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**ON THE FIRST EIGENVALUE OF THE HODGE LAPLACIAN
OF SUBMANIFOLDS**

CHRISTOS-RAENT ONTI

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ON THE FIRST EIGENVALUE OF THE HODGE LAPLACIAN OF SUBMANIFOLDS

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We prove that equality in a sharp lower bound for the first p -eigenvalue of the Hodge Laplacian on closed submanifolds in space forms can occur only on topological spheres, assuming positivity.

1. Introduction

Let M^n be a closed, connected and oriented Riemannian manifold of dimension n . For each integer $1 \leq p \leq n-1$, the Hodge–Laplace operator (or the Hodge Laplacian) acting on p -forms is defined by

$$\Delta = d\delta + \delta d : \Omega^p(M^n) \rightarrow \Omega^p(M^n),$$

where d and δ are the differential and the co-differential operators, respectively. It is well known that the spectrum of the Hodge–Laplace operator is discrete and nonnegative, and that its kernel is isomorphic to the p -th de Rham cohomology group $H^p(M^n; \mathbb{R})$. If $\lambda_{1,p}(M^n)$ denotes its lowest eigenvalue, then

$$\lambda_{1,p}(M^n) = \inf_{\omega \in \Omega^p(M^n) \setminus \{0\}} \frac{\int_M (\|d\omega\|^2 + \|\delta\omega\|^2) dM}{\int_M \|\omega\|^2 dM}.$$

Since the above is invariant by the Poincaré duality induced by the Hodge $*$ -operator, we have $\lambda_{1,p}(M^n) = \lambda_{1,n-p}(M^n)$ and thus we may assume that $p \leq n/2$. Clearly, if $\lambda_{1,p}(M^n) > 0$, then $H^p(M^n; \mathbb{R}) = H^{n-p}(M^n; \mathbb{R}) = 0$.

The Hodge Laplacian satisfies for every p -form $\omega \in \Omega^p(M^n)$ the Bochner–Weitzenböck formula

$$(1) \quad \Delta\omega = \nabla^* \nabla \omega + \mathcal{B}^{[p]}\omega,$$

where $\nabla^* \nabla$ is the connection Laplacian and $\mathcal{B}^{[p]} : \Omega^p(M^n) \rightarrow \Omega^p(M^n)$ is a certain symmetric endomorphism on the bundle of p -forms, called the *Bochner–Weitzenböck operator*. Therefore, (1) implies that lower bounds on the Bochner–Weitzenböck operator lead naturally to lower bounds on the Hodge–Laplace operator.

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In particular, from [6, Proposition 3] we get that

$$(2) \quad \text{if } \mathcal{B}^{[p]} \geq p(n-p)\Lambda \text{ for some } \Lambda > 0, \text{ then } \lambda_{1,p}(M^n) \geq p(n-p+1)\Lambda.$$

Let $f : M^n \rightarrow \tilde{M}^{n+m}$, $n \geq 3$, be an isometric immersion into a Riemannian manifold \tilde{M}^{n+m} of dimension $n+m$. The second fundamental form α_f is viewed as a section of the vector bundle $\text{Hom}(TM \times TM, N_f M)$, where $N_f M$ is the normal bundle. For each unit normal vector field $\xi \in \Gamma(N_f M)$, the associated shape operator A_ξ is given by

$$\langle A_\xi X, Y \rangle = \langle \alpha_f(X, Y), \xi \rangle, \quad X, Y \in TM.$$

Recall that the traceless part of the second fundamental form is $\Phi = \alpha_f - \langle \cdot, \cdot \rangle \mathcal{H}$, where \mathcal{H} denotes the *mean curvature vector field* given by $\mathcal{H} = (\text{tr } \alpha_f)/n$, where tr means taking the trace. Finally, by H we denote the length of the *mean curvature*, that is, $H = \|\mathcal{H}\|$.

Proposition 1 (Onti and Vlachos [5, Proposition 16]). *If the curvature operator of \tilde{M}^{n+m} is bounded from below by a constant c , then the Bochner–Weitzenböck operator of M^n , for any $1 \leq p \leq \lfloor n/2 \rfloor$, satisfies pointwise the inequality*

$$(3) \quad \min_{\substack{\omega \in \Omega^p(M^n) \\ \|\omega\|=1}} \langle \mathcal{B}^{[p]} \omega, \omega \rangle \geq \frac{p(n-p)}{n} \left(n(H^2 + c) - \frac{n(n-2p)}{\sqrt{np(n-p)}} H \|\Phi\| - \|\Phi\|^2 \right).$$

If equality holds in (3) at a point $x \in M^n$, then:

- (i) *The shape operator $A_\xi(x)$ has at most two distinct eigenvalues with multiplicities p and $n-p$ for every unit vector $\xi \in N_f M(x)$. If in addition $p < n/2$ and the eigenvalue of multiplicity $n-p$ vanishes, then $A_\xi(x) = 0$.*
- (ii) *If $H(x) \neq 0$ and $p < n/2$, then $\text{Im } \alpha_f(x) = \text{span} \{ \mathcal{H}(x) \}$.*

Therefore, if

$$\kappa_p := \min_{x \in M^n} \left\{ (H^2 + c) - \frac{n-2p}{\sqrt{np(n-p)}} H \|\Phi\| - \frac{1}{n} \|\Phi\|^2 \right\}$$

for some $1 \leq p \leq \lfloor n/2 \rfloor$, then it follows from (2) and (3) that

$$(4) \quad \lambda_{1,p}(M^n) \geq p(n-p+1)\kappa_p.$$

Inequality (4) was first proved by Savo for hypersurfaces [6, Theorem 7], and subsequently extended by Cui and Sun to submanifolds of arbitrary codimension [3, Theorem 1.1]. Those authors also showed that the inequality is sharp by providing trivial examples attaining equality. However, no characterization was given of the submanifolds for which equality holds. The aim of this note is to shed light on the case of equality in (4) assuming $\lambda_{1,p}(M^n) > 0$, when $\tilde{M}^{n+m} = \mathbb{Q}_c^{n+m}$, where \mathbb{Q}_c^{n+m} denotes the complete simply connected space form of constant sectional curvature c .

In fact, we prove that in this case equality occurs only on topological spheres. For simplicity we assume that $c \in \{0, \pm 1\}$. Thus \mathbb{Q}_c^{n+m} is the Euclidean space \mathbb{R}^{n+m} ($c = 0$), the unit sphere \mathbb{S}^{n+m} ($c = 1$), or the hyperbolic space \mathbb{H}^{n+m} ($c = -1$).

Theorem. *Let $f : M^n \rightarrow \mathbb{Q}_c^{n+m}$, $n \geq 4$, be an isometric immersion of a closed, connected and oriented Riemannian manifold. If for some $1 \leq p \leq \lfloor n/2 \rfloor$ equality holds in (4) with $\lambda_{1,p}(M^n) > 0$, then M^n is homeomorphic to the sphere \mathbb{S}^n .*

2. Proof of the Theorem

The idea of the proof is to show that M^n is a simply connected homology sphere over the integers and the proof will follow by the generalized Poincaré conjecture (Smale $n \geq 5$, Freedman $n = 4$).

Assume that for some $1 \leq p \leq \lfloor n/2 \rfloor$ equality holds in (4) with $\lambda_{1,p}(M^n) > 0$. Then Proposition 1 implies that the shape operator $A_\xi(x)$ at each point x will have at most two distinct eigenvalues of multiplicities p and $n-p$ for every unit vector $\xi \in N_f M(x)$. We claim that there exists a Morse function on M^n such that the index at each critical point is 0, p , $n-p$ or n . To this end, we distinguish two cases:

CASE $c \in \{0, 1\}$: Let $u \in \mathbb{R}^{n+m+c}$ be a vector such that the height function

$$\varphi : M^n \rightarrow \mathbb{R}, \quad \varphi(x) = \langle f_c(x), u \rangle$$

is a Morse function, where $f_c = f$ if $c = 0$, and $f_c = j \circ f$, where $j : \mathbb{S}^{n+m} \rightarrow \mathbb{R}^{n+m+1}$ denotes the standard inclusion, if $c = 1$. A direct computation gives that at a critical point x_0 of φ we have

$$u \in N_{f_c} M(x_0) \quad \text{and} \quad \text{Hess } \varphi(X, Y) = \langle \alpha_{f_c}(X, Y), u \rangle \quad \text{for all } X, Y \in T_{x_0} M.$$

Obviously, the second fundamental form of f_c has at most two distinct principal curvatures of multiplicities p and $n-p$ in every normal direction and the claim follows in this case.

CASE $c = -1$: We consider the function

$$\varphi : \mathbb{H}^{n+m} \rightarrow \mathbb{R}, \quad \varphi(x) = \frac{1}{2} r^2(x),$$

where $r(x)$ denotes the distance function issuing from some suitable choice of point $o \in \mathbb{H}^{n+m}$ to $x \in \mathbb{H}^{n+m}$. It is a standard fact that φ is smooth everywhere in \mathbb{H}^{n+m} . Let $\gamma(t)$ be a unit speed geodesic with $\gamma(0) = o$. Then, we have $\gamma'(t) = \text{grad } r(\gamma(t))$. For $X, Y \in \Gamma(T\mathbb{H})$ a direct computation gives

$$\langle X, \text{grad } \varphi \rangle = r \langle X, \text{grad } r \rangle$$

and

$$\text{Hess } \varphi(X, Y) = \langle X, \text{grad } r \rangle \langle Y, \text{grad } r \rangle + r \text{Hess } r(X, Y).$$

Consider $\tilde{\varphi} = \varphi \circ f : M^n \rightarrow (0, +\infty)$ and choose $o \in \mathbb{H}^{n+m}$ such that $\tilde{\varphi}$ is a Morse function on M^n (this is always possible). At a critical point $x_0 \in M^n$, we have $\text{grad } r(f(x_0)) \perp f_*(T_{x_0}M^n)$, that is the (unique unit speed) geodesic $\gamma(t)$ will hit $f(M^n)$ orthogonally, and

$$(5) \quad \text{Hess } \tilde{\varphi}(\tilde{X}, \tilde{Y}) = r(\text{Hess } r(f_*\tilde{X}, f_*\tilde{Y}) + \langle A_{\text{grad } r}\tilde{X}, \tilde{Y} \rangle), \quad \tilde{X}, \tilde{Y} \in T_{x_0}M^n.$$

Let $\gamma(\ell) = f(x_0)$ and consider a Jacobi field $J(t)$ along $\gamma(t)$ with $J(0) = 0$. It follows that

$$(6) \quad \text{Hess } r(J(\ell), J(\ell)) = \langle J'(\ell), J(\ell) \rangle = \frac{1}{2} \frac{d}{dt} \|J(t)\|^2|_{t=\ell}.$$

Recall that the Jacobi fields $J(t)$ in \mathbb{H}^{n+m} with $J'(0) \perp \gamma'(0)$ are given by

$$J(t) = \sinh t \cdot w(t),$$

where $w(t)$ is a parallel vector field along $\gamma(t)$ with $J'(0) = w$ and $\|w\| = 1$. Observe that $A_{\text{grad } r}(x_0)$ has at most two distinct eigenvalues, say λ and μ with multiplicities p and $n-p$, respectively. Consider an orthonormal basis $\{e_1, \dots, e_n\}$ of $T_{x_0}M^n$ such that

$$A_{\text{grad } r}(e_i) = \begin{cases} \lambda e_i & \text{for } 1 \leq i \leq p, \\ \mu e_i & \text{for } p+1 \leq i \leq n. \end{cases}$$

Let $w_i(t)$ be such that

$$w_i(\ell) = f_*(e_i), \quad 1 \leq i \leq n,$$

with corresponding Jacobi fields

$$J_i(t) = \sinh t \cdot w_i(t), \quad 1 \leq i \leq n.$$

Therefore, from (6) we obtain

$$\text{Hess } r(J_i(\ell), J_i(\ell)) = \frac{1}{2} \frac{d}{dt} \|J_i(t)\|^2|_{t=\ell} = \frac{1}{2} \sinh(2\ell), \quad \text{for all } 1 \leq i \leq n.$$

Hence (5) gives

$$\text{Hess } \tilde{\varphi}(e_i, e_i) = \begin{cases} \ell(\coth \ell + \lambda) & \text{for } 1 \leq i \leq p, \\ \ell(\coth \ell + \mu) & \text{for } p+1 \leq i \leq n, \end{cases}$$

and therefore it is now clear that $\text{Index } \tilde{\varphi}(x_0) \in \{0, p, n-p, n\}$, proving the claim.

Therefore, it follows from standard Morse theory (see [4, Theorem 3.5] or [2, Theorem 4.10]) that M^n has the homotopy type of a CW-complex with cells only in dimensions $0, p, n-p$ or n . Therefore

$$(7) \quad H_i(M^n; \mathbb{Z}) = 0 \quad \text{for all } i \neq 0, p, n-p, n.$$

Next, we claim that also

$$(8) \quad H_p(M^n; \mathbb{Z}) = H_{n-p}(M^n; \mathbb{Z}) = 0.$$

Indeed, our hypothesis implies $H^p(M^n; \mathbb{R}) = H^{n-p}(M^n; \mathbb{R}) = 0$. Hence

$$(9) \quad H_p(M^n; \mathbb{Z}) = \text{Tor}(H_p(M^n; \mathbb{Z})) \text{ and } H_{n-p}(M^n; \mathbb{Z}) = \text{Tor}(H_{n-p}(M^n; \mathbb{Z})).$$

By Poincaré duality, the universal coefficient theorem and (7), we have

$$\text{Tor}(H_p(M^n; \mathbb{Z})) \cong \text{Tor}(H_{n-p-1}(M^n; \mathbb{Z})) = 0$$

and

$$\text{Tor}(H_{n-p}(M^n; \mathbb{Z})) \cong \text{Tor}(H_{p-1}(M^n; \mathbb{Z})) = 0,$$

where \cong denotes the isomorphism of the corresponding groups. This, in combination with (9), proves (8). Hence M^n is a homology sphere over the integers.

Finally, we show that M^n is simply connected. If $p \neq 1$ this follows directly from [1, Proposition 4.5.7, p. 90], since in this case, φ has no critical points of index one. If $p = 1$, then since there are no 2-cells, it follows by the cellular approximation theorem that the inclusion of the 1-skeleton $X^{(1)} \hookrightarrow M^n$ induces isomorphism between the fundamental groups. Therefore, $\pi_1(M^n)$ is a free group on $\beta_1(M^n; \mathbb{Z}) = 0$ elements, and thus M^n is simply connected.

Therefore, M^n is a homotopy sphere and by the generalized Poincaré conjecture (Smale $n \geq 5$, Freedman $n = 4$), M^n is homeomorphic to \mathbb{S}^n , which concludes the proof of the theorem.

References

- [1] M. Audin and M. Damian, *Morse theory and Floer homology*, Springer, 2014.
- [2] J. Cheeger and D. G. Ebin, *Comparison theorems in Riemannian geometry*, North-Holland Mathematical Library **9**, North-Holland, Amsterdam, 1975.
- [3] Q. Cui and L. Sun, “Optimal lower eigenvalue estimates for Hodge-Laplacian and applications”, *J. Differential Equations* **266**:12 (2019), 8320–8343.
- [4] J. Milnor, *Morse theory*, Annals of Mathematics Studies **51**, Princeton University Press, 1963.
- [5] C.-R. Onti and T. Vlachos, “Homology vanishing theorems for pinched submanifolds”, *J. Geom. Anal.* **32**:12 (2022), Paper No. 294, 33.
- [6] A. Savo, “The Bochner formula for isometric immersions”, *Pacific J. Math.* **272**:2 (2014), 395–422.

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CHRISTOS-RAENT ONTI
DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF CYPRUS
NICOSIA
CYPRUS
onti.christos-raent@ucy.ac.cy

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EDITORS

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Vienna, Austria
matthias.aschenbrenner@univie.ac.at

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Atsushi Ichino
Department of Mathematics
Kyoto University
Kyoto 606-8502, Japan
atsushi.ichino@gmail.com

Kefeng Liu
School of Sciences
Chongqing University of Technology
Chongqing 400054, China
liu@math.ucla.edu

Sucharit Sarkar
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
sucharit@math.ucla.edu

Dimitri Shlyakhtenko
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
shlyakht@ipam.ucla.edu

Ruixiang Zhang
Department of Mathematics
University of California
Berkeley, CA 94720-3840
ruixiang@berkeley.edu

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
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