Pacific Journal of Mathematics

THE RANKIN–SELBERG INTEGRAL ON GSp_2 FOR SQUARE-FREE LEVELS

SEIJI KUGA AND MASAO TSUZUKI

Volume 342 No. 2 June 2026

PACIFIC JOURNAL OF MATHEMATICS
Vol. 342, No. 2, 2026
https://doi.org/10.2140/pjm.2026.342.299

THE RANKIN–SELBERG INTEGRAL ON $\mathbf{GSp_2}$ FOR SQUARE-FREE LEVELS

SEIJI KUGA AND MASAO TSUZUKI

We explicitly compute the Rankin–Selberg type integral introduced by Piatetski-Shapiro over adeles for vector-valued Siegel cusp forms of square-free levels $\Gamma_0(N)$. On the way, for particular test functions in the Bessel models of irreducible admissible representations, exact evaluations of the local zeta integrals are given.

1. Introduction

Lately, the Bessel periods on the symplectic similitude group $\mathsf{G} := \mathbf{GSp}_2$ of rank 2 have been gaining in importance in the arithmetic of Siegel modular forms. The notion of Bessel period and the associated local models of admissible representations of G were originally introduced by Novodvorski and Piatetski-Shapiro in their work [27; 26; 29], where a modern treatment of Andrianov's integral representation [1; 2] of the spinor L-function for Siegel modular forms was developed. Related to this, we should also mention independent work by Sugano [42], which handles vector-valued Hilbert–Siegel cusp forms on arithmetic groups defined by maximal orders of indefinite division algebras. The Bessel period also plays an essential role in a formulation of Böcherer's conjecture posed by Liu [22] in a style of the refined Gan–Gross–Prasad conjecture. This version of Böcherer's conjecture has been completely solved by Furusawa and Morimoto [9; 10].

In this paper, we compute the Rankin–Selberg type integral a la Piatetski-Shapiro for holomorphic vector-valued Siegel cusp forms of square-free levels explicitly, using results of Pitale and Schmidt [32] on local new vectors on $\mathsf{G}(\mathbb{Q}_p)$ $(p < \infty)$; since we rely on the local-global method, our result is immediately applied to nonholomorphic Siegel cusp forms of discrete series type (cf. [20; 23]).

We explain the background of our investigation for scalar-valued Siegel modular forms of weight l. Let E be an imaginary quadratic field of discriminant $D < 0$; depending on E, we define a closed $\mathbb{Q}$-subgroup $\mathsf{G}^{\#} \hookrightarrow \mathsf{G}$ as in Section 2.1 and form the Eisenstein series $E(s, \Lambda, \mu)$ $(s \in \mathbb{C})$ on $\mathsf{G}^{\#}(\mathbb{A})$ attached to an ideal class

MSC2020: primary 11F46; secondary 11F67.
Keywords: Siegel modular forms, Rankin–Selberg integral.

character Λ of E and a finite order character $\mu : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^1$ (see Section 3.1). Our focus is an asymptotic formula as $N + l \to \infty$ of the average

$$\text{(1-1)} \qquad \frac{1}{[\mathbf{Sp}_2(\mathbb{Z}) : \Gamma_0(N)]} \sum_{\varphi \in \mathscr{B}(l,N)} \langle E(s, \Lambda, \mu) \mid \varphi \rangle \, B^{E,\Lambda}(\varphi),$$

where $B^{E,\Lambda}(\varphi)$ is the Bessel period of φ in an orthonormal basis $\mathscr{B}(l, N)$ of Siegel cusp forms of weight l and square-free level $\Gamma_0(N)$, and $\langle E(s, \Lambda, \mu) \mid \varphi \rangle$ is the L^2 inner product of $E(s, \Lambda, \mu)$ and φ restricted to $\mathsf{G}^\#(\mathbb{A})$. We suppose the ramifications of $E/\mathbb{Q}$, μ, π are mutually disjoint. Actually, for application ([19]), we consider a more general average than (1-1) by requiring $\mathscr{B}(l, N)$ to consist of joint eigenforms by Hecke operators and by putting the eigenvalue of such a Hecke operator at each φ in the formation of the sum (1-1). With such generality, we obtain a completely explicit expression of (1-1) in terms of

- the spinor L-function $L(s, \pi, \mu)$ (see Section 4.6) attached to an irreducible cuspidal representation $\pi \cong \bigotimes_p \pi_p$ twisted by μ,
- an average of Fourier coefficients of a newform φ_π^0 in π, denoted by $R(\varphi_\pi^0, E, \Lambda)$ (see (4-12)) originally introduced in [18],
- an average of the local Bessel periods (5-10) of π_p, or its computed form (5-12).

Precise statements, including the vector-valued case, are given in Proposition 5.1 and Theorem 5.2. Owing to the explicit nature of our result, we are able to show that, asymptotically as the prime level N grows to infinity with the weight l being fixed, the contribution to (1-1) of the old forms, the Yoshida lifts and the Saito–Kurokawa lifts are 0 (see Theorem 5.4).

These results are necessary in our forthcoming paper [19], where we shall work out the geometric side of a relative trace formula whose spectral side is essentially (1-1). We should remark that an average similar to our computed average (5-11) without the factors $\hat{L}(s + 1/2, \pi, \mu)$ and $\hat{f}_S(\pi)$ is studied in [7, §3]. When $E = \mathbb{Q}(i)$, our average (5-11) is almost the same as the one in [5, §8] except the appearance of the local periods $\mathfrak{t}(\pi, \mu)$.

Outline of paper. In Section 2, we introduce basic objects related to algebraic subgroups of the symplectic group G and Haar measures on their p-adic, real and adelic points. In Section 3.1, the Rankin–Selberg type integral and the global Bessel function for Siegel cusp forms are recalled; after this preparation, the basic identity (Lemma 3.2) is stated, which computes the pairing $\langle E(s, \Lambda, \mu), \varphi \rangle$ in (1-1) in terms of the Mellin transform of the global Bessel function attached to φ ([29, Theorem 5.2]). We review the proof to determine a constant depending on Haar measures exactly. In Section 4, we recall the definition of the quantity $R(\varphi, E, \Lambda)$ for an ideal class character Λ, which was originally considered in [18];

we describe its behavior under the Galois conjugation $\Lambda \mapsto \Lambda^\dagger$ (Lemma 4.6). The local multiplicity one theorem for Bessel models of π_p's allows us to split $B^{E,\Lambda}(\varphi)$ for $\varphi \in \pi$ corresponding to a pure tensor in $\bigotimes_p \pi_p$ to an Euler product of local Bessel functions for π_p up to a constant. To define our newform φ_π^0 and identify this constant with $R(\varphi_\pi^0, E, \Lambda)$ (see Lemma 4.8), we fix pairs of a local Bessel functional ℓ_p on π_p and a local new vector ξ_p of π_p such that $\ell_p(\xi_p) = 1$; in Section 4.4, we explain that the main result of [32] ensures the existence of such $\{(\ell_p, \xi_p)\}_p$ when π corresponds to a new form on $\Gamma_0(N)$. In Section 6, we compute the local zeta integrals by using results of [32]. The unramified computations are known (see Section 6.2). Other cases, as well as the explicit determination of the local periods of prime level (see Section 6.8), seem to be new, so that some details are given in Sections 6.4, 6.5 and 6.6. In Section 4.6, the functional equation of the L-function is deduced from the basic identity by using Lemma 4.6. Although a "nice" functional equation of the L-function itself is known in a broader generality ([39, Lemma 1.2]) owing to Arthur's classification of the discrete spectrum, we include this section for the sake of completeness and to provide a proof free from Arthur's result, confirming that the local L-factor and the local ε-factor defined by [29] coincides with the ones proposed in [36, Tables 2 and 3]. In Section 5, we prove Proposition 5.1 and Theorem 5.2; by invoking a result by Furusawa and Morimoto [9, Theorem 8.1], we obtain an explicit expression of (1-1) solely in terms of L-functions (Theorem 5.3).

2. Preliminaries

Let V denotes the space of symmetric matrices of degree 2; we consider V as a $\mathbb{Z}$-scheme by defining

$$\mathsf{V}(R) = \left\{ \left[\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right] \mid a, b, c \in R \right\}$$

for any commutative ring R. The group $\mathbf{GL}_2(R)$ acts on this space from the right as

$$\mathsf{V}(R) \times \mathbf{GL}_2(R) \ni (T, h) \mapsto T\mathsf{s}(h) := \frac{1}{\det h} {}^t h T h \in \mathsf{V}(R). \tag{2-1}$$

Let G denote the symplectic similitude group

$$\mathsf{G} := \left\{ g \in \mathbf{GL}_4 \;\middle|\; {}^t g \left[\begin{smallmatrix} O & 1_2 \\ -1_2 & O \end{smallmatrix}\right] g = \nu(g) \left[\begin{smallmatrix} O & 1_2 \\ -1_2 & O \end{smallmatrix}\right] \quad (\exists \nu(g) \in \mathbf{GL}_1) \right\}$$

(usually denoted by $\mathbf{GSp}_2$), where $\nu : \mathsf{G} \to \mathbf{GL}_1$ is the similitude character. The center of G is denoted by Z, which coincides with the set of all the invertible scalar matrices of degree 4. Let $\mathsf{P} = \mathsf{M}\mathsf{N}$ be the Siegel parabolic subgroup of G, where

$$\mathsf{M} := \left\{ \mathsf{m}(A, c) := \left[\begin{smallmatrix} A & O \\ O & {}^t A^{-1} c \end{smallmatrix}\right] \;\middle|\; A \in \mathbf{GL}_2, c \in \mathbf{GL}_1 \right\},$$
$$\mathsf{N} := \left\{ \mathsf{n}(X) := \left[\begin{smallmatrix} 1_2 & X \\ O & 1_2 \end{smallmatrix}\right] \;\middle|\; X \in \mathsf{V} \right\}.$$

Note that $\nu(\mathsf{m}(A,c)) = c$ and $\nu(\mathsf{n}(X)) = 1$. Let

$$\mathsf{T} := \left\{\mathsf{m}\left(\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right], c\right) \;\middle|\; a,b,c \in \mathbf{GL}_1\right\},$$

which is a maximal torus of G. The Weyl group $W := N_{\mathsf{G}}(\mathsf{T})/\mathsf{T}$ of G is generated by the images of

$$s_1 := \begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0 \end{bmatrix}, \quad s_2 := \begin{bmatrix} 0&0&1&0\\0&1&0&0\\-1&0&0&0\\0&0&0&1 \end{bmatrix}.$$

2.1. ***Embedding of groups.*** Let $E = \mathbb{Q}(\sqrt{D}) \subset \mathbb{C}$ be an imaginary quadratic field of discriminant $D < 0$, so that $\sqrt{D} = i|D|^{1/2}$ with i being the imaginary unit. Let $\mathcal{O}_E$ denote the ring of integers of E. Choose a $\mathbb{Z}$-basis $\{1,\theta\}$ of $\mathcal{O}_E$ such that

$$\theta - \bar{\theta} = -\sqrt{D}, \tag{2-2}$$

where $\tau \mapsto \bar{\tau}$ denotes the nontrivial automorphism of $E/\mathbb{Q}$.

The differential ideal $\mathfrak{d}_{E/\mathbb{Q}}$, defined by $\mathfrak{d}_{E/\mathbb{Q}}^{-1} := \{\tau \in E \mid \mathrm{tr}_{E/\mathbb{Q}}(\tau\mathcal{O}_E) \subset \mathbb{Z}\}$, is $\sqrt{D}\mathcal{O}_E$; thus a symplectic $\mathbb{Z}$-bilinear form $\langle\cdot,\cdot\rangle : \mathcal{O}_E^2 \times \mathcal{O}_E^2 \to \mathbb{Z}$ is defined by

$$\langle x,y\rangle := \mathrm{tr}_{E/\mathbb{Q}}\left(\tfrac{-1}{\sqrt{D}} \det\left[\begin{smallmatrix} x_1 & y_1 \\ x_2 & y_2 \end{smallmatrix}\right]\right), \quad x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right],\ y = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \in \mathcal{O}_E^2.$$

Note that the following vectors of $\mathcal{O}_E^2$ form a symplectic $\mathbb{Z}$-basis, i.e., $\langle v_i^+, v_j^-\rangle = \delta_{ij}$ and $\langle v_i^+, v_j^+\rangle = \langle v_i^-, v_j^-\rangle = 0$ for all i,j:

$$v_1^+ := \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \quad v_2^+ := \left[\begin{smallmatrix} \theta \\ 0 \end{smallmatrix}\right], \quad v_1^- := \left[\begin{smallmatrix} 0 \\ -\bar{\theta} \end{smallmatrix}\right], \quad v_2^- := \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right].$$

For any ring R, set

$$\mathsf{G}^\#(R) := \{h \in \mathbf{GL}_2(\mathcal{O}_E \otimes_{\mathbb{Z}} R) \mid \det(h) \in R^\times\}.$$

For $h \in \mathsf{G}^\#(R)$, viewed as an R-linear automorphism of $(\mathcal{O}_E \otimes_{\mathbb{Z}} R)^2$, let $\iota_\theta(h)$ denote the 4×4-matrix representing h in the R-basis $\{v_1^+, v_2^+, v_1^-, v_2^-\}$ of $(\mathcal{O}_E \otimes_{\mathbb{Z}} R)^2$, i.e.,

$$[h(v_1^+), h(v_2^+), h(v_1^-), h(v_2^-)] = [v_1^+, v_2^+, v_1^-, v_2^-]\,\iota_\theta(h). \tag{2-3}$$

Then $\iota_\theta(h) \in \mathsf{G}(R)$ and $\nu(\iota_\theta(h)) = \det h$. Thus, we have an embedding

$$\iota_\theta : \mathsf{G}^\#(R) \to \mathsf{G}(R).$$

Let $\mathsf{B}^\#$ be the Borel subgroup of $\mathsf{G}^\#$ such that $\mathsf{B}^\#(R)$ coincides with the set of all points

$$\left[\begin{smallmatrix} a\tau & \beta \\ 0 & \tau^{-1} \end{smallmatrix}\right], \quad \tau \in (E \otimes_{\mathbb{Q}} R)^\times,\ a \in R^\times,\ \beta \in R. \tag{2-4}$$

Let $\mathsf{N}^{\#}$ denote the unipotent radical of $\mathsf{B}^{\#}$, i.e., $\mathsf{N}^{\#}(R)$ is the set of points (2-4) with $\tau = a = 1$. Set $\mathsf{Z}^{\#} := \{a1_2 \mid a \in \mathbf{GL}_1\}$; then $\mathsf{Z}^{\#}$ is a subgroup of the center of $\mathsf{G}^{\#}$ of index 2.

Let $I_\theta : E \to \mathbf{M}_2(\mathbb{Q})$ denote the matrix representation of the regular representation of the $\mathbb{Q}$-algebra E with respect to the basis $\{1, \theta\}$, i.e.,

$$[\tau, \tau\theta] = [1, \theta]\, I_\theta(\tau), \quad \tau \in E, \tag{2-5}$$

or, explicitly,

$$I_\theta(a + b\theta) = \begin{bmatrix} a & -b\mathrm{N}_{E/\mathbb{Q}}(\theta) \\ b & a + b\,\mathrm{tr}_{E/\mathbb{Q}}(\theta) \end{bmatrix}, \quad a, b \in \mathbb{Q}. \tag{2-6}$$

For $\beta = b_2 + b_3\theta \in E_R$ with $b_2, b_3 \in R$, define an element X_β of $\mathsf{V}(R)$ as

$$X_\beta := \left[\begin{smallmatrix} b_1 & b_2 \\ b_2 & b_3 \end{smallmatrix}\right] \quad \text{with } b_1 := -b_2\,\mathrm{tr}_{E/\mathbb{Q}}(\theta) - b_3\mathrm{N}_{E/\mathbb{Q}}(\theta). \tag{2-7}$$

We have

$$\iota_\theta\left(\left[\begin{smallmatrix} \tau & 0 \\ 0 & a\tau^{-1} \end{smallmatrix}\right]\right) = \mathsf{m}(I_\theta(\tau), a), \quad a \in R^\times,\ \tau \in E_R^\times, \tag{2-8}$$

$$\iota_\theta\left(\left[\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right]\right) = \mathsf{n}(X_\beta), \quad \beta \in E_R. \tag{2-9}$$

By formulas (2-6) and (2-8), we have $\iota_\theta(a1_2) = a1_4$ for ($a \in R^\times$), which implies

$$\iota_\theta(\mathsf{Z}^{\#}) = \mathsf{Z}. \tag{2-10}$$

We then have $\mathrm{N}_{E/\mathbb{Q}}(x + \theta y) = [x, y]\, T\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ ($x, y \in R$) with

$$T_\theta := \begin{bmatrix} 1 & 2^{-1}\,\mathrm{tr}_{E/\mathbb{Q}}(\theta) \\ 2^{-1}\,\mathrm{tr}_{E/\mathbb{Q}}(\theta) & \mathrm{N}_{E/\mathbb{Q}}(\theta) \end{bmatrix}. \tag{2-11}$$

Let $\mathsf{V}^{T_\theta}(\mathbb{Q})$ be the orthogonal of T_θ in $\mathsf{V}(\mathbb{Q})$ with respect to the nondegenerate quadratic form $\mathrm{tr}(XY)$ on $\mathsf{V}(\mathbb{Q})$, i.e.,

$$\mathsf{V}^{T_\theta}(\mathbb{Q}) := \{X \in \mathsf{V}(\mathbb{Q}) \mid \mathrm{tr}(T_\theta X) = 0\}.$$

Note that $\mathrm{tr}(T_\theta^2) = 1 + (\mathrm{tr}_{E/\mathbb{Q}}(\theta)/2)^2 + \mathrm{N}_{E/\mathbb{Q}}(\theta)^2 > 0$, $\det(T_\theta) = -D/4 > 0$ by (2-2). We have $\mathsf{V}^{T_\theta}(\mathbb{Q}) = \{X_\beta \mid \beta \in E\}$, and

$$\mathsf{V}(\mathbb{Q}) = \mathbb{Q}T_\theta \oplus \mathsf{V}^{T_\theta}(\mathbb{Q}), \qquad \iota_\theta(\mathsf{N}^{\#}) = \mathsf{n}(\mathsf{V}^{T_\theta}) \tag{2-12}$$

by (2-9). The group M acts on the space of rational homomomorhisms $\mathrm{Hom}(\mathsf{N}, \mathbb{G}_a)$ by the rule $\mathrm{Ad}^*(m)\chi(n) = \chi(m^{-1}nm)$ for $m \in \mathsf{M}$, $n \in \mathsf{N}$ and $\chi \in \mathrm{Hom}(\mathsf{N}, \mathbb{G}_a)$. For $T \in \mathsf{V}$, let M_T denote the stabilizer of $\chi_T : \mathsf{n}(X) \mapsto \mathrm{tr}(TX)$, and M_T° the identity component with respect to the Zariski topology. Then, for any $\mathbb{Q}$-algebra R,

$$\mathsf{M}_{T_\theta}^\circ(R) = \{\mathsf{m}(I_\theta(\tau), \mathrm{N}_{E/\mathbb{Q}}(\tau)) \mid \tau \in E_R\}.$$

Define a bilinear form on $\mathsf{V}(R)$ as $(X,Y) := -\operatorname{tr}(XY^\dagger)$ and let $\mathbf{SO}(\mathsf{V}(R))$ be the special orthogonal group of $(\cdot,\cdot)$.

2.2. ***Open compact subgroups at finite places.*** Let p be a prime number. Set $E_p := E \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $\mathscr{O}_{E,p} := \mathscr{O}_E \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Then, $\mathbf{K}_p^\# := \mathsf{G}^\#(\mathbb{Z}_p)$, which coincides with the stabilizer in $\mathsf{G}^\#(\mathbb{Q}_p)$ of the $\mathscr{O}_{E,p}$-lattice $\mathscr{O}_{E,p}^2 \subset E_p^2$, is a maximal compact subgroup of $\mathsf{G}^\#(\mathbb{Q}_p)$, and $\mathbf{K}_p := \mathsf{G}(\mathbb{Z}_p)$ is the standard maximal compact subgroup of $\mathsf{G}(\mathbb{Q}_p)$. For a nonzero ideal $\mathfrak{n} \subset \mathbb{Z}_p$, set

$$\mathbf{K}_0(\mathfrak{n}) := \left\{\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathsf{G}(\mathbb{Z}_p) \,\middle|\, C \in \mathfrak{n}\right\}.$$

For a nonzero ideal $\mathfrak{a} \subset \mathscr{O}_{E,p}$, set

$$\mathbf{K}_0^\#(\mathfrak{a}) := \left\{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathsf{G}^\#(\mathbb{Z}_p) \,\middle|\, c \in \mathfrak{a}\right\}.$$

Thus, $\mathbf{K}_0(\mathbb{Z}_p) = \mathbf{K}_p$ and $\mathbf{K}_0^\#(\mathscr{O}_{E,p}) = \mathbf{K}_p^\#$.

Lemma 2.1. *For a nonzero ideal* $\mathfrak{n} \subset \mathbb{Z}_p$, *we have* $\mathbf{K}_0^\#(\mathfrak{n}\mathscr{O}_{E,p}) = \iota_\theta^{-1}(\mathbf{K}_0(\mathfrak{n}))$.

Proof. Indeed, both $\iota_\theta^{-1}(\mathbf{K}_p)$ and $\mathbf{K}_p^\#$ coincides with the stabilizer of $\mathscr{O}_{E,p}^2 = \langle v_1^+, v_2^+, v_1^-, v_2^-\rangle_{\mathbb{Z}_p}$ in $\mathsf{G}^\#(\mathbb{Q}_p)$. Hence $\iota_\theta^{-1}(\mathbf{K}_p) = \mathbf{K}_p^\#$. In the remaining part of the proof, we suppose $\mathfrak{n} \subset p\mathbb{Z}_p$. Then for $g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathbf{K}_p$, we have $g \in \mathbf{K}_0(\mathfrak{n})$ if and only if $g(v_1^+), g(v_2^+) \in \langle v_1^+, v_2^+\rangle_{\mathbb{Z}_p} + \mathfrak{n}\langle v_1^-, v_2^-\rangle_{\mathbb{Z}_p}$. For $g = \iota_\theta(h)$ with $h = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathbf{K}_p^\#$, this last condition becomes $\left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} a\theta \\ c\theta \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathscr{O}_{E,p} \\ \mathfrak{n}\mathscr{O}_{E,p} \end{smallmatrix}\right]$, or, equivalently, $c \in \mathfrak{n}\mathscr{O}_{E,p}$. □

For $N \in \mathbb{Z}_{>0}$, define open compact subgroups $\mathbf{K}_0(N) \subset \mathsf{G}(\mathbb{A}_\mathbf{f})$ and $\mathbf{K}^\#(N\mathscr{O}_E) \subset \mathsf{G}^\#(\mathbb{A}_\mathbf{f})$ by

$$\mathbf{K}_0(N) := \prod_{p<\infty} \mathbf{K}_0(N\mathbb{Z}_p), \quad \mathbf{K}_0^\#(N\mathscr{O}_E) := \prod_{p<\infty} \mathbf{K}_0^\#(N\mathscr{O}_{E,p}).$$

2.3. ***Maximal compact subgroup at the archimedean place.*** The identity connected component of $\mathsf{G}(\mathbb{R})$ is $\mathsf{G}(\mathbb{R})^0 = \{g \in \mathsf{G}(\mathbb{R}) \mid \nu(g) > 0\}$. Set $\mathbf{K}_\infty := \mathsf{G}(\mathbb{R})^0 \cap \mathbf{O}(4)$, which is a maximal compact subgroup of $\mathsf{G}(\mathbb{R})^0$ given as

$$\mathbf{K}_\infty = \left\{\left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right] \,\middle|\, A, B \in \mathbf{M}_2(\mathbb{R}),\ A + iB \in \mathbf{U}(2)\right\}.$$

Note that $\mathbf{K}_\infty \subset \mathbf{Sp}_2(\mathbb{R})$. The action of $\mathsf{G}(\mathbb{R})^0$ on the Siegel upper-half space $\mathfrak{h}_2 := \{Z \in \mathbf{M}_2(\mathbb{C}) \mid {}^tZ = Z,\ \operatorname{Im}(Z) \gg 0\}$, denoted by $(g, Z) \mapsto g\langle Z\rangle$, is defined by the usual formula, i.e.,

$$g\langle Z\rangle := (AZ+B)(CZ+D)^{-1}, \quad g = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathsf{G}(\mathbb{R})^0, \quad Z \in \mathfrak{h}_2.$$

The stabilizer of $i1_2 \in \mathfrak{h}_2$ in $\mathsf{G}(\mathbb{R})^0$ coincides with $\mathsf{Z}(\mathbb{R})\mathbf{K}_\infty$ and the map ϕ : $\mathsf{G}(\mathbb{R})^0 \ni g \mapsto g\langle i1_2\rangle \in \mathfrak{h}_2$ induces a diffeomorphism $\mathsf{G}(\mathbb{R})^0/\mathsf{Z}(\mathbb{R})\mathbf{K}_\infty \cong \mathfrak{h}_2$. Let $\mathfrak{p}$ denote the subspace of $\mathfrak{g} := \operatorname{Lie}(\mathsf{G}(\mathbb{R})^0)$ that is mapped bijectively onto the tangent space of $\mathfrak{h}_2$ at $i1_2$ under the tangent map of ϕ at 1_4. Let $\mathfrak{p}^+$ $(\subset \mathfrak{p}_\mathbb{C})$ be the space of holomorphic tangent vectors and $\mathfrak{p}^-$ the antiholomorphic ones. Set $\gamma_\infty := \mathsf{m}(-1_2, -1) = \left[\begin{smallmatrix} -1_2 & 0 \\ 0 & 1_2 \end{smallmatrix}\right] \in \mathsf{G}(\mathbb{R})$; then $\mathbf{K}_\infty \cup \mathbf{K}_\infty\gamma_\infty$ is a maximal compact subgroup of $\mathsf{G}(\mathbb{R})$. The four vectors

$$u_1 := \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \quad u_2 := \left[\begin{smallmatrix} i \\ 0 \end{smallmatrix}\right], \quad u_3 := \left[\begin{smallmatrix} 0 \\ -i \end{smallmatrix}\right], \quad u_4 := \left[\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right]$$

in $E_{\mathbb{R}}^2$ form an $\mathbb{R}$-basis such that

$$[v_1^+, v_2^+, v_1^-, v_2^-] = [u_1, u_2, u_3, u_4]\,(b_{\mathbb{R}}^\theta)^{-1} \tag{2-13}$$

with

$$b_{\mathbb{R}}^\theta := \mathsf{m}(A_\theta, 2^{-1}\sqrt{|D|})^{-1}, \quad A_\theta := \begin{bmatrix} 1 & 2^{-1}\,\mathrm{tr}_{E/\mathbb{Q}}(\theta) \\ 0 & -2^{-1}\sqrt{|D|} \end{bmatrix} \in \mathbf{GL}_2(\mathbb{R}). \tag{2-14}$$

Note that $b_{\mathbb{R}}^\theta \in \mathsf{G}(\mathbb{R})^0$ because $\nu(b_{\mathbb{R}}^\theta)^{-1} = 2^{-1}\sqrt{|D|} > 0$. Set

$$\mathbf{K}_\infty^\# := \mathbf{U}(2) \cap \mathsf{G}^\#(\mathbb{R}) = \{k^\# \in \mathbf{U}(2) \mid \det(k^\#) = \pm 1\},$$

which is a maximal compact subgroup of $\mathsf{G}^\#(\mathbb{R})$. The identity connected component $(\mathbf{K}_\infty^\#)^0$ of $\mathbf{K}_\infty^\#$ is $\mathbf{SU}(2)$, and $\mathbf{K}_\infty^\# = (\mathbf{K}_\infty^\#)^0 \cup (\mathbf{K}_\infty^\#)^0\delta_\infty$ with $\delta_\infty := \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. A general element of $\mathbf{SU}(2)$ is written in the form

$$h = \left[\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right] \quad \text{with } a = a' + ia'', b = b' + ib'' \in \mathbb{C},\ |a|^2 + |b|^2 = 1. \tag{2-15}$$

For such an h, a computation reveals the relation

$$[hu_1, hu_2, hu_3, hu_4] = [u_1, u_2, u_3, u_4]\left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right] \quad \text{with } A := \left[\begin{smallmatrix} a' & -a'' \\ a'' & a' \end{smallmatrix}\right],\ B := \left[\begin{smallmatrix} b'' & b' \\ b' & -b'' \end{smallmatrix}\right]. \tag{2-16}$$

Lemma 2.2. *We have* $\iota_\theta^{-1}(\mathbf{K}_\infty) = (\mathbf{K}_\infty^\#)^0$. *For* $k_\infty^\# \in (\mathbf{K}_\infty^\#)^0$ *as in* (2-15), *defining* $A, B \in \mathbf{M}_2(\mathbb{R})$ *as in* (2-16), *we have*

$$\iota_\theta(k_\infty^\#) = b_{\mathbb{R}}^\theta \left[\begin{smallmatrix} A & B \\ -B & A \end{smallmatrix}\right] (b_{\mathbb{R}}^\theta)^{-1}. \tag{2-17}$$

Proof. Equation (2-17) follows directly from (2-4), (2-16) and (2-13). From

$$[hu_1, hu_2, hu_3, hu_4] = [u_1, u_2, u_3, u_4](-\gamma)$$

and $-\gamma \notin \mathbf{K}_\infty$, the assertion follows. □

2.4. ***Haar measures.*** For locally compact unimodular topological groups H relevant to us, we fix Haar measures η_H on H in the following manner. Let $\mathbb{A}$ be the adele ring of $\mathbb{Q}$ and $\psi : \mathbb{A}/\mathbb{Q} \to \mathbb{C}^1$ the basic character.

2.4.1. On the additive group $\mathbb{C}$, define $d\eta_{\mathbb{C}}(\tau) := dx\,dy = 2^{-1}|d\tau \wedge d\bar{\tau}|$ for $\tau = x + iy$ ($x, y \in \mathbb{R}$), where $dx\,dy$ is the Lebesgue measure on $\mathbb{R}^2$.

The Haar measure $\eta_{\mathbb{C}^\times}$ on $\mathbb{C}^\times$ (resp. $\eta_{\mathbb{R}^\times}$ on $\mathbb{R}^\times$) is defined by $d\eta_{\mathbb{C}^\times}(\tau) = |\tau|_{\mathbb{C}}^{-1} d\tau$ for $\tau = x + iy \in \mathbb{C}$ (resp. $|x|_{\mathbb{R}}^{-1} dx$). For each $p < \infty$, $E_p^\times$ (resp. $\mathbb{Q}_p^\times$) is endowed with the Haar measure $\eta_{E_p^\times}$ such that $\eta_{E_p^\times}(\mathscr{O}_{E,p}^\times) = 1$ (resp. $\eta_{\mathbb{Q}_p^\times}(\mathbb{Z}_p^\times) = 1$). Then, viewing $\mathbb{A}_E^\times$ as the restricted product of $E_p^\times$, we define $\eta_{\mathbb{A}_E^\times} = \prod_{p \le \infty} \eta_{E_p^\times}$. Similarly, we set $\eta_{\mathbb{A}^\times} = \prod_{p \le \infty} \eta_{\mathbb{Q}_p^\times}$.

2.4.2. For a matrix $X = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, set $X^\dagger := \left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$, so that $XX^\dagger = (\det X)\, 1_2$, and $X \mapsto X^\dagger$ induces an involution of the $\mathbb{Q}$-vector space $\mathsf{V}(\mathbb{Q})$, which is viewed as a quadratic space with the symmetric bilinear form $\operatorname{tr}(XY^\dagger)$. We endow $\mathsf{V}(\mathbb{A})$ with the self-dual Haar measure with respect to the self-duality defined by the bicharacter $(X, Y) \mapsto \psi(\operatorname{tr}(XY^\dagger))$. Let $\eta_{\mathbb{A}}$ (resp. $\eta_{\mathbb{A}_E}$) be the self-dual Haar measure on $\mathbb{A}$ (resp. $\mathbb{A}_E$) with respect to the bicharacter $(x, y) \mapsto \psi(xy)$ (resp. $(\alpha, \beta) \mapsto \psi(\operatorname{tr}_{E/\mathbb{Q}}(\alpha\bar{\beta}))$). A computation shows the relation $-\operatorname{tr}(X_\alpha X_\beta^\dagger) = \operatorname{tr}_{E/\mathbb{Q}}(\alpha\bar{\beta})$ for $\alpha, \beta \in \mathbb{A}_E$, where $X_\alpha, X_\beta \in V^{T_\theta}(\mathbb{A})$ are defined by (2-7). Thus, by the isomorphism $\mathbb{A}_E \ni \beta \mapsto X_\beta \in \mathsf{V}^{T_\theta}(\mathbb{A})$, the Haar measure $\eta_{\mathbb{A}_E}$ is transferred to a Haar measure on $\mathsf{N}^\#(\mathbb{A})$. Since $\operatorname{tr}(T_\theta T_\theta^\dagger) = -D/2 \ne 0$, we have the orthogonal direct sum decomposition $\mathsf{V}(\mathbb{Q}) = \mathbb{Q}T_\theta^\dagger \oplus \mathsf{V}^{T_\theta}(\mathbb{Q})$. By this and (2-12), write $X \in V(\mathbb{A})$ as $X = xT_\theta^\dagger + X_\beta = yT_\theta + X_\alpha$ ($x, y \in \mathbb{A}$, $\beta, \alpha \in \mathbb{A}_E$). By the definition of the Haar measures, we have $d\eta_{\mathsf{V}(\mathbb{A})}(X) = d\eta_{\mathbb{A}}(x) \otimes d\eta_{\mathbb{A}_E}(\beta)$. Since the change of variables $(x, \beta) \to (y, \alpha)$ is given as

$$y = \tfrac{-D}{2}\, x, \qquad \alpha = \beta + 2^{-1} \operatorname{tr}_{E/\mathbb{Q}}(\theta)(x + y) + (\mathrm{N}_{E/\mathbb{Q}}(\theta)x - y)\,\theta$$

and since $|-D/2|_{\mathbb{A}} = 1$, we get $d\eta_{\mathsf{V}(\mathbb{A})}(X) = d\eta_{\mathbb{A}}(y) \otimes d\eta_{\mathbb{A}_E}(\alpha)$. Through the identification $\mathsf{V}(\mathbb{A}) \ni X \mapsto \mathsf{n}(X) \in \mathsf{N}(\mathbb{A})$ and $\beta \ni \mathbb{A}_E \mapsto \left[\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right] \in \mathsf{N}^\#(\mathbb{A})$, the groups $\mathsf{N}(\mathbb{A})$ and $\mathsf{N}^\#(\mathbb{A})$ acquire Haar measures, so that the integration formula

$$\int_{\mathsf{N}(\mathbb{A})} f(n)\, dn = \int_{\mathbb{A}} \int_{\mathsf{N}^\#(\mathbb{A})} f(\mathsf{n}(xT_\theta)\, \iota_\theta(n^\#))\, dx\, dn^\# \tag{2-18}$$

holds for any $f \in L^1(\mathsf{N}(\mathbb{A}))$. Note that $\operatorname{vol}(\mathsf{N}(\mathbb{Q})\backslash\mathsf{N}(\mathbb{A})) = \operatorname{vol}(\mathsf{N}^\#(\mathbb{Q})\backslash\mathsf{N}^\#(\mathbb{A})) = 1$.

2.4.3. Let $p \le \infty$. Any element $g^\# \in \mathsf{G}^\#(\mathbb{Q}_p)$ is written as

$$g^\# = \left[\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right]\left[\begin{smallmatrix} \tau & 0 \\ 0 & \bar{\tau}a \end{smallmatrix}\right] k^\# \tag{2-19}$$

with $a \in \mathbb{Q}_p^\times$, $\tau \in E_p^\times$, $\beta \in E_p$ and $k^\# \in \mathbf{K}_p^\#$. Then, our Haar measure on $\mathsf{G}^\#(\mathbb{Q}_p)$ is symbolically defined as

$$d\eta_{\mathsf{G}^\#(\mathbb{Q}_p)}(g_p^\#) = |a|_p^2 \cdot d\eta_{E_p}(\beta)\, d\eta_{\mathbb{Q}_p^\times}(a)\, d\eta_{E_p^\times}(\tau)\, d\eta_{\mathbf{K}_p^\#}(k^\#), \tag{2-20}$$

where $d\eta_{\mathbf{K}_p^\#}(k^\#)$ is the Haar measure on $\mathbf{K}_p^\#$ with volume 1. For $p < \infty$, we have $\int_{\mathbf{K}_p^\#} \eta_{\mathsf{G}^\#(\mathbb{Q}_p)} = 1$. On the adele group $\mathsf{G}^\#(\mathbb{A})$, we use the product measure of $\eta_{\mathsf{G}^\#(\mathbb{Q}_p)}$

($p \le \infty$). Then, from (2-20), we get

$$\int_{\mathsf{G}^\#(\mathbb{A})} f(g^\#)\,\mathrm{d}g^\# = \frac{\sqrt{|D|}}{2}\int_{\mathsf{N}^\#(\mathbb{A})}\int_{\mathbb{A}^\times}\int_{\mathbb{A}_E^\times}\int_{\mathbf{K}^\#} f\left(n^\#\left[\begin{smallmatrix}\tau & 0\\ 0 & a\bar\tau\end{smallmatrix}\right]k^\#\right)|a|_{\mathbb{A}}^2\,\mathrm{d}n^\#\,\mathrm{d}\eta_{\mathbb{A}^\times}(a)\,\mathrm{d}\eta_{\mathbb{A}_E^\times}(\tau)\,\mathrm{d}k^\#, \tag{2-21}$$

where $\mathrm{d}k^\#$ is the normalized Haar measure on $\mathbf{K}^\# = \prod_{p\le\infty}\mathbf{K}_p^\#$. Note that the measure on $\mathsf{N}^\#(\mathbb{A}) \cong \mathbb{A}_E$ coincides with the $\eta_{\mathbb{A}_E}$ defined above, which equals $(\sqrt{|D|}/2)\times\prod_p \eta_{E_p}$ ($p\le\infty$) due to the formulas $\big(\prod_p \eta_{E_p}\big)(\mathbb{A}_E/E) = |D|^{1/2}/2$ [47, Chapter V, §4 Proposition 7] and $\eta_{\mathbb{A}_E}(\mathbb{A}_E/E) = 1$.

2.4.4. We fix $\eta_{\mathbf{K}_\infty}$ so that $\mathrm{vol}(\mathbf{K}_\infty) = 1$. Then we normalize $\eta_{\mathbf{Sp}_2(\mathbb{R})}$ in such a way that the quotient $\eta_{\mathbf{Sp}_2(\mathbb{R})}/\eta_{\mathbf{K}_\infty}$ corresponds to the measure $(\det Z)^{-3}\mathrm{d}X\,\mathrm{d}Y$ on $\mathfrak{h}_2 \cong \mathbf{Sp}_2(\mathbb{R})/\mathbf{K}_\infty$. Via $\mathsf{G}(\mathbb{R})^0 = \mathsf{Z}(\mathbb{R})^0\,\mathbf{Sp}_2(\mathbb{R}) \cong \mathbb{R}_{>0}\times\mathbf{Sp}_2(\mathbb{R})$, $\eta_{\mathsf{G}(\mathbb{R})}$ is defined by demanding that its restriction to $\mathsf{G}(\mathbb{R})^0$ is $\eta_{\mathbb{R}^\times}\otimes\eta_{\mathbf{Sp}_2(\mathbb{R})}$. For $p<\infty$, we fix $\eta_{\mathsf{G}(\mathbb{Q}_p)}$ by demanding $\int_{\mathbf{K}_p}\eta_{\mathsf{G}(\mathbb{Q}_p)} = 1$. Then, $\eta_{\mathsf{G}(\mathbb{A})}$ is defined to be the restricted product of $\eta_{\mathsf{G}(\mathbb{Q}_p)}$ ($p\le\infty$).

2.5. ***Idele class characters.*** Let Λ denote a character of the finite group

$$\mathrm{Cl}(E) := \mathbb{A}_E^\times/E^\times E_{\mathbb{R}}^\times\widehat{\mathscr{O}}_E^\times.$$

As is well known, this group is isomorphic to the ideal class group of E. We regard Λ as an idele class character of $E^\times$ of finite order. Since $\mathbb{A}^\times = \mathbb{Q}^\times(\mathbb{R}_{>0})\widehat{\mathbb{Z}}^\times$ is contained in the subgroup $E^\times E_{\mathbb{R}}^\times\widehat{\mathscr{O}}_E^\times$, we have

$$\Lambda|\mathbb{A}^\times = \mathbf{1}. \tag{2-22}$$

Let v be a place of E and Λ_v the v-component of Λ, i.e., $\Lambda = \bigotimes_v \Lambda_v$. We say that v is inert in $E/\mathbb{Q}$, splits in $E/\mathbb{Q}$ or ramifies in $E/\mathbb{Q}$ if $E_v := E\otimes_{\mathbb{Q}}\mathbb{Q}_p$ is an unramified field extension of $\mathbb{Q}_p$, is isomorphic to $\mathbb{Q}_p\oplus\mathbb{Q}_p$, or is a ramified field extension of $\mathbb{Q}_p$, respectively. Since Λ is supposed to be trivial on $E_{\mathbb{R}}^\times$, we have $\Lambda_\infty = \mathbf{1}$; moreover,

$$\text{If } v \text{ is inert in } E/\mathbb{Q}, \text{ then } \Lambda_v = \mathbf{1}. \tag{2-23}$$

Indeed, since Λ is supposed to be trivial on $\widehat{\mathscr{O}}_E^\times$, the restriction $\Lambda_v|\mathscr{O}_{E,v}^\times$ is trivial. If v is inert and p denote the residue characteristic of E_v, then $p\in\mathbb{Q}_p$ is a prime element of the local field E_v. We have $\Lambda_v(p) = 1$ due to (2-22). Hence, Λ_v is trivial on $p^{\mathbb{Z}}\mathscr{O}_{E,v}^\times = E_p^\times$.

Define the Galois conjugate $\Lambda^\dagger$ of Λ by setting

$$\Lambda^\dagger(\tau) := \Lambda(\bar\tau), \quad \tau\in\mathbb{A}_E^\times. \tag{2-24}$$

We have $\Lambda(\tau)\Lambda^{\dagger}(\tau) = \Lambda(\tau\bar{\tau}) = 1$ for $\tau \in \mathbb{A}_E^{\times}$ by (2-22). Hence

$$\Lambda^{\dagger} = \Lambda^{-1} = \bar{\Lambda}, \tag{2-25}$$

where $\bar{\Lambda}$ is the complex conjugate of Λ.

For an idele class character ξ of $E^{\times}$, let $\hat{L}(s,\xi)$ be the completed Hecke L-function of ξ, and $L_p(s,\xi)$ its local p-factor for $p \le \infty$. Then, $L_p(s,\xi^{\dagger}) = L_p(s,\xi)$ for any $p < \infty$. Indeed, if p is not ramified in $E/\mathbb{Q}$, the equality is trivial. Suppose that $E/\mathbb{Q}$ is ramified; if ξ_p is a ramified character of $E_p^{\times}$, then both L-factors are 1. If ξ_p is unramified, then for any prime element ϖ of E_p, we have $(\xi^{\dagger})_p(\varpi) = \xi_p(\bar{\varpi}) = \xi_p(\varpi)$, which implies the equality between local p-factors.

For any character $\mu : \mathbb{A}^{\times}/\mathbb{Q}^{\times}\mathbb{R}_{>0} \to \mathbb{C}^1$, define $\mu_E := \mu \circ \mathrm{N}_{E/\mathbb{Q}} : \mathbb{A}_E^{\times}/E^{\times} \to \mathbb{C}^1$. Then, μ_E is Galois invariant, and $\mu_E | E_{\infty}^{\times} = \mathbf{1}$. Hence, $(\Lambda\mu_E)^{\dagger} = \Lambda^{\dagger}\mu_E^{\dagger} = \Lambda^{-1}\mu_E$, and

$$\hat{L}(s,\Lambda^{-1}\mu_E) = \hat{L}(s,\Lambda^{\dagger}\mu_E) = \hat{L}(s,\Lambda\mu_E) \quad (\mathrm{Re}(s) > 1). \tag{2-26}$$

Note that $\hat{L}(s,\Lambda\mu_E)$ is holomorphic except for possible simple poles at $s = 1, 0$, which occurs if and only if both Λ and μ are trivial.

3. Eisenstein series and Rankin–Selberg integral

Let $\Lambda = \bigotimes_{p \le \infty} \Lambda_p \in \widehat{\mathrm{Cl}(E)}$ and $\mu = \bigotimes_{p \le \infty} \mu_p \in \widehat{\mathbb{A}^{\times}/\mathbb{Q}^{\times}(\mathbb{R}_{>0})^{\times}}$.

3.1. *Eisenstein series.* For details, we refer to [14, §19]; the theory on $\mathbf{GL}_2(\mathbb{A}_E)$ developed there carries over into the group $\mathsf{G}^{\#}(\mathbb{A})$ with minor modifications. For a finite-dimensional vector space V over a local field, let $\mathcal{S}(V)$ be the space of all Schwartz–Bruhat functions on V. For $p \le \infty$, $\phi \in \mathcal{S}(E_p^2)$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, we define a function $f_{\phi}^{(s,\Lambda_p,\mu_p)} : \mathsf{G}^{\#}(\mathbb{Q}_p) \to \mathbb{C}$ by

$$f_{\phi}^{(s,\Lambda_p,\mu_p)}(g^{\#}) = \mu_p(\det g^{\#})\,|\det g^{\#}|_p^{s+1} \int_{E_p^{\times}} \phi([\,0\ 1\,][\begin{smallmatrix}\tau & 0\\ 0 & \bar{\tau}\end{smallmatrix}]g^{\#})\Lambda_p\mu_{E,p}(\tau)|\tau\bar{\tau}|_p^{s+1}\,d\eta_{E_p^{\times}}(\tau). \tag{3-1}$$

When $p = \infty$, we assume that ϕ is $\mathbf{K}_{\infty}^{\#}$-finite. By local Tate theory, the function $s \mapsto f_{\phi}^{(s,\Lambda_p,\mu_p)}(g^{\#})$ is continued meromorphically to $\mathbb{C}$ in such a way that

$$\mathsf{f}_{\phi}^{(s,\Lambda_p,\mu_p)}(g^{\#}) := L_p(s+1,\Lambda_p\mu_{E,p})^{-1} \times f_{\phi}^{(s,\Lambda_p,\mu_p)}(g^{\#}) \tag{3-2}$$

is holomorphic on $\mathbb{C}$. Then, $\mathsf{f}_{\phi}^{(s,\Lambda_p,\mu_p)}$ belongs to the space $\mathscr{V}^{\#}(s,\Lambda_p,\mu_p)$ consisting of all smooth functions f on $\mathsf{G}^{\#}(\mathbb{Q}_p)$ satisfying

$$f([\begin{smallmatrix}\tau & \beta\\ 0 & a\bar{\tau}\end{smallmatrix}]g^{\#}) = \Lambda_p(\tau)^{-1}\mu_p(a)^{-1}|a|_p^{-(s+1)}f(g^{\#})$$

for any $\left[\begin{smallmatrix}\tau & \beta\\ 0 & a\bar{\tau}\end{smallmatrix}\right] \in \mathsf{B}^{\#}(\mathbb{Q}_p)$ and $g^{\#} \in \mathsf{G}^{\#}(\mathbb{Q}_p)$.

The Fourier transform $\widehat{\phi}$ of $\phi \in \mathcal{S}(E_p^2)$ is defined by

$$\widehat{\phi}(x,y) := \int_{E_p^2} \phi(u,v)\psi(\langle [\begin{smallmatrix} x \\ y \end{smallmatrix}], [\begin{smallmatrix} u \\ v \end{smallmatrix}]\rangle)\, d\eta_{E_p}(u)\, d\eta_{E_p}(v). \tag{3-3}$$

Set $w_0 := \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \in \mathsf{G}^\#(\mathbb{Q}_p)$. The standard intertwining operator

$$M(s) : \mathscr{V}^\#(s, \Lambda_p, \mu_p) \to \mathscr{V}^\#(-s, \Lambda_p^{-1}, \mu_p^{-1})$$

is defined on $\mathrm{Re}(s) > 0$ as the absolutely convergent integral

$$M(s) f(g^\#) = \int_{E_p} f\big(w_0 \left[\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right] g^\#\big)\, d\eta_{E_p}(\beta), \quad g^\# \in \mathsf{G}^\#(\mathbb{Q}_p).$$

The effect of $M(s)$ on the section $f_\phi^{(s,\Lambda_p,\mu_p)}$ is described by the Fourier transform $\widehat{\phi}$ as

$$f_{\hat{\phi}}^{(-s,\Lambda_p^{-1},\mu_p^{-1})}(g^\#) = \tag{3-4}$$
$$c_p \Lambda_p(\sqrt{D})^{-1} |D|_p^{-s+1/2} \varepsilon_p(s, \Lambda\mu_E, \psi_{E_p}) \frac{L_p(1-s, \Lambda^{-1}\mu_E^{-1})}{L_p(s, \Lambda\mu_E)} M(s) f_\phi^{(s,\Lambda_p,\mu_p)}(g^\#)$$

with $c_p = 1$ if $p < \infty$ and $c_\infty = \sqrt{|D|}/2$ if $p = \infty$, where $\varepsilon(s, \cdot, \psi_{E_p})$ denote Tate's local epsilon factor defined by the character $\psi_{E_p} := \psi_p \circ \mathrm{tr}_{E_p/\mathbb{Q}_p}$ of E_p and the associated self-dual measure on E_p as usual. Let $\mathcal{S}(\mathbb{A}_E^2)$ be the space of all the Schwartz–Bruhat functions on $\mathbb{A}_E^2$. For a decomposable element $\phi = \bigotimes_{p \le \infty} \phi_p$ in $\mathcal{S}(\mathbb{A}_E^2)$, we define

$$\mathsf{f}_\phi^{(s,\Lambda,\mu)}(g^\#) = \prod_{p \le \infty} \mathsf{f}_{\phi_p}^{(s,\Lambda_p,\mu_p)}(g_p^\#), \quad g^\# = (g_p^\#)_p \in \mathsf{G}^\#(\mathbb{A}).$$

Note that $\mathsf{f}_\phi^{(s,\Lambda,\mu)}$ is left-$\mathsf{B}^\#(\mathbb{Q})$-invariant and right-$\mathbf{K}^\#$-finite. The Eisenstein series attached to $\mathsf{f}_\phi^{(s,\Lambda,\mu)}$ is defined by the absolutely convergent series

$$E(\phi, s, \Lambda, \mu; g^\#) := \hat{L}(s+1, \Lambda\mu_E) \sum_{\delta \in \mathsf{B}^\#(\mathbb{Q}) \backslash \mathsf{G}^\#(\mathbb{Q})} \mathsf{f}_\phi^{(s,\Lambda,\mu)}(\delta g^\#), \quad g^\# \in \mathsf{G}^\#(\mathbb{A}), \tag{3-4}$$

for $\mathrm{Re}(s) > 1$. The Fourier transform $\widehat{\phi}$ of $\phi \in \mathcal{S}(\mathbb{A}_E^2)$ is defined by a formula similar to (3-3) with respect to the measure $\eta_{\mathbb{A}_E} \otimes \eta_{\mathbb{A}_E}$. The following properties of the Eisenstein series are standard.

Proposition 3.1. *Let $\phi \in \mathcal{S}(\mathbb{A}_E^2)$, $s \in \mathbb{C}$, $\Lambda \in \widehat{\mathrm{Cl}(E)}$, $\mu \in \widehat{\mathbb{A}^\times/\mathbb{Q}^\times\mathbb{R}_{>0}}$ and $g^\# \in \mathsf{G}^\#(\mathbb{A})$.*

(i) *The map $s \mapsto E(\phi, s, \Lambda, \mu; g^\#)$ ($\mathrm{Re}(s) > 1$) has a meromorphic continuation to $\mathbb{C}$, holomorphic in s unless $\Lambda\mu_E \ne \mathbf{1}$, in which case it has possible simple poles only at $s = 1, -1$. For a regular point $s \in \mathbb{C}$, the function $g^\# \mapsto E(\phi, s, \Lambda, \mu; g^\#)$ is an automorphic form on $\mathsf{G}^\#(\mathbb{Q})\mathsf{Z}^\#(\mathbb{A})\backslash\mathsf{G}^\#(\mathbb{A})$.*

(ii) *We have the functional equation* $E(\widehat{\phi}, -s, \Lambda^{-1}, \mu^{-1}; g^{\#}) = E(\phi, s, \Lambda, \mu; g^{\#})$.

(iii) *For a relatively compact subset* $\mathcal{N} \subset \mathbb{C}$ *on which* $s \mapsto E(\phi, s, \Lambda, \mu; g^{\#})$ *is regular and for a compact set* $\mathcal{U} \subset \mathsf{G}^{\#}(\mathbb{A})$, *there exist constants* $C > 0$ *and* $N > 0$ *such that*

$$|E(\phi, s, \Lambda, \mu; \left[\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right] g^{\#})| \le C\,|a|^N \quad (a \in \mathbb{R}_{>1},\ g^{\#} \in \mathcal{U},\ s \in \mathcal{N}).$$

3.2. *Rankin–Selberg integral and the basic identity.* For any cusp form φ on $\mathsf{Z}(\mathbb{A})\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$, $\Lambda \in \widehat{\mathrm{Cl}(E)}$ and $s \in \mathbb{C}$, the Rankin–Selberg integral is defined by

$$\text{(3-5)} \quad \langle E(\phi, s, \Lambda, \mu), \varphi\rangle := \int_{\mathsf{Z}^{\#}(\mathbb{A})\mathsf{G}^{\#}(\mathbb{Q})\backslash\mathsf{G}^{\#}(\mathbb{A})} E(\phi, s, \Lambda, \mu; g^{\#})\, \varphi(\iota_\theta(g^{\#}))\, \mathrm{d}g^{\#}.$$

By Proposition 3.1(iii) and the Fourier expansion (4-5), it is straightforward to show the absolute convergence of the integral for $s \in \mathbb{C}$ where the Eisenstein series is regular. For $T \in \mathsf{V}(\mathbb{Q})$, define a character $\psi_T : \mathsf{N}(\mathbb{A}) \to \mathbb{C}^1$ by

$$\psi_T(\mathsf{n}(X)) = \psi(\mathrm{tr}(TX)), \quad X \in \mathsf{V}(\mathbb{A}).$$

The (T_θ, Λ)-Bessel period of a cusp form φ on $\mathsf{Z}(\mathbb{A})\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$ is defined by the integral[1]

$$\text{(3-6)} \quad B^{T_\theta, \Lambda}(\varphi; g) := \int_{\mathbb{A}_E^\times/E^\times\mathbb{A}^\times} \Lambda(\tau)^{-1} \int_{\mathsf{N}(\mathbb{Q})\backslash\mathsf{N}(\mathbb{A})} \psi_{T_\theta}(n)^{-1} \varphi(\mathsf{m}(I_\theta(\tau), \mathrm{N}_{E/\mathbb{Q}}(\tau)) n g)\, \mathrm{d}n\, \mathrm{d}^\times\tau$$

for $g \in \mathsf{G}(\mathbb{A})$, where $\mathrm{d}^\times\tau$ denote the quotient measure on $\mathbb{A}_E^\times/\mathbb{A}^\times$ (see Section 2.4.1).

Formula (3-7) is the basic identity due to Piatetski-Shapiro [29]. To determine the constant exactly under our normalization of Haar measures, we reproduce the proof briefly.

Lemma 3.2. *Let* $\varphi : \mathsf{Z}(\mathbb{A})\backslash\mathsf{G}(\mathbb{Q})\mathsf{G}(\mathbb{A}) \to \mathbb{C}$ *be a cusp form. For* $\mathrm{Re}(s) > 1$, *we have*

$$\text{(3-7)} \quad \langle E(\phi, s, \Lambda, \mu), \varphi\rangle = \frac{\sqrt{|D|}}{2} L(s+1, \Lambda\mu_E) \int_{\mathbb{A}^\times} \int_{\mathbf{K}_\infty^{\#}\mathbf{K}_0^{\#}(\mathcal{O}_E)} \big(\mathsf{f}_\phi^{(s,\Lambda,\mu)}(k^{\#}) \mu(a)\, |a|_{\mathbb{A}}^{s-1} \times B^{T_\theta,\Lambda}(\varphi; \mathsf{m}(a1_2, a)\iota_\theta(k^{\#}))\big)\, \mathrm{d}^\times a\, \mathrm{d}k^{\#}.$$

Proof. By substituting (3-4) into (3-5),

$$L(s+1, \Lambda\mu_E)^{-1} \times \langle E(\phi, s, \Lambda, \mu), \varphi\rangle = \int_{\mathsf{Z}^{\#}(\mathbb{A})\mathsf{G}^{\#}(\mathbb{Q})\backslash\mathsf{G}^{\#}(\mathbb{A})} \sum_{\gamma \in \mathsf{B}^{\#}(\mathbb{Q})\backslash\mathsf{G}^{\#}(\mathbb{Q})} \mathsf{f}_\phi^{(s,\Lambda,\mu)}(\gamma g^{\#})\, \varphi(\iota_\theta(g^{\#}))\, \mathrm{d}g^{\#}$$

[1]Note that $\mathsf{m}(I_\theta(a), \mathrm{N}_{E/\mathbb{Q}}(a)) = a1_4$ for $a \in \mathbb{A}^\times$. Then, due to φ being $\mathsf{Z}(\mathbb{A})$-invariant, the integral $B^{T_\theta,\Lambda}(\varphi; g)$ is 0 if $\Lambda|\mathbb{A}^\times \ne 1$. This trivial vanishing does not happen if $\Lambda|\mathbb{A}^\times = \mathbf{1}$.

$$= \int_{\mathsf{Z}^\#(\mathbb{A})\mathsf{B}^\#(\mathbb{Q})\backslash \mathsf{G}^\#(\mathbb{A})} \mathsf{f}_\phi^{(s,\Lambda,\mu)}(g^\#)\,\varphi(\iota_\theta(g^\#))\,\mathrm{d}g^\#$$

$$= \frac{\sqrt{|D|}}{2}\int_{\mathbb{A}^\times/\mathbb{Q}^\times}\int_{\mathbb{A}_E^\times/\mathbb{A}^\times E^\times}\int_{\mathsf{N}^\#(\mathbb{Q})\backslash \mathsf{N}^\#(\mathbb{A})}\int_{\mathbf{K}_0^\#(\mathcal{O}_E)\mathbf{K}_\infty^\#} \mathsf{f}_\phi^{(s,\Lambda,\mu)}(k_{\mathbf{f}}^\# k_\infty^\#)|a|_{\mathbb{A}}^{-(s+1)} \times\Lambda(\tau)^{-1}\mu(a)^{-1}\varphi\big(\iota_\theta(n^\#)\,\mathsf{m}(I_\theta(\tau),\mathrm{N}_{E/\mathbb{Q}}(\tau))\left[\begin{smallmatrix} a^{-1}1_2 & 0\\ 0 & 1_2\end{smallmatrix}\right]\iota_\theta(k_{\mathbf{f}}^\# k_\infty^\#)\big) \times |a|_{\mathbb{A}}^2\,\mathrm{d}^\times a\,\mathrm{d}^\times\tau\,\mathrm{d}n^\#\,\mathrm{d}k_{\mathbf{f}}^\#\,\mathrm{d}k_\infty^\#,$$

where the last equality is proved using (2-21). By Lemma 3.3, the last expression becomes

$$\frac{\sqrt{|D|}}{2}\int_{\mathbb{A}^\times/\mathbb{Q}^\times}\int_{\mathbb{A}_E^\times/\mathbb{A}^\times E^\times}\int_{\mathsf{N}(\mathbb{Q})\backslash \mathsf{N}(\mathbb{A})}\int_{\mathbf{K}_0^\#(\mathcal{O}_E)\mathbf{K}_\infty^\#} \mathsf{f}_\phi^{(s,\Lambda,\mu)}(k_{\mathbf{f}}^\# k_\infty^\#)|a|_{\mathbb{A}}^{-(s+1)} \times\Lambda(\tau)^{-1}\mu(a)^{-1}\sum_{\alpha\in\mathbb{Q}^\times}\varphi\big(n\,\mathsf{m}(I_\theta(\tau),\mathrm{N}_{E/\mathbb{Q}}(\tau))\left[\begin{smallmatrix} \alpha a^{-1}1_2 & 0\\ 0 & 1_2\end{smallmatrix}\right]\iota_\theta(k_{\mathbf{f}}^\#)\big) \times \psi_{T_\theta}(n)^{-1}\,|a|_{\mathbb{A}}^2\,\mathrm{d}^\times a\,\mathrm{d}^\times\tau\,\mathrm{d}n\,\mathrm{d}k_{\mathbf{f}}^\#.$$

The α-summation and the a-integral over $\mathbb{A}^\times/\mathbb{Q}^\times$ are combined to yield an integral over $\mathbb{A}^\times$. The change of variables $a\mapsto a^{-1}$ and (3-6) then give the desired formula. □

Lemma 3.3. *For* $g\in\mathsf{G}(\mathbb{A})$,

$$\int_{\mathsf{N}^\#(\mathbb{Q})\backslash \mathsf{N}^\#(\mathbb{A})}\varphi(\iota_\theta(n^\#)g)\,\mathrm{d}n^\# = \sum_{\alpha\in\mathbb{Q}^\times}\int_{\mathsf{N}(\mathbb{Q})\backslash \mathsf{N}(\mathbb{A})}\varphi\big(n\left[\begin{smallmatrix} \alpha 1_2 & 0\\ 0 & 1_2\end{smallmatrix}\right]g\big)\,\psi_{T_\theta}(n)^{-1}\,\mathrm{d}n.$$

Proof. Fix $g\in\mathsf{G}(\mathbb{A})$ and define a function ϕ on $\mathbb{A}$ by

$$\phi(x) := \int_{\mathsf{N}^\#(\mathbb{Q})\backslash \mathsf{N}^\#(\mathbb{A})}\varphi(\mathsf{n}(xT_\theta)\iota_\theta(n^\#)g)\,\mathrm{d}n^\#,\quad x\in\mathbb{A}.$$

Since ϕ is a $\mathbb{Q}$-periodic smooth function on $\mathbb{A}$, it can be expanded in a Fourier series, which is absolutely and normally convergent:

$$\phi(x) = \sum_{\alpha\in\mathbb{Q}}\int_{\mathbb{A}/\mathbb{Q}}\phi(y)\psi(\alpha y)^{-1}\mathrm{d}y\times\psi(\alpha x),\quad x\in\mathbb{A}. \tag{3-8}$$

By (2-12) and (2-18),

$$\int_{\mathbb{A}/\mathbb{Q}}\phi(y)\,\psi(\alpha y)^{-1}\mathrm{d}y = \int_{\mathbb{A}/\mathbb{Q}}\int_{\mathsf{V}^{T_\theta}(\mathbb{Q})\backslash \mathsf{V}^{T_\theta}(\mathbb{A})}\varphi(\mathsf{n}(yT_\theta+Z)g)^{-1}\,\psi(\mathrm{tr}(\alpha T_\theta(yT_\theta^\vee+Z)))\,\mathrm{d}y\,\mathrm{d}Z = \int_{\mathsf{V}(\mathbb{Q})\backslash \mathsf{V}(\mathbb{A})}\varphi(\mathsf{n}(X)g)\,\psi_{T_\theta}(\mathsf{n}(\alpha X))^{-1}\,\mathrm{d}X,$$

which for $\alpha = 0$ is zero due to the cuspidality of φ. Thus, by setting $x = 0$ in (3-8),

$$\phi(0) = \sum_{\alpha\in\mathbb{Q}^\times} \int_{\mathsf{V}(\mathbb{Q})\backslash\mathsf{V}(\mathbb{A})} \varphi(\mathsf{n}(X)g)\, \psi_{T_\theta}(\mathsf{n}(\alpha X))^{-1}\, \mathrm{d}X$$
$$= \sum_{\alpha\in\mathbb{Q}^\times} \int_{\mathsf{N}(\mathbb{Q})\backslash\mathsf{N}(\mathbb{A})} \varphi(n\left[\begin{smallmatrix}\alpha 1_2 & 0\\ 0 & 1_2\end{smallmatrix}\right]g)\, \psi_{T_\theta}(n)^{-1}\, \mathrm{d}n.$$

The last equality is obtained by the change of variables $X \mapsto \alpha^{-1}X$ and by the automorphy of φ together with the relation $\mathsf{n}(\alpha^{-1}X) = \left[\begin{smallmatrix}\alpha^{-1}1_2 & 0\\ 0 & 1_2\end{smallmatrix}\right]\mathsf{n}(X)\left[\begin{smallmatrix}\alpha 1_2 & 0\\ 0 & 1_2\end{smallmatrix}\right]$. □

4. Automorphic forms and Fourier coefficients

Let $(\mathbb{Z}^2)_{\mathrm{dom}} := \{\lambda = (l_1, l_2) \in \mathbb{Z}^2 \mid l_1 \geq l_2\}$. Let $\lambda = (l_1, l_2) \in (\mathbb{Z}^2)_{\mathrm{dom}}$ and ϱ be the representation of $\mathbf{GL}_2(\mathbb{C})$ on the space V_ϱ of homogeneous polynomials in X, Y of degree $l_1 - l_2$ defined by $\varrho(h)f(X,Y) = (\det h)^{l_2} f(aX + cY, bX + dY)$ for $h = \left[\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right] \in \mathbf{GL}_2(\mathbb{C})$ and $f \in V_\varrho$. As is well-known, any irreducible rational representation of $\mathbf{GL}_2(\mathbb{C})$, up to equivalence, is obtained this way. The space V_ϱ carries an $\mathbf{SU}(2)$-invariant hermitian inner product $(\cdot \mid \cdot)_\varrho$ given by [9, (8.2.6)]. Recall $\mathbf{K}_\infty = \left\{k_\infty(u) := \left[\begin{smallmatrix}A & B\\ -B & A\end{smallmatrix}\right] \mid u = A + iB \in \mathbf{U}(2)\right\}$. For $N \in \mathbb{Z}_{>0}$, let $S_\varrho(\mathbf{K}_0(N))$ denote the space of smooth functions $\varphi : \mathsf{G}(\mathbb{A}) \to V_\varrho$ that satisfy the following conditions:

(i) $\varphi(z\gamma g) = \varphi(g)$ for $(z, \gamma, g) \in \mathsf{Z}(\mathbb{A}) \times \mathsf{G}(\mathbb{Q}) \times \mathsf{G}(\mathbb{A})$.

(ii) $\varphi(gk_{\mathbf{f}}k_\infty(u)) = \varrho(\bar{u})^{-1}\varphi(g)$ for $k_{\mathbf{f}} \in \mathbf{K}_0(N)$ and $k_\infty(u) \in \mathbf{K}_\infty$.

(iii) $R(X)\varphi = 0$ for all $X \in \mathfrak{p}^-$.

(iv) φ is bounded on $\mathsf{G}(\mathbb{A})$.

For $T \in \mathsf{V}(\mathbb{R})$ with $\det(T) \neq 0$, define the function $\mathbf{B}_\varrho^T : \mathsf{G}(\mathbb{R})^0 \to \mathrm{End}_\mathbb{C}(V_\varrho)$ by

(4-1) $$\mathbf{B}_\varrho^T(g) := \nu(g)^{(l_1+l_2)/2}\varrho(Ci + D)^{-1} \exp(2\pi i\, \mathrm{tr}(T\, g\langle i 1_2\rangle)),\quad g = \left[\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right] \in \mathsf{G}(\mathbb{R})^0.$$

This function satisfies the conditions

(4-2) $$\mathbf{B}_\varrho^T(\mathsf{m}(A,c)g) = c^{(l_1+l_2)/2}\mathbf{B}_\varrho^T(g) \circ \varrho({}^tA^{-1}\, c)^{-1},\quad \mathsf{m}(A,c) \in \mathsf{M}_T^\circ(\mathbb{R}),\ g \in \mathsf{G}(\mathbb{R})^0,$$

(4-3) $$\mathbf{B}_\varrho^T(ngk_\infty(u)) = \psi_T(n)\varrho(\bar{u})^{-1} \circ \mathbf{B}(g),\quad (n, g, u) \in \mathsf{N}(\mathbb{R}) \times \mathsf{G}(\mathbb{R})^0 \times \mathbf{U}(2),$$

(4-4) $$R(\mathfrak{p}^-)\mathbf{B}_\varrho^T = 0.$$

The function $B_{\varrho,v}^T : g \mapsto \mathbf{B}_\varrho^T(g)(v)$ with $v \in V_\varrho - (0)$ is bounded on $\mathsf{G}(\mathbb{R})^0$ if and only if $T \in \mathsf{V}(\mathbb{R})^+$, where $\mathsf{V}(\mathbb{R})^+$ denotes the set of positive definite elements in

$\mathsf{V}(\mathbb{R})$. Set

$$\gamma := \mathsf{m}(-1_2, -1) \in \mathsf{M}(\mathbb{Q}).$$

Note that $\nu(\gamma) = -1$. Set $\mathsf{V}(\mathbb{Q})^+ := \mathsf{V}(\mathbb{Q}) \cap \mathsf{V}(\mathbb{R})^+$.

Lemma 4.1. *Let $\varphi \in S_\varrho(\mathbf{K}_0(N))$. There is a unique family of vectors $a_\varphi(T; g_\mathbf{f}) \in V_\varrho$ ($T \in \mathsf{V}(\mathbb{Q})$, $g_\mathbf{f} \in \mathsf{G}(\mathbb{A}_\mathbf{f})$) such that*

$$\varphi(g_\infty g_\mathbf{f}) = \sum_{T \in \mathsf{V}(\mathbb{Q})} \mathbf{B}_\varrho^T(g_\infty)(a_\varphi(T; g_\mathbf{f})), \quad g_\infty \in \mathsf{G}(\mathbb{R})^\circ,\ g_\mathbf{f} \in \mathsf{G}(\mathbb{A}_\mathbf{f}). \tag{4-5}$$

If $T \notin \mathsf{V}(\mathbb{Q})^+$, then $a_\varphi(T; g_\mathbf{f}) = 0$ for all $g_\mathbf{f} \in \mathsf{G}(\mathbb{A}_\mathbf{f})$. For $g_\mathbf{f} \in \mathsf{G}(\mathbb{A}_\mathbf{f})$,

$$\int_{\mathsf{N}(\mathbb{A})/\mathsf{N}(\mathbb{Q})} \varphi(n g_\infty g_\mathbf{f})\, \psi_T(n)^{-1} \mathrm{d}n = \begin{cases} \mathbf{B}_\varrho^T(g_\infty)(a_\varphi(T; g_\mathbf{f})) & (g_\infty \in \mathsf{G}(\mathbb{R})^\circ), \\ 0 & (g_\infty \in \mathsf{G}(\mathbb{R}) - \mathsf{G}(\mathbb{R})^\circ). \end{cases} \tag{4-6}$$

Proof. Fix $g_\mathbf{f} \in \mathsf{G}(\mathbb{A}_\mathbf{f})$ and examine the integral, say $W(g_\infty)$, on the left side of (4-6). The function W on $\mathsf{G}(\mathbb{R})^0$ satisfies conditions (4-3) and (4-4). Hence, there is a corresponding function $F : \mathfrak{h}_2 \to V_\varrho$ determined by the relation $F(Z) = \nu(g)^{-(l_1+l_2)/2} \varrho(Ci + D)\, W(g_\infty)$, with $g_\infty = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathsf{G}(\mathbb{R})^0$, and such that $Z = (Ai + B)(Ci + D)^{-1}$. Since $F(Z)$ satisfies $F(Z + X) = \psi_T(X)\, F(Z)$ ($X \in \mathsf{V}(\mathbb{R})$), it is of the form $F(Z) = F_1(Y) \exp(2\pi i\, \mathrm{tr}(T X))$ with a V_ϱ-valued C^∞-function $F_1(Y)$. Since $F(Z)$ is holomorphic, the Cauchy–Riemann equations yield $(\mathrm{d}/\mathrm{d}y_{ij}) F_1(Y) = -2\pi t_{ij} F_1(Y)$, which is uniquely solved as

$$F_1(Y) = a_\varphi(T; g_\mathbf{f}) \exp(-2\pi\, \mathrm{tr}(T Y))$$

with a vector $a_\varphi(T; g_\mathbf{f}) \in V_\varrho$, or equivalently $W(g_\infty) = \mathbf{B}_\varrho^T(g_\infty)(a_\varphi(T; g_\mathbf{f}))$. Then, the formula in (4-5) is a consequence of the Fourier expansion of a $\mathsf{N}(\mathbb{Q})$-periodic function on $\mathsf{N}(\mathbb{A})$. Since φ is bounded on $\mathsf{G}(\mathbb{A})$, the function $W(g_\infty)$ should be bounded on $\mathsf{G}(\mathbb{R})^0$; since $g_\infty \mapsto \mathbf{B}_\varrho^T(g_\infty)(v)$, with $v \in V_\varrho - \{0\}$, is unbounded if $T \notin \mathsf{V}(\mathbb{R})^+$, we have $a_\varphi(T; g_\mathbf{f}) = 0$ for all $T \in \mathsf{V}(\mathbb{Q}) - \mathsf{V}(\mathbb{R})^+$. It remains to show the second case in (4-6). Let $g_\infty \notin \mathsf{G}(\mathbb{R})^\circ$. Decompose $\gamma \in \mathsf{M}(\mathbb{Q})$ as $\gamma_\infty \gamma_\mathbf{f}$ ($\gamma_\infty \in \mathsf{M}(\mathbb{R})$, $\gamma_\mathbf{f} \in \mathsf{M}(\mathbb{A}_\mathbf{f})$) in $\mathsf{M}(\mathbb{A})$. By the left $\mathsf{G}(\mathbb{Q})$-invariance of φ,

$$\varphi(\mathsf{n}(X) g_\infty g_\mathbf{f}) = \varphi(\gamma \mathsf{n}(X) g_\infty g_\mathbf{f}) = \varphi(\mathsf{n}(-X) \gamma_\infty g_\infty \gamma_\mathbf{f} g_\mathbf{f}).$$

Integrate this in X over $\mathsf{V}(\mathbb{Q}) \backslash \mathsf{V}(\mathbb{A})$ and make a change of variables $n \to n^{-1}$ on the way; we have

$$\int_{\mathsf{N}(\mathbb{Q}) \backslash \mathsf{N}(\mathbb{A})} \varphi(n g_\infty g_\mathbf{f})\, \psi_T(n)^{-1} \mathrm{d}n = \int_{\mathsf{N}(\mathbb{Q}) \backslash \mathsf{N}(\mathbb{A})} \varphi(n \gamma_\infty g_\infty \gamma_\mathbf{f} g_\mathbf{f})\, \psi_{-T}(n)^{-1}\, \mathrm{d}n.$$

Since $\gamma_\infty g_\infty \in \mathsf{G}(\mathbb{R})^\circ$, the last integral becomes $\mathbf{B}_\varrho^{-T}(\gamma_\infty g_\infty)(a_\varphi(-T; \gamma_\mathbf{f} g_\mathbf{f}))$ by the first case. Since $-T \notin \mathsf{V}(\mathbb{R})^+$, we have $a(-T; \gamma_\mathbf{f} g_\mathbf{f}) = 0$. ☐

The vectors $a_\varphi(T; g_{\mathbf{f}})$ in V_ϱ are referred to as the adelic Fourier coefficients of φ. As is well-known, there is a linear bijective correspondence between $\varphi \in S_\varrho(\mathbf{K}_0(N))$ and the classical V_ϱ-valued Siegel cusp forms $f(Z)$ on

$$\mathbf{\Gamma}_0(N) := \left\{ \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathbf{Sp}_2(\mathbb{Z}) \mid C \equiv 0 \pmod N \right\}$$

determined by the relation

$$f(g_\infty\langle i 1_2\rangle) = \nu(g_\infty)^{-(l_1+l_2)/2} \varrho(Ci + D)\varphi(g_\infty), \quad g_\infty = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathsf{G}(\mathbb{R})^0.$$

By the modular transformation law $f((AZ+B)(CZ+D)^{-1}) = \varrho(CZ+D)^{-1} f(Z)$ for $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \Gamma_0(N)$, since $-1_4 \in \Gamma_0(N)$, we have $f(Z) \equiv 0$; hence $S_\varrho(\mathbf{K}_0(N)) = (0)$, unless $l_1 \equiv l_2 \pmod 2$. (For details, see [36, §3.2] and [4, §4].)

Since $\mathsf{N}(\mathbb{A}) \cap \mathbf{K}_0(N) = \{\mathsf{n}(X) \mid X \in \mathsf{V}(\widehat{\mathbb{Z}})\}$, by (4-6), we have $a_\varphi(T; 1) = 0$ unless $T \in \mathcal{Q}$, where

$$\mathcal{Q} := \left\{ \left[\begin{smallmatrix} a & b/2 \\ b/2 & c \end{smallmatrix}\right] \;\middle|\; a, b, c \in \mathbb{Z} \right\}$$

is the dual lattice of $\mathsf{V}(\mathbb{Z})$. Then, (4-5) with $g_{\mathbf{f}} = 1_4$ reduces to

$$f(Z) = \sum_{T \in \mathcal{Q}^+} a_\varphi(T; 1_4) \exp(2\pi i \operatorname{tr}(TZ)), \quad Z \in \mathfrak{h}_2,$$

where $\mathcal{Q}^+ := \mathcal{Q} \cap \mathsf{V}(\mathbb{Q})^+$. This means that $a_\varphi(T) := a_\varphi(T; 1_4)$ $(T \in \mathcal{Q}^+)$ is the Fourier coefficient of $f(Z)$ in the classical sense. Set

$$\delta := \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right] \in \mathbf{GL}_2(\mathbb{Q}).$$

The map $T \mapsto -T\mathsf{s}(\delta)$ preserves the set $\mathcal{Q}^+$. If we identify $\mathcal{Q}^+$ with the set of positive definite integral binary forms $aX^2 + bXY + cY^2$, this operation corresponds to a sign change in b.

Lemma 4.2. *The set of adelic Fourier coefficients of $\varphi \in S_\varrho(\mathbf{K}_0(N))$ has the following properties*:

(4-7)
$$a_\varphi(T; \mathsf{m}(h_{\mathbf{f}}, \det h_{\mathbf{f}}) g_{\mathbf{f}} \kappa) = a_\varphi(T\mathsf{s}(h); g_{\mathbf{f}}), \;\; h \in \mathbf{GL}_2(\mathbb{Q}),\; g_{\mathbf{f}} \in \mathsf{G}(\mathbb{A}_{\mathbf{f}}),\; \kappa \in \mathbf{K}_0(N).$$

(4-8) $$a_\varphi(-T\mathsf{s}(\delta); 1) = \varrho(\delta)(a_\varphi(T; 1)), \quad T \in \mathsf{V}(\mathbb{Q}).$$

Proof. The relation in (4-7) follows from (4-6) by a simple change of variables. Let us show (4-8). We have

(4-9) $$\mathsf{n}\,(-X\mathsf{s}(\delta)) = \kappa \mathsf{n}(X)\kappa^{-1}, \quad X \in \mathsf{V}(\mathbb{A})$$

with $\kappa := \mathsf{m}(\delta, 1) \in \mathsf{G}(\mathbb{Q})$. Write $\kappa = \kappa_{\mathbf{f}}\kappa_\infty \in \mathsf{G}(\mathbb{A}_{\mathbf{f}})\,\mathsf{G}(\mathbb{R})$; then, $\kappa_{\mathbf{f}} \in \mathbf{K}_0(N)$ and

$\kappa_\infty \in \mathbf{K}_\infty$. Using (4-9) and the left $\mathsf{G}(\mathbb{Q})$-invariance of φ, we have

$$\begin{aligned}\int_{\mathsf{N}(\mathbb{Q})\backslash\mathsf{N}(\mathbb{A})} \varphi(n)\,\psi_{-T\mathsf{s}(\delta)}(n)^{-1}\,\mathrm{d}n &= \int_{\mathsf{V}(\mathbb{Q})\backslash\mathsf{V}(\mathbb{A})} \varphi(\mathsf{n}(-\mathsf{s}(\delta)X))\,\psi(\mathrm{tr}(TX))^{-1}\,\mathrm{d}X\\ &= \int_{\mathsf{V}(\mathbb{Q})\backslash\mathsf{V}(\mathbb{A})} \varphi(\kappa\mathsf{n}(X)\kappa^{-1})\,\psi(\mathrm{tr}(TX))^{-1}\,\mathrm{d}X\\ &= \varrho(\delta)\int_{\mathsf{N}(\mathbb{Q})\backslash\mathsf{N}(\mathbb{A})} \varphi(n)\psi_T(n)^{-1}\,\mathrm{d}n.\end{aligned}$$

From this and (4-6),

$$\mathbf{B}_\varrho^{-T\mathsf{s}(\delta)}(1_4)(a_\varphi(-T\mathsf{s}(\delta);1)) = \varrho(\delta)\,\mathbf{B}_\varrho^{T}(1_4)(a_\varphi(T;1)).$$

By (4-1), this leads to $\mathbf{B}_\varrho^{-T\mathsf{s}(\delta)}(1_4) = \mathbf{B}_\varrho^{T}(1_4) = e^{-2\pi\,\mathrm{tr}(T)}\,\mathrm{Id}_{V_\varrho}$. □

Define

(4-10) $$\mathcal{Q}^+_{\mathrm{prim}}(D) := \left\{T = \left[\begin{smallmatrix} a & 2^{-1}b\\ 2^{-1}b & c\end{smallmatrix}\right] \in \mathcal{Q} \;\middle|\; a>0,\ -\det T = D/4\right\}:$$

D is a fundamental discriminant, so $T \in \mathcal{Q}^+_{\mathrm{prim}}(D)$ is primitive, i.e., $\gcd(a,b,c)=1$; $\mathcal{Q}^+_{\mathrm{prim}}(D)$ is a subset of $\mathsf{V}(\mathbb{Q})$, which is stable under the action of the unimodular group $\mathbf{SL}_2(\mathbb{Z})$ induced by (2-1). The matrix T_θ defined by (2-11) belongs to $\mathcal{Q}^+_{\mathrm{prim}}(D)$. The $\mathbf{SL}_2(\mathbb{Z})$-orbit of $T \in \mathcal{Q}^+_{\mathrm{prim}}(D)$ is denoted by $[T]$.

Lemma 4.3. *For each $u \in \mathbb{A}_E^\times$, let $I_\theta(u)$ decompose in $\mathbf{GL}_2(\mathbb{A})$ as*

(4-11)
$$I_\theta(u) = \gamma_u h_u \kappa_u,\quad \gamma_u \in \mathbf{GL}_2(\mathbb{Q}),\ h_u \in \mathbf{GL}_2(\mathbb{R})^\circ,\ \kappa_u = (\kappa_{u,p})_{p<\infty} \in \mathbf{GL}_2(\widehat{\mathbb{Z}})$$

and set $T_\theta(u) := T_\theta\,\mathsf{s}(\gamma_u)$. Then,

(i) $T_\theta(u) \in \mathcal{Q}^+_{\mathrm{prim}}(D)$.

(ii) *The $\mathbf{SL}_2(\mathbb{Z})$-equivalence class of $T_\theta(u)$ does not depend on the decomposition* (4-11). *If u, u' belongs to the same $E^\times E^\times_\infty \widehat{\mathcal{O}}_E^\times$-coset, then $T_\theta(u)$ and $T_\theta(u')$ are $\mathbf{SL}_2(\mathbb{Z})$-equivalent.*

(iii) *The map $[u] \mapsto [T_\theta(u)]$ from $\mathrm{Cl}(E) = \mathbb{A}_E^\times/E^\times E^\times_\infty \widehat{\mathcal{O}}_E^\times$ to $\mathcal{Q}^+_{\mathrm{prim}}(D)/\mathbf{SL}_2(\mathbb{Z})$ is a bijection.*

Proof. This is an adelic reformulation of the classically well-known correspondence between the ideal classes and the $\mathbf{SL}_2(\mathbb{Z})$-equivalence classes of integral binary quadratic forms. The proof is straightforward. □

Since $\mathsf{m}(\mathbf{GL}_2(\widehat{\mathbb{Z}}),1) \subset \mathbf{K}_0(N)$, the relation in (4-7) shows that the function $T \mapsto a_\varphi(T) := a_\varphi(T;1_4)$ on $\mathsf{V}(\mathbb{Z})^+$ is $\mathbf{SL}_2(\mathbb{Z})$-invariant. Hence the following sum of vectors in V_ϱ is well-defined:

(4-12) $$\mathbf{R}(\varphi, E, \Lambda) := \sum_{[u]\in\mathrm{Cl}(E)} a_\varphi([T_\theta(u)])^{\mathrm{SO}}\,\Lambda(u)^{-1},$$

where $v^{\mathrm{SO}} := \int_0^\pi \varrho\big(\big[\begin{smallmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{smallmatrix}\big]\big) v \,\mathrm{d}\alpha$ for $v \in V_\varrho$. Recall the integral in (3-6) and the function in (4-1).

Lemma 4.4. *Let $\varphi \in S_\varrho(\mathbf{K}_0(N))$ and $\Lambda \in \widehat{\mathrm{Cl}(E)}_{\mathrm{prim}}$.*

(i) *For $g_\infty \in \mathsf{G}(\mathbb{R})^0$,*

$$B^{T_\theta,\Lambda}(\varphi; g_\infty) = w_D^{-1}\, \mathbf{B}_\varrho^{T_\theta}(g_\infty)(\mathbf{R}(\varphi, E, \Lambda))$$

where w_D is the number of units in $\mathfrak{o}_E$.

(ii) *Set $\sigma := \big[\begin{smallmatrix} 1 & \mathrm{tr}_{E/\mathbb{Q}}(\theta) \\ 0 & -1 \end{smallmatrix}\big]$. Then*

$$\varrho(r)(B^{T_\theta,\Lambda}(\varphi; b_{\mathbb{R}}^\theta)) = B^{T_\theta,\Lambda}(\varphi; b_{\mathbb{R}}^\theta), \quad r \in \mathbf{SO}(2), \tag{4-13}$$

$$\varrho(\delta)(B^{T_\theta,\Lambda}(\varphi; b_{\mathbb{R}}^\theta)) = B^{-T_\theta \mathsf{s}(\sigma),\Lambda}(\varphi; b_{\mathbb{R}}^\theta). \tag{4-14}$$

Proof. (i) The set $\mathbb{A}_E^\times/\mathbb{A}^\times E^\times E_{\mathbb{R}}^\times$ decomposes into a disjoint union of $\mathfrak{o}_E^\times$-orbits $[u]\mathfrak{o}_E^\times$ for different classes $[u] \in \mathrm{Cl}(E)$. Let μ denote the quotient measure on $\mathbb{A}_E^\times/\mathbb{A}^\times E^\times E_{\mathbb{R}}^\times$. For $\tau \in \mathbb{A}_E^\times$, set $\mathsf{m}_\theta(\tau) := \mathsf{m}(I_\theta(\tau), \mathrm{N}_{E/\mathbb{Q}}(\tau))$. Then, by (3-6) and (4-6), $B^{T_\theta,\Lambda}(\varphi; g_\infty)$ equals

$$\begin{aligned} \mathbf{B}_\varrho^{T_\theta}(g_\infty)\Big(\int_{\mathbb{A}_E^\times/\mathbb{A}^\times E^\times E_\infty^\times} \int_{E_\infty^\times/\mathbb{R}^\times} \mathrm{N}_{E_\infty/\mathbb{R}}(\tau_\infty)^{(l_1+l_2)/2} \varrho({}^t I_\theta(\tau_\infty)^\dagger)^{-1} a_\varphi(T_\theta; \mathsf{m}_\theta(u)) \\ \times \Lambda(u)^{-1}\, \mathrm{d}^\times\tau_\infty\, \mathrm{d}\mu(u)\Big) \\ = \mathbf{B}_\varrho^{T_\theta}(g_\infty)\Big(\sum_{[u]\in\mathrm{Cl}(E)} \int_{[u]\mathfrak{o}_E^\times} a_\varphi(T_\theta; \mathsf{m}_\theta(u))^{\mathrm{SO}}\, \Lambda(u)^{-1}\, \mathrm{d}\mu(u)\Big), \end{aligned}$$

The quotient measure $\eta_{E_\infty^\times/\mathbb{R}^\times}$ is identified with $\mathrm{d}\alpha$ under the identification

$$E_\infty^\times/\mathbb{R}^\times = \big\{\big[\begin{smallmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{smallmatrix}\big] \mid \alpha \in [0,\pi)\big\}.$$

Since $\mathsf{m}_\theta(\widehat{\mathscr{O}}_E^\times) \subset \mathbf{K}_0(N)$, by (4-6), the formula in parenthesis becomes

$$\sum_{[u]\in\mathrm{Cl}(E)} \mu([u]\mathfrak{o}_E^\times)\, a_\varphi(T_\theta; \mathsf{m}_\theta(u))\, \Lambda(u)^{-1}.$$

Let $I_\theta(u)$ decompose as in (4-11). Then, by (4-7) and Lemma 4.3, we have

$$a_\varphi(T_\theta; \mathsf{m}_\theta(u)) = a_\varphi(T_\theta \mathsf{s}(\gamma_u); 1) = a_\varphi(T_\theta(u)).$$

It remains to compute $\mu([u]\mathfrak{o}_E^\times)$. Since any fiber of the natural surjection from $\mathfrak{o}_E^\times$ onto $[u]\mathfrak{o}_E^\times$ has a simply transitive action of the group $\mathscr{O}_E^\times$, we have $\mu([u]\mathfrak{o}_E^\times) = 1/\#\mathfrak{o}_E^\times = 1/w_D$.

(ii) We have $I_\theta(e^{i\alpha}) = A_\theta^{-1}\big[\begin{smallmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{smallmatrix}\big] A_\theta$, or equivalently

$$\mathsf{m}(I_\theta(e^{i\alpha}), 1) = b_{\mathbb{R}}^\theta\, \mathrm{diag}\big(\big[\begin{smallmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{smallmatrix}\big], \big[\begin{smallmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{smallmatrix}\big]\big)(b_{\mathbb{R}}^\theta)^{-1}$$

for $\alpha \in \mathbb{R}$. Due to this and $\Lambda_\infty = 1$, formula (4-13) is proved by a change of variables $\tau \mapsto \tau u^{-1}$ in the integral (3-6). Since $b_{\mathbb{R}}^{\theta} \mathsf{m}(\delta, 1)(b_{\mathbb{R}}^{\theta})^{-1} = \mathsf{m}(\sigma, 1)$, formula (4-14) is proved in the same way as in the proof of Lemma 4.2. □

Let $\varphi \in S_{\varrho}(\mathbf{K}_0(N)) - (0)$; then $l_1 \equiv l_2 \pmod 2$ as noted above. Therefore, $\dim_{\mathbb{C}}(V_{\varrho}^{\mathbf{SO}(2)}) = 1$; indeed, $V_{\varrho}^{\mathbf{SO}(2)} = \mathbb{C}\,\mathsf{v}_{\varrho}^0$ with $\mathsf{v}_{\varrho}^0 := (X^2 + Y^2)^{(l_1 - l_2)/2}$. Set

$$\mathsf{v}_{\varrho}^{\theta} := \Big(\frac{-2}{\sqrt{|D|}}\Big)^{\frac{l_1+l_2}{2}} \varrho({}^t A_\theta)\mathsf{v}_{\varrho}^0 = \Big(\frac{-2}{\sqrt{|D|}}\Big)^{\frac{l_1-l_2}{2}} (X^2 + \mathrm{tr}_{E/\mathbb{Q}}(\theta) XY + \mathrm{N}_{E/\mathbb{Q}}(\theta) Y^2)^{\frac{l_1-l_2}{2}}.$$

Note that $\mathbf{B}_{\varrho}^{T_\theta}(b_{\mathbb{R}}^{\theta}) = \big(\frac{-2}{\sqrt{|D|}}\big)^{-(l_1+l_2)/2} e^{-2\pi\sqrt{|D|}} \varrho({}^t A_\theta)^{-1}$, so that $\mathbf{B}_{\varrho}^{T_\theta}(b_{\mathbb{R}}^{\theta})(\mathsf{v}_{\varrho}^{\theta}) = e^{-2\pi\sqrt{|D|}}\mathsf{v}_{\varrho}^0$. Thus, as a corollary to Lemma 4.4, the vector $\mathbf{R}(\varphi, E, \Lambda) \in V_{\varrho}$ is nonzero if and only if $B^{T_\theta,\Lambda}(\varphi; b_{\mathbb{R}}^{\theta}) \neq 0$, in which case there exists a unique scalar $R(\varphi, E, \Lambda)$ such that $\mathbf{R}(\varphi, E, \Lambda) = R(\varphi, E, \Lambda)\,\mathsf{v}_{\varrho}^{\theta}$, or equivalently

$$B^{T_\theta,\Lambda}(\varphi; b_{\mathbb{R}}^{\theta}) = \pi w_D^{-1} e^{-2\pi\sqrt{|D|}} R(\varphi, E, \Lambda)\mathsf{v}_{\varrho}^0. \tag{4-15}$$

Note that $(\mathsf{v}_{\varrho}^0)^{\mathrm{SO}} = \pi \mathsf{v}_{\varrho}^0$. Define a C^∞-function $B_{\varrho}^{T_\theta} : \mathsf{G}(\mathbb{R})^0 \to V_{\varrho}$ as

$$B_{\varrho}^{T_\theta}(g_\infty) := \mathbf{B}_{\varrho}^{T_\theta}(g_\infty)(\mathsf{v}_{\varrho}^{\theta}), \quad g_\infty \in \mathsf{G}(\mathbb{R})^0. \tag{4-16}$$

4.1. ***Sign condition.*** The Galois group $\mathrm{Gal}(E/\mathbb{Q})$ acts on $\mathrm{Cl}(E)$ naturally, hence on the orbit space $\mathcal{Q}_{\mathrm{prim}}^{+}(D)/\mathbf{SL}_2(\mathbb{Z})$ through the bijection in Lemma 4.3(iii). The following lemma describes the conjugate action on $\mathcal{Q}_{\mathrm{prim}}^{+}(D)/\mathbf{SL}_2(\mathbb{Z})$ explicitly. Recall the element $\sigma \in \mathbf{GL}_2(\mathbb{Z})$ defined in Lemma 4.4(ii).

Lemma 4.5. *For $u \in \mathrm{Cl}(E)$, the element $T_\theta(\bar{u})$ is $\mathbf{SL}_2(\mathbb{Z})$-equivalent to $-T_\theta(u)\mathsf{s}(\delta)$. In other words, the conjugate action on $\mathcal{Q}_{\mathrm{prim}}^{+}(D)/\mathbf{SL}_2(\mathbb{Z})$ is induced by the map $T \mapsto -T\,\mathsf{s}(\delta)$.*

Proof. The defining formula of I_θ in (2-5) yields

$$I_\theta(a + \theta b) = \begin{bmatrix} 1 & \theta \\ 1 & \bar\theta \end{bmatrix}^{-1} \begin{bmatrix} a+\theta b & 0 \\ 0 & a+\bar\theta b \end{bmatrix} \begin{bmatrix} 1 & \theta \\ 1 & \bar\theta \end{bmatrix}, \quad a, b \in \mathbb{Q}.$$

Set $t := \mathrm{tr}_{E/\mathbb{Q}}(\theta)$. Then, since $\bar\theta = t - \theta$, we have $a + b\bar\theta = a' + b'\theta$ with $a' = a + tb \in \mathbb{Q}$ and $b' = -b \in \mathbb{Q}$. Then, a computation reveals

$$I_\theta(a + b\bar\theta) = \sigma I_\theta(a + b\theta)\sigma^{-1}. \tag{4-17}$$

By the decomposition of $I_\theta(u)$ for $u \in \mathbb{A}_E^\times$ in (4-11), we may take $\gamma_{\bar u} = \sigma\gamma_u\sigma^{-1}$. Thus, $T_\theta(\bar u) = T_\theta \mathsf{s}(\gamma_{\bar u}) = T_\theta \mathsf{s}\left(\sigma\gamma_u\sigma^{-1}\right)$. A computation shows $T_\theta \mathsf{s}(\sigma) = -T_\theta$. Hence,

$$T_\theta(\bar u) = -T_\theta \mathsf{s}(\gamma_u\sigma^{-1}) = -T_\theta(u)\mathsf{s}(\sigma^{-1}) = -T_\theta(u)\mathsf{s}(\delta)\mathsf{s}\left(\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\right).$$

The last matrix is $\mathbf{SL}_2(\mathbb{Z})$-equivalent to $-T_\theta(u)\mathsf{s}(\delta)$. This completes the proof. □

For $\Lambda \in \widehat{\mathrm{Cl}(E)}$, recall from (2-24) its conjugate $\Lambda^\dagger \in \widehat{\mathrm{Cl}(E)}$.

Lemma 4.6. *Let* $\varphi \in S_{\varrho}(\mathbf{K}_0(N))$ *and* $\Lambda \in \widehat{\mathrm{Cl}(E)}$ *and suppose* $B^{T_\theta,\Lambda}(\varphi; b_{\mathbb{R}}^{\theta}) \neq 0$. *Then,*

$$R(\varphi, E, \Lambda^{\dagger}) = (-1)^{l_2} R(\varphi, E, \Lambda).$$

Proof. This is proved by Lemma 4.5 and (4-15) and by $\varrho(\delta)\mathsf{v}_{\varrho}^{0} = (-1)^{l_2}\mathsf{v}_{\varrho}^{0}$. Note that $\delta^{-1}\sigma = \left[\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right] \in \mathbf{SL}_2(\mathbb{Z})$, so that $a_{\varphi}(-T_\theta \mathsf{s}(\sigma), 1) = a_{\varphi}(-T_\theta \mathsf{s}(\delta), 1)$. □

4.2. ***Automorphic representations.*** Let $\Pi_{\mathrm{cusp}}(\mathsf{Z}\backslash\mathsf{G})$ denote the set of all those irreducible cuspidal representations of $\mathsf{G}(\mathbb{A})$ with trivial central characters, i.e., those irreducible $(\mathfrak{g}, \mathbf{K}_\infty) \times \mathsf{G}(\mathbb{A}_{\mathbf{f}})$-submodules (π, V_π) of the space of cusp forms on $\mathsf{Z}(\mathbb{A})\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})$. We endow V_π with the restriction of the L^2 inner product

$$(\varphi \mid \varphi_1)_{L^2} := \int_{\mathsf{Z}(\mathbb{A})\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})} \varphi(g)\overline{\varphi_1(g)}\,\mathrm{d}g, \qquad \varphi, \varphi_1 \in V_\pi.$$

For $\pi \in \Pi_{\mathrm{cusp}}(\mathsf{Z}\backslash\mathsf{G})$, we fix its restricted tensor decomposition $\pi \cong \bigotimes_{p\le\infty} \pi_p$ with (π_p, V_{π_p}) being an irreducible admissible unitarizable representation of $\mathsf{G}(\mathbb{Q}_p)$ with trivial central character such that a $\mathbf{K}_p$-invariant vector $\xi_p^0 \in V_{\pi_p} - (0)$ is preassigned for almost all $p < \infty$. An element $\lambda \in (\mathbb{Z}^2)_{\mathrm{dom}}$ is the highest weight of the minimal $\mathbf{K}_\infty$-type of a holomorphic discrete series representation (HDS for short) of $\mathbf{Sp}_2(\mathbb{R})$ if and only if $l_2 > 2$, in which case the Harish-Chandra parameter of the HDS is $(l_1 - 1, l_2 - 2)$. For such λ and $N \in \mathbb{Z}_{>0}$, let $\Pi_{\mathrm{cusp}}(\lambda, N)$ be a subset of $\pi \in \Pi_{\mathrm{cusp}}(\mathsf{Z}\backslash\mathsf{G})$ having the following properties:

(i) As a $(\mathfrak{g}, \mathbf{K}_\infty)$-module $\pi_\infty \cong D_{l_1-1,l_2-2} \oplus D_{-l_2+2,-l_1+1}$, where D_{m_1,m_2} is the discrete series representation of $\mathsf{G}(\mathbb{R})^0$ of Harish-Chandra parameter (m_1, m_2) with central character $\mathsf{Z}(\mathbb{R}) \cong \mathbb{R}^\times \ni z \mapsto \mathrm{sgn}(z)^{m_1+m_2+1}$.

(ii) $V_\pi^{\mathbf{K}_0(N)} \neq (0)$.

(iii) Let (ϱ, V_ϱ) be the irreducible rational representation of highest weight λ as before. The space $S_\varrho(\mathbf{K}_0(N))$ is an orthogonal direct sum of spaces $V_\pi^{\mathbf{K}_0(N)}[\varrho] := \{\varphi \in S_\varrho(\mathbf{K}_0(N)) \mid \varphi_v \in V_\pi\ (\forall v \in V_\varrho)\}$ for $\pi \in \Pi_{\mathrm{cusp}}(\lambda, N)$, where $\varphi_v(g) := (\varphi(g) \mid v)_\varrho$ for $\varphi \in S_\varrho(\mathbf{K}_0(N))$ and $v \in V_\varrho$.

Although there may be many choices of $\Pi_{\mathrm{cusp}}(\lambda, N)$, we fix one of them once and for all. For $\pi \in \Pi_{\mathrm{cusp}}(\lambda, N)$, we fix a $\mathbf{K}_\infty$-intertwining map $V_\varrho \oplus \overline{V}_\varrho \hookrightarrow D_{l_1-1,l_2-2} \oplus D_{-l_2+2,-l_1+1}$ $(\cong \pi_\infty)$ once and for all, where $\overline{V}_\varrho := \overline{\mathbb{C}} \otimes_{\mathbb{C}} V_\varrho$ is the complex conjugate of V_ϱ.

4.3. ***Basic assumptions.*** From now on, we fix a triple $(\lambda = (l_1, l_2), N, M) \in (\mathbb{Z}^2)_{\mathrm{dom}} \times \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ satisfying the following conditions:

(A-i) N is square-free.

(A-ii) $l_1 \equiv l_2 \pmod 2$, and $l_2 \in \mathbb{Z}_{\ge 3}$ so that $(l_1 - 1, l_2 - 2)$ is the Harish-Chandra parameter of an HDS.

(A-iii) N and M are coprime.

(A-iv) M is odd.

(A-v) All prime divisors of NM are inert in $E/\mathbb{Q}$.

We fix characters $\Lambda \in \widehat{\mathrm{Cl}(E)}$ and $\mu = \bigotimes_{p\le\infty} \mu_p \in \widehat{\mathbb{A}^\times/\mathbb{Q}^\times\mathbb{R}_{>0}}$ with $M := \mathrm{cond}(\mu)$ as well. As usual, the character μ induces a primitive Dirichlet character $\tilde{\mu} : (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}^1$ in such a way that $\mu(u) = \tilde{\mu}(a)^{-1}$ for all $u \in \widehat{\mathbb{Z}}^\times$ and $a \in \mathbb{Z}$ with $u - a \in M\widehat{\mathbb{Z}}$; thus $\mu_p(p) = \tilde{\mu}(p)$ for primes $p \nmid M$, and

$$\prod_{p|M} W_{\mathbb{Q}_p}(\mu_p, \psi_p) = M^{-1/2} G(\tilde{\mu}),$$

with $W_{\mathbb{Q}_p}(\mu_p, \psi_p)$ as in Definition 6.2 and $G(\tilde{\mu}) := \sum_{a\in(\mathbb{Z}/M\mathbb{Z})^\times} \tilde{\mu}(a) e^{2\pi i a/M}$, the Gauss sum of $\tilde{\mu}$. Let $\Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N)$ denote the set of all those $\pi \in \Pi_{\mathrm{cusp}}(\lambda, N)$ that satisfies the condition

(A-vi) π admits a global (T_θ, Λ)-Bessel model, i.e., there exists $\varphi \in V_\pi$ such that $B^{T_\theta,\Lambda}(\varphi; g) \neq 0$ for some $g \in \mathsf{G}(\mathbb{A})$.

We remark that the conditions listed above (when $\mu = \mathbf{1}$ and $l_1 = l_2$) are also imposed in the most part of [7]. Let $\Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N)^{\mathrm{SK}}$ denote the set of $\pi \in \Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N)$ that is the Saito–Kurokawa lift ([28]) of a cuspidal automorphic representation of $\mathbf{PGL}_2(\mathbb{A})$, which is locally described by [37] (see also [38]). From [37, §4], the set $\Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N)^{\mathrm{SK}} = \varnothing$ unless $l_1 = l_2$, i.e., V_ϱ is one dimensional. We note that all the representations $\pi \cong \bigotimes_p \pi_p$ in $\Pi_{\mathrm{cusp}}^{(T_\theta,\chi)}(l, N) \setminus \Pi_{\mathrm{cusp}}^{(T_\theta,\chi)}(l, N)^{\mathrm{SK}}$ are non-CAP (cf. [41, 17, §3.5, 30, Corollary 4.5]); then, by [48], π_p is tempered for all $p \nmid N$. By invoking [3] (see also [39]), any $\pi \in \Pi_{\mathrm{cusp}}^{(T_\theta,\chi)}(l, N) \setminus \Pi_{\mathrm{cusp}}^{(T_\theta,\chi)}(l, N)^{\mathrm{SK}}$ is either a Yoshida lift, i.e., there exists a pair of irreducible cuspidal automorphic representations (σ_1, σ_2) of $\mathbf{GL}_2(\mathbb{A})$ such that $L(s, \pi) = L(s, \sigma_1) L(s, \sigma_2)$, or a "general type", that is, there exists an irreducible cuspidal automorphic representation Π of $\mathbf{GL}_4(\mathbb{A})$ such that $L(s, \Pi, \wedge^2)$ has a pole at $s = 1$ and $L(s, \pi) = L(s, \Pi)$. Let $\Pi_{\mathrm{cusp}}^{(T_\theta,\chi)}(l, N)^{\mathrm{Y}}$ (resp. $\Pi_{\mathrm{cusp}}^{(T_\theta,\chi)}(l, N)^{\mathrm{G}}$) be the set of $\pi \in \Pi_{\mathrm{cusp}}^{(T_\theta,\chi)}(l, N)$ that is a Yoshida lift (resp. a general type). Then,

$$\Pi_{\mathrm{cusp}}^{(T_\theta,\chi)}(l, N) = \Pi_{\mathrm{cusp}}^{(T_\theta,\chi)}(l, N)^{\mathrm{G}} \cup \Pi_{\mathrm{cusp}}^{(T_\theta,\chi)}(l, N)^{\mathrm{Y}} \cup \Pi_{\mathrm{cusp}}^{(T_\theta,\chi)}(l, N)^{\mathrm{SK}} \quad \text{(disjoint union)}. \tag{4-18}$$

On the other hand, the set $\Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N)$ can be separated into the subsets consisting of all newforms (i.e., $N_\pi = N$) and all oldforms (i.e., $N_\pi \neq N$) denoted by $\Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N)^{\mathrm{new}}$ and $\Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N)^{\mathrm{old}}$ respectively. For $\bullet \in \{\mathrm{G}, \mathrm{Y}, \mathrm{SK}\}$ and $* \in \{\mathrm{new}, \mathrm{old}\}$, we set

$$\Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N)^{\bullet,*} = \Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N)^{\bullet} \cap \Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N)^{*}.$$

Let N_π be the product of primes $p < \infty$ such that π_p is not spherical. By condition (ii) in Section 4.2, we have $N_\pi \mid N$. Due to (A-i), the representations π_p for $p \mid N_\pi$ have nonzero $\mathbf{K}_0(p\mathbb{Z}_p)$-fixed vectors, which, in turn, implies that π_p is Iwahori spherical so that their isomorphism classes are listed in [36, Table 3]. (For extended tables, we refer to [34, Appendix].) Note that in [36] a different symplectic form is used to define the symplectic group; up to the adjustment for this difference, the group P_1 in [36] is our $\mathbf{K}_0(p\mathbb{Z}_p)$. In [34], all irreducible admissible representations of $\mathsf{G}(\mathbb{Q}_p)$ that admit local Bessel models are classified, and the result is conveniently summarized in [32, Table 2]. Here is a summary of what is available for our π (as in Section 4.3):

- π_∞ as a representation of $\mathsf{G}(\mathbb{R})^0$ is a direct sum of an HDS and its complex conjugate; thus, π is CAP if and only if it is a Saito–Kurokawa lift from a cuspidal representation of $\mathbf{PGL}_2(\mathbb{A})$; which happens only if $l_1 = l_2$.
- Suppose $p \nmid N_\pi$. Then, the local representation π_p is of type I and tempered if π is non-CAP, and is of type IIb when π is CAP.
- Suppose $p \mid N_\pi$. Then, the local representation π_p is either of type IIIa, in which case $\dim_{\mathbb{C}} V_{\pi_p}^{\mathbf{K}_0(p\mathbb{Z}_p)} = 2$, or of type VIb, in which case $\dim_{\mathbb{C}} V_{\pi_p}^{\mathbf{K}_0(p\mathbb{Z}_p)} = 1$; when π is CAP, π_p has to be of type VIb.

4.4. ***Bessel models.*** Let $p < \infty$. For any irreducible admissible representation (π_p, V_{π_p}) of $\mathsf{G}(\mathbb{Q}_p)$, let $(V_{\pi_p}^*)^{T_\theta,\Lambda_p}$ denote the space of all $\mathbb{C}$-linear forms $\ell : V_{\pi_p} \to \mathbb{C}$ that satisfy

$$\ell(\pi_p(\mathsf{m}(I_\theta(\tau), \mathrm{N}_{E/\mathbb{Q}}(\tau))n)\xi) = \Lambda_p(\tau)\psi_{T_\theta}(n)\,\ell(\xi), \quad \xi \in V_{\pi_p}, \tau \in E_p^\times, n \in \mathsf{N}(\mathbb{Q}_p).$$

It is known that $\dim_{\mathbb{C}}(V_\pi^*)^{T_\theta,\Lambda_p} \le 1$ [26; 33]. We say that π_p has a local (T_θ, Λ_p)-Bessel model if $(V_\pi^*)^{T_\theta,\Lambda_p} \ne (0)$; when this is the case, the space of functions of the form $g \mapsto \ell(\pi_p(g)\xi)$ with $\xi \in V_{\pi_p}$ is independent of $\ell \in (V_{\pi_p}^*)^{T_\theta,\Lambda_p} - (0)$; this space is denoted by $\mathscr{B}(T_\theta, \Lambda_p)[\pi_p]$ and is called the local (T_θ, Λ_p)-Bessel model of π_p.

When π_p is spherical, it is known that π_p has a local (T_θ, Λ_p)-Bessel model, and the space $\mathscr{B}(T_\theta, \Lambda_p)[\pi_p]$ contains a unique $\mathbf{K}_p$-invariant function B_{π_p} such that $B_{\pi_p}^0(1_4) = 1$ [42, Theorem 2-I; 6]. Let ξ_p^0 be the nonzero $\mathbf{K}_p$-fixed vector in V_{π_p}; then there exists a unique element $\ell_{\pi_p}^0 \in (V_{\pi_p}^*)^{T_\theta,\Lambda_p} - (0)$ such that $\ell_{\pi_p}^0(\pi_p(g)\xi_{\pi_p}^0) = B_{\pi_p}^0(g)$ for all $g \in \mathsf{G}(\mathbb{Q}_p)$. The pair $(\ell_{\pi_p}^0, \xi_{\pi_p}^0)$ is referred to as the unramified (T_θ, Λ_p)-Bessel datum for π_p.

Let $\pi \cong \bigotimes_{p \le \infty} \pi_p \in \Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N)$ (see Section 4.3).

Definition 4.7. A system $\{(\ell_p, \xi_p)\}_{p<\infty}$ with $\ell_p \in (V_{\pi_p}^*)^{T_\theta,\Lambda_p} - (0)$ and $\xi_p \in V_{\pi_p}^{\mathbf{K}_0(N_\pi\mathbb{Z}_p)}$, is called a (T_θ, Λ)-Bessel data for π if $(\ell_p, \xi_p) = (\ell_{\pi_p}^0, \xi_{\pi_p}^0)$ for $p \nmid N_\pi$ and $\ell_p(\xi_p) = 1$ for all $p < \infty$.

By [32, Theorem 2 8.2 and 9.3], a (T_θ, Λ)-Bessel data exists for our π. Once a (T_θ, Λ)-Bessel data $\{(\ell_p, \xi_p)\}_{p<\infty}$ is fixed, one can define $\varphi^0_{\pi,\varrho} \in V_\pi^{\mathbf{K}_0(N_\pi)}[\varrho]$ to be the V_ϱ-valued cusp form such that for any $v \in V_\varrho$, the function $(\varphi^0_{\pi,\varrho}(g) \mid v)_\varrho$ in $V_\pi \cong \bigotimes_{p\le\infty} V_{\pi_p}$ corresponds to the pure tensor $\bar{v} \otimes (\bigotimes_{p<\infty} \xi_p)$, where $\bar{v} = 1 \otimes v \in \overline{V}_\varrho \hookrightarrow V_{\pi_\infty}$. A particular choice of $\{(\ell_p, \xi_p)\}$ will be made in Section 6.8 so that $\varphi^0_{\pi,\varrho}$ corresponds to a newform on the arithmetic quotient $\Gamma_0(N)\backslash\mathfrak{h}_2$ in the sense of [7, §3.2].

Lemma 4.8. *Let $\{(\ell_p, \xi_p)\}_{p<\infty}$ be a (T_θ, Λ)-Bessel data for $\pi \in \Pi_{\mathrm{cusp}}(\lambda, N)$. Let $\phi = \bigotimes_p \phi_p \in \mathcal{S}(\mathbb{A}_E^2)$ be a decomposable element. Then, for any $\varphi \in V_\pi^{\mathbf{K}_0(N)}[\varrho]$ such that, for any $v \in V_\varrho$, φ_v corresponds to the pure tensor $v \otimes (\bigotimes_{p<\infty} v_p) \in \bigotimes_{p\le\infty} V_{\pi_p}$, we have*

$$B^{T_\theta,\Lambda}(\varphi; g) = B^{T_\theta,\Lambda}(\varphi^0_{\pi,\varrho}; g_\infty) \prod_{p<\infty} \ell_p(\pi_p(g_p)v_p), \qquad g = (g_p)_p \in \mathsf{G}(\mathbb{A}). \tag{4-19}$$

For $\mathrm{Re}(s) > 1$, $v \in V_\varrho$ *and* $b_{\mathbf{f}} = (b_p)_{p<\infty} \in \mathsf{G}(\mathbb{A}_{\mathbf{f}})$,

$$\begin{aligned}(4\text{-}20)\quad &\langle E(\phi, s, \Lambda, \mu), R(b_{\mathbf{f}} b^\theta_{\mathbb{R}})\overline{\varphi}_v\rangle \\ &\qquad = \frac{\sqrt{|D|}}{2} Z_v^{(\infty)}(\phi_\infty, \varphi^0_\pi; s, \mu_\infty, \Lambda) \prod_{p\le\infty} Z_p(\phi_p, \overline{B}_{v_p}; s, \mu_p; b_p),\end{aligned}$$

where

$$\begin{aligned} Z_v^{(\infty)}(\phi_\infty, \varphi^0_\pi; s, \mu_\infty, \Lambda) &:= \int_{\mathbf{K}^\#_\infty} \int_{\mathbb{R}^\times} f^{(s,1,\mu_\infty)}_{\phi_\infty}(k^\#_\infty)\mu_\infty(a)|a|^{s-1}_{\mathbb{R}} \\ &\qquad \times \big(v \mid B^{T_\theta,\Lambda}(\varphi^0_{\pi,\varrho}; \mathsf{m}(a1_2, a)\iota_\theta(k^\#)b^\theta_{\mathbb{R}})\big)_\varrho \,\mathrm{d}^\times a\, \mathrm{d}k^\#_\infty, \\ Z_p(\phi_p, \overline{B}_{v_p}; s, \mu_p; b_p) &:= \int_{\mathbf{K}^\#_p} \int_{\mathbb{Q}_p^\times} f^{(s,\Lambda_p,\mu_p)}_{\phi_p}(k^\#_p)\mu_p(a)|a|^{s-1}_p \\ &\qquad \times \overline{B}_{v_p}(\mathsf{m}(a_p 1_2, a_p)\iota_\theta(k^\#_p)b_p)\,\mathrm{d}^\times a_p\, \mathrm{d}k^\#_p, \end{aligned}$$

with $B_{v_p}(g_p) := \ell_p(\pi_p(g_p)v_p)$ for $g_p \in \mathsf{G}(\mathbb{Q}_p)$.

Proof. Fix $g_\infty \in \mathsf{G}(\mathbb{R})^0$ and $v \in V_\varrho$ and regard $\varphi \mapsto B^{T_\theta,\Lambda}(\varphi_v; g_\infty)$ as a linear functional on $\bigotimes_{p<\infty} V_{\pi_p}$ by the natural inclusion

$$\bigotimes_{p<\infty} V_{\pi_p} \hookrightarrow v \otimes \Big(\bigotimes_{p<\infty} V_{\pi_p}\Big) \hookrightarrow V_\pi.$$

Then, by the local multiplicity-one theorem for Bessel functionals on $\mathsf{G}(\mathbb{Q}_p)$ recalled above, there exists $C_v(g_\infty) \in \mathbb{C}$ such that

$$(v \mid B^{T_\theta,\Lambda}(\varphi; g_\infty))_\varrho = C_v(g_\infty) \prod_{p<\infty} \overline{\ell_p(v_p)} \tag{4-21}$$

for φ corresponding to $\bigotimes_{p<\infty} v_p$. To determine $C_v(g_\infty)$, set $v_p = \xi_p$ for all $p < \infty$; then by $\ell_p(\xi_p) = 1$, we get $(v \mid B^{T_\theta,\Lambda}(\varphi^0_{\pi,\varrho}; g_\infty))_\varrho = C_v(g_\infty)$. Now we apply (3-7) with T_θ, Λ and φ replaced by $-T_\theta, \Lambda^{-1}$ and $R(b_{\mathbf{f}})\varphi$. ☐

4.5. ***Gamma factor and global Bessel period.*** Recall the point $b_{\mathbb{R}}^{\theta}$ in (2-14) and the vector $v_{\varrho}^{\theta} \in V_{\varrho}$ in (4-16). Since $\nu(b_{\mathbb{R}}^{\theta}) > 0$, the point $\mathsf{m}(a1_2, a)\iota_{\theta}(k_{\infty}^{\#})b_{\mathbb{R}}^{\theta}$ with $a \in \mathbb{R}^{\times}$ and $k_{\infty}^{\#} \in \mathbf{K}_{\infty}^{\#}$ belongs to $\mathsf{G}(\mathbb{R})^0$ if and only if $\operatorname{sgn}(a)\operatorname{sgn}(k_{\infty}^{\#}) = +1$, where $\operatorname{sgn}(k_{\infty}^{\#}) := \operatorname{sgn}(\nu(\iota_{\theta}(k_{\infty}^{\#})))$. Thus, by (3-6) and (4-6), we have

$$B^{T_{\theta},\Lambda}(\varphi_{\pi}^{0}; \mathsf{m}(a1_2, a)\iota_{\theta}(k_{\infty}^{\#})b_{\mathbb{R}}^{\theta}) = 0$$

unless $\operatorname{sgn}(a)\operatorname{sgn}(k_{\infty}^{\#}) = +1$. By Lemma 4.4 and (4-16),

$$B^{T_{\theta},\Lambda}(\varphi_{\pi}^{0}; g_{\infty}) = \pi w_D^{-1} R(\varphi_{\pi}^{0}, E, \Lambda)\, B_{\varrho}^{T_{\theta}}(g_{\infty}), \quad g_{\infty} \in \mathsf{G}(\mathbb{R})^0.$$

Substituting this, we have that $Z_{v}^{(\infty)}(\phi_{\infty}, \varphi_{\pi}^{0}; s, \Lambda)$ equals

$$2\pi w_D^{-1}\, \overline{R(\varphi_{\pi}^{0}, E, \Lambda)} \int_{(\mathbf{K}_{\infty}^{\#})^0} \int_0^{\infty} |a|_{\mathbb{R}}^{s-1} f_{\phi_{\infty}}^{(s,1,\mu_{\infty})}(k_{\infty}^{\#})(v \mid B_{\varrho}^{T_{\theta}}(\mathsf{m}(a1_2, a)\iota_{\theta}(k_{\infty}^{\#})b_{\mathbb{R}}^{\theta}))_{\varrho}\, \mathrm{d}^{\times}a\, \mathrm{d}k_{\infty}^{\#}. \tag{4-22}$$

Now we specify ϕ_{∞}. Recall that the highest weight of ϱ is $\lambda = (l_1, l_2)$ and that $l_1 - l_2 \in 2\mathbb{Z}_{>0}$. Set $d := l_1 - l_2$ and define

$$\mathsf{f}_{\varrho}(u) := (d+1)\frac{(\varrho(C\bar{u}C^{-1})\mathsf{v}_{\varrho}^{0} \mid \mathsf{v}_{\varrho}^{0})_{\varrho}}{(\mathsf{v}_{\varrho}^{0} \mid \mathsf{v}_{\varrho}^{0})_{\varrho}}, \qquad u \in (\mathbf{K}_{\infty}^{\#})^0,$$

with $C := \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & 1 \\ i & -i \end{smallmatrix}\right]$. If $u = \left[\begin{smallmatrix} a & -\bar{b} \\ b & \bar{a} \end{smallmatrix}\right] \in (\mathbf{K}_{\infty}^{\#})^0 = \mathbf{SU}(2)$ with $a = a' + ia''$, $b = b' + ib''$ and $A, B \in \mathbf{Mat}_2(\mathbb{R})$ defined as in (2-16), so that $(b_{\mathbb{R}}^{\theta})^{-1}\iota_{\theta}(u)b_{\mathbb{R}}^{\theta} = k_{\infty}(A + iB)$ (Lemma 2.2), then a computation reveals $C\bar{u}C^{-1} = A - iB$. Thus, the automorphism $u \mapsto C\bar{u}C^{-1}$ of $\mathbf{SU}(2)$ brings the subgroup

$$\mathsf{B}^{\#}(\mathbb{R}) \cap (\mathbf{K}_{\infty}^{\#})^0 = \left\{\left[\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right] \mid a \in \mathbb{C}^1\right\}$$

to the subgroup $\mathbf{SO}(2)$, which fixes v_{ϱ}^{0}. Hence, f_{ϱ} is left-$\mathsf{B}^{\#}(\mathbb{R}) \cap (\mathbf{K}_{\infty}^{\#})^0$-invariant. Thus, since $\mathsf{G}^{\#}(\mathbb{R}) = \mathsf{B}^{\#}(\mathbb{R})(\mathbf{K}_{\infty}^{\#})^0$, there exists a unique element $\mathsf{f}_{\varrho}^{(s)}$ of $\mathcal{V}^{\#}(s, 1, \mu_{\infty})$ such that $\mathsf{f}_{\varrho}^{(s)}|\mathbf{K}_{\infty}^{\#} = \mathsf{f}_{\varrho}$. Define $\phi_{\infty} \in \mathcal{S}(\mathbb{C}^2)$ by

$$\phi_{\infty}(x, y) := (d+1)|D|^{-d/4} \sum_{j=0}^{d/2} \binom{d/2}{j}^2 (x\bar{x})^j (-y\bar{y})^{d/2-j}\, \tfrac{2}{\pi} \exp\Big(-\tfrac{2\pi}{\sqrt{|D|}}(x\bar{x} + y\bar{y})\Big). \tag{4-23}$$

Then, noting that $\varrho(C)^{-1}\mathsf{v}_{\varrho}^{0} = (XY)^{d/2}$ and using formulas (8,2,4), (8.2.5) and (8.2.6) of [9], we easily confirm the relation

$$\phi_{\infty}([0, \bar{\tau}]u) = f_{\varrho}(u)\big||D|^{-d/4}|\tau|_{\mathbb{C}}^{d/2} \exp\Big(-\tfrac{2\pi}{\sqrt{|D|}}|\tau|_{\mathbb{C}}\Big), \quad \tau \in \mathbb{C}^{\times},\ u \in \mathbf{SU}(2).$$

Then, by computing the integral in (3-1) for $g_{\infty} \in \mathbf{SU}(2)$, we get the relation

$$f_{\phi_{\infty}}^{(s,1,\mu_{\infty})} = |D|^{\frac{s+1}{2}}(-1)^{\frac{l_1-l_2}{2}}\, \Gamma_{\mathbb{C}}\big(s + \tfrac{l_1-l_2}{2} + 1\big)\, \mathsf{f}_{\varrho}^{(s)}. \tag{4-24}$$

For $u \in (\mathbf{K}_\infty^\#)^0$ and A, B as above, by (4-1), (4-16) and Lemma 2.2, we have

$$\big(v \mid B_\varrho^{T_\theta}(\mathsf{m}(a1_2, a)\iota_\theta(u)b_\mathbb{R}^\theta)\big)_\varrho = (-1)^{\frac{l_1+l_2}{2}} a^{\frac{l_1+l_2}{2}} \exp\big(-2\sqrt{|D|}\pi\, a\big)\,\big(v \mid \varrho(C\bar{u}C^{-1})^{-1}\mathrm{v}_\varrho^0\big)_\varrho.$$

By using the orthogonality of matrix coefficients on $\mathbf{SU}(2)$, the integral in (4-22) is computed as

$$|D|^{\frac{s}{2}}(-1)^{l_1}\Gamma_\mathbb{C}\big(s+\tfrac{l_1-l_2}{2}+1\big)(v \mid \mathrm{v}_\varrho^0)_\varrho \int_0^\infty a^{s+\frac{l_1+l_2}{2}-1}\exp\big(-2\sqrt{|D|}\pi\, a\big)\,\mathrm{d}^\times a$$
$$= \tfrac{1}{2}|D|^{\frac{1}{2}(1-\frac{l_1+l_2}{2})}(-1)^{l_1}(v \mid \mathrm{v}_\varrho^0)_\varrho \Gamma_\mathbb{C}\big(s+\tfrac{l_1-l_2}{2}+1\big)\Gamma_\mathbb{C}\big(s+\tfrac{l_1+l_2}{2}-1\big)$$

for $\mathrm{Re}(s)+\frac{l_1+l_2}{2} > 1$. Recall $L(s,\pi_\infty) = \Gamma_\mathbb{C}\big(s+\frac{l_1-l_2}{2}+\frac{1}{2}\big)\Gamma_\mathbb{C}\big(s+\frac{l_1+l_2}{2}-\frac{3}{2}\big)$. Thus,

$$\text{(4-25)}\quad Z_v^{(\infty)}(\phi_\infty, \varphi_\pi^0; s, \mu_\infty, \Lambda) = \pi w_D^{-1}\,\overline{R(\varphi_\pi^0, E, \Lambda)}(v \mid \mathrm{v}_\varrho^0)_\varrho(-1)^{l_1}|D|^{\frac{1}{2}(1-\frac{l_1+l_2}{2})}L\big(s+\tfrac{1}{2}, \pi_\infty\big).$$

The formula in [46, 7.23 Lemma] yields

$$\text{(4-26)}\qquad M(s)\mathsf{f}_\varrho^{(s)} = \frac{\pi}{s-d/2}\prod_{j=1}^{d/2}\frac{s-d/2+j-1}{s+d/2-j}\,\mathsf{f}_\varrho^{(-s)}.$$

Combining this with (3-4) and (4-24), we easily deduce

$$\text{(4-27)}\qquad f_{\hat{\phi}_\infty}^{(-s,1,\mu_\infty)} = \tfrac{|D|}{4}(-1)^{\frac{l_1-l_2}{2}} f_{\phi_\infty}^{(-s,1,\mu_\infty)}.$$

4.6. ***The spinor L-function and its functional equation.*** Let $\pi \cong \bigotimes_v \pi_v$ be a cuspidal automorphic representation of $\mathsf{G}(\mathbb{A})$ with the trivial central character: then, $\bar{\pi}_p \cong \pi_p^\vee \cong \pi_p$ for all $p < \infty$ by [43, Proposition 2.3]. The twist π_μ of π by an idele class character $\mu : \mathbb{A}^\times/\mathbb{Q}^\times \to \mathbb{C}^1$ is defined on the space V_π of π as $\pi_\mu(g) := \pi(g)\cdot(\mu\circ\nu)(g)$ for $g \in \mathsf{G}(\mathbb{A})$, so that the central character of π_μ is μ^2. Let $\pi \in \Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N)$ and μ be as in Section 4.3, so that $\lambda = (l_1, l_2) \in \mathbb{Z}^2$, $l_1 \equiv l_2 \pmod 2$, $l_1 \geq l_2 > 2$ and the ramification loci of π and μ are disjoint. We define the spinor L-function of π twisted by μ as the Euler product

$$L(s,\pi,\mu) := \prod_{p<\infty} L(s, (\pi_\mu)_p), \quad \mathrm{Re}(s) > \tfrac{5}{2},$$

with $L(s,(\pi_\mu)_p)$ the local L-factor listed in [34, Table A.8]. Define

$$\hat{L}(s,\pi,\mu) := \Gamma_\mathbb{C}\big(s+\tfrac{l_1-l_2}{2}+\tfrac{1}{2}\big)\Gamma_\mathbb{C}\big(s+\tfrac{l_1+l_2}{2}-\tfrac{3}{2}\big)\,L(s,\pi,\mu).$$

Using Proposition 3.1 and Lemmas 4.8 and 4.6, combining (4-25), (4-27) and the computations of the local zeta integrals for cases 1, 4, 5, and 6 in table (6-8), we obtain a meromorphic continuation of $L(s,\pi,\mu)$ to $\mathbb{C}$ as well as the functional equation

$$\hat{L}(s,\pi,\mu)=\varepsilon(s,\pi,\mu)\,\hat{L}(1-s,\pi,\bar{\mu})$$

with

$$\varepsilon(s,\pi,\mu):=(-1)^{l_2}\tilde{\mu}(N_\pi^2)\left(\frac{G(\tilde{\mu})}{\sqrt{M}}\right)^4(M^4N_\pi^2)^{1/2-s},$$

as expected. Moreover $\hat{L}(s,\pi,\mu)$ is holomorphic except for possible simple poles at $s=3/2,-1/2$, which does not occur when μ is nontrivial. By (4-26), $M(s)\mathsf{f}_\varrho^{(s)}$ has a simple zero at $s=1$ if $l_1>l_2$, which in turn implies the holomorphy at $s=1$ of the global intertwining operator applied to the section $f_\phi^{(s)}$ as well as the Eisenstein series $E(\phi,s,\Lambda,\mu)$ for ϕ_∞ as above. Hence, by (4-20) with an appropriate ϕ, $\hat{L}(s,\pi,\mu)$ is holomorphic at $s=3/2$ when $l_1>l_2$. If μ is real-valued, then

$$G(\tilde{\mu})/\sqrt{M}\in\{1,i\}\quad\text{and}\quad\varepsilon(1/2,\pi,\mu)=(-1)^{l_2},$$

so that $L(1/2,\pi,\mu)=0$ unless l_2 is even.

5. Spectral average of Rankin–Selberg integrals

The space $S_\varrho(\mathbf{K}_0(N))$ is endowed with the Hermitian inner product associated to the norm $\int_{\mathsf{Z}(\mathbb{A})\mathsf{G}(\mathbb{Q})\backslash\mathsf{G}(\mathbb{A})}(\varphi(g)\mid\varphi(g))_\varrho\,\mathrm{d}g$ ($\varphi\in S_\varrho(\mathbf{K}_0(N))$). Let $\mathscr{H}(\mathsf{G}(\mathbb{Q}_p)\,/\!\!/\,\mathbf{K}_p)$ be the Hecke algebra for $(\mathsf{G}(\mathbb{Q}_p),\mathbf{K}_p)$ for $p<\infty$. For any finite set S of primes p that is prime to N, define

$$\mathscr{H}_S:=\bigotimes_{p\in S}\mathscr{H}(\mathsf{G}(\mathbb{Q}_p)\,/\!\!/\,\mathbf{K}_p).$$

The $\mathbb{C}$-algebra $\mathscr{H}_S$ acts on the finite-dimensional Hilbert space $S_\varrho(\mathbf{K}_0(N))$ normally by

$$[R(f_S)\varphi](g)=\int_{\mathsf{G}(\mathbb{Q}_S)}\varphi(gx_S)f_S(x_S)\,\mathrm{d}x_S,\quad g\in\mathsf{G}(\mathbb{A}),\ f_S\in\mathscr{H}_S,\ \varphi\in S_l(\mathbf{K}_0(N)),$$

where $\mathsf{G}(\mathbb{Q}_S):=\prod_{p\in S}\mathsf{G}(\mathbb{Q}_p)$. Let μ and M be as in Section 4.3. We define the Schwartz–Bruhat function $\phi\in\mathcal{S}(\mathbb{A}_E^2)$ associated with μ as

$$\phi=\prod_{p\le\infty}\phi_p,\quad \phi_p(x,y)=\begin{cases}\mathbb{1}_{\mathscr{O}_{E_p}}(x)\mathbb{1}_{\mathscr{O}_{E_p}}(y) & (p<\infty,\ p\nmid M),\\ \mathbb{1}_{p^e\mathscr{O}_{E_p}}(x)\mathbb{1}_{1+p^e\mathscr{O}_{E_p}}(y) & (p<\infty,\ p^e\parallel M,\ e\ge1),\end{cases}$$

with ϕ_∞ as in (4-23). In this section, we investigate the averages

$$\text{(5-1)}\quad \mathbb{I}^{(s)}(\lambda,N,f_S):=\frac{1}{[\mathbf{K_f}:\mathbf{K}_0(N)]}\sum_{\varphi\in\mathscr{B}(\lambda,N)}\langle E(\phi,s,\Lambda,\mu),R(b)R(f_S)\overline{\varphi_{\mathsf{v}_\varrho^0}}\rangle\,B^{T_\theta,\Lambda}(\varphi_{\mathsf{v}_\varrho^0};b_{\mathbb{R}}^\theta),$$

where $\mathsf{v}_\varrho^0\in V_\varrho^{\mathbf{SO}(2)}$ is the vector from Section 4 and $b=(b_p)_{p\le\infty}\in\mathsf{G}(\mathbb{A})$ is defined by

$$b_p = \begin{cases} 1_4 & (p<\infty, p \nmid NM), \\ \eta_p & (p<\infty, p \mid N), \\ b_p^M := \left[\begin{smallmatrix} p^e 1_2 & T_\theta^\dagger \\ 0 & 1_2 \end{smallmatrix}\right] & (p<\infty, p^e \parallel M, e \geq 1), \\ b_{\mathbb{R}}^\theta & (p=\infty). \end{cases} \tag{5-2}$$

Here

$$\eta_p := \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & p & 0 & 0 \\ -p & 0 & 0 & 0 \end{bmatrix} \in \mathsf{G}(\mathbb{Q}_p)$$

is the Atkin–Lehner element, $\mathbf{K_f} := \mathbf{K}_0(1)$, and $\mathscr{B}(\lambda, N) = \bigcup_{\pi \in \Pi_{\mathrm{cusp}}(\lambda, N)} \mathscr{B}_\pi(\lambda, N)$ is an orthonormal basis of $S_\varrho(\mathbf{K}_0(N))$ with $\mathscr{B}_\pi(\lambda, N)$ an orthonormal basis of $V_\pi^{\mathbf{K}_0(N)}[\varrho]$. The sum is independent of the choice of an orthonormal basis and can be written as $\mathbb{I}^{(s)}(\Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N), f_S)$, where for any subset $X \subset \Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N)$ we define

$$\mathbb{I}^{(s)}(X, f_S) = \frac{1}{[\mathbf{K_f} : \mathbf{K}_0(N)]} \sum_{\pi \in X} \hat{f}_S(\pi_S) \times \sum_{\varphi \in \mathscr{B}_\pi(\lambda, N)} \langle E(\phi, s, \Lambda, \mu), R(b)\overline{\varphi_{\mathsf{v}_\varrho^0}} \rangle \, B^{T_\theta,\Lambda}(\varphi_{\mathsf{v}_\varrho^0}; b_{\mathbb{R}}^\theta), \tag{5-3}$$

where $\hat{f}_S(\pi_S)$ is the spherical Fourier transforms of f_S at $\pi_S := \bigotimes_{p \in S} \pi_p$, which is defined as the eigenvalue of the operator $\pi_S(f_S)$ on the $\mathbf{K}_S = \prod_{p \in S} \mathbf{K}_p$-fixed vectors $\pi_S^{\mathbf{K}_S} \cong \mathbb{C}$. For $\bullet \in \{\mathrm{T}, \mathrm{G}, \mathrm{Y}, \mathrm{SK}\}$ and $* \in \{\mathrm{new}, \mathrm{old}\}$, we define $\mathbb{I}^{(s)}(\lambda, N, f_S)^\bullet$, $\mathbb{I}^{(s)}(\lambda, N, f_S)^*$, and $\mathbb{I}^{(s)}(\lambda, N, f_S)^{\bullet,*}$ to be $\mathbb{I}^{(s)}(X, f_S)$ with $X = \Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N)^\bullet$, $\Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N)^*$ and $\Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N)^{\bullet,*}$, respectively. Then, due to (4-18), the average (5-1) has the expression

$$\mathbb{I}^{(s)}(\lambda, N, f_S) = \mathbb{I}^{(s)}(\lambda, N, f_S)^{\mathrm{G,new}} + \mathbb{I}^{(s)}(\lambda, N, f_S)^{\mathrm{Y,new}} + \mathbb{I}^{(s)}(\lambda, N, f_S)^{\mathrm{SK,new}} + \mathbb{I}^{(s)}(\lambda, N, f_S)^{\mathrm{old}}. \tag{5-4}$$

5.1. ***A construction of orthonormal basis.*** Let $\pi \cong \bigotimes_{p \leq \infty} \pi_p$ be an element of the set $\Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N)$ (see Section 4.3). We fix a (T_θ, Λ)-Bessel data $\{(\ell_p, \xi_p)\}_{p<\infty}$ of π (see Section 4.4) once and for all. Set $\varphi_\pi^0(g) := (\varphi_{\pi,\varrho}^0(g) \mid \mathsf{v}_\varrho^0)_\varrho$, which is an element of $V_\pi^{\mathbf{K}_0(N)}$. Since π is a unitary representation, all of its factors π_p are unitarizable. For each $p < \infty$, we can uniquely fix a $\mathsf{G}(\mathbb{Q}_p)$-invariant inner product $(\cdot \mid \cdot)_p$ on V_{π_p} by demanding $(\xi_p \mid \xi_p)_p = 1$; for $p = \infty$, we fix a $\mathsf{G}(\mathbb{R})$-invariant inner product so that its pullback to $V_\varrho \hookrightarrow \pi_\infty$ (cf. property (i) in Section 4.2) coincides with the inner product $(\cdot \mid \cdot)_\varrho$ of V_ϱ. Let φ_π^0 be the global new form attached to $\{\xi_p\}_{p<\infty}$ (see Section 4.4). Then,

$$\frac{(\varphi \mid \varphi)_{L^2}}{(\varphi_\pi^0 \mid \varphi_\pi^0)_{L^2}} = (v|v)_\varrho \prod_{p<\infty} (v_p|v_p)_p \tag{5-5}$$

for any $\varphi \in V_\pi$ that corresponds to $v \otimes \big(\bigotimes_{p<\infty} v_p\big)$ with $v \in V_\varrho$. As below, for $p<\infty$, fix an orthonormal basis

$$\mathscr{B}(\pi_p, \mathbf{K}_0(N\mathbb{Z}_p))$$

of $V_{\pi_p}^{\mathbf{K}_0(N\mathbb{Z}_p)}$ in such a way that

$$\mathscr{B}(\pi_p, \mathbf{K}_0(N\mathbb{Z}_p)) = \{\xi_{\pi_p}^0\} \quad \text{if } p \nmid N.$$

Then, a pure tensor of the form

$$(\varphi_\pi^0 \mid \varphi_\pi^0)_{L^2}^{-1/2} \times \bigotimes_{p<\infty} v_p, \quad v_p \in \mathscr{B}(\pi_p, \mathbf{K}_0(N\mathbb{Z}_p)), \tag{5-6}$$

yields an element $\varphi \in V_\pi^{\mathbf{K}_0(N)}[l]$ such that for $v \in V_\varrho$ the element $\varphi_v \in V_\pi$ corresponds to v tensored with (5-6). The set of functions φ obtained in this way from (5-6) will be denoted by $\mathscr{B}_\pi(\lambda, N)$. If $\pi \in \Pi_{\text{cusp}}(\lambda, N)$ does not satisfy condition (A-iv) in Section 4.3, then we fix arbitrary orthonormal basis $\mathscr{B}_\pi(\lambda, N)$ of the space $V_\pi^{\mathbf{K}_0(N)}[\lambda]$. Let $\mathcal{B}(\lambda, N)$ be the union of the sets $\mathscr{B}_\pi(\lambda, N)$ for $\pi \in \Pi_{\text{cusp}}(\lambda, N)$; then, by (ii) in Section 4.2, the set $\mathscr{B}(\lambda, N)$ is an orthonormal basis of $S_\varrho(\mathbf{K}_0(N))$.

5.2. ***Computation of the average.*** Let φ correspond to a pure tensor as in (5-6). Then, by (4-20) and the computations recalled in Section 4.5 and Section 6.2,

$$\begin{aligned}(5\text{-}7)\quad &\langle E(\phi, s, \Lambda, \mu), R(b)\overline{\varphi_{\mathrm{v}_\varrho^0}}\rangle \\ &= (\varphi_\pi^0 \mid \varphi_\pi^0)_{L^2}^{-1/2}\, \pi\, w_D^{-1}\, \overline{R(\varphi_\pi^0, E, \Lambda)}\, \hat{L}\big(s+\tfrac12, \pi, \mu\big) \\ &\quad \times (\mathrm{v}_\varrho^0 \mid \mathrm{v}_\varrho^0)_\varrho (-1)^{l_2} 2^{-2} |D|^{\frac12(4-\frac{l_1+l_2}{2})} \prod_{p|N} Z_p^*(\phi_p, \bar{B}_{v_p}; s, \mu_p; b_p),\end{aligned}$$

where

$$Z_p^*(\phi_p, \bar{B}_{v_p}; s, \mu_p; b_p) := L\big(s+\tfrac12, \pi_p, \mu_p\big)^{-1} Z_p(\phi_p, \bar{B}_{v_p}; s, \mu_p; b_p)$$

is the normalized local zeta integral. Moreover, by (4-21) and Lemma 4.4,

$$\begin{aligned}(5\text{-}8)\quad & B^{T_\theta,\Lambda}(\varphi_{\mathrm{v}_\varrho^0}; b_{\mathbb{R}}^\theta) = \\ & \langle \varphi_\pi^0 \mid \varphi_\pi^0\rangle_{L^2}^{-1/2}\, \pi\, w_D^{-1} R(\varphi_\pi^0, E, \Lambda)\, (B_\varrho^{T_\theta}(b_{\mathbb{R}}^\theta) \mid \mathrm{v}_\varrho^0)_\varrho \prod_{p|N} \ell_p(v_p).\end{aligned}$$

Note that

$$B_\varrho^{T_\theta}(b_{\mathbb{R}}^\theta) = (-1)^{\frac{l_1+l_2}{2}} \exp(-2\pi\sqrt{|D|})\, \mathrm{v}_\varrho^0$$

by (4-1) and (2-14). From (5-3), (5-7) and (5-8), we get:

Proposition 5.1. *Let $f_S \in \mathscr{H}_S$. Then, $\mathbb{I}^{(s)}(\lambda, N, f_S)$ equals*

$$\frac{2^{-2}\pi^2|D|^{\frac{1}{2}(3-\frac{l_1+l_2}{2})}e^{-2\pi\sqrt{|D|}}}{w_D^2[\mathbf{K_f}:\mathbf{K}_0(N)]}M^{s-6}\zeta_M(1)\zeta_M(4)\tilde{\mu}(2D)G(\tilde{\mu})$$
$$\times (\mathsf{v}_\varrho^0 \mid \mathsf{v}_\varrho^0)_\varrho^2 \sum_{\pi\in\Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda,N)} \hat{f}_S(\pi_S)\frac{|R(\varphi_\pi^0, E, \Lambda)|^2}{(\varphi_\pi^0|\varphi_\pi^0)_{L^2}}\hat{L}\big(s+\tfrac{1}{2},\pi,\mu\big)$$
$$\times \prod_{p<\infty}\mathbb{I}^{(s)}_{\pi_p,\mathbf{K}_0(N\mathbb{Z}_p)}(\xi_p,\phi_p,\Lambda_p,\mu_p;b_p), \tag{5-9}$$

where ξ_p for $p<\infty$ is from the fixed (T_θ,Λ)-Bessel data $\{(\ell_p,\xi_p)\}_{p<\infty}$ of π, and

$$\mathbb{I}^{(s)}_{\pi_p,\mathbf{K}_0(N\mathbb{Z}_p)}(\xi_p,\phi_p,\Lambda_p,\mu_p;b_p) := \sum_{v\in\mathscr{B}(\pi_p,\mathbf{K}_0(N\mathbb{Z}_p))} Z_p^*(\phi_p,\bar{B}_{v_p};s,\mu_p,b_p)\,\ell_p(v). \tag{5-10}$$

We also use the notation $\zeta_M(s)$ to denote $\prod_{p|M}(1-p^{-s})^{-1}$.

Note that if π_p with $p\nmid N$ is unramified and $(\ell^0_{\pi_p},\xi^0_{\pi_p})$ is the unramified Bessel datum of π_p, then (5-10) is 1. For other cases, we compute (5-10) in Section 6.8 completely. Substituting them, we obtain

Theorem 5.2. *Let $\lambda=(l_1,l_2)$, N, μ, $\tilde{\mu}$ and M be as in Section 4.3. Let S be a finite set of prime numbers relatively prime to DMN. Then, for any $f_S=\bigotimes_{p\in S} f_p\in\mathscr{H}_S$, the value $\mathbb{I}^{(s)}(\lambda,N,f_S)$ equals*

$$\frac{2^{-2}\pi^2|D|^{\frac{1}{2}(3-\frac{l_1+l_2}{2})}e^{-2\pi\sqrt{|D|}}}{w_D^2[\mathbf{K_f}:\mathbf{K}_0(N)]}$$
$$\times M^{s-6}\zeta_M(1)\zeta_M(4)\tilde{\mu}(2D)G(\tilde{\mu})N^{s-1}\tilde{\mu}(N)\prod_{p|N}(1+p^{-2})^{-1}$$
$$\times (\mathsf{v}_\varrho^0|\mathsf{v}_\varrho^0)_\varrho^2 \sum_{\pi\in\Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda,N)} \hat{f}_S(\pi_S)\frac{|R(\varphi_\pi^0,E,\Lambda)|^2}{(\varphi_\pi^0|\varphi_\pi^0)_{L^2}}\hat{L}\big(s+\tfrac{1}{2},\pi,\mu\big)\,\mathfrak{t}^{(s)}(\pi,\mu), \tag{5-11}$$

with $\mathfrak{t}^{(s)}(\pi,\mu)=\prod_{p|N}\mathfrak{t}^{(s)}(\pi_p,\mu_p)$ and $\mathfrak{t}^{(s)}(\pi_p,\mu_p)$ defined as

$$\begin{cases} 1 & \text{if } p\mid N_\pi \text{ and } \pi_p \text{ is of type VIb},\\ 2 & \text{if } p\mid N_\pi \text{ and } \pi_p \text{ is of type IIIa},\\ 2(p-1)p^{-5}L(1,\pi_p,\mathrm{Std}) & \\ \qquad\times\big(1-\frac{\mu_p(p)p^{-s}}{p+1}\,\mathrm{tr}(p^{-1}T_{1,0}+\eta_p|\pi_p^{\mathbf{K}_0(p\mathbb{Z}_p)})+\mu_p^2(p)p^{-2s}\big) & \\ \qquad \text{if } p\mid \frac{N}{N_\pi}. \end{cases} \tag{5-12}$$

To refine (5-11) by considering $\mathbb{I}^{(s)}(\lambda, N, f_S)^{\bullet,\mathrm{new}}$ ($\bullet \in \{\mathrm{G}, \mathrm{Y}, \mathrm{SK}\}$), we use an explicit formula of the quantity $|R(\varphi_\pi^0, E, \Lambda)|^2/(\varphi_\pi^0|\varphi_\pi^0)_{L^2}$ first proved by Dickson, Pitale, Saha, and Schmidt [7, Theorems 1.13 and 3.13] when $l_1 = l_2$ under the refined GGP conjecture for Bessel periods posed by Liu [22]. Thanks to a theorem by Furusawa and Morimoto [10, Theorem 8.1], the formula in [7, Theorems 1.13 and 3.13] is extended to vector-valued forms unconditionally so that it can be applied to (5-11).

Theorem 5.3. *Let the notation and assumptions be as in Theorem 5.2.*

(i) $\mathbb{I}^{(s)}(\lambda, N, f_S)^{\mathrm{G,new}}$ *equals*

$$\frac{2^{\#S(N)+2l_1-6}\pi^2|D|^{\frac{l_1+l_2+2}{4}}e^{-2\pi\sqrt{|D|}}}{[\mathbf{K_f}:\mathbf{K}_0(N)]} \times M^{s-6}\zeta_M(1)\zeta_M(4)\tilde{\mu}(2D)G(\tilde{\mu})N^{s-1}\tilde{\mu}^{-1}(N)\prod_{p\in S(N)}(1+p^{-1}) \times \sum_{\pi\in\Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda,N)^{\mathrm{G,new}}} \hat{f}_S(\pi_S)\frac{\hat{L}(\frac{1}{2},\pi\times\mathcal{A}I(\Lambda^{-1}))}{\hat{L}(1,\pi;\mathrm{Ad})}\hat{L}\big(s+\tfrac{1}{2},\pi,\mu\big),$$

where $S(N)$ is the set of all the prime divisors of N, and $\mathcal{A}I(\Lambda^{-1})$ is the automorphic induction to $\mathbf{GL}_2(\mathbb{A})$ from the character Λ^{-1} of $\mathbb{A}_E^\times/E^\times$.

(ii) *If N has an even number of prime divisors, then $\mathbb{I}^{(s)}(\lambda, N, f_S)^{\mathrm{Y,new}} = 0$. If N has an odd number of prime divisors, then $\mathbb{I}^{(s)}(\lambda, N, f_S)^{\mathrm{Y,new}}$ equals*

$$\frac{2^{\#S(N)+2l_1-7}\pi^2|D|^{\frac{l_1+l_2+2}{4}}e^{-2\pi\sqrt{|D|}}}{[\mathbf{K_f}:\mathbf{K}_0(N)]} \times M^{s-6}\zeta_M(1)\zeta_M(4)\tilde{\mu}(2D)G(\tilde{\mu})N^{s-1}\tilde{\mu}^{-1}(N)\prod_{p\in S(N)}(1+p^{-1}) \times \sum_{\substack{\pi_1\in\Pi_{\mathbf{PGL}_2,\mathrm{cusp}}(l_1+l_2-2,N)^{\mathrm{new}}\\ \pi_2\in\Pi_{\mathbf{PGL}_2,\mathrm{cusp}}(l_1-l_2+2,N)^{\mathrm{new}}}} \bigg(\hat{f}_S(\mathrm{Y}(\pi_1,\pi_2)_S) \times \frac{\hat{L}\big(\frac{1}{2},\pi_1\times\mathcal{A}I(\Lambda^{-1})\big)\hat{L}\big(\frac{1}{2},\pi_2\times\mathcal{A}I(\Lambda^{-1})\big)\hat{L}\big(s+\frac{1}{2},\pi_1\times\mu\big)\hat{L}\big(s+\frac{1}{2},\pi_2\times\mu\big)}{\hat{L}(1,\pi_1;\mathrm{Ad})\hat{L}(1,\pi_2;\mathrm{Ad})\hat{L}(1,\pi_1\times\pi_2)}\bigg),$$

where $\Pi_{\mathbf{PGL}_2,\mathrm{cusp}}(k,N)^{\mathrm{new}}$ is the set of all irreducible cuspidal representations of $\mathbf{PGL}_2(\mathbb{A})$ associated to holomorphic newforms of weight k and level N and $\mathrm{Y}(\pi_1,\pi_2)\in\Pi_{\mathrm{cusp}}(\lambda,N)^{\mathrm{new}}$ denotes the Yoshida lift of π_1 and π_2.

(iii) *If* $\Lambda \neq \mathbb{1}$ *or* $l_1 > l_2$, *then* $\mathbb{I}^{(s)}(\lambda, N, f_S)^{\mathrm{SK,new}} = 0$. *If* $\Lambda = \mathbb{1}$ *and* $l_1 = l_2\ (=: l)$, *then* $\mathbb{I}^{(s)}((l,l), N, f_S)^{\mathrm{SK,new}}$ *equals*

$$\frac{3\cdot 2^{2l-3}\pi^2|D|^{(l+2)/2}e^{-2\pi\sqrt{|D|}}}{[\mathbf{K_f}:\mathbf{K}_0(N)]}\cdot\frac{s}{4\pi}$$
$$\times M^{s-6}\zeta_M(1)\zeta_M(4)\tilde{\mu}(2D)G(\tilde{\mu})N^{s-1}\tilde{\mu}^{-1}(N)\prod_{p\in S(N)}(1+p^{-1})\cdot\hat{L}(s+1,\mu)\hat{L}(s,\mu)$$
$$\times\sum_{\pi_0\in\Pi^{(T_\theta,1)}_{\mathbf{PGL}_2,\mathrm{cusp}}(2l-2,N)^{\mathrm{new}}}\hat{f}_S(\mathrm{SK}(\pi_0)_S)\frac{\hat{L}(\frac{1}{2},\pi_0\times\chi_D)\hat{L}(1,\chi_D)^2}{\hat{L}(\frac{3}{2},\pi_0)\hat{L}(1,\pi_0;\mathrm{Ad})}\hat{L}\big(s+\tfrac{1}{2},\pi_0\times\mu\big),$$

where $\Pi^{(T_\theta,1)}_{\mathbf{PGL}_2,\mathrm{cusp}}(2l-2,N)^{\mathrm{new}}$ *is the set of all* $\pi_0\in\Pi_{\mathbf{PGL}_2,\mathrm{cusp}}(2l-2,N)^{\mathrm{new}}$ *such that the Saito–Kurokawa lift* $\mathrm{SK}(\pi_0)$ *of* π_0 *has the* $(T_\theta,\mathbb{1})$*-Bessel model, and* χ_D *is the Kronecker character of modulo* D.

Proof. The equalities in (i) and (ii) are a direct corollary to Theorem 5.2 and [9, Theorem 8.1] (for the scalar case we refer to [7, Theorems 1.13 and 3.14]); note that the polynomial $Q_{S,\varrho}$ (with $S=T_\theta$) in [9, (8.2.16)] equals

$$(-1)^{\frac{l_1-l_2}{2}}\Big(\frac{2}{\sqrt{|D|}}\Big)^{\frac{l_1+l_2}{2}}\mathrm{v}^{\theta}_{\varrho},$$

our $R(\varphi,E,\Lambda)$ equals the quantity $w_D\big(\frac{\sqrt{|D|}}{2}\big)^{-\frac{l_1+l_2}{2}}\pi\,\mathcal{B}_\Lambda(\varphi;E)/(Q_{S,\varrho},Q_{S,\varrho})_{l_1-l_2}$ defined by [9, (8.2.17)], and $(Q_{S,\varrho},Q_{S,\varrho})_{l_1-l_2}=\big(\frac{|D|}{4}\big)^{-(l_1+l_2)/2}(\mathrm{v}^0_\varrho,\mathrm{v}^0_\varrho)_{l_1-l_2}=\big(\frac{|D|}{4}\big)^{-(l_1+l_2)/2}(\mathrm{v}^0_\varrho\mid\mathrm{v}^0_\varrho)_\varrho$. We also note the formula

$$\hat{L}(s,\pi,\mu)=\hat{L}(s,\pi_1\times\mu)\hat{L}(s,\pi_2\times\mu),$$

where $\pi=Y(\pi_1,\pi_2)$. This is obtained immediately by comparing the local factors. It is noted in [7, p.296] that each local representation π_p of $\pi\cong\bigotimes_p\pi_p\in\Pi^{(T_\theta,\Lambda)}_{\mathrm{cusp}}(\lambda,N)^{\mathrm{Y,new}}$ for $p\mid N$ is of type VIb. Finally, we check statement (iii). As quoted in [7, Theorem 3.11], the (T_θ,Λ)-Bessel periods of the Saito–Kurokawa lifts are zero unless Λ is trivial due to Qiu. By comparing the local L-functions ([38, Theorem 5.2(ii)], cf. [28, Theorem 3.1]), we have

$$\hat{L}(s,\pi,\mu)=\frac{1}{4\pi}(s-1/2)\,\hat{L}(s,\pi_0\times\mu)\hat{L}(s+1/2,\mu)\hat{L}(s-1/2,\mu),\tag{5-13}$$

where $\pi=\mathrm{SK}(\pi_0)$. By [38] and the definition, the set $\Pi^{(T_\theta,1)}_{\mathrm{cusp}}((l,l),N)^{\mathrm{SK}}$ corresponds bijectively to the set $\Pi^{(T_\theta,1)}_{\mathbf{PGL}_2,\mathrm{cusp}}(2l-2,N)$ via the Saito–Kurokawa lifting if $l\geq 3$. Note that the condition $l\geq 3$ is ensured by assumption (A-ii) in Section 4.3. □

By (5-4), $\mathbb{I}^{(s)}(\lambda,N,f_S)-\mathbb{I}^{(s)}(\lambda,N,f_S)^{\mathrm{G,new}}$ is the sum of $\mathbb{I}^{(s)}(\lambda,N,f_S)^{\mathrm{Y,new}}$, $\mathbb{I}^{(s)}(\lambda,N,f_S)^{\mathrm{SK,new}}$ and $\mathbb{I}^{(s)}(\lambda,N,f_S)^{\mathrm{old}}$. Our main focus is their values at $s=0$.

For those, we have the following upper bounds in the level aspect. For $f_S \in \mathscr{H}_S$, let $\|f_S\|_1 := \int_{\mathsf{G}(\mathbb{Q}_S)} |f_S(g_S)|\,\mathrm{d}g_S$ denote its L^1-norm.

Theorem 5.4. *Let l, μ, $\tilde{\mu}$, M, S be as in Theorem 5.2.*

(i) *Let λ be fixed as before and N an inert prime in $E/\mathbb{Q}$. Then, there exist constants $C_{D,\lambda,\mu}, C'_{D,\lambda,\mu} > 0$ depending only on D, l, μ, and f_S such that*

$$|\mathbb{I}^{(0)}(\lambda, N, f_S)^{\mathrm{old}}| < C_{D,\lambda,\mu}\|f_S\|_1 N^{-8}, \tag{5-14}$$

$$|\mathbb{I}^{(0)}(\lambda, N, f_S)^{\mathrm{Y,new}}| < C'_{D,\lambda,\mu}\|f_S\|_1 N^{-\frac{3}{2}}. \tag{5-15}$$

(ii) *Let N be 1 or an inert prime in $E/\mathbb{Q}$. When $\mu \neq \mathbf{1}$, $\mathbb{I}^{(0)}(l, N, f_S)^{\mathrm{SK,new}} = 0$. When $\mu = \mathbf{1}$, there exists a constant $C > 0$ such that*

$$|\mathbb{I}^{(0)}((l,l), N, f_S)^{\mathrm{SK,new}}| < C|D|^{\frac{l}{2}} l^{\frac{1}{2}} \|f_S\|_1 N^{-\frac{3}{2}}. \tag{5-16}$$

Proof. We have the inequality $|\hat{f}_S(\pi_S)| \le \|f_S\|_1$ for all irreducible unitary representation π_S of $\mathsf{G}(\mathbb{Q}_S)$. Because N is a prime, one can check that the summation range of $\mathbb{I}^{(0)}(\lambda, N, f_S)^{\mathrm{old}}$ is $\Pi_{\mathrm{cusp}}^{(T_\theta,\chi)}(\lambda, N)^{\mathrm{old}} = \Pi_{\mathrm{cusp}}^{(T_\theta,\chi)}(\lambda, 1)$, which is independent of N. From the value of $\mathfrak{t}(\pi_p, \mu_p)$ in (5-12) for $p = N$, combined with the temperedness of π_N due to [48] and the matrix of $T_{1,0}$ in [31, Table 3], we get $\mathfrak{t}(\pi_N, \mu_N) = O(N^{-4})$. This and the equation $[\mathbf{K_f} : \mathbf{K}_0(N)] = N^3(1+N^{-2})(1+N^{-1})$ for prime N yield the bound (5-14).

Next we treat the average for Yoshida lifts. For $\pi_1 \in \Pi_{\mathbf{PGL}_2,\mathrm{cusp}}(l_1+l_2-2, N)^{\mathrm{new}}$ and $\pi_2 \in \Pi_{\mathbf{PGL}_2,\mathrm{cusp}}(l_1-l_2+2, N)^{\mathrm{new}}$, we need the lower bound

$$L(1, \pi_1 \times \pi_2) \gg_{l_1,l_2} \exp(-C\sqrt{\log N})$$

uniform in N. This is a special case of Lemma 5.5, because π_1 and π_2 are everywhere tempered by Deligne's estimate and by the fact that the local p-components of π_i for $p \mid N$ are the (twisted) Steinberg representation, which is tempered. Note that $\pi_1 \not\cong \pi_2$ due to the weight condition. Now, we deduce the inequality (5-15) bounding the sum from above by a product of the average considered in [8, Theorem 1.1]; to do this, we invoke the subconvexity bound

$$L\left(\tfrac{1}{2}, \pi\right) = O(C(\pi)^{\frac{1}{4}-\delta}) \quad (\exists\, \delta > 0) \tag{5-17}$$

for automorphic cuspidal representations π of $\mathbf{GL}_2(\mathbb{A})$ ([24]) and the nonnegativity of $L\left(\frac{1}{2}, \pi_i \times \mathcal{AI}(\Lambda^{-1})\right) = L\left(\frac{1}{2}, \mathrm{BC}_{E/\mathbb{Q}}(\pi_i) \otimes \Lambda^{-1}\right)$ for $i = 1, 2$ due to [15].

Let us prove (5-16); by Theorem 5.3(ii), we may assume $\lambda = (l, l)$. Suppose $\mu \neq \mathbf{1}$; then, by Theorem 5.3(ii), $\mathbb{I}^{(s)}((l,l), N, f_S)^{\mathrm{SK,new}} \times s^{-1}$ is entire, hence $\mathbb{I}^{(0)}((l,l), N, f_S)^{\mathrm{SK,new}} = 0$ because $\hat{L}(s, \mu)$, as well as $\hat{L}(s, \pi_0 \times \chi_D)$, is entire. In the rest of the proof, we assume $\mu = \mathbf{1}$. Then, $s\hat{L}(s, \mu)\hat{L}(s+1, \mu)$ has a simple pole at $s = 0$. By [37, Theorem 3.1 and Table 2], $\mathrm{SK}(\pi_0)$ with $\pi_0 \in \Pi_{\mathrm{cusp},\mathbf{PGL}_2}(2l-2, N)$ has the global (T_θ, Λ)-Bessel model only if the sign of the functional equation of

π_0 is -1 so that $\hat{L}(s+1/2,\pi_0)$ has a zero at $s=0$. Hence, Theorem 5.3(ii) gives us the following majorant of $\mathbb{I}^{(0)}((l,l),N,f_S)^{\mathrm{SK,new}}$:

$$\frac{2^3\pi^{\frac{5}{2}}(l-1)^{-2}}{[\mathbf{K_f}:\mathbf{K}_0(N)]}\cdot\frac{\Gamma(l)\|f_S\|_1}{\Gamma(l-\frac{1}{2})}|D|^{\frac{l}{2}}\sum_{\pi_0\in\Pi^{(T_\theta,\mathbf{1})}_{\mathbf{PGL}_2,\mathrm{cusp}}(2l-2,N)}\frac{L(\frac{1}{2},\pi_0\times\chi_D)\,|L'(\frac{1}{2},\pi_0)|}{L(1,\pi_0;\mathrm{Ad})}.$$

To estimate this, we invoke a subconvexity bound $L(1/2,\pi_0\times\chi_D)=O((l^2N)^{\frac{1}{4}-\delta})$ for some $\delta>0$ from (5-17) and a lower bound

$$L(1,\pi_0;\mathrm{Ad})>c_0\exp\bigl(-c_1\sqrt{\log(1+l^2N)}\bigr)$$

for some constant $c_0,c_1>0$ which is known by [12, Theorem 0.1] (see also the remark after [21, Corollary 7]). The lower bound implies $L(1,\pi_0;\mathrm{Ad})^{-1}=O((l^2N)^{2\delta})$. By a common argument, we derive a subconvexity bound for the central derivative $L'(1/2,\pi_0)=O((l^2N)^{\frac{1}{4}-\delta})$ from the bound $L(1/2,\pi_0|\cdot|_{\mathbb{A}}^{it})=O((l^2N(1+|t|))^{1/4-\delta})$ that follows from (5-17). Finally, (5-16) follows using the uniform bound $\#\Pi_{\mathbf{PGL}_2,\mathrm{cusp}}(2l-2,N)=O(lN)$ and the asymptotic $\frac{\Gamma(l)}{\Gamma(l-\frac{1}{2})}\sim l^{\frac{1}{2}}$ $(l\to\infty)$. □

Lemma 5.5. *Let π_1 and π_2 be irreducible cuspidal automorphic representations of* $\mathbf{PGL}_n$ *such that $C(\pi_1),C(\pi_2)\le Q$ with $Q>2$. We assume that $\pi_1\not\cong\pi_2$ and both of them are self-dual and tempered everywhere. Then,*

$$L(1,\pi_1\times\pi_2)\gg\exp(-C\sqrt{\log Q})$$

with an absolute constant $C>0$.

Proof. We recall the argument indicated in [35] (attributed originally to [25]), which eliminates a possibility of Siegel zeros of the L-function $L(s,\pi_1\times\pi_2)$. Fix $t\in\mathbb{R}$. Then, $L(s,\Pi\times\widetilde{\Pi})$ with Π being the isobaric sum

$$\Pi:=\pi_1\boxplus(\pi_1\times|\cdot|_{\mathbb{A}}^{it})\boxplus(\pi_1\times|\cdot|_{\mathbb{A}}^{-it})\boxplus\pi_2\boxplus(\pi_2\times|\cdot|_{\mathbb{A}}^{it})\boxplus(\pi_2\times|\cdot|_{\mathbb{A}}^{-it})$$

is a Dirichlet series with nonnegative coefficients [13, Lemma a]. Moreover, by [16, Proposition 9.4] and the self-duality of π_1 and π_2, one expresses $L(s,\Pi\times\widetilde{\Pi})$ as

$$L(s,\Pi\times\widetilde{\Pi})=\prod_{1\le i,j\le 2}\begin{bmatrix}L(s,\pi_i\times\pi_j)^3L(s-it,\pi_i\times\pi_j)^2L(s+it,\pi_i\times\pi_j)^2\\ \times L(s-2it,\pi_i\times\pi_j)L(s+2it,\pi_i\times\pi_j)\end{bmatrix}^2,$$

which shows that $L(s,\Pi\times\widetilde{\Pi})$ has a pole of order 6 at $s=1$ (due to $\pi_1\not\cong\pi_2$), and has a zero of order 8 at $s=\sigma$ if $L(\sigma+it,\pi_i\times\pi_j)=0$ $(i,j=1,2)$. Hence, by [11, p. 178, Lemma], one can show that $L(s,\pi_1\times\pi_2)$ has no zeros on the interval $\bigl(1-\frac{C_0}{\log M},1\bigr)$ for some constant $C_0>0$ with

$$M=(1+|t|)^{24}C(\pi_2\times\pi_2)^{18}C(\pi_1\times\pi_1)^9C(\pi_2\times\pi_2)^2,$$

where $C(\pi_i \times \pi_j)$ is the analytic conductor of $L(s, \pi_i \times \pi_j)$. By [13, Lemma b], we have $C(\pi_i \times \pi_j) \ll Q^{2n}$. Thus, for an absolute constant $C_0' > 0$, $L(s, \pi_1 \times \pi_2)$ is zero-free on the region $1 - \frac{C_0'}{\log Q(1+|\mathrm{Im}(s)|)} < \mathrm{Re}(s) < 1$. Now, we apply the argument of [21, Corollary 7] to the L-function $L(s, \pi_1 \times \pi_2)$. Since π_1, π_2 are assumed to be tempered everywhere, the optimal bound of the lambda function $\Lambda_{\mathcal{A}}(n)$ for $\mathcal{A} = \pi_1 \times \pi_2$ is available so that the automorphy of $\pi_1 \times \pi_2$ is not necessary, which simplifies the proof to get the lower bound of $L(1, \pi_1 \times \pi_2)$. $\square$

As a consequence of Theorem 5.4, we get

$$\mathbb{I}^{(0)}(\lambda, N, f_S)^{\mathrm{G,new}} = \mathbb{I}^{(0)}(\lambda, N, f_S) + O_{\Lambda,\lambda,\mu}(\|f_S\|_1 N^{-\frac{3}{2}}). \tag{5-18}$$

6. Computation of local zeta integrals for p-adic fields

In this section, let F be a nonarchimedean local field of characteristic 0, $\mathcal{O}$ the integer ring of F, $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$, ϖ a generator of $\mathfrak{p}$ and $q = \#(\mathcal{O}/\mathfrak{p})$. Let $|\cdot|$ denote the normalized absolute value of F, i.e., $|\varpi| = q^{-1}$. Fix a nontrivial additive character $\psi : F \to \mathbb{C}^1$ with $\mathrm{cond}(\psi) := \min\{n \in \mathbb{Z}\ ;\ \psi|_{\varpi^n\mathcal{O}} = 1\} = 0$.

We compute the local zeta integral à la Piatetski-Shapiro [29] for several representations, taking particular test functions; as a result, we determine the local L-factors and the local ε-factors in [29] to confirm that they coincides with the expected ones listed in [34, Tables A8 and A9]. As explained in Section 4.3, for a particular global application in mind, we only deal with representations of types I, IIb, IIIa and VIb (but allowing the central characters to be nontrivial when we are concerned with newvectors.)

6.1. ***Local zeta integral for Bessel models.*** We first review some generalities on local zeta integrals and then recall results from [32] on explicit formulas of Bessel functions for Iwahori spherical representations of G, which are possible local components of $\pi \cong \otimes_p \pi_p \in \Pi_{\mathrm{cusp}}^{(T_\theta,\Lambda)}(\lambda, N)$.

Let $K = \mathsf{G}(\mathcal{O})$ be a standard maximal compact subgroup of $\mathsf{G}(F)$ and

$$K_0(\mathfrak{p}^e) := \left\{\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in K \;\middle|\; C \equiv 0 \pmod{\mathfrak{p}^e}\right\}$$

be the congruence subgroup of level $\mathfrak{p}^e$.

For a symmetric matrix $T = \left[\begin{smallmatrix} \mathsf{a} & \mathsf{b}/2 \\ \mathsf{b}/2 & \mathsf{c} \end{smallmatrix}\right] \in \mathrm{Sym}_2(F)$ such that $\mathsf{d} := \mathsf{b}^2 - 4\mathsf{ac} \neq 0$, set $\xi = \left[\begin{smallmatrix} \mathsf{b}/2 & \mathsf{c} \\ -\mathsf{a} & -\mathsf{b}/2 \end{smallmatrix}\right]$. Let $L := F \oplus F\xi$ be the two-dimensional F-algebra, $\mathcal{O}_L$ be its integer ring, and $\mathfrak{p}_L := \mathfrak{p}\mathfrak{o}_L$. Define the additive character ψ_L on L by $\psi_L := \psi \circ \mathrm{tr}_{L/F}$. The maps

$$L \ni x + y\xi \mapsto x + \tfrac{\sqrt{\mathsf{d}}}{2}y \in F(\sqrt{\mathsf{d}}) \qquad (\mathsf{d} \notin (F^\times)^2),$$
$$L \ni x + y\xi \mapsto \left(x + \tfrac{\sqrt{\mathsf{d}}}{2}y, x - \tfrac{\sqrt{\mathsf{d}}}{2}y\right) \in F \oplus F \quad (\mathsf{d} \in (F^\times)^2),$$

are isomorphisms of F-algebras. We assume that

(i) $\mathsf{a} \in \mathscr{O}^\times$ and $\mathsf{b}, \mathsf{c} \in \mathscr{O}$.

(ii) If $\mathsf{d} \notin (F^\times)^2$, then $\sqrt{\mathsf{d}}\mathscr{O}_L$ is the different ideal of L/F.

(iii) If $\mathsf{d} \in (F^\times)^2$, then $\mathsf{d} \in \mathscr{O}^\times$.

We can check that

$$L^\times = \left\{\tau \in \mathbf{GL}_2(F) \,\middle|\, {}^t\tau T \tau = \det\tau \cdot T\right\}.$$

The map $L \ni \tau \mapsto \tau^\dagger \in L$ is the nontrivial F-automorphism on L. We set $\theta_0 = \mathsf{b}/2 - \xi \in \mathscr{O}_L$, then $\{\mathsf{a}, \theta_0\}$ is an $\mathscr{O}$-basis of $\mathscr{O}_L$. Let $\langle\cdot,\cdot\rangle$ denote the symplectic form on L^2 over F defined by

$$\langle x, y\rangle := \mathrm{tr}_{L/F}(-(2\xi)^{-1}(x_1y_2 - x_2y_1)), \quad x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right], y = \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] \in L^2.$$

We set

$$\begin{aligned} G^\#(F) &:= \left\{g \in \mathbf{GL}_2(L) \,\middle|\, \det g \in F^\times\right\}, \\ B^\#(F) &:= \left\{\left[\begin{smallmatrix} \tau & \beta \\ 0 & a\tau^\dagger \end{smallmatrix}\right] \,\middle|\, \tau \in L^\times, \beta \in L, a \in F^\times\right\}, \\ K^\# &:= G^\#(F) \cap \mathbf{GL}_2(\mathscr{O}_L), \\ K_0^\#(\mathfrak{p}^e) &:= G^\#(F) \cap \left[\begin{smallmatrix} \mathscr{O}_L & \mathscr{O}_L \\ \mathfrak{p}_L^e & \mathscr{O}_L \end{smallmatrix}\right]. \end{aligned}$$

Since $\langle g^\# x, g^\# y\rangle = \det g^\# \langle x, y\rangle$ for any $x, y \in L^2$ and $g^\# \in \mathsf{G}(F)$, we get a natural embedding $G^\#(F) \ni g \mapsto \iota(g^\#) \in \mathsf{G}(F)$. More precisely, $\iota(g^\#)$ is the representation matrix of the action $L^2 \ni x \mapsto g^\# x \in L^2$ with respect to an F-basis $\left\{\left[\begin{smallmatrix} \mathsf{a} \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} \theta_0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ -\mathsf{a}^{-1}\theta_0^\dagger \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right\}$. A computation yields

$$\iota\left(\left[\begin{smallmatrix} \tau & 0 \\ 0 & a\tau^\dagger \end{smallmatrix}\right]\right) = \mathsf{m}(\tau, a\det\tau),$$

$$\iota\left(\left[\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} 1_2 & X_\beta \\ 0 & 1_2 \end{smallmatrix}\right], \quad \beta = \beta_2\mathsf{a} + \beta_3\theta_0 \in L, \quad X_\beta := \left[\begin{smallmatrix} -\mathsf{a}^{-1}\mathsf{b}\beta_2 - \mathsf{a}^{-1}\mathsf{c}\beta_3 & \beta_2 \\ \beta_2 & \beta_3 \end{smallmatrix}\right], \tag{6-1}$$

$$\iota\left(\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\right) = \begin{bmatrix} 0 & 0 & -\mathsf{a}^{-2}\mathsf{b} & \mathsf{a}^{-1} \\ 0 & 0 & \mathsf{a}^{-1} & 0 \\ 0 & \mathsf{a} & 0 & 0 \\ \mathsf{a} & \mathsf{b} & 0 & 0 \end{bmatrix}. \tag{6-2}$$

We note that $K^\# = \iota^{-1}(K)$ and $K_0^\#(\mathfrak{p}^e) = \iota^{-1}(K_0(\mathfrak{p}^e))$. We call

$$R := \left\{\iota\left(\left[\begin{smallmatrix} \tau & 0 \\ 0 & \tau^\dagger \end{smallmatrix}\right]\right)n \,\middle|\, \tau \in L^\times, n \in \mathsf{N}(F)\right\}$$

the Bessel subgroup of $\mathsf{G}(F)$ (with respect to T). For a character $\Lambda : L^\times \to \mathbb{C}^1$, we can check that the map

$$R \ni \iota\left(\left[\begin{smallmatrix} \tau & 0 \\ 0 & \tau^\dagger \end{smallmatrix}\right]\right)\mathsf{n}(X) \mapsto \Lambda(\tau)\psi(\mathrm{tr}\,(TX)) \in \mathbb{C}^1$$

defines a character on R. We denote this character by $\Lambda \otimes \psi_T$.

Let (π, V_π) be an irreducible admissible representation of $\mathsf{G}(F)$. Let $(V_\pi^*)^{T,\Lambda}$ denote the space of all $\mathbb{C}$-linear forms $\ell : V_\pi \to \mathbb{C}$ that satisfies

$$\ell(\pi(r)\xi) = \Lambda \otimes \psi_T(r)\,\ell(\xi), \quad \xi \in V_\pi,\ r \in R.$$

Then, $\dim_{\mathbb{C}}(V_\pi^*)^{T,\Lambda} \le 1$ [26; 33]. We say that π has a local (T, Λ)-Bessel model if $(V_\pi^*)^{T,\Lambda} \ne (0)$. In this case, we can define the local (T, Λ)-Bessel model of π as in Section 4.4. Moreover, when π is spherical and ψ_T and Λ are unramified, we define the unramified (T, Λ)-Bessel datum $(\ell_\pi^0, \xi_\pi^0) \in (V_\pi^*)^{T,\Lambda} \times V_\pi^K$ for π as in Section 4.4, so that

$$B_\pi^0(g) = \ell_\pi^0(\pi(g)\xi_\pi^0), \quad g \in \mathsf{G}(\mathbb{Q}_p),$$

is the unramified Bessel function such that $B_\pi^0(1_4) = 1$. Recall that all irreducible admissible representations of $\mathsf{G}(F)$ that admit local Bessel models are classified in [34], and the result is conveniently summarized in [32, Table 2]. For our cases, as we mentioned in Section 4.4, we may assume that π is of type I, IIb, IIIa, or VIb. We quote some of its properties from [32, Table 2]:

(6-3)

type	π	$\dim V_\pi^K$	$\dim V_\pi^{K_0(\mathfrak{p})}$	cent. char.
I	$\chi \times \chi' \rtimes \sigma$	1	4	$\chi\chi'\sigma^2$
IIb	$\chi 1_{\mathbf{GL}_2} \rtimes \sigma$	1	3	$\chi^2\sigma^2$
IIIa	$\chi \rtimes \sigma_{\mathrm{St}_{\mathbf{GL}_2}}$	0	2	$\chi\sigma^2$
VIb	$\tau(T, \lvert\cdot\rvert^{-\frac{1}{2}}\sigma)$	0	1	σ^2

Here, χ, χ', and σ are unramified quasicharacters on $F^\times$ such that representations are irreducible and admit (T, Λ)-Bessel models. For later use we set $\alpha = \chi(\varpi)$, $\beta = \chi'(\varpi)$, and $\gamma = \sigma(\varpi)$. According to [34, Table A2], for type IIIa and type VIb, the corresponding representations are unitarizable if and only if the inducing quasicharacters are unitary, whereas, for type I, the unitarity of the inducing quasicharacters is equivalent to the corresponding representations being unitarizable and tempered. In what follows, having a global application in mind, we suppose the unitarity of all the inducing characters, i.e., $|\alpha| = |\beta| = |\gamma| = 1$.

Similarly to (3-1), for a Schwartz–Bruhat function $\phi \in \mathcal{S}(L^2)$, a character μ on $F^\times$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we define a function on $G^\#(F)$ by

$$f_\phi^{(s,\Lambda,\mu)}(g^\#) = \mu(\det g^\#)|\det g^\#|^{s+1} \int_{L^\times} \phi([\,0\ 1\,][\begin{smallmatrix}\tau & 0\\ 0 & \tau^\dagger\end{smallmatrix}]g^\#)\Lambda\mu_L(\tau)|\tau\tau^\dagger|^{s+1} \mathrm{d}^\times\tau,$$

where $\mu_L(\tau) := \mu(\tau\tau^\dagger)$, $\tau \in L^\times$ and $\mathrm{d}^\times\tau$ is the normalized Haar measure on $L^\times$ such that $\mathrm{vol}(\mathscr{O}_L) = 1$. For an irreducible smooth admissible representation π of $\mathsf{Z}(F)\backslash\mathsf{G}(F)$, admitting (T, Λ)-Bessel model and $B \in \mathscr{B}(T, \Lambda)[\pi]$, define the zeta integral by

(6-4)
$$Z(\phi, B, s, \mu; g) = \int_{F^\times}\int_{K^\#} B(\mathsf{m}(a1_2, a)\iota(k^\#)g)\mu(a)\,|a|^{s-1} f_\phi^{(s,\Lambda,\mu)}(k^\#)\mathrm{d}k^\#\mathrm{d}^\times a$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. It is shown that $Z(\phi, B, \mu; g)$ is a rational function in q^{-s} [40, Lemma 5.3.1]. Then, the local L-function $L(s, \pi, \mu)$ is defined to be the unique function such that $L(s,\pi,\mu)^{-1}$ is a polynomial in q^{-s} with constant term 1 and such that $Z(\phi,B,s,\mu,1)/L(s,\pi,\mu)\in\mathbb{C}[q^s,q^{-s}]$ for all ϕ and B ([29]; see also [40, Proposition 5.3.3]). For later use, we need the following:

Lemma 6.1. *Suppose* Λ *is unramified. Let* $B\in\mathscr{B}(T,\Lambda)^{K_0(\mathfrak{p})}$ *and* $\phi=\mathbb{1}_{\mathfrak{o}_L\oplus\mathfrak{o}_L}$*. Then, for* $\mathrm{Re}(s)\gg 0$,

$$Z(\phi,B,s,\mu;\eta)=\frac{L(s+1,\Lambda\mu_L)}{q^2+1}\Big\{\sum_{l=0}^{\infty}\eta B(h(l,0))\mu(\varpi)^l q^{-l(s-1)} + q^{s+1}\mu(\varpi)^{-1}\sum_{l=0}^{\infty}B(h(l,0))\mu(\varpi)^l q^{-l(s-1)}\Big\}, \tag{6-5}$$

where ηB *denotes the right-translation of* B *by* $\eta:=\begin{bmatrix} & & & -1\\ & & 1 & \\ & \varpi & & \\ -\varpi & & & \end{bmatrix}$.

Proof. It is easy to check the equality $f_\phi^{(s,\Lambda,\mu)}(k^\#)=L(s+1,\Lambda\mu_L)$ for all $k^\#\in K^\#$. By considering the left $K_0^\#(\mathfrak{p})$-coset decomposition of $K^\#$,

$$K^\#=K_0^\#(\mathfrak{p})\sqcup\Big(\bigsqcup_{\xi\in\mathscr{O}_L/\mathfrak{p}_L}\left[\begin{smallmatrix}1&\xi\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]K_0^\#(\mathfrak{p})\Big), \tag{6-6}$$

the zeta integral $Z(\phi,B,s,\mu;\eta)$ becomes $[K^\#:K_0^\#(\mathfrak{p})]^{-1}L(s+1,\Lambda\mu_L)$ times

$$\int_{F^\times}\Big(\eta B(\mathsf{m}(a1_2,a))+\sum_{\xi\in\mathscr{O}_L/\mathfrak{p}_L}\eta B\big(\mathsf{m}(a1_2,a)\iota(\left[\begin{smallmatrix}1&\xi\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right])\big)\Big)\mu(a)|a|^{s-1}\,\mathrm{d}^\times a.$$

For $a\in\varpi^l\mathfrak{o}^\times$ ($l\in\mathbb{Z}$), by the left R-equivariance and the right $K_0(\mathfrak{p})$-invariance of B, we have

$$B\big(\mathsf{m}(a1_2,a)\iota(\left[\begin{smallmatrix}1&\xi\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right])\big)=\psi(a\,\mathrm{tr}(T_\theta X_\xi))B(h(l+1,0))$$

due to the relation $\mathsf{m}(a1_2,a)\iota(\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right])\eta\in h(l+1,0)K_0(\mathfrak{p})$, which is easily confirmed by (6-2). Note that $\mathrm{tr}(T_\theta X_\xi)=0$. Hence, the ξ-sum becomes $q^2\times B(h(l+1,0))$ because $\#(\mathfrak{o}_L/\mathfrak{p}_L)=q^2$. From this and the vanishing of $B(h(l,0))$ for $l<0$ — see (6-10) — we get (6-5). □

By [29, Proposition 3.2], there exists an entire function $\varepsilon(\pi,s,\mu,\psi)$ such that the local functional equation

$$\varepsilon(\pi,s+1/2,\mu,\psi)\frac{Z(\phi,B,s,\mu;g)}{L(s+1/2,\pi,\mu)}=\frac{Z(\widehat{\phi},B^\dagger,-s,\mu^{-1};g)}{L(-s+1/2,\hat{\pi},\mu^{-1})} \tag{6-7}$$

holds. Here, we have defined $B^\dagger\in\mathscr{B}(T,\Lambda^\dagger)[\pi]$ by setting

$$B^\dagger(g):=B\left(\begin{bmatrix}1&\mathsf{a}^{-1}\mathsf{b}&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&-\mathsf{a}^{-1}\mathsf{b}&1\end{bmatrix}g\right)\quad(g\in\mathsf{G}(F)),$$

we have denoted by $\Lambda^\dagger$ the Galois conjugate of Λ defined by $\Lambda^\dagger(\tau):=\Lambda(\tau^\dagger)$ for

$\tau \in L^\times$, and $\widehat{\phi}$ is the Fourier transform of $\phi \in \mathcal{S}(L^2)$, defined as

$$\widehat{\phi}(x,y) = \int_{L^2} \phi(u,v)\psi_L\big(\langle \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]\rangle\big)\, \mathrm{d}u\mathrm{d}v$$

$\mathrm{d}u$ and $\mathrm{d}v$ being the Haar measures on L such that $\mathrm{vol}(\mathscr{O}_L) = 1$.

We shall get explicit evaluations of the zeta integrals $Z(\phi, B, s, \mu; g)$ for unramified Λ and particular choices of ϕ, B and g as shown in the next table, which allow us to determine $\varepsilon(s, \pi, \mu, \psi)$:

(6-8)

	type	ϕ	B	μ	L/F	g
case 1	I or IIb	$\mathbb{1}_{\mathscr{O}_L \oplus \mathscr{O}_L}$	K-fixed	unramified	–	1_4
case 2	I or IIb	$\mathbb{1}_{\mathfrak{p}_L^e \oplus 1+\mathfrak{p}_L^e}$	K-fixed	ramified	inert	b
case 3	I or IIb	$\widehat{\mathbb{1}_{\mathfrak{p}_L^e \oplus 1+\mathfrak{p}_L^e}}$	K-fixed	ramified	inert	b
case 4	I or IIb	$\mathbb{1}_{\mathscr{O}_L \oplus \mathscr{O}_L}$	$K_0(\mathfrak{p})$-fixed	unramified	inert	η
case 5	IIIa	$\mathbb{1}_{\mathscr{O}_L \oplus \mathscr{O}_L}$	$K_0(\mathfrak{p})$-fixed	unramified	inert	η
case 6	VIb	$\mathbb{1}_{\mathscr{O}_L \oplus \mathscr{O}_L}$	$K_0(\mathfrak{p})$-fixed	unramified	inert	η

Here, the integer e is the conductor of μ defined to be the minimum nonnegative integer n satisfying $\mu|_{(1+\mathfrak{p}^n)\cap\mathfrak{o}^\times} = 1$, $b \in \mathsf{G}(F)$ is given by

(6-9) $$b := \left[\begin{smallmatrix} \varpi^e 1_2 & T^\dagger \\ 0 & 1_2 \end{smallmatrix}\right],$$

and L/F is said to be inert if it is an unramified field extension. For $l, m \in \mathbb{Z}$, set

$$h(\ell,m) := \begin{bmatrix} \varpi^{\ell+2m} & & & \\ & \varpi^{\ell+m} & & \\ & & 1 & \\ & & & \varpi^m \end{bmatrix} \in \mathsf{G}(F).$$

As in [32], for a smooth representation (π, V_π) of $\mathsf{G}(F)$, we define an operator $T_{\ell,m} \in \mathrm{End}(V_\pi)$ by

$$T_{\ell,m} v := \mathrm{vol}(K_0(\mathfrak{p}))^{-1} \int_{K_0(\mathfrak{p})h(\ell,m)K_0(\mathfrak{p})} \pi(g)v\, \mathrm{d}g, \quad v \in V_\pi,$$

with $\mathrm{d}g$ the Haar measure on $\mathsf{G}(F)$ such that $\mathrm{vol}(K) = 1$, and the Atkin–Lehner involution by

$$\eta v := \pi(\eta)v, \quad \eta := \begin{bmatrix} & & & -1 \\ & & 1 & \\ & \varpi & & \\ -\varpi & & & \end{bmatrix}, \quad v \in V_\pi.$$

If π is irreducible and has the (T, Λ)-Bessel models, then Lemma 4.4(i) of [32] gives for any $B \in \mathscr{B}(T, \Lambda)[\pi]^{K_0(\mathfrak{p})}$ the support conditions

(6-10) $$B(h(\ell,m)) = B(h(\ell,m)s_2) = 0 \qquad \text{if } \ell < 0,,$$

(6-11) $$B(h(\ell,m)s_1s_2) = B(h(\ell,m)s_2s_1s_2) = 0 \quad \text{if } \ell < -1.$$

6.2. ***Unramified computation (case 1).*** When π is spherical, π is of type I or IIb in our cases. Recall that there exists a unique element $B^0_\pi \in \mathscr{B}(T,\Lambda)[\pi]^K$ such that $B^0_\pi(1_4)=1$. Then, the following formula for an unramified μ in its greatest generality is due to Sugano [42, Theorem 2.1]:

$$Z(\phi, B^0_\pi, s, \mu; 1_4) = L(s+1/2, \pi, \mu), \quad \mathrm{Re}(s) \gg 0.$$

In [20, Proposition 5.9], a proof of this formula is given when $L = F \oplus F$ and π is of type I based on the explicit formula of B^0_π due to [6].

6.3. ***Unramified computation (cases 2 and 3).*** Now we assume that $e > 0$ and that $T \in \mathrm{Sym}^2(F)$ satisfies the following conditions:

- L/F is an unramified field extension.
- $T \in \mathbf{GL}_2(\mathscr{O}), \quad -\frac{\mathfrak{d}}{2} = \mathrm{tr}(TT^\dagger) \in \mathscr{O}^\times.$

Then ϖ remains a prime element in $\mathfrak{p}_L$. Let b be as in (6-9), and set $\phi = \mathbb{1}_{\mathfrak{p}_L^e \oplus 1+\mathfrak{p}_L^e} \in \mathcal{S}(L^2)$. Our goal in this subsection is to calculate the zeta integrals $Z(\phi, B^0_\pi, s, \mu; b)$ and $Z(\hat{\phi}, B^0_\pi, s, \mu; b)$ explicitly. We note that $Z(\phi, B^0_\pi, s, \mu; 1_4)$ must be zero because the a-integral in (6-4) vanishes.

Definition 6.2 (root number). Let ψ and μ be as above, we define the Gauss sum $W_F(\mu,\psi)$ by

$$W_F(\mu,\psi) := q^{-\frac{e}{2}} \mu(\varpi)^{-e} \sum_{\alpha \in (\mathscr{O}/\mathfrak{p}^e)^\times} \psi(\varpi^{-e}\alpha)\mu(\alpha).$$

This sum has absolute value 1 and is independent of the choice of ϖ. The following lemma is more or less well-known [44, (2.18)].

Lemma 6.3. *Let ψ and μ be as above. For $n \in \mathbb{Z}$,*

$$\int_{\mathscr{O}^\times} \psi(\varpi^n a)\mu(a)\mathrm{d}^\times a = \begin{cases} q^{-\frac{e}{2}+1}(q-1)^{-1}\mu(\varpi)^e W_F(\mu,\psi) & (n=-e), \\ 0 & (n \neq -e). \end{cases}$$

Lemma 6.4. *Let L/F be an unramified quadratic extension. For any character μ of F with $e := \mathrm{cond}(\mu) > 0$,*

$$W_L(\mu_L, \psi_L) = (-1)^e W_F(\mu,\psi)^2.$$

Proof. For a virtual representation V of the Weil group $\mathfrak{W}_F$ of F, let $\varepsilon(s,V,\psi)$ denote the local epsilon factor à la Deligne–Langlands [45]. By [45, (3.4.8)], we have

$$\varepsilon(s, \mathrm{Ind}_{\mathfrak{W}_L}^{\mathfrak{W}_F} \rho, \psi) = \varepsilon(s,\rho,\psi_L) \tag{6-12}$$

for any ρ of degree 0. Any character μ of $F^\times$ is viewed as a one-dimensional representation of $\mathfrak{W}_F$ by $F^\times \cong \mathfrak{W}_F^{\mathrm{ab}}$. If $\mu_L := \mu \circ \mathrm{N}_{L/F}$, then $\mathrm{Ind}_{\mathfrak{W}_L}^{\mathfrak{W}_F} \mu_L \cong \mu \oplus \mu\eta_{L/F}$. If μ is unramified, then $\varepsilon(s,\mu,\psi)=1$; since L/F is unramified, so is the character $\eta_{L/F}$. From these observations, we see that equality (6-12) holds true when $\rho = \mathbf{1}$. Thus, (6-12) is true when $\rho = \mu$ with $e > 0$. Since $\varepsilon(s, \mu\eta_{L/F}^j, \psi) = q^{-e(s-1/2)} W_F(\bar{\mu}\eta_{L/F}^j, \psi)$ for $j = 0, 1$ and $\varepsilon(s, \mu_L, \psi_L) = q^{-2e(s-1/2)} W_L(\bar{\mu}_L, \psi_L)$, we obtain $W_L(\mu_L, \psi_L) = W_F(\mu, \psi) W_F(\mu\eta_{L/F}, \psi)$. Since $e > 0$ and $\eta_{L/F}(\varpi) = -1$, we have

$$W_F(\mu\eta_{L/F}, \psi) = \eta_{L/F}(\varpi)^e W_F(\mu, \psi) = (-1)^e W_F(\mu, \psi). \qquad \square$$

Lemma 6.5. *For $k^\# = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in K^\#$, we have*

$$f_\phi^{(s,\Lambda,\mu)}(k^\#) = \begin{cases} q^{-2e+2}(q^2-1)^{-1}\mu_L(d)^{-1}\mu(\det k^\#) & (c \in \mathfrak{p}_L^e), \\ 0 & (\text{otherwise}), \end{cases}$$

$$f_{\hat{\phi}}^{(s,\Lambda,\mu)}(k^\#) = \begin{cases} q^{e(2s-3)+2}(q^2-1)^{-1}\Lambda(\varpi)^{-e}\mu_L(c)^{-1} W_L(\mu_L, \psi_L) & (c \in \mathscr{O}_L^\times), \\ 0 & (c \notin \mathscr{O}_L^\times). \end{cases}$$

Proof. These equations immediately follow by Lemma 6.3. $\square$

Lemma 6.6. *For $u \in \mathscr{O}^\times$,*

$$\sum_{\substack{\eta \in \mathscr{O}_L/\mathfrak{p}_L^e \\ u + \mathrm{N}_{L/F}(\eta) \in \mathscr{O}^\times}} \mu(u + \mathrm{N}_{L/F}(\eta)) = (-1)^e q^e \mu(u).$$

Proof. Because L/F is an unramified extension, the norm map $\mathrm{N}_{L/F} : \mathscr{O}_L^\times \to \mathscr{O}^\times$ is surjective. Thus, there exists $v \in \mathscr{O}_L^\times$ such that $u = \mathrm{N}_{L/F}(v)$. By replacing η by $v\eta$, without loss of generality, we may assume that $u = 1$.

For an integer i with $0 \le i \le e+1$, we set

$$\overline{\mathscr{O}_{L,i}} = \begin{cases} \{\eta \in (\mathscr{O}_L/\mathfrak{p}_L^e)^\times \mid 1 + \mathrm{N}_{L/F}(\eta) \not\equiv 0 \pmod{\mathfrak{p}}\} & (i = 0), \\ \mathfrak{p}_L^i/\mathfrak{p}_L^e & (0 < i \le e), \\ \varnothing & (i = e+1) \end{cases}$$

and

$$\overline{\mathscr{U}_i} = \begin{cases} \mathscr{O}^\times/(1+\mathfrak{p}^e) & (i = 0), \\ (1+\mathfrak{p}^i)/(1+\mathfrak{p}^e) & (0 < i \le e), \\ \varnothing & (i = e+1). \end{cases}$$

When $0 \le i < e/2$, we can check that

$$\overline{\mathscr{O}_{L,i}} \setminus \overline{\mathscr{O}_{L,i+1}} \ni \eta \pmod{\mathfrak{p}_L^e} \mapsto 1 + N_{L/F}(\eta) \pmod{\mathfrak{p}^e} \in \overline{\mathscr{U}_{2i}} \setminus \overline{\mathscr{U}_{2i+1}}$$

is a surjective map having the fiber of cardinality $q^{e-1}(q+1)$ at each point.

When $e/2 \le i \le e$, it is immediate that

$$\mu(1 + \mathrm{N}_{L/F}(\eta)) = 1, \quad \eta \in \overline{\mathscr{O}_{L,i}}.$$

Hence

$$\begin{aligned}\sum_{\substack{\eta\in\mathscr{O}_L/\mathfrak{p}_L^e\\ 1+\mathrm{N}_{L/F}(\eta)\in\mathscr{O}^\times}} &\mu(1+\mathrm{N}_{L/F}(\eta))\\ &=\sum_{i=0}^{e}\sum_{\eta\in\overline{\mathscr{O}_{L,i}\setminus\mathscr{O}_{L,i+1}}}\mu(1+\mathrm{N}_{L/F}(\eta))\\ &=q^{e-1}(q+1)\sum_{0\le i<e/2}\;\sum_{\varepsilon\in\overline{\mathscr{U}_{2i}}\setminus\overline{\mathscr{U}_{2i+1}}}\mu(\varepsilon)+\sum_{e/2\le i\le e}\#(\overline{\mathscr{O}_{L,i}}\setminus\overline{\mathscr{O}_{L,i+1}}).\end{aligned}$$

Then the lemma follows from

$$\sum_{\varepsilon\in\overline{\mathscr{U}_{2i}}\setminus\overline{\mathscr{U}_{2i+1}}}\mu(\varepsilon)=\begin{cases}0 & \left(0\le i<\frac{e-1}{2}\right)\\ -1 & \left(i=\frac{e-1}{2}\right)\end{cases}\quad\text{and}\quad\sum_{e/2\le i\le e}\#(\overline{\mathscr{O}_{L,i}}\setminus\overline{\mathscr{O}_{L,i+1}})=q^{2e-2\lceil e/2\rceil},$$

where $\lceil\cdot\rceil:\mathbb{R}\to\mathbb{Z}$ is the ceiling function. □

Lemma 6.7. *Let $B\in\mathscr{B}(T,\Lambda)[\pi]^K$ and $\eta\in\mathscr{O}$ with $\frac14\mathsf{a}^6\mathsf{d}+\mathrm{N}_{L/F}(\eta)\in\varpi^j\mathscr{O}^\times$ $(0\le j\le e)$.*

(i) *If we set $Y_\eta=-\mathsf{a}^2T^\dagger+X_\eta$, then we have*

$$\det Y_\eta=-\tfrac14\mathsf{a}^4\mathsf{d}-\mathsf{a}^{-2}\mathrm{N}_{L/F}(\eta).$$

(ii) *There exist $\tau\in\mathscr{O}_L^\times$ and $A\in\mathbf{GL}_2(\mathscr{O})$ such that*

$$Y_\eta=\tau\left[\begin{smallmatrix}\varpi^j&0\\0&1\end{smallmatrix}\right]A.$$

(iii) *For $n\in\mathbb{Z}$ and $a\in\mathscr{O}^\times$,*

$$\begin{aligned}&B\left(\mathsf{m}(\varpi^n a1_2,\varpi^n a)\iota\left(\left[\begin{smallmatrix}0&-1\\1&\eta^\dagger\end{smallmatrix}\right]\right)b\right)\\ &\qquad=\psi\left(-\varpi^n a\cdot\tfrac12\mathsf{a}^4\mathsf{d}\left(\tfrac14\mathsf{a}^6\mathsf{d}+\mathrm{N}_{L/F}(\eta)\right)^{-1}\right)\cdot B\left(h(e+n-2j,j)\right).\end{aligned}$$

Proof. (i) Noting that $\det T=-\mathsf{d}/4$, $\det X_\eta=-\mathsf{a}^{-2}\mathrm{N}_{L/F}(\eta)$, and $\mathrm{tr}(TX_\eta)=\mathrm{tr}(X_\eta T^\dagger)=0$, we obtain

$$\begin{aligned}\det Y_\eta&=\tfrac12\,\mathrm{tr}(Y_\eta Y_\eta^\dagger)=\tfrac12\,\mathrm{tr}(\mathsf{a}^4TT^\dagger+X_\eta^\dagger X_\eta)\\ &=\mathsf{a}^4\det T+\det(X_\eta)=-\tfrac14\mathsf{a}^4\mathsf{d}-\mathsf{a}^{-2}\mathrm{N}_{L/F}(\eta).\end{aligned}$$

(ii) By (i) and the assumption that $\frac14\mathsf{a}^6\mathsf{d}+\mathrm{N}_{L/F}(\eta)\in\varpi^j\mathscr{O}^\times$, we have $Y_\eta\in\mathbf{GL}_2(F)$. The disjoint union $\mathbf{GL}_2(F)=\bigsqcup_{m\ge0}L^\times\left[\begin{smallmatrix}\varpi^m&0\\0&1\end{smallmatrix}\right]\mathbf{GL}_2(\mathscr{O})$ (see [42, Lemma 2–4]) implies that there exist $m\in\mathbb{Z}_{\ge0}$, $\ell\in\mathbb{Z}$, $\tau\in\mathscr{O}_L^\times$, and $A\in\mathbf{GL}_2(\mathscr{O})$ such that

$$Y_\eta=(\varpi^\ell\tau)\left[\begin{smallmatrix}\varpi^m&0\\0&1\end{smallmatrix}\right]A=\tau\left[\begin{smallmatrix}\varpi^{m+\ell}&0\\0&\varpi^\ell\end{smallmatrix}\right]A.$$

We now prove that $m=j$ and $\ell=0$. Because $\tau\in\mathbf{GL}_2(\mathscr{O})$, the Smith normal form of Y_η is $\left[\begin{smallmatrix}\varpi^{m+\ell}&0\\0&\varpi^\ell\end{smallmatrix}\right]$. Hence, by the theory of the Smith normal form, we only have to show that the elements of Y_η are coprime. This follows from the identity $\mathrm{tr}(Y_\eta T)=\frac12\mathsf{a}^2\mathsf{d}$ and the assumption $\frac12\mathsf{d}\in\mathscr{O}^\times$.

(iii) By using (6-1), we can check that

$$\iota\left(\left[\begin{smallmatrix}1&0\\-\eta^\dagger&1\end{smallmatrix}\right]\right)=\left[\begin{smallmatrix}1_2&0\\X_\eta^\dagger&1_2\end{smallmatrix}\right],\quad \iota\left(\left[\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right]\right)b\iota\left(\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]\right)=\left[\begin{smallmatrix}1_2&0\\-\mathsf{a}^2T&\varpi^e 1_2\end{smallmatrix}\right].$$

Hence we have

$$\mathsf{m}(\varpi^n a 1_2,\varpi^n a)\iota\left(\left[\begin{smallmatrix}0&-1\\1&\eta^\dagger\end{smallmatrix}\right]\right)b=\left[\begin{smallmatrix}\varpi^n a1_2&0\\Y_\eta^\dagger&\varpi^e 1_2\end{smallmatrix}\right]\iota\left(\left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right]\right).$$

By (ii), we may choose elements $\tau\in\mathcal{O}_L^\times$ and $A\in\mathbf{GL}_2(\mathcal{O})$ with $Y_\eta=\tau\left[\begin{smallmatrix}\varpi^j&0\\0&1\end{smallmatrix}\right]A$. Then, a computation shows that

$$\left[\begin{smallmatrix}\varpi^n a1_2&0\\Y_\eta^\dagger&\varpi^e 1_2\end{smallmatrix}\right]=\mathsf{n}(\varpi^n a(\det Y_\eta)^{-1}Y_\eta)\left[\begin{smallmatrix}\tau&0\\0&{}^t\tau^\dagger\end{smallmatrix}\right]h(e+n-2j,j)k,$$

where $k=\left[\begin{smallmatrix}A&0\\0&{}^tA^\dagger\end{smallmatrix}\right]\left[\begin{smallmatrix}a\varpi^j(\det Y_\eta)^{-1}1_2&0\\0&1_2\end{smallmatrix}\right]\left[\begin{smallmatrix}0&-1_2\\1_2&\varpi^e(\det Y_\eta)^{-1}Y_\eta\end{smallmatrix}\right]\in K$. Since Λ is unramified and B is K-invariant, we have the desired equality. □

Proposition 6.8. *Suppose that π is of type I or IIb. Let $\mu:F^\times\to\mathbb{C}^1$ and $\Lambda:L^\times\to\mathbb{C}^1$ be characters such that cond$(\mu)=e>0$ and Λ is unramified. Suppose that $T=\left[\begin{smallmatrix}\mathsf{a}&\mathsf{b}/2\\\mathsf{b}/2&\mathsf{c}\end{smallmatrix}\right]\in\mathrm{Sym}^2(F)\cap\mathbf{GL}_2(\mathcal{O})$ satisfies that L/F is an unramified field extension and $2d\in\mathcal{O}^\times$. When $\phi=1_{\mathfrak{p}_L^e\oplus 1+\mathfrak{p}_L^e}$ and* $\mathrm{Re}(s)>1$, *we have*

$$Z(\phi,B_\pi^0,s,\mu;b)=q^{e(s-\frac{11}{2})+5}(q^4-1)^{-1}(q-1)^{-1}\mu(-2^{-1}\mathsf{d})^{-1}W_F(\mu,\psi),$$

$$Z(\hat\phi,B_\pi^0,s,\mu;b)=(-1)^e q^{e(3s-\frac{11}{2})+5}(q^4-1)^{-1}(q-1)^{-1}\Lambda(\varpi)^{-e}\mu(-2^{-1}\mathsf{a}^2)\times W_L(\mu_L,\psi_L)W_F(\mu,\psi).$$

Proof. Let $K^\#(\mathfrak{p}^e)=\{k^\#\in K^\#\mid k^\#\equiv 1_2 \pmod{\mathfrak{p}_L^e}\}$ be the principal congruence subgroup of level $\mathfrak{p}_L^e$. If $\phi\in\mathcal{S}(L^2)$ is right $K^\#(\mathfrak{p}^e)$-invariant, the function

$$K^\#\ni k^\#\mapsto f_\phi^{(s,\Lambda,\mu)}(k^\#)\int_{F^\times}B_\pi^0(\mathsf{m}(a1_2,a)\iota(k^\#)b)\mu(a)|a|^{s-1}\in\mathbb{C}$$

is left-$B^\#(\mathcal{O})$-invariant and right $K^\#(\mathfrak{p}^e)$-invariant because $b^{-1}\iota(K^\#(\mathfrak{p}^e))b\subset K$. Since $K^\#(\mathfrak{p}^e)$ is a normal subgroup of $K^\#$ and $B^\#(\mathcal{O})K^\#(\mathfrak{p}^e)=K_0^\#(\mathfrak{p}^e)$, we have

$$\text{(6-13)}\quad \begin{aligned}&Z(\phi,B_\pi^0,s,\mu;b)\\&=\sum_{[\gamma]\in B^\#(\mathcal{O})\backslash K^\#/K^\#(\mathfrak{p}^e)}\mathrm{vol}(B^\#(\mathcal{O})\gamma K^\#(\mathfrak{p}^e))f_\phi^{(s,\Lambda,\mu)}(\gamma)\Upsilon^{(s,\mu)}(\gamma)\\&=[K^\#:K_0^\#(\mathfrak{p}^e)]^{-1}\sum_{[\gamma]\in K_0^\#(\mathfrak{p}^e)\backslash K^\#}f_\phi^{(s,\Lambda,\mu)}(\gamma)\Upsilon^{(s,\mu)}(\gamma)\end{aligned}$$

where we set $\Upsilon^{(s,\mu)}(\gamma):=\int_{F^\times}B_\pi^0(\mathsf{m}(a1_2,a)\iota(\gamma)b)\,\mu(a)\,|a|^{s-1}\mathrm{d}^\times a$ to make room.

A complete set of representatives for $K^\#(\mathfrak{p}^e)\backslash K^\#$ is given by

$$\text{(6-14)}\qquad \left[\begin{smallmatrix}1&0\\\xi&1\end{smallmatrix}\right]\ (\xi\in\mathfrak{p}_L/\mathfrak{p}_L^e),\quad \left[\begin{smallmatrix}0&-1\\1&\eta^\dagger\end{smallmatrix}\right]\ (\eta\in\mathcal{O}_L/\mathfrak{p}_L^e).$$

Now we prove the first equation. By Lemma 6.5, when γ runs through the above elements, we have $f_\phi^{(s,\Lambda,\mu)}(1_2)=q^{-2e+2}(q^2-1)^{-1}$ and $f_\phi^{(s,\Lambda,\mu)}(\gamma)=0$ if $\gamma\neq 1$.

By substituting this in (6-13), noting that $[K^\# : K_0^\#(\mathfrak{p}^e)] = q^{2e-2}(q^2+1)$, and Lemma 6.3, we have

$$\begin{aligned}
&Z(\phi, B_\pi^0, s, \mu; b)\\
&= q^{-4e+4}(q^4-1)^{-1} \int_{F^\times} B_\pi^0\left(\left[\begin{smallmatrix} a\varpi^e 1_2 & aT^\dagger \\ 0 & 1_2 \end{smallmatrix}\right]\right)\mu(a)|a|^{s-1}\mathrm{d}^\times a\\
&= q^{-4e+4}(q^4-1)^{-1} \int_{F^\times} B_\pi^0\left(\left[\begin{smallmatrix} a\varpi^e 1_2 & 0 \\ 0 & 1_2 \end{smallmatrix}\right]\right)\psi(a\cdot \mathrm{tr}(TT^\dagger))\mu(a)|a|^{s-1}\mathrm{d}^\times a\\
&= q^{-4e+4}(q^4-1)^{-1}\mu(-2^{-1}\mathsf{d})^{-1}\\
&\qquad\times \sum_{n\in\mathbb{Z}} B_\pi^0\left(\left[\begin{smallmatrix} \varpi^{e+n} 1_2 & 0 \\ 0 & 1_2 \end{smallmatrix}\right]\right)\mu(\varpi)^n q^{-n(s-1)} \int_{\mathcal{O}^\times}\psi(\varpi^n a)\mu(a)\mathrm{d}^\times a\\
&= q^{e(s-\frac{11}{2})+5}(q^4-1)^{-1}(q-1)^{-1}\mu(-2^{-1}\mathsf{d})^{-1}W_F(\mu,\psi).
\end{aligned}$$

Next, we prove the second equation. When we replace ϕ by $\widehat{\phi}$ in (6-13) and consider the representative (6-14), we can ignore the contribution of ξ-terms in the summation from Lemma 6.5. Hence, by noting the K-invariance of B, we have

$$\begin{aligned}
\text{(6-15)}\quad Z(\widehat{\phi}, B_\pi^0, s, \mu; b) &= q^{e(2s-5)+4}(q^4-1)^{-1}\Lambda(\varpi)^{-e}W_L(\mu_L,\psi_L)\\
&\quad\times \sum_{\eta\in\mathcal{O}_L/\mathfrak{p}_L^e}\int_{F^\times} B_\pi^0(\mathsf{m}(a1_2,a)\iota(\left[\begin{smallmatrix} 0 & -1 \\ 1 & \eta^\dagger \end{smallmatrix}\right])b)\mu(a)|a|^{s-1}\mathrm{d}^\times a.
\end{aligned}$$

By replacing η by $\eta+\varpi^e u$ for some $u\in\mathcal{O}_L$ if we need, we may assume that $\mathsf{a}^6\mathsf{d}+\mathrm{N}_{L/F}(\eta)\in\varpi^j\mathcal{O}_L^\times$ with $0\le j\le e$. Then, Lemmas 6.3 and 6.7(iii) imply

$$\begin{aligned}
&\int_{F^\times} B_\pi^0(\mathsf{m}(a1_2,a)\iota(\left[\begin{smallmatrix} 0 & -1 \\ 1 & \eta^\dagger \end{smallmatrix}\right])b)\mu(a)|a|^{s-1}\mathrm{d}^\times a\\
&= \sum_{n\in\mathbb{Z}} B_\pi^0(h(e+n-2j,j))\mu(\varpi)^n q^{-n(s-1)}\\
&\qquad\times\int_{\mathcal{O}^\times}\psi\left(-\varpi^n a\cdot\tfrac{1}{2}\mathsf{a}^4\mathsf{d}\left(\tfrac{1}{4}\mathsf{a}^6\mathsf{d}+\mathrm{N}_{L/F}(\eta)\right)^{-1}\right)\mu(a)\mathrm{d}^\times a\\
&= \mu\left(-2\mathsf{a}^{-4}\mathsf{d}^{-1}\varpi^{-j}\left(\tfrac{1}{4}\mathsf{a}^6\mathsf{d}+\mathrm{N}_{L/F}(\eta)\right)\right)\\
&\qquad\times\sum_{n\in\mathbb{Z}} B_\pi^0(h(e+n-2j,j))\mu(\varpi)^n q^{-n(s-1)}\int_{\mathcal{O}^\times}\psi(\varpi^{n-j}a)\mu(a)\mathrm{d}^\times a\\
&= q^{-\frac{e}{2}+1+(e-j)(s-1)}(q-1)^{-1}\mu(-2\mathsf{a}^{-4}\mathsf{d}^{-1})\mu\left(\tfrac{1}{4}\mathsf{a}^6\mathsf{d}+\mathrm{N}_{L/F}(\eta)\right)\\
&\qquad W_F(\mu,\psi)\cdot B_\pi^0(h(-j,j)).
\end{aligned}$$

The last expression is equal to 0 unless $j=0$ from (6-10). By substituting this into (6-15) and Lemma 6.6, we have that $Z(\phi, B_\pi^0, s, \mu; b)$ equals

$$\begin{aligned}
&q^{e(3s-\frac{13}{2})+5}(q^4-1)^{-1}(q-1)^{-1}\Lambda(\varpi)^{-e}\mu(-2\mathsf{a}^{-4}\mathsf{d}^{-1})\\
&\qquad\times W_L(\mu_L,\psi_L)W_F(\mu,\psi)\sum_{\substack{\eta\in\mathcal{O}_L/\mathfrak{p}_L^e\\ \mathsf{a}^6\mathsf{d}+\mathrm{N}_{L/F}(\eta)\in\mathcal{O}^\times}}\mu\left(\tfrac{1}{4}\mathsf{a}^6\mathsf{d}+\mathrm{N}_{L/F}(\eta)\right)\\
&= (-1)^e q^{e(3s-\frac{11}{2})+5}(q^4-1)^{-1}(q-1)^{-1}\Lambda(\varpi)^{-e}\mu(-2^{-1}\mathsf{a}^2)W_L(\mu_L,\psi_L)W_F(\mu,\psi).
\end{aligned}$$

□

Corollary 6.9. *Let π, T, μ, and Λ be as in Proposition 6.8. Then*

$$\varepsilon(\pi, s, \mu, \psi) = (-1)^e q^{4e(\frac{1}{2}-s)} \Lambda(\varpi)^{-e} \mu(-\mathsf{a}^{-2}\mathsf{d}) \overline{W_L(\mu_L, \psi_L) W_F(\mu, \psi)^2}$$
$$= q^{4e(\frac{1}{2}-s)} \Lambda(\varpi)^{-e} \mu(-\mathsf{a}^{-2}\mathsf{d}) \overline{W_F(\mu, \psi)^4}.$$

Proof. The first equality follows from Proposition 6.8 and (6-7); the second equality is due to Lemma 6.4. □

6.4. ***Computation for old forms of type* I *and* IIb (*case* 4).** Now we assume that π has the trivial central character. Then, π has the (T, Λ)-Bessel model if and only if $\Lambda = 1$. Recall that the dimension of $V_\pi^{K_0(\mathfrak{p})}$ is equal to 4 or 3 according as π is of type I or type IIb.

Proposition 6.10. *Let π be a smooth admissible irreducible representation of type I or type IIb with trivial central character. Suppose that $\mu : F^\times \to \mathbb{C}^1$ is an unramified character. When $\phi = \mathbb{1}_{\mathscr{O}_L \oplus \mathscr{O}_L}$ and $\mathrm{Re}(s) > 1$, we have*

$$Z(\phi, B, s, \mu; \eta) = \tfrac{1}{q^2+1} L(s + 1/2, \pi, \mu)$$
$$\times\big[\eta B + q^{-1} T_{1,0} B + \big(\mu(\varpi)^{-1} q^{s+1} + \mu(\varpi) q^{-s+1} - \mathrm{tr}((q^{-1} T_{1,0} + \eta)|_{V_\pi^{K_0(\mathfrak{p})}})\big) B\big](1_4)$$

for $B \in \mathscr{B}(T, 1)[\pi]^{K_0(\mathfrak{p})}$.

Proof. We will give a proof of the case that π is of type I. In the case of type IIb, the proof is similar. Let $\{B_j^*\}_{1 \le j \le 4} \subset \mathscr{B}(T, 1)[\pi]^{K_0(\mathfrak{p})}$ be an eigenbasis of $T_{1,0}$ with eigenvalues $\{\lambda_j\}_{1 \le j \le 4}$. Then, we suffice to check the desired equation only for $B = B_j^*$ $(1 \le j \le 4)$. Let $(\eta_{ij})_{1 \le i, j \le 4}$ be the representation matrix of $\eta|_{K_0(\mathfrak{p})}$ with respect to $\{B_j^*\}_{1 \le j \le 4}$. Then, (6-21) shows that

$$B_j^*(h(l+1, 0)) = \lambda_j q^{-3} B_j^*(h(l, 0)), \quad l \in \mathbb{Z}_{\ge 0}.$$

By applying Lemma 6.1 to $B = B_j^*$, we obtain

$$Z(\phi, B_j^*, s, \mu; \eta) = \frac{L(s+1, \mu_L)}{q^2+1} \left(\sum_{i=1}^4 \frac{\eta_{ij} B_i^*(1_4)}{1 - \lambda_i \mu(\varpi) q^{-(s+2)}} + \frac{\mu(\varpi)^{-1} q^{s+1} B_j^*(1_4)}{1 - \lambda_j \mu(\varpi) q^{-(s+2)}} \right).$$

Thus, we may put

$$Z(\phi, B_j^*, s, \mu; \eta) = \frac{1}{q^2+1} \prod_{i=1}^4 (1 - \lambda_i q^{-2} X)^{-1} \frac{A_0 + A_1 X + A_2 X^2 + A_3 X^3 + A_4 X^4}{X(1 - q^{-2} X^2)}.$$

with $X = \mu(\varpi) q^{-s}$ for some $A_i \in \mathbb{C}$ $(0 \le i \le 4)$.

By noting that $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} = \{\alpha\beta\gamma q^{3/2}, \beta\gamma q^{3/2}, \alpha\gamma q^{3/2}, \gamma q^{3/2}\}$ from [34, Table A.8], we have

$$(q^2+1) \cdot \frac{Z(\phi, B_j^*, s, \mu; 1_4)}{L(s+1/2, \pi, \mu)} = \frac{A_0 + A_1 X + A_2 X^2 + A_3 X^3 + A_4 X^4}{X(1 - q^{-2} X^2)}. \tag{6-16}$$

A direct computation shows that

$$A_0 = qB_j^*(1_4), \tag{6-17}$$

$$A_1 = \eta B_j^*(1_4) + q^{-1}\operatorname{tr}(T_{1,0}|_{V_\pi^{K_0(\mathfrak{p})}}) \cdot B_j^*(1_4) - q^{-1} \cdot T_{1,0} B_j^*(1_4). \tag{6-18}$$

The functional equation (6-7) and the relation $\varepsilon(\pi, s, \mu, \psi) = 1$ (see Theorem 4.4 in [29]) show that the right side of (6-16) is invariant under the transformation $X \to X^{-1}$. By comparing coefficients, we have

$$A_2 = q^{-2}(q^2-1)A_0, \quad A_3 = -q^{-2}A_1, \quad A_4 = -q^{-2}A_0. \tag{6-19}$$

Substituting (6-17), (6-18), and (6-19) into (6-16), the right side of (6-16) becomes

$$\big[\eta B_j^* + q^{-1} T_{1,0} B_j^* + \big(\mu(\varpi)^{-1} q^{s+1} + \mu(\varpi) q^{-s+1} - \operatorname{tr}(q^{-1} T_{1,0}|_{V_\pi^{K_0(\mathfrak{p})}})\big) B_j^*\big](1_4).$$

□

6.5. ***Computation for newforms of type* IIIa *(case* 5*).*** In this subsection, we do not assume the triviality of the central character. By [32, §9.1], there exists a basis $\{B_1, B_2\}$ of $\mathscr{B}(T, \Lambda)[\pi]^{K_0(\mathfrak{p})}$ such that

$$\begin{aligned} T_{1,0}B_1 &= \alpha\gamma q B_1, & T_{1,0}B_2 &= \gamma q B_2, \\ T_{0,1}B_1 &= \alpha\gamma^2(\alpha q + 1) q B_1, & T_{0,1}B_2 &= \alpha\gamma^2(\alpha^{-1} q + 1) q B_2, \\ \eta B_1 &= \alpha\gamma B_2, & \eta B_2 &= \gamma B_1. \end{aligned} \tag{6-20}$$

We now consider the values of B_1 and B_2 at the identity element. Lemma 9.1 of [32] ensures their nonvanishing and $\Lambda(\varpi) = \alpha\gamma^2$ by the condition of central character of π.

Lemma 6.11. *Let $T \in \mathrm{Sym}^2(F)$ such that L/F is an unramified field extension. Suppose that $\Lambda : L^\times \to \mathbb{C}^1$ is an unramified character. Then*

$$B_2(1_4) = \alpha^{-1} B_1(1_4) \neq 0.$$

Proof. For $B \in \mathscr{B}(T, \Lambda)[\pi]^{K_0(\mathfrak{p})}$ and any nonnegative integer l, by Lemmas 5.1 and 5.3 in [32], we get the following relations:

$$T_{1,0}B(h(l,0)) = q^3 B(h(l+1,0)), \tag{6-21}$$

$$T_{1,0}B(h(l,0)s_2) = q^2(q-1)B(h(l+1,0)) + q^2 B(h(l-1,1)s_1s_2), \tag{6-22}$$

$$\begin{aligned} T_{1,0}B(h(l,0)s_2s_1s_2) &= q^2(q-1)B(h(l+1,0)) \\ &\quad + \Lambda(\varpi)B(h(l-1,0)s_2s_1s_2) + (q^2-1)B(h(l-1,1)s_1s_2), \end{aligned} \tag{6-23}$$

$$T_{0,1}B(h(l,0)) = q^3(q+1)B(h(l,1)), \tag{6-24}$$

$$\begin{aligned} T_{0,1}B(h(l,0)s_2) &= q^4 B(h(l,1)s_1s_2) + q\Lambda(\varpi)B(h(l-2,1)s_1s_2) \\ &\quad + q^2(q-1)B(h(l,1)) + \begin{cases} -q\Lambda(\varpi)B(1_4) & (l=0), \\ q(q-1)\Lambda(\varpi)B(h(l,0)) & (l \geq 1). \end{cases} \end{aligned} \tag{6-25}$$

Set $B = B_1$. By putting $l = 0$ in the above relations, applying the relation (6-20), and using the equation that $B_2(h(l,0)) = \gamma B_1(h(l-1,0)s_2s_1s_2)$ for $l \geq 0$, we have

$$\alpha\gamma q B_1(s_2) = \alpha\gamma(q-1)B_1(1_4)+q^2B_1(h(-1,1)s_1s_2), \tag{6-26}$$

$$\Lambda(\varpi)\gamma^{-1}(q^{-1}-1)B_2(1_4) = \alpha\gamma(q-1)B_1(1_4)+(q^2-1)B_1(h(-1,1)s_1s_2), \tag{6-27}$$

$$\Lambda(\varpi)(\alpha q+1)qB_1(s_2) = q^4B_1(h(0,1)s_1s_2) + \frac{\Lambda(\varpi)(\alpha q^2-\alpha q-q^2-1)}{q+1}B_1(1_4). \tag{6-28}$$

Similarly, by putting $l = 1$ in (6-23), we have

$$\Lambda(\varpi)q^{-3}(1-q)B_2(1_4) = \Lambda(\varpi)\alpha q^{-2}(q-1)B_1(1_4)+(q^2-1)B_1(h(0,1)s_1s_2).$$

From this last equation, together with (6-26), (6-27), (6-28), we obtain $B_2(1_4) = \alpha^{-1}B_1(1_4)$. The nonvanishing of B_1 at 1_4 follows from Theorem 9.3 of [32]. □

Proposition 6.12. *Suppose that π is of type IIIa. Let $T \in \mathrm{Sym}^2(F)$ be such that L/F is an unramified field extension. Suppose that $\mu : F^\times \to \mathbb{C}^1$ and $\Lambda : L^\times \to \mathbb{C}^1$ are unramified characters. When $\phi = 1_{\mathcal{O}_L\oplus\mathcal{O}_L}$ and $\mathrm{Re}(s) > 1$, we have*

$$Z(\phi, B, s, \mu; \eta) = \frac{\Lambda(\varpi)^{-1}\mu(\varpi)^{-1}q^{s+1}}{q^2+1}L(s+1/2, \pi, \mu)\cdot B(1_4), \quad B \in \mathscr{B}(s, \Lambda)[\pi]^{K_0(\mathfrak{p})}. \tag{6-29}$$

Proof. By substituting $B = B_i$ in (6-5) for $i = 1, 2$, the statement is immediate for $B = B_i$ by (6-20), (6-21), and Lemma 6.11. Since B_1 and B_2 span $\mathscr{B}(T, \Lambda)[\pi]^{K_0(\mathfrak{p})}$, we complete the proof. □

Corollary 6.13. *Let π, T, μ, and Λ be as in Proposition 6.12, then*

$$\varepsilon(\pi, s, \mu, \psi) = \mu(\varpi)^2q^{2(\frac{1}{2}-s)}.$$

6.6. ***Computation for newforms of type* VI*b* (*case* 6).** Suppose that π is of type VIb; we do not assume the triviality of the central character. By [32, Table 3], any element $B \in \mathscr{B}(T, \Lambda)[\pi]^{K_0(\mathfrak{p})}$ satisfies

$$T_{1,0}B = \gamma q B, \quad \eta B = \gamma B. \tag{6-30}$$

By [32, Theorem 9.3], we have $B(1_4) \neq 0$ if and only if $B \neq 0$.

Proposition 6.14. *Suppose that π is of type VIb. Let $T \in \mathrm{Sym}^2(F)$ be such that L/F is an unramified field extension. Suppose that $\mu : F^\times \to \mathbb{C}^1$ and $\Lambda : L^\times \to \mathbb{C}^1$ are unramified characters. When $\phi = 1_{\mathcal{O}_L\oplus\mathcal{O}_L}$ and $\mathrm{Re}(s) > 1$, we have*

$$Z(\phi, B, s, \mu; \eta) = \frac{\Lambda(\varpi)^{-1}\mu(\varpi)^{-1}q^{s+1}}{q^2+1}L(s+1/2, \pi, \mu)\cdot B(1_4), \quad B \in \mathscr{B}(T, \Lambda)[\pi]^{K_0(\mathfrak{p})}.$$

Proof. Similar to that of Proposition 6.12, using relations (6-21)–(6-25). □

Corollary 6.15. *Let π, T, μ, and Λ be as in Proposition 6.14. Then*

$$\varepsilon(\pi, s, \mu, \psi) = \mu(\varpi)^2 q^{2(\frac{1}{2}-s)}.$$

6.7. *The values of Bessel models at $\mathbf{1_4}$ for old forms.* For quasicharacters χ, χ', and σ on $F^\times$ which are trivial on $\mathscr{O}^\times$, the representation $(\chi \times \chi' \rtimes \sigma, V)$ can be realized as the right regular representation of the space of all smooth functions f on $\mathsf{G}(F)$ satisfying

$$f\left(\begin{bmatrix} a & * & * & * \\ 0 & b & * & * \\ 0 & 0 & ca^{-1} & 0 \\ 0 & 0 & * & cb^{-1} \end{bmatrix} g\right) = |a^2 b||c|^{-\frac{3}{2}} \chi(a)\chi'(b)\sigma(c) f(g), \quad a, b, c \in F^\times,\ g \in \mathsf{G}(F).$$

The space V has a unique K-invariant element $f_K \in V$ such that $f_K(1_4) = 1$. Let $I = K \cap \begin{bmatrix} \mathscr{O} & \mathfrak{p} & \mathscr{O} & \mathscr{O} \\ \mathscr{O} & \mathscr{O} & \mathscr{O} & \mathscr{O} \\ \mathfrak{p} & \mathfrak{p} & \mathscr{O} & \mathscr{O} \\ \mathfrak{p} & \mathfrak{p} & \mathfrak{p} & \mathscr{O} \end{bmatrix}$ be the Iwahori subgroup of K. For $w \in W$, let $f_w \in V^I$ be the unique element such that $f_w|_K = \mathbb{1}_{IwI}$.

By [31, Section 2], the space V^I is 8-dimensional space spanned by

$$f_{1_4}, \quad f_{s_1}, \quad f_{s_2}, \quad f_{s_2 s_1}, \quad f_{s_1 s_2 s_1}, \quad f_{s_1 s_2}, \quad f_{s_1 s_2 s_1 s_2}, \quad f_{s_2 s_1 s_2}.$$

A basis of the 4-dimensional space $V^{K_0(\mathfrak{p})}$ is given by

$$f_1^{\mathrm{I}} := f_{1_4} + f_{s_1}, \ f_2^{\mathrm{I}} := f_{s_2} + f_{s_2 s_1}, \ f_3^{\mathrm{I}} := f_{s_1 s_2 s_1} + f_{s_1 s_2}, \ f_4^{\mathrm{I}} := f_{s_1 s_2 s_1 s_2} + f_{s_2 s_1 s_2}.$$

Recall that π is isomorphic to $\chi \times \chi' \rtimes \sigma$ or $\chi 1_{\mathbf{GL_2}} \rtimes \sigma$ for some characters χ, χ', and σ on $F^\times$ according as π is of type I or IIb. The representation $(\chi 1_{\mathbf{GL_2}} \rtimes \sigma, V)$ can be realized as a subrepresentation of $|\cdot|^{1/2}\chi \times |\cdot|^{-1/2}\chi \rtimes \sigma$, and a basis of 3-dimensional space $V^{K_0(\mathfrak{p})}$ is given by

$$f_1^{\mathrm{II}b} := f_{1_4} + f_{s_1}, \ \ f_2^{\mathrm{IIb}} := f_{s_2} + f_{s_2 s_1} + f_{s_1 s_2 s_1} + f_{s_1 s_2}, \ \ f_3^{\mathrm{IIb}} := f_{s_1 s_2 s_1 s_2} + f_{s_2 s_1 s_2}.$$

The representation matrices of operators $T_{1,0}$ and η on $V_\pi^{K_0(\mathfrak{p})}$ are given in [31, Table 3] and [31, Lemma 2.1].

Now we assume that π has the trivial central character. Let (ℓ_π^0, f_K) be the Bessel data for π defined above. For $\bullet \in \{\mathrm{I}, \mathrm{IIb}\}$, define a basis $\{B_i^\bullet\}_{1 \le i \le \dim V_\pi^{K_0(\mathfrak{p})}}$ of $V_\pi^{K_0(\mathfrak{p})}$ by

$$B_i^\bullet(g) := \ell_\pi^0(\pi(g) f_i^\bullet), \quad g \in \mathsf{G}(F).$$

We need the values of these functions at $g = 1_4$, which are given as follows.

Proposition 6.16. *Let π be a smooth admissible irreducible K-spherical representation of type I or IIb, and $\{B_i^\bullet\}$ be as above for $\bullet \in \{\mathrm{I}, \mathrm{IIb}\}$.*

(i) *If π is of type I,*

$$B_1^{\mathrm{I}}(1_4)=\frac{\alpha\beta}{(q-\alpha)(q-\beta)},\quad B_2^{\mathrm{I}}(1_4)=\frac{-q\beta}{(q-\alpha)(q-\beta)},$$
$$B_3^{\mathrm{I}}(1_4)=\frac{-q\alpha}{(q-\alpha)(q-\beta)},\quad B_4^{\mathrm{I}}(1_4)=\frac{q^2}{(q-\alpha)(q-\beta)}.$$

(ii) *If π is of type IIb,*

$$B_1^{\mathrm{IIb}}(1_4)=\frac{\alpha^2}{(q^{1/2}-\alpha)(q^{3/2}-\alpha)},\quad B_2^{\mathrm{IIb}}(1_4)=\frac{-q^{1/2}(1+q)\alpha}{(q^{1/2}-\alpha)(q^{3/2}-\alpha)},$$
$$B_3^{\mathrm{IIb}}(1_4)=\frac{q^2}{(q^{1/2}-\alpha)(q^{3/2}-\alpha)}.$$

Proof. Set $d:=\dim V_\pi^{K_0(\mathfrak{p})}$. By the construction of the elements $\{f_i^\bullet\}_{1\le i\le d}$, we have $B_\pi^0=\sum_{i=1}^d B_i^\bullet$. For any $\ell\in\mathbb{Z}_{\ge 0}$, we let $T_{1,0}^\ell$ act on the both sides of this equation and then evaluate at $g=1_4$ using (6-21); thus,

$$B_\pi^0(h(\ell,0))=q^{-3\ell}\sum_{i=1}^d (T_{1,0}^\ell B_i^\bullet)(1_4).$$

The value on the left-hand side is given in [6, Corollary 1.8]; by [31, Table 3], we have a system of linear equations among $B_i^\bullet(1_4)$, which is solved easily. □

6.8. ***Local periods.*** For an irreducible admissible unitalizable $K_0(\mathfrak{p})$-spherical representation (π,V_π) on $\mathsf{G}(F)$ having the (T,Λ)-Bessel model, we fix a pair $(\ell^\pi,\xi^\pi)\in(V_\pi^*)^{T,\Lambda}\times V_\pi^{K_0(\mathfrak{p})}$ satisfying $\ell_\pi(\xi_\pi)=1$. When π is K-spherical, we choose (ℓ^π,ξ^π) as the unramified Bessel datum (ℓ_0^π,ξ_0^π). Then, there exists a unique $\mathsf{G}(F)$-invariant inner product $\langle\cdot|\cdot\rangle$ on V_π such that $\langle\xi^\pi|\xi^\pi\rangle=1$. For $s\in\mathbb{C}$, $\phi\in\mathcal{S}(L^2)$, $g\in\mathsf{G}(F)$, and a character μ on $F^\times$, we define the local period of π as

$$\mathbb{I}_{\pi,K_0(\mathfrak{p})}^{(s)}(\xi^\pi,\phi,\Lambda,\mu;g)=\sum_{B\in\mathcal{B}((V_\pi^*)^{T,\Lambda},K_0(\mathfrak{p}))} Z^*(\phi,B,s,\mu;g)\overline{B(1_4)},$$

where

$$Z^*(\phi,B,s,\mu;g):=L(s+1/2,\pi,\mu)^{-1}Z(\phi,B,s,\mu;g)$$

is the normalized zeta integral and $\mathcal{B}((V_\pi^*)^{T,\Lambda},K_0(\mathfrak{p}))$ is an orthonormal basis of $\mathscr{B}(T,\Lambda)[\pi]^{K_0(\mathfrak{p})}$ with respect to the inner product on $\mathscr{B}(T,\Lambda)[\pi]$ via the $\mathsf{G}(F)$-isomorphism

$$V_\pi\ni\xi\mapsto(g\mapsto\ell_\pi(\pi(g)\xi))\in\mathscr{B}(T,\Lambda)[\pi]. \tag{6-31}$$

6.8.1. *Local periods for type I and IIb.* When π is of type I or IIb, we recall that the representation space can be realized as in Section 6.7. We choose $\xi^\pi=\xi_0^\pi=f_K$. Then, since we are dealing with induced representations from unitary characters,

the standard inner product $\langle f|g\rangle := \int_K f(k)\overline{g(k)}dk$ $(f, g \in V_\pi)$ becomes $\mathsf{G}(F)$-invariant and satisfies $\langle \xi_0^\pi|\xi_0^\pi\rangle = 1$.
For $\bullet \in \{\mathrm{I}, \mathrm{IIb}\}$, the set $\{\langle B_i^\bullet|B_i^\bullet\rangle^{-\frac{1}{2}} B_i^\bullet\}_{1\le i\le \dim V_\pi^{K_0(\mathfrak{p})}}$ forms an orthonormal basis of $\mathscr{B}(T,\Lambda)[\pi]^{K_0(\mathfrak{p})}$. By [36, Remark 2.1.3], the values $\langle B_i^\bullet|B_i^\bullet\rangle$ are given as follows:

$$\langle B_i^{\mathrm{I}}|B_i^{\mathrm{I}}\rangle = q^{i-1}(q+1) \quad (1\le i\le 4),$$
$$\langle B_1^{\mathrm{IIb}}|B_1^{\mathrm{IIb}}\rangle = (q+1),\quad \langle B_2^{\mathrm{IIb}}|B_2^{\mathrm{IIb}}\rangle = q(q+1)^2,\quad \langle B_3^{\mathrm{IIb}}|B_3^{\mathrm{IIb}}\rangle = q^3(q+1).$$

By Propositions 6.10 and 6.16, and the matrix representation of $T_{1,0}$ and η given in [31, Table 3] and [31, Lemma 2.1], we can find the local period by a direct calculation.[2]

Proposition 6.17. *Let π be a smooth admissible irreducible unitary representation of type I or type IIb with trivial central character. Suppose that Λ and μ are unramified. When $\phi = \mathbb{1}_{\mathscr{O}_L\oplus\mathscr{O}_L}$, we have*

$$\mathbb{I}^{(s)}_{\pi,K_0(\mathfrak{p})}(\xi_0^\pi,\phi,\Lambda,\mu;\eta) = \frac{2(q-1)}{q^5(q^2+1)}L(1,\pi;\mathrm{Std}) \times \big(\mu(\varpi)^{-1}q^{s+1} + \mu(\varpi)q^{-s+1} - \tfrac{1}{q+1}\,\mathrm{tr}(T_{1,0}+q\eta|_{V_\pi^{K_0(\mathfrak{p})}})\big),$$

where $L(s,\pi;\mathrm{Std})$ is the standard L-function of π defined by

$$L(s,\pi;\mathrm{Std}) = (1-\alpha q^{-s})^{-1}(1-\beta q^{-s})^{-1}(1-\alpha^{-1}q^{-s})^{-1}(1-\beta^{-1}q^{-s})^{-1}(1-q^{-s})^{-1}$$

or

$$L(s,\pi;\mathrm{Std}) = (1-\alpha q^{-s+\frac{1}{2}})^{-1}(1-\alpha q^{-s-\frac{1}{2}})^{-1}(1-\alpha^{-1}q^{-s+\frac{1}{2}})^{-1}(1-\alpha^{-1}q^{-s-\frac{1}{2}})^{-1}(1-q^{-s})^{-1},$$

according as π is of type I or IIb.

6.8.2. *Local period for type IIIa.* Suppose that π is of type IIIa. Then, as remarked before, $|\alpha| = |\gamma| = 1$ and $\alpha \ne 1$. By Lemma 6.11, there exists a basis $\{B_i\}_{i=1,2}$ of $\mathscr{B}(T,\Lambda)[\pi]^{K(\mathfrak{p})}$, unique up to constant, satisfying the relations (6-20) and $B_2(1_4) = \alpha^{-1}B_1(1_4) \ne 0$. Thus, we can uniquely fix $\{B_i^{\mathrm{IIIa}}\}_{i=1,2}$ by requiring $B_1^{\mathrm{IIIa}}(1_4) = 1$ and the datum (ℓ_π, ξ^π) by $\ell_\pi(\xi^\pi) = B_1^{\mathrm{IIIa}}$ to form (6-31) and induce the $\mathsf{G}(F)$-invariant inner product on $\mathscr{B}(T,\Lambda)[\pi]$ such that $\langle B_1^{\mathrm{IIIa}}|B_1^{\mathrm{IIIa}}\rangle = 1$. Since $\langle B_1|B_2\rangle$ vanishes by [7, Lemma 2.11], we then have $\langle B_2^{\mathrm{IIIa}}|B_2^{\mathrm{IIIa}}\rangle = 1$ and $\langle B_1^{\mathrm{IIIa}}|B_2^{\mathrm{IIIa}}\rangle = 0$ due to $|\alpha| = 1$. The following proposition is immediate from Proposition 6.12.

Proposition 6.18. *Let π be a smooth admissible irreducible unitary representation of type IIIa. Suppose that Λ and μ are unramified. When $\phi = \mathbb{1}_{\mathscr{O}_L\oplus\mathscr{O}_L}$,*

$$\mathbb{I}^{(s)}_{\pi,K_0(\mathfrak{p})}(\xi^\pi,\phi,\Lambda,\mu;\eta) = \frac{2\Lambda(\varpi)^{-1}\mu(\varpi)^{-1}q^{s+1}}{q^2+1}.$$

[2]We have used MATHEMATICA 13.

6.8.3. *Local period for type VIb.* Suppose that π is of type VIb. We recall that $\dim(V_\pi^{K_0(\mathfrak{p})}) = \dim(\mathscr{B}(T,\Lambda)^{K_0(\mathfrak{p})}) = 1$; thus, by [32, Theorem 9.3], there exists a unique element $B^{\mathrm{VIb}} \in \mathscr{B}(T,\Lambda)^{K_0(\mathfrak{p})}$ satisfying $B^{\mathrm{VIb}}(1_4) = 1$. We fix the datum (ℓ_π, ξ^π) by setting $\ell_\pi(\xi^\pi) = B^{\mathrm{VIb}}$. The following result is immediate from Proposition 6.14.

Proposition 6.19. *Let π be a smooth admissible irreducible unitary representation of type VIb. Suppose that Λ and μ are unramified. When $\phi = \mathbb{1}_{\mathscr{O}_L \oplus \mathscr{O}_L}$,*

$$\mathbb{I}^{(s)}_{\pi, K_0(\mathfrak{p})}(\xi^\pi, \phi, \Lambda, \mu; \eta) = \frac{\Lambda(\varpi)^{-1}\mu(\varpi)^{-1}q^{s+1}}{q^2+1}.$$

Acknowledgments

The authors thank the referee for a careful reading and constructive suggestions.

Kuga was supported by JSPS KAKENHI Grant Number 23KJ1931. Tsuzuki was supported by JSPS KAKENHI Grant Number 20K03565.

References

[1] A. N. Andrianov, "Dirichlet series with Euler product in the theory of Siegel modular forms of genus two", *Trudy Mat. Inst. Steklov.* **112** (1971), 73–94, 386. In Russian; translated in *Proc. Steklov Inst. Math.* **112** (1971), 70–93.

[2] A. N. Andrianov, "Euler products that correspond to Siegel's modular forms of genus 2", *Uspehi Mat. Nauk* **29**:3(177) (1974), 43–110. In Russian; translated in *Russ. Math. Surveys* **29**:3, 1974, 45–116.

[3] J. Arthur, *The endoscopic classification of representations: orthogonal and symplectic groups*, AMS Colloquium Publications **61**, American Mathematical Society, 2013.

[4] M. Asgari and R. Schmidt, "Siegel modular forms and representations", *Manuscripta Math.* **104**:2 (2001), 173–200.

[5] V. Blomer, "Spectral summation formulae for GSp(4) and moments of spinor L-functions", *J. Eur. Math. Soc.* (*JEMS*) **21**:6 (2019), 1751–1774.

[6] D. Bump, S. Friedberg, and M. Furusawa, "Explicit formulas for the Waldspurger and Bessel models", *Israel J. Math.* **102** (1997), 125–177.

[7] M. Dickson, A. Pitale, A. Saha, and R. Schmidt, "Explicit refinements of Böcherer's conjecture for Siegel modular forms of squarefree level", *J. Math. Soc. Japan* **72**:1 (2020), 251–301.

[8] B. Feigon and D. Whitehouse, "Averages of central L-values of Hilbert modular forms with an application to subconvexity", *Duke Math. J.* **149**:2 (2009), 347–410.

[9] M. Furusawa and K. Morimoto, "Refined global Gross–Prasad conjecture on special Bessel periods and Böcherer's conjecture", *J. Eur. Math. Soc.* **23**:4 (2021), 1295–1331.

[10] M. Furusawa and K. Morimoto, "On the Gross–Prasad conjecture with its refinement for $(\mathrm{SO}(5), \mathrm{SO}(2))$ and the generalized Böcherer conjecture", *Compos. Math.* **160**:9 (2024), 2115–2202.

[11] D. Goldfeld, J. Hoffstein, and D. Lieman, "Appendix: An effective zero-free region", *Ann. of Math.* (2) **140**:1 (1994), 177–181.

[12] J. Hoffstein and P. Lockhart, "Coefficients of Maass forms and the Siegel zero", *Ann. of Math.* (2) **140**:1 (1994), 161–176.

[13] J. Hoffstein and D. Ramakrishnan, "Siegel zeros and cusp forms", *Internat. Math. Res. Notices* **1995**:6 (1995), 279–308.

[14] H. Jacquet, *Automorphic forms on* GL(2), *II*, Lecture Notes in Mathematics **278**, Springer, 1972.

[15] H. Jacquet and N. Chen, "Positivity of quadratic base change L-functions", *Bull. Soc. Math. France* **129**:1 (2001), 33–90.

[16] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, "Rankin–Selberg convolutions", *Amer. J. Math.* **105**:2 (1983), 367–464.

[17] H. H. Kim, S. Wakatsuki, and T. Yamauchi, "An equidistribution theorem for holomorphic Siegel modular forms for GSp_4 and its applications", *J. Inst. Math. Jussieu* **19**:2 (2020), 351–419.

[18] E. Kowalski, A. Saha, and J. Tsimerman, "Local spectral equidistribution for Siegel modular forms and applications", *Compos. Math.* **148**:2 (2012), 335–384.

[19] S. Kuga and M. Tsuzuki, "An asymptotic formula of spectral average of central L-values on $\mathbf{GSp}(2)$ for square free levels", preprint, 2024.

[20] F. Lemma, "On higher regulators of Siegel threefolds, II: The connection to the special value", *Compos. Math.* **153**:5 (2017), 889–946.

[21] X. Li, "Upper bounds on L-functions at the edge of the critical strip", *Int. Math. Res. Not.* **2010**:4 (2010), 727–755.

[22] Y. Liu, "Refined global Gan–Gross–Prasad conjecture for Bessel periods", *J. Reine Angew. Math.* **717** (2016), 133–194.

[23] D. Loeffler, V. Pilloni, C. Skinner, and S. L. Zerbes, "Higher Hida theory and p-adic L-functions for GSp_4", *Duke Math. J.* **170**:18 (2021), 4033–4121.

[24] P. Michel and A. Venkatesh, "The subconvexity problem for GL_2", *Publ. Math. Inst. Hautes Études Sci.* **111** (2010), 171–271.

[25] C. J. Moreno, "Analytic proof of the strong multiplicity one theorem", *Amer. J. Math.* **107**:1 (1985), 163–206.

[26] M. E. Novodvorskii, "Uniqueness theorems for generalized Bessel models", *Mat. Sb.* (*N.S.*) **90**(132) (1973), 275–287, 326. In Russian; translated in *Math. USSR Sbornik* **19**:2, 1973, 275–286.

[27] M. E. Novodvorskii and I. I. Piateckii-Shapiro, "Generalized Bessel models for the symplectic group of rank 2", *Mat. Sb.* (*N.S.*) **90**(132) (1973), 246–256. In Russian; translated in *Math. USSR Sbornik* **19**:2, 1973, 243–255.

[28] I. I. Piatetski-Shapiro, "On the Saito–Kurokawa lifting", *Invent. Math.* **71**:2 (1983), 309–338.

[29] I. I. Piatetski-Shapiro, "L-functions for GSp_4", *Pacific J. Math.* (special issue) (1997), 259–275.

[30] A. Pitale and R. Schmidt, "Ramanujan-type results for Siegel cusp forms of degree 2", *J. Ramanujan Math. Soc.* **24**:1 (2009), 87–111.

[31] A. Pitale and R. Schmidt, "Bessel models for GSp(4): Siegel vectors of square-free level", preprint (long version of previous entry), 2013, available at https://sites.math.unt.edu/~schmidt/papers/BSI.pdf.

[32] A. Pitale and R. Schmidt, "Bessel models for GSp(4): Siegel vectors of square-free level", *J. Number Theory* **136** (2014), 134–164.

[33] D. Prasad and R. Takloo-Bighash, "Bessel models for GSp(4)", *J. Reine Angew. Math.* **655** (2011), 189–243.

[34] B. Roberts and R. Schmidt, *Local newforms for* GSp(4), Lecture Notes in Mathematics **1918**, Springer, 2007.

[35] P. Sarnak, "Nonvanishing of L-functions on $\mathfrak{R}(s) = 1$", pp. 719–732 in *Contributions to automorphic forms, geometry, and number theory*, Johns Hopkins Univ. Press, Baltimore, MD, 2004.

[36] R. Schmidt, "Iwahori-spherical representations of GSp(4) and Siegel modular forms of degree 2 with square-free level", *J. Math. Soc. Japan* **57**:1 (2005), 259–293.

[37] R. Schmidt, "The Saito–Kurokawa lifting and functoriality", *Amer. J. Math.* **127**:1 (2005), 209–240.

[38] R. Schmidt, "On classical Saito–Kurokawa liftings", *J. Reine Angew. Math.* **604** (2007), 211–236.

[39] R. Schmidt, "Packet structure and paramodular forms", *Trans. Amer. Math. Soc.* **370**:5 (2018), 3085–3112.

[40] R. Schmidt and L. Tran, "Zeta integrals for GSp(4) via Bessel models", *Pacific J. Math.* **296**:2 (2018), 437–480.

[41] D. Soudry, "The CAP representations of $\mathrm{GSp}(4, \mathbb{A})$", *J. Reine Angew. Math.* **383** (1988), 87–108.

[42] T. Sugano, "On holomorphic cusp forms on quaternion unitary groups of degree 2", *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **31**:3 (1985), 521–568.

[43] R. Takloo-Bighash, "L-functions for the p-adic group GSp(4)", *Amer. J. Math.* **122**:6 (2000), 1085–1120.

[44] J. T. Tate, Jr., *Fourier analysis in number fields and Hecke's zeta-functions*, Ph.D. thesis, Princeton University, 1950, available at https://www.proquest.com/docview/304411725.

[45] J. Tate, "Number theoretic background", pp. 3–26 in *Automorphic forms, representations and L-functions* (Corvallis, OR, 1977), vol. 2, Proc. Sympos. Pure Math. **33**(2), Amer. Math. Soc., 1979.

[46] N. R. Wallach, "Representations of reductive Lie groups", pp. 71–86 in *Automorphic forms, representations and L-functions* (Corvallis, OR, 1977), vol. 1, Proc. Sympos. Pure Math. **33**(1), Amer. Math. Soc., 1979.

[47] A. Weil, *Basic number theory*, Grundl. Math. Wissen. **144**, Springer, 1967.

[48] R. Weissauer, *Endoscopy for* GSp(4) *and the cohomology of Siegel modular threefolds*, Lecture Notes in Mathematics **1968**, Springer, 2009.

Received July 21, 2024. Revised November 22, 2025.

SEIJI KUGA
FACULTY OF SCIENCE AND TECHNOLOGY
SOPHIA UNIVERSITY
TOKYO
JAPAN
s-kuga-2g7@sophia.ac.jp

MASAO TSUZUKI
FACULTY OF SCIENCE AND TECHNOLOGY
SOPHIA UNIVERSITY
TOKYO
JAPAN
m-tsuduk@sophia.ac.jp

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

EDITORS

PRODUCTION

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2026 is US $710/year for the electronic version, and $965/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 1945-5844 electronic, 0030-8730 printed) at Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Mathematical Sciences Publishers, 2000 Allston Way # 59, Berkeley, CA 94701-4004.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 342 No. 2 June 2026