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**ON MULTILINEAR MAXIMAL OPERATORS
ALONG HOMOGENEOUS CURVES**

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Suppose that

$$\vec{\gamma}(t) := (\gamma_1(t), \dots, \gamma_n(t)) = (a_1 t^{d_1}, \dots, a_n t^{d_n}), \quad 1 \leq d_1 < \dots < d_n \in \mathbb{Z}, \quad a_i \neq 0$$

is a homogeneous polynomial curve. We prove that whenever $p_1, \dots, p_n > 1$ and $1/p = \sum_{j=1}^n 1/p_j \leq 1$, there exists an absolute constant $0 < C = C_{p_1, \dots, p_n} < \infty$ such that

$$\left\| \sup_{r>0} \frac{1}{r} \int_0^r \prod_{i=1}^n |f_i(x - \gamma_i(t))| dt \right\|_{L^p(\mathbb{R})} \leq C \cdot \prod_{i=1}^n \|f_i\|_{L^{p_i}(\mathbb{R})}.$$

Our main tool is a smoothing estimate, adapted from work of Kosz, Mirek, Peluse, Wan, and Wright.

1. Introduction

The study of multilinear maximal functions dates back to celebrated work of Lacey [11], who proved the following theorem.

Theorem 1.1. *Suppose that $p_1, p_2 > 1$, and that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} < \frac{3}{2}$. Then there exists an absolute constant $0 < C_{p_1, p_2} < \infty$ such that*

$$\begin{aligned} \|B^{\vec{\gamma}}(f_1, f_2)\|_{L^p(\mathbb{R})} &:= \left\| \sup_{r>0} \frac{1}{r} \int_0^r |f(x - \gamma_1(t))g(x - \gamma_2(t))| dt \right\|_{L^p(\mathbb{R})} \\ &\leq C_{p_1, p_2} \|f_1\|_{L^{p_1}(\mathbb{R})} \|f_2\|_{L^{p_2}(\mathbb{R})}, \end{aligned}$$

whenever $\gamma_i(t) = a_i t$ and $a_1 \neq a_2$ are nonzero.

The key property of this operator is its modulation invariance, which necessitated an approach using time-frequency analysis, building off ideas of Lacey and Thiele in their work on the bilinear Hilbert transform [12; 13]; this method was later adapted to handle multilinear extensions, see [2].

On the other hand, when the modulation invariance embedded in $\vec{\gamma}$ is eliminated, different techniques can be used. This was first explored in the singular integral context in [14; 16], with subsequent work of [15] establishing the following; see also [5].

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Theorem 1.2. *Suppose that $\gamma_1(t) = t$, $\gamma_2(t) = P(t)$, where $P(t)$ is a polynomial of degree d which vanishes to degree ≥ 2 at the origin. Then whenever $p_1, p_2 > 1$ and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p} < \frac{d}{d-1}$, there exists an absolute constant $0 < C_{p_1, p_2} < \infty$ such that*

$$\|B^{\vec{\gamma}}(f_1, f_2)\|_{L^p(\mathbb{R})} \leq C_{p_1, p_2} \|f_1\|_{L^{p_1}(\mathbb{R})} \|f_2\|_{L^{p_2}(\mathbb{R})}.$$

The key ingredient in establishing [Theorem 1.2](#) was a *Sobolev estimate*, a representative case of which is stated below.

Proposition 1.3 (special case). *Suppose that $\gamma_1(t) = t$, $\gamma_2(t) = t^2$. There exist absolute constants $0 < c < C < \infty$ such that if \hat{f}_i vanishes outside $\{|\xi| \leq 2^{l+ik}\}$ for some i , then*

$$\left\| \int_0^1 f_1(x - \gamma_1(2^{-k}t)) f_2(x - \gamma_2(2^{-k}t)) dt \right\|_{L^1(\mathbb{R})} \leq C 2^{-cl} \|f_1\|_{L^2(\mathbb{R})} \|f_2\|_{L^2(\mathbb{R})}.$$

In other words, the only obstruction to the estimate

$$\left\| \int_0^1 f_1(x - \gamma_1(2^{-k}t)) f_2(x - \gamma_2(2^{-k}t)) dt \right\|_{L^1(\mathbb{R})} \ll \|f_1\|_{L^2(\mathbb{R})} \|f_2\|_{L^2(\mathbb{R})},$$

where \ll informally stands for “much smaller than”, arises from zero-frequency considerations; note that the modulation invariance from [Theorem 1.1](#) precludes such an argument. Aside from their utility in studying the operators $\{B^{\vec{\gamma}}\}$, Sobolev estimates have found a wide use in problems in Euclidean Ramsey theory, dating back to the work of Bourgain [1], with more recent contributions found in [3; 4; 6; 9; 10] among others. The current state of the art for operators of the form $\{B^{\vec{\gamma}}\}$ is essentially due to Hu and Lie [7], who addressed trilinear formulations

$$\vec{\gamma} = (P_1(t), P_2(t), P_3(t))$$

provided P_i are distinct degree polynomials which vanish at different rates at 0; see [Observation 1.2\(i\)](#) of [7] and [5, Remark 2]. their key input was a trilinear analogue of [Proposition 1.3](#). We state below a slightly stronger version of their estimate [5, Theorem 3.1]. Their estimate has an additional dependence on k , which we drop here as justified by [Proposition 1.6](#) below.

Proposition 1.4 (special case). *Suppose that $\gamma_i(t) = t^i$, $1 \leq i \leq 3$. There exist absolute constants $0 < c < C < \infty$ such that if \hat{f}_i vanishes outside $\{|\xi| \leq 2^{l+ik}\}$ for some i , then*

$$\left\| \int_0^1 \prod_{i=1}^3 f_i(x - \gamma_i(2^{-k}t)) dt \right\|_{L^1(\mathbb{R})} \leq C 2^{-cl} \prod_{i=1}^3 \|f_i\|_{L^3(\mathbb{R})}.$$

The goal of this paper is to address multilinear analogues of $B^{\vec{\gamma}}$ under the simplifying assumption that our curves are homogeneous polynomials.

Specifically, we will be concerned with polynomial curves

$$\vec{\gamma}(t) := (\gamma_1(t), \dots, \gamma_n(t)) = (a_1 t^{d_1}, \dots, a_n t^{d_n}), \quad 1 \leq d_1 < \dots < d_n \in \mathbb{Z}, \quad a_i \neq 0.$$

Our main result is this:

Theorem 1.5. *Suppose that $p_1, \dots, p_n > 1$, and that*

$$\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_n} \leq 1.$$

Then there exists an absolute constant $0 < C_{p_1, \dots, p_n; \vec{\gamma}} < \infty$ such that

$$\left\| \sup_{r>0} \frac{1}{r} \int_0^r \prod_{i=1}^n |f_i(x - \gamma_i(t))| dt \right\|_{L^p(\mathbb{R})} \leq C_{p_1, \dots, p_n; \vec{\gamma}} \prod_{i=1}^n \|f_i\|_{L^{p_i}(\mathbb{R})}.$$

As might be expected, the key input in proving [Theorem 1.5](#) is the following multilinear Sobolev estimate:

Proposition 1.6. *Suppose that $\gamma_i(t) = a_i t^{d_i}$, $1 \leq i \leq n$ with integer exponents $1 \leq d_1 < d_2 < \dots < d_n \in \mathbb{Z}$, $a_i \neq 0$. There exist absolute constants $0 < c < C < \infty$ such that if \hat{f}_i vanishes outside $\{|\xi| \leq 2^{l+d_i k}\}$ for some i , then*

$$\left\| \int_0^1 \prod_{i=1}^n f_i(x - \gamma_i(2^{-k}t)) dt \right\|_{L^1(\mathbb{R})} \leq C 2^{-cl} \prod_{i=1}^n \|f_i\|_{L^{p_i}(\mathbb{R})}.$$

To establish [Proposition 1.6](#), we adapt a recent result of Kosz, Mirek, Peluse, Wan, and Wright [\[8\]](#); with [Proposition 1.6](#) in hand, [Theorem 1.5](#) readily presents. While it is reasonable to expect that an analogue of [Proposition 1.6](#), and thus [Theorem 1.5](#), should hold for more general distinct-degree polynomial curves γ_i which vanish to distinct degrees at the origin, the argument to deduce [Proposition 1.6](#) from the results of [\[8\]](#) crucially relies on homogeneity, and does not readily adapt to the more general setting.

Notation. Throughout, we let φ denote various mean-one Schwartz functions, normalized in some sufficiently large seminorm. The precise choice of φ might differ from line to line. Similarly, we use ψ to denote a similar function, but with

$$(1.7) \quad \mathbf{1}_{|\xi| \approx 1} \leq \hat{\psi} \leq \mathbf{1}_{|\xi| \approx 1}$$

and such that

$$(1.8) \quad \sum_l \psi(\xi/2^l) = \mathbf{1}_{\xi \neq 0}.$$

We use the notation

$$\phi_k(x) := 2^k \phi(2^k x)$$

to denote L^1 -normalized dilations, and let

$$(1.9) \quad B_k^{\vec{\gamma}}(f_1, \dots, f_n)(x) := B_k(f_1, \dots, f_n)(x) := \int_0^1 \prod_{i=1}^n f_i(x - \gamma_i(2^{-k}t)) dt.$$

Below, we will regard $0 \neq a_1, \dots, a_n = O(1)$ as arbitrary but fixed, and will abbreviate

$$(1.10) \quad D := D(\vec{\gamma}) := d_1 + \dots + d_n.$$

Asymptotic notation. We will make use of the modified Vinogradov notation. We use $X \lesssim Y$, or $Y \gtrsim X$, to denote the estimate $X \leq CY$ for an absolute constant C . We use $X \approx Y$ as shorthand for $Y \lesssim X \lesssim Y$. We also make use of big-O notation: we let $O(Y)$ denote a quantity that is $\lesssim Y$. We let $f(t) := o_{t \rightarrow a}(X(t))$ denote a quantity such that $|f(t)|/X(t) \rightarrow 0$ as $t \rightarrow a$.

If we need C to depend on a parameter, we shall indicate this by subscripts, thus for instance $X \lesssim_p Y$ denotes the estimate $X \leq C_p Y$ for some C_p depending on p . We analogously define $O_p(Y)$.

2. Sobolev estimates

The main goal of this section is to prove the following Sobolev estimate.

Proposition 2.1. *There exists an absolute $c > 0$ such that the following single scale estimate holds whenever $s_i \geq 0$:*

$$\|B_k(\psi_{kd_1+s_1} * f_1, \dots, \psi_{kd_n+s_n} * f_n)\|_{L^1(\mathbb{R})} \lesssim 2^{-cs} \prod_{i=1}^n \|f_i\|_{L^n(\mathbb{R})},$$

where $s := \max\{s_i\}$.

The proof of [Proposition 1.3](#) will be accomplished via a projection argument, anchored by [\[8, Theorem 6.1\]](#) in the case where $\gamma_i(t) = a_i t^{d_i}$, and $\mathbb{K} = \mathbb{R}$, which dictates that the operator

$$A_{-k}(F_1, \dots, F_n)(x_1, \dots, x_n) := \frac{1}{2^k} \int_0^{2^k} \prod_{i=1}^n F_i(x_1, \dots, x_i - a_i t^{d_i}, \dots, x_n) dt$$

satisfies nontrivial norm estimates whenever some \hat{F}_i vanishes in $|\xi_i| \leq 2^{-kd_i} \delta^{-1}$.

Lemma 2.2 (special case of Theorem 6.1 of [\[8\]](#)). *In the above setting, suppose that some \hat{F}_i vanishes in $|\xi_i| \leq 2^{-kd_i} \delta^{-1}$. Then there exists some absolute $c > 0$ such that*

$$(2.3) \quad \|A_{-k}(F_1, \dots, F_n)\|_{L^1(\mathbb{R}^n)} \lesssim_n (\delta^c + 2^{-kc}) \prod_{i=1}^n \|F_i\|_{L^n(\mathbb{R}^n)},$$

provided $k \geq 0$.

We now remove the dependence on k on the right side of (2.3), and address the case where $k \leq 0$. Specifically, we prove that for all k

$$(2.4) \quad \|A_{-k}(F_1, \dots, F_n)\|_{L^1(\mathbb{R}^n)} \lesssim_n \delta^c \prod_{i=1}^n \|F_i\|_{L^n(\mathbb{R}^n)}$$

whenever some \hat{F}_i vanishes in $|\xi_i| \leq 2^{-kd_i} \delta^{-1}$; for concreteness, suppose that this index is j .

To do so, introduce the operator

$$D_\lambda F(x_1, \dots, x_n) := F(\lambda^{d_1} x_1, \dots, \lambda^{d_n} x_n)$$

and choose $2^{k_0} \gg \delta^{-1}$. If we define

$$G_i(x_1, \dots, x_n) := D_{2^{k-k_0}} F_i(x_1, \dots, x_n),$$

then \hat{G}_i vanishes on $|\xi_j| \leq 2^{-k_0 d_j} \delta^{-1}$, so

$$\|A_{-k_0}(G_1, \dots, G_n)\|_{L^1(\mathbb{R}^n)} \lesssim \delta^c \prod_{i=1}^n \|G_i\|_{L^n(\mathbb{R}^n)}.$$

But now

$$\begin{aligned} & D_{2^{k_0-k}}(A_{-k_0}(G_1, \dots, G_n))(x_1, \dots, x_n) \\ &= 2^{-k_0} \int_0^{2^{k_0}} \prod_{i=1}^n G_i(2^{d_1(k_0-k)} x_1, \dots, 2^{d_i(k_0-k)} x_i + a_i t^{d_i}, \dots, 2^{d_n(k_0-k)} x_n) dt \\ &= 2^{-k} \int_0^{2^k} \prod_{i=1}^n G_i(2^{d_1(k_0-k)} x_1, \dots, 2^{d_i(k_0-k)} x_i + a_i 2^{d_i(k_0-k)} t^{d_i}, \dots, 2^{d_n(k_0-k)} x_n) dt \\ &= A_{-k}(D_{2^{k_0-k}} G_1, \dots, D_{2^{k_0-k}} G_n)(x) \\ &= A_{-k}(F_1, \dots, F_n)(x), \end{aligned}$$

so the result follows from changing variables.

Proof of Proposition 2.1. Set $g_i := \psi_{kd_i+s_i} * f_i$, let $\vec{1} := n^{-1/2}(1, \dots, 1) \in \mathbb{R}^n$, and let $\varphi : \mathbb{R}^n \rightarrow \mathbb{C}$ be a bump function with compactly supported Fourier transform that is constant along $\mathbb{R}\vec{1}$. For each

$$2^{-100D|k|-100s} \gg \epsilon > 0$$

sufficiently small (see (1.10)), define

$$F_i(x_1, \dots, x_n) := \epsilon^{1-1/n} \varphi(\epsilon x) g_i(x \cdot \vec{1})$$

so that $\|F_i\|_{L^n(\mathbb{R}^n)} \approx \|g_i\|_{L^n(\mathbb{R})}$ and \hat{F}_i is supported in an $O(\epsilon)$ neighborhood of

$$\{\xi \vec{1} : \hat{g}_i(\xi) \neq 0\};$$

in particular, for some j , \hat{F}_j vanishes when $|\xi| \lesssim 2^{kd_j+s}$. So,

$$\|A_k(F_1, \dots, F_n)\|_{L^1(\mathbb{R}^n)} \lesssim 2^{-cs} \prod_{i=1}^n \|g_i\|_{L^n(\mathbb{R})}.$$

On the other hand,

$$\begin{aligned} & \|A_k(F_1, \dots, F_n)(x_1, \dots, x_n)\|_{L^1(\mathbb{R}^n)} \\ &= \left\| 2^k \int_0^{2^{-k}} \left(\prod_{i=1}^n g_i(x \cdot \vec{1} - a_i t^{d_i}) \right) \cdot \left(\epsilon^{n-1} \prod_{i=1}^n \varphi(\epsilon x - \epsilon a_i t^{d_i} \vec{e}_i) \right) dt \right\|_{L^1(\mathbb{R}^n)} \\ &= \int_{(\mathbb{R}\vec{1})^\perp} \int_{\mathbb{R}} \left| 2^k \int_0^{2^{-k}} \left(\prod_{i=1}^n g_i(y - a_i t^{d_i}) \right) \left(\epsilon^{n-1} \prod_{i=1}^n \varphi(\epsilon z - \epsilon a_i t^{d_i} \vec{e}_i) \right) dt \right| dy dz \end{aligned}$$

using a change of variables (so in particular, z_1, \dots, z_{n-1} form an orthogonal basis for $(\mathbb{R}\vec{1})^\perp$); above, \vec{e}_i is the i -th coordinate vector.

By Taylor expansion,

$$\begin{aligned} & \int_{(\mathbb{R}\vec{1})^\perp} \int_{\mathbb{R}} \left| 2^k \int_0^{2^{-k}} \left(\prod_{i=1}^n g_i(y - a_i t^{d_i}) \right) \left(\epsilon^{n-1} \prod_{i=1}^n \varphi(\epsilon z - \epsilon a_i t^{d_i} \vec{e}_i) \right) dt \right| dy dz \\ &= \|\varphi^n\|_{L^1((\mathbb{R}\vec{1})^\perp)} \cdot \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \left| 2^k \int_0^{2^{-k}} \prod_{i=1}^n g_i(y - a_i t^{d_i}) dt \right| dy \\ &\quad + O(2^{O(k)}\epsilon) \int_{\mathbb{R}^{n-1}} \left(\int_{\mathbb{R}} 2^k \int_0^{2^{-k}} \prod_{i=1}^n |g_i(y - a_i t^{d_i})| dt dy \right) \epsilon^{n-1} (1 + \epsilon|z|)^{-100} dz \\ &= \|\varphi^n\|_{L^1((\mathbb{R}\vec{1})^\perp)} \|B_k(g_1, \dots, g_n)(x)\|_{L^1(\mathbb{R})} + O_k\left(\epsilon \prod_{i=1}^n \|g_i\|_{L^n(\mathbb{R})}\right). \end{aligned}$$

The result follows by sending $\epsilon \downarrow 0$. □

The following proposition will be used to complement [Proposition 1.3](#) via interpolation; before we can prove it, we state a lemma.

Proposition 2.5. *Suppose $n \geq 2$, $p_1, \dots, p_n > 1$ and $\frac{1}{p} = \sum_i \frac{1}{p_i} \leq 1$. Then*

$$\left\| \sup_k B_k(\psi_{kd_1+s_1} * f, \dots, \psi_{kd_n+s_n} * f) \right\|_{L^p(\mathbb{R})} \lesssim s^n \prod_{i=1}^n \|f\|_{L^{p_i}(\mathbb{R})},$$

where $s := \max\{s_i\}$.

Lemma 2.6. *Suppose that φ is a Schwartz function. For $t \in \mathbb{R}$, consider the maximal function*

$$M^t f(x) := \sup_j |\varphi_j * f|(x - 2^{-j}t).$$

There exists an absolute constant $0 < C < \infty$ (independent of t) such that

$$\|M^t f\|_{L^{1,\infty}(\mathbb{R})} \leq C \log(2 + |t|) \|f\|_{L^1(\mathbb{R})}.$$

Proof. Since the maximal function is trivially bounded on $L^\infty(\mathbb{R})$ we may use vector-valued Calderón–Zygmund theory; see [\[17, §1\]](#), for example. In particular,

it suffices to show that

$$\int_{|x| \geq 10|y|} \sum_j |\varphi_j(x - 2^{-j}t - y) - \varphi_j(x - 2^{-j}t)| \lesssim \log |t|.$$

By dilation invariance we can normalize $|y| \approx 1$. When $2^j \leq 100^{-1}$ we use the mean-value theorem; when $100^{-1} \leq 2^j \lesssim |t|$ we use a single-scale estimate; and when $2^j \gg |t|$ we use the decay of φ . \square

Proof of Proposition 2.5. We have the pointwise bound

$$\begin{aligned} & |B_k(\psi_{kd_1+s_1} * f_1, \dots, \psi_{kd_n+s_n} * f_n)(x)| \\ &= \left| \int_0^1 \left(\int_{\mathbb{R}^n} \prod_{i=1}^n \psi_{kd_i+s_i}(x - a_i 2^{-kd_i} t^{d_i} - u_i) f_i(u_i) du \right) dt \right| \\ &\leq \int_0^1 \prod_{i=1}^n M^{2^{s_i} a_i t^{d_i}} f_i(x) dt, \end{aligned}$$

where M^t is the maximal function from Lemma 2.6 with φ chosen to be ψ . So

$$\begin{aligned} \left\| \sup_k |B_k(\psi_{kd_1+s_1} * f_1, \dots, \psi_{kd_n+s_n} * f_n)| \right\|_{L^p(\mathbb{R})} &\lesssim \int_0^1 \left\| \prod_{i=1}^n M^{2^{s_i} a_i t^{d_i}} f_i \right\|_{L^p(\mathbb{R})} dt \\ &\lesssim s^n \prod_{i=1}^n \|f_i\|_{L^{p_i}(\mathbb{R})}, \end{aligned}$$

as desired. \square

3. The proof of Theorem 1.5

We will prove that

$$(3.1) \quad \left\| \sup_k |B_k(f_1, \dots, f_n)| \right\|_{L^p(\mathbb{R})} \lesssim \prod_{i=1}^n \|f_i\|_{L^{p_i}(\mathbb{R})}.$$

Theorem 1.5 follows from (3.1), because averaging operators are positive and therefore the lacunary supremum in (3.1) dominates the full supremum in Theorem 1.5 up to a factor of two.

The proof is by induction. Thus, we will assume that for all $m < n$

$$\left\| \sup_k |B_k(f_1, \dots, f_m)| \right\|_{L^p(\mathbb{R})} \lesssim \prod_{i=1}^m \|f_i\|_{L^{p_i}(\mathbb{R})},$$

whenever $\vec{\gamma}$ is a homogeneous polynomial curve, $p_1, \dots, p_m > 1$, and $\frac{1}{p_1} + \dots + \frac{1}{p_m} = \frac{1}{p} \leq 1$, with the $n = 1$ case following from Hardy–Littlewood and convexity.

By Taylor expansion, for each k , we have the decomposition

$$(3.2) \quad \begin{aligned} & B_k(f_1, \dots, f_n) \\ &= \sum_{j \geq 0} \frac{1}{j!} ((\partial^j \varphi)_{kd_n} * f_n) B_{k, \neq n}^{(j)}(f_1, \dots, f_{n-1}) + B_k(f_1, \dots, f_{n-1}, f_n - \varphi_{kd_n} * f_n), \end{aligned}$$

where

$$B_{k, \neq n}^{(j)}(f_1, \dots, f_{n-1})(x) := 2^k \int_0^{2^{-k}n-1} \prod_{i=1}^{n-1} f_i(x - a_i t^{d_i}) (-a_n 2^{kd_n} t^{d_n})^j dt,$$

and $\partial^j \varphi$ satisfies all of the same Schwartz normalizations as φ up to factors of C^j , $C = O(1)$.

Thus, by iterating (3.2), induction and convexity, it suffices to prove that

$$\left\| \sup_k |B_k(f_1 - \varphi_{kd_1} * f_1, \dots, f_n - \varphi_{kd_n} * f_n)| \right\|_{L^p(\mathbb{R})} \lesssim \prod_{i=1}^n \|f_i\|_{L^{p_i}(\mathbb{R})}$$

in the above range of p, p_i .

Define

$$B_{k,s}(f_1, \dots, f_n) := \sum_{\substack{s_i \geq 0 \\ \max\{s_i\} = s}} B_k(\psi_{kd_1+s_1} * f_1, \dots, \psi_{kd_n+s_n} * f_n).$$

Then, by Proposition 1.3, we have the bound

$$\begin{aligned} \left\| \sup_k |B_{k,s}(f_1, \dots, f_n)| \right\|_{L^1(\mathbb{R})} &\leq \sum_k \left\| B_{k,s}(f_1, \dots, f_n) \right\|_{L^1(\mathbb{R})} \\ &\leq 2^{-cs} \sum_{0 \leq s_i \leq s} \sum_k \prod_{i=1}^n \|\psi_{kd_i+s_i} * f_i\|_{L^n(\mathbb{R})} \\ &\leq 2^{-cs} \sum_{s_i \leq s} \prod_{i=1}^n \left(\sum_k \|\psi_{kd_i+s_i} * f_i\|_{L^n(\mathbb{R})}^n \right)^{1/n} \\ &= 2^{-cs} \sum_{s_i \leq s} \prod_{i=1}^n \left\| \left(\sum_k |\psi_{kd_i+s_i} * f_i|^n \right)^{1/n} \right\|_{L^n(\mathbb{R})} \\ &\leq 2^{-cs} \sum_{s_i \leq s} \prod_{i=1}^n \|Sf_i\|_{L^n(\mathbb{R})}, \end{aligned}$$

where

$$Sf := \left(\sum_k |\psi_k * f|^2 \right)^{1/2}$$

is the Littlewood–Paley square function, which is bounded on $L^p(\mathbb{R})$, $1 < p < \infty$. So

$$(3.3) \quad \sum_k \|B_{k,s}(f_1, \dots, f_n)\|_{L^1(\mathbb{R})} \lesssim n s^n 2^{-cs} \prod_{i=1}^n \|f_i\|_{L^n(\mathbb{R})}.$$

We will interpolate this with Proposition 2.5, to see that whenever $p_1, \dots, p_n > 1$, $\frac{1}{p} = \sum_i \frac{1}{p_i} \leq 1$, there exists an absolute $c = c_{p_1, \dots, p_n; p} > 0$ such that

$$(3.4) \quad \left\| \sup_k |B_{k,s}(f_1, \dots, f_n)| \right\|_{L^p(\mathbb{R})} \lesssim s^n 2^{-cs} \prod_{i=1}^n \|f_i\|_{L^{p_i}(\mathbb{R})}.$$

A final sum over $s \geq 1$ completes the proof, assuming (3.4) holds.

We now give the details of the interpolation argument. It is enough to prove (3.4) for nonnegative functions f_i , and by monotone convergence we may assume that

they take only finitely many values. We normalize them so that $\|f_i\|_{L^{p_i}(\mathbb{R})} = 1$. By monotone convergence it is enough to prove (3.4) with the supremum extended only over finitely many values k , at which point we can choose a measurable function $k(x)$ assigning to every x the maximizer. Thus, it suffices to estimate

$$\|B_{k(x),s}(f_1, \dots, f_n)(x)\|_{L^p(\mathbb{R})} = \sup_{\|h\|_{L^{p'}(\mathbb{R})} \leq 1} \int_{\mathbb{R}} h(x)\gamma(x)B_{k(x),s}(f_1, \dots, f_n)(x) dx,$$

where $\gamma(x) \in \{z \in \mathbb{C} : |z| = 1\}$ denotes the argument of $B_{k(x)}(f_1, \dots, f_n)(x)$:

$$\gamma(x) = \begin{cases} \frac{B_{k(x),s}(f_1, \dots, f_n)(x)}{|B_{k(x),s}(f_1, \dots, f_n)(x)|} & \text{if } B_{k(x),s}(f_1, \dots, f_n)(x) \neq 0, \\ 1 & \text{if } B_{k(x),s}(f_1, \dots, f_n)(x) = 0. \end{cases}$$

Replacing h by $|h|$ makes this expression larger. Using also monotone convergence, we may assume that h is a nonnegative simple function.

Now pick exponents $q_1, \dots, q_n > 1$, $\frac{1}{q} = \sum_i \frac{1}{q_i} \leq 1$ such that there exists $\theta \in (0, 1)$ with

$$\frac{1}{p_i} = \frac{1-\theta}{n} + \frac{\theta}{q_i}, \quad i = 1, \dots, n.$$

Define for complex z with $\Re z \in [0, 1]$

$$f_i^{(z)}(x) = \mathbf{1}_{f_i(x) \neq 0} \exp\left(\left(z \frac{p_i}{q_i} + (1-z) \frac{p_i}{n}\right) \log f_i(x)\right)$$

and

$$h^{(z)}(x) = \mathbf{1}_{h(x) \neq 0} \exp\left(z \frac{p'}{q'} \log h(x)\right).$$

These functions are well defined since f_i and h are nonnegative. For fixed x the functions $f_i^{(z)}(x)$ and $h^{(z)}(x)$ are all bounded and analytic in the interior of the strip $\Re z \in [0, 1]$, because f_i and h are simple functions. The claim (3.4) now follows from applying the Hadamard three lines theorem to the function $F(z) = \int_{\mathbb{R}} h^{(z)}(x)\gamma(x)B_{k(x),s}(f_1^{(z)}, \dots, f_n^{(z)})(x) dx$, utilizing (3.3) when $\Re z = 0$ and Proposition 2.5 when $\Re z = 1$. This completes the proof.

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