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THE PLAIN SPHERE NUMBER OF A LINK

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Let L be a link in S^3. We consider a natural class of meridional presentations for $\pi_1(S^3 \setminus L)$ in which the relations are witnessed by certain embedded two-spheres easily observed in a fixed diagram of L. The Wirtinger presentation is a special case. We prove that the bridge number of L equals the smallest number of generators of $\pi_1(S^3 \setminus L)$ over all such presentations.

1. Introduction

We introduce a new definition of bridge number of a link. The definition arises in the study of the meridional rank conjecture (MRC), which asks whether the bridge number of a link L equals the smallest number of meridional generators of $\pi_1(S^3 \setminus L)$. The conjecture, posed by Cappell and Shaneson [17, Problem 1.11], has been established in a variety of cases [1; 2; 3; 6; 7; 8; 9; 10; 11; 14; 19]. The analogous statement is also shown to hold for some knotted spheres in S^4 [16].

In [5], it is shown that the bridge number equals the minimal number of meridional generators over all presentations that allow only iterated Wirtinger relations in a fixed diagram. Our main result is that equality persists after significantly generalizing the presentations considered.

Denote the *bridge number* and *meridional rank* of a link L by $\beta(L)$ and $\mu(L)$, respectively. In [5], the authors introduce a new invariant, the *Wirtinger number* of a link, denoted $\omega(L)$. The Wirtinger number is defined in terms of a “coloring game” played in a fixed diagram of L; see Definition 2.1. Performing a coloring move at a crossing c reflects the fact that the Wirtinger meridians of the overstrand and one understrand at c generate the Wirtinger meridian of the second understrand at c. Hence, a valid coloring sequence for D demonstrates that the Wirtinger meridians of the initially colored strands, or seeds (Definition 2.4), generate $\pi_1(S^3 \setminus L)$.

By starting with a diagram in minimal bridge position and choosing the strands containing the local maxima as seeds, we easily see that $\beta(L) \geq \omega(L)$; moreover, by definition, $\omega(L) \geq \mu(L)$. In [5], it is shown that in fact $\beta(L) = \omega(L)$. We now prove that the bridge number equals the smallest number of meridional generators

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of $\pi_1(S^3 \setminus L)$ across a considerably more general (though still "visible" in a single diagram) class of meridional presentations.

The *plain sphere number* of a link L (Definition 2.2) is the smallest number of Wirtinger meridians that generate $\pi_1(S^3 \setminus L)$ using only relations witnessed by certain embedded two-spheres, which we now describe. Consider a link diagram D, and let γ denote an embedded circle in the plane of projection, such that γ avoids neighborhoods of crossings and meets the interior of strands of D transversely. Further assume that γ meets D in exactly n points and that, of those, precisely one is contained in a given strand, s, of D. Then, the Wirtinger meridians of the remaining strands that meet γ generate the Wirtinger meridian of s. To see this, cap off the simple closed curve γ with two disks, $D^2_\pm$, whose interiors are disjoint from the plane of projection and such that D^2_+ (resp. D^2_-) is above (resp. below) the plane. We refer to $D^2_+ \cup_\gamma D^2_-$ as a *plain sphere* — it is indeed in plain sight — and we say that the sphere witnesses a relation in the group. That is, the product of (appropriately oriented) Wirtinger meridians of strands intersecting γ, taken in the order determined by γ, is trivial. The Wirtinger relation at any crossing c corresponds to a circle γ equal to the boundary of a small neighborhood of c. Clearly, $\beta(L) = \omega(L) \geq \rho(L) \geq \mu(L)$.

Theorem 1.1. *Let L be a link in S^3. The plain sphere number of L equals the bridge number of L.*

The theorem is proved in Section 3. In Section 2 we give the formal definition of $\rho(L)$, recall the definition of $\omega(L)$, and establish some terminology. The short Section 4 provides an example contrasting the plain sphere and Wirtinger numbers of a diagram, and describes a procedure for computing $\rho(D)$ for a fixed diagram. In Section 5 we describe some plain spheres in words.

2. Preliminaries

Recall that if L is a link in $\mathbb{R}^3$ and $P : \mathbb{R}^3 \to \mathbb{R}^2$ is the standard projection map given by $P(x, y, z) = (x, y)$, then $P(L)$ is a *link projection* if $P|_L$ is a regular projection. Hence, a link projection is a finite four-valent graph in the plane, and we refer to the vertices of this graph as crossings. A *link diagram* is a knot projection together with labels at each crossing that indicate which strand goes over and which goes under. By standard convention, these labels take the form of deleting parts of the under-arc at every crossing, and thus we think of a link diagram as a disjoint union of closed arcs, or *strands*, in the plane, together with instructions for how to connect these strands to form a union of simple closed curves in $\mathbb{R}^3$. At times we will refer to the knot projection $P(D)$ corresponding to a given knot diagram D, where $P(D)$ is obtained from D by forgetting the under- and over- information at crossings.

Let D be a diagram of a link L with n crossings. Denote by $s(D)$ the set of strands $s_1, s_2, \ldots, s_n$ and let $v(D)$ denote the set of crossings $c_1, c_2, \ldots, c_n$. Two strands s_i and s_j of D are *adjacent* if s_i and s_j are the understrands of some crossing in D. The diagram of the unknot with a single crossing is the unique knot diagram up to planar isotopy for which there exists a strand s_i of D that is adjacent to itself. In all cases we consider, adjacent arcs are understood to be distinct.

Definition 2.1. We call D *Wirtinger k-colorable* if we have specified a set A of strands of D and a nested sequence of subsets $A = A_0 \subset A_1 \subset \cdots \subset A_{|s(D)|-|A|} = s(D)$ such that the following hold:

(1) $|A| = k$.

(2) $A_{i+1} \setminus A_i = \{s_j\}$ for some strand s_j in D.

(3) Whenever $A_{i+1} \setminus A_i = \{s_j\}$, s_j is adjacent to s_i at some crossing $c \in v(D)$, and $s_i \in A_i$.

(4) The over-strand s_k at c is an element of A_i.

When D is k-colorable in the above sense, the Wirtinger meridians of the strands in A generate all Wirtinger meridians in the diagram, using only Wirtinger relations at crossings.

Definition 2.2. We call D *plain sphere k-colorable* if we have specified a set A of strands of D and a nested sequence of subsets $A = A_0 \subset A_1 \subset \cdots \subset A_{|s(D)|-|A|} = s(D)$ such that the following hold:

(1) $|A| = k$.

(2) $A_{i+1} \setminus A_i = \{s_j\}$ for some strand s_j in D.

(3) Whenever $A_{i+1} \setminus A_i = \{s_j\}$, there exists an embedded circle L_{i+1} in the plane of projection such that: L_{i+1} is transverse to the projection $P(D)$; L_{i+1} is disjoint from small neighborhoods of crossings in D; $|L_{i+1} \cap s_j| = 1$; and the points of intersection between L_{i+1} and strands of D are all contained in A_{i+1}.

We refer to the circles L_i as *loops*.

Condition (3) precisely guarantees that the Wirtinger meridian of s_j is generated by the Wirtinger meridians of the strands of D that have nontrivial intersection with L_{i+1}, other than s_j itself. This is expressed by a relation in $\pi_1(S^3 \setminus L)$, witnessed by a plain sphere as defined in the introduction: an embedded two-sphere S^2_j that intersects the plane of projection in the loop L_{i+1}. The existence of a valid coloring sequence as above shows that the Wirtinger meridians of the strands in A generate all Wirtinger meridians in the diagram, using only relations witnessed by plain two-spheres.

Remark 2.3. We could almost regard the spheres S^2_j in the above description as simultaneously and disjointly embedded in S^3. Indeed we will show later that the loops L_i can be assumed pairwise nonintersecting. And, each S^2_j is the boundary union of two embedded disks, on opposite sides of the plane of projection, whose interiors do not meet. However, since the upper disk D^2_{j+} of each S^2_j contains the basepoint, the spheres do, in fact, intersect at a point.

Definition 2.4. When a diagram D can be colored by a valid sequence of Wirtinger coloring moves (resp. plain sphere coloring moves) as in Definition 2.1 (resp. Definition 2.2), the elements of A are called the *seed strands*, or simply *seeds*, for the coloring.

The minimum value of k such that D is Wirtinger k-colorable is the *Wirtinger number* of D, denoted $\omega(D)$. Similarly, the minimum value of k such that D is plain sphere k-colorable is the *plain sphere number* of D, denoted $\rho(D)$. We use $\omega(D)$ and $\rho(D)$ to define invariants of L.

Definition 2.5. Let $L \subset S^3$ be a link. The *Wirtinger number* of L, denoted $\omega(L)$, is the minimal value of $\omega(D)$ over all diagrams D of L. Similarly, the *plain sphere number* of L, denoted $\rho(L)$, is the minimal value of $\rho(D)$ over all diagrams D of L.

We define a number of auxiliary terms related to Definitions 2.1, 2.2 and 2.5. These terms will be used extensively in the lemmas and theorems that follow. If D is a Wirtinger k-colorable diagram with $A_i \setminus A_{i-1} = \{s_j\}$, we say that the strand s_j is colored at *stage* i by a *Wirtinger coloring move*. Similarly, If D is a plain sphere k-colorable diagram with $A_i \setminus A_{i-1} = \{s_j\}$, we say that the strand s_j is colored at *stage* i by a *plain sphere coloring move* or, for short, a *loop coloring move*. A *partially plain sphere colored* link diagram D is a nested collection of sets $A = A_0 \subset A_1 \subset \cdots \subset A_r$ that meet all of the requirements of Definition 2.2 with the exception that A_r may be a proper subset of $s(D)$. Additionally, if D is a plain sphere k-colorable diagram, then a *plain sphere coloring sequence* for D is an ordered set of loops $\mathcal{L} = (L_1, L_2, \ldots, L_{|s(D)|-k})$ that are as in condition (3) of Definition 2.2. Finally, if L_i is a loop in a plain sphere coloring sequence for D that is isotopic (via an isotopy that is transverse to the link projection) to the boundary of a regular neighborhood of a crossing of D, then we call L_i a *Wirtinger loop*. Clearly, any plain sphere coloring move performed using a Wirtinger loop is also achievable via a Wirtinger coloring move. In other words, a Wirtinger coloring sequence is merely a special case of a plain sphere coloring sequence. This shows that $\rho(D) \le \omega(D)$ for all link diagrams D and hence $\rho(L) \le \omega(L)$ for all links L in S^3.

An easy example of a plain sphere coloring move that is not a Wirtinger move can be found in any diagram that is not visually prime. Namely, the connected

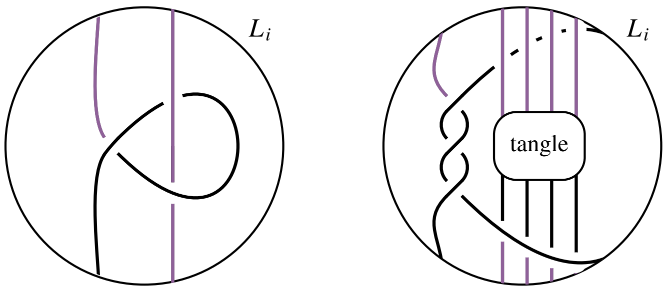

Figure 1. Possible plain sphere coloring moves at stage i of a coloring sequence which do not reduce to sequences of Wirtinger moves in the given tangle diagrams. On the left is the move used in Figure 6; on the right is a generalization. The oval contains an arbitrary n-strand tangle T_n, pictured in the case $n = 4$. The remaining tangle strand can be replaced by any one-strand tangle; as long as all but one of the points in $L_i \cap D$ are colored before stage i, the move will be valid.

sum sphere is a plain sphere intersecting two strands whose meridians cobound an annulus in the link complement. Two additional plain sphere coloring moves that are not Wirtinger moves are depicted in Figure 1. We refer to such loop moves as nontrivial loop moves. In Figure 9 of [4] there is an example of a minimal diagram D of a nonprime knot K such that $\rho(D) = \beta(D) = 5$ while $\omega(D)$=6. A prime knot diagram D with $\rho(D) < \omega(D)$ is given in Figure 6 (page 230). This exhibits a nontrivial plain sphere move that does not come from a connected sum sphere. For a family of plain sphere moves see Figure 1.

3. Proof

We prove Theorem 1.1 by showing that the plain sphere number of L equals the Wirtinger number of L. The result then follows from [5], where it is shown that the Wirtinger number and bridge number are equal. The equality $\rho(L) = \omega(L)$ relies on the following lemmas.

Lemma 3.1. *Let D be a link diagram. The plain sphere number of D can be realized by a collection of disjoint circles. That is, if $\rho(D) = n$, then there exists a set of n seed strands in D, together with an ordered set $\mathcal{L} = (L_1, \ldots, L_{|s(D)|-n})$ of disjointly embedded circles, each transverse to D and disjoint from small neighborhoods of all crossings, such that $\mathcal{L}$ defines a valid plain sphere coloring sequence for D.*

Proof. Let $\mathcal{L}$ be an ordered collection of loops realizing the plain sphere number of D. That is, $\mathcal{L}$ gives a coloring sequence for D starting from n seeds. After a small perturbation if needed, we may assume that the pairwise intersections of circles in $\mathcal{L}$ are disjoint from D. We define the *complexity* of $\mathcal{L}$ to be the (unsigned) count of intersection points between pairs of circles in $\mathcal{L}$:

$$\mathfrak{c}(\mathcal{L}) := \sum_{\substack{L_i, L_j \in \mathcal{L} \\ i \neq j}} |L_i \cap L_j|.$$

Without loss of generality we may assume that $\mathcal{L}$ has minimal complexity over all coloring sequences for D starting with n seeds. (Recall that $\rho(D) \leq \omega(D)$, and the Wirtinger number allows us to define a plain sphere coloring sequence where the loops are pairwise disjoint. Thus, it is clear that the complexity of a coloring sequence can be chosen to be zero after potentially increasing the number of seeds. What we show here is that $\mathcal{L}$ can be chosen to realize $\rho(D)$ while also satisfying $\mathfrak{c}(\mathcal{L}) = 0$.)

If $\mathfrak{c}(\mathcal{L}) \neq 0$, we can find $L_i, L_r \in \mathcal{L}$ such that $i \neq r$ and $L_i \cap L_r \neq \varnothing$. Select L_i with the property that it is the last loop in the ordered set $\mathcal{L}$ that has nonempty intersection with other loops in $\mathcal{L}$. That is, for all $k > i$, L_k is disjoint from the other loops in $\mathcal{L}$.

We will prove the statement by contradiction; we will show that as long as $\mathfrak{c}(\mathcal{L}) \neq 0$, the coloring sequence $\mathcal{L}$ does not in fact minimize complexity over all plain sphere coloring sequences starting with n seeds. We will do this by producing a new plain sphere coloring sequence in which we replace L_i with another circle L_i^*, such that L_i^* defines a valid plain sphere coloring move at stage i and colors the same strand as L_i. (This implies that all subsequent moves $L_{i+1}, \ldots, L_{|s(D)|-n}$ in $\mathcal{L}$ can be performed without modification.) Lastly, we will show that $\mathfrak{c}\big((L_1, \ldots, L_{i-1}, L_i^*, L_{i+1}, \ldots, L_{|s(D)|-n})\big) < \mathfrak{c}(\mathcal{L})$.

In order to simplify the exposition, assume from now on that we have already carried out the coloring moves determined by $L_1, L_2, \ldots, L_{i-1}$, and we are at stage i of the coloring process. That is, the coloring moves determined by $L_1, \ldots, L_{i-1}$ have been performed, and now L_i determines a valid plain sphere coloring move.

Denote by E_i the disk bounded by L_i in the plane of projection, and denote by $\mathcal{A}$ the set of components of intersection between E_i and loops in $\mathcal{L}$:

$$\mathcal{A} := E_i \cap \Big(\bigcup_{\substack{L_j \in \mathcal{L} \\ j \neq i}} L_j \Big).$$

Note that $\mathcal{A}$ is the union of properly embedded 1-manifolds (i.e. arcs and loops) in E_i. Consider the possibility that $\mathcal{A}$ contains a loop L_j. Our assumptions about L_i imply that either L_j is disjoint from all other loops in $\mathcal{L}$ or all points in $L_j \cap D$ have already been colored at stage i of the coloring sequence. With this in mind, we turn our attention to components of $\mathcal{A}$ that are arcs. The boundaries of the arcs in $\mathcal{A}$ cut $L_i = \partial E_i$ into a collection of arcs $\mathcal{A}'$, and moreover the arcs in $\mathcal{A}$ cut E_i into a collection of disks $\mathcal{D}$ (some of which may contain closed components of

$\mathcal{A}$ in their interiors). Observe that each arc in $\mathcal{A}$ is contained in a circle in $\mathcal{L}$ that intersects L_i. Therefore, coloring moves determined by these circles have already been performed. This ensures that all points of intersection between D and arcs in $\mathcal{A}'$, with the exception of one point — the point being colored at stage i, using the loop L_i — are colored before stage i of the loop coloring sequence.

Since L_i is a circle in $\mathcal{L}$, we know that $L_i \cap D$ is a set of isolated points. Since L_i defines a coloring move, the intersection $L_i \cap D$ contains at least two points. At stage i of the coloring sequence, L_i determines a valid coloring move. Therefore, only one of the points in $L_i \cap D$ is uncolored. Denote this point by x and the arc in $\mathcal{A}'$ containing x by a. Then a is contained on the boundary of one of the disks in $\mathcal{D}$, seen in Figure 2. Denote this disk by E_i^* and observe that $\partial E_i^* =: L_i^*$ is the union of a and other subarcs of $\mathcal{A}$. In particular, every point of intersection between D and L_*^i lies on an arc contained in either $\mathcal{A}$ or $\mathcal{A}'$. It follows from the above that, at stage i of the sequence, x is the only point in $L_i^* \cap D$ that is not colored. Thus,

$$\mathcal{L}^* := (L_1, \ldots, L_{i-1}, L_i^*, L_{i+1}, \ldots, L_{|s(D)|-n})$$

is a valid plain sphere coloring sequence starting from n seeds. (To satisfy transversality conditions in the above, we use L_i^* to denote the boundary of a slightly smaller disk contained in the interior of E_i^*.)

Lastly, we check that $\mathfrak{c}(\mathcal{L}) < \mathfrak{c}(\mathcal{L}^*)$. By construction, L_i^* is the boundary of a slightly shrunken copy of the disk E_i^*. Therefore, L_i^* itself is disjoint from $\mathcal{L}$. By contrast, L_i has the property that $L_i \cap L_j \neq \varnothing$ for some $j \neq i$. Therefore, as claimed, replacing L_i by L_i^* has the effect of strictly decreasing the complexity of the plain sphere coloring sequence. □

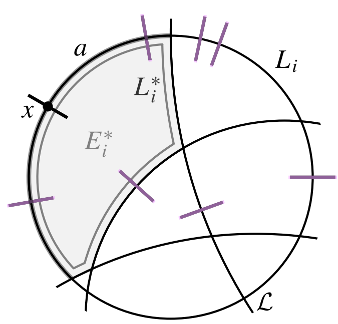

Figure 2. A region in the plane containing a link diagram at stage i of the coloring process. Subarcs of the diagram are represented by short line segments, and circles in $\mathcal{L}$ that intersect L_i are represented by arcs. The loop L_i, which colors the strand containing the point x, can be replaced with the loop L_i^*, which is disjoint from all other loops in $\mathcal{L}$ and represents a valid plain sphere coloring move at stage i for the same strand.

Lemma 3.2. *Let D be a diagram of a link L with $\rho(D) = n$. There exists a diagram D' of L such that the conclusion of Lemma 3.1 holds, and such that there exists a set of n seeds for D' and a valid coloring sequence $\mathcal{L}'$ for D', starting from those seeds, with the property that all innermost loops of $\mathcal{L}'$ are Wirtinger loops.*

Proof. Since $\rho(D) = n$, there exists a coloring sequence $\mathcal{L}$ for D starting from n seeds. Using Lemma 3.1, after possibly modifying $\mathcal{L}$, we can assume that the circles contained in $\mathcal{L}$ are disjoint.

Let L_i be an innermost loop in $\mathcal{L}$. This means that L_i bounds a disk E_i whose interior is disjoint from $\mathcal{L}$. As before, we consider L_i at stage i of the coloring sequence. That is, among the points in the intersection $L_i \cap D$, only one point, denoted x, is not colored. Let s be the strand of D containing x and let $\partial(s) = \{a, b\}$. By the definition of a coloring move, L_i intersects s transversely in exactly one point. Therefore, a and b are contained in the two different components of $\mathbb{R}^2 \backslash L_i$. Without loss of generality, assume $a \in E_i$, $b \notin E_i$. Denote by c the crossing in D incident to a. See Figure 3. The key fact here is that $s \cap E_i$ is the *only arc* in $D \cap E_i$ that is not colored at stage i. This is a consequence of the following two observations: x is the only noncolored point in $\partial E_i \cap D$, so all other strands intersecting ∂E_i are colored; and L_i is innermost, so no strand entirely contained in E_i° can be colored via another circle in $\mathcal{L}$. Therefore, no strand in E_i° can change color after stage i. So, indeed, at stage i, of all components of $D \cap E_i$ only $s \cap E_i$ is not colored.

Case 1: Assume that s is not the overstrand at c. From the above discussion, we know that the overstrand and second understand at c are both colored before stage i. Therefore, we can color s via a Wirtinger move at the crossing c. In particular, we can replace the loop L_i in $\mathcal{L}$ by a circle bounding a small neighborhood of c in the plane. This defines a valid coloring sequence $\mathcal{L}'$ in which the innermost loop L_i was replaced by one that satisfies the conclusions of the lemma.

Case 2: Assume that s is the overstrand at c. We also know that c is the crossing where s terminates. Hence, s is both the overstrand and an understrand at c. In $P(D)$, the planar projection determined by D, s is the union of edges of $P(D)$

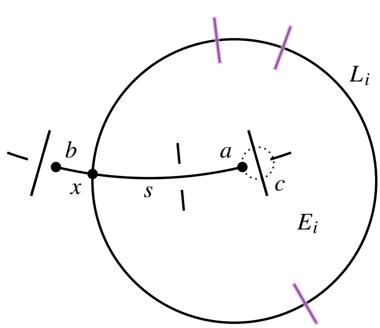

Figure 3. Exactly one uncolored strand s intersects the loop $L_i \in \mathcal{L}$ at stage i.

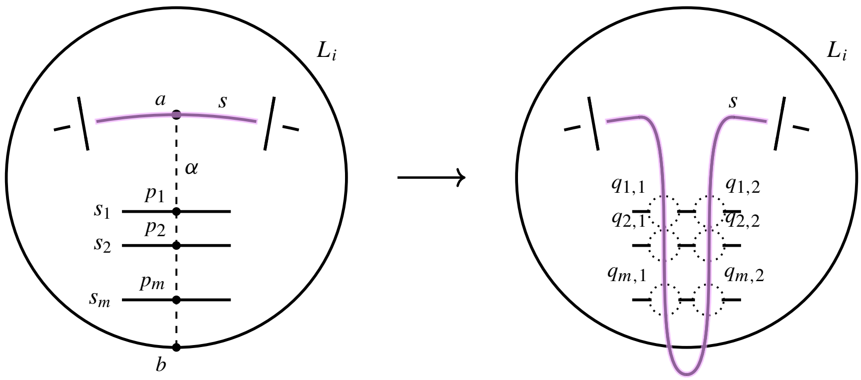

Figure 4. An isotopy to ensure no seed strand s is contained entirely in the interior of any disk E_i in the plane with $\partial(E_i) = L_i$. After a sequence of m Reidemeister II moves, $2m$-many Wirtinger coloring loops are added in appropriate places to the sequence $\mathcal{L}$ in order to obtain $\mathcal{L}'$.

and, since s is both the overstrand and understrand at c, we can conclude that the union of a subset of the edges in $P(D)$ constitutes an embedded loop γ in E_i. Moreover, γ bounds a disk $G \subset E_i$ such that $\gamma = \partial G$ is contained in the projection of the strand s. Hence, there is an isotopy of D along G creating a diagram D' in which crossing c has been resolved in the direction that preserves the number of components of the link. Note that the image of the strand s under this isotopy is contained in a strand of D' that intersects L_i in a second, necessarily colored, point. This eliminates the move determined by L_i from the coloring sequence.

One of the two cases will apply to each innermost loop in $\mathcal{L}$. Therefore, each innermost loop L_i may be removed after an isotopy contained in E_i or replaced by a Wirtinger loop. The result is a coloring sequence starting from n seeds in which all innermost loops represent Wirtinger moves. □

Definition 3.3. Let D be a partially colored link diagram and $\mathcal{L}$ an ordered set of disjoint circles transverse to D and representing valid coloring sequence moves on D. Let loop C be in $\mathcal{L}$ and let G be the disk bounded in the plane by C. A loop C is *depth-two* if: there exists at least one loop in $\mathcal{L}$ that is contained in the interior of G; and all loops contained in the interior of G are innermost loops of $\mathcal{L}$. For $n > 2$, the definition is inductive in the natural way: a loop C is *depth-n* if disk G contains at least one depth-$(n-1)$ loop and no nested sequence of n or more loops.

Lemma 3.4. *Let D be a diagram of a link L with $\rho(D) = n$. There exists a diagram D' of L and a valid coloring sequence $\mathcal{L}'$ for D' starting from n seeds, such that all the conclusions of Lemma 3.2 are satisfied and, moreover, if L_i is any loop in $\mathcal{L}'$, then no seed for the coloring sequence is entirely contained in the interior of the disk E_i bounded by L_i in the plane of projection.*

Proof. By Lemma 3.2 we can assume that $\rho(D) = n$ and that $\mathcal{L}$ is a plain sphere coloring sequence starting with seeds $s_1, s_2, \ldots, s_n$ such that all innermost loops in $\mathcal{L}$ are Wirtinger loops. If $L_i \in \mathcal{L}$ represents a Wirtinger move, then without loss of generality we may assume that L_i is the boundary of a small neighborhood of a crossing in D, and the desired conclusion automatically holds for L_i. In general, however, E_i may contain multiple crossings of D and potentially entire strands. Assume that a seed strand s is contained in E_i°. Define α to be an embedded arc in E_i, transverse to D, disjoint from small neighborhoods of crossings of D, and with the property that $\partial\alpha =: \{a, b\}$ has $a \in s^\circ$ and $b \in (L_i \setminus D)$. See Figure 4. We may use α to guide an isotopy of s in the plane of projection to produce a new strand such that the image of a after the isotopy lies outside of E_i, in a small neighborhood of b. The isotopy can be thought of as a sequence of Reidemeister II moves in which s passes over every strand it encounters along α.

Let D_1 be the diagram that results from the above isotopy. We claim that $\rho(D_1) = n$ and that D_1 admits a coloring sequence that is obtained from $\mathcal{L}$ by inserting $2m$ additional loops representing Wirtinger moves, where $m := |\alpha \cap D|$. More specifically, let $\alpha \cap D =: \{p_1, \ldots, p_m\}$. The additional Wirtinger loops will be interspersed across $\mathcal{L}$ according to when the strands in D containing the p_j are colored. See Figure 4 for the isotopy of s and the new strands and crossings created.

Note that each point p_j corresponds to a Reidemeister II move performed while isotoping s along α. Thus, in a neighborhood of p_j, two new crossings — call them $q_{j,1}$ and $q_{j,2}$ — are created during the isotopy, and at both crossings s is the overstrand. Let s_j denote the strand of D containing p_j. (After the isotopy, s_j is subdivided into three strands by $q_{j,1}$ and $q_{j,2}$.) At some stage in the coloring sequence $\mathcal{L}$ for D, the strand s_j is colored. This implies that, at the corresponding stage of the coloring sequence for D_1, one of the understrands at either $q_{j,1}$ or $q_{j,2}$ is colored. But the overstrand at these crossings, s, is a seed, so it is colored as well. Thus, we can perform two consecutive Wirtinger moves at these crossings and color all three strands into which s_j has been subdivided. As a result, any further coloring moves given by $\mathcal{L}$ whose validity relies on s_j being colored will be valid. Repeating this procedure for each $j \in \{1, \ldots, m\}$ produces the desired coloring sequence for D_1.

In sum, after an isotopy supported in a neighborhood of an arc, we ensured that the seed strand s is no longer contained in the interior of E_i. Moreover, we produced a coloring sequence, with the same number of seeds, for the diagram resulting from this isotopy. Repeating this procedure for every instance where a seed is contained entirely within the interior of one of the disks E_i, we arrive at the desired diagram D' and valid coloring sequence $\mathcal{L}'$ from n seeds. □

Before presenting the next lemma, we review some standard properties of tangles. An *m-strand tangle* is a collection of m disjoint arcs properly embedded in a 3-ball.

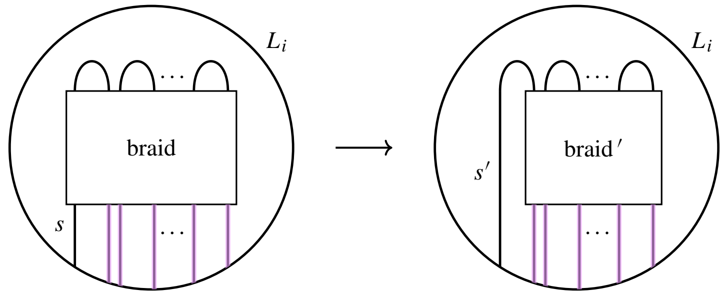

Figure 5. An isotopy at stage i of the coloring sequence of a rational m-strand tangle supported in disk E_i with $\partial(E_i) = L_i$. After isotopy, starting with $2m-1$ colored strands that intersect L_i, all strands that meet E_i can be colored using only Wirtinger moves.

An m-strand tangle is *rational* if all arcs of the tangle can be simultaneously isotoped into the boundary of the 3-ball.

Remark 3.5. Given a tangle R properly embedded in $D^2 \times [0, 1]$ such that $\partial R \subset (\partial D^2) \times [0, 1]$, R has a tangle diagram, analogous to a knot diagram, achieved by projecting R onto $D^2 \times \{0\}$. Let $x \in D^2$ denote the point with coordinates (0, 1), where D^2 is identified with the unit disk in the plane. If R is a rational tangle embedded in $D^2 \times [0, 1]$ such that $\partial R \subset \partial D^2 \times [0, 1]$, then after planar isotopy, and possibly Reidemeister moves supported in the interior of $D^2 \times \{0\}$, R has a tangle diagram as in the left image in Figure 5. In brief, this is true since the disks of parallelism for the arcs in R can all be isotoped to be disjoint from the disk that is a regular neighborhood of $(D^2 \times \{1\}) \cup (D^2 \times \{0\}) \cup (\{x\} \times [0, 1])$ in $\partial(D^2 \times [0, 1])$. Isotoping the arcs of R along these disks of parallelism until they almost lie in $\partial(D^2 \times [0, 1])$ results in a diagram of R that lies in an annular neighborhood of $\partial D^2 \times \{0\}$ in $D^2 \times \{0\}$ such that the projection of R is disjoint from a neighborhood of $\{x\} \times \{0\}$ and each arc of the projection has exactly one maximum with respect to the radial height function on $D^2 \times \{0\}$ (where we think of height as increasing as we approach the center of the disk). After a planar isotopy, this diagram becomes the diagram of the left image in Figure 5.

Remark 3.6. Any tangle R properly embedded in $D^2 \times [0, 1]$ such that each arc of R has a unique local maximum (or each has a unique local minimum) with respect to projection on the [0, 1] component of $D^2 \times [0, 1]$ is a rational tangle. In brief, this is true since if an arc α in R has a unique local maximum at height $a \in [0, 1]$, then there is a disk of parallelism between the subarc of α above $D^2 \times \{a - \epsilon\}$ and an arc in $D^2 \times \{a - \epsilon\}$. Since every arc in R contains a unique local maximum, this disk can be extended downward by moving both endpoints along the arc until it becomes a disk of parallelism between α and a subarc in $\partial(D^2 \times [0, 1])$. After generating a disk of parallelism in this way for each arc of R, we can ensure that these disks are pairwise disjoint by an application of a standard innermost disk and outermost arc argument. Thus, R is rational.

Lemma 3.7. *Let D be a diagram of a link L with $\rho(D) = n$, and let $\mathcal{L} = (L_1, L_2, \dots, L_{|s(D)|-n})$ be a plain sphere coloring sequence for D, satisfying the conclusions of Lemma 3.4. Let L_i be a depth-two loop in $\mathcal{L}$ and denote by E_i the disk in the plane of projection bounded by L_i. There exists an isotopy of D, supported in a small neighborhood of E_i, such that the resulting diagram D' has a coloring sequence $\mathcal{L}'$ with n seeds and such that L_i is replaced by a collection of innermost loops.*

Proof. By assumption, $\mathcal{L}$ is a coloring sequence for D starting from n seeds and satisfying the conclusions of Lemma 3.4. Since L_i is a depth-two loop, there exists at least one loop of $\mathcal{L}$ contained in E_i and moreover all such loops are innermost. Since the coloring sequence satisfies the conclusions of Lemma 3.2, all loops contained in E_i are Wirtinger loops. Since the conclusions of Lemma 3.4 hold as well, we also have that no seed for the coloring sequence $\mathcal{L}$ is entirely contained in E_i. Let s be the unique strand of D that intersects L_i and is colored at stage i in the coloring process.

Recall that $P : \mathbb{R}^3 \to \mathbb{R}^2$ denotes the standard projection map and that $P(L)$ is the regular projection that gives rise to D. We can assume that $L \subset \mathbb{R}^2 \times [0, 1]$. If $L_i \cap D = 2m$, then, since no seed is entirely contained in E_i, $E_i \times [0, 1] = P^{-1}(E_i) \cap (\mathbb{R}^2 \times [0, 1])$ is a 3-ball containing a properly embedded m-strand tangle $T = L \cap E_i \times [0, 1]$. In particular, the lack of seeds in E_i implies T contains no simple closed curves. Double $E_i \cap D$ along ∂E_i to obtain a link diagram D^{dbl} of the link L^{dbl} where L^{dbl} is obtained by doubling T along $\partial E_i \times [0, 1]$. Since E_i does not entirely contain any seed strands and all loops contained in E_i are Wirtinger loops, the set of $2m$ strands of D^{dbl} that intersect ∂E_i, call them $\sigma_1, \dots, \sigma_{2m}$, are a collection of seed strands for a Wirtinger coloring of D^{dbl}. By Theorem 1.2 of [4], there is an isotopy of L^{dbl} achieved by modifying only the z-coordinate of points on L^{dbl} and preserving the projection $P(L^{dbl})$ at all times after which L^{dbl} has exactly $2m$ local maxima with respect to projection to the z-axis and each of these maxima project to exactly one of $\sigma_1 \cap \partial E_i, \dots, \sigma_{2m} \cap \partial E_i$. This isotopy restricts to a proper isotopy of T in $E_i \times [0, 1]$ after which each strand of T contains a unique local minimum with respect to the z-axis. Hence, by Remark 3.6, T must be a rational m-strand tangle. Since T is rational, then, by Remark 3.5, there is a sequence of Reidemeister moves supported in E_i that produces a new diagram D^* as in Figure 5, left (see previous page). That is, $T \cap E_i$ is obtained from a braid by taking the plat closure to one side. By replacing the point x in Remark 3.5 by a point in L_i close to $s \cap L_i$ in the clockwise direction, we can ensure that s is the leftmost strand entering the braid box from below, as in the left image in Figure 5.

Furthermore, it is well known that there is an isotopy of T fixing ∂T that pulls one arc out of the braid box, shown in Figure 5; see, for example, the Claim within the proof of Theorem 1 in [20]. After isotopy, call the resulting link diagram D' (Figure 5, right) and let s' be the strand containing the image of s. Notice that since all strands entering the braid box from below are colored at stage i, then all strands of D' that meet E_i can be colored via Wirtinger loops, even s'.

Thus, after an isotopy supported in a neighborhood of E_i, we can replace L_i by a collection of Wirtinger loops. In particular, we eliminated a depth-two loop from the coloring sequence. □

Corollary 3.8. *If L is a link and D is a diagram of L with $\rho(D) = \rho(L) = n$, then there exists a diagram D', equivalent to D, such that $\rho(D') = \omega(D') = n$.*

Proof. Let D be a link diagram and $\mathcal{L}$ a plain sphere coloring sequence for D starting with n seeds. By repeated application of Lemma 3.7, we can eventually eliminate all depth-two loops. Eliminating a single loop as described in the lemma may not immediately reduce the number of depth-two loops as it could simultaneously produce a new depth-two loop from a previous depth-three loop; however, the procedure can be iterated as many times as needed.

After the necessary amount of persistence, we arrive at a diagram D', equivalent to D, which admits a plain sphere coloring sequence $\mathcal{L}'$, also with n seeds, such that $\mathcal{L}'$ contains no depth-two loops. Since D was chosen to be minimal with respect to plain sphere number, then $\rho(D') = n$. But this in fact implies that all loops in $\mathcal{L}'$ are innermost. By Lemma 3.2, all moves in the coloring sequence $\mathcal{L}'$ are Wirtinger moves. Therefore, $n \geq \omega(D')$. But we also have, by definition, that $\omega(D') \geq \rho(D') = n$. □

Proof of Theorem 1.1. Let L be a link in S^3 with $\rho(L) = n$. We wish to show that $\omega(L) = n$ as well. Let D be a diagram realizing $\rho(L)$. This implies that D admits a plain sphere coloring sequence starting from n seeds. By Corollary 3.8, there exists a diagram D' of L that can be colored starting with n seeds and using only Wirtinger moves. Therefore, $n \geq \omega(D') \geq \omega(L) \geq \rho(L) = n$. It then follows from [5] that $\beta(L) = \omega(L) = n$. □

4. Example and computations

4.1. ***Example.*** We compute the plain sphere number of a minimal diagram of the knot $K = 14n1527$, shown in Figure 6. The bridge number of K is 3 and equals the plain sphere number of the pictured diagram. The Wirtinger number of the diagram is 4. We thank Nathan Dunfield for detecting this example and sharing it with us.

4.2. ***Computing the plain sphere number of diagrams.*** Let D be a link diagram in S^2, $P(D)$ be the projection corresponding to D. Recall that a region of D is the

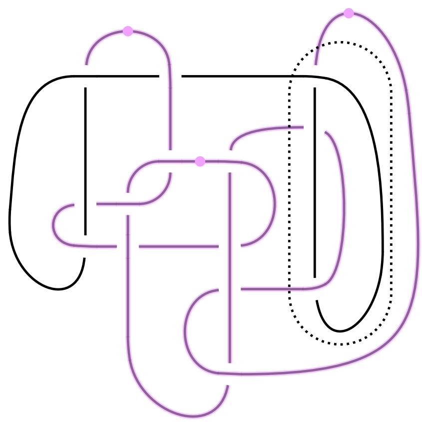

Figure 6. A partially colored minimal diagram D of the knot $K = 14n1527$ with $\beta(K) = \rho(D) = 3$ and $\omega(D) = 4$. Seed strands are marked with dots. Colored strands are purple. At this stage of the coloring sequence, only Wirtinger moves have been performed, and no further Wirtinger moves are possible. The dotted circle L_8 represents a loop coloring move. (There are several alternative loop coloring moves.) After coloring the black strand intersecting L_8, the rest of the diagram can be colored by Wirtinger moves.

closure of a connected component of $S^2 \setminus P(D)$. Note that the boundary of a region is a cycle in $P(D)$ where $P(D)$ is viewed as a 4-valent graph. Let Γ_D denote the dual graph to D.

Definition 4.1. Let L be an embedded loop in S^2 that is transverse to D and is disjoint from neighborhoods of crossings of D. We say L is *tight* if for every region R of D such that $L \cap R \neq \varnothing$, $L \cap R$ is an arc properly embedded in R with endpoints on distinct edges of $P(D)$ in ∂R.

Note that every tight loop in S^2 is isotopic in S^2 to an embedded loop in Γ_D via an isotopy that is transverse to the edges of $P(D)$ and disjoint from the vertices of $P(D)$.

Lemma 4.2. *Let D be a link diagram. There exists a minimal plain sphere coloring sequence for D consisting entirely of tight loops.*

Proof. Given a plain sphere coloring sequence $\mathcal{L}$ for a diagram D, define

$$\tau(\mathcal{L}) = \sum_{L \in \mathcal{L}} |L \cap D|.$$

Let $\mathcal{L}$ be a minimal plain sphere coloring sequence such that the loops in $\mathcal{L}$ are disjointly embedded. Assume that $\mathcal{L}$ minimizes $\tau(\mathcal{L})$ over all such sequences. By Definition 2.2, for every $L_i \in \mathcal{L}$ and for every region R of D, if $L_i \cap R \neq \varnothing$, then $L_i \cap R$ is a collection of arcs.

Suppose toward a contradiction that there exists $L_i \in \mathcal{L}$ and a region R of D such that $L_i \cap R$ consists of two or more arcs. If L_i colors the strand s of D, by definition, $L_i \cap s$ is a single point. Let γ_1 and γ_2 be two distinct arcs of $L_i \cap R$ that are contained in the boundary of the same component of $R \setminus L_i$. Call the closure of this component C. Let α be an arc properly embedded in C with one endpoint in γ_1 and one endpoint in γ_2. Let N be a rectangular I-fibered neighborhood of α in C such that $\partial N = \beta_1 \cup \beta_2 \cup \delta_1 \cup \delta_2$ where β_i is embedded in γ_i and $\delta_i^\circ \subset C^\circ$. Surger L_i along α in the standard way by removing β_1 and β_2 and gluing in δ_1 and δ_2. This results in two loops in S^2, L_i' and L_i'', both of which are transverse to D and disjoint from neighborhoods of crossings in D. Both L_i' and L_i'' have nontrivial intersection with D, so

$$|L_i' \cap D| < |L_i \cap D| \quad \text{and} \quad |L_i'' \cap D| < |L_i \cap D|.$$

Exactly one of L_i' and L_i'' have nontrivial intersection with s, say L_i' does. All other points of intersection of D with L_i' and L_i'' are contained in colored strands. Then $\mathcal{L}' = (L_1, \dots, L_{i-1}, L_i', L_{i+1}, \dots, L_m)$ is a minimal plain sphere coloring sequence with $\tau(\mathcal{L}') < \tau(\mathcal{L})$, a contradiction. □

Consequently, $\rho(D)$ can be realized by a sequence of loops represented by embedded cycles in Γ_D, an approach suggested by Nathan Dunfield. In particular, computing $\rho(D)$ for a fixed link diagram D is algorithmic. The forthcoming version 3.3 of SnapPy [13] will include a feature computing $\rho(D)$. A computation performed using this feature shows that over 600 diagrams in the knot table through 16 crossings have plain sphere number strictly less than Wirtinger number (Nathan Dunfield, private communication, 2025).

We conclude by pointing out that our plain spheres generalize the connected sum spheres that appear in the study of the visual primeness of links; see [18; 12; 15]. Of course, not all relators in the group of a link are necessarily witnessed by plain spheres in a given diagram D. This is readily seen to be the case even when D is a diagram of the unknot. Since any finite set of meridians of a link can be realized as Wirtinger meridians in some diagram, one can regard the meridional rank conjecture as positing equality between the bridge number and a certain elusive sphere number of links, allowing immersed spheres that more easily evade the eye.

5. Figures

Caught in tumbleweed, in perpetual slumber
They roll: round, squished, immersed, cucumber
Elusive, blistering, fierce
Faint! Diabolical! Spheres!
Does each breed breed the bridge number?

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