathsf{t}(\boldsymbol{t}_q)$.

Suppose that $z \in \mathsf{con}(\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p)$ with $z \neq \mathsf{h}(\boldsymbol{s}_0), \mathsf{t}(\boldsymbol{s}_p)$, so that $\mathsf{h}(\boldsymbol{s}_0)z \hookrightarrow \boldsymbol{u}_2$. Then by Lemma 6.15(i), the longest $\{\overline{\mathsf{h}(\boldsymbol{s}_0)}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ can only be $\mathsf{h}(\boldsymbol{s}_0)z$ or $z^*(\mathsf{h}(\boldsymbol{s}_0))^*$. But since $\boldsymbol{v}_2$ is a bipartite word (see Remark 6.9(iv)) that contains the variable $\mathsf{h}(\boldsymbol{t}_0) = \mathsf{h}(\boldsymbol{s}_0)$, it cannot contain the variable $(\mathsf{h}(\boldsymbol{s}_0))^*$. Hence the longest $\{\overline{\mathsf{h}(\boldsymbol{s}_0)}, \bar{z}\}$-subsequence of $\boldsymbol{v}_2$ must be $\mathsf{h}(\boldsymbol{s}_0)z$, so that $z \in \mathsf{con}(\boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q)$. Therefore, the inclusion $\mathsf{con}(\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p) \subseteq \mathsf{con}(\boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q)$ holds. The reverse inclusion $\mathsf{con}(\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p) \supseteq \mathsf{con}(\boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q)$ is established by a symmetrical argument, so that $\mathsf{con}(\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p) = \mathsf{con}(\boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q)$. Further, since $\mathsf{h}(\boldsymbol{s}_0) = \mathsf{h}(\boldsymbol{t}_0)$ and $\mathsf{t}(\boldsymbol{s}_p) = \mathsf{t}(\boldsymbol{t}_q)$ by (a), it is easy to show by Lemma 6.15 that $\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p = \boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q$. Hence

(b) $\boldsymbol{s}_0\boldsymbol{s}_1 \cdots \boldsymbol{s}_p = \boldsymbol{t}_0\boldsymbol{t}_1 \cdots \boldsymbol{t}_q = z_1 z_2 \cdots z_r$ for some distinct $z_1, z_2, \ldots, z_r \in \mathcal{X} \cup \mathcal{X}^*$,

where $z_1 = \mathsf{h}(\boldsymbol{s}_0) = \mathsf{h}(\boldsymbol{t}_0)$ and $z_r = \mathsf{t}(\boldsymbol{s}_p) = \mathsf{t}(\boldsymbol{t}_q)$.

Now it follows from Lemma 6.14 that $p = 0$ if and only if $q = 0$, so there are two cases: $p = q = 0$ and $p, q \geq 1$. If $p = q = 0$, then $\boldsymbol{u}_2 = \boldsymbol{s}_0 = \boldsymbol{t}_0 = \boldsymbol{v}_2$, so the proof is complete. Hence it remains to assume $p, q \geq 1$.

Seeking a contradiction, suppose that $\boldsymbol{s}_0 \neq \boldsymbol{t}_0$. Then by (b), either $\boldsymbol{s}_0$ is a proper prefix of $\boldsymbol{t}_0$ or $\boldsymbol{t}_0$ is a proper prefix of $\boldsymbol{s}_0$. By symmetry, suppose that $\boldsymbol{s}_0$ is a proper prefix of $\boldsymbol{t}_0$, so that $\boldsymbol{s}_0 = z_1 z_2 \cdots z_k$ and $\boldsymbol{t}_0 = z_1 z_2 \cdots z_\ell$ with $k < \ell \leq r$. Then

$$\boldsymbol{u}_2 = z_1 z_2 \cdots z_k \cdot \boldsymbol{c}_1 \cdot z_{k+1} z_{k+2} \cdots$$

and

$$\boldsymbol{v}_2 = z_1 z_2 \cdots z_k \cdot z_{k+1} z_{k+2} \cdots z_\ell \cdot \boldsymbol{d}_1 \cdot z_{\ell+1} z_{\ell+2} \cdots .$$

Since $\boldsymbol{c}_1$ is an ordered B_0-block, it begins with y^2 for some $y \in \mathcal{X} \cup \mathcal{X}^*$. Then $y^2 z_{k+1} \hookrightarrow \boldsymbol{u}_2$ but $y^2 z_{k+1} \not\hookrightarrow \boldsymbol{v}_2$ and $z_{k+1}^*(y^*)^2 \not\hookrightarrow \boldsymbol{v}_2$, which is impossible in view of Lemma 6.16(i). Therefore, $\boldsymbol{s}_0 = z_1 z_2 \cdots z_k = \boldsymbol{t}_0$, so that

$$\boldsymbol{u}_2 = z_1 z_2 \cdots z_k \cdot \boldsymbol{c}_1 \cdot z_{k+1} z_{k+2} \cdots$$

and

$$\boldsymbol{v}_2 = z_1 z_2 \cdots z_k \cdot \boldsymbol{d}_1 \cdot z_{k+1} z_{k+2} \cdots .$$

Seeking a contradiction, suppose that $\mathsf{con}(\boldsymbol{c}_1) \neq \mathsf{con}(\boldsymbol{d}_1)$, say $y \in \mathsf{con}(\boldsymbol{c}_1) \setminus \mathsf{con}(\boldsymbol{d}_1)$. Then since $\boldsymbol{c}_1$ is an ordered B_0-block, $\boldsymbol{c}_1 = \boldsymbol{a} y^2 \boldsymbol{b}$ for some $\boldsymbol{a}, \boldsymbol{b} \in F^1_{\mathsf{inv}}(\mathcal{X})$. Hence $y^2 z_{k+1} \hookrightarrow \boldsymbol{u}_2$ but $y^2 z_{k+1} \not\hookrightarrow \boldsymbol{v}_2$ and $z_{k+1}^*(y^*)^2 \not\hookrightarrow \boldsymbol{v}_2$, which is impossible in view of Lemma 6.16(i). Therefore, $\mathsf{con}(\boldsymbol{c}_1) = \mathsf{con}(\boldsymbol{d}_1)$. Since $\boldsymbol{c}_1$ and $\boldsymbol{d}_1$ are ordered B_0-blocks, we have $\boldsymbol{c}_1 = \boldsymbol{d}_1$.

Without loss of generality, assume that $p \leq q$. The arguments in the previous two paragraphs can be repeated to show that $\boldsymbol{c}_i = \boldsymbol{d}_i$ and $\boldsymbol{s}_i = \boldsymbol{t}_i$ and for all $i = 1, 2, \ldots, p-1$. Hence

$$\boldsymbol{u}_2 = \boldsymbol{s}_0 \left(\prod_{i=1}^{p-1} (\boldsymbol{c}_i \boldsymbol{s}_i) \right) \boldsymbol{c}_p \boldsymbol{s}_p \quad \text{and} \quad \boldsymbol{v}_2 = \boldsymbol{s}_0 \left(\prod_{i=1}^{p-1} (\boldsymbol{c}_i \boldsymbol{s}_i) \right) \boldsymbol{d}_p \boldsymbol{t}_p \boldsymbol{d}_{p+1} \boldsymbol{t}_{p+1} \cdots \boldsymbol{d}_q \boldsymbol{t}_q .$$

Arguments that are dual to those from the previous two paragraphs (with the use of Lemma 6.17 instead of Lemma 6.16) can be repeated to show that $\boldsymbol{s}_p = \boldsymbol{t}_q$ and then $\boldsymbol{c}_p = \boldsymbol{d}_q$. It follows from (b) that $p = q$ and $\boldsymbol{u}_2 = \boldsymbol{v}_2$. □

7. Involution semigroups of order up to four

Multiplication tables of involution semigroups of order up to four are given in this section. For a more compact presentation, the column/row headers are omitted from each multiplication table. For instance, the involution semigroup $S = \{1, 2, 3\}$ given by the multiplication table on the left is abbreviated to the array on the right:

S	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2
x	1	2	3
x^*	1	3	2

→

1	2	3
2	3	1
3	1	2
1	3	2

Up to isomorphism, there are three involution semigroups of order two and 15 involution semigroups of order three, all of which are commutative [20, Section 4]; see Table 2.

Up to isomorphism, there are 83 involution semigroups of order four; see Table 3.

1 1	1 1	1 2
1 1	1 2	2 1
1 2	1 2	1 2

1 1 1	1 1 1	1 1 1	1 1 1	1 1 1
1 1 1	1 1 1	1 1 1	1 1 1	1 1 2
1 1 1	1 1 1	1 1 2	1 1 3	1 2 3
1 2 3	1 3 2	1 2 3	1 2 3	1 2 3

1 1 1	1 1 1	1 1 1	1 1 1	1 1 1
1 2 1	1 2 1	1 2 2	1 2 2	1 2 3
1 1 3	1 1 3	1 2 2	1 2 3	1 3 2
1 2 3	1 3 2	1 2 3	1 2 3	1 2 3

1 1 3	1 1 3	1 2 2	1 2 3	1 2 3
1 1 3	1 2 3	2 1 1	2 3 1	2 3 1
3 3 1	3 3 1	2 1 1	3 1 2	3 1 2
1 2 3	1 2 3	1 2 3	1 2 3	1 3 2

Table 2. The three involution semigroups of order two and the 15 of order three.

1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1
1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1
1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 2	1 1 1 2	1 1 1 2	1 1 1 2	1 1 1 3
1 1 1 1	1 1 1 1	1 1 1 2	1 1 1 4	1 1 1 4	1 1 2 1	1 1 2 1	1 1 2 1	1 1 2 2	1 1 2 3	1 1 3 4
1 2 3 4	1 2 4 3	1 2 3 4	1 2 3 4	1 3 2 4	1 2 4 3	1 2 3 4	1 2 4 3	1 2 3 4	1 2 3 4	1 2 3 4

1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1
1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1
1 1 1 3	1 1 2 1	1 1 2 1	1 1 2 1	1 1 2 1	1 1 2 2	1 1 2 2	1 1 3 1	1 1 3 1	1 1 3 3	1 1 3 3
1 2 1 4	1 1 1 2	1 1 1 2	1 1 1 4	1 1 2 2	1 1 2 2	1 1 2 2	1 1 1 4	1 1 1 4	1 1 3 3	1 1 3 4
1 3 2 4	1 2 3 4	1 2 4 3	1 2 3 4	1 2 4 3	1 2 3 4	1 2 4 3	1 2 3 4	1 2 4 3	1 2 3 4	1 2 3 4

1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1
1 1 1 1	1 1 1 2	1 1 1 2	1 1 1 2	1 1 1 2	1 1 1 2	1 1 1 2	1 1 1 2	1 1 1 2	1 1 2 2	1 1 2 2
1 1 3 4	1 1 1 2	1 1 1 3	1 1 1 3	1 1 2 3	1 1 3 1	1 1 3 3	1 2 3 1	1 2 3 2	1 2 3 3	1 2 3 3
1 1 4 3	1 2 2 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 1 4	1 2 3 4	1 1 1 4	1 1 1 4	1 2 3 3	1 2 3 4
1 2 3 4	1 2 3 4	1 2 3 4	1 3 2 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 4 3	1 2 4 3	1 2 3 4	1 2 3 4

1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1
1 1 2 2	1 2 1 1	1 2 1 1	1 2 1 1	1 2 1 1	1 2 1 1	1 2 1 2	1 2 1 2	1 2 2 2	1 2 2 2	1 2 2 2
1 2 3 4	1 1 3 1	1 1 3 1	1 1 3 3	1 1 3 3	1 1 3 4	1 1 3 3	1 1 3 3	1 2 2 2	1 2 2 2	1 2 2 2
1 2 4 3	1 1 1 4	1 1 1 4	1 1 3 3	1 1 3 4	1 1 4 3	1 2 3 4	1 2 3 4	1 2 2 2	1 2 2 2	1 2 2 3
1 2 3 4	1 2 3 4	1 2 4 3	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 3 2 4	1 2 3 4	1 2 4 3	1 2 3 4

1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1
1 2 2 2	1 2 2 2	1 2 2 2	1 2 2 2	1 2 2 2	1 2 2 2	1 2 2 2	1 2 2 4	1 2 2 4	1 2 3 3	1 2 3 4
1 2 2 2	1 2 2 3	1 2 3 2	1 2 3 2	1 2 3 3	1 2 3 3	1 2 3 4	1 2 2 4	1 2 3 4	1 3 2 2	1 3 4 2
1 2 2 4	1 2 3 4	1 2 2 4	1 2 2 4	1 2 3 3	1 2 3 4	1 2 4 3	1 4 4 2	1 4 4 2	1 3 2 2	1 4 2 3
1 2 3 4	1 2 3 4	1 2 3 4	1 2 4 3	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4

1 1 1 1	1 1 1 4	1 1 1 4	1 1 1 4	1 1 1 4	1 1 1 4	1 1 1 4	1 1 1 4	1 1 1 4	1 1 1 4	1 1 1 4
1 2 3 4	1 1 1 4	1 1 1 4	1 1 1 4	1 1 1 4	1 1 2 4	1 2 1 4	1 2 1 4	1 2 2 4	1 2 2 4	1 2 3 4
1 3 4 2	1 1 1 4	1 1 1 4	1 1 2 4	1 1 3 4	1 2 3 4	1 1 3 4	1 1 3 4	1 2 2 4	1 2 3 4	1 3 2 4
1 4 2 3	4 4 4 1	4 4 4 1	4 4 4 1	4 4 4 1	4 4 4 1	4 4 4 1	4 4 4 1	4 4 4 1	4 4 4 1	4 4 4 1
1 2 4 3	1 2 3 4	1 3 2 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 3 2 4	1 2 3 4	1 2 3 4	1 2 3 4

1 1 3 3	1 1 3 3	1 1 3 3	1 1 3 3	1 1 3 3	1 1 3 3	1 1 3 4	1 1 3 4	1 1 3 4	1 1 3 4	1 2 2 2
1 1 3 3	1 1 3 3	1 2 3 3	1 2 3 4	1 2 3 4	2 2 4 4	1 1 3 4	1 1 3 4	1 2 3 4	1 2 3 4	2 1 1 1
3 3 1 1	3 3 1 1	3 3 1 1	3 3 1 1	3 3 1 1	1 1 3 3	3 3 4 1	3 3 4 1	3 3 4 1	3 3 4 1	2 1 1 1
3 3 1 1	3 3 1 2	3 3 1 1	3 4 1 1	3 4 1 2	2 2 4 4	4 4 1 3	4 4 1 3	4 4 1 3	4 4 1 3	2 1 1 1
1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4	1 3 2 4	1 2 3 4	1 2 4 3	1 2 3 4	1 2 4 3	1 2 3 4

1 2 2 2	1 2 2 4	1 2 3 4	1 2 3 4	1 2 3 4	1 2 3 4
2 1 1 1	2 4 4 1	2 1 4 3	2 1 4 3	2 1 4 3	2 1 4 3
2 1 1 1	2 4 4 1	3 4 1 2	3 4 1 2	3 4 2 1	3 4 2 1
2 1 1 1	4 1 1 2	4 3 2 1	4 3 2 1	4 3 1 2	4 3 1 2
1 2 4 3	1 2 3 4	1 2 3 4	1 2 4 3	1 2 3 4	1 2 4 3

Table 3. The 83 involution semigroups of order four.

Acknowledgements

The authors are grateful to the reviewers for their insightful reports and suggestions.

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Received March 11, 2025. Revised January 13, 2026.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 342 No. 1 May 2026
