Pacific Journal of Mathematics

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Volume 342 No. 2 June 2026

PACIFIC JOURNAL OF MATHEMATICS
Vol. 342, No. 2, 2026
https://doi.org/10.2140/pjm.2026.342.259

SYMPLECTIC AUTOMORPHISMS OF A SURFACE WITH GENUS TWO FIBRATION AND THEIR ACTION ON CH_0

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Let S be a complex smooth projective surface with a genus two fibration, and $\mathrm{Aut}_s(S)$ the group of symplectic automorphisms, fixing every holomorphic 2-forms (if any) on S. Based on the work of Jin-Xing Cai, we show that, if $\chi(\mathcal{O}_S) \geq 5$, then $|\mathrm{Aut}_s(S)| \leq 2$. Then we verify, under some conditions, that $\mathrm{Aut}_s(S)$ acts trivially on the Albanese kernel $\mathrm{CH}_0(S)_{\mathrm{alb}}$ of the 0-th Chow group, which is predicted by a conjecture of Bloch and Beilinson. As a consequence, if an automorphism $\sigma \in \mathrm{Aut}(S)$ acts trivially on $H^{i,0}(S)$ for $0 \leq i \leq 2$, then it also acts trivially on $\mathrm{CH}_0(S)_{\mathrm{alb}}$.

1. Introduction

Throughout the paper, we work over the complex numbers field $\mathbb{C}$.

Let X be a n-dimensional connected smooth projective variety over $\mathbb{C}$. When studying the automorphism group $\mathrm{Aut}(X)$, it is natural to look at the induced action of $\mathrm{Aut}(X)$ on the cohomology groups that are naturally attached to X, such as $H^*(X, R)$ with R an abelian group or $H^*(X, \Omega_X^p)$ for $0 \leq p \leq \dim X$, where Ω_X^p is the coherent sheaf of holomorphic p-forms on X. On the other hand, there is also an induced action of $\mathrm{Aut}(X)$ on the Chow groups $\mathrm{CH}_*(X)$, which is a more refined invariant than the cohomology groups. A deep conjecture of Bloch and Beilinson predicts that, roughly speaking, there is a natural decreasing filtration

MSC2020: 14J50, 14J29, 14C15.

Keywords: surface of general type, fibration of genus two, symplectic automorphism, Chow group, Bloch–Beilinson conjecture.

$F^\bullet \mathrm{CH}_*(X)_{\mathbb{Q}}$ on the $\mathrm{CH}_*(X)_{\mathbb{Q}} := \mathrm{CH}_*(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that the induced action of $\mathrm{Aut}(X)$ on the graded pieces of $F^\bullet \mathrm{CH}_*(X)_{\mathbb{Q}}$ is determined by the action of $\mathrm{Aut}(X)$ on the cohomology groups $H^*(X, \mathbb{Q})$. We refer to [Voi03, Section 11.2.2] for a precise statement of the Bloch–Beilinson conjecture.

For a smooth projective surface S, it is clear that $\mathrm{CH}_2(S) \cong \mathbb{Z}$ and $\mathrm{CH}_1(S) \cong \mathrm{Pic}(S)$. Thus the focus is on the still mysterious 0-th Chow group $\mathrm{CH}_0(S)$. There is a natural filtration on $\mathrm{CH}_0(S)$:

$$0 \subset \mathrm{CH}_0(S)_{\mathrm{alb}} \subset \mathrm{CH}_0(S)_{\mathrm{hom}} \subset \mathrm{CH}_0(S) \tag{1-1}$$

where $\mathrm{CH}_0(S)_{\mathrm{hom}}$ is the kernel of the degree map $\deg : \mathrm{CH}_0(S) \to \mathbb{Z}$, and $\mathrm{CH}_0(S)_{\mathrm{alb}}$ is the kernel of the Albanese map $\mathrm{alb}_S : \mathrm{CH}_0(S)_{\mathrm{hom}} \to \mathrm{Alb}(S)$. A famous result of Mumford [Mum69] says, if the geometric genus $p_g(S) > 0$, then there is no uniform integer $d > 0$ such that any $\alpha \in \mathrm{CH}_0(S)_{\mathrm{hom}}$ can be written as $\alpha = \alpha^+ - \alpha^- \in \mathrm{CH}_0(S)$ with both of α^+ and α^- effective and $\deg \alpha^+ = \deg \alpha^- = d$. It is thus legitimate to call $\mathrm{CH}_0(S)_{\mathrm{hom}}$ and also the Albanese kernel $\mathrm{CH}_0(S)_{\mathrm{alb}}$ infinite-dimensional.

The filtration (1-1) is supposed to be the filtration in the aforementioned Bloch–Beilinson conjecture, and as a consequence, we have

Conjecture 1.1 [Voi03, Conjecture 11.19]. *Let S and T be smooth projective surfaces, Γ a cycle of codimension 2 in $S \times T$ such that the map $[\Gamma]^* : H^{2,0}(T) \to H^{2,0}(S)$ vanishes. Then the map $\Gamma_* : \mathrm{CH}_0(S)_{\mathrm{alb}} \to \mathrm{CH}_0(T)_{\mathrm{alb}}$ vanishes.*

By taking $T = S$ with $p_g(S) = 0$, and $\Gamma = \Delta_S \subset S \times S$ the diagonal in Conjecture 1.1, we recover Bloch's initial conjecture.

Conjecture 1.2 ([Blo75]; see also [Voi03, Conjecture 11.2]). *Let S be a smooth projective surface. If $p_g(S) = 0$ then $\mathrm{CH}_0(S)_{\mathrm{alb}} = 0$.*

Bloch's conjecture has been verified in various special cases (see [DL23, page 444] for a discussion), but is widely open in general.

We are interested in the induced action of automorphisms on $\mathrm{CH}_0(S)$. So, take $T = S$, and $\Gamma := \Gamma_\sigma - \Delta_S$, where Γ_σ is the graph of an automorphism σ in a subgroup $G \subset \mathrm{Aut}(S)$, and we obtain from Conjecture 1.1 the following

Conjecture 1.3 (cf. [Voi12, Conjecture 1.2]). *Let S be a smooth projective surface, and G a group of automorphisms of S acting trivially on $H^{2,0}(S)$. Then G acts trivially on $\mathrm{CH}_0(S)_{\mathrm{alb}}$.*

Define the group $\mathrm{Aut}_s(S)$ of *symplectic automorphisms*

$$\mathrm{Aut}_s(S) := \{\sigma \in \mathrm{Aut}(S) \mid \sigma \text{ induces the trivial action on } H^{2,0}(S) = H^0(S, K_S)\}.$$

and the group $\mathrm{Aut}_{\mathcal{O}}(S)$ of *$\mathcal{O}$-cohomologically trivial automorphisms*

$$\mathrm{Aut}_{\mathcal{O}}(S) := \{\sigma \in \mathrm{Aut}(S) \mid \sigma \text{ induces the trivial action on } H^i(S, \mathcal{O}_S) \text{ for any } i\}.$$

By Serre duality, we have $H^0(S, K_S) \cong H^2(S, \mathcal{O}_S)^\vee$, and hence $\mathrm{Aut}_{\mathcal{O}}(S) \subset \mathrm{Aut}_s(S)$; the two groups coincide if the irregularity $q(S) := \dim H^1(S, \mathcal{O}_S) = 0$. The automorphisms in $\mathrm{Aut}_s(S)$ are those fixing a general holomorphic 2-form (if any) on S, and hence called symplectic.

Conjecture 1.3 amounts to saying that $\mathrm{Aut}_s(S)$ acts trivially on $\mathrm{CH}_0(S)_{\mathrm{alb}}$.

For surfaces with $p_g(S) > 0$, Conjecture 1.3 is known to hold in certain cases:

- S is an abelian surface and $G = \mathrm{Aut}_s(S)$ [BKL76, Theorems A.1 and A.7; Paw19, Corollary 1.5];
- S is a K3 surface, and $G \subset \mathrm{Aut}_s(S)$ is finite [Voi12; Huy12];
- $S = K(A)$ is the Kummer K3 surface associated to an abelian surface A, and G is generated by $\sigma \in \mathrm{Aut}_s(S)$ that lifts to a group automorphism $\tilde{\sigma}$ of A [Paw19, Theorem 1.10];
- S is a K3 surface with an elliptic fibration $f : S \to B$, and G preserves the fibration structure f [DL23, Theorem 1.5];
- S is a K3 surface with either the Picard number $\rho(S) \geq 3$ or S admitting a Jacobian fibration, and $G = \mathrm{Aut}_s(S)$ [LYZ23, Theorem 1.3];
- S has Kodaira dimension one and $G = \mathrm{Aut}_{\mathcal{O}}(S)$ [DL23, Theorem 5.10];
- S has Kodaira dimension one with $(p_g, q) \notin \{(1, 1),\ (2, 2)\}$, and $G = \mathrm{Aut}_s(S)$ [DL23, Theorem 1.3];
- S has a finite-dimensional Chow motive in the sense of Kimura [Kim05] (this is the case if S has an isotrivial fibration), and $G \subset \mathrm{Aut}_s(S)$ is finite [DL23, Lemma 5.9].

In this paper, we investigate Conjecture 1.3 for surfaces of general type with a genus two fibration. Recall that a fibration of genus g on a smooth projective surface S means a morphism $f : S \to B$ onto a smooth projective curve B with connected fibers of genus g.

Theorem 1.4. *Let S be a surface of general type with a genus two fibration $f : S \to B$ and $\chi(\mathcal{O}_S) \geq 5$. Suppose that* $\mathrm{Aut}_s(S)$ *is nontrivial.*

(i) $|\mathrm{Aut}_s(S)| = 2$.

(ii) *If the canonical map $\phi_{K_S} : S \dashrightarrow \mathbb{P}^{p_g(S)-1}$ is composed with a pencil, then* $\mathrm{Aut}_s(S)$ *preserves every fiber of f, and acts trivially on* $\mathrm{CH}_0(S)_{\mathrm{alb}}$.

(iii) *If the canonical map $\phi_{K_S} : S \dashrightarrow \mathbb{P}^{p_g(S)-1}$ is generically finite onto its image $T := \phi_{K_S}(S)$, then* $\mathrm{Aut}_{\mathcal{O}}(S)$ *is trivial, and* $\mathrm{Aut}_s(S)$ *acts trivially on* $\mathrm{CH}_0(S)_{\mathrm{alb}}$ *unless*

(a) $q(S) = g(B) \geq \chi(\mathcal{O}_S) - 3$,

(b) *a smooth model of the quotient surface* $S/\langle\sigma\tau\rangle$ *is of general type with* $p_g = q = 0$, *where* σ *is the generator of* $\mathrm{Aut}_s(S)$ *and* τ *is the hyperelliptic fibration of* f, *and*

(c) f *is not isotrivial.*

The bound $|\mathrm{Aut}_s(S)| \le 2$ of Theorem 1.4(i) is implicit in Jin-Xing Cai's work [Cai06a; Cai06b] on the automorphism group H of genus two fibrations acting trivially on $H^2(S, \mathbb{Q})$. Specifically, by Hodge decomposition, one has $H \subset \mathrm{Aut}_s(S)$, and his proof that $|H| \le 2$ uses only this fact.

Cai also constructed various fibered surfaces of genus two with an involution acting trivially on $H^2(S, \mathbb{Q})$ in [Cai06a; Cai06b; Cai07]. Only one series of them, namely [Cai06b, Example 3.5], satisfies all of the conditions (a)–(c) of Theorem 1.4, and we do not know whether or not $\mathrm{Aut}_s(S)$ acts trivially on $\mathrm{CH}_0(S)_{\mathrm{alb}}$ for S in this example.

Theorem 1.4 follows from Propositions 3.4, Proposition 3.7, and Corollary 2.4. Let us explain the ideas of the proofs. Following [Cai06a; Cai06b], the proof of Theorem 1.4(i) is based on the results of Xiao on genus two fibrations [Xiao85]. Specifically, let $G \subset \mathrm{Aut}(S)$ be the subgroup generated by $\mathrm{Aut}_s(S)$ together with the hyperelliptic involution τ. Then the canonical map ϕ_{K_S} of S factors through the quotient map $S \to S/G$ (Lemma 3.1), and the explicit bounds for ϕ_{K_S} of Xiao (Theorem 3.3) give the bound $|G| \le 4$ and hence $|\mathrm{Aut}_s(S)| \le 2$.

Once the bound on $|\mathrm{Aut}_s(S)|$ is established, we have

$$G = \{\mathrm{id}_S,\ \sigma,\ \tau,\ \sigma\tau\} \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

We can then decompose $\mathrm{CH}_0(S)_{\mathrm{alb},\mathbb{Q}}$ into eigenspaces with respect to the G-action, and see that $\mathrm{Aut}_s(S)$ acts trivially on $\mathrm{CH}_0(S)_{\mathrm{alb}}$ as soon as $\mathrm{CH}_0(S/\langle\sigma\tau\rangle)_{\mathrm{alb}} = 0$ (Lemma 2.6). Since $p_g(S/\langle\sigma\tau\rangle) = 0$, the Albanese kernel $\mathrm{CH}_0(S/\langle\sigma\tau\rangle)_{\mathrm{alb}}$ vanishes if (a smooth model of) the quotient surface $S/\langle\sigma\tau\rangle$ is not of general type by [BKL76]. The bulk of our arguments is devoted to verify the latter condition, once the surface $f : S \to B$ is not isotrivial but violates one of the conditions (a) and (b) in Theorem 1.4(iii). The case of isotrivial fibrations has been settled once and for all by applying Kimura's finite-dimensionality of such surfaces (Corollary 2.4).

Assuming Conjecture 1.2 holds, then $\mathrm{CH}_0(S/\langle\sigma\tau\rangle)_{\mathrm{alb}} = 0$, and we are done again (Proposition 3.8).

Concerning the condition $\chi(\mathcal{O}_S) \ge 5$ in the theorem, we remark that minimal surfaces of general type with $\chi(\mathcal{O}_S) < 5$ form (only) a bounded family. This condition is used mainly to ensure that the canonical map of S is well-behaved and that S has at most one genus two fibration on it.

Finally, as a consequence of Theorem 1.4, Conjecture 1.3 holds for the subgroup $\mathrm{Aut}_{\mathcal{O}}(S) \subset \mathrm{Aut}_s(S)$ for surfaces of general type with a genus two fibration, whose invariants are not so small.

Corollary 1.5. *Let S be a surface of general type with a genus two fibration $f : S \to B$ and $\chi(\mathcal{O}_S) \geq 5$. Then* $\mathrm{Aut}_{\mathcal{O}}(S)$ *acts trivially on* $\mathrm{CH}_0(S)_{\mathrm{alb}}$.

As pointed out by a referee, since there is an $\mathrm{Aut}_{\mathcal{O}}(S)$-fixed point $x \in S$, $\mathrm{Aut}_{\mathcal{O}}(S)$ even acts trivially on the whole $\mathrm{CH}_0(S)$, provided that $q(S) = 0$.

Outline. We recall in Section 2 some relevant notions and facts such as fibration-preserving automorphisms, the induced action on the Albanese variety and the 0-th Chow group, as well as useful criteria for the triviality of the induced action on the Albanese kernel. In Section 3, we focus on genus two fibrations, first using the canonical map to bound the symplectic automorphism group $\mathrm{Aut}_s(S)$, and then verifying the triviality of its induced action on the Albanese kernel as stated in Theorem 1.4.

Notation and conventions. Let S be smooth projective surface over $\mathbb{C}$.

- For a coherent sheaf $\mathcal{F}$ on S, we denote $h^i(S, \mathcal{F}) := \dim_{\mathbb{C}} H^i(S, \mathcal{F})$, and $\mathcal{F}^\vee := \mathcal{H}om_{\mathcal{O}_S}(\mathcal{F}, \mathcal{O}_S)$, the dual of $\mathcal{F}$.
- For $0 \leq i \leq 2$, Ω^i_S denotes the sheaf of i-forms on S.
- ω_S denotes the canonical sheaf of S, which can be identified with Ω^2_S, and K_S denotes a canonical divisor of S.
- The following numerical invariants are attached to S:
  - the *geometric genus* $p_g(S) := h^0(S, \omega_S)$;
  - the *irregularity* $q(S) := h^1(S, \mathcal{O}_S) = h^0(S, \Omega^1_S)$;
  - the *holomorphic Euler characteristic* $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$.

For a singular projective surface T, say with rational singularities, its geometric genus $p_g(T)$, irregularity $q(T)$ are defined as the corresponding invariants of its smooth model.

For a finite group G and an element $\sigma \in G$, their orders are denoted by $|G|$ and $|\sigma|$ respectively. An action of G on a set X is called *trivial* if $\sigma(x) = x$ for any $\sigma \in G$ and any $x \in X$.

2. Preliminaries

2.1. ***Fibered surfaces and their automorphisms.*** Let S be a normal projective surface, and $f : S \to B$ a fibration onto a smooth projective curve, that is, f is a surjective morphism with connected fibers. The fibration is called

- *of genus g* if its general fiber has genus g;
- *hyperelliptic* (resp. *elliptic*, resp. *a $\mathbb{P}^1$-fibration*) if its smooth fibers are hyperelliptic (resp. elliptic, resp. $\mathbb{P}^1$);
- *isotrivial* if its smooth fibers are mutually isomorphic.

We call $q_f := q(S) - g(B)$ the *relative irregularity* of f.

The subgroup $\mathrm{Aut}_f(S)$ of *fibration-preserving automorphisms* is defined as follows:

$$\mathrm{Aut}_f(S) := \{\sigma \in \mathrm{Aut}(S) \mid \sigma \text{ maps fibers of } f \text{ to fibers}\}$$

There is an induced action of $\mathrm{Aut}_f(S)$ on B, that is, a homomorphism $r : \mathrm{Aut}_f(S) \to \mathrm{Aut}(B)$, such that for any $\sigma \in \mathrm{Aut}_f(S)$ there is an induced automorphism $\sigma_B = r(\sigma) \in \mathrm{Aut}(B)$ such that the following diagram is commutative:

$$\begin{array}{ccc} S & \xrightarrow{\sigma} & S \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow{\sigma_B} & B \end{array}$$

The elements of $\mathrm{Aut}_B(S) := \ker r$ are called *fiber-preserving automorphisms*.

Lemma 2.1. *Let $f : S \to B$ be a non-isotrivial fibration of genus $g \geq 2$. Then $\mathrm{Aut}_f(S)$ is a finite group.*

Proof. Consider the moduli map associated to f

$$\mu : B \to \overline{\mathcal{M}}_g,$$

where $\overline{\mathcal{M}}_g$ is the (compact) moduli space of stable curves of genus g. Since f is not isotrivial, μ is generically finite, and the image $\mathrm{Aut}_f(S)|_B$ of $r : \mathrm{Aut}_f(S) \to \mathrm{Aut}(B)$ has order at most $\deg \mu$. It follows that

(2-1)
$$|\mathrm{Aut}_f(S)| = |\mathrm{Aut}_B(S)| \cdot |\mathrm{Aut}_f(S)|_B| \leq |\mathrm{Aut}(F)| \cdot |\mathrm{Aut}_f(S)|_B| \leq |\mathrm{Aut}(F)| \cdot \deg \mu$$

where F is a general fiber of f. Since $g(F) = g \geq 2$, $\mathrm{Aut}(F)$ is a finite group, and hence $\mathrm{Aut}_f(S)$ is also a finite group by (2-1). ☐

2.2. ***The induced action on the Albanese variety and the 0-th Chow group.*** The Chow group of a normal projective variety X is the group of rational equivalence classes of algebraic cycles on X; we refer to [Voi03, Chapter 9] for the basic properties of Chow groups.

Let X be a smooth projective variety. Fixing a base point $x_0 \in X$, we may define a group homomorphism

$$\mathrm{alb} : \mathrm{CH}_0(X)_{\mathrm{hom}} \to \mathrm{Alb}(X) = H^0(X, \Omega_X^1)^\vee / H_1(X, \mathbb{Z}), \quad \sum_i n_i[x_i] \mapsto \sum_i n_i \Big[\int_{x_0}^{x_i}\Big],$$

where $\int_{x_0}^{x_i} : H^0(X, \Omega_X^1) \to \mathbb{C}$ maps a holomorphic 1-form η to the integral $\int_{x_0}^{x_i} \eta$, determined up to $\int_\gamma \eta$ for a closed loop γ on X. The homomorphism alb does not depend on the choice of the base point x_0.

Any automorphism group $G \subset \mathrm{Aut}(X)$ has an induced action on $\mathrm{CH}_0(X)$ and $\mathrm{Alb}(X)$ as follows: For $\sigma \in G$, $\sum_i n_i[x_i] \in \mathrm{CH}_0(X)$, and $\left[\int_{x_0}^{x}\right] \in \mathrm{Alb}(X)$,

$$\sigma_*\Big(\sum_i n_i[x_i]\Big) = \sum_i n_i[\sigma(x_i)], \quad \sigma_*\Big(\Big[\int_{x_0}^{x}\Big]\Big) = \Big[\int_{\sigma(x_0)}^{\sigma(x)}\Big] = \Big[\int_{x_0}^{\sigma(x)}\Big] - \Big[\int_{x_0}^{\sigma(x_0)}\Big].$$

Note that G acts by group automorphisms on $\mathrm{CH}_0(X)$ and $\mathrm{Alb}(X)$, and the homomorphism alb is G-equivariant. The G-action extends in an obvious way to $\mathrm{CH}_0(X)_{\mathbb{Q}} := \mathrm{CH}_0(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathrm{Alb}(X)_{\mathbb{Q}} := \mathrm{Alb}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let Z be a normal projective birational model of X, with at most rational singularities. Then we have natural identifications (see [JLZ23, Section 2.1] for a discussion for the Albanese variety of a singular variety):

$$\mathrm{Alb}(Z) \cong \mathrm{Alb}(X), \quad \mathrm{CH}_0(Z) \cong \mathrm{CH}_0(X).$$

The following basic but important fact about the G-actions on $\mathrm{Alb}(X)_{\mathbb{Q}}$ and $\mathrm{CH}_0(X)_{\mathbb{Q}}$ should be well-known, but we write a proof of it for completeness (cf. [Lat21, Lemma 1.6]).

Lemma 2.2. *Let X be a smooth projective variety and $G \subset \mathrm{Aut}(X)$ a finite subgroup. Let Y be a normal projective birational model with rational singularities of the quotient variety X/G. Then*

$$\mathrm{Alb}(X)^G_{\mathbb{Q}} \cong \mathrm{Alb}(Y)_{\mathbb{Q}}, \quad \mathrm{CH}_0(X)^G_{\mathrm{alb},\mathbb{Q}} \cong \mathrm{CH}_0(Y)_{\mathrm{alb},\mathbb{Q}}, \quad \mathrm{CH}_0(X)^G_{\mathrm{hom},\mathbb{Q}} \cong \mathrm{CH}_0(Y)_{\mathrm{hom},\mathbb{Q}}.$$

Proof. Since X/G and Y have at most rational singularities, there are natural identifications

$$\mathrm{Alb}(X/G) = \mathrm{Alb}(Y),$$
$$\mathrm{CH}_0(X/G)_{\mathrm{hom},\mathbb{Q}} \cong \mathrm{CH}_0(Y)_{\mathrm{hom},\mathbb{Q}},$$
$$\mathrm{CH}_0(X/G)_{\mathrm{alb},\mathbb{Q}} \cong \mathrm{CH}_0(Y)_{\mathrm{alb},\mathbb{Q}}.$$

Thus we may assume that $Y = X/G$.

Recall that

$$\mathrm{Alb}(X)_{\mathbb{Q}} = \frac{H^0(X, \Omega^1_X)^\vee \otimes_{\mathbb{Z}} \mathbb{Q}}{H_1(X, \mathbb{Q})},$$

and hence we have natural isomorphisms of $\mathbb{Q}$-vector spaces

$$\text{(2-2)} \quad \mathrm{Alb}(X)^G_{\mathbb{Q}} \cong \frac{\big(H^0(X, \Omega^1_X)^G\big)^\vee \otimes_{\mathbb{Z}} \mathbb{Q}}{H_1(X, \mathbb{Q})^G} \cong \frac{H^0\big(Y, \Omega^1_Y\big)^\vee \otimes_{\mathbb{Z}} \mathbb{Q}}{H_1(Y, \mathbb{Q})} \cong \mathrm{Alb}(X/G)_{\mathbb{Q}},$$

where Y is a smooth projective model of X/G.

By the universal property of the Albanese morphism and by the flatness of $\mathbb{Q}$ as a $\mathbb{Z}$-module, the quotient map $\pi : X \to X/G$ induces the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{CH}_0(X)_{\mathrm{alb},\mathbb{Q}} & \longrightarrow & \mathrm{CH}_0(X,\mathbb{Q})_{\mathrm{hom},\mathbb{Q}} & \longrightarrow & \mathrm{Alb}(X)_{\mathbb{Q}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*} & & \\ 0 & \longrightarrow & \mathrm{CH}_0(X/G)_{\mathrm{alb},\mathbb{Q}} & \longrightarrow & \mathrm{CH}_0(X/G)_{\mathrm{hom},\mathbb{Q}} & \longrightarrow & \mathrm{Alb}(X/G)_{\mathbb{Q}} & \longrightarrow & 0 \end{array}$$

Restricting to the G-invariant parts of the $\mathbb{Q}$-vector spaces in the first row, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{CH}_0(X)^G_{\mathrm{alb},\mathbb{Q}} & \longrightarrow & \mathrm{CH}_0(X,\mathbb{Q})^G_{\mathrm{hom},\mathbb{Q}} & \longrightarrow & \mathrm{Alb}(X)^G_{\mathbb{Q}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & \mathrm{CH}_0(X/G)_{\mathrm{alb},\mathbb{Q}} & \longrightarrow & \mathrm{CH}_0(X/G)_{\mathrm{hom},\mathbb{Q}} & \longrightarrow & \mathrm{Alb}(X/G)_{\mathbb{Q}} & \longrightarrow & 0 \end{array}$$

and we have seen in (2-2) that γ is an isomorphism.

For a point $x \in X$, let $\bar{x}$ be its image in X/G. Any 0-cycle on X/G is rationally equivalent to one with support outside the branch locus $\mathcal{B}$ of π [Voi12, Fact 3.3]. For $[z] \in \mathrm{CH}_0(X/G)_{\mathrm{hom},\mathbb{Q}}$, we may thus assume that $z = \sum_i n_i \bar{x}_i$ with $\bar{x}_i \notin \mathcal{B}$ for any i, and one sees that β has an inverse

$$\beta^{-1}([z]) = \frac{1}{|G|} \sum_i \sum_{g \in G} [g(x_i)].$$

Therefore, β is an isomorphism.

By the Five Lemma, α is also an isomorphism. □

2.3. *Useful criteria for a symplectic automorphism to act trivially on* $\mathbf{CH_0(S)_{alb}}$.

Lemma 2.3 (cf. [DL23, Lemma 5.9]). *Let S be a smooth projective surface. Assume that the Chow motive $\mathfrak{h}(S)$ of S is finite-dimensional in the sense of Kimura* [Kim05] *and O'Sullivan. Then any symplectic automorphism of finite order of S acts trivially on* $\mathrm{CH}_0(S)_{\mathrm{alb}}$.

In particular:

Corollary 2.4. *Let $f : S \to B$ be an isotrivially fibered surface. Then any finite order symplectic automorphism of S acts trivially on* $\mathrm{CH}_0(S)_{\mathrm{alb}}$.

Proof. Since f is an isotrivial fibration, S is birational to $(\widetilde{B} \times F)/G$, where $\widetilde{B}$ is a smooth projective curve dominating B and F is a general fiber of f. Therefore, S is dominated by $\widetilde{B} \times F$ and hence has finite-dimensional Chow motive by [Kim05]. Now apply Lemma 2.3. □

Corollary 2.5. *Let $f : S \to B$ be a fibered surface of genus g such that $q_f = g$. Then any finite order symplectic automorphism of S acts trivially on* $\mathrm{CH}_0(S)_{\mathrm{alb}}$.

Proof. Since $g = q_f$, the fibration f is isotrivial by [Bea82], and hence the assertion follows from Corollary 2.4. □

If a surface S has several commuting involutions, then we have the following criterion for the triviality of the $\mathrm{Aut}_s(S)$-action on the Albanese kernel.

Lemma 2.6. *Let S be a smooth projective surface. Let $G = \langle\sigma, \tau\rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$ be a subgroup of* $\mathrm{Aut}(S)$. *Suppose that* $\mathrm{CH}_0(S/\langle\tau\rangle)_{\mathrm{alb}} = 0$. *Then σ acts trivially on* $\mathrm{CH}_0(S)_{\mathrm{alb}}$ *if and only if* $\mathrm{CH}_0(S/\langle\sigma\tau\rangle)_{\mathrm{alb}} = 0$.

Proof. Since $\mathrm{CH}_0(S)_{\mathrm{alb}}$ has no torsion [Roj80], we may work on the Albanese kernels with $\mathbb{Q}$-coefficients. Since $\mathrm{CH}_0(S/\langle\tau\rangle)_{\mathrm{alb}} = 0$, τ acts as -1 on $\mathrm{CH}_0(S)_{\mathrm{alb},\mathbb{Q}}$. Thus σ acts as identity on $\mathrm{CH}_0(S)_{\mathrm{alb},\mathbb{Q}}$ if and only if $\sigma\tau$ acts as -1 on $\mathrm{CH}_0(S)_{\mathrm{alb},\mathbb{Q}}$, which is equivalent to the vanishing of the $(\sigma\tau)$-invariant part $\mathrm{CH}_0(S)^{\sigma\tau}_{\mathrm{alb},\mathbb{Q}} \cong \mathrm{CH}_0(S/\langle\sigma\tau\rangle)_{\mathrm{alb},\mathbb{Q}}$. □

If a surface with vanishing geometric genus is not of general type, then it has trivial Albanese kernel [BKL76]. Thus we may obtain from Lemma 2.6 the following statement.

Corollary 2.7 [DL23, Lemma 5.1]. *Let S be a smooth projective surface. Let $G = \langle\sigma, \tau\rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$ be a subgroup of* $\mathrm{Aut}(S)$ *such that* (*the smooth models of*) *the quotient surfaces $S/\langle\tau\rangle$ and $S/\langle\sigma\tau\rangle$ are not of general type and have vanishing geometric genus. Then σ acts trivially on* $\mathrm{CH}_0(S)_{\mathrm{alb}}$.

3. Symplectic automorphisms of surfaces with a genus two fibration

For a smooth projective surface S with $p_g(S) > 0$, one may use the canonical system $|K_S|$ to define a rational map $\phi_{K_S} : S \dashrightarrow \mathbb{P}^{p_g(S)-1}$, called the *canonical map* of S. This map is a main tool in our study of symplectic automorphisms of surfaces with a genus two fibration, due to the following observation.

Lemma 3.1. *Let S be a smooth projective surface of general type with a hyperelliptic fibration $f : S \to B$, and $G \subset$* $\mathrm{Aut}(S)$ *the subgroup generated by the hyperelliptic involution $\tau \in$* $\mathrm{Aut}_B(S)$ *together with the symplectic automorphism group* $\mathrm{Aut}_s(S)$. *Suppose that $p_g := p_g(S) > 0$, so that the canonical map $\phi_{K_S} : S \dashrightarrow \mathbb{P}^{p_g-1}$ is well-defined. Then ϕ_{K_S} factors through the quotient map $\pi : S \to S/G$, that is, there is a rational map $\varphi : S/G \dashrightarrow \mathbb{P}^{p_g-1}$ such that $\phi_{K_S} = \varphi \circ \pi$.*

Proof. Since τ acts as -1 on $H^0(S, K_S)$ while $\mathrm{Aut}_s(S)$ acts trivially on $H^0(S, K_S)$, ϕ_{K_S} is a G-equivariant map with G acting trivially on the target. The assertion of the lemma follows. □

Before recalling the fundamental results of Xiao on the canonical map of a surface of general type with a genus two fibration, we need to introduce some notation first.

Notation 3.2. Let S be a surface of general type, and $f : S \to B$ a genus two fibration. We abbreviate the Euler characteristic $\chi(\mathcal{O}_S)$, the geometric genus $p_g(S)$, the irregularity $q(S)$, the genus $g(B)$ of the base curve B by $\chi,\ p_g,\ q,\ b$. Let $M \subset f_*\omega_S$ be an invertible subsheaf with maximal degree and $e := 2\deg M - \deg f_*\omega_S$. Then we have $\deg f_*\omega_S = \chi + 3(b-1)$ (see the proof of [Xiao85, théorème 2.1]) and hence

$$\deg M = \tfrac{1}{2}(e + \deg f_*\omega_S) = \tfrac{1}{2}(e + \chi + 3(b-1)). \tag{3-1}$$

As in Lemma 3.1, we will denote by τ the hyperelliptic involution of f, and let $G \subset \mathrm{Aut}(S)$ be the subgroup generated by $\mathrm{Aut}_s(S)$ and τ. Note that τ acts as -1 on $H^0(S, K_S)$ and hence $\tau \notin \mathrm{Aut}_s(S)$ as soon as $p_g(S) > 0$.

Theorem 3.3 [Xiao85, pages 71–73, corollaire 1 and proposition 5.2]. *Let S be a smooth projective surface of general type with a genus two fibration $f : S \to B$, as in Notation 3.2.*

(i) *If $p_g \geq 3$ and the image of the canonical map ϕ_{K_S} is a curve, then ϕ_{K_S} factors through f, and there are three possibilities for the numerical invariants*:
  (a) $q = b = 0$ *and* $e = p_g$;
  (b) $b = 0, q = 1$ *and* $e = p_g + 1$;
  (c) $q = b = 1$ *and* $e = p_g$.

(ii) *If $\chi(\mathcal{O}_S) \geq 4$ and the image of the canonical map is a surface, then* $\deg \phi_{K_S} \in \{2, 4\}$. *Moreover, if* $\deg \phi_{K_S} = 4$, *then*
  (a) $p_g(S) \leq 2b + 2$;
  (b) $T := \phi_{K_S}(S)$ *is either a rational surface or a cone over an elliptic curve.*

Now we discuss $\mathrm{Aut}_s(S)$ for a fibered surface $f : S \to B$ of genus two according to the behavior of the canonical map, given by Theorem 3.3.

Proposition 3.4. *Let S be a smooth projective surface of general type with a genus two fibration $f : S \to B$ such that $p_g(S) \geq 3$. Suppose that the image of the canonical map ϕ_{K_S} is a curve.*

(i) $\mathrm{Aut}_s(S) \subset \mathrm{Aut}_B(S)$, *that is, each symplectic automorphism of S is fiber-preserving.*

(ii) $\mathrm{Aut}_s(S)$ *has order at most* 2, *and it acts trivially on* $\mathrm{CH}_0(S)_{\mathrm{alb}}$.

Proof. By Theorem 3.3(i), ϕ_{K_S} factors through f. Therefore, the fibration f is canonical, and every automorphism of S preserves it. In other words, we have $\mathrm{Aut}(S) = \mathrm{Aut}_f(S)$.

By (3-1), we have

$$\deg M = \begin{cases} p_g - 1 & \text{in cases (a) and (b) of Theorem 3.3(i),} \\ p_g & \text{in case (c) of Theorem 3.3(i).} \end{cases}$$

In all cases, $\deg M \geq 2b+1$ and hence M is a very ample invertible sheaf on B. By counting dimensions, we have $H^0(B, M) \cong H^0(B, f_*\omega_S) \cong H^0(S, \omega_S)$, and hence the canonical map ϕ_{K_S} factors as

$$\phi_{K_S} : S \xrightarrow{\ f\ } B \xhookrightarrow{\ \phi_M\ } \mathbb{P}^{p_g-1}$$

where ϕ_M is the embedding defined by the complete linear system $|M|$.

The morphisms ϕ_{K_S}, f, and ϕ_M are all G-equivariant. Since $\mathrm{Aut}_s(S)$ acts trivially on $\mathbb{P}^{p_g-1}$ and B embeds into $\mathbb{P}^{p_g-1}$, the induced action of $\mathrm{Aut}_s(S)$ on B is trivial, and hence $\mathrm{Aut}_s(S) \subset \mathrm{Aut}_B(S)$. This proves (i).

(ii) follows from (i) and the next result. □

Lemma 3.5. *Let S be a smooth projective surface, and $f : S \to B$ a fibration of genus two. If $p_g(S) > 0$, then $\mathrm{Aut}_B(S) \cap \mathrm{Aut}_s(S)$ has order at most 2, and it acts trivially on $\mathrm{CH}_0(S)_{\mathrm{alb}}$.*

Proof. Let F be a general fiber of f. Then $\mathrm{Aut}_B(S)$ injects into $\mathrm{Aut}(F)$, which is finite. Suppose that $\sigma \in \mathrm{Aut}_B(S) \cap \mathrm{Aut}_s(S) \setminus \{\mathrm{id}_S\}$. Then

$$p_g(S/\langle\sigma\rangle) = p_g(S) > 0.$$

This implies that $g(F/\langle\sigma\rangle) = 1$, since it is smaller than 2 but cannot be 0, due to the positivity of p_g. By the Riemann–Hurwitz formula, we have

$$2g(F) - 2 = |\sigma|\Big(2g(F/\langle\sigma\rangle) - 2 + \sum_i \Big(1 - \frac{1}{r_i}\Big)\Big). \tag{3-2}$$

Since the automorphism group $\langle\sigma\rangle$ generated by σ is abelian, the quotient map $F \to F/\langle\sigma\rangle$ has at least two branch points and hence $\sum_i \big(1 - \frac{1}{r_i}\big) \geq 1$. Then the equality (3-2) implies that $|\sigma| \leq 2$.

The hyperelliptic involution $\tau \in \mathrm{Aut}_B(S)$ commutes with σ, and they generate a Klein group $G := \langle\sigma, \tau\rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$. Since the induced fibrations $S/\langle\sigma\tau\rangle \to B$ and $S/\langle\tau\rangle \to B$ have genus 1 and 0 respectively, so both surfaces are not of general type, and they have vanishing p_g, we may conclude by Corollary 2.7 that σ acts trivially on $\mathrm{CH}_0(S)_{\mathrm{alb}}$. □

Lemma 3.6. *Let S be a smooth projective surface of general type with a genus two fibration $f : S \to B$. If $\chi(\mathcal{O}_S) \geq 5$, then $\mathrm{Aut}(S) = \mathrm{Aut}_f(S)$, that is, every automorphism of S preserves the fibration f.*

Proof. This is because f is the unique genus two fibration on S by [Xiao85, proposition 6.4 and théorème 6.5], and hence preserved by any automorphism of S. □

Proposition 3.7. *Let S be a smooth projective surface of general type with a genus two fibration $f : S \to B$, as in Notation 3.2. Suppose that $\chi(\mathcal{O}_S) \geq 5$, that the canonical map ϕ_{K_S} is generically finite onto its image $T := \phi_{K_S}(S)$, and that $\mathrm{Aut}_s(S)$ is nontrivial.*

(i) $\mathrm{Aut}_s(S)$ *has order* 2, *and preserves the fibration* f.

(ii) $S \dashrightarrow T$ *is birationally a* $(\mathbb{Z}/2\mathbb{Z})^2$*-cover, i.e., the induced extension* $K(S)/K(T)$ *of function fields is Galois with group isomorphic to* $(\mathbb{Z}/2\mathbb{Z})^2$.

(iii) $b \geq 2$, *and the induced action of* $\mathrm{Aut}_s(S)$ *on* B *is not trivial.*

(iv) $\mathrm{Aut}_{\mathcal{O}}(S)$ *is trivial.*

(v) *If* $q_f > 0$, *then* $\mathrm{Aut}_s(S)$ *acts trivially on* $\mathrm{CH}_0(S)_{\mathrm{alb}}$.

Proof. By Lemma 3.1, ϕ_{K_S} factors through the quotient map $S \to S/G$, where $G \subset \mathrm{Aut}(S)$ is the subgroup generated by $\mathrm{Aut}_s(S)$ and the hyperelliptic involution τ, which does not lie in $\mathrm{Aut}_s(S)$. Therefore,

$$2|\mathrm{Aut}_s(S)| \leq |G| \leq \deg \phi_{K_S} \in \{2, 4\}$$

where we used the bound on $\deg \phi_{K_S}$ from Theorem 3.3(ii). Since $\mathrm{Aut}_s(S)$ is nontrivial by assumption, we infer that

$$\deg \phi_{K_S} = 4, \quad |\mathrm{Aut}_s(S)| = 2, \quad G \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

By Lemma 3.6, $\mathrm{Aut}_s(S)$ preserves the fibration f. Thus (i) is proved.

For (ii), note that ϕ_{K_S} has degree 4 and factors through the quotient map $S \to S/\langle \tau, \sigma \rangle$, which also has degree 4. Thus the induced map $S/\langle \tau, \sigma \rangle \dashrightarrow T$ is birational, and (ii) follows.

(iii) By Theorem 3.3(ii), we have

$$p_g \leq 2b + 2. \tag{3-3}$$

It follows that

$$b \geq p_g - b - 2 \geq p_g - q - 2 = \chi - 3 \geq 2$$

where the last inequality is because $\chi \geq 5$ by assumption.

By Theorem 3.3(ii), T is either a rational surface or a cone over an elliptic curve. This implies that the smooth models of T (and of S/G) have irregularity at most 1. Thus, the dimension of $H^0(S, \Omega_S^1)^G$ is at most 1. On the other hand, since $S/\langle \tau \rangle$ is a $\mathbb{P}^1$-fibration over B, we have

$$\dim H^0(S, \Omega_S^1)^\tau = b \geq 2.$$

It follows that $H^0(S, \Omega^1_S)^G \subsetneq H^0(S, \Omega^1_S)^\tau$, and hence σ does not act trivially on $H^0(S, \Omega^1_S) \cong H^1(S, \mathcal{O}_S)^\vee$. Therefore, the automorphism $\sigma_B \in \mathrm{Aut}(B)$ induced by σ is not the identity. This finishes the proof of (iii).

(iv) We have $\sigma \notin \mathrm{Aut}_{\mathcal{O}}(S)$ by (iii), and hence $\mathrm{Aut}_{\mathcal{O}}(S)$ is trivial.

(v) Note that $q_f \leq 2$ in any case. If $q_f = 2$, then the assertion follows directly from Corollary 2.5.

Thus we may assume that $q_f = 1$, and there is a one-form

$$\eta_0 \in H^0(S, \Omega^1_S) \setminus f^* H^0(B, \Omega_B),$$

which we may assume to be an eigenvector with eigenvalue λ under the action of σ. Then

$$H^0(S, \Omega^1_S) = f^* H^0(\Omega^1_B) \oplus \mathbb{C} \cdot \eta_0,$$

both summands of which are invariant under the action of $G = \langle \sigma, \tau \rangle$. Define a $\mathbb{C}$-linear map

$$\wedge\eta_0 : H^0(B, \Omega^1_B) \to H^0(S, K_S), \quad \eta \mapsto f^*\eta \wedge \eta_0, \tag{3-4}$$

which is easily seen to be injective and G-equivariant. Since σ acts trivially on $H^0(S, K_S)$ and hence also trivially on the subspace $f^* H^0(B, \Omega^1_B)) \wedge \eta_0$, it follows that σ acts as the scalar multiplication by λ^{-1} on the whole $H^0(B, \Omega^1_B)$. Thus the canonical map $\phi_{K_B} : B \to \mathbb{P}^{b-1}$ factors through the quotient map $B \to B/\langle \sigma_B \rangle$. Since $b \geq 2$, we infer that B is a hyperelliptic curve and σ_B is its hyperelliptic involution, acting as -1 on $H^0(B, \Omega^1_B)$. It follows that $\lambda = -1$, and the eigenvalues of the action are as in the following table:

	$f^* H^0(\Omega^1_B)$	$\mathbb{C} \cdot \eta_0$
σ	-1	-1
τ	1	-1

It follows that

$$p_g(S/\langle \sigma\tau \rangle) = 0, \quad q(S/\langle \sigma\tau \rangle) = 1$$

and hence $S/\langle \sigma\tau \rangle$ is not of general type [Bea96, Chapter VI]. By Corollary 2.7, σ acts trivially on $\mathrm{CH}_0(S)_{\mathrm{alb}}$. □

Finally, we make an observation that Conjecture 1.2 (for surfaces with $p_g = 0$) implies Conjecture 1.1 for surfaces of general type with a genus two fibration and $\chi(\mathcal{O}_S) \geq 5$.

Proposition 3.8. *Let S be a smooth projective surface of general type with a genus two fibration and $\chi(\mathcal{O}_S) \geq 5$. Assume that Conjecture 1.2 holds. Then $\mathrm{Aut}_s(S)$ acts trivially on $\mathrm{CH}_0(S)_{\mathrm{alb}}$.*

Proof. We may assume that $\mathrm{Aut}_s(S)$ is nontrivial. By Theorem 1.4, we may assume that the canonical map ϕ_{K_S} is generically finite, and $\mathrm{Aut}_s(S)$ has order two. Let σ be the generator of $\mathrm{Aut}_s(S)$, and $\tau \in \mathrm{Aut}_B(S)$ the hyperelliptic involution. Then $\langle\sigma, \tau\rangle \cong (\mathbb{Z}/2\mathbb{Z})^2$, and since $S/\langle\tau\rangle \to B$ is a $\mathbb{P}^1$-fibration, one has $\mathrm{CH}_0(S/\langle\tau\rangle)_{\mathrm{alb}} = 0$. Note also that $p_g(S/\langle\sigma\tau\rangle) = 0$.

Assuming Conjecture 1.2 holds, we have $\mathrm{CH}_0(S/\langle\sigma\tau\rangle)_{\mathrm{alb}} = 0$, and hence σ acts trivially on $\mathrm{CH}_0(S)_{\mathrm{alb}}$ by Lemma 2.6. □

Acknowledgements

We thank Professor Jin-Xing Cai for communications on some questions related to the paper, and the referees for their careful reading and useful suggestions. Wenfei Liu was supported by the NSFC (no. 12571046).

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Received January 20, 2025. Revised May 25, 2025.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 342 No. 2 June 2026