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SUM OF THE SQUARES OF THE p'-CHARACTER DEGREES

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We study the sum of the squares of the irreducible character degrees not divisible by some prime p of a finite group, and its relationship with the corresponding quantity in a p-Sylow normalizer. This led us to study a recent conjecture by E. Giannelli, which we prove for $p = 2$ and in some other cases.

1. Introduction

There are some strikingly simple and seemingly innocent statements that follow from the recently proven McKay conjecture [CS24] (or from the techniques developed to prove it) whose validity appears to resist any elementary justification. For instance, W. Feit was interested in proving that if P is an abelian Sylow p-subgroup of G, then $k(G) \geq k(\mathbf{N}_G(P))$, where $k(G)$ is the number of conjugacy classes of the finite group G [F80]. Although this inequality now follows as a consequence of [CS24], no alternative proof is currently known. (Equality happens if and only if $P \trianglelefteq G$ by the Itô–Michler theorem.)

This current project — long awaited by the third author — was expected to provide another example of such phenomena. And yet, it does not. What was anticipated to be a theorem remains, for now, a conjecture. As usual, $\mathrm{Irr}_{p'}(G)$ is the set of irreducible complex characters of the finite group G whose degree is not divisible by the prime p, and $P' = [P, P]$ is the derived subgroup of the group P.

Conjecture A. *Let G be a finite group and $P \in \mathrm{Syl}_p(G)$. Then*

$$\sum_{\chi \in \mathrm{Irr}_{p'}(G)} \chi(1)^2 \geq |\mathbf{N}_G(P) : P'|,$$

with equality if and only if $\mathbf{N}_G(P)$ has a normal complement in G.

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The McKay theorem establishes that there exists a bijection $f : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\boldsymbol{N}_G(P))$. One way to prove Conjecture A, perhaps the most obvious, is to show that we can choose f to satisfy the additional condition that $f(\chi)(1) \leq \chi(1)$ for all $\chi \in \mathrm{Irr}_{p'}(G)$. This turns out to be a conjecture by E. Giannelli [G25], and it does not appear to follow easily from the proof of the McKay conjecture. In Theorem 3.5 below we carry out the standard reduction of the McKay conjecture now incorporating Giannelli's strengthening, which is therefore reduced now to a question on simple groups. This question is verified in a number of important cases in Sections 4 and 5 below. In Theorem 5.2, we completely prove it for $p = 2$, and this establishes:

Theorem B. *Conjecture A holds for $p = 2$.*

Can Conjecture A be proven independently of Giannelli's strengthening of McKay? We do not know the answer to this question. Characters of p'-degree in a group tend to have much larger degree than those in the normalizer of a Sylow p-subgroup, but so far, no effective strategy has been developed to exploit this observation.

Theorem C below, whose proof is rather involved — but does not use the classification of finite simple groups — takes care of the equality in Conjecture A (assuming Giannelli's conjecture), and might have independent interest.

Theorem C. *Let G be a finite group, p a prime, and $P \in \mathrm{Syl}_p(G)$. Then all irreducible characters of p'-degree of $\boldsymbol{N}_G(P)$ extend to G if and only if there is $K \trianglelefteq G$ such that $K\boldsymbol{N}_G(P) = G$ and $\boldsymbol{N}_K(P) = 1$.*

This extends [I86, Theorem B], where it is required that *all* the irreducible characters of $\boldsymbol{N}_G(P)$ extend to G. As we shall explain, in order to prove Theorem C, we shall need to establish a relative version of it with respect to a normal subgroup. We find it surprising that a character restriction type of theorem admits a relative version, since these versions regarding group structure occur very rarely.

There are some variations of Conjecture A which we do not attempt here. Among others, it seems reasonable to replace P' by the Frattini subgroup $\Phi(P)$ and the set of p'-degree irreducible characters by the so-called *almost p-rational* characters of p'-degree. Proving this, however, seems much more complicated.

Concerning Giannelli's conjecture, let us mention here that in general it is not always possible to find McKay bijections $f : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\boldsymbol{N}_G(P))$ such that $f(\chi)$ is an irreducible constituent of $\chi_{\boldsymbol{N}_G(P)}$, as shown by A_5 for $p = 2$, or $\mathrm{GL}_2(3)$ for $p = 3$.

A final but important remark is in order: Why do we care about the sum of the squares of the irreducible character degrees not divisible by p? First of all, this number would be the dimension of any complex algebra affording the p'-degree irreducible characters. Also, of the famous list of problems by Richard Brauer [B63], Problem 2 asks when non-isomorphic finite groups G and H have isomorphic group algebras $\mathbb{C}G$ and $\mathbb{C}H$. If G has a normal p-complement, Isaacs proved that H has a normal p-complement too [N18, Theorem 7.8]. The question of whether the normality of a Sylow p-subgroup P of G is recognized by $\mathbb{C}G$ remains open. In [N04a], it is suggested that, perhaps, this happens if and only if

$$\Big(\sum_{\chi\in\mathrm{Irr}_{p'}(G)} \chi(1)^2\Big)_{p'} = |G|_{p'}\,,$$

where $n_{p'} = n/n_p$ with n_p denoting the largest power of the prime p dividing the integer n.

2. Proof of Theorem C

Our notation for ordinary characters follows [I06] and [N18]. In this section we prove Theorem C. In order to do that, we need to prove a relative version of it, that allows us to use induction. This makes the proof more complicated. We start with some preliminary results.

Lemma 2.1. *Let $N \trianglelefteq G$ and let $\theta \in \mathrm{Irr}(N)$ be G-invariant. Then*

$$\ker\theta = \bigcap_{\chi\in\mathrm{Irr}(G|\theta)} \ker\chi\,.$$

Proof. It is clear that $\ker\theta$ is contained in $\ker\chi$ for every $\chi \in \mathrm{Irr}(G|\theta)$, since $\chi_N = e_\chi\theta$ for $\chi \in \mathrm{Irr}(G|\theta)$. Suppose that $x \in \bigcap_{\chi\in\mathrm{Irr}(G|\theta)} \ker\chi$. Then $\theta^G(x) = \theta^G(1) = |G:N|\theta(1) \neq 0$, and therefore $x \in N$. Since θ is G-invariant, we have $\theta^G(x) = |G:N|\theta(x)$, and thus $\theta(x) = \theta(1)$, as wanted. □

Lemma 2.2. *Suppose that $K \trianglelefteq G$, $H \leq G$, $N = K\cap H$ and $G = KH$. Let $\gamma \in \mathrm{Irr}(K)$ be G-invariant, and let $\theta \in \mathrm{Irr}(N)$ be an irreducible H-invariant constituent of γ_N. Suppose that restriction defines a bijection $\mathrm{Irr}(G|\gamma) \to \mathrm{Irr}(H|\theta)$. Then $\gamma_N = \theta$.*

Proof. We have

$$\sum_{\chi\in\mathrm{Irr}(G|\gamma)} (\chi(1)/\gamma(1))^2 = |G:K| \quad\text{and}\quad \sum_{\tau\in\mathrm{Irr}(H|\theta)} (\tau(1)/\theta(1))^2 = |H:N|.$$

By hypothesis,

$$\begin{aligned}\sum_{\tau\in\mathrm{Irr}(H|\theta)} (\tau(1)/\theta(1))^2 &= \sum_{\chi\in\mathrm{Irr}(G|\gamma)} (\chi(1)/\theta(1))^2\\ &= (\gamma(1)/\theta(1))^2 \sum_{\chi\in\mathrm{Irr}(G|\gamma)} (\chi(1)/\gamma(1))^2.\end{aligned}$$

Since $|H:N| = |G:K|$, we deduce that $\gamma_N = \theta$. □

Lemma 2.3. *Suppose that $G = KP$, where K is a normal p-complement and $P \in \mathrm{Syl}_p(G)$. Let $\gamma \in \mathrm{Irr}(G)$. If $P \subseteq \ker\gamma$, then $\gamma_{\boldsymbol{C}_K(P)}$ is irreducible.*

Proof. Since $\ker\gamma \trianglelefteq G$, we have $[K, P]P \subseteq \ker\gamma$. Thus γ_K is an irreducible character of K with $[K, P]$ contained in its kernel. By coprime action (for instance, Lemma 4.28 of [I08]), we have $\boldsymbol{C}_K(P)[K, P] = K$ and thus $\gamma_{\boldsymbol{C}_K(P)}$ is irreducible. □

Lemma 2.4. *Suppose that $N \trianglelefteq G$, $P \in \mathrm{Syl}_p(G)$. Let $N \le K_i \trianglelefteq G$ be complementing $N\boldsymbol{N}_G(P)/N$ for $i = 1, 2$ in G/N. Then $K_1 = K_2$.*

Proof. By working in G/N, we may assume that $N = 1$. Then G/K_i has a normal Sylow p-subgroup. Let $L = K_1 \cap K_2$. Then G/L has a normal Sylow p-subgroup. Hence, $L\boldsymbol{N}_G(P) = G$. Then $K_i = K_i \cap L\boldsymbol{N}_G(P) = L(K_i \cap \boldsymbol{N}_G(P)) = L$, as wanted. □

In several occasions, we shall use Tate's theorem in the following form.

Theorem 2.5. *Suppose that G is a finite group, $P \in \mathrm{Syl}_p(G)$, and $K \trianglelefteq G$. If $K \cap P = K \cap P'$, then K has a normal p-complement.*

Proof. We may assume that $G = KP$. In the notation of [I06, Theorem 6.31], we have $P \cap \boldsymbol{A}^p(G) = \boldsymbol{A}^p(P) = P'$, where $\boldsymbol{A}^p(G)$ is the smallest normal subgroup L of G such that G/L is an abelian p-group. It then follows that G has a normal p-complement by [I06, Theorem 6.31]. Therefore, K has a normal p-complement. □

Let $N \trianglelefteq G$ and let $\theta \in \mathrm{Irr}(N)$ be G-invariant. We let

$$\mathrm{Irr}_{p',\mathrm{rel}}(G|\theta) := \{\chi \in \mathrm{Irr}(G|\theta) \mid p \text{ does not divide } \chi(1)/\theta(1)\}.$$

The following easily implies Theorem C (when N is trivial).

Theorem 2.6. *Let $N \trianglelefteq G$ and let $\theta \in \mathrm{Irr}(N)$ be G-invariant. Let $P \in \mathrm{Syl}_p(G)$. Assume that θ extends to NP. The following statements are equivalent.*

(i) *Every $\chi \in \mathrm{Irr}_{p',\mathrm{rel}}(N\boldsymbol{N}_G(P)|\theta)$ extends to G.*

(ii) *Every $\chi \in \mathrm{Irr}(N\boldsymbol{N}_G(P)|\theta)$ extends to G.*

(iii) *There is a normal complement K/N to $\boldsymbol{N}_G(P)N/N$ in G/N such that θ has an extension $\hat{\theta} \in \mathrm{Irr}(K)$.*

(iv) *There is a normal complement K/N to $\boldsymbol{N}_G(P)N/N$ in G/N such that θ has a G-invariant extension $\hat{\theta} \in \mathrm{Irr}(K)$.*

Proof. First we prove that (iv) and (iii) are equivalent. We only have to prove that (iii) implies (iv). Since $PN \cap K = N$, we have that K/N is a group of order not divisible by p. By Sylow theory, $\boldsymbol{N}_G(PN) = \boldsymbol{N}_G(P)N$, and therefore $\boldsymbol{C}_{K/N}(P) = \boldsymbol{C}_{K/N}(PN/N) \subseteq K/N \cap \boldsymbol{N}_{G/N}(PN/N) = 1$. Let $\eta \in \mathrm{Irr}(K)$ be an extension of θ, and let Δ be the set of extensions of θ to K. We have that

$\Delta = \{\lambda\eta \mid \lambda \in \mathrm{Irr}(K/N)$ is linear$\}$, by Gallagher's theorem [I06, Corollary 6.17]. Therefore $|\Delta|$ is not divisible by p. Since P acts on Δ by conjugation, because θ is G-invariant, by counting we have that there is some P-invariant $\hat{\theta} \in \Delta$ extending θ. Since $\boldsymbol{C}_{K/N}(P) = 1$, we have that $\hat{\theta}$ is the unique P-invariant character of K lying over θ (using [I06, Theorem 13.31 and Problem 13.10]. We shall use this argument several times). We claim that $\hat{\theta}$ is also G-invariant. It is enough to show that $\hat{\theta}$ is $\boldsymbol{N}_G(P)$-invariant. If $n \in \boldsymbol{N}_G(P)$, then $\hat{\theta}^n$ is a P-invariant extension of $\theta^n = \theta$. By uniqueness, $\hat{\theta}^n = \hat{\theta}$, as wanted.

To prove that (iv) implies (ii), we apply [N18, Lemma 6.8(d)].

Since it is clear that (ii) implies (i), to complete the proof of the theorem, it is enough to show that (i) implies (iii). We argue this by induction on $|G : N|$. Recall that $N\boldsymbol{N}_G(P)/N = \boldsymbol{N}_{G/N}(PN/N)$. By using the theory of character triple isomorphisms (see [I06, Chapter 11]), we may assume that $N \subseteq \boldsymbol{Z}(G)$. Hence, $N \subseteq \boldsymbol{N}_G(P)$. In particular, θ is linear. Write $\theta = \theta_p\theta_{p'}$, where θ_p has order a power of p and $\theta_{p'}$ has order not divisible by p. Since θ extends to PN by hypothesis, therefore so does θ_p, which is a power of θ. Hence, θ_p extends to some linear character $\nu \in \mathrm{Irr}(G)$ (by [I06, Theorem 6.26]). If $\chi \in \mathrm{Irr}_{p',\mathrm{rel}}(\boldsymbol{N}_G(P)|\theta_{p'})$, then $\chi\nu_{\boldsymbol{N}_G(P)} \in \mathrm{Irr}_{p',\mathrm{rel}}(\boldsymbol{N}_G(P)|\theta)$. If $\eta \in \mathrm{Irr}(G)$ extends $\chi\nu_{\boldsymbol{N}_G(P)}$, then $\eta\nu^{-1}$ extends χ. Therefore, we may assume that θ is a linear character of p'-order. By modding out by $\ker\theta$, we may assume that N is a p'-group and that θ is faithful. Hence, our hypothesis is that every $\chi \in \mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)|\theta)$ extends to G. We want to show that there is $K \trianglelefteq G$ complementing $\boldsymbol{N}_G(P)/N$ in G, and that θ extends to K. For each $\tau \in \mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)|\theta)$, fix $\tilde{\tau} \in \mathrm{Irr}(G)$ an extension of τ to G.

If $P' \subseteq X \leq P$ and $X \trianglelefteq \boldsymbol{N}_G(P)$, we claim that there exists $L \trianglelefteq G$ such that $L \cap \boldsymbol{N}_G(P) = NX$. Let $\tilde{\theta} = \theta \times 1_X$. Notice that all $\mathrm{Irr}(\boldsymbol{N}_G(P)|\tilde{\theta})$ have p'-degree because $\boldsymbol{N}_G(P)/X$ has an abelian normal Sylow p-subgroup P/X. Let

$$U = \bigcap_{\tau \in \mathrm{Irr}(\boldsymbol{N}_G(P)|\tilde{\theta})} \ker\tilde{\tau} \trianglelefteq G.$$

Then, using Lemma 2.1, we have $U \cap \boldsymbol{N}_G(P) = \ker\tilde{\theta} = X$. Let $L = UN \trianglelefteq G$. Therefore $L \cap \boldsymbol{N}_G(P) = NX$, as claimed. Notice that in this case, $X = P \cap L \in \mathrm{Syl}_p(L)$.

By letting $X = P'$ in the claim in the previous paragraph of this proof, let $L \trianglelefteq G$ such that $L \cap \boldsymbol{N}_G(P) = NP'$. Since $P \cap L = P'$, by Tate's theorem (Theorem 2.5), L has a normal p-complement K. Also, P' is a Sylow p-subgroup of L. Let $W = K\boldsymbol{N}_G(P) = L\boldsymbol{N}_G(P)$. Notice that $\boldsymbol{C}_{K/N}(P) = 1$ since $\boldsymbol{N}_K(P) = N$, and therefore over θ there is a unique P-invariant $\eta \in \mathrm{Irr}(K)$ (again using [I06, Theorem 13.31 and Problem 13.10]). By uniqueness, notice that η is $\boldsymbol{N}_G(P)$-invariant. We have

$$|\mathrm{Irr}_{p'}(W|\theta)| = |\mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)|\theta)|$$

by the relative McKay conjecture for p-solvable groups [N18, Theorem 10.26]. (Notice that W is indeed p-solvable, so we have not used the solution of the general McKay conjecture.) We claim that $\mathrm{Irr}_{p'}(W|\theta)=\{\tilde{\tau}_W \mid \tau\in\mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)|\theta)\}$. Indeed, let $f:\mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)|\theta)\to\mathrm{Irr}_{p'}(W|\theta)$ given by $f(\tau)=\tilde{\tau}_W$. Since $(\tilde{\tau})_{\boldsymbol{N}_G(P)}=\tau$, we do have $f(\tau)\in\mathrm{Irr}_{p'}(W|\theta)$. Since f is clearly injective, it is necessarily bijective and the claim follows. Since η is the only P-invariant irreducible character of K over θ, it easily follows that $\mathrm{Irr}_{p'}(G|\theta)=\mathrm{Irr}_{p'}(G|\eta)$ and $\mathrm{Irr}_{p'}(W|\theta)=\mathrm{Irr}_{p'}(W|\eta)$. Indeed, if $\chi\in\mathrm{Irr}_{p'}(G|\theta)$, then χ_K has some P-invariant irreducible constituent, using that χ has p'-degree. All irreducible constituents of χ_K lie over θ, so we deduce that this P-invariant constituent should be η. Notice then that every $\mathrm{Irr}_{p'}(W|\eta)$ extends to G. We claim now that η is G-invariant. First notice that since η is P-invariant and K is a p'-group, η extends to KP (using Corollary 6.2 of [N18]). Hence, there is some p'-degree irreducible character of W over η. Therefore, there exists $\tau\in\mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)|\theta)$ such that $\tilde{\tau}_W$ lies over η. Since η is invariant in W, it follows that $W\le G_\eta$. If ϵ is the Clifford correspondent of $\tilde{\tau}$ over η, it follows that $\epsilon^G=\tilde{\tau}$. Since $\tilde{\tau}_{G_\eta}$ is irreducible, necessarily $G_\eta=G$. Hence, if $K>N$, we can apply induction to $\mathrm{Irr}_{p'}(W|\eta)$ with respect to G.

Suppose first that $P'\trianglelefteq G$, and assume next that $P'>1$. Working in $\bar{G}=G/P'$ and using induction, we conclude that there is $P'\le R\trianglelefteq G$ such that $R\boldsymbol{N}_G(P)=G$, and $R\cap\boldsymbol{N}_G(P)=NP'$, and that $\theta\times 1_{P'}$ extends to $\gamma\in\mathrm{Irr}(R)$. Since $P\cap R=P'$, we have that R has a normal p-complement S by Tate's theorem, and Sylow p-subgroup P'. This normal p-complement complements $\boldsymbol{N}_G(P)/N$ in G/N. Since $\gamma_N=\theta$, we have that γ_S extends θ and we are done, in the case that $P'\trianglelefteq G$ and $P'>1$. Assume now that P is abelian. Hence, all $\mathrm{Irr}(\boldsymbol{N}_G(P)|\theta)=\mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)|\theta)$ extend to G, by hypothesis. By the claim in the fourth paragraph of this proof (letting $X=P$), let $Y\trianglelefteq G$ such that $Y\cap\boldsymbol{N}_G(P)=NP$. Then $P\in\mathrm{Syl}_p(Y)$ and $G=Y\boldsymbol{N}_G(P)$ by the Frattini argument. Now, $P\subseteq\boldsymbol{Z}(\boldsymbol{N}_Y(P))$, and by Burnside's p-complement theorem (Theorem 5.13 of [I08]), Y has a normal p-complement Q. Then $Q\boldsymbol{N}_G(P)=G$ and $Q\cap\boldsymbol{N}_G(P)=N$. Notice that G is p-solvable. Since $\boldsymbol{C}_{Q/N}(P)=1$, let $\mu\in\mathrm{Irr}(Q)$ be the unique P-invariant over θ. By uniqueness, μ is $\boldsymbol{N}_G(P)$-invariant, and therefore G-invariant. Also $\mathrm{Irr}_{p'}(G|\theta)=\mathrm{Irr}_{p'}(G|\mu)=\mathrm{Irr}(G|\mu)$, using that μ extends to $QP=Y$, Gallagher [I06, Corollary 6.17], and the fact that P is abelian. By the relative version of the McKay conjecture (in p-solvable groups; [N18, Theorem 10.26]), we have

$$|\mathrm{Irr}(G|\mu)|=|\mathrm{Irr}_{p'}(G|\mu)|=|\mathrm{Irr}_{p'}(G|\theta)|=|\mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)|\theta)|=|\mathrm{Irr}(\boldsymbol{N}_G(P)|\theta)|.$$

By a previous argument, we see that $\mathrm{Irr}(G|\mu)=\{\tilde{\tau}\mid\tau\in\mathrm{Irr}(\boldsymbol{N}_G(P)|\theta)\}$, and we see that restriction defines a bijection $\mathrm{Irr}(G|\mu)\to\mathrm{Irr}(\boldsymbol{N}_G(P)|\theta)$. By Lemma 2.2, we have $\mu_N=\theta$, and we are done. Therefore we may assume that P' is not normal in G. Hence, using the notation of the fifth paragraph of this proof, we may assume that

$K > N$. (Otherwise, $L = NP' = N \times P'$ and necessarily $P' \trianglelefteq G$.) Every $\mathrm{Irr}_{p'}(W|\eta)$ extends to G. By induction, there exists $K \le E \trianglelefteq G$ complementing W/K and some $\rho \in \mathrm{Irr}(E)$ such that $\rho_K = \eta$. Notice that E complements $\boldsymbol{N}_G(P)/N$. In particular $\boldsymbol{C}_{E/N}(P) = 1$, and ρ is the only P-invariant over θ. Hence $\mathrm{Irr}_{p'}(G|\theta) = \mathrm{Irr}_{p'}(G|\rho)$. By a previous argument, we know that ρ is $\boldsymbol{N}_G(P)$-invariant, and therefore G-invariant. We only need to show that $\rho_N = \theta$. Recall that G is p-solvable. Hence $|\mathrm{Irr}_{p'}(G|\rho)| = |\mathrm{Irr}_{p'}(G|\theta)| = |\mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)|\theta)|$ (again by [N18, Theorem 10.26]), and therefore we deduce that

$$\mathrm{Irr}_{p'}(G|\rho) = \{\tilde{\tau} \mid \tau \in \mathrm{Irr}(\boldsymbol{N}_G(P)|\theta)\}.$$

Using that E is a p'-group, let $\hat{\rho} \in \mathrm{Irr}(EP)$ be an extension of ρ. Since $\hat{\rho}$ has p'-degree and $EP \trianglelefteq G$ (because $E\boldsymbol{N}_G(P) = G$), it follows that $\hat{\rho}$ lies under some p'-irreducible character of G, call it χ. Hence $\hat{\rho}$ lies under some $\tilde{\tau} = \chi$ for some $\tau \in \mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)|\theta)$. However $\tau_{P'}$ is a multiple of $1_{P'}$. Thus $\hat{\rho}_{P'}$ is a multiple of $1_{P'}$. Write $C = \boldsymbol{C}_E(P')$. By Lemma 2.3 applied to EP', we have that $\varphi = \hat{\rho}_C = \rho_C$ is irreducible. Hence, restriction defines a bijection $\mathrm{Irr}(G|\rho) \to \mathrm{Irr}(\boldsymbol{N}_G(P')|\varphi)$, by [N18, Lemma 6.8(d)]. Notice that $\mathrm{Irr}(\boldsymbol{N}_G(P')|\varphi) = \mathrm{Irr}(\boldsymbol{N}_G(P')|\theta)$ since $\boldsymbol{C}_{C/N}(P) = 1$. By the p-solvable case of the McKay conjecture [N18, Theorem 10.26] we know that $|\mathrm{Irr}_{p'}(\boldsymbol{N}_G(P')|\theta)| = |\mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)|\theta)|$. Therefore

$$\mathrm{Irr}_{p'}(\boldsymbol{N}_G(P')|\theta) = \left\{\tilde{\tau}_{\boldsymbol{N}_G(P')} | \tau \in \mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)|\theta)\right\}.$$

Since $\boldsymbol{N}_G(P') < G$, by induction, $\boldsymbol{N}_G(P)/N$ has a normal complement in $\boldsymbol{N}_G(P')/N$, which by Lemma 2.4 has to be C and that θ extends to C. By the first paragraph of this proof, θ has a P-invariant extension to C. Since $\boldsymbol{C}_{C/N}(P) = 1$, then this P-invariant extension should be $\varphi = \rho_C$, and therefore ρ extends θ, as desired. □

3. The McKay conjecture and inequality between character degrees

The following refinement of the McKay conjecture has been proposed in [G25].

Conjecture 3.1. *Let G be a finite group, p a prime and $P \in \mathrm{Syl}_p(G)$. Then there is a bijection*

$$^* : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\boldsymbol{N}_G(P))$$

with

$$\chi^*(1) \le \chi(1)$$

for all $\chi \in \mathrm{Irr}_{p'}(G)$.

We observe now that if Conjecture 3.1 is true, then so is Conjecture A.

Proposition 3.2. *Let G be a finite group, let p be a prime and let $P \in \mathrm{Syl}_p(G)$. If Conjecture 3.1 is true for G then*

$$\sum_{\chi \in \mathrm{Irr}_{p'}(G)} \chi(1)^2 \geq |\boldsymbol{N}_G(P) : P'|$$

with equality if and only if $\boldsymbol{N}_G(P)$ has a normal complement in G.

Proof. Let $\Omega : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\boldsymbol{N}_G(P))$ be the bijection from Conjecture 3.1. Then

$$\sum_{\chi \in \mathrm{Irr}_{p'}(G)} \chi(1)^2 \geq \sum_{\chi \in \mathrm{Irr}_{p'}(G)} \Omega(\chi)(1)^2 = \sum_{\psi \in \mathrm{Irr}_{p'}(\boldsymbol{N}_G(P))} \psi(1)^2 = |\boldsymbol{N}_G(P) : P'|$$

and the inequality part follows. If

$$\sum_{\chi \in \mathrm{Irr}_{p'}(G)} \chi(1)^2 = |\boldsymbol{N}_G(P) : P'|$$

then for every $\chi \in \mathrm{Irr}_{p'}(G)$ we have $\Omega(\chi)(1) = \chi(1)$. We claim that this implies that restriction is a bijection $\mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\boldsymbol{N}_G(P))$. Write $H = \boldsymbol{N}_G(P)$. Suppose that $a_1 = 1 < \cdots < a_k$ are the degrees of $\mathrm{Irr}_{p'}(G)$, and therefore, of $\mathrm{Irr}_{p'}(H)$, occurring with multiplicities $m_1, \ldots, m_k$, respectively. Let $\{\psi_1, \ldots, \psi_t\}$ and $\{\chi_1, \ldots, \chi_t\}$ be the irreducible characters in $\mathrm{Irr}_{p'}(H)$ and $\mathrm{Irr}_{p'}(G)$ of maximal degree a_k, where $t = m_k$. Now, $(\psi_i)^G$ contains an irreducible character of p'-degree of G of degree at least $\psi_i(1)$, using that $(\psi_i)^G(1)$ is not divisible by p. Necessarily, ψ_i contains some χ_j. Then $(\chi_j)_H = \psi_i$, and we can reorder so that $(\chi_i)_H = \psi_i$. Suppose that $\{\delta_1, \ldots, \delta_s\}$ and $\{\eta_1, \ldots, \eta_s\}$ are the irreducible characters in $\mathrm{Irr}_{p'}(H)$ and $\mathrm{Irr}_{p'}(G)$ of the next degree a_{k-1}, with $s = m_{k-1}$. Again $(\delta_i)^G$ contains a p'-irreducible character τ with degree at least a_{k-1}. If $\tau(1) = a_k$, then we know that τ_H is irreducible and then $\tau_H = \delta_i$. This is impossible. Hence, $(\delta_i)^G$ contains some η_j, which necessarily extends δ_i. Reordering, we may assume that $(\eta_i)_H = \delta_i$. We proceed like this until the claim is proven.

We may now apply Theorem 2.6 with $N = 1$ to conclude. □

The purpose of this section is to give a reduction of Conjecture 3.1 to simple groups. The following is a slight refinement of the inductive McKay condition, and will be the condition we impose on quasisimple groups in our reduction. We use the *central isomorphism* relation $\geq_c$ from [N18, Definition 10.14], which generalizes the notion of a character triple isomorphism.

Conjecture 3.3. *Let S be a quasisimple group with cyclic center, $P \in \mathrm{Syl}_p(S)$ and $A = \mathrm{Aut}(S)_P$. Then there is an A-stable subgroup $\boldsymbol{N}_S(P) \leq M < S$ and an A-equivariant bijection*

$$\Psi : \mathrm{Irr}_{p'}(S) \to \mathrm{Irr}_{p'}(M)$$

with

$$\Psi(\chi)(1) \leq \chi(1) \quad \textit{and} \quad (S \rtimes A_\chi, S, \chi) \geq_c (M \rtimes A_\chi, M, \Psi(\chi))$$

for every $\chi \in \mathrm{Irr}_{p'}(S)$.

Theorem 3.4. *Let $K \trianglelefteq G$ be perfect, and assume $K/\mathbf{Z}(K) \cong X^n$ where X is a finite simple group and $\mathbf{Z}(K)$ cyclic. Assume every quasisimple group S with $S/\mathbf{Z}(S) \cong X$ and $\mathbf{Z}(S)$ cyclic satisfies Conjecture 3.3. Let $Q \in \mathrm{Syl}_p(G)$ and $R = Q \cap K$. Then there exists a $\mathbf{N}_G(R)$-stable subgroup $\mathbf{N}_K(R) \leq H < K$ and a $\mathbf{N}_G(R)$-equivariant bijection*

$$\Psi : \mathrm{Irr}_{p'}(K) \to \mathrm{Irr}_{p'}(H)$$

such that

$$\Psi(\theta)(1) \leq \theta(1) \quad \textit{and} \quad (G_\theta, K, \theta) \geq_c (H\mathbf{N}_G(R)_\theta, H, \Psi(\theta))$$

for all $\theta \in \mathrm{Irr}_{p'}(K)$.

Proof. K is a central product $S_1 \cdots S_n$ where $\mathbf{Z}(S_i) = \mathbf{Z}(K)$, each S_i is perfect and $S_i/\mathbf{Z}(S_i) \cong X$. Use the proof of [N18, Theorem 10.25] using the S_i's instead of the universal covering group of X, and noticing that the construction of the bijection Ψ satisfies the degree inequality if one assumes Conjecture 3.3. Note that the subgroup H is constructed by taking the product of the subgroups M appearing in the statement of Conjecture 3.3. □

The next result follows the proof of [N18, Theorem 10.26]. We sketch it for the reader's convenience. Recall that we say a nonabelian finite simple group S is *involved* in G if there exist $H \trianglelefteq K \leq G$ with $K/H \cong S$.

Theorem 3.5. *Assume that, for every simple group X of order divisible by p involved in G, Conjecture 3.3 holds whenever S is a quasisimple group with $S/\mathbf{Z}(S) \cong X$ and $\mathbf{Z}(S)$ is cyclic. Let $Z \trianglelefteq G$, $P \in \mathrm{Syl}_p(G)$, $\lambda \in \mathrm{Irr}(Z)$ and assume λ is P-invariant. Then there is a bijection*

$$\Omega : \mathrm{Irr}_{p'}(G|\lambda) \to \mathrm{Irr}_{p'}(Z\mathbf{N}_G(P)|\lambda)$$

with $\Omega(\chi)(1) \leq \chi(1)$ for all $\chi \in \mathrm{Irr}_{p'}(G)$.

Proof. We argue by induction on $|G : Z|$. Since $|G : G_\lambda| \geq |Z\mathbf{N}_G(P) : (Z\mathbf{N}_G(P))_\lambda|$, by the Clifford correspondence we may assume λ is G-invariant. By character triple isomorphisms we may assume Z is central and cyclic and λ is linear and faithful. Let L/Z be a chief factor of G/Z. Let Δ be a G-transversal on the set of P-invariant characters in $\mathrm{Irr}(L|\lambda)$ lying under some $\chi \in \mathrm{Irr}_{p'}(G)$ and notice that [N18, Lemma 9.3] implies that Δ is also a $\mathbf{N}_G(P)$-transversal on the P-invariant characters in $\mathrm{Irr}(L|\lambda)$ lying under some $\chi \in \mathrm{Irr}_{p'}(L\mathbf{N}_G(P))$. This implies that

$$\mathrm{Irr}_{p'}(G|\lambda) = \bigsqcup_{\theta \in \Delta} \mathrm{Irr}_{p'}(G|\theta)$$

and

$$\mathrm{Irr}_{p'}(L\mathbf{N}_G(P)|\lambda) = \bigsqcup_{\theta \in \Delta} \mathrm{Irr}_{p'}(L\mathbf{N}_G(P)|\theta).$$

By induction, for every $\mu \in \Delta$ there is a bijection

$$\Omega_\mu : \mathrm{Irr}_{p'}(G|\mu) \to \mathrm{Irr}_{p'}(L\boldsymbol{N}_G(P)|\mu)$$

satisfying $\Omega_\mu(\chi)(1) \le \chi(1)$ for every $\chi \in \mathrm{Irr}_{p'}(G|\mu)$. We define

$$\Omega_1 : \mathrm{Irr}_{p'}(G|\lambda) \to \mathrm{Irr}_{p'}(L\boldsymbol{N}_G(P)|\lambda)$$

by $\Omega_1(\chi) = \Omega_\mu(\chi)$ if $\chi \in \mathrm{Irr}(G|\mu)$ and we have Ω_1 is a bijection satisfying $\Omega_1(\chi)(1) \le \chi(1)$ for every $\chi \in \mathrm{Irr}_{p'}(G|\lambda)$. If $L\boldsymbol{N}_G(P) < G$ then by induction we have a bijection

$$\Omega_0 : \mathrm{Irr}_{p'}(L\boldsymbol{N}_G(P)|\lambda) \to \mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)|\lambda)$$

also satisfying $\Omega_0(\psi)(1) \le \psi(1)$ for every $\psi \in \mathrm{Irr}_{p'}(G|\lambda)$. We take $\Omega := \Omega_0 \circ \Omega_1$ and this bijection satisfies the desired properties. Thus we may assume $L\boldsymbol{N}_G(P) = G$. In particular, L/Z is not a p-group.

If L/Z is a p'-group then we write $Z = Z_p \times Z'_p$ where $Z_p \in \mathrm{Syl}_p(Z)$ and $\mu = \lambda_{Z_{p'}}$ and $\nu = \lambda_{Z_p}$. Let $K \trianglelefteq L$ be a p-complement of L. If Δ is a G-transversal on the set of P-invariant characters of $\mathrm{Irr}(K|\mu)$ and $^* : \mathrm{Irr}_P(K) \to \mathrm{Irr}(C)$ is the P-Glauberman correspondence, where $C = \boldsymbol{C}_K(P)$, then notice that this correspondence restricts to a bijection $^* : \mathrm{Irr}_P(K|\mu) \to \mathrm{Irr}(C|\mu)$. We have

$$\mathrm{Irr}_{p'}(G|\lambda) = \bigsqcup_{\theta \in \Delta} \mathrm{Irr}_{p'}(G|\theta \times \nu)$$

and

$$\mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)|\lambda) = \bigsqcup_{\theta \in \Delta} \mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)|\theta^* \times \nu)$$

again applying [N18, Lemma 9.3]. By [T08, Theorem 6.5] and [T09, Theorem 7.12] the character triples (G_θ, K, θ) and $(\boldsymbol{N}_G(P)_{\theta^*}, C, \theta^*)$ are isomorphic, which implies that there is a bijection

$$\Phi_\theta : \mathrm{Irr}(G_\theta|\theta) \to \mathrm{Irr}(\boldsymbol{N}_G(P)_{\theta^*}|\theta^*)$$

preserving character degree ratios (and in particular, restricting to p'-degree characters). Since $\theta^*(1) \le \theta(1)$, we have $\Phi_\theta(\psi)(1) \le \psi(1)$ for all $\psi \in \mathrm{Irr}_{p'}(G_\theta|\theta)$. Further, it follows from the definition of character triple isomorphism that Φ_θ restricts to a bijection

$$\Phi_\theta : \mathrm{Irr}_{p'}(G_\theta|\theta \times \nu) \to \mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)_{\theta^*}|\theta^* \times \nu)$$

with $\Phi_\theta(\psi)(1) \le \psi(1)$ for all $\psi \in \mathrm{Irr}_{p'}(G_\theta|\theta \times \nu)$. Since $G_\theta \cap \boldsymbol{N}_G(P) = \boldsymbol{N}_G(P)_{\theta^*}$, using the Clifford correspondence we find a bijection

$$\hat{\Phi}_\theta : \mathrm{Irr}_{p'}(G|\theta \times \nu) \to \mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)|\theta^* \times \nu)$$

with $\hat{\Phi}_\theta(\chi)(1) \leq \chi(1)$ for all $\chi \in \mathrm{Irr}_{p'}(G_\theta|\theta \times \nu)$ and we are done by defining

$$\Omega : \mathrm{Irr}_{p'}(G|\lambda) \to \mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)|\lambda)$$

by $\Omega(\chi) := \hat{\Phi}_\theta(\chi)$ where $\theta \in \Delta$ lies below χ.

Therefore we may assume L/Z is semisimple. Let $K = L'$, $Z_1 = Z \cap K = \boldsymbol{Z}(K)$, and $\nu = \lambda_{Z_1}$. Then K is perfect and satisfies the hypotheses of Theorem 3.4. Let $R = P \cap K$. There exists a $\boldsymbol{N}_G(R)$-stable subgroup $\boldsymbol{N}_K(R) \leq H < K$ and an $\boldsymbol{N}_G(R)$-equivariant bijection

$$\Psi : \mathrm{Irr}_{p'}(K) \to \mathrm{Irr}_{p'}(H)$$

satisfying $\Psi(\eta)(1) \leq \eta(1)$ and inducing central character triple isomorphisms. By the definition of $\geq_c$ we see that Ψ restricts to a bijection

$$\Psi : \mathrm{Irr}_{p'}(K|\nu) \to \mathrm{Irr}_{p'}(H|\nu)$$

with the same properties. The character triple isomorphisms induce bijections

$$\Phi_\mu : \mathrm{Irr}_{p'}(G_\mu|\mu) \to \mathrm{Irr}_{p'}(H\boldsymbol{N}_G(R)_{\Psi(\mu)}|\Psi(\mu))$$

which satisfy $\Phi_\mu(\chi)(1) \leq \chi(1)$, and send characters over the central product $\mu \cdot \lambda \in \mathrm{Irr}_{p'}(L)$ to characters over $\Psi(\mu) \cdot \lambda \in \mathrm{Irr}_{p'}(HZ)$. Again, $|G : G_\mu| \geq |H\boldsymbol{N}_G(R) : M\boldsymbol{N}_G(R)_{\Psi(\mu)}|$ so by the Clifford correspondence and the above remark we may find a bijection

$$\hat{\Phi}_\mu : \mathrm{Irr}_{p'}(G|\mu \cdot \lambda) \to \mathrm{Irr}_{p'}(H\boldsymbol{N}_G(R)|\Psi(\mu) \cdot \lambda)$$

satisfying $\hat{\Phi}_\mu(\chi)(1) \leq \chi(1)$ for all $\chi \in \mathrm{Irr}_{p'}(G|\mu \cdot \lambda)$. Now by taking transversals over the P-invariant characters in $\mathrm{Irr}_{p'}(K|\nu)$ that lie under characters $\chi \in \mathrm{Irr}_{p'}(G)$ and arguing as before we obtain a bijection

$$\Omega_0 : \mathrm{Irr}_{p'}(G|\lambda) \to \mathrm{Irr}_{p'}(H\boldsymbol{N}_G(R)|\lambda)$$

satisfying $\Omega_0(\chi)(1) \leq \chi(1)$ for every $\chi \in \mathrm{Irr}_{p'}(G|\lambda)$. Since $H < K$ and $L\boldsymbol{N}_G(P) = G$ we have $\boldsymbol{N}_G(P) \leq H\boldsymbol{N}_G(R) < G$ so, by induction we have a bijection

$$\Omega_1 : \mathrm{Irr}_{p'}(H\boldsymbol{N}_G(R)|\lambda) \to \mathrm{Irr}_{p'}(\boldsymbol{N}_G(P)|\lambda)$$

satisfying $\Omega_1(\psi)(1) \leq \psi(1)$ for all $\psi \in \mathrm{Irr}_{p'}(H\boldsymbol{N}_G(R)|\lambda)$ and the result follows by taking $\Omega := \Omega_1 \circ \Omega_0$. □

Conjecture 3.1 can be recovered by setting $Z = 1$ in Theorem 3.5. By arguing as in Proposition 3.2 and using Theorem 3.5 we obtain a relative version of Conjecture A. If $N \trianglelefteq G$ and $\theta \in \mathrm{Irr}(N)$ recall that we write

$$\mathrm{Irr}_{p',\mathrm{rel}}(G|\theta) = \{\chi \in \mathrm{Irr}(G|\theta) \mid p \text{ does not divide } \chi(1)/\theta(1)\}.$$

Proposition 3.6. *Let G be a finite group and $P \in \mathrm{Syl}_p(G)$. Let $N \trianglelefteq G$, and let $\theta \in \mathrm{Irr}(N)$ be G-invariant such that θ extends to NP. Assume Conjecture 3.3 holds for every covering group of every simple group of order divisible by p involved in G. Then*

$$\sum_{\chi \in \mathrm{Irr}_{p',\mathrm{rel}}(G|\theta)} (\chi(1)/\theta(1))^2 \geq \sum_{\tau \in \mathrm{Irr}_{p',\mathrm{rel}}(N\mathbf{N}_G(P)|\theta)} (\tau(1)/\theta(1))^2,$$

with equality if and only if there is a normal complement K/N to $\mathbf{N}_G(P)N/N$ in G/N such that θ has an extension $\hat{\theta} \in \mathrm{Irr}(K)$.

4. Quasisimple groups

In this section and the next, we present evidence supporting Conjecture 3.3. We verify the conjecture for, among other cases, all quasisimple groups of exceptional Lie type with respect to all primes, as well as for all groups of Lie type defined in characteristic equal to the given prime p. These results, in particular, allow us to confirm the conjecture for all quasisimple groups when $p = 2$.

In the cases mentioned, we prove a slightly stronger version of Conjecture 3.3:

Conjecture 4.1. *Let S be a quasisimple group with cyclic center, $P \in \mathrm{Syl}_p(S)$ and $A = \mathrm{Aut}(S)_P$. Then there exists an A-equivariant bijection*

$$\Psi : \mathrm{Irr}_{p'}(S) \to \mathrm{Irr}_{p'}(\mathbf{N}_S(P))$$

such that

$$\chi(1) \geq \Psi(\chi)(1) \quad \textit{and} \quad (S \rtimes A_\chi, S, \chi) \geq_c (\mathbf{N}_S(P) \rtimes A_\chi, \mathbf{N}_S(P), \Psi(\chi))$$

for every $\chi \in \mathrm{Irr}_{p'}(S)$.

The existence of an A-equivariant bijection Ψ from $\mathrm{Irr}_{p'}(S)$ to $\mathrm{Irr}_{p'}(\mathbf{N}_S(P))$ satisfying the so-called *central isomorphism*

$$(S \rtimes A_\chi, S, \chi) \geq_c (\mathbf{N}_S(P) \rtimes A_\chi, \mathbf{N}_S(P), \Psi(\chi))$$

(of character triples) has been established in several papers and was completed in [CS24] by Cabanes and Späth. In fact, by [CS24, Theorem B], we now know that such bijection, which we shall refer to as a *McKay-good* bijection, exists for all finite groups. It is the extra *degree condition* $\Psi(\chi)(1) \leq \chi(1)$ that we need to consider in this paper.

Following the literature, we say that a quasisimple group S satisfies the *inductive McKay condition* if it satisfies Conjecture 3.3, possibly excluding the degree condition. Note that there are other equivalent formulations of the inductive McKay conditions; see, for example, [IMN07, §10]. In some cases, we will work with this version of the inductive condition. We also note that if Conjecture 4.1 holds for a quasisimple group S (with or without cyclic center), then it also holds for every quotient of S by a central subgroup. For further discussion, we refer the reader to [CS24, Remark 2.9].

Notation 4.2. Let G be a finite group.

(i) $d(G) := \min\{\chi(1) : \chi \in \mathrm{Irr}(G), \chi(1) > 1\}$ is the smallest nontrivial degree of a (complex) irreducible character of G. By convention, $d(G) = 1$ if G is abelian.

(ii) $m_p(G) := \min\{\chi(1) : \chi \in \mathrm{Irr}_{p'}(G), \chi(1) > 1\}$ is the smallest nontrivial degree of an irreducible p'-character of G. By convention, $m_p(G) = 1$ if G has no non-linear p'-degree irreducible character.

(iii) $b_p(G) := \max\{\chi(1) : \chi \in \mathrm{Irr}_{p'}(G)\}$ is the largest degree of an irreducible p'-character of G.

(iv) We will use $m(G)$ and $b(G)$, respectively, for $m_p(G)$ and $b_p(G)$, whenever p is implicitly known or the presence of p is not important.

Hypothesis 4.3. $m(S) \geq b(\boldsymbol{N}_S(P))$, where S is a quasisimple group, p is a prime, and $P \in \mathrm{Syl}_p(S)$.

Proposition 4.4. *Fix a quasisimple group S and a prime p. If Hypothesis 4.3, or the stronger condition $d(S) \geq b(\boldsymbol{N}_S(P))$, is true for S, then so is Conjecture 4.1 for S.*

Proof. If $m(S) \geq b(\boldsymbol{N}_S(P))$, then any bijection Ψ from $\mathrm{Irr}_{p'}(S)$ to $\mathrm{Irr}_{p'}(\boldsymbol{N}_S(P))$ sending 1_S to $1_{\boldsymbol{N}_S(P)}$ automatically satisfies the degree condition $\Psi(\chi)(1) \leq \chi(1)$ for every $\chi \in \mathrm{Irr}_{p'}(S)$. □

Hypothesis 4.3, unfortunately, fails quite often. For instance, in general, it fails when S is a cover of an alternating or a simple classical group (in characteristic not equal to p). It also fails for certain groups of Lie type in characteristic p (see the proof of Proposition 4.11).

4.1. *Groups of Lie type in characteristic p.* The failure of Hypothesis 4.3 when S is a group of Lie type in characteristic p arises in the case of unitary groups. This case requires additional work. Our notation for finite simple groups (and related ones) follows [C85; Atl].

Low-degree irreducible representations of the special unitary groups $\mathrm{SU}_n(q)$ ($q = p^f$ is a power of a prime p and $n \geq 3$, excluding $(n, q) = (3, 2)$) are studied in [TZ96, §4] and [LOST10, §6.1]. Among these are the so-called irreducible Weil characters, denoted by $\zeta^i_{n,q}$ for $0 \leq i \leq q$. The characters $\zeta^i_{n,q}$ with $i > 0$ have degree $(q^n - (-1)^n)/(q + 1)$, while $\zeta^0_{n,q}$ has degree $(q^n + q(-1)^n)/(q + 1)$. In fact, these $\zeta^i_{n,q}$ account for all nontrivial characters of $\mathrm{SU}_n(q)$ with degree at most $d(\mathrm{SU}_n(q)) + 1$. Hence, if p is the defining characteristic of the group, we have

$$m_p(\mathrm{SU}_n(q)) = (q^n - (-1)^n)/(q + 1).$$

For our purposes, we need to construct a non-Weil character of the unitary groups that is invariant under the automorphism groups.

To do so, we first describe the Weil characters of $G := \mathrm{SU}_n(q)$ via the notion of Lusztig series and semisimple characters, as follows. The set $\mathrm{Irr}(G)$ of irreducible characters of G is partitioned into the Lusztig series $\mathcal{E}(G, s)$, where s runs over a complete set of representatives of conjugacy classes of semisimple elements of $G^* := \mathrm{PGU}_n(q)$ (see [GM20, Theorem 2.6.2]). Each $\mathcal{E}(G, s)$ contains one or more special members called semisimple characters whose degrees are equal to $|G^* : \boldsymbol{C}_{G^*}(s)|_{p'}$ (see [GM20, Definition 2.6.9]).

Centralizers of semisimple elements in classical groups are well known (see [C81; Franceschi] for instance). We recall here the needed result for $\mathrm{GU}_n(q)$.

Lemma 4.5. *Let $G = \mathrm{GU}_n(q)$ and $s \in G$ be a semisimple element. For a monic polynomial $g(t) = t^d + a_{d-1}t^{d-1} + \cdots + a_1 t + a_0 \in \mathbb{F}_{q^2}(t)$ not equal to t, write $g^*(t) := t^d + (a_1/a_0)^q t^{d-1} + \cdots + (a_{d-1}/a_0)^q t + (1/a_0)^q$. Let $f(t)$ be the characteristic polynomial of s and assume that its decomposition into irreducible monic polynomials over $\mathbb{F}_{q^2}$ is*

$$f(t) = \pm \prod_{i=1}^{a} f_i(t)^{n_i} \times \prod_{j=1}^{b} (g_j(t) g_j^*(t))^{m_j},$$

where $f_i = f_i^*$ and $g_j \neq g_j^*$ for every i and j and they are pairwise distinct. Let $d_i := \deg(f_i)$ and $e_j := \deg(g_j)$. Then

$$\boldsymbol{C}_G(s) \cong \prod_{i=1}^{a} \mathrm{GU}_{n_i}(q^{d_i}) \times \prod_{j=1}^{b} \mathrm{GL}_{m_j}(q^{2e_j}).$$

Remark 4.6. Irreducible factors of the characteristic polynomial $f(t)$ of a semisimple element in $\mathrm{GU}_n(q)$ are subject to certain restrictions. For example, a polynomial g is a factor if and only if g^* is also a factor. Additionally, any factor g satisfying $g = g^*$ must have odd degree.

Let δ be a generator of the multiplicative group $\mathbb{F}_{q^2}^\times$. For $1 \leq i \leq q$, consider the semisimple element $s_i \in G^*$ to be the image of the diagonal matrix

$$\tilde{s}_i := \mathrm{diag}(\delta^{i(q-1)}, 1^{n-1}) \in \widetilde{G} := \mathrm{GU}_n(q)$$

under the natural projection $\pi : \widetilde{G} \to G^*$. For general s, $\mathcal{E}(G, s)$ may contain more than one semisimple character, but we claim that, each $\mathcal{E}(G, s_i)$ contains a unique one.

Note that $\widetilde{G}$ is self-dual (in the sense of [C85, §4.3]) and we may identify $\widetilde{G}$ with its dual group. As the center of the ambient algebraic group of $\widetilde{G}$ is connected, the series $\mathcal{E}(\widetilde{G}, \tilde{s}_i)$ contains a unique semisimple character [GM20, p. 171]. We denote this character by ψ_i. Then $\psi_i(1) = |\widetilde{G} : \boldsymbol{C}_{\widetilde{G}}(\tilde{s}_i)|_{p'}$. By Lemma 4.5,

$$\boldsymbol{C}_{\widetilde{G}}(\tilde{s}_i) \cong \mathrm{GU}_{n-1}(q) \times \mathrm{GU}_1(q).$$

The choice of eigenvalues of s_i's implies that $\boldsymbol{C}_{G^*}(s_i) = \boldsymbol{C}_{\widetilde{G}}(\tilde{s}_i)/\boldsymbol{Z}\widetilde{G}$. (To see this, let $C := \pi^{-1}(\boldsymbol{C}_{G^*}(s_i))$.) Then the mapping $\tau : C \to \boldsymbol{Z}\widetilde{G}$ defined by $gsg^{-1} = \tau(g)s$ is a homomorphism with $\operatorname{Ker}\tau = \boldsymbol{C}_{\widetilde{G}}(\tilde{s}_i)$. However, the fact that s and gsg^{-1} have the same eigenvalues forces $\tau(g) = 1$ for every $g \in C$.) It follows that

$$|G^* : \boldsymbol{C}_{G^*}(s_i)|_{p'} = |\widetilde{G} : \boldsymbol{C}_{\widetilde{G}}(\tilde{s}_i)|_{p'} = \frac{q^n - (-1)^n}{q+1},$$

so that semisimple characters in $\mathcal{E}(G, s_i)$ have the same degree as the (only) semisimple character ψ_i in $\mathcal{E}(\widetilde{G}, \tilde{s}_i)$. By [GM20, Corollary 2.6.18], semisimple characters in $\mathcal{E}(G, s_i)$ are irreducible constituents of the restriction of ψ_i from $\widetilde{G}$ to G^*. We deduce that $\mathcal{E}(G, s_i)$ has a unique semisimple character, as claimed. This is the Weil character $\zeta_{n,q}^i$ mentioned above.

Lemma 4.7. *Assume the above notation. Then a Weil character $\zeta_{n,q}^i$ for some $1 \le i \le q$ is invariant under all field automorphisms of G if and only if $i(p-1)$ is divisible by $q+1$. In particular, if $p = 2$, then every $\zeta_{n,q}^i$ is moved by some automorphism of G.*

Proof. The character $\zeta_{n,q}^i$ is invariant under all field automorphisms of G if and only if $\delta^{i(q-1)} = \delta^{ip(q-1)}$, which means that $i(q-1) \equiv ip(q-1) (\bmod q^2 - 1)$, and the first statement follows. When $p = 2$, each $\zeta_{n,q}^i$ (for $1 \le i \le q$) is moved by some field automorphisms of G, as desired. □

Lemma 4.8. *Let $G = \mathrm{SU}_n(q)$ where $n \ge 3$ is odd, $(n, q+1) = 1$, and $q = p^f$ for some prime p. Then G possesses an irreducible character χ of degree not divisible by p such that χ is* Aut(G)*-invariant and $\chi(1) > q^{n-1}$.*

Proof. By the hypothesis, G is simple and self-dual. By [B09, Lemma 5], p'-degree irreducible characters of G are precisely the semisimple characters of G, which in turn can be labeled by conjugacy classes of semisimple elements of $G^* := \mathrm{PGU}_n(q) \cong G$.

It is well known that Aut(G) permutes the Lusztig series of G. In our situation, by identifying the automorphisms of G with the corresponding automorphisms of G^* under the natural isomorphism, every $\varphi \in \mathrm{Aut}(G)$ maps $\mathcal{E}(G, s)$ to $\mathcal{E}(G, \varphi(s))$ (see [T18, Proposition 7.2]). Since each $\mathcal{E}(G, s)$ contains a unique semisimple character (using again [GM20, p. 171] and the fact that the ambient algebraic group of $\mathrm{PGU}_n(q)$ has connected center), of degree $|G^* : \boldsymbol{C}_{G^*}(s)|_{p'}$, it follows that a p'-degree irreducible characters of G is Aut(G)-invariant if and only if the semisimple conjugacy class labeling it is Aut(G^*)-invariant. Therefore, the result follows if we are able to produce a semisimple element $s \in G^*$ such that its G^*-conjugacy class is $\mathrm{Aut}(G^*)$-invariant and $|G^* : \boldsymbol{C}_{G^*}(s)|_{p'} > q^{n-1}$. Recall that $\mathrm{Out}(G^*) \cong C_{2f}$ is the cyclic group of order $2f$ consisting of the field automorphisms of G^*.

Suppose first that q is odd. Let s be the image of a diagonal matrix $\tilde{s} \in \widetilde{G} := \mathrm{GU}_n(q)$ with spectrum $\mathrm{Spec}(\tilde{s}) = \{1, \ldots, 1, -1, -1\}$ under the projection $\pi : \widetilde{G} \to G^*$ mentioned earlier. Note that n is odd. Similar arguments as above show that $\boldsymbol{C}_{\widetilde{G}}(\tilde{s})$ is the complete inverse image of $\boldsymbol{C}_{G^*}(s)$. Moreover, $\boldsymbol{C}_{\widetilde{G}}(\tilde{s}) \cong \mathrm{GU}_{n-2}(q) \times \mathrm{GU}_2(q)$. It follows that

$$|G^* : \boldsymbol{C}_{G^*}(s)|_{p'} = |\widetilde{G} : \boldsymbol{C}_{\widetilde{G}}(\tilde{s})|_{p'} = \frac{(q^n+1)(q^{n-1}-1)}{(q^2-1)(q+1)}.$$

Observe that, as $\tilde{s}$ is invariant under all field automorphisms, s is $\mathrm{Aut}(G^*)$-invariant. The required character χ can be taken to be the (only) semisimple character in the Lusztig series associated to s.

Assume now that $q = 2^f$ with f even. We then take $\tilde{s}$ to be a diagonal matrix with $\mathrm{Spec}(\tilde{s}) = \{\delta^{(q^2-1)/3}, \delta^{2(q^2-1)/3}, 1^{n-2}\}$ instead, where δ, as before, is a generator of $\mathbb{F}_{q^2}^\times$. Again, by Lemma 4.5, we have $\boldsymbol{C}_{\widetilde{G}}(\tilde{s}) \cong \mathrm{GU}_{n-2}(q) \times \mathrm{GL}_1(q^2)$ and $\boldsymbol{C}_{\widetilde{G}}(\tilde{s})$ is the complete inverse image of $\boldsymbol{C}_{G^*}(s)$. (This is clear when $n > 3$. When $n = 3$, the image of the homomorphism $\tau : \pi^{-1}(\boldsymbol{C}_{G^*}(s)) \to \boldsymbol{Z}\widetilde{G}$ defined by $gsg^{-1} = \tau(g)s$ is contained in $\{\mathrm{Id}, \delta^{(q^2-1)/3}\mathrm{Id}, \delta^{2(q^2-1)/3}\mathrm{Id}\}$. However, both $\delta^{(q^2-1)/3}\mathrm{Id}$ and $\delta^{2(q^2-1)/3}\mathrm{Id}$ have order 3, which does not divide $|\boldsymbol{Z}\widetilde{G}| = q+1$. Therefore we still have $\tau(g = \mathrm{Id}$ for all $g \in \pi^{-1}(\boldsymbol{C}_{G^*}(s))$.) We then have

$$|G^* : \boldsymbol{C}_{G^*}(s)|_{p'} = \frac{(q^n+1)(q^{n-1}-1)}{(q^2-1)(q-1)}.$$

It is straightforward to check that this is greater than q^{n-1}, and we conclude as in the previous case.

We solve the remaining case $q = 2^f$ with f odd by a somewhat different argument, working with the ambient algebraic group of G and its associated Frobenius map instead. So let $\boldsymbol{G}$ be a simple algebraic group of adjoint type of type A, defined over an algebraically closed field of characteristic 2. Let F_f be the standard Frobenius map on $\boldsymbol{G}$, raising all matrix entries to the 2^f-th power and ρ the inverse transpose. Set $F := F_f \circ \rho$. Then $G = G^*$ is precisely the group of F-fixed points in $\boldsymbol{G}$. Let $F_0 := F_1 \circ \rho$. As f is odd, we have $F = F_f \circ \rho = F_0^f$. Note that F_1 induces a generator, say σ_{F_1}, for $\mathrm{Out}(G) \cong C_{2f}$; on the other hand, F_0 induces $\sigma_{F_0} \in \mathrm{Out}(G)$ of order f and $\langle \sigma_{F_0} \rangle = \langle (\sigma_{F_0})^2 \rangle = \langle (\sigma_{F_1})^2 \rangle$. By the work of Brunat [B09, Proposition 2], the number of σ_{F_0}-invariant irreducible characters of G of odd degree is equal to the number of semisimple characters of $\boldsymbol{G}^{F_0} = \mathrm{PGU}_n(2) \cong \mathrm{SU}_n(2)$. (Note that, by the assumptions, f is odd and $(n, 2^f+1) = 1$. In particular, n is coprime to 3.) This number, in turn, is 2^{n-1}, by [C85, Corollary 8.3.6]. These 2^{n-1} characters are permuted by σ_{F_1}, forming orbits of size 1 or 2. The trivial character forms its own orbit. Consequently, at least one of the nontrivial characters, say χ, is fixed by σ_{F_1} – that is, χ is $\mathrm{Out}(G)$-invariant. By Lemma 4.7, χ cannot be a Weil character. Moreover, under the assumptions on n, q, and f, we have $n \geq 5$. It then follows from [TZ96, Table V] that $\chi(1) \geq (q^n+1)(q^{n-1}-q^2)/(q+1)(q^2-1)$, and therefore $\chi(1) > q^{n-1}$. This concludes the proof. □

Lemma 4.9. *Let $G = \boldsymbol{G}^F$ be the group of fixed points of a simple algebraic group $\boldsymbol{G}$ of simply connected type over an algebraically closed field of characteristic p, under a Steinberg endomorphism F of $\boldsymbol{G}$, and assume that $G/\boldsymbol{Z}G$ is simple. Let $P \in \mathrm{Syl}_p(G)$. Then*

$$b(\boldsymbol{N}_G(P)) \leq |T/\boldsymbol{Z}G|,$$

where $T := \boldsymbol{T}^F$ and $\boldsymbol{T}$ is a maximally split F-stable maximal torus of $\boldsymbol{G}$.

Proof. Let $\boldsymbol{B}$ be an F-stable Borel subgroup of $\boldsymbol{G}$ containing $\boldsymbol{T}$. Let Φ be the root system of $\boldsymbol{G}$ with respect to $\boldsymbol{T}$ and $\boldsymbol{B}$. Let Φ^+ and $\Delta = \{\alpha_i : i \in I\}$ be the corresponding set of positive roots and simple roots, respectively. Also, let $\boldsymbol{X}_\alpha$ denote the root subgroup associated to each $\alpha \in \Phi$.

Let $\boldsymbol{U} := \prod_{\alpha\in\Phi^+} \boldsymbol{X}_\alpha$, which is the unipotent radical of $\boldsymbol{B}$. According to [MT11, Corollary 24.11], we have $P := \boldsymbol{U}^F \in \mathrm{Syl}_p(G)$ and $\boldsymbol{B}^F = \boldsymbol{N}_G(P)$. Furthermore,

$$\boldsymbol{N}_G(P) = P \rtimes T,$$

where $T := \boldsymbol{T}^F$. At this point we have $b(\boldsymbol{N}_G(P)) \leq |\boldsymbol{N}_G(P)/P| = |T|$ but, in many cases, this upper bound for $b(\boldsymbol{N}_G(P))$ is not sufficient. We need to know more details about the action of T on $\mathrm{Irr}(P/P')$.

Let $\boldsymbol{U}_c := \prod_{\alpha\in\Phi^+\setminus\Delta} \boldsymbol{X}_\alpha$. As explained in [C85, §2.9], $\boldsymbol{U}_c$ is normal in $\boldsymbol{U}$ and $\boldsymbol{U}/\boldsymbol{U}_c = \prod_{i\in I} \boldsymbol{X}_{\alpha_i}$. The endomorphism F naturally acts on the roots, and thus on the root subgroups, given by $F(\boldsymbol{X}_\alpha) = \boldsymbol{X}_{F(\alpha)}$. As $\boldsymbol{B}$ and $\boldsymbol{T}$ are F-stable, F permutes the positive roots, as well as the simple roots. Both $\boldsymbol{U}$ and $\boldsymbol{U}_c$ are therefore F-stable. Let ρ denote the permutation on I induced from the action of F on the simple roots, and $\mathcal{O}$ be the set of ρ-orbits on I. For each such an orbit J, let $\boldsymbol{X}_J := \prod_{i\in J} \boldsymbol{X}_{\alpha_i}$, which is an F-stable group. Further, let $X_J := \boldsymbol{X}_J^F$. We then have

$$\boldsymbol{U}^F/{\boldsymbol{U}_c}^F = \prod_{J\in\mathcal{O}} X_J. \tag{4-1}$$

Note that each $\boldsymbol{X}_\alpha$ is normalized by $\boldsymbol{T}$ (see [C85, p. 18]). Thus $\boldsymbol{U}^F$ and ${\boldsymbol{U}_c}^F$ are both normalized by $T = \boldsymbol{T}^F$, and so T acts on the factor group $\boldsymbol{U}^F/{\boldsymbol{U}_c}^F$. In fact, each direct factor X_J with $J \in \mathcal{O}$ of $\boldsymbol{U}^F/{\boldsymbol{U}_c}^F$ is normalized by T. Remark that $\boldsymbol{Z}G \leq T$ (see the proof of [MT11, Lemma 24.12]) and $\boldsymbol{Z}G$ acts trivially on $\boldsymbol{U}^F$. Therefore the maximal size of a T-orbit on $\boldsymbol{U}^F/{\boldsymbol{U}_c}^F$ is at most $|T/\boldsymbol{Z}G|$.

Assume from now on that $G \notin \{\mathrm{Sp}_{2n}(2), F_4(2), G_2(3)\}$. (For these exceptions, a Sylow 2-subgroup of G is self-normalizing and the desired inequality is immediate.)

Then, according to [B09, Lemma 5], ${\boldsymbol{U}_c}^F$ is the derived subgroup P' of $P = \boldsymbol{U}^F$. As the actions of T on P/P' and $\operatorname{Irr}(P/P')$ are isomorphic, $|T/\boldsymbol{Z}G|$ is also an upper bound for the sizes of the T-orbits on $\operatorname{Irr}(P/P')$. Moreover, the restriction of every irreducible character of $P/P' \rtimes T$ to P/P' is multiplicity-free, as T is abelian and $|T|$ is coprime to p. The result now follows. □

Lemma 4.10. *Assume the hypotheses of Lemma 4.9. Then $S = G/\boldsymbol{Z}G$ satisfies Hypothesis 4.3, unless $G = \mathrm{SU}_n(q)$ with $n \geq 3$ odd and $(n, q+1) = 1$.*

Proof. Clearly, $m(G) \leq m(S)$ and $b(\boldsymbol{N}_G(Q)) \geq b(\boldsymbol{N}_S(P))$ for $P \in \mathrm{Syl}_p(S)$ and $Q \in \mathrm{Syl}_p(G)$. Therefore, if Hypothesis 4.3 holds for G, then it also holds for S. Sylow p-subgroups and their normalizers of a finite reductive group in characteristic p are best described through the framework of Borel subgroups and their unipotent radicals in the ambient algebraic group. For convenience, we shall use the same P for a Sylow p-subgroup of G (in fact, $P \cong Q$, see [MT11, p. 214]).

Note that the group X_J in the proof of Lemma 4.9 is isomorphic to $\mathbb{F}_{q^{|J|}}$ and $|X_J| = q^{|J|}$, where q is the absolute value of all eigenvalues of F on the character group of an F-stable maximal torus of $\boldsymbol{G}$. Furthermore, as noted in [C85, p. 74], we have

$$|T| = \prod_{J \in \mathcal{O}} (q^{|J|} - 1).$$

Lemma 4.9 therefore implies that

$$b(\boldsymbol{N}_G(P)) \leq \frac{\prod_{J \in \mathcal{O}} (q^{|J|} - 1)}{|\boldsymbol{Z}G|}.$$

The ρ-action on the set of the simple roots, defined in the proof of Lemma 4.9, for all the relevant $(\boldsymbol{G}, F)$ is described in [C85, p. 37], allowing one to easily determine the sizes of ρ-orbits. Generally, $|J| \in \{1, 2, 3\}$ for all $(\boldsymbol{G}, F)$ and when G is untwisted, $|J| = 1$ for all J. On the other hand, the values of the smallest nontrivial p'-degree $m(G)$ can be read off from [TZ96] and [N10] when G is of classical type and those for exceptional types can be found in [Lü01].

Consider $G = \mathrm{SL}_2(q)$ with q odd. Then $|T| = q-1$ and so $b(\boldsymbol{N}_G(P)) \leq (q-1)/2$. On the other hand, $m(G) = (q-1)/2$ and so Hypothesis 4.3 is satisfied. (In this case, in fact, $b(\boldsymbol{N}_G(P)) = m(G) = (q-1)/2$. Each element t of $T \cong \mathbb{F}_q^\times$ acts on $P \cong \mathbb{F}_q^+$ by mapping u to ut^2, and so the stabilizer in T of any nontrivial (linear) character of P is precisely the order-2 subgroup of T. Therefore $\boldsymbol{N}_G(P)$ has $q-1$ linear characters and four characters of degree $(q-1)/2$.) Similarly, for $G = \mathrm{SL}_2(q)$ with q even, one has $m(G) = q - 1 = |T|$ and Hypothesis 4.3 is still valid.

Let $G = \mathrm{SL}_n(q)$ with $n \geq 3$. Here $|T| = (q-1)^{n-1}$ and $m(G) = (q^n - 1)/(q-1)$, unless $(n, q) = (4, 3)$. Note that $m(\mathrm{SL}_4(3)) = 26$. In any case, we have $|T| \leq m(G)$, and Hypothesis 4.3 is verified.

Let $G = \mathrm{SU}_n(q)$ with $n \geq 3$. Then

$$|T| = \begin{cases} (q^2-1)^{(n-1)/2} & \text{if } n \text{ is odd,} \\ (q^2-1)^{(n-2)/2}(q-1) & \text{if } n \text{ is even.} \end{cases}$$

and, as mentioned,

$$m(G) = (q^n - (-1)^n)/(q+1).$$

We now can verify that Hypothesis 4.3 holds when n is even or n is odd and $|\boldsymbol{Z}G| = (n, q+1) > 1$.

For $G = \mathrm{Sp}_{2n}(q)$ or $\mathrm{Spin}_{2n+1}(q)$ with $n \geq 2$, we have $|T| = (q-1)^n$ and $b(\boldsymbol{N}_G(P)) \leq (q-1)^n/(2, q-1)$. One can verify from Table 1 that, $d(G)$, which serves as a lower bound for $m(G)$, is always at least $(q-1)^n/(2, q-1)$. Similarly, for $G = \mathrm{Spin}^{\epsilon}_{2n}(q)$ with $\epsilon \in \{\pm\}$ and $n \geq 4$, we have $b(\boldsymbol{N}_G(P)) \leq (q-1)^{n-1}(q-\epsilon 1)/(4, q^n - \epsilon 1)$, while $m(G) \geq (q^n-1)(q^{n-1}-1)/2(q+1)$, by [N10, Theorems 1.3 and 1.4]. Hypothesis 4.3 is again satisfied.

Let $G = {}^2B_2(q^2)$ or ${}^2G_2(q^2)$, where $q^2 = 2^{2n+1}$ or $q^2 = 3^{2n+1}$, respectively. In these cases, $|T| = q^2 - 1$, which is less than $m(G)$. (For type ${}^2B_2$, $m(G)$ is at least $\sqrt{1/2}\,q(q^2-1)$, while for type ${}^2G_2$, $m(G) = q^4 - q^2 + 1$.) Similarly, when $G = {}^2F_4(q^2)$, we have $|T| = (q^2-1)^2$ and $m(G) \geq \sqrt{1/2}\,q^9(q^2-1)$. In all these cases, the required inequality holds.

For the remaining exceptional-type groups, we always have $|T| \leq q^{rk(G)}$, where $rk(G)$ is the semisimple rank of $\boldsymbol{G}$. However, the lower bound for the smallest nontrivial irreducible representation of G, as shown in Table 1, confirms that $m(G) > q^{rk(G)}$. □

Proposition 4.11. *Conjecture 4.1 is true when S is a quotient of a non-exceptional covering group of a simple group of Lie type in characteristic p.*

Proof. We assume that S is not the Tits group ${}^2F_4(2)'$, since Hypothesis 4.3 can be verified directly for this group using GAP 4.11.0 (http://www.gap-system.org). By Lemma 4.10 and Proposition 4.4, it remains to consider only the case where $G = \mathrm{SU}_n(q)$ with $n \geq 3$ odd and $(n, q+1) = 1$.

In this case, G can be viewed as a group of adjoint type and hence, by [C85, §2.9],

$$T \cong \prod_{J \in \mathcal{O}} T_J,$$

the direct product of cyclic groups $T_J \cong C_{q^2-1}$. (Here, every orbit J, defined in the proof of Lemma 4.9, has length 2.) Further, the action of T on $P/P' \cong \prod_{J \in \mathcal{O}} X_J$ (see (4-1)) is a "product" action, in the way that T_J acts trivially on $X_{J'}$ if $J \neq J'$ and transitively on $X_J \setminus \{1\}$. Therefore T has a (unique) regular orbit on P/P', as well as on $\mathrm{Irr}(P/P')$, implying that $b(\boldsymbol{N}_G(P)) = |T|$ and $\boldsymbol{N}_G(P)$ has a unique p'-degree irreducible character of degree $|T| = (q^2-1)^{(n-1)/2}$. We shall denote this character by τ. Every other p'-degree irreducible character of $\boldsymbol{N}_G(P)$ restricts trivially to at least one of X_J's, and hence has degree at most $|T|/(q^2-1)$.

Note that, as $(n, q+1) = 1$, G is simple and the group, say A, of outer automorphisms of G is cyclic and stabilizes (the unipotent subgroup) P. Moreover, a bijection between $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\boldsymbol{N}_G(P))$ is McKay-good if and only if it is A-equivariant. The existence of such a bijection was established in [B09; S12]. In other words, we know that, for every subgroup $B \le A$, the numbers of B-fixed characters in $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(\boldsymbol{N}_G(P))$ are the same. Note that the above-mentioned character τ of $\boldsymbol{N}_G(P)$ is A-invariant (due to its uniqueness property). Let ξ be the character of G produced by Lemma 4.8; in particular, ξ is p'-degree and A-invariant. Now the numbers of B-fixed characters in $\mathrm{Irr}_{p'}(G)\setminus\{\xi\}$ and $\mathrm{Irr}_{p'}(\boldsymbol{N}_G(P))\setminus\{\tau\}$ are the same for every $B \le A$. Using [I06, Lemma 13.23], one can construct an A-equivariant bijection from $\mathrm{Irr}_{p'}(G)\setminus\{\xi\}$ to $\mathrm{Irr}_{p'}(\boldsymbol{N}_G(P))\setminus\{\tau\}$, which can be extended to an A-equivariant bijection $\Psi : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\boldsymbol{N}_G(P))$ such that $\Psi(\xi) = \tau$.

We claim that $\chi(1) \ge \Psi(\chi)(1)$ for all $\chi \in \mathrm{Irr}_{p'}(G)$. First, observe that

$$\xi(1) > q^{n-1} > (q^2-1)^{(n-1)/2} = \tau(1).$$

On the other hand, when $\chi \ne \xi$, we have $\Psi(\chi) \ne \tau$, and so $\Psi(\chi)(1) \le |T|/(q^2-1) = (q^2-1)^{(n-3)/2}$, which implies that

$$\chi(1) \ge \frac{q^n-(-1)^n}{q+1} \ge \Psi(\chi)(1),$$

and this finishes the proof. □

4.2. ***Groups of Lie type in characteristic not equal to p.***

Lemma 4.12. *Let $H \le G$ be such that $|G : H|$ is not divisible by p. Then $b(H) \le b(G)$.*

Proof. Let $\varphi \in \mathrm{Irr}_{p'}(H)$ such that $\varphi(1) = b(H)$. By Frobenius reciprocity, φ is contained in χ_H for any irreducible constituent χ of φ^G. Since $\varphi^G(1) = \varphi(1)|G:H|$ is not divisible by p, at least one of those constituents has p'-degree. □

Proposition 4.13. *Conjecture 4.1 is true when S is a non-exceptional covering group of a simple group of exceptional Lie type in characteristic different from p.*

Proof. Assume for now that S is not of Suzuki or Ree type. As before, S is a quotient of $G := \boldsymbol{G}^F$, where $\boldsymbol{G}$ is a simple algebraic group of simply connected type and F is a Steinberg endomorphism on $\boldsymbol{G}$. Let q be the absolute value of all eigenvalues of F on the character group of an F-stable maximal torus of $\boldsymbol{G}$. We know that p does not divide q, and we may assume that p divides $|G|$.

We require some d-Harish-Chandra theory, particularly the concept of Sylow d-tori, which was first introduced by Broué and Malle in [BM92]. (For a detailed

G	conditions	$d(G)$
$\mathrm{SL}_2(q)$	$q \geq 5$	$(q-1)/(2, q-1)$
$\mathrm{SL}_n(q)$	$n \geq 3$ $(n, q) \notin \{(3, 2), (3, 4), (4, 2), (4, 3)\}$	$(q^n - 1)/(q-1)$
$\mathrm{SU}_n(q)$	$n \geq 3$ odd; $(n, q) \neq (3, 2)$ $n \geq 4$ even; $(n, q) \notin \{(4, 2), (4, 3)\}$	$(q^n - q)/(q+1)$ $(q^n - 1)/(q+1)$
$\mathrm{Sp}_{2n}(q)$	$n \geq 2$; q odd $n \geq 2$; q even; $(n, q) \neq (2, 2)$	$(q^n - 1)/2$ $(q^n - 1)(q^n - q)/2(q+1)$
$\mathrm{Spin}_{2n+1}(q)$	$n \geq 3$; $q > 3$ odd $n \geq 3$; $q = 3$	$(q^{2n} - 1)/(q^2 - 1)$ $(q^n - 1)(q^n - q)/(q^2 - 1)$
$\mathrm{Spin}^+_{2n}(q)$	$n \geq 4$; $q > 3$ $n \geq 4$; $q \in \{2, 3\}$; $(n, q) \neq (4, 2)$	$(q^n - 1)(q^{n-1} + q)/(q^2 - 1)$ $(q^n - 1)(q^{n-1} - 1)/(q^2 - 1)$
$\mathrm{Spin}^-_{2n}(q)$	$n \geq 4$	$(q^n + 1)(q^{n-1} - q)/(q^2 - 1)$
${}^2B_2(q^2)$	$q^2 = 2^{2f+1} \geq 8$	$\sqrt{1/2}q(q^2 - 1)$
${}^2G_2(q^2)$	$q^2 = 3^{2f+1} \geq 27$	$q^4 - q^2 + 1$
${}^2F_4(q^2)$	$q^2 = 2^{2f+1} \geq 8$	$\sqrt{1/2}q^9(q^2 - 1)$
$G_2(q)$	$q \geq 3$	$\geq q^3 - 1$
${}^3D_4(q)$		$\geq q^3(q^2 - 1)$
$F_4(q)$		$\geq q^8 + q^4 + 1$
$E_6(q)_{sc}$		$\geq q^9(q^2 - 1)$
${}^2E_6(q)_{sc}$		$\geq q^9(q^2 - 1)$
$E_7(q)_{sc}$		$\geq q^{15}(q^2 - 1)$
$E_8(q)$		$\geq q^{27}(q^2 - 1)$

Table 1. Values or bounds for the minimal nontrivial degree of ordinary characters of finite reductive groups of simply connected type [TZ96; Lü01].

account, see also [GM20, §3.5].) Define e as the multiplicative order of q modulo p if $p > 2$ or if $p = 2$ and $q \equiv 1 (\bmod 4)$. For $p = 2$ and $q \equiv -1 (\bmod 4)$, let $e := 2$. As $p \mid |G|$, we know that $\Phi_e(q) \mid |G|$, where Φ_e is the e-th cyclotomic polynomial. Let $\boldsymbol{S}_e$ be a Sylow e-torus of $\boldsymbol{G}$. It is, by definition, an F-stable torus of $\boldsymbol{G}$ whose order polynomial is the maximal power of Φ_e dividing the generic order of $\boldsymbol{G}$ (see [GM20, p. 259]).

Let $\boldsymbol{L}_e := \boldsymbol{C}_{\boldsymbol{G}}(\boldsymbol{S}_e)$, known as a (minimal) e-split Levi subgroup of $\boldsymbol{G}$. It is F-stable (see [GM20, p. 258]) and we write $L_e := \boldsymbol{L}_e^F$. Note that, by the conjugation property of Sylow d-tori, we have $\boldsymbol{N}_G(\boldsymbol{S}_e) = \boldsymbol{N}_G(\boldsymbol{L}_e)$. The quotient $W_{\boldsymbol{G}}(\boldsymbol{L}_e) := \boldsymbol{N}_G(\boldsymbol{L}_e)/\boldsymbol{L}_e^F$ is referred to as the *relative Weyl group* of $\boldsymbol{L}_e$ in $\boldsymbol{G}$.

According to [M07, Proposition 5.21], $\boldsymbol{N}_G(\boldsymbol{S}_e)$ contains a Sylow p-subgroup, say P, of G. In fact, by [M07, Theorem 7.8], $\boldsymbol{N}_G(\boldsymbol{S}_e)$ contains $\boldsymbol{N}_G(P)$, unless

$p = 3$ and $G = G_2(q)$ with $q \equiv 2, 4, 5$, or $7(\bmod 9)$. For now, let us exclude this exception. It follows from Lemma 4.12 that

$$b(\boldsymbol{N}_G(P)) \leq b(\boldsymbol{N}_G(\boldsymbol{L}_e)).$$

Suppose first that e is a regular number for $(\boldsymbol{G}, F)$, which means that $\boldsymbol{L}_e$ is a maximal torus of $\boldsymbol{G}$ (see [GM20, p. 259]). Then $L_e := \boldsymbol{L}_e^F$ is abelian, and thus $b(\boldsymbol{N}_G(\boldsymbol{L}_e)) \leq |\boldsymbol{N}_G(\boldsymbol{L}_e)/L_e|$, by [I06, Corollary 11.29]. In summary, if e is a regular number for $(\boldsymbol{G}, F)$ and G is not of type G_2, then

$$b(\boldsymbol{N}_G(P)) \leq |W_{\boldsymbol{G}}(\boldsymbol{L}_e)|.$$

Consider the case when the power of Φ_e in the order polynomial of $\boldsymbol{G}$ is precisely 1, or equivalently, where $W_{\boldsymbol{G}}(\boldsymbol{L}_e)$ is cyclic (see [GM20, Proposition 3.5.12]). As computed in [BM93, Table 8.1], with one exception at type E_8 and $e = 30$, we have $|W_{\boldsymbol{G}}(\boldsymbol{L}_e)| \leq 24$ for all relevant G and e. One can easily check that the minimal character degree $d(G)$ of G, displayed in Table 1, is always at least 26, and so we are done. For the exception, we have $|W_{\boldsymbol{G}}(\boldsymbol{L}_e)| = 30$, while $d(G) \geq q^{27}(q^2 - 1)$ and so Hypothesis 4.3 is satisfied. On the other hand, non-cyclic relative Weyl groups of minimal e-split Levi subgroups for exceptional types are available in [GM20, Table 3.2]. We have verified that $|W_{\boldsymbol{G}}(\boldsymbol{L}_e)|$ remains at most $d(G)$, except for the specific cases discussed in the next paragraph.

Consider $G = E_8(q)$, for instance. Then $|W_{\boldsymbol{G}}(\boldsymbol{L}_e)| \leq d(G)$ unless $e = 2$, $q = 2$, and $p = 3$. For the exception, $W_{\boldsymbol{G}}(\boldsymbol{L}_e)$ is isomorphic to the Weyl group $W(E_8) = C_2.GO_8^+(2)$ of E_8, which has order 696729600, while the smallest nontrivial degree $d(E_8(2))$ of $E_8(2)$ is 545925250. However, since $b_3(\boldsymbol{N}_G(\boldsymbol{L}_e))$ divides $|W_{\boldsymbol{G}}(\boldsymbol{L}_e)|$, by [I06, Corollary 11.29], we have $b_3(\boldsymbol{N}_G(\boldsymbol{L}_e)) \leq 696729600_{3'} = 2867200 < d(G)$, and the result follows as before. Other exceptions occur for $(G, q, p, e) = (E_7(q)_{sc}, 2, 3, 2)$ or $({}^2E_6(2)_{sc}, 2, 3, 2)$, but the arguments are entirely similar.

Next we consider the case where e is not a regular number for $(\boldsymbol{G}, F)$. This includes $e = 5$ for type E_6; $e = 10$ for ${}^2E_6$; $e \in \{4, 5, 8, 10, 12\}$ for E_7; and $e \in \{7, 9, 14, 18\}$ for E_8. Note that L_e is no longer abelian. But, as $S_e := \boldsymbol{S}_e^F$ is abelian, we shall use the bound

$$b(\boldsymbol{N}_G(P)) \leq b(\boldsymbol{N}_G(\boldsymbol{L}_e)) \leq |\boldsymbol{N}_G(\boldsymbol{L}_e) : S_e| = |W_{\boldsymbol{G}}(\boldsymbol{L}_e)||L_e : S_e|$$

instead.

Let $G = E_7(q)_{sc}$ and $e = 4$. Then $|\boldsymbol{N}_G(\boldsymbol{L}_e)/L_e| = 96$ and L_e has type $\Phi_4^2 A_1(q)^3$ (see [GM20, Table 3.3]). Thus $|\boldsymbol{N}_G(\boldsymbol{L}_e) : S_e| \leq 96q^3(q^2 - 1)^3$, which is smaller than $d(E_7(q)_{sc})$, as desired. In the remaining cases, $W_{\boldsymbol{G}}(\boldsymbol{L}_e)$ is cyclic and its order together with the structure of L_e are again given in [BM93, Table 8.1]. Note that, in this case, the Sylow e-torus S_e has order $\Phi_e(q)$. It is now straightforward to check that $|W_{\boldsymbol{G}}(\boldsymbol{L}_e)||L_e : S_e| \leq d(G)$ for all relevant G and e.

For Suzuki and Ree groups, Broué and Malle introduced an adapted version of Φ_e, denoted by $\Phi_e^{(p)}$, which are cyclotomic polynomials over $\mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\sqrt{3})$ (see [M07, §8]). For the Tits group, we can verify Hypothesis 4.3 directly using GAP; therefore, we assume that $S \neq {}^2F_4(2)'$. With this, there exists a Sylow $\Phi_e^{(p)}$-torus $\boldsymbol{S}$ of $\boldsymbol{G}$ such that $N_G(\boldsymbol{S}) \geq N_G(P)$ for some Sylow p-subgroup P of G, unless

(i) $p = 2$ and $G = {}^2G_2(q)$, or

(ii) $p = 3$ and $G = {}^2F_4(q)$ with $q \equiv 2, 5 (\mod 9)$.

(See [M07, Theorem 8.4].) The case when such a Sylow $\Phi_e^{(p)}$-torus exists can be argued similarly as above. (Alternatively, one can use [IMN07, §16 and §17] and [A98] to achieve the result for these groups.) The case $p = 2$ and $G = {}^2G_2(q)$ follows from the proof of the McKay inductive conditions for the group (see [IMN07, §17]). When $p = 3$ and ${}^2F_4(q)$, according to [A98, (2B)], we have $|N_G(P)/P| \leq 48$, and the same reasoning applies.

Finally, the case $p = 3$ and $G_2(q)$ (that we previously excluded) follows from An's proof of the Alperin-McKay conjecture for $G_2(q)$. (See [A94, p. 190] where it was shown that $|N_G(P)/P|$ is bounded above by 16, which is smaller than $d(G_2(q))$.) □

4.3. *Exceptional covering groups.* Here we deal with *exceptional covering groups* of finite simple groups. These include 3-fold and 6-fold covers of A_6 and A_7, covers of sporadic simple groups (by convention), and certain covers of simple groups of Lie type with a non-generic Schur multiplier (see [MT11, Table 24.3]). Here, a (perfect central) cover S of a simple group of Lie type X is called exceptional if it is not a quotient of the finite reductive group of simply connected type covering X; in particular, S is a proper cover of X.

Proposition 4.14. *Conjecture 4.1 is true when S is an exceptional covering group.*

Proof. The existence of a McKay-good bijection for exceptional covering groups was established by Malle [M08]. We note that, except the single case $S = 2 \cdot \mathrm{PSL}_3(4)$, the group of outer automorphisms stabilizing $P \in \mathrm{Syl}_p(S)$ is either trivial or cyclic of prime order, and so any bijection that respects the action of those outer automorphisms is McKay-good. Building on Malle's work, we show that in most cases, Hypothesis 4.3 holds, ensuring that any such bijection automatically satisfies the additional degree condition. In the remaining cases, it turns out that S has precisely one character of degree smaller than $b(\boldsymbol{N}_S(P))$. In these instances, it suffices to identify a corresponding (p'-degree) character of $\boldsymbol{N}_S(P)$ with smaller degree that is invariant under the action of the outer automorphisms.

Throughout we let $X := S/Z$ and $Q := PZ/Z \in \mathrm{Syl}_p(X)$, where $Z := \boldsymbol{Z}S$. Of course we may assume that $p \mid |S|$.

First suppose that $|S|_p = p$; in particular, p is odd. In this case, work of Dade [D66; D96] provides a natural bijection between the irreducible characters in any block B of S and those in its Brauer correspondent b, and this bijection satisfies the required degree condition. We may assume that B has full defect. When $|S|_p = p$, Dade's bijection is described in Lemmas 4.7–4.10 of [D96] and is known, in particular, to preserve decomposition numbers. Hence it suffices to verify the desired degree inequality for the corresponding bijection between the Brauer characters of B and b given in [D96, Lemma 4.7].

This bijection between Brauer characters is defined as follows. The Green correspondence sends the isomorphism classes of finite-dimensional non-projective indecomposable kS-modules M belonging to B bijectively onto the isomorphism classes of finite-dimensional non-projective indecomposable $k\mathbf{N}_S(P)$-modules $\widetilde{M}$, where k is a suitable residue field of characteristic p (see [F82, III.5]). Moreover, this correspondence satisfies the degree inequality $\dim(M) \geq \dim(\widetilde{M})$. If M is simple, the socle $S(\widetilde{M})$ of $\widetilde{M}$ lies in a uniquely determined isomorphism class of simple $k\mathbf{N}_S(P)$-modules. Dade's bijection then sends the Brauer character afforded by M to that afforded by $S(\widetilde{M})$, and therefore also satisfies the degree inequality. (We also note that in a recent preprint [Li25], Linckelmann proved that, when a defect group of B is cyclic in general, there exists a perfect isometry between $\mathbb{Z}\,\mathrm{Irr}(B)$ and $\mathbb{Z}\,\mathrm{Irr}(b)$ with the degree condition.) Furthermore, Koshitani and Späth [KS16] proved that Dade's bijection fulfills the inductive Alperin–McKay condition, and hence is McKay-good. We assume from now on that $p^2 \mid |S|$.

I) Consider the sporadic groups. The structure of the quotient group $\mathbf{N}_X(Q)/Q'$ is given in [W98]. When $X \in \{Fi_{24}', B, M\}$, we have checked that $m(S) \geq |Z||\mathbf{N}_X(Q)/Q| \geq |\mathbf{N}_S(P)/P|$, thereby confirming that Hypothesis 4.3 holds. For the remaining sporadic groups, computations using GAP reveal that Hypothesis 4.3 fails in the following cases. We include here the relevant values of $m(S)$ and $b(\mathbf{N}_S(P))$.

$$(S, p, b(\mathbf{N}_S(P)), m(S)) \in \{(Co_2, 5, 24, 23), (Co_3, 3, 32, 23), (Co_3, 5, 24, 23), (McL, 5, 24, 22), (3 \cdot McL, 5, 24, 22)\}.$$

When $S \in \{Co_2, Co_3\}$, as the outer automorphism group of S is trivial, any bijection (from $\mathrm{Irr}_{p'}(S)$ to $\mathrm{Irr}_{p'}(\mathbf{N}_S(P))$) is McKay-good. In these cases, S has a unique irreducible p'-degree character, say ψ, of degree smaller than $b(\mathbf{N}_S(P))$. Furthermore, we observe that $\mathbf{N}_S(P)$ is not perfect. (Indeed, $\mathbf{N}_S(P)/P' \cong 5^2 \rtimes (4 \cdot \mathsf{S}_4)$ when $S = Co_2$, and $\mathbf{N}_S(P)/P' \cong \mathsf{S}_3 \times (3^2 : SD_{16})$ when $S = Co_3$.) Consequently, $\mathbf{N}_S(P)$ has a nontrivial linear character, say τ. It then follows that any bijection sending ψ to τ satisfies the required degree condition.

Let $(S, p) \in \{(McL, 5), (3 \cdot McL, 5)\}$. In both cases, the outer automorphism group $\mathrm{Out}(S) \cong C_2$ normalizes P. Moreover, S has exactly one irreducible character of degree less than $b(N_S(P)) = 24$, and this character is necessarily $\mathrm{Out}(S)$-invariant. Suppose first that $S = McL$. By [Atl], $N_S(P) = P \rtimes D$, where P is an extra-special group of order 5^3 and exponent 5, and $D = C_3 \rtimes C_8$. The group D acts Frobeniusly on $\mathrm{Irr}(P/P') \cong C_5 \times C_5$, and hence $N_S(P)$ has a unique irreducible character of degree 24. This is the only $5'$-degree character of $N_S(P)$ of degree greater than $m(S) = 22$. All other $5'$-degree irreducible characters of $N_S(P)$ are trivial on P, and thus can be viewed as characters of D, with degrees 1 and 2. Now, the required bijection can be constructed in the way that this degree-24 character corresponds to (any) $\mathrm{Out}(S)$-invariant irreducible character of S of degree greater than 22. (The group McL indeed has several such characters.) When $S = 3 \cdot McL$, again $\mathrm{Irr}_{5'}(N_S(P))$ contains a single member of degree 24. As in the case $S = McL$, this is the only $5'$-degree character of $N_S(P)$ of degree greater than $m(S) = 22$, and the same reasoning applies.

II) The simple groups of Lie type with a non-generic Schur multiplier are:

$$\mathrm{PSL}_2(4), \mathrm{PSL}_2(9), \mathrm{PSL}_3(2), \mathrm{PSL}_3(4), \mathrm{PSL}_4(2), \mathrm{PSU}_4(2), \mathrm{PSU}_4(3),$$
$$\mathrm{PSU}_6(2), {}^2B_2(8), \mathrm{PO}_7(3), \mathrm{PSp}_6(2), \mathrm{PO}_8^+(2), G_2(3), G_2(4), F_4(2), \text{ and } {}^2E_6(2).$$

(See [MT11, Table 24.3].)

Consider the (only) exceptional cover $S = 2 \cdot F_4(2)$ of $X = F_4(2)$. When $p = 2$, as all the faithful irreducible characters of S have even degree ([Atl]), the problem is reduced to the simple group $F_4(2)$, which was already solved in Proposition 4.11. When $p = 3$, the Sylow normalizer $N_X(Q)$ of X is contained in $\mathrm{PSL}_4(3).2_2$ and lifts to the direct product $C_2 \times N_X(Q)$ in S, as described in [M08, §4]. From the proof of Proposition 4.11, the maximal $3'$-degree of the Sylow 3-normalizer of $\mathrm{SL}_4(3)$ is at most $(3-1)^3/2 = 4$, implying $b(N_S(P)) = b(N_X(Q)) \leq 8$. When $p = 5$, $N_X(Q)$ is an extension of C_5^2 with the complex reflection group G_8, which has the structure $C_4.\mathsf{S}_4$. In this case, $N_S(P) = Z \times N_X(Q)$, and elementary character theory yields $b(N_S(P)) \leq 24$. Similarly, for $p = 7$, $N_X(Q)$ is an extension of C_7^2 with the complex reflection group G_5 of structure $C_6.\mathsf{A}_4$. Here, it can be shown that $b(N_S(P)) \leq 48$. As $m(S) = 52$ ([Atl]), Hypothesis 4.3 holds for all the relevant p.

Consider $X = {}^2E_6(2)$. This group has two exceptional covers $2 \cdot X$ and $6 \cdot X$ with cyclic center. When $p = 2$, as in the case $S = 2 \cdot F_4(2)$, the problem reduces to handling the covers X and $3 \cdot X$, which are actually non-exceptional covers of X, and we are done by Proposition 4.11. Consider $p = 3$. Then $N_X(Q)$ is contained in $\Omega_7(3)$ and is lifted to the direct product $C_2 \times N_X(Q)$ in $2 \cdot X$. The proof of Proposition 4.11 shows that the maximal $3'$-degree of the Sylow 3-normalizer of $\Omega_7(3)$ is at most 4, and therefore $b(N_S(P)) = b(N_X(Q)) \leq 4$ for $S = 2 \cdot X$. For the six-fold cover $S = 6 \cdot X$, as the $3'$-degree irreducible characters of both S and $\mathbf{N}_S(P)$ are trivial on $\mathbf{Z}S_3$, we still have $b(\mathbf{N}_S(P)) = b(\mathbf{N}_{S/\mathbf{Z}S_3}(P/\mathbf{Z}S_3)) \leq 4$. When $p = 5$, [M08, Table 2] shows that the largest $5'$-degree of the Sylow normalizer of both $2 \cdot X$ and $6 \cdot X$ (and of X itself as well) is 48. For $p = 7$, as mentioned in [M08, §4], the Sylow 7-normalizer of X is contained in $F_4(2)$ and moreover, $F_4(2)$ is lifted to $2 \cdot F_4(2)$ in $2 \cdot X$ and to $3 \times 2 \cdot F_4(2)$ in $6 \cdot X$. It follows from the previous paragraph that $b(\mathbf{N}_S(P)) \leq 48$, whether S is the double or 6-fold cover of X. On the other hand, the smallest nontrivial degree of $6 \cdot {}^2E_6(2)$ is 1938 (GAP). Hypothesis 4.3 again holds true in this case.

Finally, using GAP, we have checked that Hypothesis 4.3 also holds for all other exceptional covers of the simple groups of Lie type listed above, as well as for the 3-fold and 6-fold covers of A_6 and A_7. The proof is complete. □

A few remarks are in order. First, Propositions 4.11, 4.13, and 4.14 together imply that Conjecture 4.1 holds for every quasisimple group S of exceptional Lie type and for every prime p. Second, our arguments show that, in all cases considered, Hypothesis 4.3 holds for (S, p), except possibly in the following situations: $S = PSU_n(q)$ with $n \geq 3$ odd, $(n, q+1) = 1$, and q a power of p; when S is sporadic and $|S|_p = p$; or when $(S, p) \in \{(Co_2, 5), (Co_3, 3), (Co_3, 5), (McL, 5), (3 \cdot McL, 5)\}$.

5. Odd-degree characters

In this section we confirm Conjectures 4.1, 3.3, and 3.1 for $p = 2$, thereby proving Theorem B.

Theorem 5.1. *Conjecture 4.1 is true for all quasisimple groups S and $p = 2$.*

Proof. By Propositions 4.14, 4.11, and 4.13, and noting that Sylow 2-subgroups of an alternating group or its double cover are self-normalizing [O76], we only need to consider classical groups over fields of odd characteristic. Furthermore, we may assume that S is a non-exceptional covering group of the simple group $X := S/\mathbf{Z}S$. As before, we use G for the finite reductive group of simply connected type such that $G/\mathbf{Z}G = X$, and so S is a certain quotient of G. By Proposition 4.4, the existence of a required bijection is guaranteed if we are able to show that

$$b(\mathbf{N}_G(Q)) \leq m(G) \tag{5-1}$$

for $Q \in \mathrm{Syl}_2(G)$.

Let $X = \mathrm{PSL}_2(q)$ with $5 \leq q$ odd, and hence $G = \mathrm{SL}_2(q)$. Assume first that $q \equiv \pm 3 (\bmod 8)$. Then Q is the quaternion group of order 8 and $\mathbf{N}_G(Q)$ is isomorphic to $\mathrm{SL}_2(3)$ (see [IMN07, §15E]). We have $b(\mathbf{N}_G(Q)) = 3$, while $m(G) = (q-1)/2$ if $q \equiv 3 (\bmod 8)$ and $m(G) = (q+1)/2$ if $q \equiv -3 (\bmod 8)$, and so (5-1) is satisfied. When $q \equiv \pm 1 (\bmod 8)$, Q is self-normalizing and the inequality is trivial.

Let $X = \mathrm{PSL}_n^{\pm}(q)$ with $n \geq 3$. Here, as usual, we use the superscript $+$ for linear groups, while $-$ for unitary groups. Then $G = \mathrm{SL}_n^{\pm}(q)$. Let $\widetilde{G} := \mathrm{GL}_n^{\pm}(q)$, $R \in \mathrm{Syl}_2(\widetilde{G})$, and take $Q := R \cap G$. By [K05, Theorem 1], we have

$$\mathbf{N}_{\widetilde{G}}(Q) = \mathbf{N}_{\widetilde{G}}(R) = R\mathbf{C}_{\widetilde{G}}(R).$$

The structure of Sylow normalizers in $\widetilde{G}$ was determined by Carter and Fong in [CF64, Lemma 6 and Theorem 4], as follows

$$\mathbf{N}_{\widetilde{G}}(R) \cong R \times (C_{(q\mp 1)_{2'}})^t,$$

where $(q \mp 1)_{2'}$ is the odd part of $q \mp 1$ and t is the number of terms in the 2-adic expansion of n. It follows that

$$\mathbf{N}_{\widetilde{G}}(Q)/Q' = \mathbf{N}_{\widetilde{G}}(R)/Q' \cong (R/Q') \times (C_{(q\mp 1)_{2'}})^t.$$

Let $\boldsymbol{b}(M)$ denote the largest degree of an irreducible character of a finite group M. We have

$$\boldsymbol{b}(\mathbf{N}_{\widetilde{G}}(Q)/Q') = \boldsymbol{b}(R/Q') \leq |R : Q| \leq q+1.$$

Here, the inequality in the middle follows from [I06, Corollary 11.29] and the fact that Q/Q' is an abelian normal subgroup of R/Q'. The last inequality follows from $|R : Q| = |R : (R \cap G)| = |RG : G| \leq |\widetilde{G} : G| \leq q+1$. Since $\mathbf{N}_G(Q)/Q'$ is a normal subgroup of $\mathbf{N}_{\widetilde{G}}(Q)/Q'$, we deduce that

$$b(\mathbf{N}_G(Q)) = \boldsymbol{b}(\mathbf{N}_G(Q)/Q') \leq \boldsymbol{b}(\mathbf{N}_{\widetilde{G}}(Q)/Q') \leq q+1.$$

The desired inequality $b(\mathbf{N}_G(Q)) \leq m(G)$ then follows immediately from the bound provided in Table 1.

Next, consider $X = PSp_{2n}(q)$ with $n \geq 2$ and q odd. Then $G = \mathrm{Sp}_{2n}(q)$ with $\mathbf{Z}G$ being cyclic of order 2. The Sylow 2-subgroup Q is self-normalizing in G when $q \equiv \pm 1 (\bmod 8)$; otherwise, $|\mathbf{N}_G(Q)/Q| = 3^t$ where t is the number of terms in the 2-adic expansion of n (see [CF64, Theorem 4]). In the former case, Hypothesis 4.3 is trivial. In the latter, we have

$$b(\mathbf{N}_G(Q)) \leq |\mathbf{N}_G(Q)/Q| = 3^t < (q^n - 1)/2 = d(G) \leq m(G),$$

and we are done again.

Lastly, consider $X = \Omega_{2n+1}(q)$ with $n \geq 3$ or $X = P\Omega_{2n}^{\pm}(q)$ with $n \geq 4$ (q again is odd). Here $G = \mathrm{Spin}_{2n+1}(q)$ or $\mathrm{Spin}_{2n}^{\pm}(q)$, respectively. According to [CF64, Theorem 5], Sylow 2-subgroups of G are self-normalizing, and we conclude as before. □

Theorem 5.2. *Conjecture 3.1 is true when* $p = 2$.

Proof. This follows from Theorems 5.1 and 3.5. □

As said in the introduction, Theorem B immediately follows from Theorem 5.2.

We conclude with some remarks. To complete the proof of Conjecture A and Conjecture 3.1 for all primes p, it remains to verify Conjecture 3.3 for quasisimple classical groups S in characteristic different from p, as well as for covers of alternating groups. This appears to be a nontrivial problem, since for these groups the normalizer $N_S(P)$ typically has many irreducible p'-characters whose degrees exceed the minimal p'-degree of S.

Conjecture 3.1 has now been established for symmetric groups by Giannelli [G25]. It may be possible to adapt the methods of [G25] to prove Conjecture 3.3 for alternating groups.

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