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NON-BRAID-POSITIVE HYPERBOLIC L-SPACE KNOTS

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An L-space knot is a knot that admits a positive Dehn surgery yielding an L-space. Many known hyperbolic L-space knots are braid positive, meaning they can be represented as the closure of a positive braid. Recently, Baker and Kegel showed that the hyperbolic L-space knot $o9\_30634$ from Dunfield's census is not braid-positive, and they constructed infinitely many candidates for hyperbolic L-space knots that may not be braid-positive. However, it remains unproven whether their examples are genuinely non-braid-positive. In this paper, we construct infinitely many hyperbolic L-space knots that are not braid-positive, and are distinct from those considered by Baker and Kegel.

1. Introduction

An *L-space* is a rational homology 3–sphere whose (hat version) Heegaard Floer homology has rank equal to the order of its first homology. A knot is called an *L-space knot* if it admits a positive Dehn surgery yielding an L-space. L-space knots were originally motivated by the study of knots admitting lens space surgeries [14], and they continue to be an active subject of research.

In this paper, we consider the braid positivity of L-space knots. A knot or link is said to be *braid-positive* if it can be expressed as the closure of a positive braid. Many known L-space knots are braid-positive; for example, positive torus knots are L-space knots, and then they are braid-positive. On the other hand, it is known that not all L-space knots are braid-positive; for example, the (2, 3)-cable of the right-handed trefoil is not braid-positive (see Example 1 of [1]). However, the existence of non-braid-positive hyperbolic L-space knots remained an open problem for some time (see Problem 31.2 of [5], Question 2 of [1]).

Recently, Baker and Kegel examined Dunfield's list of 632 hyperbolic L-space knots. They showed that all but one of these knots are braid-positive, and that the exceptional knot, $o9\_30634$, is not braid-positive [2]. Furthermore, they constructed infinitely many candidates for non-braid-positive hyperbolic L-space knots, although they could not prove that any of their examples are definitely non-braid-positive.

MSC2020: 57K10, 57K18.
Keywords: L-space knot, braid-positive.

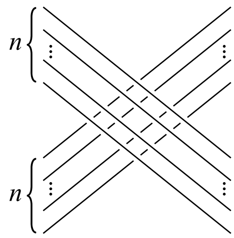

Figure 1. The braid X_n.

The purpose of this paper is to give infinitely many hyperbolic L-space knots that are not braid-positive. While all candidates by Baker and Kegel are represented as closures of 4-braids, we construct the infinite family of such knots by increasing the number of strands in the braid. For each $n \geq 2$, we define a knot K_n as the closure of a $2n$-braid, as follows. Let $[\varepsilon_1 i_1, \varepsilon_2 i_2, \dots, \varepsilon_m i_m]$ denote the braid $\sigma_{i_1}^{\varepsilon_1}\sigma_{i_2}^{\varepsilon_2}\cdots\sigma_{i_m}^{\varepsilon_m}$, where σ_i is the i-th standard generator of the degree $2n$ braid group and $\varepsilon_j = \pm 1$. Define the braid

$$X_n = [n, n-1, n+1, n-2, n, n+2, \dots, 1, 3, \dots, 2n-1, \\ 2n-2, \dots, 4, 2, \dots, n+1, n-1, n],$$

illustrated in Figure 1. Let K_n ($n \geq 1$) be a knot represented by the closure of $2n$-braid

$$\beta_n = X_n^3 \cdot [-1, -2, \dots, -(n-1), n, n-1, n-2, \dots, 2, 1, 1, 2, 3, \dots, n],$$

shown in Figure 2. Note that K_2 coincides with the knot $o9\_30634$, and K_1 is the $(2, 5)$-torus knot. Although the K_n for odd n do not appear in the statement of Theorem 1.1, we use them in the proof.

Theorem 1.1. *When n is even, K_n is a non-braid-positive hyperbolic L-space knot.*

Since the knots $\{K_n\}$ are mutually distinct (see Lemma 2.8), we have the following corollary.

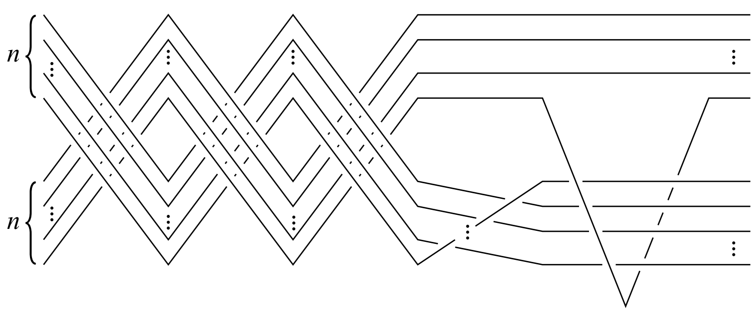

Figure 2. The knot K_n is the closure of this braid.

Corollary 1.2. *There exist infinitely many non-braid-positive hyperbolic L-space knots.*

When n is odd, we also expect K_n to be a non-braid positive hyperbolic L-space knot. However, in this case, the criterion we use for braid positivity failed to detect it.

2. Non-braid positivity

In this section, we prove that K_n is non-braid-positive when n is even. Our proof is based on Ito's criterion using the HOMFLY polynomial [8].

2.1. ***HOMFLY polynomial and its zeroth coefficient polynomial.*** For an oriented link L, the HOMFLY polynomial $P_L(v, z)$ is a two-variable Laurent polynomial defined by the skein relation

$$v^{-1}P_{L_+}(v, z) - vP_{L_-}(v, z) = zP_{L_0}(v, z),$$

together with $P_U(v, z) = 1$, where U is the unknot. Here, the links (or diagrams) L_+, L_- and L_0 coincide outside a small 3-ball, and inside the 3-ball, they differ as in Figure 3. (Throughout, all braids are assumed to be oriented from left to right.)

In [8], Ito provided a criterion for the braid positivity of a link using the HOMFLY polynomial. We state the result in the case of knots. Let

$$\widetilde{P}_K(\alpha, z) = (-\alpha)^{-g(K)}P_K(v, z)|_{-v^2=\alpha},$$

where $g(K)$ denotes the genus of a knot K.

Theorem 2.1 [8, Theorem 2]. *If K is a braid-positive knot, then $\widetilde{P}_K(\alpha, z)$ is positive, that is, all nonzero coefficients of $\widetilde{P}_K(\alpha, z)$ are positive integers.*

The HOMFLY polynomial $P_L(v, z)$ can be expressed in the form

$$P_L(v, z) = (v^{-1}z)^{-\#L+1}\sum_{i=0} p_L^i(v)z^{2i},$$

where $\#L$ is the number of components of the link L. The polynomial $p_L^i(v)$ is called the *i-th coefficient* (*HOMFLY*) *polynomial* of L. In this paper, we focus on the *zeroth coefficient polynomial* $p_L^0(v)$. It is known that this polynomial satisfies several important properties; see, for example [9, Section 2] and [19, Section 2].

The zeroth coefficient polynomial $p_L^0(v)$ satisfies the skein relation

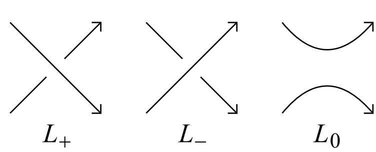

Figure 3. The small 3-ball in the diagrams of L_+, L_- and L_0.

$$v^{-2}p^0_{L_+}(v) - p^0_{L_-}(v) = \begin{cases} p^0_{L_0}(v) & (\delta = 0), \\ 0 & (\delta = 1), \end{cases} \tag{2-1}$$

where $\delta = \frac{1}{2}(\#L_+ - \#L_0 + 1) \in \{0, 1\}$. In particular, $\delta = 0$ if L_+ is a knot. Furthermore, for a two-component link $L = k_1 \cup k_2$, we have

$$p^0_L(v) = (v^{-2} - 1)v^{2\cdot \mathrm{lk}(k_1,k_2)} p^0_{k_1}(v) p^0_{k_2}(v), \tag{2-2}$$

where $\mathrm{lk}(k_1, k_2)$ is the linking number of k_1 and k_2. In particular, when L_+ is a knot (hence L_- is also a knot and L_0 is a two-component link $k_1 \cup k_2$), the skein relation (2-1) can be rewritten as

$$p^0_{L_-}(v) = v^{-2}p^0_{L_+}(v) + (1 - v^{-2})v^{2\cdot \mathrm{lk}(k_1,k_2)} p^0_{k_1}(v) p^0_{k_2}(v). \tag{2-3}$$

2.2. *Degree of p^0_L for a braid-positive link L.* For a braid β, let $L(\beta)$ denote the closure of β. A positive braid β is a *minimal positive braid* if the number of strands of β is minimum among all the positive braid representative of $L(\beta)$. The following proposition is useful when applying the skein relation to a braid-positive link.

Proposition 2.2 [21, Lemma 2]. *Let L be a braid-positive link that is not an unlink. Then there exists a positive braid β such that L is the closure of a positive braid of the form $\sigma_i^2\beta$ for some i. Moreover, such a positive braid representative $\sigma_i^2\beta$ of L can be taken so that it is a minimal positive braid.*

Lemma 2.3. *Let β be a positive n-braid, and let e be the number of crossings of β. Then the degree of the zeroth coefficient polynomial satisfies*

$$\deg p^0_{L(\beta)}(v) \le n + e - \#L(\beta).$$

Proof. We prove this lemma by induction on e. For an unlink U, we have $p^0_U(v) = (v^{-2} - 1)^{\#U-1}$, so $\deg p^0_U(v) = 0$.

Assume that $L(\beta)$ is not an unlink. By Proposition 2.2, we can write $L(\beta) = L(\sigma_i^2\beta')$ for some i, where β' is a positive n-braid with $e-2$ crossings. (If necessary, we may increase the number of crossings to realize β' as an n-braid.)

Applying the skein relation (2-1), we get

$$p^0_{L(\beta)}(v) = p^0_{L(\sigma_i^2\beta')}(v) = v^2 p^0_{L(\beta')}(v) + \begin{cases} v^2 p^0_{L(\sigma_i\beta')}(v) & (\delta = 0), \\ 0 & (\delta = 1). \end{cases}$$

Note that $\#L(\beta') = \#L(\beta)$, and $\#L(\sigma_i\beta') = \#L(\beta) + 1$ if $\delta = 0$. By the assumption of induction, we have

$$\deg v^2 p^0_{L(\beta')}(v) \le 2 + (n + e - 2 - \#L(\beta')) = n + e - \#L(\beta),$$

and if $\delta = 0$,

$$\deg v^2 p^0_{L(\sigma_i\beta')}(v) \le 2 + (n + e - 1 - \#L(\sigma_i\beta')) = n + e - \#L(\beta).$$

The claim follows. □

Definition 2.4. A positive n-braid β is said to be *sharp* if

$$\deg p_L^0(\beta)(v) = n + e - \#L(\beta),$$

where e is the number of crossings of β.

We now state several lemmas concerning the notion of sharpness.

Lemma 2.5. *If positive braid $\sigma_i^2\beta$ is sharp, then at least one of β and $\sigma_i\beta$ is sharp.*

Proof. The claim follows immediately from the proof of Lemma 2.3. □

Lemma 2.6. *If a positive braid β is sharp, then it is a minimal positive braid.*

Proof. Let β be a positive n-braid that is not minimal, and let β' be a minimal positive braid representative of $L(\beta)$ with n' strands ($n' < n$). The number of crossings of β is $n - \chi(L(\beta))$ and that of β' is $n' - \chi(L(\beta))$ [18]. Here $\chi(L)$ is the maximal Euler characteristic among all compact, connected, oriented surfaces whose boundary is a link L.

By Lemma 2.3,

$$\deg p_{L(\beta)}^0(v) \leq n' + n' - \chi(L(\beta)) - \#L(\beta) < n + n - \chi(L(\beta)) - \#L(\beta).$$

This implies that β is not sharp. □

Lemma 2.7. *Let $L = k_1 \cup k_2$ be a two-component link represented as the closure of a positive braid β. Suppose that each component knot k_1 and k_2 can also be represented as the closure of positive braids β_1 and β_2, respectively. If β is sharp, then both β_1 and β_2 are sharp.*

Proof. Let n, n_1, n_2 be the numbers of strands, and e, e_1, e_2 be the numbers of crossings of β, β_1 and β_2, respectively. Then,

$$n = n_1 + n_2, \ \ e = e_1 + e_2 + \operatorname{lk}(k_1, k_2).$$

By (2-2) and Lemma 2.3, we have

$$\deg p_L^0 \leq 2 \cdot \operatorname{lk}(k_1, k_2) + (n_1 + e_1 - 1) + (n_2 + e_2 - 1) = n + e - 2.$$

Equality holds if and only if β_1 and β_2 are sharp. Since β is sharp, equality holds, and the result follows. □

2.3. ***Non-braid positivity for K_n.*** We are ready to prove that K_n is not braid-positive when n is even.

Lemma 2.8. *For $n \geq 2$, the top term of $p_{K_n}^0(v)$ is $(-1)^n v^{3n^2+3n}$.*

Proof. We prove this by induction on n. Using Sage [17], one can confirm that the top term of $p_{K_2}^0(v)$ is v^{18}.

Throughout the following, we identify each braid with its closure diagram in the skein tree.

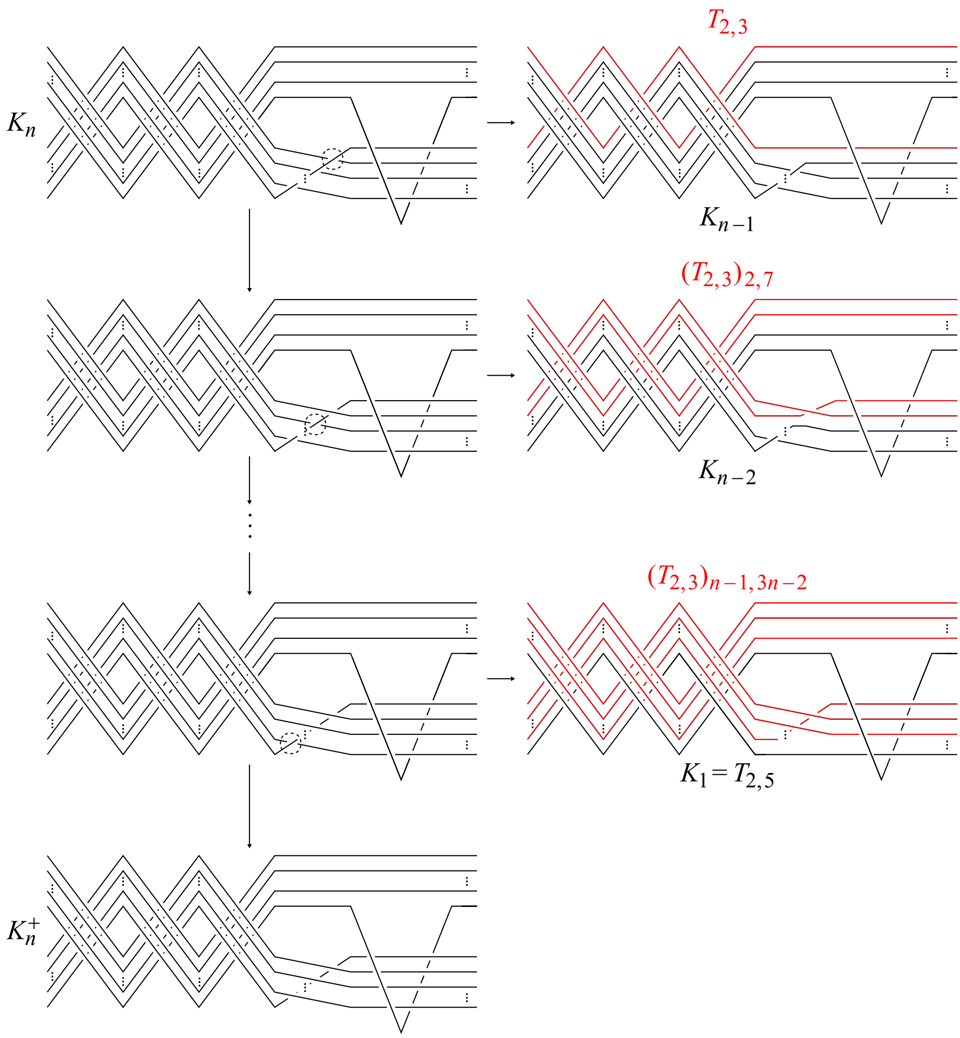

Figure 4. The skein tree for the knot K_n. Consider the crossing change and the smoothing at the crossing indicated by the dashed circle.

Assume that $n \geq 3$. From the skein tree shown in Figure 4 and equation (2-3), we obtain

$$\begin{aligned} p^0_{K_n}(v) &= (1 - v^{-2}) v^{2(3(n-1)\cdot 1+1)} p^0_{T_{2,3}}(v) p^0_{K_{n-1}}(v) \\ &\quad + (1 - v^{-2}) v^{-2} \cdot v^{2(3(n-2)\cdot 2+2)} p^0_{(T_{2,3})_{2,7}}(v) p^0_{K_{n-2}}(v) + \cdots \\ &\quad + (1 - v^{-2}) v^{-2(n-2)} \cdot v^{2(3\cdot 1\cdot (n-1)+n-1)} p^0_{(T_{2,3})_{n-1,3n-2}}(v) p^0_{K_1}(v) \\ &\quad + v^{-2(n-1)} p^0_{K_n^+}(v) \\ &= \sum_{k=1}^{n-1} v^{-2(k-1)} (1 - v^{-2}) v^{2(3(n-k)\cdot k+k)} p^0_{(T_{2,3})_{k,3k+1}}(v) p^0_{K_{n-k}}(v) + v^{-2(n-1)} p^0_{K_n^+}(v), \end{aligned}$$

where $(T_{2,3})_{k,3k-1}$ denotes the $(k, 3k-1)$-cable of the right-handed trefoil and K_n^+ is the braid-positive knot shown at the bottom in Figure 4.

By the induction assumption and the fact $\deg p^0_{K_1}(v) = \deg p^0_{T_{2,5}}(v) = 6$, we have $\deg p^0_{K_i} = 3i^2 + 3i$ for $1 \le i \le n-1$. Since $(T_{2,3})_{k,3k+1}$ is represented by the closure of the positive braid $X_k^3 \cdot [1, 2, \ldots, k-1]$, Lemma 2.3 gives

$$\begin{aligned}&\deg\bigl(v^{-2(k-1)}(1-v^{-2})v^{2(3(n-k)\cdot k+k)}p^0_{(T_{2,3})_{k,3k+1}}(v)\,p^0_{K_{n-k}}(v)\bigr)\\ &\le -2(k-1)+2(3(n-k)k+k)+(2k+3k^2+(k-1)-1)+3(n-k)^2+3(n-k)\\ &= 3n^2+3n.\end{aligned}$$

Equality holds if and only if the positive braid $X_k^3 \cdot [1, 2, \ldots, k-1]$ is sharp.

Claim 2.9. *For $k \ge 2$, the positive braid $X_k^3 \cdot [1, 2, \ldots, k-1]$ is not sharp.*

Proof. We consider the skein tree

$$\begin{array}{ccccccccccc} D_0 & \to & D_1 & \to & \cdots & \to & D_i & \to & D_{i+1} & \to & \cdots \to D_k, \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\ D_0^s & & D_1^s & & & & D_i^s & & D_{i+1}^s & & \end{array}$$

where D_0 is the braid $X_k^3 \cdot [1, 2, \ldots, k-1]$ and each D_i $(i = 1, \ldots, k)$ and D_i^s $(i = 0, \ldots, k-1)$ are as illustrated in Figures 5 and 6. The right arrows correspond to crossing changes, and the down arrows correspond to smoothings. Note that all links in the skein tree are braid-positive links.

D_k contains exactly one occurrence of the generator σ_{2k-1}, and is hence a nonminimal positive braid. Each D_i^s $(0 \le i \le k-1)$ is also nonminimal, as shown in Figures 7 and 8. Thus, by Lemmas 2.5 and 2.6, the claim follows. □

On the other hand, K_n^+ is represented as the closure of the positive braid

$$X_n^3 \cdot [1, 2, \ldots, n-1, n, n-1, n-2, \ldots, 2, 1, 1, 2, 3, \ldots, n].$$

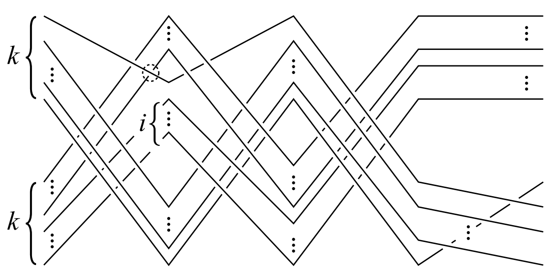

Figure 5. The diagram D_i $(0 \le i \le k)$. Performing a crossing change at the crossing indicated by the dashed circle, followed by a Reidemeister II move, results in the diagram D_{i+1}. Alternatively, smoothing the crossing produces the diagram D_i^s.

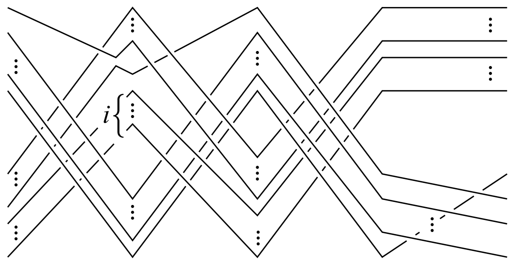

Figure 6. The diagram D_i^s $(0 \leq i \leq k-1)$.

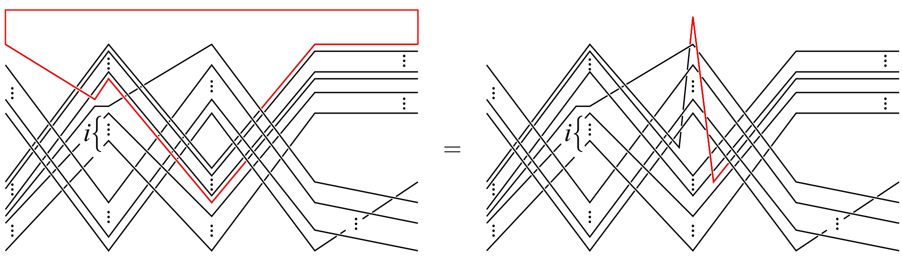

Figure 7. The positive braid D_i^s $(0 \leq i \leq k-2)$ is not minimal.

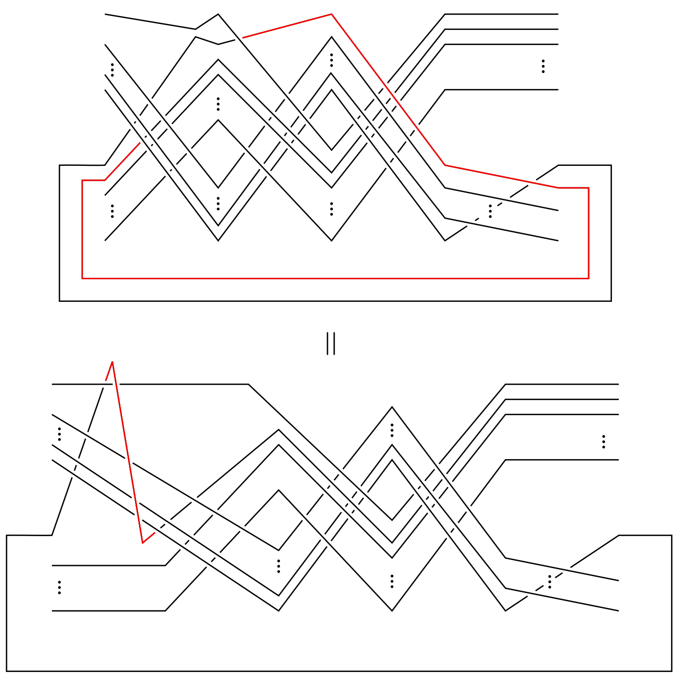

Figure 8. The positive braid D_{k-1}^s is not minimal.

Lemma 2.3 gives

$$\deg v^{-2(n-1)} p^0_{K_n^+}(v) \leq -2(n-1) + 2n + (3n^2 + n - 1 + 2n) - 1 = 3n^2 + 3n.$$

Equality holds if and only if the positive braid

$$X_n^3 \cdot [1, 2, \dots, n-1, n, n-1, n-2, \dots, 2, 1, 1, 2, 3, \dots, n]$$

is sharp again.

Claim 2.10. *For* $n \geq 3$, *the positive braid*

$$X_n^3 \cdot [1, 2, \dots, n-1, n, n-1, n-2, \dots, 2, 1, 1, 2, 3, \dots, n]$$

is not sharp.

Proof. We consider the skein tree

$$\begin{array}{ccccccccccccc} E_0 & \to & E_1 & \to & \cdots & \to & E_i & \to & E_{i+1} & \to & \cdots & \to & E_n, \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \\ E_0^s & & E_1^s & & & & E_i^s & & E_{i+1}^s & & & & \end{array}$$

where E_0 is the braid described in the claim, and E_n is $X_n^3 \cdot [1, 2, \dots, n-1]$, which is not sharp by Claim 2.9. The diagrams E_i and E_i^s are illustrated in Figures 9 and 10.

As shown in Figure 10, the diagram E_i^s consists of two components k_1 and k_2, represented by the positive braids γ_i^1 and γ_i^2 respectively. Figure 11 shows that γ_i^1 $(1 \leq i \leq n-1)$ is not minimal, and hence nonsharp by Lemma 2.6. Then, Lemma 2.7 implies that E_i^s is not sharp for $1 \leq i \leq n-1$.

For $i = 0$, we observe that γ_0^1 is a 1-braid (thus sharp), so we analyze γ_0^2. We perform a crossing change and smoothing at the crossing marked in Figure 12. The resulting diagrams (Figures 13 and 14) demonstrate that γ_0^2 is not sharp, and hence E_0^s is not sharp. □

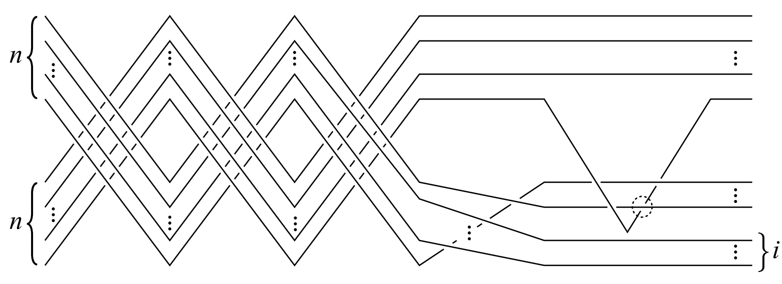

Figure 9. The diagram E_i $(0 \leq i \leq n)$. Performing a crossing change at the crossing indicated by the dashed circle, results in the diagram E_{i+1}. Alternatively, smoothing the crossing produces the diagram E_i^s.

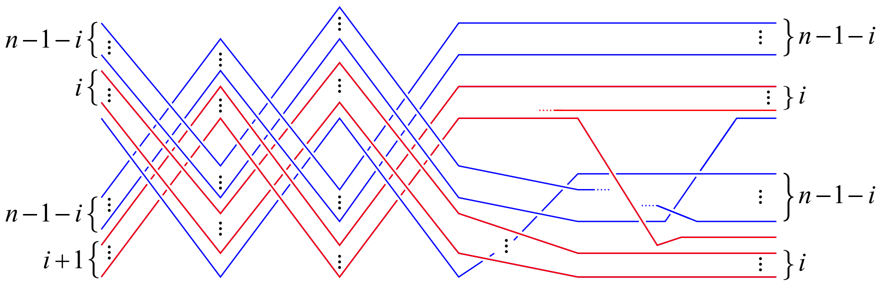

Figure 10. The diagram E_i^s $(0 \le i \le n-1)$ represents the two-component link $k_1 \cup k_2$. k_1 is represented by the positive braid γ_i^1 (red), and k_2 by γ_i^2 (blue).

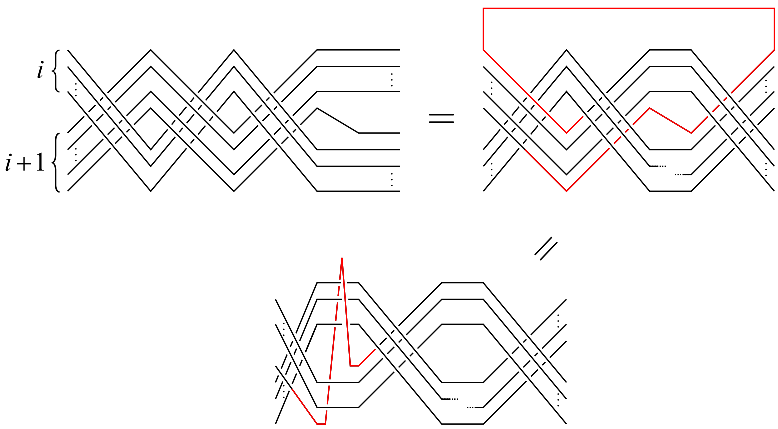

Figure 11. The positive braid γ_i^1 $(1 \le i \le n-1)$ is not minimal. The first deformation is a conjugation that moves the X-shaped part on the left side of the braid to the right.

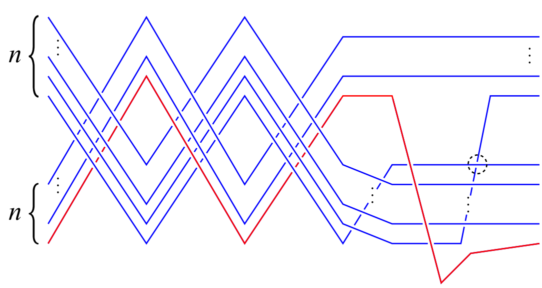

Figure 12. The diagram E_0^s. The red line represents γ_0^1, and the blue line represents γ_0^2. The crossing change at the crossing indicated by the dashed circle changes γ_0^2 into the diagram as in Figure 13. Alternatively, the smoothing yields the diagram as in Figure 14.

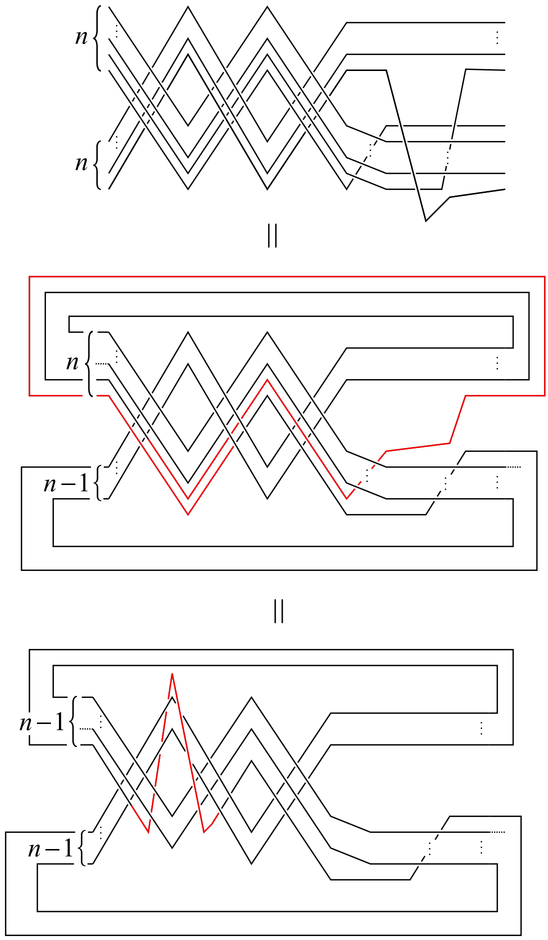

Figure 13. The diagram obtained by the crossing change of γ_0^2. As indicated, the corresponding positive braid is not minimal, hence not sharp.

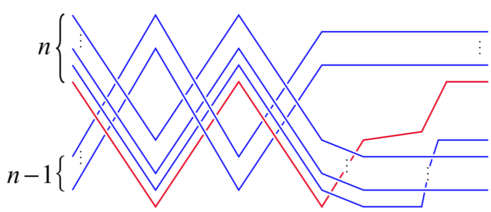

Figure 14. The diagram obtained by the smoothing of γ_0^2. The blue line is the positive braid $X_{n-1}^3[1, 2, \ldots, n-2]$, which is not sharp by Claim 2.9.

By Claims 2.9 and 2.10, the top term of $p_{K_n}^0(v)$ arises from the term

$$(1 - v^{-2})v^{2(3(n-1)\cdot 1+1)} p_{T_{2,3}}^0(v) p_{K_{n-1}}^0(v),$$

which evaluates to $v^{2(3(n-1)\cdot 1+1)}(-v^4)(-1)^{n-1}v^{3(n-1)^2+3(n-1)} = (-1)^n v^{3n^2+3n}$. The proof of Lemma 2.8 is complete. □

Lemma 2.11. *When n is even, the genus of K_n is given by*

$$g(K_n) = \tfrac{3}{2}n^2 - \frac{1}{2}n + 1.$$

Proof. By [13] and Proposition 3.1 below, K_n is an L-space knot, and hence fibered. A quasi-positive Seifert surface for K_n can be constructed by attaching $3n^2+n+1$ bands to $2n$ disks, where one of the bands corresponds to the braid

$$[-1, -2, \dots, -(n-1), n, n-1, \dots, 2, 1]$$

and the remaining bands correspond to the other positive crossings. By [16], the surface is incompressible and therefore serves as the fiber surface of K_n. A direct calculation then yields the genus $g(K_n) = \frac{3}{2}n^2 - \frac{1}{2}n + 1$. □

Proposition 2.12. *If n is even, then K_n is not braid-positive.*

Proof. Set $n = 2k$. By Lemmas 2.8 and 2.11, the top term of $(-\alpha)^{-g(K_{2k})} p^0_{K_{2k}}(v)|_{-v^2=\alpha}$ is

$$(-\alpha)^{-(6k^2-k+1)}(-1)^{2k}(-\alpha)^{6k^2+3k} = (-\alpha)^{4k-1} = -\alpha^{4k-1}.$$

Since this is the term of z^0 in $\widetilde{P}_{K_n}(\alpha, z)$, $\widetilde{P}_{K_n}(\alpha, z)$ is a non-positive polynomial. Therefore, Theorem 2.1 implies that K_n is not braid-positive. □

3. L-space surgery

In this section, we prove that K_n admits a Dehn surgery yielding an L-space if n is even. Throughout, we set $n = 2k$. We apply the Montesinos trick [12]: for a strongly invertible link L in S^3, the manifold obtained by Dehn surgery on L is the double branched cover of a link ℓ arising from a tangle replacement on the axis. See [2; 20] for details.

Figure 15 illustrates a strongly invertible position of the link $K \cup C_1 \cup C_2 \cup C_3$, where performing (-1)-surgery on C_1 and 1-surgery on both C_2 and C_3 transforms K into K_{2k}. Note that r-surgery on K corresponds to $(8k^2+r)$-surgery on K_{2k}.

Proposition 3.1. *$(12k^2+2k)$-surgery on K_{2k} yields an L-space.*

Proof. When $k = 1$, the 14-surgery on $K_2 = o9\_30634$ yields an L-space [2]. Assume that $k \geq 2$. Consider the quotient of the link in Figure 15 under the involution around the axis, as shown in Figure 16. Note that the surgery coefficient on K is $4k^2+2k$, and the writhe of K in the diagram is $4k^2+2k+1$. Hence, $(4k^2+2k)$-surgery on K corresponds to a tangle replacement by the (-1)-tangle.

Figure 17 illustrates a deformation of the quotient. Performing the indicated

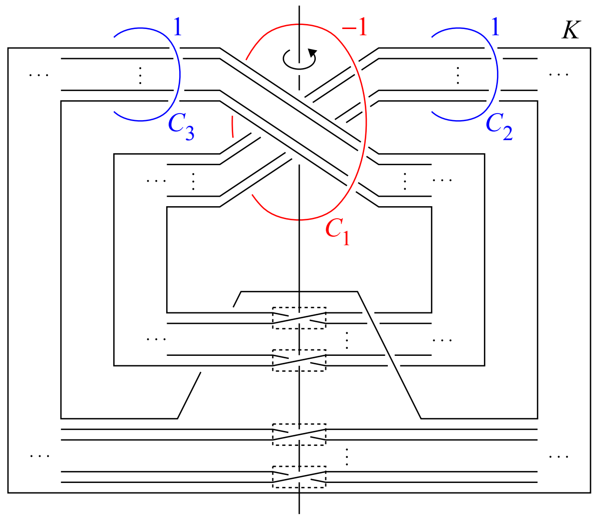

Figure 15. A strongly invertible position of the link $K \cup C_1 \cup C_2 \cup C_3$. The dashed boxes total $2k-1$: k in the upper part and $k-1$ in the lower.

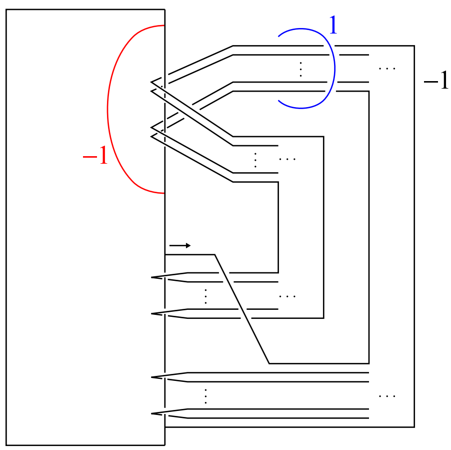

Figure 16. The quotient by the involution around the axis.

surgeries and tangle replacements on the middle right portion of the figure yields the link diagram in the bottom portion, which we denote by ℓ.

Claim 3.2. *The double branched cover of the link ℓ is an L-space.*

Proof. Using the Goeritz matrix derived from a checkerboard coloring of the diagram of ℓ (Figure 17, bottom), we compute $\det \ell = 12k^2 + 2k$. By smoothing the $2k-1$ crossings indicated by the dashed box in that figure, as shown in Figure 18, we obtain the links (or knots) ℓ^i_∞ $(i = 1, \ldots, 2k-1)$ and a knot ℓ_0.

A computation gives $\det \ell^i_\infty = 12k^2 + 2k - (6k+1)i$ and $\det \ell_0 = 6k+1$. Hence,

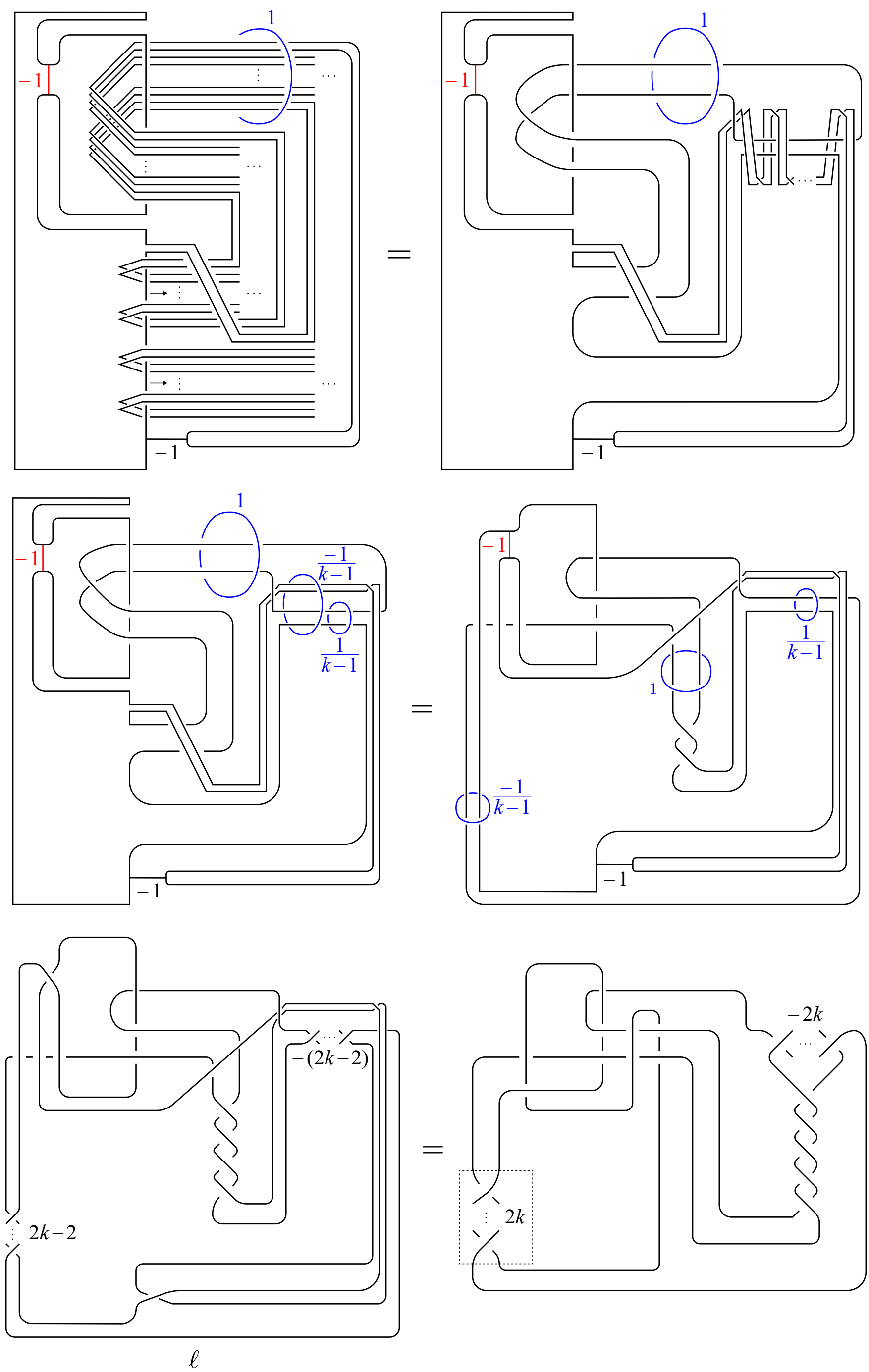

Figure 17. Three deformation steps for the quotient from Figure 16. In the bottom part, integers indicate the number of half-twists: right-handed if positive, left-handed otherwise.

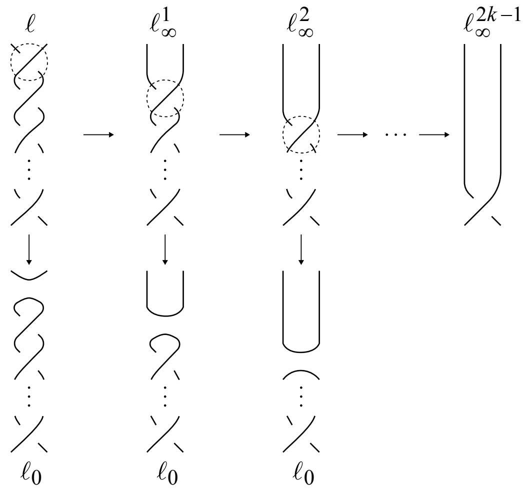

Figure 18. Smoothing the crossings indicated by the dashed circle.

$$\det \ell_\infty^i = \det \ell_\infty^{i+1} + \det \ell_0 \quad \text{for } i = 1, \ldots, 2k-2, \quad \text{and} \quad \det \ell = \det \ell_\infty^1 + \det \ell_0.$$

By [14, Proposition 2.1] and [15, Proposition 2.1], it suffices to show that the double branched covers of ℓ_0 and ℓ_∞^{2k-1} are both L-spaces.

As shown in Figure 19, the knot ℓ_0 is the Montesinos knot $M\left(-\frac{2}{3}, \frac{1}{2}, \frac{2k}{6k-1}\right)$, which is not quasi-alternating by [6]. Its double branched cover is the Seifert fibered

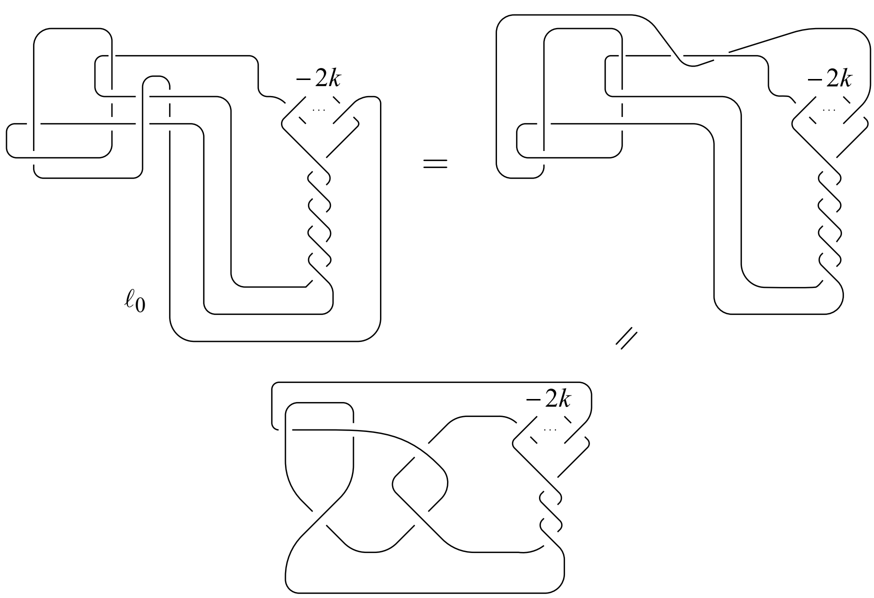

Figure 19. ℓ_0 is a Montesinos knot.

space

$$M\big(0; -\tfrac{2}{3}, \tfrac{1}{2}, \tfrac{2k}{6k-1}\big) = M\big(-1; \tfrac{1}{2}, \tfrac{2k}{6k-1}, \tfrac{1}{3}\big),$$

following the convention of [11], consistent with [2; 20].

According to [10; 11], such a Seifert fibered space $M(-1; r_1, r_2, r_3)$ with $1 \geq r_1 \geq r_2 \geq r_3 \geq 0$ is an L-space if and only if there are no relatively prime integers $m > a > 0$ satisfying $mr_1 < a < m(1 - r_2)$ and $mr_3 < 1$. For $r_1 = \frac{1}{2}$, $r_2 = \frac{2k}{6k-1}$, and $r_3 = \frac{1}{3}$, the condition $mr_3 < 1$ with $m > a > 0$ gives $a = 1$, $m = 2$, but this violates $mr_1 < a$. Thus, the double branched cover of ℓ_0 is an L-space.

Similarly, Figure 20 shows that ℓ_∞^{2k-1} is the Montesinos knot $M\big(\frac{2}{5}, -\frac{1}{2}, \frac{2k}{14k-1}\big)$,

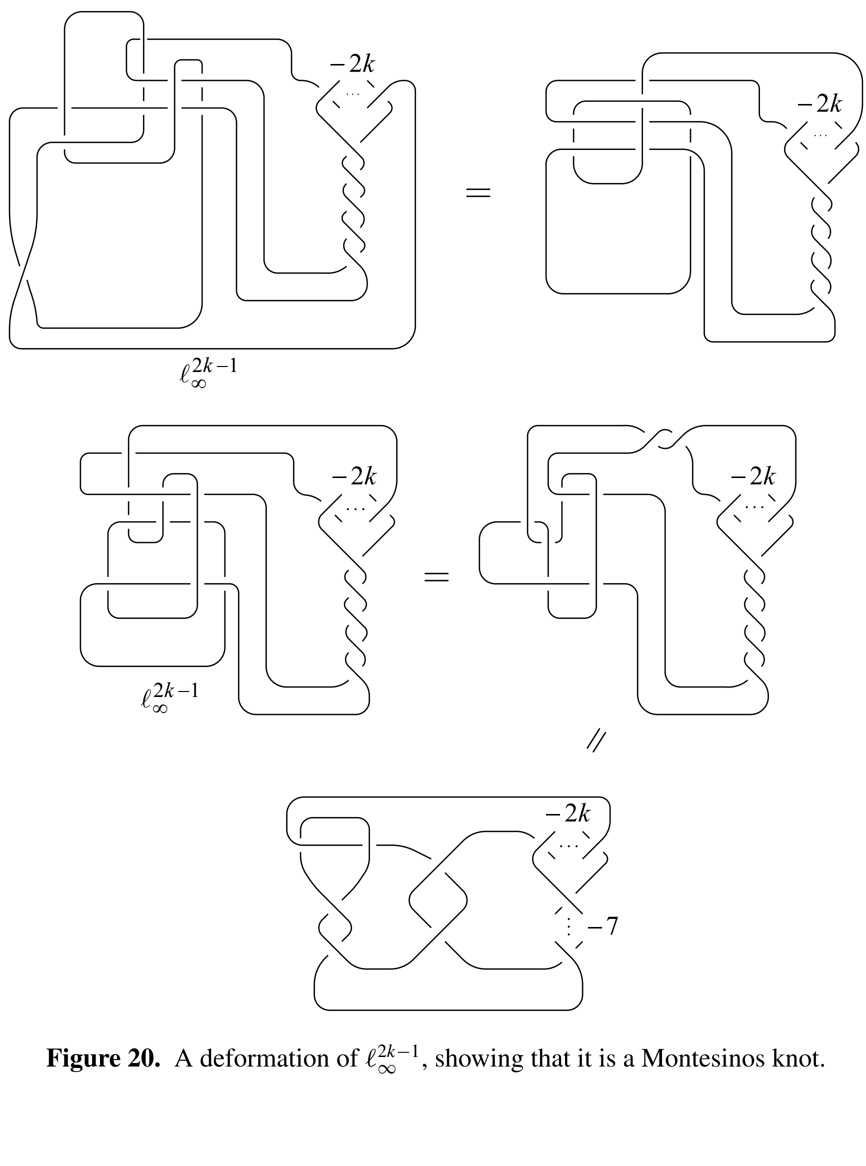

Figure 20. A deformation of ℓ_∞^{2k-1}, showing that it is a Montesinos knot.

and its double branched cover is

$$M\left(0; \tfrac{2}{5}, -\tfrac{1}{2}, \tfrac{2k}{14k-1}\right) = M\left(-1; \tfrac{1}{2}, \tfrac{2}{5}, \tfrac{2k}{14k-1}\right).$$

Set $r_1 = \frac{1}{2}$, $r_2 = \frac{2}{5}$, and $r_3 = \frac{2k}{14k-1}$. The condition $mr_3 < 1$ implies $m < 7 - \frac{1}{2k}$, so $m = 2, 3, \dots, 6$. But for these values, there exists no integer a satisfying $\frac{1}{2}m < a < \frac{3}{5}m$. Therefore, the double branched cover of ℓ_∞^{2k-1} is an L-space. □

By the Montesinos trick and Claim 3.2, the proof of Proposition 3.1 is complete. □

4. Hyperbolicity

To complete the proof of Theorem 1.1, we show that K_n is hyperbolic. To this end, we prove that the braid β_n representing K_n, regarded as an element of the mapping class group of a punctured 2-disk, is pseudo-Anosov in the sense of the Nielsen–Thurston classification. We apply the criterion of Bestvina and Handel [3], see also [4].

Let D be a 2-disk, and let $P = \{p_1, \dots, p_k\}$ be a set of k punctures. Take k small circles c_i $(i = 1, \dots, k)$, each centered at p_i, such that the interior of c_i contains no other punctures. Choose a finite graph G embedded in D such that it is homotopy equivalent to $D \setminus P$, and contains $C = \{c_1, \dots, c_k\}$ as a subgraph. We allow G to have loops, but assume that it has no vertices of valence 1 or 2. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of G, respectively. A *walk* τ in G is a finite sequence alternating between vertices and edges $(v_1, e_1, v_2, e_2, \dots, v_\ell, e_\ell, v_{\ell+1})$, where $v_1, \dots, v_{\ell+1} \in V(G)$ and $e_1, \dots, e_\ell \in E(G)$ such that the endpoints of e_i are v_i and v_{i+1}. There is no confusion in denoting this walk by $\tau = e_1 e_2 \cdots e_\ell$. Let $W(G)$ be the set of all walks in G.

A homeomorphism f on D that preserves the set of punctures P induces a *graph map*

$$g \colon (V(G), W(G)) \to (V(G), W(G)),$$

which preserves the set C set-wise. (The definition of a graph map involves the notion of a *fibered surface* associated with G, but we omit those details.)

Definition 4.1. A graph map g is said to be *efficient* if there are no integers $m \geq 1$ and edges $e \in E(G)$ such that

$$g^m(e) = \cdots e_i e_i \cdots$$

for some $e_i \in E(G)$. If such integer $m \geq 1$ and an edge $e \in E(G)$ exist, then we say that $g^m(e)$ has a *back track*.

Let $\mathrm{Vec}(G)$ be the real vector space spanned by $E(G)$. Each walk in G determines an element of $\mathrm{Vec}(G)$ by mapping to the linear combination in which the coefficient of each e_i is its multiplicity in the walk. For a graph map g, we define its transition matrix T_g as the linear transformation

$$\mathcal{T}_g\colon \mathrm{Vec}(G) \to \mathrm{Vec}(G)$$

that maps each edge e to $g(e)$, regarded as an element of $\mathrm{Vec}(G)$.

Let C^{pre} be the set of edges $e \in E(G)$ such that $g^m(e)$ is contained in C for some $m \geq 1$, and define the set of *real edges* by

$$E^{\mathrm{re}}(G) = E(G) \setminus (C \cup C^{\mathrm{pre}}).$$

By the definition of g and C^{pre}, the transition matrix $\mathcal{T}_g$ has the block form

$$\mathcal{T}_g = \begin{pmatrix} \mathcal{C} & \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{C}^{\mathrm{pre}} & \mathcal{D} \\ 0 & 0 & \mathcal{T}_g^{\mathrm{re}} \end{pmatrix},$$

where $\mathcal{C}$, $\mathcal{C}^{\mathrm{pre}}$ and $\mathcal{T}_g^{\mathrm{re}}$ are the transition matrices associated with C, C^{pre} and $E^{\mathrm{re}}(G)$.

Recall that a nonnegative square matrix M is called *irreducible* if, for any indices (i, j), there exists $m \geq 1$ such that the (i, j)-entry of M^m is positive. By the Perron–Frobenius theorem, such a matrix M has a real, positive eigenvalue greater than the absolute values of all other eigenvalues. This is called the *Perron–Frobenius eigenvalue* and is denoted by $\lambda(M)$ (it coincides with the spectral radius of M).

Theorem 4.2 [3]. *Let f be a homeomorphism on D that preserves the set of punctures P, and g be an induced graph map for f. Suppose that*

- *g is efficient, and*
- *the transition matrix $\mathcal{T}_g^{\mathrm{re}}$ with respect to the real edges is irreducible with $\lambda(\mathcal{T}_g^{\mathrm{re}}) > 1$.*

Then the mapping class of f (up to isotopy) is pseudo-Anosov with dilatation equal to $\lambda(\mathcal{T}_g^{\mathrm{re}})$.

Lemma 4.3. *The mapping class corresponding to the braid*

$$\beta_n = X_n^3 \cdot [-1, -2, \dots, -(n-1), n, n-1, n-2, \dots, 2, 1, 1, 2, 3, \dots, n] \ (n \geq 2)$$

is pseudo-Anosov.

Proof. When $n = 2$, the braid β_2 is pseudo-Anosov (confirmed using Sage [17]). Assume that $n \geq 3$.

Consider the braid

$$\beta_n' = [n, n-1, n-2, \dots, 2, 1, 1, 2, 3, \dots, n] \cdot X_n^3 \cdot [-1, -2, \dots, -(n-1)],$$

which is conjugate to β_n.

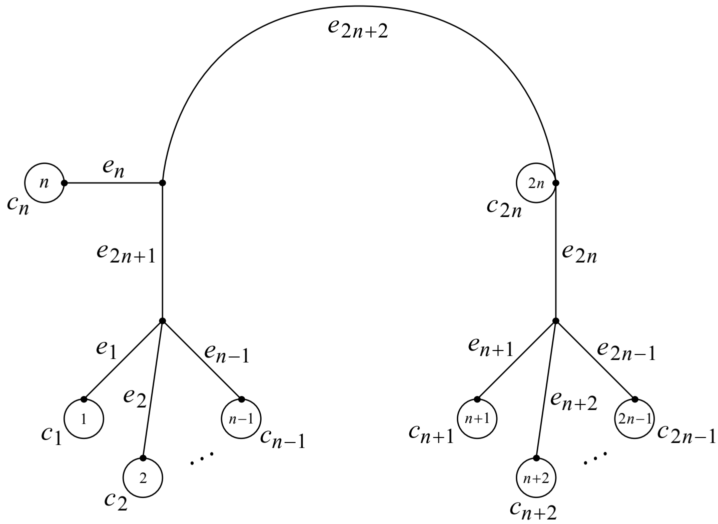

Figure 21. The graph G embedded in $D \setminus \{p_1, \ldots, p_{2n}\}$. Each integer inside a circle c_i represents a puncture.

Let G be a graph embedded in $D \setminus \{p_1, \ldots, p_{2n}\}$, as shown in Figure 21. (Note that this graph was chosen so that Theorem 4.2 can be applied.) Figures 22 and 23 illustrate a deformation of G induced by the braid β'_n. The induced graph map g acts on the real edges as follows:

$$\begin{aligned}
g(e_i) &= e_{n+1+i} \text{ for } i = 1, \ldots, n-1, \\
g(e_{n+i}) &= e_i \text{ for } i = 1, \ldots, n-1, \\
g(e_n) &= e_{2n+1} e_{n-1} c_{n-1} \cdots e_{2n} c_{2n} e_{2n} e_{n+1}, \\
g(e_{2n}) &= e_{2n+1}, \\
g(e_{2n+1}) &= e_{n+1} c_{n+1} e_{n+1} \cdots e_{2n} c_{n-1} e_{2n+2}, \text{ and} \\
g(e_{2n+2}) &= e_n c_n e_n \cdots e_{n-1} e_{2n+1} e_n.
\end{aligned}$$

Note that $e_1, \ldots, e_{2n+2}$ are real edges, and $g(e)$ has no back tracks for any real edge e.

Suppose that $g^k(e)$ has a back track for some $e \in E^{\mathrm{re}}(G)$ and $k \geq 1$. Then it must occur at e_n, that is, $g^k(e)$ contains a subwalk of the form

$$g^k(e) = \cdots e_{2n+1} e_n e_n e_{2n+1} \cdots .$$

This implies that $g^{k-1}(e)$ contains a subwalk of the form

$$g^{k-1}(e) = \cdots e_{2n+2} e_{2n} \cdots \quad \text{or} \quad \cdots e_{2n} e_{2n+2} \cdots ,$$

which does not occur by Figure 23. Hence, g is efficient.

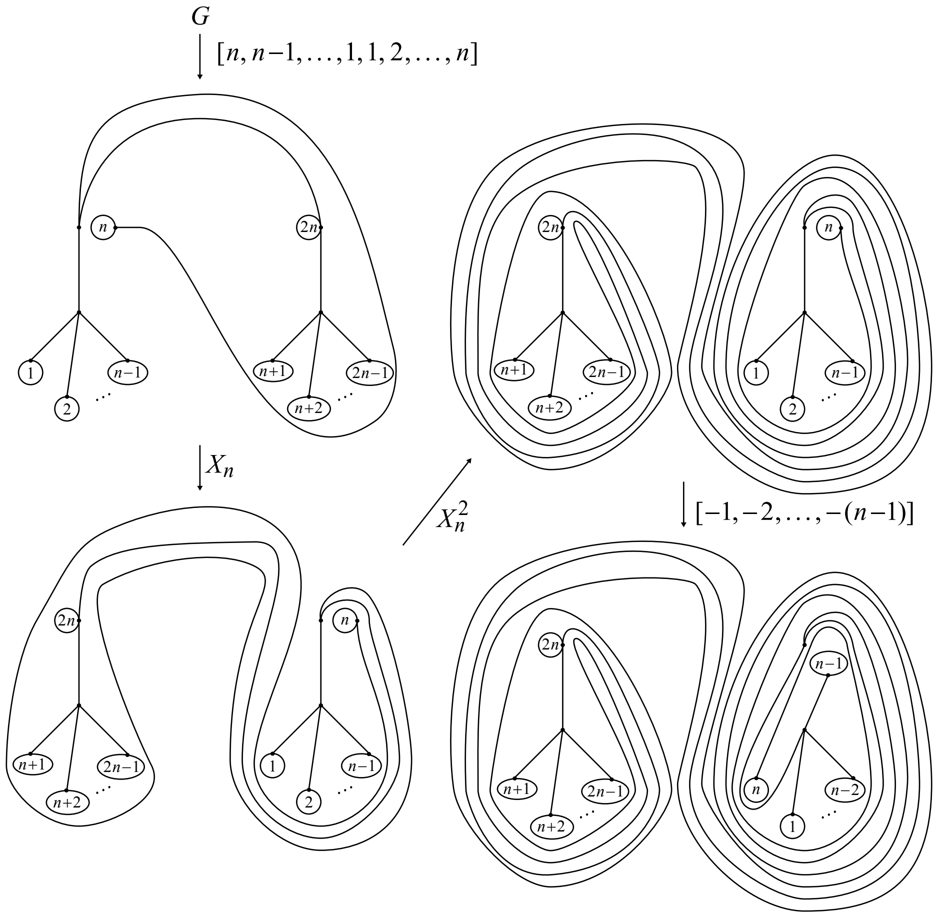

Figure 22. The deformation of G induced by the braid β'_n.

Since $g(e_n)$ passes through all real edges of G, and for any real edge e, there exists $k \geq 1$ such that $g^k(e)$ passes through e_n, it follows that some power $(\mathcal{T}_g^{\text{re}})^m$ of the transition matrix has all entries that are positive integers. We can choose m such that the trace of $(\mathcal{T}_g^{\text{re}})^m$ is greater than $\#E^{\text{re}}(G)$. Therefore, $(\mathcal{T}_g^{\text{re}})$ is irreducible and satisfies $\lambda(\mathcal{T}_g^{\text{re}}) > 1$. By Theorem 4.2, β'_n is pseudo-Anosov, and hence so is β_n. □

Proposition 4.4. *K_n $(n \geq 2)$ is a hyperbolic knot.*

Proof. Note that K_n can be obtained by the closure of the braid $X_n \beta_n X_n^{-1}$, and $X_n \beta_n X_n^{-1}$ is also pseudo-Anosov by Lemma 4.3.

By Figure 24, the Dehornoy floor of $X_n \beta_n X_n^{-1}$ is greater than 1 (see [7] for definition of the Dehornoy floor). Therefore, by Theorem 1.3 of [7], the closure of β_n, namely K_n, is a hyperbolic knot. □

Proof of Theorem 1.1. By Propositions 2.12, 3.1 and 4.4, we have the conclusion. □

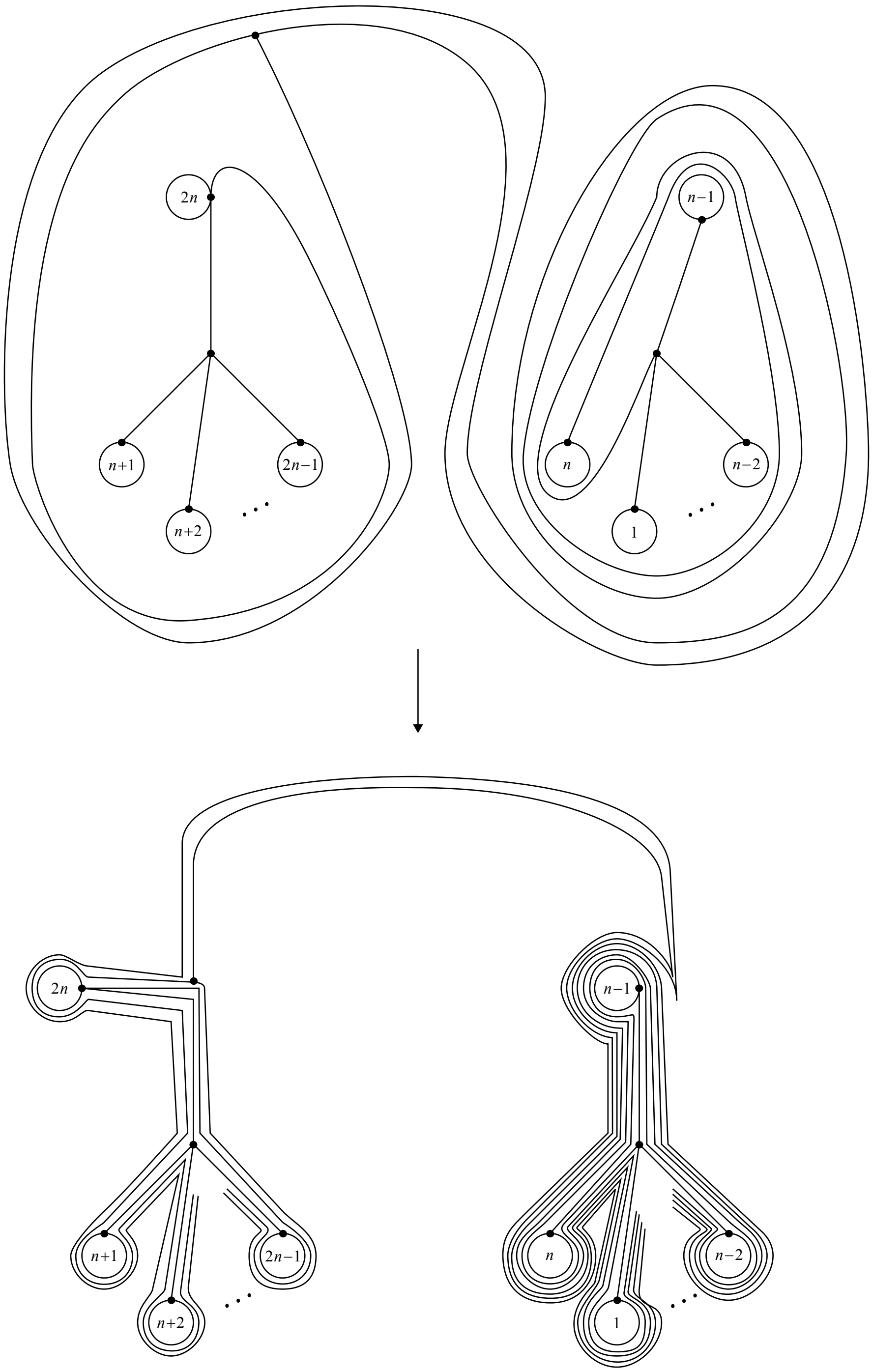

Figure 23. Continued from Figure 22. Here, the deformation from the right bottom in Figure 22 to the bottom of this figure is by an isotopy.

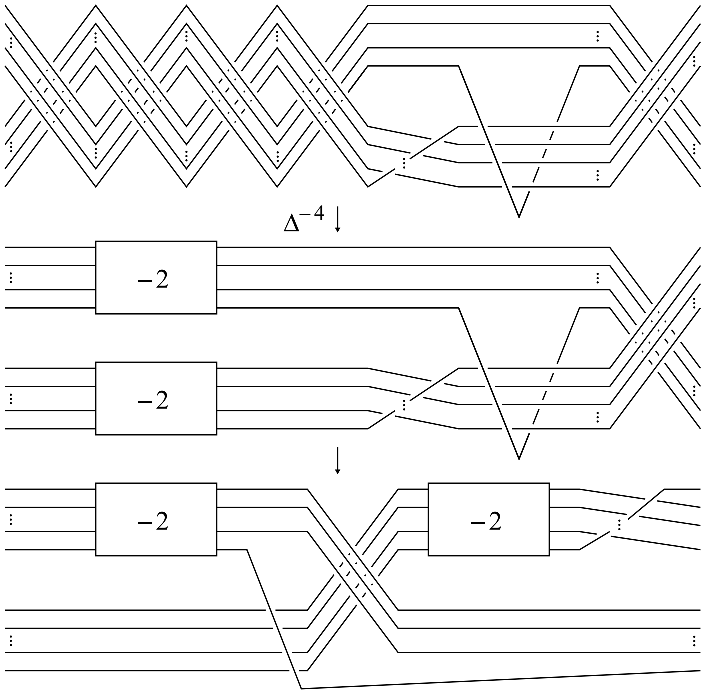

Figure 24. Top: the braid $X_n \beta_n X_n^{-1}$. Middle: the braid obtained by rewriting the braid $\Delta^{-4} X_n \beta_n X_n^{-1}$, where Δ is the Garside fundamental $2n$-braid. Here the box with -2 indicates two left-handed full-twists. Bottom: the braid obtained by further modifying the middle braid. Since this braid is σ_1–positive, we have $\Delta^4 <_D X_n \beta_n X_n^{-1}$ where $<_D$ is the Dehornoy order.

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