*Pacific Journal of Mathematics*

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Volume 342 No. 2 June 2026

PACIFIC JOURNAL OF MATHEMATICS
Vol. 342, No. 2, 2026
https://doi.org/10.2140/pjm.2026.342.207

# ON MULTILINEAR MAXIMAL OPERATORS ALONG HOMOGENEOUS CURVES

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**Suppose that**

$$\vec{\gamma}(t) := (\gamma_1(t), \dots, \gamma_n(t)) = (a_1 t^{d_1}, \dots, a_n t^{d_n}), \;\; 1 \le d_1 < \cdots < d_n \in \mathbb{Z}, \;\; a_i \neq 0$$

**is a homogeneous polynomial curve. We prove that whenever** $p_1, \dots, p_n > 1$ **and** $1/p = \sum_{j=1}^n 1/p_j \le 1$**, there exists an absolute constant** $0 < C = C_{p_1,\dots,p_n} < \infty$ **such that**

$$\left\| \sup_{r>0} \frac{1}{r} \int_0^r \prod_{i=1}^n |f_i(x - \gamma_i(t))| \, dt \right\|_{L^p(\mathbb{R})} \le C \cdot \prod_{i=1}^n \|f_j\|_{L^{p_j}(\mathbb{R})}.$$

**Our main tool is a smoothing estimate, adapted from work of Kosz, Mirek, Peluse, Wan, and Wright.**

## 1. Introduction

The study of multilinear maximal functions dates back to celebrated work of Lacey [11], who proved the following theorem.

**Theorem 1.1.** *Suppose that* $p_1, p_2 > 1$, *and that* $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} < \frac{3}{2}$. *Then there exists an absolute constant* $0 < C_{p_1,p_2} < \infty$ *such that*

$$\begin{aligned} \|B^{\vec{\gamma}}(f_1, f_2)\|_{L^p(\mathbb{R})} &:= \left\| \sup_{r>0} \frac{1}{r} \int_0^r \left| f(x - \gamma_1(t)) g(x - \gamma_2(t)) \right| dt \right\|_{L^p(\mathbb{R})} \\ &\le C_{p_1,p_2} \|f_1\|_{L^{p_1}(\mathbb{R})} \|f_2\|_{L^{p_2}(\mathbb{R})}, \end{aligned}$$

*whenever* $\gamma_i(t) = a_i t$ *and* $a_1 \neq a_2$ *are nonzero.*

The key property of this operator is its modulation invariance, which necessitated an approach using time-frequency analysis, building off ideas of Lacey and Thiele in their work on the bilinear Hilbert transform [12; 13]; this method was later adapted to handle multilinear extensions, see [2].

On the other hand, when the modulation invariance embedded in $\vec{\gamma}$ is eliminated, different techniques can be used. This was first explored in the singular integral context in [14; 16], with subsequent work of [15] establishing the following; see also [5].

*MSC2020:* 42A99.

*Keywords:* multilinear maximal operators, Littlewood–Paley theory, smoothing inequalities.

**Theorem 1.2.** *Suppose that $\gamma_1(t) = t,\ \gamma_2(t) = P(t)$, where $P(t)$ is a polynomial of degree $d$ which vanishes to degree $\geq 2$ at the origin. Then whenever $p_1, p_2 > 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} < \frac{d}{d-1}$, there exists an absolute constant $0 < C_{p_1,p_2} < \infty$ such that*

$$\|B^{\vec{\gamma}}(f_1, f_2)\|_{L^p(\mathbb{R})} \leq C_{p_1,p_2} \|f_1\|_{L^{p_1}(\mathbb{R})} \|f_2\|_{L^{p_2}(\mathbb{R})}.$$

The key ingredient in establishing Theorem 1.2 was a *Sobolev estimate*, a representative case of which is stated below.

**Proposition 1.3** (special case). *Suppose that $\gamma_1(t) = t,\ \gamma_2(t) = t^2$. There exist absolute constants $0 < c < C < \infty$ such that if $\hat{f}_i$ vanishes outside $\{|\xi| \leq 2^{l+ik}\}$ for some $i$, then*

$$\left\| \int_0^1 f_1(x - \gamma_1(2^{-k}t)) f_2(x - \gamma_2(2^{-k}t))\, dt \right\|_{L^1(\mathbb{R})} \leq C 2^{-cl} \|f_1\|_{L^2(\mathbb{R})} \|f_2\|_{L^2(\mathbb{R})}.$$

In other words, the only obstruction to the estimate

$$\left\| \int_0^1 f_1(x - \gamma_1(2^{-k}t)) f_2(x - \gamma_2(2^{-k}t))\, dt \right\|_{L^1(\mathbb{R})} \ll \|f_1\|_{L^2(\mathbb{R})} \|f_2\|_{L^2(\mathbb{R})},$$

where $\ll$ informally stands for "much smaller than", arises from zero-frequency considerations; note that the modulation invariance from Theorem 1.1 precludes such an argument. Aside from their utility in studying the operators $\{B^{\vec{\gamma}}\}$, Sobolev estimates have found a wide use in problems in Euclidean Ramsey theory, dating back to the work of Bourgain [1], with more recent contributions found in [3; 4; 6; 9; 10] among others. The current state of the art for operators of the form $\{B^{\vec{\gamma}}\}$ is essentially due to Hu and Lie [7], who addressed trilinear formulations

$$\vec{\gamma} = (P_1(t), P_2(t), P_3(t))$$

provided $P_i$ are distinct degree polynomials which vanish at different rates at 0; see Observation 1.2(i) of [7] and [5, Remark 2]. their key input was a trilinear analogue of Proposition 1.3. We state below a slightly stronger version of their estimate [5, Theorem 3.1]. Their estimate has an additional dependence on $k$, which we drop here as justified by Proposition 1.6 below.

**Proposition 1.4** (special case). *Suppose that $\gamma_i(t) = t^i,\ 1 \leq i \leq 3$. There exist absolute constants $0 < c < C < \infty$ such that if $\hat{f}_i$ vanishes outside $\{|\xi| \leq 2^{l+ik}\}$ for some $i$, then*

$$\| \int_0^1 \prod_{i=1}^3 f_i(x - \gamma_i(2^{-k}t))\, dt \|_{L^1(\mathbb{R})} \leq C 2^{-cl} \prod_{i=1}^3 \|f_i\|_{L^3(\mathbb{R})}.$$

The goal of this paper is to address multilinear analogues of $B^{\vec{\gamma}}$ under the simplifying assumption that our curves are homogeneous polynomials.

Specifically, we will be concerned with polynomial curves

$$\vec{\gamma}(t) := (\gamma_1(t), \dots, \gamma_n(t)) = (a_1 t^{d_1}, \dots, a_n t^{d_n}), \quad 1 \le d_1 < \cdots < d_n \in \mathbb{Z}, \ a_i \neq 0.$$

Our main result is this:

**Theorem 1.5.** *Suppose that* $p_1, \dots, p_n > 1$, *and that*

$$\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_n} \le 1.$$

*Then there exists an absolute constant* $0 < C_{p_1,\dots,p_n;\vec{\gamma}} < \infty$ *such that*

$$\Big\| \sup_{r>0} \frac{1}{r} \int_0^r \prod_{i=1}^n |f_i(x - \gamma_i(t))| \, dt \Big\|_{L^p(\mathbb{R})} \le C_{p_1,\dots,p_n;\vec{\gamma}} \prod_{i=1}^n \|f_i\|_{L^{p_i}(\mathbb{R})}.$$

As might be expected, the key input in proving Theorem 1.5 is the following multilinear Sobolev estimate:

**Proposition 1.6.** *Suppose that* $\gamma_i(t) = a_i t^{d_i}$, $1 \le i \le n$ *with integer exponents* $1 \le d_1 < d_2 < \cdots < d_n \in \mathbb{Z}$, $a_i \neq 0$. *There exist absolute constants* $0 < c < C < \infty$ *such that if* $\hat{f}_i$ *vanishes outside* $\{|\xi| \le 2^{l+d_i k}\}$ *for some* $i$, *then*

$$\Big\| \int_0^1 \prod_{i=1}^n f_i(x - \gamma_i(2^{-k} t)) \, dt \Big\|_{L^1(\mathbb{R})} \le C 2^{-cl} \prod_{i=1}^n \|f_i\|_{L^n(\mathbb{R})}.$$

To establish Proposition 1.6, we adapt a recent result of Kosz, Mirek, Peluse, Wan, and Wright [8]; with Proposition 1.6 in hand, Theorem 1.5 readily presents. While it is reasonable to expect that an analogue of Proposition 1.6, and thus Theorem 1.5, should hold for more general distinct-degree polynomial curves $\gamma_i$ which vanish to distinct degrees at the origin, the argument to deduce Proposition 1.6 from the results of [8] crucially relies on homogeneity, and does not readily adapt to the more general setting.

***Notation.*** Throughout, we let $\varphi$ denote various mean-one Schwartz functions, normalized in some sufficiently large seminorm. The precise choice of $\varphi$ might differ from line to line. Similarly, we use $\psi$ to denote a similar function, but with

$$\mathbf{1}_{|\xi| \approx 1} \le \hat{\psi} \le \mathbf{1}_{|\xi| \approx 1} \tag{1.7}$$

and such that

$$\sum_l \psi(\xi/2^l) = \mathbf{1}_{\xi \neq 0}. \tag{1.8}$$

We use the notation

$$\phi_k(x) := 2^k \phi(2^k x)$$

to denote $L^1$-normalized dilations, and let

$$B_k^{\vec{\gamma}}(f_1,\dots,f_n)(x) := B_k(f_1,\dots,f_n)(x) := \int_0^1 \prod_{i=1}^n f_i(x-\gamma_i(2^{-k}t))\,dt. \tag{1.9}$$

Below, we will regard $0 \neq a_1,\dots,a_n = O(1)$ as arbitrary but fixed, and will abbreviate

$$D := D(\vec{\gamma}) := d_1 + \cdots + d_n. \tag{1.10}$$

*Asymptotic notation.* We will make use of the modified Vinogradov notation. We use $X \lesssim Y$, or $Y \gtrsim X$, to denote the estimate $X \leq CY$ for an absolute constant $C$. We use $X \approx Y$ as shorthand for $Y \lesssim X \lesssim Y$. We also make use of big-O notation: we let $O(Y)$ denote a quantity that is $\lesssim Y$. We let $f(t) := o_{t\to a}(X(t))$ denote a quantity such that $|f(t)|/X(t) \to 0$ as $t \to a$.

If we need $C$ to depend on a parameter, we shall indicate this by subscripts, thus for instance $X \lesssim_p Y$ denotes the estimate $X \leq C_p Y$ for some $C_p$ depending on $p$. We analogously define $O_p(Y)$.

## 2. Sobolev estimates

The main goal of this section is to prove the following Sobolev estimate.

**Proposition 2.1.** *There exists an absolute $c > 0$ such that the following single scale estimate holds whenever $s_i \geq 0$:*

$$\big\| B_k(\psi_{kd_1+s_1} * f_1,\dots,\psi_{kd_n+s_n} * f_n)\big\|_{L^1(\mathbb{R})} \lesssim 2^{-cs} \prod_{i=1}^n \|f_i\|_{L^n(\mathbb{R})},$$

*where $s := \max\{s_i\}$.*

The proof of Proposition 1.3 will be accomplished via a projection argument, anchored by [8, Theorem 6.1] in the case where $\gamma_i(t) = a_i t^{d_i}$, and $\mathbb{K} = \mathbb{R}$, which dictates that the operator

$$A_{-k}(F_1,\dots,F_n)(x_1,\dots,x_n) := \frac{1}{2^k}\int_0^{2^k} \prod_{i=1}^n F_i(x_1,\dots,x_i - a_i t^{d_i},\dots,x_n)\,dt$$

satisfies nontrivial norm estimates whenever some $\hat{F}_i$ vanishes in $|\xi_i| \leq 2^{-kd_i}\delta^{-1}$.

**Lemma 2.2** (special case of Theorem 6.1 of [8]). *In the above setting, suppose that some $\hat{F}_i$ vanishes in $|\xi_i| \leq 2^{-kd_i}\delta^{-1}$. Then there exists some absolute $c > 0$ such that*

$$\big\| A_{-k}(F_1,\dots,F_n)\big\|_{L^1(\mathbb{R}^n)} \lesssim_n (\delta^c + 2^{-kc}) \prod_{i=1}^n \|F_i\|_{L^n(\mathbb{R}^n)}, \tag{2.3}$$

*provided $k \geq 0$.*

We now remove the dependence on $k$ on the right side of (2.3), and address the case where $k \le 0$. Specifically, we prove that for all $k$

$$\|A_{-k}(F_1, \ldots, F_n)\|_{L^1(\mathbb{R}^n)} \lesssim_n \delta^c \prod_{i=1}^{n} \|F_i\|_{L^n(\mathbb{R}^n)} \tag{2.4}$$

whenever some $\hat{F}_i$ vanishes in $|\xi_i| \le 2^{-kd_i}\delta^{-1}$; for concreteness, suppose that this index is $j$.

To do so, introduce the operator

$$D_\lambda F(x_1, \ldots, x_n) := F(\lambda^{d_1} x_1, \ldots, \lambda^{d_n} x_n)$$

and choose $2^{k_0} \gg \delta^{-1}$. If we define

$$G_i(x_1, \ldots, x_n) := D_{2^{k-k_0}} F_i(x_1, \ldots, x_n),$$

then $\hat{G}_i$ vanishes on $|\xi_j| \le 2^{-k_0 d_j}\delta^{-1}$, so

$$\|A_{-k_0}(G_1, \ldots, G_n)\|_{L^1(\mathbb{R}^n)} \lesssim \delta^c \prod_{i=1}^{n} \|G_i\|_{L^n(\mathbb{R}^n)}.$$

But now

$$\begin{aligned}
&D_{2^{k_0-k}}\big(A_{-k_0}(G_1, \ldots, G_n)\big)(x_1, \ldots, x_n) \\
&\quad = 2^{-k_0} \int_0^{2^{k_0}} \prod_{i=1}^{n} G_i(2^{d_1(k_0-k)} x_1, \ldots, 2^{d_i(k_0-k)} x_i + a_i t^{d_i}, \ldots, 2^{d_n(k_0-k)} x_n)\, dt \\
&\quad = 2^{-k} \int_0^{2^k} \prod_{i=1}^{n} G_i(2^{d_1(k_0-k)} x_1, \ldots, 2^{d_i(k_0-k)} x_i + a_i 2^{d_i(k_0-k)} t^{d_i}, \ldots, 2^{d_n(k_0-k)} x_n) \\
&\quad = A_{-k}(D_{2^{k_0-k}} G_1, \ldots, D_{2^{k_0-k}} G_n)(x) \\
&\quad = A_{-k}(F_1, \ldots, F_n)(x),
\end{aligned}$$

so the result follows from changing variables.

*Proof of Proposition 2.1.* Set $g_i := \psi_{kd_i+s_i} * f_i$, let $\vec{1} := n^{-1/2}(1, \ldots, 1) \in \mathbb{R}^n$, and let $\varphi : \mathbb{R}^n \to \mathbb{C}$ be a bump function with compactly supported Fourier transform that is constant along $\mathbb{R}\vec{1}$. For each

$$2^{-100D|k|-100s} \gg \epsilon > 0$$

sufficiently small (see (1.10)), define

$$F_i(x_1, \ldots, x_n) := \epsilon^{1-1/n} \varphi(\epsilon x) g_i(x \cdot \vec{1})$$

so that $\|F_i\|_{L^n(\mathbb{R}^n)} \approx \|g_i\|_{L^n(\mathbb{R})}$ and $\hat{F}_i$ is supported in an $O(\epsilon)$ neighborhood of

$$\{\xi\vec{1} : \hat{g}_i(\xi) \ne 0\};$$

in particular, for some $j$, $\hat{F}_j$ vanishes when $|\xi| \lesssim 2^{kd_j+s}$. So,

$$\big\|A_k(F_1, \ldots, F_n)\big\|_{L^1(\mathbb{R}^n)} \lesssim 2^{-cs} \prod_{i=1}^{n} \|g_i\|_{L^n(\mathbb{R})}.$$

On the other hand,

$$
\begin{aligned}
&\big\| A_k(F_1, \dots, F_n)(x_1, \dots, x_n) \big\|_{L^1(\mathbb{R}^n)} \\
&\quad = \bigg\| 2^k \int_0^{2^{-k}} \Big( \prod_{i=1}^n g_i(x \cdot \vec{1} - a_i t^{d_i}) \Big) \cdot \Big( \epsilon^{n-1} \prod_{i=1}^n \varphi(\epsilon x - \epsilon a_i t^{d_i} \vec{e}_i) \Big) \, dt \bigg\|_{L^1(\mathbb{R}^n)} \\
&\quad = \int_{(\mathbb{R}\vec{1})^\perp} \int_{\mathbb{R}} \bigg| 2^k \int_0^{2^{-k}} \Big( \prod_{i=1}^n g_i(y - a_i t^{d_i}) \Big) \Big( \epsilon^{n-1} \prod_{i=1}^n \varphi(\epsilon z - \epsilon a_i t^{d_i} \vec{e}_i) \Big) \, dt \bigg| \, dy \, dz
\end{aligned}
$$

using a change of variables (so in particular, $z_1, \dots, z_{n-1}$ form an orthogonal basis for $(\mathbb{R}\vec{1})^\perp$); above, $\vec{e}_i$ is the $i$-th coordinate vector.

By Taylor expansion,

$$
\begin{aligned}
&\int_{(\mathbb{R}\vec{1})^\perp} \int_{\mathbb{R}} \bigg| 2^k \int_0^{2^{-k}} \Big( \prod_{i=1}^n g_i(y - a_i t^{d_i}) \Big) \Big( \epsilon^{n-1} \prod_{i=1}^n \varphi(\epsilon z - \epsilon a_i t^{d_i} \vec{e}_i) \Big) \, dt \bigg| \, dy \, dz \\
&\quad = \|\varphi^n\|_{L^1((\mathbb{R}\vec{1})^\perp)} \cdot \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \bigg| 2^k \int_0^{2^{-k}} \prod_{i=1}^n g_i(y - a_i t^{d_i}) \, dt \bigg| \, dy \\
&\qquad + O(2^{O(k)}\epsilon) \int_{\mathbb{R}^{n-1}} \Big( \int_{\mathbb{R}} 2^k \int_0^{2^{-k}} \prod_{i=1}^n \big| g_i(y - a_i t^{d_i}) \big| \, dt \, dy \Big) \epsilon^{n-1} (1 + \epsilon |z|)^{-100} \, dz \\
&\quad = \|\varphi^n\|_{L^1((\mathbb{R}\vec{1})^\perp)} \big\| B_k(g_1, \dots, g_n)(x) \big\|_{L^1(\mathbb{R})} + O_k\Big( \epsilon \prod_{i=1}^n \|g_i\|_{L^n(\mathbb{R})} \Big).
\end{aligned}
$$

The result follows by sending $\epsilon \downarrow 0$. □

The following proposition will be used to complement Proposition 1.3 via interpolation; before we can prove it, we state a lemma.

**Proposition 2.5.** *Suppose $n \geq 2$, $p_1, \dots, p_n > 1$ and $\frac{1}{p} = \sum_i \frac{1}{p_i} \leq 1$. Then*

$$
\big\| \sup_k \big| B_k(\psi_{kd_1+s_1} * f, \dots, \psi_{kd_n+s_n} * f) \big| \big\|_{L^p(\mathbb{R})} \lesssim s^n \prod_{i=1}^n \|f\|_{L^{p_i}(\mathbb{R})},
$$

*where $s := \max\{s_i\}$.*

**Lemma 2.6.** *Suppose that $\varphi$ is a Schwartz function. For $t \in \mathbb{R}$, consider the maximal function*

$$
M^t f(x) := \sup_j |\varphi_j * f|(x - 2^{-j} t),
$$

*There exists an absolute constant $0 < C < \infty$ (independent of $t$) such that*

$$
\|M^t f\|_{L^{1,\infty}(\mathbb{R})} \leq C \log(2 + |t|) \|f\|_{L^1(\mathbb{R})}.
$$

*Proof.* Since the maximal function is trivially bounded on $L^\infty(\mathbb{R})$ we may use vector-valued Calderón–Zygmund theory; see [17, §1], for example. In particular, it suffices to show that

$$\int_{|x|\geq 10|y|} \sum_j \left|\varphi_j(x-2^{-j}t-y)-\varphi_j(x-2^{-j}t)\right| \lesssim \log|t|.$$

By dilation invariance we can normalize $|y|\approx 1$. When $2^j \leq 100^{-1}$ we use the mean-value theorem; when $100^{-1}\leq 2^j \lesssim |t|$ we use a single-scale estimate; and when $2^j \gg |t|$ we use the decay of $\varphi$. □

*Proof of Proposition 2.5.* We have the pointwise bound

$$\begin{aligned}\big|B_k(\psi_{kd_1+s_1} * f_1, \dots, \psi_{kd_n+s_n} * f_n)(x)\big| \\ &= \left|\int_0^1 \Big(\int_{\mathbb{R}^n} \prod_{i=1}^n \psi_{kd_i+s_i}(x-a_i 2^{-kd_i}t^{d_i}-u_i) f_i(u_i)\,du\Big)\,dt\right| \\ &\leq \int_0^1 \prod_{i=1}^n M^{2^{s_i}a_i t^{d_i}} f_i(x)\,dt,\end{aligned}$$

where $M^t$ is the maximal function from Lemma 2.6 with $\varphi$ chosen to be $\psi$. So

$$\begin{aligned}\Big\|\sup_k \big|B_k(\psi_{kd_1+s_1} * f_1, \dots, \psi_{kd_n+s_n} * f_n)\big|\Big\|_{L^p(\mathbb{R})} &\lesssim \int_0^1 \Big\|\prod_{i=1}^n M^{2^{s_i}a_i t^{d_i}} f_i\Big\|_{L^p(\mathbb{R})}\,dt \\ &\lesssim s^n \prod_{i=1}^n \|f_i\|_{L^{p_i}(\mathbb{R})},\end{aligned}$$

as desired. □

## 3. The proof of Theorem 1.5

We will prove that

$$\Big\|\sup_k |B_k(f_1,\dots,f_n)|\Big\|_{L^p(\mathbb{R})} \lesssim \prod_{i=1}^n \|f_i\|_{L^{p_i}(\mathbb{R})}. \tag{3.1}$$

Theorem 1.5 follows from (3.1), because averaging operators are positive and therefore the lacunary supremum in (3.1) dominates the full supremum in Theorem 1.5 up to a factor of two.

The proof is by induction. Thus, we will assume that for all $m<n$

$$\Big\|\sup_k |B_k(f_1,\dots,f_m)|\Big\|_{L^p(\mathbb{R})} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R})},$$

whenever $\vec{\gamma}$ is a homogeneous polynomial curve, $p_1,\dots,p_m>1$, and $\frac{1}{p_1}+\cdots+\frac{1}{p_m}=\frac{1}{p}\leq 1$, with the $n=1$ case following from Hardy–Littlewood and convexity.

By Taylor expansion, for each $k$, we have the decomposition

$$\begin{aligned}&B_k(f_1,\dots,f_n) \\ &\quad= \sum_{j\geq 0}\frac{1}{j!}\big((\partial^j\varphi)_{kd_n} * f_n\big) B^{(j)}_{k,\neq n}(f_1,\dots,f_{n-1}) + B_k(f_1,\dots,f_{n-1},f_n-\varphi_{kd_n} * f_n),\end{aligned} \tag{3.2}$$

where

$$B^{(j)}_{k,\neq n}(f_1,\dots,f_{n-1})(x) := 2^k \int_0^{2^{-k}} \prod_{i=1}^{n-1} f_i(x - a_i t^{d_i})\ (-a_n 2^{kd_n} t^{d_n})^j\, dt,$$

and $\partial^j \varphi$ satisfies all of the same Schwartz normalizations as $\varphi$ up to factors of $C^j$, $C = O(1)$.

Thus, by iterating (3.2), induction and convexity, it suffices to prove that

$$\Big\| \sup_k |B_k(f_1 - \varphi_{kd_1} * f_1, \dots, f_n - \varphi_{kd_n} * f_n)| \Big\|_{L^p(\mathbb{R})} \lesssim \prod_{i=1}^n \|f_i\|_{L^{p_i}(\mathbb{R})}$$

in the above range of $p$, $p_i$.

Define

$$B_{k,s}(f_1,\dots,f_n) := \sum_{\substack{s_i \geq 0\\ \max\{s_i\}=s}} B_k(\psi_{kd_1+s_1} * f_1, \dots, \psi_{kd_n+s_n} * f_n).$$

Then, by Proposition 1.3, we have the bound

$$\begin{aligned} \Big\| \sup_k |B_{k,s}(f_1,\dots,f_n)| \Big\|_{L^1(\mathbb{R})} &\leq \sum_k \Big\| B_{k,s}(f_1,\dots,f_n) \Big\|_{L^1(\mathbb{R})} \\ &\leq 2^{-cs} \sum_{0\leq s_i \leq s} \sum_k \prod_{i=1}^n \|\psi_{kd_i+s_i} * f_i\|_{L^n(\mathbb{R})} \\ &\leq 2^{-cs} \sum_{s_i\leq s} \prod_{i=1}^n \Big( \sum_k \|\psi_{kd_i+s_i} * f_i\|^n_{L^n(\mathbb{R})} \Big)^{1/n} \\ &= 2^{-cs} \sum_{s_i\leq s} \prod_{i=1}^n \Big\| \Big( \sum_k |\psi_{kd_i+s_i} * f_i|^n \Big)^{1/n} \Big\|_{L^n(\mathbb{R})} \\ &\leq 2^{-cs} \sum_{s_i\leq s} \prod_{i=1}^n \|Sf_i\|_{L^n(\mathbb{R})}, \end{aligned}$$

where

$$Sf := \Big( \sum_k |\psi_k * f|^2 \Big)^{1/2}$$

is the Littlewood–Paley square function, which is bounded on $L^p(\mathbb{R})$, $1<p<\infty$. So

$$\sum_k \|B_{k,s}(f_1,\dots,f_n)\|_{L^1(\mathbb{R})} \lesssim_n s^n 2^{-cs} \prod_{i=1}^n \|f_i\|_{L^n(\mathbb{R})}. \tag{3.3}$$

We will interpolate this with Proposition 2.5, to see that whenever $p_1,\dots,p_n > 1$, $\frac{1}{p} = \sum_i \frac{1}{p_i} \leq 1$, there exists an absolute $c = c_{p_1,\dots,p_n;p} > 0$ such that

$$\Big\| \sup_k |B_{k,s}(f_1,\dots,f_n)| \Big\|_{L^p(\mathbb{R})} \lesssim s^n 2^{-cs} \prod_{i=1}^n \|f_i\|_{L^{p_i}(\mathbb{R})}. \tag{3.4}$$

A final sum over $s \geq 1$ completes the proof, assuming (3.4) holds.

We now give the details of the interpolation argument. It is enough to prove (3.4) for nonnegative functions $f_i$, and by monotone convergence we may assume that they take only finitely many values. We normalize them so that $\|f_i\|_{L^{p_i}(\mathbb{R})} = 1$. By monotone convergence it is enough to prove (3.4) with the supremum extended only over finitely many values $k$, at which point we can choose a measurable function $k(x)$ assigning to every $x$ the maximizer. Thus, it suffices to estimate

$$\big\|B_{k(x),s}(f_1,\dots,f_n)(x)\big\|_{L^p(\mathbb{R})} = \sup_{\|h\|_{L^{p'}(\mathbb{R})}\le 1} \int_{\mathbb{R}} h(x)\gamma(x)B_{k(x),s}(f_1,\dots,f_n)(x)\,dx,$$

where $\gamma(x) \in \{z \in \mathbb{C} : |z| = 1\}$ denotes the argument of $B_{k(x)}(f_1,\dots,f_n)(x)$:

$$\gamma(x) = \begin{cases} \dfrac{\overline{B_{k(x),s}(f_1,\dots,f_n)(x)}}{|B_{k(x),s}(f_1,\dots,f_n)(x)|} & \text{if } B_{k(x),s}(f_1,\dots,f_n)(x) \neq 0, \\ 1 & \text{if } B_{k(x),s}(f_1,\dots,f_n)(x) = 0. \end{cases}$$

Replacing $h$ by $|h|$ makes this expression larger. Using also monotone convergence, we may assume that $h$ is a nonnegative simple function.

Now pick exponents $q_1,\dots,q_n > 1$, $\frac{1}{q} = \sum_i \frac{1}{q_i} \le 1$ such that there exists $\theta \in (0, 1)$ with

$$\frac{1}{p_i} = \frac{1-\theta}{n} + \frac{\theta}{q_i}, \qquad i = 1,\dots,n.$$

Define for complex $z$ with $\Re z \in [0, 1]$

$$f_i^{(z)}(x) = \mathbf{1}_{f_i(x)\neq 0} \exp\Big(\Big(z\frac{p_i}{q_i} + (1-z)\frac{p_i}{n}\Big)\log f_i(x)\Big)$$

and

$$h^{(z)}(x) = \mathbf{1}_{h(x)\neq 0} \exp\Big(z\frac{p'}{q'}\log h(x)\Big).$$

These functions are well defined since $f_i$ and $h$ are nonnegative. For fixed $x$ the functions $f_i^{(z)}(x)$ and $h^{(z)}(x)$ are all bounded and analytic in the interior of the strip $\Re z \in [0, 1]$, because $f_i$ and $h$ are simple functions. The claim (3.4) now follows from applying the Hadamard three lines theorem to the function $F(z) = \int_{\mathbb{R}} h^{(z)}(x)\gamma(x)B_{k(x),s}(f_1^{(z)},\dots,f_n^{(z)})(x)\,dx$, utilizing (3.3) when $\Re z = 0$ and Proposition 2.5 when $\Re z = 1$. This completes the proof.

## References

[1] J. Bourgain, "A nonlinear version of Roth's theorem for sets of positive density in the real line", *J. Analyse Math.* **50** (1988), 169–181. MR

[2] C. Demeter, T. Tao, and C. Thiele, "Maximal multilinear operators", *Trans. Amer. Math. Soc.* **360**:9 (2008), 4989–5042. MR

[3] P. Durcik and J. Roos, "A new proof of an inequality of Bourgain", *Math. Res. Lett.* **31**:4 (2024), 1047–1060. MR

[4] P. Durcik, S. Guo, and J. Roos, "A polynomial Roth theorem on the real line", *Trans. Amer. Math. Soc.* **371**:10 (2019), 6973–6993. MR

[5] A. Gaitan and V. Lie, “The boundedness of the (sub)bilinear maximal function along “non-flat” smooth curves”, *J. Fourier Anal. Appl.* **26**:4 (2020), art. id. 69, 33 pp. MR

[6] G.-D. Hong, “Polynomial Szemerédi for sets with large Hausdorff dimension on the torus”, preprint, 2025. arXiv 2507.14407

[7] B. Hu and V. Lie, “On the curved trilinear Hilbert transform”, preprint, 2023. arXiv 2308.10706

[8] D. Kosz, M. Mirek, S. Peluse, R. Wan, and J. Wright, “The multilinear circle method and a question of Bergelson”, preprint, 2024. arXiv 2411.09478

[9] B. Krause, “A non-linear Roth theorem for fractals of sufficiently large dimension”, preprint, 2019. arXiv 1904.10562

[10] B. Krause, “On polynomial progressions inside sets of large dimension”, preprint, 2025. arXiv 2508.04680

[11] M. T. Lacey, “The bilinear maximal functions map into $L^p$ for $2/3 < p \leq 1$”, *Ann. of Math.* (2) **151**:1 (2000), 35–57. MR

[12] M. Lacey and C. Thiele, “$L^p$ estimates on the bilinear Hilbert transform for $2 < p < \infty$”, *Ann. of Math.* (2) **146**:3 (1997), 693–724. MR

[13] M. Lacey and C. Thiele, “On Calderón’s conjecture”, *Ann. of Math.* (2) **149**:2 (1999), 475–496. MR

[14] X. Li, “Bilinear Hilbert transforms along curves, I: The monomial case”, *Anal. PDE* **6**:1 (2013), 197–220. MR

[15] X. Li and L. Xiao, “Uniform estimates for bilinear Hilbert transforms and bilinear maximal functions associated to polynomials”, *Amer. J. Math.* **138**:4 (2016), 907–962. MR

[16] V. Lie, “On the boundedness of the bilinear Hilbert transform along “non-flat” smooth curves”, *Amer. J. Math.* **137**:2 (2015), 313–363. MR

[17] E. M. Stein, *Harmonic analysis*: *real-variable methods*, *orthogonality*, *and oscillatory integrals*, Princeton Mathematical Series **43**, Princeton University Press, 1993. MR

Received August 14, 2025. Revised December 2, 2025.

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The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 342 No. 2 June 2026