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LIOUVILLE THEOREMS AND NEW GRADIENT ESTIMATES FOR POSITIVE SOLUTIONS TO $\Delta_p u + au^q = 0$ ON A COMPLETE MANIFOLD

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We use the Saloff-Coste Sobolev inequality and the Nash–Moser iteration method to study the local and global behaviors of positive solutions to the nonlinear elliptic equation $\Delta_p u + au^q = 0$ defined on a complete Riemannian manifold (M, g) with Ricci lower bound, where $p > 1$ is a constant and $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the usual p-Laplace operator. Under certain assumptions on a, p and q, we derive some gradient estimates and Liouville type theorems for positive solutions to the above equation. In particular, under certain assumptions on a, p and q we show whether or not the exact Cheng–Yau log-gradient estimates for the positive solutions to $\Delta_p u + au^q = 0$ on (M, g) with Ricci lower bound hold true is equivalent to whether or not the positive solutions to this equation fulfill Harnack inequality, and hence some new Cheng–Yau log-gradient estimates are established.

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1. Introduction

Gradient estimates are a fundamental technique in the study of partial differential equations on a Riemannian manifold. They can be used to deduce Liouville-type theorems [1; 2; 23; 12; 7; 19], to derive Harnack inequalities [23; 12], to infer local and global behavior of solutions, to study the geometry of manifolds [4; 20; 12; 11], and so on.

On the other hand, it is well-known that Liouville's theorem has had a huge impact across many fields, such as complex analysis, partial differential equations, geometry, probability, discrete mathematics and complex and algebraic geometry. The impact of the Liouville theorem has been even larger as the starting point of many further developments. For details on the Liouville properties of harmonic functions and some related theory of function on a manifold we refer to an expository paper [5] written by T. H. Colding (see also [4]).

In this paper, we are concerned with the equation

$$\Delta_p u + au^q = 0 \tag{1-1}$$

defined on a complete Riemannian manifold (M, g) equipped with a metric g, where $p > 1$, a, q are constants and

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$$

is the usual p-Laplace operator.

For simplicity, we will focus on equation (1-1) and try to establish some new gradient estimates on the positive solutions to this equation. Now we recall some relative results in the previous literature with the equation.

In the case that M is an Euclidean space, this equation was studied by Serrin and Zou in [15] and some Liouville theorems and universal estimates were established. Very recently, J. He, together with one of us (Wang) and G. Wei, [9] adopted a new way to employ Nash–Moser iteration to study the gradient estimates of this equation on a complete Riemannian manifold.

The new estimate

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} &+ au^{q-1} \\ &\le \frac{2n}{2-n\max\{0,q-1\}}\left(\frac{C_1^2(n-1)(1+\sqrt{\kappa}R)+C_2}{R^2} + 2\kappa + \frac{2nC_1^2}{(2+n\max\{0,q-1\})R^2}\right)\end{aligned} \tag{1-2}$$

was obtained in [13] in the case $p=2$ if the Ricci curvature of the domain manifold satisfies $\mathrm{Ric}_g \ge -(n-1)\kappa$ and $q < \frac{n+2}{n}$. Obviously, this is a stronger estimate than the logarithmic gradient estimate (also see [10]). Wang and Wei [19] also derived Cheng–Yau-type gradient estimates for positive solutions to $\Delta u + u^q = 0$ under the assumption

$$q \in \Big(-\infty,\ \frac{n+1}{n-1} + \frac{2}{\sqrt{n(n-1)}}\Big).$$

Shortly afterward, the authors of [9] extended the Cheng–Yau estimate to the range

$$q \in \Big(-\infty,\ \frac{n+3}{n-1}\Big).$$

Recently, Z. Lu extended the estimate (1-2) in [13] to the range $q \in \big(-\infty,\ \frac{n+3}{n-1}\big)$.

The first goal of this paper is to give gradient estimates for positive solutions with positive lower bounds to (1-1), different from the exact log-gradient estimate.

As a second goal we try to answer two natural questions:

- *Is the value $\frac{n+3}{n-1}(p-1)$ above optimal for deriving the exact Cheng–Yau estimates for a positive C^1 solution to (1-1) on a complete manifold with Ricci curvature bounded below*?
- *Does the exact Cheng–Yau estimate hold true if u is a C^1 smooth positive solution to (1-1) that satisfies the standard Harnack inequality*?

Inspired by [8; 9; 21; 22], in the present paper we use the Nash–Moser iteration method to study the gradient estimate and the Liouville property of equation (1-1), defined on a complete Riemannian manifold.

Statement of main results. By a solution u of (1-1) in an (arbitrary) domain Ω we mean a positive solution $u \in C^1(\Omega) \cap C^3(\widetilde{\Omega})$, where $\widetilde{\Omega} = \{x \in \Omega \mid |\nabla u(x)| \neq 0\}$. Any solution of (1-1) satisfies $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,\ 1)$ (see [6; 16; 17], for example). Moreover, u is in fact smooth in $\widetilde{\Omega}$.

For brevity we define

$$h := \beta(p-1)\Big[\frac{p-n}{(n-1)^2}\beta + \frac{2}{n-1}\Big],$$

$$\phi_\beta := \begin{cases} \sup_{B(x_0,R)} u & \text{if } 0 < \beta < 2, \\ 1 & \text{if } \beta = 2, \\ \inf_{B(x_0,R)} u & \text{if } \beta > 2. \end{cases}$$

We suppose that β satisfies the condition

$$\beta \in \begin{cases} \big(0,\ \frac{2(n-1)}{n-p}\big) & \text{if } 1 < p < n, \\ (0,\ +\infty) & \text{if } p \geq n. \end{cases} \tag{1-3}$$

Now, we state our main results.

Theorem 1.1. *Let $p > 1$ and let (M, g) be an n-dimensional $(n \geq 3)$ complete manifold with* $\mathrm{Ric} \geq -(n-1)\kappa$, *where κ is a nonnegative constant. Assume u is* *a positive solution to equation (1-1) on the geodesic ball $B(x_0, 2R) \subset M$. If the constants a, p and q satisfy either*

$$\frac{\beta}{2}\cdot\frac{n+1}{n-1}(p-1)-h^{1/2} < q < \frac{\beta}{2}\cdot\frac{n+1}{n-1}(p-1)+h^{1/2} \quad (a \neq 0)$$

or

$$a\Big[\frac{\beta}{2}\cdot\frac{n+1}{n-1}(p-1)-q\Big] \geq 0,$$

where β is a constant satisfying (1-3), then there exists a positive constant $\mathcal{C} = \mathcal{C}(n, p, q, \beta)$ such that

$$\sup_{B(x_0,R/2)} \frac{|\nabla u|^2}{u^\beta} \leq \mathcal{C}\frac{1+\kappa R^2}{R^2}\phi_\beta^{2-\beta}. \tag{1-4}$$

If $\beta = 2$, we have

$$\frac{\beta}{2}\cdot\frac{n+1}{n-1}(p-1)+h^{1/2} = \frac{n+3}{n-1}(p-1) \quad \text{and} \quad \frac{\beta}{2}\cdot\frac{n+1}{n-1}(p-1)-h^{1/2} = p-1,$$

so this case recovers the conclusion of Theorem 1.1 in [9]. At the same time, from (1-4) we can infer that

$$\sup_{B(x_0,R/2)} |\nabla u|^2 \leq \mathcal{C}\frac{1+\kappa R^2}{R^2}\phi_\beta^2 = \mathcal{C}\frac{1+\kappa R^2}{R^2}\sup_{B(x_0,R)} u^2,$$

if p and q satisfy the assumptions of Theorem 1.1 with $\beta \in (0,\ 2)$.

For convenience, we define

$$\Psi(I) := \sup_{\beta\in I}\Big[\frac{\beta}{2}\cdot\frac{n+1}{n-1}(p-1)+h^{1/2}\Big] \quad \text{and} \quad \Gamma(I) := \inf_{\beta\in I}\Big[\frac{\beta}{2}\cdot\frac{n+1}{n-1}(p-1)-h^{1/2}\Big].$$

If $2 \in I$, we obviously have

$$\Psi(I) \geq \frac{n+3}{n-1}(p-1) \quad \text{and} \quad \Gamma(I) \leq p-1.$$

So, we always have

$$\Psi\Big(\Big(0,\ \frac{2(n-1)}{n-p}\Big)\Big) \geq \frac{n+3}{n-1}(p-1),$$

since $p > 1$, and hence $2 \in \big(0,\ \frac{2(n-1)}{n-p}\big)$.

We then obtain the following consequences of Theorem 1.1:

Corollary 1.2. *Let $p > 1$ and let (M, g) be an n-dimensional $(n \geq 3)$ complete manifold with $\mathrm{Ric} \geq -(n-1)\kappa$, where κ is a nonnegative constant. Assume u is a positive solution to equation (1-1) on the geodesic ball $B(x_0, 2R) \subset M$. Assume also that the constants a, p and q satisfy one of the following five conditions*:

- $a > 0,\ p \geq n$ *and* $q \in \mathbb{R}$.
- $a > 0,\ 1 < p < n$ *and* $q < \Psi\big((0,\ \frac{2(n-1)}{n-p})\big)$.

- $a < 0,\ p \ge n$ *and* $q > \Gamma((0,\ +\infty))$.
- $a < 0,\ 1 < p < n$ *and* $q > \Gamma\big((0,\ \frac{2(n-1)}{n-p})\big)$.
- $a = 0,\ p > 1$.

Then there exist positive constants $\mathcal{C} = \mathcal{C}(n, p, q)$ *and* $\beta = \beta(n, p, q) \in (0,\ +\infty)$ *such that*

$$\sup_{B(x_0, R/2)} \frac{|\nabla u|^2}{u^\beta} \le \mathcal{C}\frac{1 + \kappa R^2}{R^2}\phi_\beta^{2-\beta}.$$

Note that in case 2 ($a > 0$ and $1 < p < n$), if there exists a point $\beta_0 \in \big(0,\ \frac{2(n-1)}{n-p}\big)$ such that

$$\Psi\bigg(\Big(0,\ \frac{2(n-1)}{n-p}\Big)\bigg) = \bigg[\frac{\beta}{2}\cdot\frac{n+1}{n-1}(p-1) + h^{1/2}\bigg]\bigg|_{\beta=\beta_0},$$

then the condition $q < \Psi\big(\big(0,\ \frac{2(n-1)}{n-p}\big)\big)$ can be relaxed to $q \le \Psi\big(\big(0,\ \frac{2(n-1)}{n-p}\big)\big)$. In the other four cases, we can obtain similar conclusions.

Further, if $1 < p < n$, it is easy to see that

$$\frac{n+1}{n-p}(p-1) \le \Psi\bigg(\Big(0,\ \frac{2(n-1)}{n-p}\Big)\bigg).$$

Usually, this is a strict inequality; for instance, if we let $n = 3$ and $p = 2$, then

$$\frac{\beta}{2}\cdot\frac{n+1}{n-1}(p-1) + h^{1/2} = \beta + \sqrt{\beta(1-\beta/4)} \tag{1-5}$$

and $\beta \in (0,\ 4)$. Hence, we can check that (1-5) attains its maximum at an interior point $\beta_0 = 2 + 4/\sqrt{5} \in (0, 4)$. Therefore, we get

$$\Psi((0, 4)) = 2 + \sqrt{5} > 4 = \tfrac{3+1}{3-2}\cdot(2-1).$$

But we also have

$$\Psi((0, 4)) = 2 + \sqrt{5} > \tfrac{n+3}{n-1}(p-1) = 3.$$

This indicates that for $q \ge \frac{n+3}{n-1}(p-1)$ one also derives the gradient estimate.

Corollary 1.3. *Let* $p > 1$ *and let* (M, g) *be an n-dimensional* $(n \ge 3)$ *complete manifold with* $\mathrm{Ric} \ge -(n-1)\kappa$, *where* κ *is a nonnegative constant. Assume u is a positive solution to equation* (1-1) *on the geodesic ball* $B(x_0, 2R) \subset M$. *If the constants* a, p *and* q *satisfy*

$$a < 0 \quad \text{and} \quad q > \Gamma((0, 2]),$$

then there exist positive constants $\mathcal{C} = \mathcal{C}(n, p, q)$ *and* $\beta = \beta(n, p, q) \in (0, 2]$ *such that*

$$\sup_{B(x_0, R/2)} \frac{|\nabla u|^2}{u^\beta} \le \mathcal{C}\frac{1 + \kappa R^2}{R^2}\sup_{B(x_0, R)} u^{2-\beta}.$$

By using Theorem 1.1, we then reach the following conclusion.

Corollary 1.4. *Let $p > 1$ and let (M, g) be an n-dimensional $(n \geq 3)$ complete manifold with* $\mathrm{Ric} \geq -(n-1)\kappa$, *where κ is a nonnegative constant. Assume that u is a positive solution to equation* (1-1) *on the geodesic ball $B(x_0, 2R) \subset M$ and that*

$$u(x) \geq b > 0, \quad x \in B(x_0, R).$$

Assume also that the constants a, p and q satisfy one of the following conditions:

- $a > 0$, $p \geq n$ *and* $q \in \mathbb{R}$.
- $a > 0$, $1 < p < n$ *and* $q < \Psi\big(\big[2, \frac{2(n-1)}{n-p}\big)\big)$.
- $a < 0$, $p \geq n$ *and* $q > \Gamma([2, +\infty))$.
- $a < 0$, $1 < p < n$ *and* $q > \Gamma\big(\big[2, \frac{2(n-1)}{n-p}\big)\big)$.
- $a = 0$, $p > 1$.

Then there exist positive constants $\mathcal{C} = \mathcal{C}(n, p, q, b)$ and $\beta = \beta(n, p, q) \in [2, +\infty)$, such that

$$\sup_{B(x_0, R/2)} \frac{|\nabla u|^2}{u^\beta} \leq \mathcal{C}\frac{1+\kappa R^2}{R^2}.$$

The next result states that whether or not the exact Cheng–Yau log-gradient estimates for the positive solutions to equation (1-1) hold true is equivalent to whether or not the positive solutions to (1-1) fulfill Harnack's inequality.

Theorem 1.5. *Let $p > 1$ and (M, g) be an n-dimensional $(n \geq 3)$ complete manifold with* $\mathrm{Ric} \geq -(n-1)\kappa$, *where κ is a nonnegative constant. Assume u is a positive solution to equation* (1-1) *on the geodesic ball $B(x_0, 2R) \subset M$ satisfying the Harnack inequality*

$$\sup_{B(x_0, R)} u \leq l \inf_{B(x_0, R)} u.$$

Assume also that the constants a, q and p satisfy one of the following conditions:

- $a > 0$, $p \geq n$ *and* $q \in \mathbb{R}$.
- $a > 0$, $1 < p < n$ *and* $q < \Psi\big(\big(0, \frac{2(n-1)}{n-p}\big)\big)$.
- $a < 0$, $p \geq n$ *and* $q > \Gamma((0, +\infty))$.
- $a < 0$, $1 < p < n$ *and* $q > \Gamma\big(\big(0, \frac{2(n-1)}{n-p}\big)\big)$.
- $a = 0$, $p > 1$.

Then there exists a positive constant $\mathcal{C} = \mathcal{C}(n, p, q, l)$ such that

$$\sup_{B(x_0, R/2)} \frac{|\nabla u|^2}{u^2} \leq \mathcal{C}\frac{1+\kappa R^2}{R^2}.$$

Corollary 1.6. *Let $p > 1$ and let (M, g) be an n-dimensional $(n \geq 3)$ complete manifold with nonnegative Ricci curvature. Assume u is a positive solution to equation (1-1) on any given geodesic ball $B(x_0, 2R) \subset M$, and that it satisfies the Harnack inequality*

$$\sup_{B(x_0,R)} u \leq l \inf_{B(x_0,R)} u,$$

where l is independent of u and R. Assume also that the constants a, q and p satisfy one of the following conditions:

- $a > 0,\ p \geq n$ *and* $q \in \mathbb{R}$.
- $a > 0,\ 1 < p < n$ *and* $q < \Psi\big((0, \frac{2(n-1)}{n-p})\big)$.
- $a < 0,\ p \geq n$ *and* $q > \Gamma((0, +\infty))$.
- $a < 0,\ 1 < p < n$ *and* $q > \Gamma\big((0, \frac{2(n-1)}{n-p})\big)$.
- $a = 0,\ p > 1$.

Then there exist a positive constant $\mathcal{C} = \mathcal{C}(n, p, q, l)$ such that

$$\sup_{B(x_0,R/2)} \frac{|\nabla u|^2}{u^2} \leq \frac{\mathcal{C}}{R^2}.$$

Conversely, if the above log-*gradient estimate holds true, then, for any given $B(x_0, R) \subset M$, there holds*

$$\sup_{B(x_0,R)} u \leq l \inf_{B(x_0,R)} u,$$

where l is independent of u and R.

If we consider (1-1) in $\mathbb{R}^n$ $(n \geq 3)$, we can achieve the following conclusion.

Corollary 1.7. *Assume u is a positive solution to equation (1-1) on the ball $B(x_0, 2R) \subset \mathbb{R}^n$. Assume also that the constants a, q and p satisfy one of the following conditions*:

- $a > 0,\ 1 < p < n,\ p \neq q$ *and* $q \in \big(p-1, \frac{(p-1)n}{n-p}\big)$.
- $a > 0,\ p \geq n, q \neq p$ *and* $q \in (0, +\infty)$.
- $a \geq 1$ *and* $1 < p = q < n < p^2$.
- $a \geq 1$ *and* $p = q \geq n$.

Then there exist a positive constant $\mathcal{C} = \mathcal{C}(n, p, q, a)$ such that

$$\sup_{B(x_0,R/2)} \frac{|\nabla u|^2}{u^2} \leq \frac{\mathcal{C}}{R^2}.$$

The corollary tells us that $\frac{n+3}{n-1}(p-1)$ is not an optimal bound for deriving an exact Cheng–Yau-type log-gradient estimate, since

$$\sup_{B(x_0,R/2)} \frac{|\nabla u|^2}{u^2} \leq \frac{\mathcal{C}}{R^2}$$

in the case $a>0,\ p\geq n,\ q\neq p$ and $q\in(0,\ +\infty)$.

The next result is a direct consequence of Corollary 1.7.

Theorem 1.8. *Assume u is a positive solution to equation (1-1) on $\mathbb{R}^n$ and $n\geq 3$. Assume also that the constants a, p and q satisfy one of the following conditions*:

- $a>0,\ 1<p<n,\ p\neq q$ *and* $q\in\left(p-1,\ \frac{(p-1)n}{n-p}\right)$.
- $a>0,\ p\geq n,\ p\neq q$ *and* $q\in(0,\ +\infty)$.
- $a\geq 1$ *and* $1<p=q<n<p^2$.
- $a\geq 1$ *and* $p=q\geq n$.

then (1-1) admits no positive solution.

In the above conclusions, we always suppose that $\dim M=n\geq 3$. In fact, for the case $\dim M=2$ we can also obtain similar conclusions. Since the proofs are similar to the case $\dim M\geq 3$, we will not give details.

Main ideas of proof and the organization of paper. In order to give the gradient estimates, we consider the linearized operator $\mathcal{L}_p$ of the p-Laplace operator at a solution u, and let $\mathcal{L}_p$ act on an auxiliary function given by

$$F(u)=\frac{|\nabla u|^2}{u^\beta}, \quad \beta>0.$$

The use of such an auxiliary function is inspired by the gradient estimates established in [18] for another equation related to Ricci solitons. Then, we need to establish some suitable pointwise estimate of $\mathcal{L}_p(F)$ using the techniques of Cheng and Yau [3; 23], so that we can take a Nash–Moser iteration scheme to give the L^∞-norm of $F(u)$. Saloff-Coste's Sobolev inequalities play an important role in our arguments.

Outline. In Section 2, we recall some background and establish important lemmas, which will play a key role in the Nash–Moser iteration process. In Section 3, the main body of this paper, we prove the gradient estimates. In Section 4, we give the proofs of the main theorem and its corollaries.

2. Preliminaries

Throughout this paper, we let (M, g) be an n-dimensional Riemannian manifold ($n\geq 3$), and ∇ denotes the Levi-Civita connection corresponding to the metric g.

We denote the volume form on (M, g) by

$$d\,\mathrm{vol} = \sqrt{\det(g_{ij})}\, dx_1 \wedge \cdots \wedge dx_n,$$

where $(x_1, \ldots, x_n)$ is a local coordinate chart, and for simplicity we usually omit the volume form of integral over M.

Definition 2.1. We say that $u \in C^1(M) \cap W^{1,p}_{loc}(M)$ is a weak solution of (1-1) if for all $\psi \in W^{1,p}_0(M)$ we have

$$\int_M |\nabla u|^{p-2} \langle \nabla u, \nabla \psi \rangle = \int_M a u^q \psi.$$

Next, we recall the Saloff-Coste Sobolev inequalities (see [14]), which shall play a key role in our proof of the main theorems.

Lemma 2.2 (Saloff-Coste [14]). *Let (M, g) be a complete manifold with* Ric $\geq -(n-1)\kappa$. *For $n > 2$, there exists a positive constant C_n, depending only on n, such that for all $B \subset M$ of radius R and volume V we have for $h_1 \in C^\infty_0(B)$*

$$\|h_1\|^2_{L^{2n/(n-2)}(B)} \leq \exp\{C_n(1 + \sqrt{\kappa} R)\} V^{-2/n} R^2 \Big(\int_B |\nabla h_1|^2 + R^{-2} h_1^2 \Big).$$

For $n = 2$, the above inequality holds with n replaced by any fixed $n' > 2$.

Now we consider the linearization operator $\mathcal{L}_p$ of p-Laplace operator:

$$\mathcal{L}_p(\psi) = \mathrm{div}\big[f^{p/2-1} A\big(\nabla \psi\big)\big], \tag{2-1}$$

where $f = |\nabla u|^2$ and

$$A\,(\nabla \psi) = \nabla \psi + (p-2) f^{-1} \langle \nabla \psi, \nabla u \rangle \nabla u. \tag{2-2}$$

We first derive an useful expression of $\mathcal{L}_p(f)$.

Lemma 2.3. *The equality*

$$\mathcal{L}_p(f) = \big(\tfrac{p}{2} - 1\big) f^{p/2-2} |\nabla f|^2 + 2 f^{p/2-1} \big(|\nabla \nabla u|^2 + \mathrm{Ric}(\nabla u, \nabla u) \big) + 2 \langle \nabla \Delta_p u, \nabla u \rangle$$

holds pointwise in $\{x : f(x) > 0\}$.

Proof. By the definition of A in (2-2), we have

$$A(\nabla f) = \nabla f + (p-2) f^{-1} \langle \nabla f, \nabla u \rangle \nabla u. \tag{2-3}$$

Combining (2-1) and (2-3), we obtain

$$\mathcal{L}_p(f)=\left(\tfrac{p}{2}-1\right)f^{p/2-2}|\nabla f|^2+f^{p/2-1}\Delta f+(p-2)\left(\tfrac{p}{2}-2\right)f^{p/2-3}\langle\nabla f,\nabla u\rangle^2 + (p-2)f^{p/2-2}\langle\nabla\langle\nabla f,\nabla u\rangle,\nabla u\rangle+(p-2)f^{p/2-2}\langle\nabla f,\nabla u\rangle\Delta u. \tag{2-4}$$

At the same time, by the definition of the p-Laplacian, we have

$$\begin{aligned}2\langle\nabla\Delta_p u,\nabla u\rangle &= (p-2)\left(\tfrac{p}{2}-2\right)f^{p/2-3}\langle\nabla f,\nabla u\rangle^2+(p-2)f^{p/2-2}\langle\nabla\langle\nabla f,\nabla u\rangle,\nabla u\rangle\\ &\quad+(p-2)f^{p/2-2}\langle\nabla f,\nabla u\rangle\Delta u+2f^{p/2-1}\langle\nabla\Delta u,\nabla u\rangle.\end{aligned} \tag{2-5}$$

We combine (2-4) and (2-5) to obtain

$$\begin{aligned}\mathcal{L}_p(f)&=\\ &\left(\tfrac{p}{2}-1\right)f^{p/2-2}|\nabla f|^2+f^{p/2-1}\Delta f+2\langle\nabla\Delta_p u,\nabla u\rangle-2f^{p/2-1}\langle\nabla\Delta u,\nabla u\rangle.\end{aligned} \tag{2-6}$$

From (2-6) and the Bochner formula

$$\tfrac{1}{2}\Delta f=|\nabla\nabla u|^2+\mathrm{Ric}(\nabla u,\nabla u)+\langle\nabla\Delta u,\nabla u\rangle$$

we get

$$\mathcal{L}_p(f)=\left(\tfrac{p}{2}-1\right)f^{p/2-2}|\nabla f|^2+2f^{p/2-1}\left(|\nabla\nabla u|^2+\mathrm{Ric}(\nabla u,\nabla u)\right)+2\langle\nabla\Delta_p u,\nabla u\rangle.$$

Thus, we finish the proof. □

In the last section, we are going to use the following lemmas. We denote by B_R a ball with radius R and let Ω be a domain in $\mathbb{R}^n$.

Lemma 2.4 [15, Theorem 4.1(a)]. *Suppose $n>m$ and $s\in\left(m,\frac{m(n-1)}{n-m}\right)$. Let ω be a nonnegative weak solution of the differential inequality*

$$\omega^{s-1}\le-\Delta_m\omega\le\Lambda\omega^{s-1}\quad in\ \Omega, \tag{2-7}$$

for some constant $\Lambda>1$. Then there is a constant $\mathcal{C}=\mathcal{C}(n,m,s,\Lambda)>0$ such that

$$\sup\nolimits_{B_R}\omega(x)\le\mathcal{C}\inf\nolimits_{B_R}\omega(x). \tag{2-8}$$

Lemma 2.5 [15, Theorem 4.3(a)]. *Let $n\le m$. Assume the hypotheses of Lemma 2.4, except that the condition $s\in\left(m,\frac{m(n-1)}{n-m}\right)$ is replaced by $s\in(1,+\infty)$, that is, $\frac{m(n-1)}{n-m}=+\infty$. Then (2-8) is valid with $\mathcal{C}=\mathcal{C}(n,m,s,\Lambda)>0$.*

3. Gradient estimates

3.1. ***Estimate for the linearized operator of p-Laplace.*** First, we need to give the pointwise estimate of $\mathcal{L}_p(F)$, where

$$F=\frac{f}{u^\beta}\quad(\beta>0)$$

and $\mathcal{L}_p$ is the linearized operator of p-Laplacian at u.

Lemma 3.1. *The equality*

$$\mathcal{L}_p(F) = u^{-\beta}\mathcal{L}_p(f) + \beta(\beta+1)(p-1)u^{-\beta-2}f^{p/2+1} - \beta\left(1+\tfrac{p}{2}\right)(p-1)u^{-\beta-1}f^{p/2-1}\langle\nabla f,\nabla u\rangle - \beta(p-1)u^{-\beta-1}f^{p/2}\Delta u$$

holds pointwise in $\{x : f(x) > 0\}$.

Proof. By the definition of A in (2-2), we have

$$A(\nabla F) = u^{-\beta}A(\nabla f) - \beta u^{-\beta-1}f A(\nabla u), \tag{3-1}$$

$$A(\nabla u) = (p-1)\nabla u, \tag{3-2}$$

$$A(\nabla f) = \nabla f + (p-2)f^{-1}\langle\nabla u,\nabla f\rangle\nabla u. \tag{3-3}$$

Combining (2-1) and (3-1), we obtain

$$\mathcal{L}_p(F) = \operatorname{div}\left[u^{-\beta}f^{p/2-1}A(\nabla f)\right] - \beta\operatorname{div}\left[u^{-\beta-1}f^{p/2}A(\nabla u)\right]. \tag{3-4}$$

Direct computation shows that

$$\operatorname{div}\left[u^{-\beta}f^{p/2-1}A(\nabla f)\right] = -\beta u^{-\beta-1}f^{p/2-1}\langle A(\nabla f),\nabla u\rangle + u^{-\beta}\operatorname{div}\left[f^{p/2-1}A(\nabla f)\right] \tag{3-5}$$

and

$$\operatorname{div}\left[u^{-\beta-1}f^{p/2}A(\nabla u)\right] = -(\beta+1)u^{-\beta-2}f^{p/2}\langle A(\nabla u),\nabla u\rangle + \tfrac{p}{2}u^{-\beta-1}f^{p/2-1}\langle A(\nabla u),\nabla f\rangle + u^{-\beta-1}f^{p/2}\operatorname{div}A(\nabla u). \tag{3-6}$$

By substituting (2-1) and (3-3) into (3-5), we have

$$\operatorname{div}\left[u^{-\beta}f^{p/2-1}A(\nabla f)\right] = -\beta(p-1)u^{-\beta-1}f^{p/2-1}\langle\nabla f,\nabla u\rangle + u^{-\beta}\mathcal{L}_p(f). \tag{3-7}$$

Substituting (3-2) into (3-6) leads to

$$\operatorname{div}\left[u^{-\beta-1}f^{p/2}A(\nabla u)\right] = -(\beta+1)(p-1)u^{-\beta-2}f^{p/2+1} + \tfrac{p}{2}(p-1)u^{-\beta-1}f^{p/2-1}\langle\nabla f,\nabla u\rangle + (p-1)u^{-\beta-1}f^{p/2}\Delta u. \tag{3-8}$$

Now, we plug (3-7) and (3-8) into (3-4) to derive the required equality, and hence finish the proof of Lemma 3.1. □

Lemma 3.2. *Let u be a positive solution of equation (1-1) in $\Omega \subset M$. Then*

$$\begin{aligned}\mathcal{L}_p(F) = {} & \left(\tfrac{p}{2}-1\right)u^{\beta}f^{p/2-2}|\nabla F|^2 \\ & + 2a\left[\tfrac{\beta}{2}(p-1)-q\right]u^{q-\beta-1}f - p\beta u^{-1}f^{p/2-1}\langle\nabla F,\nabla u\rangle \\ & + 2u^{-\beta}f^{p/2-1}\left(|\nabla\nabla u|^2 + \operatorname{Ric}(\nabla u,\nabla u)\right) + \left[-\tfrac{1}{2}p\beta^2 + (p-1)\beta\right]u^{-\beta-2}f^{p/2+1}\end{aligned} \tag{3-9}$$

holds pointwise in $\{x \in \Omega : f(x) > 0\}$.

Proof. By summarizing Lemma 2.3 and Lemma 3.1 we can achieve that

$$
\begin{aligned}
\text{(3-10)}\quad \mathcal{L}_p(F) = {} & u^{-\beta} \\
& \times\Big[\big(\tfrac{p}{2}-1\big)f^{p/2-2}|\nabla f|^2+2f^{p/2-1}\big(|\nabla\nabla u|^2+\mathrm{Ric}(\nabla u,\nabla u)\big)+2\langle\nabla\Delta_p u,\nabla u\rangle\Big] \\
& +\beta(\beta+1)(p-1)u^{-\beta-2}f^{p/2+1} \\
& -\beta\big(1+\tfrac{p}{2}\big)(p-1)u^{-\beta-1}f^{p/2-1}\langle\nabla f,\nabla u\rangle-\beta(p-1)u^{-\beta-1}f^{p/2}\Delta u.
\end{aligned}
$$

Since $F = f/u^\beta$, we can infer that

$$\nabla f = \beta u^{-1} f\nabla u + u^\beta \nabla F.$$

Hence, we have

$$\text{(3-11)}\qquad \langle\nabla f,\nabla u\rangle = \beta u^{-1}f^2 + u^\beta\langle\nabla F,\nabla u\rangle,$$

$$\text{(3-12)}\qquad |\nabla f|^2 = \beta^2u^{-2}f^3 + 2\beta u^{\beta-1}f\langle\nabla F,\nabla u\rangle + u^{2\beta}|\nabla F|^2.$$

Substituting (1-1), (3-11) and (3-12) into (3-10), we obtain

$$
\begin{aligned}
\text{(3-13)}\quad \mathcal{L}_p(F) = {} & \big(\tfrac{p}{2}-1\big)u^\beta f^{p/2-2}|\nabla F|^2 - 2aqu^{q-\beta-1}f \\
& + 2u^{-\beta}f^{p/2-1}\big(|\nabla\nabla u|^2+\mathrm{Ric}(\nabla u,\nabla u)\big) \\
& + \big[-\tfrac{1}{2}(p^2-2p+2)\beta^2+(p-1)\beta\big]\,u^{-\beta-2}f^{p/2+1} \\
& - \tfrac{1}{2}(p^2-p+2)\beta u^{-1}f^{p/2-1}\langle\nabla F,\nabla u\rangle-\beta(p-1)u^{-\beta-1}f^{p/2}\Delta u.
\end{aligned}
$$

From (1-1) and the equality

$$\Delta_p u = \mathrm{div}\big(f^{p/2-1}\nabla u\big) = \big(\tfrac{p}{2}-1\big)f^{p/2-2}\langle\nabla f,\nabla u\rangle + f^{p/2-1}\Delta u,$$

it is easy to verify that

$$\text{(3-14)}\qquad \Delta u = \big(1-\tfrac{p}{2}\big)\,f^{-1}\langle\nabla f,\nabla u\rangle - au^q f^{1-\frac{p}{2}}.$$

Substituting (3-11) into the above (3-14) yields

$$\text{(3-15)}\qquad \Delta u = \big(1-\tfrac{p}{2}\big)\,u^\beta f^{-1}\langle\nabla F,\nabla u\rangle + \big(1-\tfrac{p}{2}\big)\,\beta u^{-1}f - au^q f^{1-\frac{p}{2}}.$$

Hence, we substitute (3-15) into (3-13) to derive the required equality. Thus we complete the proof of Lemma 3.2. □

Lemma 3.3. *Let $\alpha > 1$ and let u be a positive solution of equation (1-1) in $\Omega\subset M$. Then, the following holds pointwise in $\{x\in\Omega : f(x) > 0\}$:*

$$
\begin{aligned}
\text{(3-16)}\quad & \frac{1}{\alpha}F^{2-\alpha}\mathcal{L}_p(F^\alpha) \\
& = \big(\alpha+\tfrac{p}{2}-2\big)f^{p/2-1}|\nabla F|^2+(\alpha-1)(p-2)f^{p/2-2}\langle\nabla F,\nabla u\rangle^2 \\
& \quad + 2u^{-2\beta}f^{p/2}\big(|\nabla\nabla u|^2+\mathrm{Ric}(\nabla u,\nabla u)\big)-\beta pu^{-\beta-1}f^{p/2}\langle\nabla F,\nabla u\rangle \\
& \quad + 2a\big[\tfrac{\beta}{2}(p-1)-q\big]u^{q-2\beta-1}f^2+\big[-\tfrac{p}{2}\beta^2+(p-1)\beta\big]u^{-2\beta-2}f^{p/2+2}
\end{aligned}
$$

Proof. By the definition of A in (2-2), we have

$$A(\nabla(F^\alpha)) = \alpha F^{\alpha-1} A(\nabla F) \tag{3-17}$$

and

$$\langle A(\nabla F), \nabla F\rangle = |\nabla F|^2 + (p-2) f^{-1}\langle \nabla F, \nabla u\rangle^2. \tag{3-18}$$

Combining (2-1), (3-17) and (3-18), we obtain

$$\begin{aligned} \mathcal{L}_p(F^\alpha) &= \alpha \operatorname{div}\bigl[F^{\alpha-1} f^{p/2-1} A(\nabla F)\bigr] \\ &= \alpha(\alpha-1) F^{\alpha-2} f^{p/2-1}\langle A(\nabla F), \nabla F\rangle + \alpha F^{\alpha-1}\mathcal{L}_p(F) \\ &= \alpha(\alpha-1) F^{\alpha-2} f^{p/2-1}\bigl[|\nabla F|^2 + (p-2) f^{-1}\langle\nabla F, \nabla u\rangle^2\bigr] + \alpha F^{\alpha-1}\mathcal{L}_p(F). \end{aligned} \tag{3-19}$$

In view of (3-9) and (3-19), we can derive the required equality and complete the proof of Lemma 3.3. □

Next, we need to consider the pointwise estimate of $\mathcal{L}_p(F^\alpha)$. We begin with two lemmas.

Lemma 3.4. *Let $a \neq 0$, $\alpha > 1$ and let u be a positive solution of (1-1) in $\Omega \subset M$ with $\mathrm{Ric} \geq -(n-1)\kappa$. Set*

$$H = \beta(p-1)\bigl[\tfrac{p-n}{2(n-1)}\beta + 1\bigr] - \frac{\bigl[\frac{\beta}{2}\frac{n+1}{n-1}(p-1) - q\bigr]^2}{\frac{2}{n-1} - \frac{p-1}{(n-1)^2}\bigl[\alpha + \frac{p-n}{2(n-1)}\bigr]^{-1}}. \tag{3-20}$$

Then we have, pointwise in $\{x \in \Omega : f(x) > 0\}$,

$$\frac{1}{\alpha} F^{2-\alpha}\mathcal{L}(F^\alpha) \geq -2(n-1)\kappa u^{-2\beta} f^{p/2+1} + (p-1)\beta \frac{p-n}{n-1} u^{-\beta-1} f^{p/2}\langle \nabla F, \nabla u\rangle + H u^{-2\beta-2} f^{p/2+2}. \tag{3-21}$$

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal frame of TM on a domain with $f \neq 0$ such that $e_1 = \nabla u / |\nabla u|$. We infer the equalities

$$4\sum_{i=1}^n u_{1i}^2 = f^{-1}|\nabla f|^2, \quad u_{11} = \frac{\langle \nabla f, \nabla u\rangle}{2f}, \tag{3-22}$$

$$\Delta_p u = (p-1) f^{p/2-1} u_{11} + f^{p/2-1}\sum_{i=2}^n u_{ii}. \tag{3-23}$$

Substituting (3-11) and (3-12) into the two equations in (3-22) leads respectively to

$$4\sum_{i=1}^n u_{1i}^2 = u^{2\beta} f^{-1}|\nabla F|^2 + \beta^2 u^{-2} f^2 + 2\beta u^{\beta-1}\langle \nabla F, \nabla u\rangle \tag{3-24}$$

and

$$2u_{11} = \beta u^{-1} f + u^\beta f^{-1}\langle \nabla F, \nabla u\rangle. \tag{3-25}$$

Combining (1-1) and (3-23), we obtain

$$\Big(\sum_{i=2}^n u_{ii}\Big)^2 = \big[(p-1)u_{11} + af^{1-p/2}u^q\big]^2. \tag{3-26}$$

By substituting (3-25) into (3-26), we have

$$\Big(\sum_{i=2}^n u_{ii}\Big)^2 = \big[\tfrac{p-1}{2}\beta u^{-1}f + \tfrac{p-1}{2}u^\beta f^{-1}\langle\nabla F,\nabla u\rangle + af^{1-\frac{p}{2}}u^q\big]^2. \tag{3-27}$$

By omitting some nonnegative terms in $|\nabla\nabla u|^2$ and using Cauchy's inequality, we arrive at

$$|\nabla\nabla u|^2 \ge \sum_{i=1}^n u_{1i}^2 + \sum_{i=2}^n u_{ii}^2 \ge \sum_{i=1}^n u_{1i}^2 + \frac{1}{n-1}\Big(\sum_{i=2}^n u_{ii}\Big)^2. \tag{3-28}$$

We plug (3-24) and (3-27) into (3-28) to obtain

$$\begin{aligned}|\nabla\nabla u|^2 \ge{}& \tfrac14 u^{2\beta}f^{-1}|\nabla F|^2 + \tfrac14\beta^2u^{-2}f^2 + \tfrac12\beta u^{\beta-1}\langle\nabla F,\nabla u\rangle\\ &+ \tfrac{1}{n-1}\big[\tfrac{p-1}{2}\beta u^{-1}f + \tfrac{p-1}{2}u^\beta f^{-1}\langle\nabla F,\nabla u\rangle + af^{1-\frac{p}{2}}u^q\big]^2.\end{aligned} \tag{3-29}$$

By expanding the last term of (3-29), we obtain

$$\begin{aligned}|\nabla\nabla u|^2 \ge{}& \tfrac14 u^{2\beta}f^{-1}|\nabla F|^2 + \tfrac{(p-1)^2}{4(n-1)}u^{2\beta}f^{-2}\langle\nabla F,\nabla u\rangle^2 + \tfrac{\beta^2}{4}\big[1+\tfrac{(p-1)^2}{n-1}\big]u^{-2}f^2\\ &+ \tfrac{\beta}{2}\big[1+\tfrac{(p-1)^2}{n-1}\big]u^{\beta-1}\langle\nabla F,\nabla u\rangle + \tfrac{a^2}{n-1}u^{2q}f^{2-p} + a\tfrac{p-1}{n-1}\beta u^{q-1}f^{2-p/2}\\ &+ a\tfrac{p-1}{n-1}\beta u^{q-1}f^{2-p/2} + a\tfrac{p-1}{n-1}u^{q+\beta}f^{-\frac{p}{2}}\langle\nabla F,\nabla u\rangle.\end{aligned}$$

Using this inequality and $\mathrm{Ric}\ge -(n-1)\kappa$, we have

$$\begin{aligned}&2u^{-2\beta}f^{p/2}\big(|\nabla\nabla u|^2 + \mathrm{Ric}(\nabla u,\nabla u)\big)\\ &\ge -2(n-1)\kappa u^{-2\beta}f^{p/2+1} + \tfrac12 f^{p/2-1}|\nabla F|^2 + \beta\big[1+\tfrac{(p-1)^2}{n-1}\big]u^{-\beta-1}f^{p/2}\langle\nabla F,\nabla u\rangle\\ &\quad+ \tfrac{\beta^2}{2}\big[1+\tfrac{(p-1)^2}{n-1}\big]u^{-2\beta-2}f^{p/2+2} + \tfrac{(p-1)^2}{2(n-1)}f^{p/2-2}\langle\nabla F,\nabla u\rangle^2 + \tfrac{2a^2}{n-1}u^{2q-2\beta}f^{2-p/2}\\ &\quad+ a\tfrac{2(p-1)}{n-1}\beta u^{q-2\beta-1}f^2 + a\tfrac{2(p-1)}{n-1}u^{q-\beta}\langle\nabla F,\nabla u\rangle.\end{aligned} \tag{3-30}$$

Substituting (3-30) into (3-16) yields

$$\begin{aligned}&\tfrac{1}{\alpha}F^{2-\alpha}\mathcal{L}_p(F^\alpha)\\ &\quad\ge -2(n-1)\kappa u^{-2\beta}f^{p/2+1} + 2a\big[\tfrac{\beta}{2}\tfrac{n+1}{n-1}(p-1) - q\big]u^{q-2\beta-1}f^2\\ &\qquad+ \big(\alpha+\tfrac{p}{2}-\tfrac32\big)f^{p/2-1}|\nabla F|^2 + \big[(\alpha-1)(p-2) + \tfrac{(p-1)^2}{2(n-1)}\big]f^{p/2-2}\langle\nabla F,\nabla u\rangle^2\\ &\qquad+ (p-1)\beta\big[\tfrac{p-n}{2(n-1)}\beta+1\big]u^{-2\beta-2}f^{p/2+2} + (p-1)\beta\tfrac{p-n}{n-1}u^{-\beta-1}f^{p/2}\langle\nabla F,\nabla u\rangle\\ &\qquad+ \tfrac{2a^2}{n-1}u^{2q-2\beta}f^{2-p/2} + a\tfrac{2(p-1)}{n-1}u^{q-\beta}\langle\nabla F,\nabla u\rangle.\end{aligned} \tag{3-31}$$

By using $p > 1$, $\alpha > 1$ and

$$f^{p/2-1}|\nabla F|^2 \geq f^{p/2-2}\langle \nabla F, \nabla u\rangle^2, \tag{3-32}$$

we arrive at

$$\begin{aligned}\left(\alpha + \tfrac{p}{2} - \tfrac{3}{2}\right) f^{p/2-1}|\nabla F|^2 + \left[(\alpha-1)(p-2) + \tfrac{(p-1)^2}{2(n-1)}\right] f^{p/2-2}\langle \nabla F, \nabla u\rangle^2 A \\ \geq (p-1)\left[\alpha + \tfrac{p-n}{2(n-1)}\right] f^{p/2-2}\langle \nabla F, \nabla u\rangle^2.\end{aligned} \tag{3-33}$$

By substituting (3-33) into (3-31), we obtain

$$\begin{aligned}\tfrac{1}{\alpha} F^{2-\alpha}\mathcal{L}_p(F^\alpha) \geq & -2(n-1)\kappa u^{-2\beta} f^{p/2+1} + 2a\left[\tfrac{\beta}{2}\tfrac{n+1}{n-1}(p-1) - q\right] u^{q-2\beta-1} f^2 \\ & + (p-1)\left[\alpha + \tfrac{p-n}{2(n-1)}\right] f^{p/2-2}\langle \nabla F, \nabla u\rangle^2 \\ & + (p-1)\beta\left[\tfrac{p-n}{2(n-1)}\beta + 1\right] u^{-2\beta-2} f^{p/2+2} \\ & + (p-1)\beta\tfrac{p-n}{n-1} u^{-\beta-1} f^{p/2}\langle \nabla F, \nabla u\rangle + \tfrac{2a^2}{n-1} u^{2q-2\beta} f^{2-p/2} \\ & + a\tfrac{2(p-1)}{n-1} u^{q-\beta}\langle \nabla F, \nabla u\rangle.\end{aligned} \tag{3-34}$$

Noting that $\alpha > 1$ and using the inequality $a_1^2 - 2a_1a_2 \geq -a_2^2$, we infer

$$\begin{aligned}(p-1)\left[\alpha + \tfrac{p-n}{2(n-1)}\right] f^{p/2-2}\langle \nabla F, \nabla u\rangle^2 + a\tfrac{2(p-1)}{n-1} u^{q-\beta}\langle \nabla F, \nabla u\rangle \\ \geq -\tfrac{a^2}{(n-1)^2}(p-1)\left[\alpha + \tfrac{p-n}{2(n-1)}\right]^{-1} u^{2q-2\beta} f^{2-p/2}.\end{aligned} \tag{3-35}$$

We substitute (3-35) into (3-34) to obtain

$$\begin{aligned}\tfrac{1}{\alpha} F^{2-\alpha}\mathcal{L}(F^\alpha) \geq & -2(n-1)\kappa u^{-2\beta} f^{p/2+1} + 2a\left[\tfrac{\beta}{2}\tfrac{n+1}{n-1}(p-1) - q\right] u^{q-2\beta-1} f^2 \\ & + (p-1)\beta\left[\tfrac{p-n}{2(n-1)}\beta + 1\right] u^{-2\beta-2} f^{p/2+2} \\ & + (p-1)\beta\tfrac{p-n}{n-1} u^{-\beta-1} f^{p/2}\langle \nabla F, \nabla u\rangle \\ & + \left\{\tfrac{2a^2}{n-1} - \tfrac{a^2}{(n-1)^2}(p-1)\left[\alpha + \tfrac{p-n}{2(n-1)}\right]^{-1}\right\} u^{2q-2\beta} f^{2-p/2}.\end{aligned} \tag{3-36}$$

Using $\alpha > 1$ and the inequality $a_1^2 - 2a_1a_2 \geq -a_2^2$ again, we obtain

$$\begin{aligned}\left\{\tfrac{2a^2}{n-1} - \tfrac{a^2}{(n-1)^2}(p-1)\left[\alpha + \tfrac{p-n}{2(n-1)}\right]^{-1}\right\} u^{2q-2\beta} f^{2-p/2} \\ + 2a\left[\tfrac{\beta}{2}\tfrac{n+1}{n-1}(p-1) - q\right] u^{q-2\beta-1} f^2 \\ \geq -\frac{\left[\frac{\beta}{2}\frac{n+1}{n-1}(p-1) - q\right]^2}{\frac{2}{n-1} - \frac{p-1}{(n-1)^2}\left[\alpha + \frac{p-n}{2(n-1)}\right]^{-1}} u^{-2\beta-2} f^{p/2+2}.\end{aligned} \tag{3-37}$$

Now, we plug (3-37) into (3-36) to deduce the desired inequality and hence complete the proof of Lemma 3.4. □

Lemma 3.5. *Let $\alpha > 1$ and let u be a positive solution of equation (1-1) in $\Omega \subset M$ with* $\mathrm{Ric} \geq -(n-1)\kappa$. *Then we have, pointwise in $\{x \in \Omega : f(x) > 0\}$,*

$$\frac{1}{\alpha} F^{2-\alpha}\mathcal{L}_p(F^\alpha) \geq -2(n-1)\kappa u^{-2\beta} f^{p/2+1} + \beta(p-1)\frac{p-n}{n-1} u^{-\beta-1} f^{p/2}\langle \nabla F, \nabla u\rangle + 2a\big[\tfrac{\beta}{2}\tfrac{n+1}{n-1}(p-1) - q\big] u^{q-2\beta-1} f^2 + \frac{\beta}{2}(p-1)\big(2 + \tfrac{p-n}{n-1}\beta\big) u^{-2\beta-2} f^{p/2+2} \tag{3-38}$$

Proof. By using the inequality $(a_1+a_2)^2 \geq a_1^2 + 2a_1a_2$, we have

$$\big[\tfrac{p-1}{2}\beta u^{-1} f + \tfrac{p-1}{2} u^\beta f^{-1}\langle \nabla F, \nabla u\rangle + a f^{1-\frac{p}{2}} u^q\big]^2 \geq \frac{(p-1)^2}{4}\beta^2 u^{-2} f^2 + (p-1)\beta u^{-1} f\big(\tfrac{p-1}{2} u^\beta f^{-1}\langle \nabla F, \nabla u\rangle + a f^{1-\frac{p}{2}} u^q\big). \tag{3-39}$$

Substituting (3-39) into (3-29), we have

$$|\nabla\nabla u|^2 \geq \frac{1}{4} u^{2\beta} f^{-1} |\nabla F|^2 + \frac{\beta^2}{4}\big[1 + \tfrac{(p-1)^2}{n-1}\big] u^{-2} f^2 + \frac{\beta}{2}\big[1 + \tfrac{(p-1)^2}{n-1}\big] u^{\beta-1}\langle \nabla F, \nabla u\rangle + a\beta\frac{p-1}{n-1} u^{q-1} f^{2-p/2}. \tag{3-40}$$

By using (3-40) and the assumption $\mathrm{Ric} \geq -(n-1)\kappa$, we have

$$2u^{-2\beta} f^{p/2}\big(|\nabla\nabla u|^2 + \mathrm{Ric}(\nabla u, \nabla u)\big) \geq -2(n-1)\kappa u^{-2\beta} f^{p/2+1} + \frac{1}{2} f^{p/2-1}|\nabla F|^2 + \frac{\beta^2}{2}\big[1 + \tfrac{(p-1)^2}{n-1}\big] u^{-2\beta-2} f^{p/2+2} + \beta\big[1 + \tfrac{(p-1)^2}{n-1}\big] u^{-\beta-1} f^{p/2}\langle \nabla F, \nabla u\rangle + 2a\beta\frac{p-1}{n-1} u^{q-2\beta-1} f^2. \tag{3-41}$$

We plug (3-41) into (3-16) to derive

$$\frac{1}{\alpha} F^{2-\alpha}\mathcal{L}_p(F^\alpha) \geq -2(n-1)\kappa u^{-2\beta} f^{p/2+1} + \beta(p-1)\frac{p-n}{n-1} u^{-\beta-1} f^{p/2}\langle \nabla F, \nabla u\rangle + \big(\alpha + \tfrac{p}{2} - \tfrac{3}{2}\big) f^{p/2-1}|\nabla F|^2 + (\alpha-1)(p-2) f^{p/2-2}\langle \nabla F, \nabla u\rangle^2 + 2a\big[\tfrac{\beta}{2}\tfrac{n+1}{n-1}(p-1) - q\big] u^{q-2\beta-1} f^2 + \frac{\beta}{2}(p-1)\big(2 + \tfrac{p-n}{n-1}\beta\big) u^{-2\beta-2} f^{p/2+2}. \tag{3-42}$$

Noting that $p > 1$, $\alpha > 1$ and $f^{p/2-1}|\nabla F|^2 \geq f^{p/2-2}\langle \nabla F, \nabla u\rangle^2$, we arrive at

$$\big(\alpha + \tfrac{p}{2} - \tfrac{3}{2}\big) f^{p/2-1}|\nabla F|^2 + (\alpha-1)(p-2) f^{p/2-2}\langle \nabla F, \nabla u\rangle^2 \geq \big(\alpha - \tfrac{1}{2}\big)(p-1) f^{p/2-2}\langle \nabla F, \nabla u\rangle^2. \tag{3-43}$$

In view of (3-42) and (3-43), we can derive the desired inequality and complete the proof of Lemma 3.5. □

By using Lemmas 3.4 and 3.5, we can achieve the following pointwise estimate of $\mathcal{L}_p(F^\alpha)$.

Lemma 3.6. *Let u be a positive solution of equation (1-1) in $\Omega \subset M$ with* $\mathrm{Ric} \geq -(n-1)\kappa$. *Set*

$$h = \beta(p-1)\big[\tfrac{p-n}{(n-1)^2}\beta + \tfrac{2}{n-1}\big] \tag{3-44}$$

and suppose that β *satisfies*

$$\beta \in \begin{cases} \left(0, \frac{2(n-1)}{n-p}\right) & \text{if } 1 < p < n, \\ (0, +\infty) & \text{if } p \geq n. \end{cases} \tag{3-45}$$

If the constants a, p *and* q *satisfy either*

$$\frac{\beta}{2} \cdot \frac{n+1}{n-1}(p-1) - h^{1/2} < q < \frac{\beta}{2} \cdot \frac{n+1}{n-1}(p-1) + h^{1/2} \quad (a \neq 0)$$

or

$$a\left[\frac{\beta}{2} \cdot \frac{n+1}{n-1}(p-1) - q\right] \geq 0,$$

then there exists $\alpha > 1$ *such that*

$$\begin{aligned} \tfrac{1}{\alpha}\mathcal{L}(F^\alpha) \geq{}& -2(n-1)\kappa u^{(p/2-1)\beta} F^{\alpha+p/2-1} \\ &+ Au^{(p/2)\beta-2} F^{\alpha+p/2} - Bu^{(\beta/2)(p-1)-1} F^{\alpha+(p-3)/2}|\nabla F| \end{aligned} \tag{3-46}$$

pointwise in $\{x \in \Omega : f(x) > 0\}$, *where* A *is a positive constant and*

$$B = (p-1)\beta \frac{|n-p|}{n-1}.$$

Proof. Case 1: $\frac{\beta}{2} \cdot \frac{n+1}{n-1}(p-1) - h^{1/2} < q < \frac{\beta}{2} \cdot \frac{n+1}{n-1}(p-1) + h^{1/2}$, with $a \neq 0$.

This implies $\lim_{\alpha \to +\infty} H > 0$, where H is defined in (3-20). Thus, we can choose α_0 large enough that for any $\alpha > \alpha_0$,

$$H > 0. \tag{3-47}$$

Furthermore, we have

$$\begin{aligned} (p-1)\beta \tfrac{p-n}{n-1} u^{-\beta-1} f^{p/2}\langle \nabla F, \nabla u\rangle &\geq -(p-1)\beta \tfrac{|n-p|}{n-1} u^{-\beta-1} f^{(p+1)/2}|\nabla F| \\ &= -Bu^{-\beta-1} f^{(p+1)/2}|\nabla F|. \end{aligned} \tag{3-48}$$

Combining (3-21) and (3-48), we can infer that

$$\begin{aligned} \tfrac{1}{\alpha} F^{2-\alpha}\mathcal{L}(F^\alpha) \geq{}& -2(n-1)\kappa u^{-2\beta} f^{p/2+1} \\ &- Bu^{-\beta-1} f^{(p+1)/2}|\nabla F| + Hu^{-2\beta-2} f^{p/2+2} \quad (a \neq 0). \end{aligned} \tag{3-49}$$

Combining (3-47) and (3-49) leads to (3-46).

Case 2: $a\left[\frac{\beta}{2}\frac{n+1}{n-1}(p-1) - q\right] \geq 0$..

In this case we have

$$2a\left[\tfrac{\beta}{2}\tfrac{n+1}{n-1}(p-1) - q\right] u^{q-2\beta-1} f^2 \geq 0. \tag{3-50}$$

Combining (3-38), (3-48) and (3-50), we obtain

$$\begin{aligned} \tfrac{1}{\alpha} F^{2-\alpha}\mathcal{L}_p(F^\alpha) \geq{}& -2(n-1)\kappa u^{-2\beta} f^{p/2+1} - Bu^{-\beta-1} f^{(p+1)/2}|\nabla F| \\ &+ \tfrac{\beta}{2}(p-1)\left(2 + \tfrac{p-n}{n-1}\beta\right) u^{-2\beta-2} f^{p/2+2}. \end{aligned} \tag{3-51}$$

Since β satisfies (3-45), it is easy to see that

$$\frac{\beta}{2}(p-1)\left(2+\frac{p-n}{n-1}\beta\right)>0. \tag{3-52}$$

Combining the above, we complete the proof of Lemma 3.6. □

3.2. *Deducing the main integral inequality.*

Lemma 3.7. *Let M be a complete manifold with* $\mathrm{Ric}\geq -(n-1)\kappa$ *and let u be a positive solution of equation (1-1) in $B_{2R}(x_0)\subset M$. Then, there exist constants t large enough and $\mu_1>0$ such that*

$$\exp\{-C_n(1+\sqrt{\kappa}R)\}V^{2/n}R^{-2}\left\|F^{\frac{p}{4}+\frac{\alpha-1}{2}+\frac{t}{2}}\eta\right\|^2_{L^{2n/(n-2)}(\Omega)}+\mu t\phi_\beta^{\beta-2}\int_\Omega F^{t+\alpha+p/2}\eta^2$$
$$\leq\left((n-1)\mu_1 t\kappa+R^{-2}\right)\int_\Omega F^{t+\alpha+p/2-1}\eta^2+\mu_1\int_\Omega F^{t+p/2+\alpha-1}|\nabla\eta|^2,$$

where $\Omega=B_R(x_0)$, $\eta\in C_0^\infty(\Omega,\mathbb{R})$ is a nonnegative function and V is the volume of $B_R(x_0)$.

Proof. We choose a geodesic ball $\Omega=B_R(x_0)\subset M$ and a test function $\xi\cdot u^\lambda=F_\epsilon^t\eta^2\cdot u^\lambda$, where $\eta\in C_0^\infty(\Omega,\mathbb{R})$ is nonnegative, $F_\epsilon=(F-\epsilon)^+$, $\epsilon>0$, $t>1$ and $\lambda\in\mathbb{R}$ are to be determined later. It follows from (2-1) that

$$\begin{aligned}
\frac{1}{\alpha}\int_\Omega &\mathcal{L}_p(F^\alpha)\cdot\xi\cdot u^\lambda\\
&=-\int_\Omega\left\langle\nabla(\xi u^\lambda),\ f^{p/2-1}F^{\alpha-1}[\nabla F+(p-2)f^{-1}\langle\nabla u,\nabla F\rangle\nabla u]\right\rangle\\
&=-\int_\Omega f^{p/2-1}F^{\alpha-1}u^\lambda\langle\nabla F,\nabla\xi\rangle-\lambda\int_\Omega u^{\lambda-1}f^{p/2-1}F^{\alpha-1}\langle\nabla F,\nabla u\rangle\xi\\
&\quad-(p-2)\int_\Omega f^{p/2-2}F^{\alpha-1}u^\lambda\langle\nabla F,\nabla u\rangle\langle\nabla u,\nabla\xi\rangle\\
&\quad-(p-2)\lambda\int_\Omega f^{p/2-1}F^{\alpha-1}u^{\lambda-1}\langle\nabla F,\nabla u\rangle\xi\\
&=-\int_\Omega f^{p/2-1}F^{\alpha-1}u^\lambda\langle\nabla F,\nabla\xi\rangle-(p-1)\lambda\int_\Omega f^{p/2-1}F^{\alpha-1}u^{\lambda-1}\langle\nabla F,\nabla u\rangle\xi\\
&\quad-(p-2)\int_\Omega f^{p/2-2}F^{\alpha-1}u^\lambda\langle\nabla F,\nabla u\rangle\langle\nabla u,\nabla\xi\rangle.
\end{aligned}$$

Since $\xi=F_\epsilon^t\eta^2$, we can achieve that

$$\begin{aligned}
\frac{1}{\alpha}\int_\Omega &\mathcal{L}_p(F^\alpha)\cdot F_\epsilon^t\eta^2\cdot u^\lambda\\
&=-\int_\Omega f^{p/2-1}F^{\alpha-1}u^\lambda\left\langle\nabla F,\ tF_\epsilon^{t-1}\eta^2\nabla F+2F_\epsilon^t\eta\nabla\eta\right\rangle\\
&\quad-(p-1)\lambda\int_\Omega f^{p/2-1}F^{\alpha-1}u^{\lambda-1}\langle\nabla F,\nabla u\rangle F_\epsilon^t\eta^2\\
&\quad-(p-2)\int_\Omega f^{p/2-2}F^{\alpha-1}u^\lambda\langle\nabla F,\nabla u\rangle\left\langle\nabla u,\ tF_\epsilon^{t-1}\eta^2\nabla F+2F_\epsilon^t\eta\nabla\eta\right\rangle
\end{aligned} \tag{3-53}$$

$$
\begin{aligned}
= -t\int_\Omega & u^{(p/2-1)\beta+\lambda} F^{p/2+\alpha-2} F_\epsilon^{t-1} |\nabla F|^2 \eta^2 \\
& - 2\int_\Omega u^{(p/2-1)\beta+\lambda} F^{p/2+\alpha-2} F_\epsilon^{t} \langle \nabla F, \nabla \eta\rangle \eta \\
& - (p-1)\lambda \int_\Omega u^{(p/2-1)\beta+\lambda-1} F^{p/2+\alpha-2} F_\epsilon^{t} \langle \nabla F, \nabla u\rangle \eta^2 \\
& - (p-2)t \int_\Omega u^{(p/2-2)\beta+\lambda} F^{p/2+\alpha-3} F_\epsilon^{t-1} \langle \nabla F, \nabla u\rangle^2 \eta^2 \\
& - 2(p-2) \int_\Omega u^{(p/2-2)\beta+\lambda} F^{p/2+\alpha-3} F_\epsilon^{t} \langle \nabla F, \nabla u\rangle \langle \nabla u, \nabla \eta\rangle \eta.
\end{aligned}
$$

Combining (3-46) and (3-53), we achieve

$$
\begin{aligned}
-t\int_\Omega & u^{(p/2-1)\beta+\lambda} F^{p/2+\alpha-2} F_\epsilon^{t-1} |\nabla F|^2 \eta^2 \\
& -2\int_\Omega u^{(p/2-1)\beta+\lambda} F^{p/2+\alpha-2} F_\epsilon^{t} \langle \nabla F, \nabla \eta\rangle \eta \\
& - (p-1)\lambda \int_\Omega u^{(p/2-1)\beta+\lambda-1} F^{p/2+\alpha-2} F_\epsilon^{t} \langle \nabla F, \nabla u\rangle \eta^2 \\
& - (p-2)t \int_\Omega u^{(p/2-2)\beta+\lambda} F^{p/2+\alpha-3} F_\epsilon^{t-1} \langle \nabla F, \nabla u\rangle^2 \eta^2 \\
& - 2(p-2) \int_\Omega u^{(p/2-2)\beta+\lambda} F^{p/2+\alpha-3} F_\epsilon^{t} \langle \nabla F, \nabla u\rangle \langle \nabla u, \nabla \eta\rangle \eta \\
& \geq -2(n-1)\kappa \int_\Omega u^{(p/2-1)\beta+\lambda} F^{\alpha+p/2-1} F_\epsilon^{t} \eta^2 \\
& \quad + A\int_\Omega u^{(p/2)\beta-2+\lambda} F^{\alpha+p/2} F_\epsilon^{t}\eta^2 - B\int_\Omega u^{\beta/2(p-1)-1+\lambda} F^{\alpha+(p-3)/2} |\nabla F| F_\epsilon^{t}\eta^2.
\end{aligned}
$$

From this, we obtain

$$
\begin{aligned}
\text{(3-54)}\quad 2(n-1)\kappa & \int_\Omega u^{(p/2-1)\beta+\lambda} F^{\alpha+p/2-1} F_\epsilon^{t} \eta^2 \\
& - 2\int_\Omega u^{(p/2-1)\beta+\lambda} F^{p/2+\alpha-2} F_\epsilon^{t} \langle \nabla F, \nabla \eta\rangle \eta \\
& - 2(p-2) \int_\Omega u^{(p/2-2)\beta+\lambda} F^{p/2+\alpha-3} F_\epsilon^{t} \langle \nabla F, \nabla u\rangle \langle \nabla u, \nabla \eta\rangle \eta \\
& + \big(B + (p-1)|\lambda|\big) \int_\Omega u^{(\beta/2)(p-1)-1+\lambda} F^{\alpha+u(p-3)/2} |\nabla F| F_\epsilon^{t}\eta^2 \\
\geq t & \int_\Omega u^{(p/2-1)\beta+\lambda} F^{p/2+\alpha-2} F_\epsilon^{t-1} |\nabla F|^2 \eta^2 \\
& + (p-2)t\int_\Omega u^{(p/2-2)\beta+\lambda} F^{p/2+\alpha-3} F_\epsilon^{t-1} \langle \nabla F, \nabla u\rangle^2 \eta^2 \\
& + A\int_\Omega u^{(p/2)\beta-2+\lambda} F^{\alpha+p/2} F_\epsilon^{t}\eta^2.
\end{aligned}
$$

Set

$$
\begin{aligned}
\text{(3-55)}\quad L_p = t\int_\Omega & u^{(p/2-1)\beta+\lambda} F^{p/2+\alpha-2} F_\epsilon^{t-1} |\nabla F|^2 \eta^2 \\
& + (p-2)t\int_\Omega u^{(p/2-2)\beta+\lambda} F^{p/2+\alpha-3} F_\epsilon^{t-1} \langle \nabla F, \nabla u\rangle^2 \eta^2.
\end{aligned}
$$

Then we need to consider two cases:

Case i: $p \geqslant 2$. Here we get from (3-34) that

$$L_p \geq t \int_\Omega u^{(p/2-1)\beta+\lambda} F^{p/2+\alpha-2} F_\epsilon^{t-1} |\nabla F|^2 \eta^2. \tag{3-56}$$

Case ii: $1 < p < 2$. Here, again from (3-34), we get

$$\begin{aligned} L_p &= t \int_\Omega u^{(p/2-1)\beta+\lambda} F^{p/2+\alpha-2} F_\epsilon^{t-1} |\nabla F|^2 \eta^2 \\ &\qquad - (2-p) t \int_\Omega u^{(p/2-2)\beta+\lambda} F^{p/2+\alpha-3} F_\epsilon^{t-1} \langle \nabla F, \nabla u\rangle^2 \eta^2 \\ &\geq t \int_\Omega u^{(p/2-1)\beta+\lambda} F^{p/2+\alpha-2} F_\epsilon^{t-1} |\nabla F|^2 \eta^2 \\ &\qquad - (2-p) t \int_\Omega u^{(p/2-1)\beta+\lambda} F^{p/2+\alpha-2} F_\epsilon^{t-1} |\nabla F|^2 \eta^2 \\ &= (p-1) t \int_\Omega u^{(p/2-1)\beta+\lambda} F^{p/2+\alpha-2} F_\epsilon^{t-1} |\nabla F|^2 \eta^2. \end{aligned} \tag{3-57}$$

Set

$$\theta(p) = \begin{cases} p-1 & \text{if } 1 < p < 2; \\ 1 & \text{if } p \geq 2. \end{cases} \tag{3-58}$$

Combining (3-55), (3-56), (3-57) and (3-58) yields

$$L_p \geq \theta(p) t \int_\Omega u^{(p/2-1)\beta+\lambda} F^{p/2+\alpha-2} F_\epsilon^{t-1} |\nabla F|^2 \eta^2. \tag{3-59}$$

By combining (3-54) and (3-59), we have

$$\begin{aligned} &2(n-1)\kappa \int_\Omega u^{(p/2-1)\beta+\lambda} F^{\alpha+p/2-1} F_\epsilon^t \eta^2 \\ &\quad - 2 \int_\Omega u^{(p/2-1)\beta+\lambda} F^{p/2+\alpha-2} F_\epsilon^t \langle \nabla F, \nabla\eta\rangle \eta \\ &\quad -2(p-2) \int_\Omega u^{(p/2-2)\beta+\lambda} F^{p/2+\alpha-3} F_\epsilon^t \langle \nabla F, \nabla u\rangle \langle \nabla u, \nabla\eta\rangle \eta \\ &\quad + \big(B + (p-1)|\lambda|\big) \int_\Omega u^{(\beta/2)(p-1)-1+\lambda} F^{\alpha+(p-3)/2} |\nabla F| F_\epsilon^t \eta^2 \\ &\geq \theta(p) t \int_\Omega u^{(p/2-1)\beta+\lambda} F^{p/2+\alpha-2} F_\epsilon^{t-1} |\nabla F|^2 \eta^2 + A \int_\Omega u^{(p/2)\beta-2+\lambda} F^{\alpha+p/2} F_\epsilon^t \eta^2. \end{aligned}$$

By letting $\epsilon \to 0^+$, we obtain

$$\begin{aligned} &2(n-1)\kappa \int_\Omega u^{(p/2-1)\beta+\lambda} F^{t+\alpha+p/2-1} \eta^2 \\ &\qquad - 2 \int_\Omega u^{(p/2-1)\beta+\lambda} F^{t+p/2+\alpha-2} \langle \nabla F, \nabla\eta\rangle \eta \\ &\qquad - 2(p-2) \int_\Omega u^{(p/2-2)\beta+\lambda} F^{t+p/2+\alpha-3} \langle \nabla F, \nabla u\rangle \langle \nabla u, \nabla\eta\rangle \eta \\ &\qquad + \big(B + (p-1)|\lambda|\big) \int_\Omega u^{(\beta/2)(p-1)-1+\lambda} F^{t+\alpha+(p-3)/2} |\nabla F| \eta^2 \\ &\quad \geq \theta(p) t \int_\Omega u^{(p/2-1)\beta+\lambda} F^{t+p/2+\alpha-3} |\nabla F|^2 \eta^2 + A \int_\Omega u^{(p/2)\beta-2+\lambda} F^{t+\alpha+p/2} \eta^2. \end{aligned} \tag{3-60}$$

By the absolute value inequality and the Cauchy inequality, we have

$$\begin{aligned}(B+(p-1)|\lambda|)\int_\Omega u^{(\beta/2)(p-1)-1+\lambda}F^{t+\alpha+(p-3)/2}|\nabla F|\eta^2 \\ \le \frac{\theta(p)}{4}t\int_\Omega u^{(p/2-1)\beta+\lambda}F^{t+p/2+\alpha-3}|\nabla F|^2\eta^2 \\ + \frac{1}{\theta(p)t}\big(B+(p-1)|\lambda|\big)^2\int_\Omega u^{(p/2)\beta-2+\lambda}F^{t+\alpha+p/2}\eta^2,\end{aligned}\tag{3-61}$$

$$\begin{aligned}-2\int_\Omega u^{(p/2-1)\beta+\lambda}F^{t+p/2+\alpha-2}\langle\nabla F,\nabla\eta\rangle\eta \\ \le 2\int_\Omega u^{(p/2-1)\beta+\lambda}F^{t+p/2+\alpha-2}|\nabla F||\nabla\eta|\eta \\ \le \frac{\theta(p)}{4}t\int_\Omega u^{(p/2-1)\beta+\lambda}F^{t+p/2+\alpha-3}|\nabla F|^2\eta^2 \\ + \frac{4}{\theta(p)t}\int_\Omega u^{(p/2-1)\beta+\lambda}F^{t+p/2+\alpha-1}|\nabla\eta|^2,\end{aligned}\tag{3-62}$$

$$\begin{aligned}-2(p-2)\int_\Omega u^{(p/2-2)\beta+\lambda}F^{t+p/2+\alpha-3}\langle\nabla F,\nabla u\rangle\langle\nabla u,\nabla\eta\rangle\eta \\ \le 2|p-2|\int_\Omega u^{(p/2-1)\beta+\lambda}F^{t+p/2+\alpha-2}|\nabla F||\nabla\eta|\eta \\ \le \frac{\theta(p)}{4}t\int_\Omega u^{(p/2-1)\beta+\lambda}F^{t+p/2+\alpha-3}|\nabla F|^2\eta^2 \\ + \frac{4}{\theta(p)t}(p-2)^2\int_\Omega u^{(p/2-1)\beta+\lambda}F^{t+p/2+\alpha-1}|\nabla\eta|^2.\end{aligned}\tag{3-63}$$

Substituting (3-61), (3-62) and (3-63) into (3-60), we obtain

$$\begin{aligned}\frac{\theta(p)}{4}t\int_\Omega u^{(p/2-1)\beta+\lambda}F^{t+p/2+\alpha-3}|\nabla F|^2\eta^2 + A\int_\Omega u^{(p/2)\beta-2+\lambda}F^{t+\alpha+p/2}\eta^2 \\ \le 2(n-1)\kappa\int_\Omega u^{(p/2-1)\beta+\lambda}F^{t+\alpha+p/2-1}\eta^2 \\ + \frac{1}{\theta(p)t}\big(B+(p-1)|\lambda|\big)^2\int_\Omega u^{(p/2)\beta-2+\lambda}F^{t+\alpha+p/2}\eta^2 \\ + \frac{4}{\theta(p)t}\big(1+(p-2)^2\big)\int_\Omega u^{(p/2-1)\beta+\lambda}F^{t+p/2+\alpha-1}|\nabla\eta|^2.\end{aligned}\tag{3-64}$$

Now we choose t large enough that

$$\frac{1}{\theta(p)t}\big(B+(p-1)|\lambda|\big)^2 \le \frac{A}{2}.\tag{3-65}$$

It follows from (3-64) and (3-65) that

$$\begin{aligned}\frac{\theta(p)}{4}t\int_\Omega u^{(p/2-1)\beta+\lambda}F^{t+p/2+\alpha-3}|\nabla F|^2\eta^2 + \frac{A}{2}\int_\Omega u^{(p/2)\beta-2+\lambda}F^{t+\alpha+p/2}\eta^2 \\ \le 2(n-1)\kappa\int_\Omega u^{(p/2-1)\beta+\lambda}F^{t+\alpha+p/2-1}\eta^2 \\ + \frac{4}{\theta(p)t}\big(1+(p-2)^2\big)\int_\Omega u^{(p/2-1)\beta+\lambda}F^{t+p/2+\alpha-1}|\nabla\eta|^2.\end{aligned}$$

By letting $\lambda = \beta\left(1-\frac{p}{2}\right)$, we obtain

$$\text{(3-66)}\quad \frac{\theta(p)}{4}t\int_\Omega F^{t+p/2+\alpha-3}|\nabla F|^2\eta^2 + A/2\int_\Omega u^{\beta-2}F^{t+\alpha+p/2}\eta^2 \le 2(n-1)\kappa\int_\Omega F^{t+\alpha+p/2-1}\eta^2 + \frac{4}{\theta(p)t}(1+(p-2)^2)\int_\Omega F^{t+p/2+\alpha-1}|\nabla\eta|^2.$$

On the other hand, we have

$$\text{(3-67)}\quad |\nabla(F^{p/4+(\alpha-1+t)/2}\eta)|^2 = \left|\tfrac{1}{4}(p+2t+2\alpha-2)F^{p/4+(t+\alpha-3)/2}\eta\nabla F + F^{p/4+(t-a-1)/2}\nabla\eta\right|^2 \le \tfrac{1}{8}(p+2t+2\alpha-2)^2F^{p/2+t+\alpha-3}|\nabla F|^2\eta^2 + 2F^{p/2+t+\alpha-1}|\nabla\eta|^2.$$

Substituting (3-67) into (3-66) gives

$$\text{(3-68)}\quad \frac{2\theta(p)t}{(p+2t+2\alpha-2)^2}\int_\Omega\left|\nabla(F^{p/4+(\alpha-1+t)/2}\eta)\right|^2 + \frac{A}{2}\int_\Omega u^{\beta-2}F^{t+\alpha+p/2}\eta^2 \le 2(n-1)\kappa\int_\Omega F^{t+\alpha+p/2-1}\eta^2 + \left\{\frac{4}{\theta(p)t}(1+(p-2)^2) + \frac{4\theta(p)t}{(p+2t+2\alpha-2)^2}\right\}\int_\Omega F^{t+p/2+\alpha-1}|\nabla\eta|^2.$$

At the same time, the Saloff-Coste Sobolev inequality implies

$$\text{(3-69)}\quad \exp\left\{-C_n(1+\sqrt{\kappa}R)\right\}V^{2/n}R^{-2}\left\|F^{p/4+(\alpha-1+t)/2}\eta\right\|^2_{L^{2n/(n-2)}(\Omega)} \le \int_\Omega\left|\nabla(F^{p/4+(\alpha-1+t)/2}\eta)\right|^2 + R^{-2}\int_\Omega F^{p/2+\alpha+t-1}\eta^2.$$

Now, we substitute (3-69) into (3-68) to obtain

$$\frac{2\theta(p)t}{(p+2t+2\alpha-2)^2}\exp\left\{-C_n(1+\sqrt{\kappa}R)\right\}V^{2/n}R^{-2}\left\|F^{p/4+(\alpha-1+t)/2}\eta\right\|^2_{L^{2n/(n-2)}(\Omega)} \le -\frac{A}{2}\int_\Omega u^{\beta-2}F^{t+\alpha+p/2}\eta^2 + \left[2(n-1)\kappa + \frac{2\theta(p)t}{(p+2t+2\alpha-2)^2R^2}\right]\int_\Omega F^{t+\alpha+p/2-1}\eta^2 + \left\{4/\theta(p)t\big(1+(p-2)^2\big) + \frac{4\theta(p)t}{(p+2t+2\alpha-2)^2}\right\}\int_\Omega F^{t+\frac{p}{2}+\alpha-1}|\nabla\eta|^2.$$

We divide both sides of this inequality by $\dfrac{2\theta(p)t}{(p+2t+2\alpha-2)^2}$ to obtain

$$\text{(3-70)}\quad \exp\left\{-C_n(1+\sqrt{\kappa}R)\right\}V^{2/n}R^{-2}\left\|F^{p/4+(\alpha-1+t)/2}\eta\right\|^2_{L^{2n/(n-2)}(\Omega)} + \frac{A(p+2t+2\alpha-2)^2}{4\theta(p)t}\int_\Omega u^{\beta-2}F^{t+\alpha+p/2}\eta^2 \le \left[(n-1)\frac{(p+2t+2\alpha-2)^2}{\theta(p)t}\kappa + \frac{1}{R^2}\right]\int_\Omega F^{t+\alpha+p/2-1}\eta^2 + \left\{\frac{2(p+2t+2\alpha-2)^2}{\theta^2(p)t^2}(1+(p-2)^2)+2\right\}\int_\Omega F^{t+p/2+\alpha-1}|\nabla\eta|^2.$$

Set

$$\mu_1 = \sup_{t\in[1,\infty)} \frac{2(p+2t+2\alpha-2)^2}{\theta^2(p)t^2}(1+(p-2)^2)+2, \quad \mu = A \inf_{t\in[1,\infty)} \frac{(p+2t+2\alpha-2)^2}{4\theta(p)t^2}.$$

Then μ_1 and μ are both finite positive constants. Combining their definitions with (3-70) yields

$$\exp\{-C_n(1+\sqrt{\kappa}R)\}V^{2/n}R^{-2}\|F^{p/4+(\alpha-1+t)/2}\eta\|^2_{L^{2n/(n-2)}(\Omega)} + \mu t\int_\Omega u^{\beta-2}F^{t+\alpha+p/2}\eta^2 \leq ((n-1)\mu_1 t\kappa + R^{-2})\int_\Omega F^{t+\alpha+p/2-1}\eta^2 + \mu_1\int_\Omega F^{t+p/2+\alpha-1}|\nabla\eta|^2. \tag{3-71}$$

Set

$$\phi_\beta = \begin{cases} \sup_\Omega u & \text{if } 0<\beta<2, \\ 1 & \text{if } \beta = 2, \\ \inf_\Omega u & \text{if } \beta > 2. \end{cases} \tag{3-72}$$

With this notation we obtain an inequality identical to (3-71), except that the summand on the second line is replaced by

$$\mu t\,\phi_\beta^{\beta-2}\int_\Omega F^{t+\alpha+p/2}\eta^2. \tag{3-73}$$

This is sufficient to prove the lemma. □

3.3. L^{β_1}-bound of gradient in a geodesic ball with radius $3R/4$.

Lemma 3.8. *Let (M, g) be an n-dimensional $(n \geq 3)$ complete manifold with* $\mathrm{Ric} \geq -(n-1)\kappa$, *where κ is a nonnegative constant. Furthermore, suppose that a, q, p and β satisfy the conditions stated in Lemma 3.6. Let*

$$\beta_1 = \frac{n}{n-2}\cdot\frac{p+2t_0+2\alpha-2}{2}. \tag{3-74}$$

If u is a positive solution to equation (1-1) on the geodesic ball $B(x_0, 2R) \subset M$, then for t_0 large enough there exists $\mathcal{C} = \mathcal{C}(n, p, q, \beta) > 0$ such that

$$\|F\|_{L^{\beta_1}(B(x_0,3R/4))} \leq \mathcal{C}V^{1/\beta_1}(\kappa+R^{-2})\phi^{2-\beta}, \tag{3-75}$$

where V is the volume of the geodesic ball $B_R(x_0)$ and ϕ is defined in (3-72).

Proof. As observed at the end of the previous proof, we have

$$\exp\{-C_n(1+\sqrt{\kappa}R)\}V^{2/n}R^{-2}\|F^{p/4+(\alpha-1+t_0)/2}\eta\|^2_{L^{2n/(n-2)}(\Omega)} + \mu t_0\phi_\beta^{\beta-2}\int_\Omega F^{t_0+\alpha+p/2}\eta^2 \leq ((n-1)\mu_1 t_0\kappa + R^{-2})\int_\Omega F^{t_0+\alpha+p/2-1}\eta^2 + \mu_1\int_\Omega F^{t_0+p/2+\alpha-1}|\nabla\eta|^2, \tag{3-76}$$

where $t = t_0$ satisfies (3-65). Define

$$\Omega_1 := \Big\{x : F \geq \Big(2(n-1)\kappa\frac{\mu_1}{\mu} + \frac{2}{\mu t_0 R^2}\Big)\phi^{2-\beta},\ x \in \Omega\Big\}. \tag{3-77}$$

Then we have

$$\big((n-1)\mu_1 t_0\kappa + R^{-2}\big)\int_{\Omega_1} F^{t_0+\alpha+p/2-1}\eta^2 \leq \frac{\mu}{2}t_0\phi_\beta^{\beta-2}\int_\Omega F^{t_0+\alpha+p/2}\eta^2. \tag{3-78}$$

Set $\Omega_2 := \Omega \setminus \Omega_1 = \Big\{x : F < \Big(2(n-1)\kappa\frac{\mu_1}{\mu} + \frac{2}{\mu t_0 R^2}\Big)\phi^{2-\beta},\ x \in \Omega\Big\}$. Then

$$\begin{aligned}\big((n-1)\mu_1 t_0\kappa + R^{-2}\big)&\int_{\Omega_2} F^{t_0+\alpha+p/2-1}\eta^2 \\ &\leq \big((n-1)\mu_1 t_0\kappa + R^{-2}\big)\Big\{\Big(2(n-1)\kappa\frac{\mu_1}{\mu} + \frac{2}{\mu t_0 R^2}\Big)\phi^{2-\beta}\Big\}^{t_0+\alpha+p/2-1}V,\end{aligned} \tag{3-79}$$

where V is the volume of $\Omega = B(x_0, R)$. Combining (3-78) and (3-79), we obtain

$$\begin{aligned}\big((n-1)\mu_1 t_0\kappa + R^{-2}\big)&\int_\Omega F^{t_0+\alpha+p/2-1}\eta^2 - \frac{\mu}{2}t_0\phi_\beta^{\beta-2}\int_\Omega F^{t_0+\alpha+p/2}\eta^2 \\ &\leq \big((n-1)\mu_1 t_0\kappa + R^{-2}\big)\Big\{\Big[2(n-1)\kappa\frac{\mu_1}{\mu} + \frac{2}{\mu t_0 R^2}\Big]\phi^{2-\beta}\Big\}^{t_0+\alpha+p/2-1}V.\end{aligned} \tag{3-80}$$

We set $\Omega_1 = B(x_0, 3R/4)$ and choose $\eta_1 \in C_0^\infty(\Omega)$ satisfying

$$0 \leq \eta_1 \leq 1, \qquad \eta_1 \equiv 1 \quad \text{in } \Omega_1, \qquad |\nabla\eta_1| \leq C/R,$$

and let

$$\eta = \eta_1^{t_0+p/2+\alpha}.$$

Then, we have

$$\begin{aligned}\mu_1\int_\Omega F^{t_0+p/2+\alpha-1}|\nabla\eta|^2 &= \mu_1\big(t_0+\tfrac{p}{2}+\alpha\big)^2\int_\Omega F^{t_0+p/2+\alpha-1}\eta_1^{p+2\alpha-2+2t_0}|\nabla\eta_1|^2 \\ &\leq \mu_1\big(t_0+\tfrac{p}{2}+\alpha\big)^2\frac{C^2}{R^2}\int_\Omega F^{t_0+p/2+\alpha-1}\eta_1^{p+2\alpha-2+2t_0}.\end{aligned} \tag{3-81}$$

By Hölder's inequality, (3-81) can be written as

$$\begin{aligned}\mu_1&\int_\Omega F^{t_0+p/2+\alpha-1}|\nabla\eta|^2 \\ &\leq \mu_1\big(t_0+\tfrac{p}{2}+\alpha\big)^2\frac{C^2}{R^2}\Big(\int_\Omega F^{t_0+p/2+\alpha}\eta_1^{2t_0+p+2\alpha}\Big)^{\frac{2t_0+p+2\alpha-2}{2t_0+p+2\alpha}}V^{\frac{2}{2t_0+p+2\alpha}} \\ &= \mu_1\big(t_0+\tfrac{p}{2}+\alpha\big)^2\frac{C^2}{R^2}\Big(\int_\Omega F^{t_0+p/2+\alpha}\eta^2\Big)^{\frac{2t_0+p+2\alpha-2}{2t_0+p+2\alpha}}V^{\frac{2}{2t_0+p+2\alpha}}.\end{aligned} \tag{3-82}$$

By using Young's inequality, we can write (3-82) as

$$\begin{aligned}\mu_1\int_\Omega F^{t_0+p/2+\alpha-1}|\nabla\eta|^2 &\leq \tfrac{1}{2}\mu t_0\phi^{\beta-2}\int_\Omega F^{t_0+p/2+\alpha}\eta^2 \\ &+ \tfrac{2}{2t_0+p+2\alpha}\Big[\mu_1\big(t_0+\tfrac{p}{2}+\alpha\big)^2\tfrac{C^2}{R^2}\Big]^{t_0+p/2+\alpha}\Big[\tfrac{2(2t_0+p+2\alpha-2)}{(2t_0+p+2\alpha)\mu t_0}\phi^{2-\beta}\Big]^{t_0+p/2+\alpha-1}V.\end{aligned} \tag{3-83}$$

Substituting (3-80) and (3-83) into (3-76) gives

$$\begin{aligned}(3\text{-}84)\quad &\exp\left\{-C_n(1+\sqrt{\kappa}R)\right\} V^{2/n} R^{-2} \left\| F^{p/4+(\alpha-1+t_0)/2}\eta\right\|^2_{L^{2n/(n-2)}(\Omega)} \\ &\le \left((n-1)\mu_1 t_0\kappa + R^{-2}\right)\left\{\left[2(n-1)\kappa\tfrac{\mu_1}{\mu} + \tfrac{2}{\mu t_0 R^2}\right]\phi^{2-\beta}\right\}^{t_0+\alpha+p/2-1} V \\ &\quad + \tfrac{2}{2t_0+p+2\alpha}\left[\mu_1\left(t_0+\tfrac{p}{2}+\alpha\right)^2\tfrac{C^2}{R^2}\right]^{t_0+p/2+\alpha}\left[\tfrac{2(2t_0+p+2\alpha-2)}{(2t_0+p+2\alpha)\mu t_0}\phi^{2-\beta}\right]^{t_0+p/2+\alpha-1} V.\end{aligned}$$

Taking $\dfrac{2}{2t_0+p+2\alpha-2}$ powers on both sides of (3-84), we obtain

$$\begin{aligned}&\|F\|_{L^{\beta_1}(B(x_0,3R/4))} \\ &\le \left\| F^{p/4+(\alpha-1+t_0)/2}\eta\right\|^{4/2t_0+p+2\alpha-2}_{L^{2n/(n-2)}(\Omega)} \\ &\le \exp\left\{\frac{2C_n(1+\sqrt{\kappa}R)}{2t_0+p+2\alpha-2}\right\} V^{1/\beta_1} \\ &\quad\times\Bigg\{\left((n-1)\mu_1 t_0 R^2\kappa+1\right)\left(\left[2(n-1)\kappa\tfrac{\mu_1}{\mu}+\tfrac{2}{\mu t_0 R^2}\right]\phi^{2-\beta}\right)^{t_0+\alpha+p/2-1} \\ &\qquad + \frac{2R^2}{2t_0+p+2\alpha}\left[\mu_1\left(t_0+\tfrac{p}{2}+\alpha\right)^2\tfrac{C^2}{R^2}\right]^{t_0+p/2+\alpha} \\ &\qquad\qquad\times\left[\tfrac{2(2t_0+p+2\alpha-2)}{(2t_0+p+2\alpha)\mu t_0}\phi^{2-\beta}\right]^{t_0+p/2+\alpha-1}\Bigg\}^{\frac{2}{2t_0+p+2\alpha-2}},\end{aligned}$$

where

$$\beta_1 = \frac{n}{n-2}\cdot\frac{p+2t_0+2\alpha-2}{2}.$$

Using the fact that $(a_1+a_2)^{b_1} \le 2^{b_1}(a_1^{b_1}+a_2^{b_1})$, valid for $a_i \ge 0$ and $b_1 > 0$, we infer from the preceding inequality that

$$(3\text{-}85)\qquad \|F\|_{L^{\beta_1}(B(x_0,3R/4))} \le \exp\left\{\frac{2C_n(1+\sqrt{\kappa}R)}{2t_0+p+2\alpha-2}\right\} V^{1/\beta_1}(I_1+I_2),$$

where

$$I_2 := 2^{2/2t_0+p+2\alpha-2}\left(\frac{2}{2t_0+p+2\alpha}\right)^{\frac{2}{2t_0+p+2\alpha-2}}\left(\mu_1(t_0+\tfrac{p}{2}+\alpha)^2C^2\right)^{\frac{2t_0+p+2\alpha}{2t_0+p+2\alpha-2}} \frac{2(2t_0+p+2\alpha-2)}{(2t_0+p+2\alpha)\mu t_0}\phi^{2-\beta}R^{-2}$$

will be estimated later, and

(3-86)

$$\begin{aligned}I_1 &:= 2^{\frac{2}{2t_0+p+2\alpha-2}}\left((n-1)\mu_1 t_0 R^2\kappa+1\right)^{\frac{2}{2t_0+p+2\alpha-2}}\left(2(n-1)\kappa\tfrac{\mu_1}{\mu}+\tfrac{2}{\mu t_0 R^2}\right)\phi^{2-\beta} \\ &= 2^{\frac{2t_0+p+2\alpha}{2t_0+p+2\alpha-2}}\left((n-1)\mu_1 t_0 R^2\kappa+1\right)^{\frac{2t_0+p+2\alpha}{2t_0+p+2\alpha-2}} t_0^{-1}\mu^{-1}R^{-2}\phi^{2-\beta} \\ &\le 2^{\frac{2t_0+p+2\alpha}{2t_0+p+2\alpha-2}}\left((n-1)\mu_1 t_0+1\right)^{\frac{2t_0+p+2\alpha}{2t_0+p+2\alpha-2}}(1+R^2\kappa)^{\frac{2t_0+p+2\alpha}{2t_0+p+2\alpha-2}} t_0^{-1}\mu^{-1}R^{-2}\phi^{2-\beta} \\ &= 2^{\frac{2t_0+p+2\alpha}{2t_0+p+2\alpha-2}}\left((n-1)\mu_1+t_0^{-1}\right)^{\frac{2}{2t_0+p+2\alpha-2}}(1+R^2\kappa)^{\frac{2t_0+p+2\alpha}{2t_0+p+2\alpha-2}}\mu^{-1}R^{-2}\phi^{2-\beta}.\end{aligned}$$

Noticing that

$$\lim_{t_0\to+\infty} 2^{\frac{2t_0+p+2\alpha}{2t_0+p+2\alpha-2}}\big[(n-1)\mu_1+t_0^{-1}\big]^{\frac{2}{2t_0+p+2\alpha-2}}=2,$$

we can verify that

$$\sup_{t_0\in[1,+\infty)} 2^{\frac{2t_0+p+2\alpha}{2t_0+p+2\alpha-2}}\big[(n-1)\mu_1+t_0^{-1}\big]^{\frac{2}{2t_0+p+2\alpha-2}}<+\infty. \tag{3-87}$$

Combining (3-86) and (3-87) leads to

$$I_1\le \mathcal{C}_{I_1}(1+R^2\kappa)^{\frac{2t_0+p+2\alpha}{2t_0+p+2\alpha-2}}\mu^{-1}R^{-2}\phi^{2-\beta}, \tag{3-88}$$

where $\mathcal{C}_{I_1}$ is a positive constant which depends only on n, p and q.

Similarly, we have

$$I_2\le \mathcal{C}_{I_2}\mu^{-1}t_0R^{-2}\phi^{2-\beta}, \tag{3-89}$$

where $\mathcal{C}_{I_2}$ is a positive constant which depends only on n, p and q.

Substituting (3-88) and (3-89) into (3-85), we obtain

$$\begin{aligned}\|F\|_{L^{\beta_1}(B(x_0,3R/4))}\le{}&\exp\left\{\frac{2C_n(1+\sqrt{\kappa}R)}{2t_0+p+2\alpha-2}\right\}\\&\times V^{1/\beta_1}\Big[\mathcal{C}_{I_1}(1+\kappa R^2)^{\frac{2t_0+p+2\alpha}{2t_0+p+2\alpha-2}}+\mathcal{C}_{I_2}t_0\Big]R^{-2}\mu^{-1}\phi^{2-\beta}.\end{aligned}$$

We can derive from this inequality that

$$\begin{aligned}\|F\|_{L^{\beta_1}(B(x_0,3R/4))}\le{}&\mathcal{C}_1\exp\left\{\frac{2C_n(1+\sqrt{\kappa}R)}{2t_0+p+2\alpha-2}\right\}\\&V^{1/\beta_1}\Big[(1+\kappa R^2)^{\frac{2t_0+p+2\alpha}{2t_0+p+2\alpha-2}}+t_0\Big]R^{-2}\phi^{2-\beta},\end{aligned} \tag{3-90}$$

where $\mathcal{C}_1:=\max\{\mathcal{C}_{I_1},\mathcal{C}_{I_2}\}\mu^{-1}$ is a positive constant depending only on n, β, p, q.

Now, let t_0 satisfy (3-65) and

$$(1+\kappa R^2)\le t_0\le \mathcal{C}_0(1+\kappa R^2), \tag{3-91}$$

where $\mathcal{C}_0=\mathcal{C}_0(n,p,q,\beta)$ is a positive constant. Combining (3-90) and (3-91) gives

$$\|F\|_{L^{\beta_1}(B(x_0,3R/4))}\le \mathcal{C}V^{1/\beta_1}(\kappa+R^{-2})\phi^{2-\beta},$$

where $\mathcal{C}$ is a positive constant that depends only on n, β, p, q. This completes the proof of Lemma 3.8. □

3.4. *Moser iteration for positive solutions of* (1-1).

Lemma 3.9. *Let (M,g) be an n-dimensional $(n\ge 3)$ complete manifold with* $\mathrm{Ric}\ge-(n-1)\kappa$, *where κ is a nonnegative constant. Suppose that a, q, p and β satisfy the same conditions as in Lemma 3.6. Let*

$$\beta_1=\frac{n}{n-2}\cdot\frac{p+2t_0+2\alpha-2}{2}.$$

If u is a positive solution to equation (1-1) on the geodesic ball $B(x_0, 2R) \subset M$, then for t_0 large enough there exists $\mathcal{C} = \mathcal{C}(n, p, q) > 0$ such that

$$\|F\|_{L^\infty(B(x_0,R/2))} \le \mathcal{C} V^{-1/\beta_1} \|F\|_{L^{\beta_1}(\Omega_1)},$$

where V is the volume of the geodesic ball $B_R(x_0)$.

Proof. Recall the integral inequality (3-71) from the proof of Lemma 3.8 (page 417). By dropping the second nonnegative term in that inequality, we obtain

$$\exp\left\{-C_n(1+\sqrt{\kappa}R)\right\} V^{2/n} R^{-2} \left\| F^{p/4+(\alpha-1+t)/2}\eta \right\|^2_{L^{2n/(n-2)}(\Omega)} \le \left((n-1)\mu_1 t\kappa + R^{-2}\right) \int_\Omega F^{t+\alpha+p/2-1}\eta^2 + \mu_1 \int_\Omega F^{t+p/2+\alpha-1}|\nabla\eta|^2, \tag{3-92}$$

Then we set

$$r_m := \frac{R}{2} + \frac{R}{4^m} \quad \text{and} \quad \Omega_m := B(x_0, r_m),$$

and then choose $\eta_m \in C_0^\infty(\Omega_m)$ satisfying

$$0 \le \eta_m \le 1, \qquad \eta_m \equiv 1 \quad in\ B(x_0, r_{m+1}), \quad |\nabla\eta_m| \le C\frac{4^m}{R}.$$

Replacing η by η_m in (3-92), we can easily verify that

$$\exp\left\{-C_n(1+\sqrt{\kappa}R)\right\} V^{2/n} R^{-2} \left\| F^{p/4+(\alpha-1+t)/2} \right\|^2_{L^{2n/(n-2)}(\Omega_{m+1})} \le \left((n-1)\mu_1 t\kappa + R^{-2}\right) \int_{\Omega_m} F^{t+p/2+\alpha-1} + \mu_1 \frac{C^2 16^m}{R^2} \int_{\Omega_m} F^{t+p/2+\alpha-1}. \tag{3-93}$$

Next, we choose

$$\beta_1 = \left(t_0 + \tfrac{p}{2} + \alpha - 1\right) \frac{n}{n-2} \quad \text{and} \quad \beta_{m+1} = \frac{n\beta_m}{n-2},$$

and let $t = t_m$ such that

$$t_m + \tfrac{p}{2} + \alpha - 1 = \beta_m.$$

Then

$$\exp\left\{-C_n(1+\sqrt{\kappa}R)\right\} V^{2/n} \left(\int_{\Omega_{m+1}} F^{\beta_{m+1}}\right)^{(n-2)/n} \le \left((n-1)\mu_1 t_m R^2\kappa + 1 + \mu_1 C^2 16^m\right) \int_{\Omega_m} F^{\beta_m}. \tag{3-94}$$

Taking $1/\beta_m$ powers on both sides of (3-94), we obtain

$$\|F\|_{L^{\beta_{m+1}}(\Omega_{m+1})} \le \exp\left\{\frac{C_n(1+\sqrt{\kappa}R)}{\beta_m}\right\} V^{-2/(n\beta_m)} \left((n-1)\mu_1 t_m \kappa R^2 + 1 + \mu_1 C^2 16^m\right)^{1/\beta_m} \|F\|_{L^{\beta_m}(\Omega_m)}. \tag{3-95}$$

Keeping the definition of t_m in mind, from (3-95) we deduce that

$$\|F\|_{L^{\beta_{m+1}}(\Omega_{m+1})} \le \exp\Big\{\frac{C_n(1+\sqrt{\kappa}R)}{\beta_m}\Big\} V^{-2/n\beta_m} 16^{m/\beta_m} \times \big((n-1)\mu_1\big(t_0+\tfrac{p}{2}+\alpha-1\big)\kappa R^2+1+\mu_1 C^2\big)^{1/\beta_m} \|F\|_{L^{\beta_m}(\Omega_m)}.$$

Noting that

$$\sum_{m=1}^{\infty}\frac{1}{\beta_m}=\frac{n}{2\beta_1} \quad \text{and} \quad \sum_{m=1}^{\infty}\frac{m}{\beta_m}=\frac{n^2}{4\beta_1},$$

we have

$$\begin{aligned}\|F\|_{L^\infty(B(x_0,R/2))} &\le \exp\Big\{\frac{nC_n(1+\sqrt{\kappa}R)}{2\beta_1}\Big\} V^{-1/\beta_1} 16^{n^2/(4\beta_1)} \\ &\qquad \times \big((n-1)\mu_1\,\big(t_0+\tfrac{p}{2}+\alpha-1\big)\kappa R^2+1+\mu_1 C^2\big]^{n/(2\beta_1)} \|F\|_{L^{\beta_1}(\Omega_1)} \\ &\le \mathcal{C}_3 \exp\Big\{\frac{nC_n(1+\sqrt{\kappa}R)}{2t_0+p+2\alpha-2}\Big\} V^{-1/\beta_1}(1+\kappa R^2)^{n/(2\beta_1)}\|F\|_{L^{\beta_1}(\Omega_1)},\end{aligned}$$

where $\mathcal{C}_3=\mathcal{C}_3(n,p,q)$ is a positive constant. In view of (3-65) ($t=t_0$) and (3-91), it is not difficult to see that

$$\|F\|_{L^\infty(B(x_0,R/2))} \le \mathcal{C} V^{-\frac{1}{\beta_1}} \|F\|_{L^{\beta_1}(\Omega_1)},$$

where $\mathcal{C}=\mathcal{C}(n,p,q)$ is a positive constant. □

4. Proof of the main theorem and its consequences

Theorem 1.1 follows easily from Lemmas 3.8 and 3.9. Therefore, we omit its proof, but give those of Corollary 1.2, Theorem 1.5 and Corollary 1.7. We omit those of Corollary 1.6 (very easy) and those of Corollaries 1.3 and 1.4 (similar to that of Corollary 1.2).

Proof of Corollary 1.2. By using Theorem 1.1, we only need to confirm that the constants a, q and p satisfy either

$$\frac{\beta}{2}\cdot\frac{n+1}{n-1}(p-1)-h^{1/2} < q < \frac{\beta}{2}\cdot\frac{n+1}{n-1}(p-1)+h^{1/2} \quad (a\neq 0)$$

or

$$a\Big[\frac{\beta}{2}\cdot\frac{n+1}{n-1}(p-1)-q\Big]\ge 0. \tag{4-1}$$

Here, we only check case 1; the others are similar.

Case 1: $a>0$, $p\geqslant n$ and $q\in\mathbb{R}$. Since $p\ge n$, we can see that $\beta\in(0,+\infty)$ by using (1-3). Furthermore, since $a>0$, we can verify that (4-1) is equivalent to

$$q\le\frac{\beta}{2}\cdot\frac{n+1}{n-1}(p-1). \tag{4-2}$$

Hence, for any fixed p ($p \geq n$), n and $q \in \mathbb{R}$, we can make (4-2) be true by letting β large enough. Therefore, we complete the proof of case 1. □

Proof of Theorem 1.5. By using Corollary 1.2, we know that there exist positive constants $\mathcal{C} = \mathcal{C}(n, p, q)$ and $\beta = \beta(n, p, q) \in (0, \infty)$, such that

$$\sup_{B(x_0,R/2)} \frac{|\nabla u|^2}{u^\beta} \leq \mathcal{C} \frac{1+\kappa R^2}{R^2} \phi_\beta^{2-\beta}. \tag{4-3}$$

Since

$$\phi_\beta = \begin{cases} \sup_{B(x_0,R)} u & \text{if } 0 < \beta < 2, \\ 1 & \text{if } \beta = 2, \\ \inf_{B(x_0,R)} u & \text{if } \beta > 2, \end{cases} \tag{4-4}$$

we consider three cases:

Case 1: $\beta \in (0, 2)$. Combining (4-3) and (4-4), we have the estimate

$$\sup_{B(x_0,R/2)} \frac{|\nabla u|^2}{u^\beta} \leq \mathcal{C} \frac{1+\kappa R^2}{R^2} \sup_{B(x_0,R)} u^{2-\beta}. \tag{4-5}$$

Multiplying both sides of (4-5) by $\sup\limits_{B(x_0,R)} u^{\beta-2}$ leads to

$$\sup_{B(x_0,R/2)} \frac{|\nabla u|^2}{u^\beta} \sup_{B(x_0,R)} v^{\beta-2} \leq \mathcal{C} \frac{1+\kappa R^2}{R^2} \sup_{B(x_0,R)} u^{2-\beta} \sup_{B(x_0,R)} u^{\beta-2}. \tag{4-6}$$

Since $\sup_{B(x_0,R)} u \leq l \inf_{B(x_0,R)} u$, we see that

$$\sup_{B(x_0,R)} u^{2-\beta} \sup_{B(x_0,R)} u^{\beta-2} = \Big(\sup_{B(x_0,R)} u\Big)^{2-\beta} \Big(\inf_{B(x_0,R)} u\Big)^{\beta-2} \leq l^{2-\beta}. \tag{4-7}$$

Furthermore,

$$\sup_{B(x_0,R/2)} \frac{|\nabla u|^2}{u^2} \leq \sup_{B(x_0,R/2)} \frac{|\nabla u|^2}{u^\beta} \sup_{B(x_0,R/2)} u^{\beta-2} \leq \sup_{B(x_0,R/2)} \frac{|\nabla u|^2}{u^\beta} \sup_{B(x_0,R)} u^{\beta-2}. \tag{4-8}$$

Now, substituting (4-7) and (4-8) into (4-6) leads to

$$\sup_{B(x_0,R/2)} \frac{|\nabla u|^2}{u^2} \leq \mathcal{C} l^{2-\beta} \frac{1+\kappa R^2}{R^2}.$$

Thus we finish the proof of case 1.

Case 2: $\beta = 2$. Combining (4-3) and (4-4), we have the estimate

$$\sup_{B(x_0,R/2)} \frac{|\nabla u|^2}{u^2} \leq \mathcal{C} \frac{1+\kappa R^2}{R^2}.$$

Therefore, we complete the proof of case 2.

Case 3: $\beta > 2$. The proof of case 3 is similar to case 1. Hence, we omit it. □

Now, we turn to (1-1) in $\mathbb{R}^n$ and give the proof of Corollary 1.7, which uses Lemmas 2.4 and 2.5.

Proof of Corollary 1.7. In order to use Lemmas 2.4 and 2.5, we need to consider separately the four cases from the statements. We give the proofs in cases 1 and 3 only; in cases 2 and 4 they are similarly to those of 1 and 3 respectively.

Case 1: $a > 0$, $1 < p < n$, $p \neq q$ and $q \in \big(p-1, (p-1)n/(n-p)\big)$. Define $u = a^{1/(p-q-1)}w$. Then w satisfies $\Delta_p w + w^q = 0$, thanks to (1-1). By using Lemma 2.4, we know that for any $1 < p < n$ and $q \in \big(p-1, \frac{(p-1)n}{n-p}\big)$, the Harnack inequality

$$\sup_{B_R} w(x) \leq \mathcal{C} \inf_{B_R} w(x) \tag{4-9}$$

holds true, where $\mathcal{C} = \mathcal{C}(n, p, q)$ is a positive constant. Combining Theorem 1.5 and (4-9), we deduce that for any $1 < p < n$ and $q \in \big(p-1, \frac{(p-1)n}{n-p}\big)$, the estimate

$$\sup_{B(x_0,R/2)} \frac{|\nabla w|^2}{w^2} \leq \frac{\mathcal{C}}{R^2} \tag{4-10}$$

holds true, where $\mathcal{C} = \mathcal{C}(n, p, q)$ is a positive constant. Substituting $w = a^{1/(q-p+1)}u$ into (4-10), we obtain

$$\sup_{B(x_0,R/2)} \frac{|\nabla u|^2}{u^2} \leq \frac{\mathcal{C}}{R^2},$$

where $\mathcal{C} = \mathcal{C}(n, p, q)$ is a positive constant. This finishes the proof of case 1.

Case 3: $a \geqslant 1$ and $1 < p = q < n < p^2$. Since $p = q$ and $1 < p < n < p^2$, we know that $q \in \big(p-1, \frac{(p-1)n}{n-p}\big)$. By using Lemma 2.4, we know that for any $1 < p = q < n < p^2$ the Harnack inequality

$$\sup_{B_R} u(x) \leq \mathcal{C} \inf_{B_R} u(x) \tag{4-11}$$

holds true, where $\mathcal{C} = \mathcal{C}(n, p, q, a)$ is a positive constant. By using Theorem 1.5 and (4-11), we achieve the estimate

$$\sup_{B(x_0,R/2)} \frac{|\nabla u|^2}{u^2} \leq \frac{\mathcal{C}}{R^2},$$

where $\mathcal{C} = \mathcal{C}(n, p, q, a)$ is a positive constant. This finishes the proof of case 3. □

Proof of Theorem 1.8. By using Corollary 1.7, we know that if the constants a, q and p satisfy one of the four conditions listed in the theorem, then the estimate

$$\sup_{B(x_0,R/2)} \frac{|\nabla u|^2}{u^2} \leq \frac{\mathcal{C}}{R^2}$$

holds true, where $\mathcal{C} = \mathcal{C}(n, p, q, a)$ is a positive constant. By letting $R \to +\infty$, we conclude that $\sup_{\mathbb{R}^n} |\nabla u| = 0$. Hence, u is a constant. But, no positive constant is a positive solution to (1-1). □

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