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We study a strong openness property for singular Hermitian vector bundles (E, h) that are Griffiths-semipositive.

1. Introduction

The strong openness conjecture is an important subject in complex geometry. It was completely solved by Q. Guan and X. Zhou as follows:

Theorem 1.1 [11, Theorem 1.1]. *Let Δ^n be the unit polydisc, and let (L, φ) be a singular Hermitian line bundle on Δ^n. Assume that the associated curvature satisfies $i\Theta_{L,\varphi} \geqslant 0$, and that F is a holomorphic section of L which satisfies*

$$\int_{\Delta^n} |F|^2 e^{-\varphi} d\lambda < \infty,$$

where $d\lambda$ is the Lebesgue measure. After shrinking Δ^n if necessary, there exists a positive ε such that

$$\int_{\Delta^n} |F|^2 e^{-(1+\varepsilon)\varphi} d\lambda < \infty.$$

Equivalently, the following equality concerning multiplier ideal sheaves holds:

$$\bigcup_{\varepsilon>0} \mathscr{I}((1+\varepsilon)\varphi) = \mathscr{I}(\varphi). \tag{1}$$

It has led to fruitful developments, such as [3; 1; 2; 13; 9; 12]. Among them, we mention the following variant for later use.

Corollary 1.1 [3, Corollary B.2]. *Let φ, ψ be plurisubharmonic functions on a domain $U \subseteq \mathbb{C}^n$. Assume that ψ has analytic singularities. Then for any compact subset $K \Subset U$, there exists a positive ε such that*

$$e^{\psi-\varphi} \in L^1_{\mathrm{loc}}(U, K) \quad \text{if and only if} \quad e^{\psi-(1+\varepsilon)\varphi} \in L^1_{\mathrm{loc}}(U, K).$$

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Here $L^1_{\mathrm{loc}}(U, K)$ *denotes the set of Lebesgue measurable functions* f *defined in a neighborhood of* K *in* U *such that* $|f|$ *is locally integrable in a neighborhood of every point* $x \in K$.

It is natural to ask if the *strong openness property* holds for the higher-rank vector bundles. There are some discussions in [16; 18; 19], and we should make a brief review first. For an arbitrary holomorphic vector bundle E and a singular Hermitian metric h, they considered the multiplier submodule sheaf $\mathcal{E}(h)$ defined as

$$\mathcal{E}(h)_x := \{F_x \in E_x \,|\, |F_x|^2_h \text{ is integrable around } x\}.$$

Suppose that $\{h_j\}$ is a sequence of singular metrics decreasing to h. Then, under a certain Nakano-type positivity (on $\{h_j\}$ or h), they concluded that

$$\bigcup_j \mathcal{E}(h_j) = \mathcal{E}(h). \tag{2}$$

In this paper, we investigate this problem from another point of view. More precisely, we apply the deep relationship between the Hermitian metrics on E and its tautological line bundle $\mathcal{O}_E(1)$, to reduce everything to the rank one case. This theory is known as the Finsler geometry, which will be recalled later. Then we obtain that

Theorem 1.2. *Let* E *be a holomorphic vector bundle of rank* r *over* Δ^n, *and let* h *be a singular Hermitian metric on* E *such that* (E, h) *is Griffiths semipositive. Assume that* $\det h$ *has analytic singularities. If* F *is a holomorphic section of* E *satisfying*

$$\int_{\Delta^n} |F|^2_h d\lambda < \infty,$$

then after shrinking Δ^n *if necessary, there exists a positive* ε (*independent of* F) *such that*

$$\int_{\Delta^n} |F|^{2(1+\varepsilon)}_h d\lambda < \infty.$$

In principle, the Griffiths-type positivity is strictly weaker than the Nakano-type positivity. But we should also notice that [16; 18; 19] made no restrictions on the singularity of $\det h$. Hence, there is no direct relationship between [16; 18; 19] and Theorem 1.2.

Next, let us consider general singular metrics. Remember that in the line bundle case, if we furthermore suppose that $\mathscr{I}(\varphi) = \mathcal{O}_{\Delta^n}$, the equality (1) is then called the *openness property*. It was proved by [4], and certainly can be seen as a special case of Theorem 1.1. We generalize it to the higher-rank vector bundles as follows:

Theorem 1.3. *Let* E *be a holomorphic vector bundle of rank* r *over* Δ^n_R, *where* Δ^n_R *is a polydisc with radius* R *slightly larger than* 1. *Let* h *be a singular Hermitian* *metric on E such that (E, h) is Griffiths semipositive, and let φ be the metric on $\mathcal{O}_E(1)$ induced by h. Assume that*

$$\mathscr{I}((r+1)\varphi) = \mathcal{O}_{\mathbb{P}(E^*)}. \tag{3}$$

Then there exist positive numbers $\hat{R}$ and ε such that

$$\int_{\Delta^n_{\hat{R}}} |F|_h^{2(1+\varepsilon)} d\lambda < \infty$$

for any holomorphic section F of E.

As we will see in Section 4, (3) implies that

$$\int_{\Delta^n_R} |F|_h^2 d\lambda < \infty$$

for any F, namely

$$\mathcal{E}(h) = E.$$

But the converse seems not obvious.

This paper is organized as follows: in Section 2 we will review the basic materials of singular metrics and Finsler geometry, and in Section 3 we give the proof of Theorem 1.2. In the end we prove Theorem 1.3.

2. Preliminary

Let X be a complex manifold of dimension n, and let $p : E \to X$ be a holomorphic vector bundle over X of rank r.

2.1. *Singular Hermitian metrics.*

Definition 2.1 [5; 21; 22]. A singular Hermitian metric on E is a map h that associates to every point $x \in X$ a singular Hermitian inner product $|\cdot|_{h,x} : E_x \to [0, +\infty]$ on the complex vector space E_x, subject to the following two conditions:

1. h is finite and positive definite almost everywhere, meaning that for all x outside a set of Lebesgue measure zero, $|\cdot|_{h,x}$ is a Hermitian inner product on E_x;
2. h is measurable, meaning that the function

   $$|F|_h : U \to [0, +\infty], \quad x \mapsto |F(x)|_{h,x},$$

   is measurable for any open $U \subseteq X$ and $F \in \Gamma(U, E)$.

Definition 2.2 [22]. Let h be a singular Hermitian metric on E, which canonically induces a singular metric h^* on the dual bundle E^*.

(1) (E, h) is called Griffiths-seminegative (or negatively curved) if, for any (local) holomorphic section F of E, the function $\log |F|_h^2$ is plurisubharmonic.

(2) (E, h) is called Griffiths-semipositive (or positively curved) if (E^*, h^*) is Griffiths-seminegative.

Proposition 2.1 [22, Proposition 1.3]. *Let E be a holomorphic vector bundle over Δ^n, equipped with a singular Hermitian metric h.*

(1) (See also [5], Proposition 3.1.) *If (E, h) is Griffiths-semipositive, then over any smaller polydisk there is a sequence of smooth, Griffiths positive metrics $\{h_\nu\}$ increasing pointwise to h.*

(2) *If (E, h) is Griffiths-seminegative, then* $\log\det h$ *is a plurisubharmonic function.*

Definition 2.3 [6]. A plurisubharmonic function φ is said to have analytic singularities if φ can be written locally as

$$\varphi = \alpha\log\bigl(|f_1|^2 + \cdots + |f_N|^2\bigr) + v,$$

where $\alpha \in \mathbb{R}_+$, v is a bounded function and the f_j are holomorphic functions.

When (E, h) is a Griffiths-semipositive singular Hermitian vector bundle, the function $-\log\det h$ is plurisubharmonic by Proposition 2.1(2). Then we can ask it has analytic singularities, and briefly say that $\det h$ has analytic singularities. Obviously, it not necessarily implies that h itself has analytic singularities.

2.2. *Finsler geometry revisited.* We only collect the necessary materials. One could refer to [8; 15] for a sophisticated comprehension.

Let $x = (x^1, \dots, x^n)$ be a local coordinate system in X and let $w = (w^0, \dots, w^{r-1})$ be the fiber coordinate system defined by a local holomorphic frame

$$W = \{W^0, \dots, W^{r-1}\}$$

of E. Let h be a smooth Hermitian metric on E. We write

$$h_i = \frac{\partial h}{\partial x^i}, \quad h_{\bar{j}} = \frac{\partial h}{\partial \bar{x}^j}, \quad h_{i\bar{j}} = \frac{\partial^2 h}{\partial x^i \partial \bar{x}^j}, \quad h_{i\alpha} = \frac{\partial^2 h}{\partial x^i \partial w^\alpha},$$

and so on, to denote the differentiation with respect to $x^i, \bar{x}^j$ $(1 \leqslant i, j \leqslant n)$ and $w^\alpha, \bar{w}^\beta$ $(0 \leqslant \alpha, \beta \leqslant r-1)$.

Denote by $q : \mathbb{P}(E) \to X$ the natural projection from the projectivized bundle to the ambient space, and h induces a Hermitian metric q^*h on q^*E. Then as a subbundle,

$$\mathcal{O}_{\mathbb{P}(E)}(-1) := \bigl\{((x, [w]), Z) \in q^*E \,|\, Z = \lambda w, \lambda \in C\bigr\}$$

inherits a metric from q^*E, whose weight function is denoted by $-\psi$. Accordingly, we can define the metrics ψ, $-\varphi$ and φ on $\mathcal{O}_{\mathbb{P}(E)}(1)$, $\mathcal{O}_{\mathbb{P}(E^*)}(-1)$ and $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ respectively in an obvious way. By definition, one easily verifies: when (E, h) is

Griffiths-semipositive, $(\mathcal{O}_{\mathbb{P}(E^*)}(1), \varphi)$ is semipositive. For this reason, we usually denote $\mathcal{O}_{\mathbb{P}(E^*)}(1)$ by $\mathcal{O}_E(1)$, and call it the tautological line bundle of E.

Remark 2.1. Following the same procedure, a singular Hermitian metric h on E induces singular metrics on $\mathcal{O}_{\mathbb{P}(E)}(-1)$, $\mathcal{O}_{\mathbb{P}(E)}(1)$, $\mathcal{O}_E(-1)$ and $\mathcal{O}_E(1)$ respectively.

Moreover, if (E, h) is Griffiths-semipositive, $\mathcal{O}_E(1)$ is pseudo-effective equipped with the corresponding metric as is shown in [21], Proposition 2.3.5.

Next let us recall the celebrated curvature formula in [15]. Remember that ψ is the weight function of the metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ induced by h. We first expand $i\partial\bar{\partial}\psi$ on $\mathbb{P}(E)$ as follows:

$$i\partial\bar{\partial}\psi = i\big(g_{i\bar{j}}dx^i \wedge d\bar{x}^j + g_{i\bar{\beta}}dx^i \wedge d\bar{e}^\beta + g_{\alpha\bar{j}}de^\alpha \wedge d\bar{x}^j + g_{\alpha\bar{\beta}}de^\alpha \wedge d\bar{e}^\beta\big).$$

Here $(w^0, \dots, w^{r-1})$ lifts as $(e^1, \dots, e^{r-1})$, and we adopt the summation convention of Einstein. Note if we furthermore ask the holomorphic frame W to be normal with respect h at x, $ig_{\alpha\bar{\beta}}de^\alpha \wedge d\bar{e}^\beta$ is just the Fubini–Study metric on $\mathbb{P}(E)_x$. Thus the matrix $[g_{\alpha\bar{\beta}}]$ is invertible everywhere. Denote by $[g^{\bar{\beta}\alpha}]$ the inverse matrix. Then we can define the conformal basis by

$$\{\delta e^\alpha = de^\alpha + g^{\bar{\beta}\alpha}g_{i\bar{\beta}}dx^i, dx^i\}.$$

It is shown in [15] that on this basis we can rewrite $i\partial\bar{\partial}\psi$ as

$$i\partial\bar{\partial}\psi = -\Psi + \omega_{FS}, \tag{4}$$

where

$$\Psi = i\Theta_{\alpha\bar{\beta}i\bar{j}}\frac{e^\alpha\bar{e}^\beta}{h_{\sigma\bar{\tau}}e^\sigma\bar{e}^\tau}dx^i \wedge d\bar{x}^j \quad \text{and} \quad \omega_{FS} = ig_{\alpha\bar{\beta}}\delta e^\alpha \wedge \delta\bar{e}^\beta.$$

Restricted on each $\mathbb{P}(E)_x$, ω_{FS} is just the Fubini–Study metric up to a coordinate transform.

We should also make a brief explanation of Ψ. $\Theta_{\alpha\bar{\beta}i\bar{j}}$ is the curvature tensor of E associated with h. Moreover, remember that $(w^0, \dots, w^{r-1})$ lifts to $(e^1, \dots, e^{r-1})$. Locally, say on $\{w^0 \neq 0\}$, we have $e^\alpha = w^\alpha/w^0$ for $\alpha = 1, \dots, r-1$. In view of this relationship, Ψ is interpreted as

$$\Psi = i\Theta_{\alpha\bar{\beta}i\bar{j}}\frac{w^\alpha\bar{w}^\beta}{h_{\sigma\bar{\tau}}w^\sigma\bar{w}^\tau}dx^i \wedge d\bar{x}^j.$$

Since both $\Theta_{\alpha\bar{\beta}i\bar{j}}$ and $h_{\sigma\bar{\tau}}$ are independent of w^α and $w^\alpha\bar{w}^\beta/(h_{\sigma\bar{\tau}}w^\sigma\bar{w}^\tau)$ is homogeneous of degree 0, this expression is indeed a function of $(e^1, \dots, e^{r-1})$. Therefore Ψ is well-defined.

In practice, we will apply (4) to the tautological bundle $(\mathcal{O}_E(1), \varphi)$ on $\mathbb{P}(E^*)$. It enables us to establish a crucial isometry:

$$K^{-1}_{\mathbb{P}(E^*)/X} \simeq \mathcal{O}_E(r) \otimes \pi^* \det E^*.$$

3. Strong openness

We briefly recall the L^2-representation in [20; 17] for a (singular) Hermitian metric of a vector bundle, which helps us to reduce everything to the line bundle case; then we apply Corollary 1.1 to obtain our desired estimate.

We will work with the following setup: let X be a complex manifold, and let E be a holomorphic vector bundle of rank r over X. Let $\pi : \mathbb{P}(E^*) \to X$ be the natural projection. For a Hermitian metric h on E, we denote by φ the weight function of the induced metric h_L on $\mathcal{O}_E(1)$. Suppose that h is smooth, then $i\partial\bar\partial\varphi$ is a Fubini–Study-type metric along each fiber $\mathbb{P}(E^*)_x$. We briefly denote it by $\omega_x := (i\partial\bar\partial\varphi)|_{\mathbb{P}(E^*)_x}$, then the volume $\mathrm{Vol}(\mathbb{P}(E^*)_x)$ against ω_x equals 1 for every x. This is standard in Finsler geometry, and one could refer to [7, Lemma 2.1] for a beautiful explanation.

3.1. ***The L^2-representation.*** We have a canonical isomorphism $\pi_*\mathcal{O}_E(1) \simeq E$ [14, Chapter II, Proposition 7.11]). Hence, for any local holomorphic section F of E, $|F|^2 e^{-\varphi}$ is understood in an obvious way. Combining with [20; 17], we obtain:

Proposition 3.1. *Up to a multiplication of a constant, for every $x \in X$ and $F \in E_x$ we have*

$$|F|^2_{h,x} = \int_{\mathbb{P}(E^*)_x} |F|^2 e^{-\varphi}\omega_x^{r-1}. \tag{5}$$

Furthermore, φ induces an isometry between the canonical isomorphism:

$$K^{-1}_{\mathbb{P}(E^*)/X} \simeq \mathcal{O}_E(r) \otimes \pi^* \det E^*.$$

Namely, if $(e^1, \dots, e^{r-1})$ is a fiber coordinate system of $\mathbb{P}(E^*)$, then

$$\omega_x^{r-1} = i^{r-1}\pi^* \det h^* \cdot e^{-r\varphi} de^1 \wedge d\bar e^1 \wedge \cdots \wedge de^{r-1} \wedge d\bar e^{r-1} \tag{6}$$

up to a constant.

Moreover, when (E, h) is a Griffiths-semipositive singular Hermitian vector bundle, the integral of the right hand side of (5) *is still well-defined, and* (5) *holds almost everywhere.*

Proof. The smooth version of the L^2-representation (5) is nothing but a reformulation of [17], Theorem 7.1. In fact, if we take $k = 1$ and $F = \mathcal{O}_X$, (3.3) in [17] defines a metric f on E via $\pi_*\mathcal{O}_E(1) \simeq E$, which is exactly

$$\frac{1}{(r-1)!}\int_{\mathbb{P}(E^*)_x} |F|^2 e^{-\varphi}\omega_x^{r-1}.$$

Then (7.3) there implies that $f = h$ up to multiplication by $\frac{(r+1)^{r-1}}{r!}$, which is the desired conclusion.

As for (6), it is enough to show that the associated curvature forms of both sides are equal. For this purpose, we apply (4). Keep the notation there. Without loss of generality, we can make $\{W^0, \dots, W^{r-1}\}$ normal for h^* at a fixed point x. Then at $(x; e^1, \dots, e^{r-1})$, ω_x is just the standard Fubini–Study metric on $\mathbb{P}^{r-1}$ and

$$\Psi = i\Theta_{\alpha\bar{\beta}i\bar{j}} \frac{w^\alpha \bar{w}^\beta}{|w|^2} dx^i \wedge d\bar{x}^j.$$

Taking the integral against ω_x^{r-1}, we obtain

$$\begin{aligned}\pi_*\Psi &= i\Theta_{\alpha\bar{\beta}i\bar{j}} \Big(\int_{\mathbb{P}(E^*)_x} \frac{w^\alpha \bar{w}^\beta}{|w|^2} \omega_x^{r-1} \Big) dx^i \wedge d\bar{x}^j = i\Theta_{\alpha\bar{\beta}i\bar{j}} \frac{\delta_{\alpha\bar{\beta}}}{r} dx^i \wedge d\bar{x}^j \\ &= \frac{i}{r} \sum_\alpha \Theta_{\alpha\bar{\alpha}i\bar{j}} dx^i \wedge d\bar{x}^j = \frac{i}{r} \Theta_{\det E^*, \det h^*}\end{aligned}$$

at x, hence everywhere. On the other hand, remember that $\int_{\mathbb{P}(E^*)_x} \omega_x^{r-1} = 1$,

$$\int_{\mathbb{P}(E^*)/X} i\pi^*\Theta_{\det E^*, \det h^*} \wedge \omega_x^{r-1} = i\Theta_{\det E^*, \det h^*}.$$

Here $\int_{\mathbb{P}(E^*)/X} \cdot \wedge \omega_x^{r-1}$ refers to taking the integral along fibers. It exactly implies that there exists an element γ with

$$\Psi = \frac{i}{r}\pi^*\Theta_{\det E^*, \det h^*} + \gamma \quad \text{and} \quad \int_{\mathbb{P}(E^*)/X} \gamma \wedge \omega_x^{r-1} = 0. \tag{7}$$

Combining with the fact that ω_{FS} is the Fubini–Study metric along each fiber of π and the canonical isomorphism

$$K^{-1}_{\mathbb{P}(E^*)/X} \simeq \mathcal{O}_E(r) \otimes \pi^* \det E^*,$$

we conclude from (7) that

$$i\Theta_{\mathcal{O}_E(r), r\varphi} = ir\partial\bar{\partial}\varphi = -r\Psi + r\omega_{\mathrm{FS}} = -i\pi^*\Theta_{\det E^*, \det h^*} + i\Theta_{K^{-1}_{\mathbb{P}(E^*/X)}, \omega_x^{r-1}}.$$

The proof of (6) is then finished. Observe that by setting

$$(G, h_G) := (\mathcal{O}_E(r+1) \otimes \pi^* \det E^*, h_L^{r+1} \otimes \pi^* \det h^*),$$

we can interpret (5) as

$$|F|^2_{h,x} = \int_{\mathbb{P}(E^*)_x} |F|^2_{h_G} \tag{8}$$

in view of (6). The definition of $\int_{\mathbb{P}(E^*)_x} |F|^2_{h_G}$ is based on

$$\mathcal{O}_E(1) \simeq K_{\mathbb{P}(E^*)/X} \otimes \mathcal{O}_E(r+1) \otimes \pi^* \det E^*,$$

and is carefully explained in [20], Sect. 2.2.

Whereas the singular version of (5) is nothing but a reformulation of [20], Proposition 3.1. Indeed, by [20] the integral of the right hand side of (8) is still well-defined when h is singular and (E, h) is Griffiths-semipositive. Moreover, (8) holds outside $V := \{\det h = \infty\}$. Note V must be a set of measure zero since h is finite almost everywhere, which concludes the last assertion. □

We next restate and prove Theorem 1.2:

Theorem 3.1. *Let E be a holomorphic vector bundle of rank r over Δ^n, and let h be a singular Hermitian metric on E such that (E, h) is Griffiths-semipositive. Assume that* $\det h$ *has analytic singularities. If F is a holomorphic section of E satisfying*

$$\int_{\Delta^n} |F|_h^2 d\lambda < \infty,$$

then after shrinking Δ^n if necessary, there exists a positive ε (independent of F) such that

$$\int_{\Delta^n} |F|_h^{2(1+\varepsilon)} d\lambda < \infty.$$

Proof. Keep the notations of Proposition 3.1. Due to (the singular version of) (8) and Fubini's theorem, it is enough to show that there exists a positive ε such that

$$\int_{\Delta^n} \Big(\int_{\mathbb{P}(E^*)_x} |F|_{h_G}^2 \Big)^{1+\varepsilon} d\lambda < \infty.$$

By hypothesis, $\pi^* \det h^* = e^{\psi}$, where ψ is a plurisubharmonic function with analytic singularities. So we have

$$\begin{aligned}
\int_{\Delta^n} \Big(\int_{\mathbb{P}(E^*)_x} |F|_{h_G}^2 \Big)^{1+\varepsilon} d\lambda &= \int_{\Delta^n} (\det h^*)^{1+\varepsilon} \Big(\int_{\mathbb{P}(E^*)_x} |F|^2 e^{-(r+1)\varphi} \Big)^{1+\varepsilon} d\lambda \\
&\leqslant C \int_{\Delta^n} (\det h^*)^{1+\varepsilon} \Big(\int_{\mathbb{P}(E^*)_x} (|F|^2 e^{-(r+1)\varphi})^{1+\varepsilon} \Big) d\lambda \\
&= C \int_{\Delta^n} \int_{\mathbb{P}(E^*)_x} e^{(1+\varepsilon)(\log |F|^2 + \psi) - (1+\varepsilon)(r+1)\varphi} d\lambda.
\end{aligned}$$

The inequality is a simple application of Hölder's inequality, and C is a universal positive constant.

On the other hand, by assumption F a priori satisfies

$$\int_{\Delta^n} \int_{\mathbb{P}(E^*)_x} e^{\log |F|^2 + \psi - (r+1)\varphi} d\lambda < \infty.$$

So after shrinking Δ^n if necessary, by Corollary 1.1 there exists a positive ε_0 such that

$$\int_{\Delta^n} \int_{\mathbb{P}(E^*)_x} e^{\log |F|^2 + \psi - (1+\varepsilon_0)(r+1)\varphi} d\lambda < \infty. \tag{9}$$

Since $e^{\varepsilon_0(\log|F|^2+\psi)}$ is bounded, (9) implies that

$$\int_{\Delta^n}\Big(\int_{\mathbb{P}(E^*)_x}|F|^2_{h_G}\Big)^{1+\varepsilon_0}d\lambda<\infty.$$

In the end, notice that [23], Theorem 1.2 indicates that $\mathcal{E}(h)$ is coherent hence is locally finitely generated in our situation. So we can even obtain a uniform constant ε_0 for all holomorphic sections of $\mathcal{E}(h)$. The proof is complete. □

4. Openness

Now we would like to remove the restriction in Theorem 3.1 that $\det h$ has analytic singularities. A natural idea is to approximate h by a sequence of smooth metrics $\{h_\nu\}$, whose existence is due to Proposition 2.1. Then we apply Theorem 3.1 on each h_ν (which gives a corresponding ε_ν), and take the limit to obtain the desired estimate. For this purpose, we need a universal lower bound for these ε_ν. The related topic is called the effectiveness of the *strong openness property*. It is much involved even in the line bundle case (see [10], Theorem 1.3). As a compromise, we will instead apply the following effective version of the *openness property* in [4].

The line bundle case revisited.

Theorem 4.1 [4, Theorem 1.1]. *Let φ be a plurisubharmonic function in B^n with $\varphi\leqslant 0$. Assume that*

$$\int_{B^n}e^{-\varphi}d\lambda<\infty.$$

Then there is a number $\varepsilon>0$ such that

$$\int_{B^n/2}e^{-(1+\varepsilon)\varphi}d\lambda<\infty.$$

Here B^n and $B^n/2$ refer to the n-dimensional balls of radii 1 *and* 1/2 *respectively. Moreover, ε can be taken so that*

$$\varepsilon\geqslant\frac{\delta_n}{\int_{B^n}e^{-\varphi}d\lambda},$$

where δ_n depends only on the dimension.

Let us briefly recall some crucial steps in the proof of Theorem 4.1 for readers' benefit. For any $s\geqslant 0$ and holomorphic function h, let

$$\varphi_s:=\max(\varphi+s,0)$$

and

$$\|h\|_s^2:=\int_{B^n/2}|h|^2e^{-2\varphi_s}d\lambda.$$

The factor 2 in the exponent is necessary, and we define the norm $\|\cdot\|_s$ here over $B^n/2$ (rather than B^n in [4]) for our purposes. Then a simple variant of Proposition 2.1 in [4] indicates that for $0 < \varepsilon < 1$,

$$\int_{B^n/2} e^{-(1+\varepsilon)\varphi} d\lambda = a_\varepsilon \int_0^\infty e^{(1+\varepsilon)s} \|1\|_s^2 ds + b_\varepsilon \mathrm{Vol}(B^n). \tag{10}$$

Here

$$a_\varepsilon = \frac{1-\varepsilon^2}{2} \quad \text{and} \quad b_\varepsilon = \frac{1-\varepsilon}{2^{n+1}}.$$

Pick a universal constant δ_n so that if w is holomorphic and

$$\int_{B^n} |w|^2 d\lambda \leqslant 4\delta_n,$$

then

$$\sup_{B^n/2} |w| \leqslant \tfrac{1}{10}.$$

The existence of such a δ_n is due to the mean value property of holomorphic functions. In particular, δ_n depends only on the dimension.

Now $\int_{B^n} e^{-\varphi} d\lambda < \infty$ by hypothesis,

$$\int_0^\infty e^s \left(\int_{B^n} e^{-2\varphi_s} \right) ds \leqslant \int_{B^n} e^{-\varphi} d\lambda < \infty$$

due to (the original version of) Proposition 2.1 in [4]. Then by Theorem 3.3 of [4], for any $\varepsilon > 0$ and $s > \frac{1}{2\varepsilon}$ there is a holomorphic function h_s such that

$$\int_{B^n} |1 - h_s|^2 d\lambda \leqslant 4\varepsilon \int_{B^n} e^{-\varphi} d\lambda, \tag{11}$$

and

$$\int_{B^n} |h_s|^2 e^{-2\varphi_s} d\lambda \leqslant e^{-(1+2\varepsilon)s} \mathrm{Vol}(B^n). \tag{12}$$

Take

$$\varepsilon = \frac{\delta_n}{\int_{B^n} e^{-\varphi} d\lambda}.$$

Then (11) implies that the holomorphic function $1 - h_s$ satisfies

$$\int_{B^n} |1 - h_s|^2 d\lambda \leqslant 4\delta_n,$$

hence $\sup_{B^n/2} |1 - h_s| \leqslant \frac{1}{10}$. Therefore h_s satisfies

$$1 < 2|h_s|^2$$

on $B^n/2$. Multiply by $e^{-2\varphi_s}$ and integrate on $B^n/2$, we obtain from this inequality and (12) that

$$\|1\|_s^2 \leqslant 2\int_{B^n/2} |h_s|^2 e^{-2\varphi_s} d\lambda \leqslant 2\int_{B^n} |h_s|^2 e^{-2\varphi_s} d\lambda \leqslant 2e^{-(1+2\varepsilon)s}\mathrm{Vol}(B^n). \tag{13}$$

Now multiply the both sides of (13) by $e^{(1+\varepsilon)s}$ and integrate from 0 to infinity. Combining with (10) we get

$$\begin{aligned}
&\int_{B^n/2} e^{-(1+\varepsilon)\varphi} d\lambda \\
&\leqslant a_\varepsilon \int_0^{\frac{1}{2\varepsilon}} e^{(1+\varepsilon)s}\|1\|_s^2 ds + a_\varepsilon \int_{\frac{1}{2\varepsilon}}^{\infty} e^{(1+\varepsilon)s} 2e^{-(1+2\varepsilon)s}\mathrm{Vol}(B^n) ds + b_\varepsilon \mathrm{Vol}(B^n) \\
&\leqslant a_\varepsilon \mathrm{Vol}(B^n/2) \int_0^{\frac{1}{2\varepsilon}} e^{(1+\varepsilon)s} ds + a_\varepsilon \mathrm{Vol}(B^n) \int_{\frac{1}{2\varepsilon}}^{\infty} e^{-\varepsilon s} ds + b_\varepsilon \mathrm{Vol}(B^n) \\
&= \frac{a_\varepsilon}{2^n}\mathrm{Vol}(B^n) \frac{e^{\frac{1+\varepsilon}{2\varepsilon}} - 1}{1+\varepsilon} + a_\varepsilon \mathrm{Vol}(B^n) \frac{1}{\varepsilon\sqrt{e}} + b_\varepsilon \mathrm{Vol}(B^n) \\
&= \left(\frac{a_\varepsilon (e^{\frac{1+\varepsilon}{2\varepsilon}} - 1)}{2^n(1+\varepsilon)} + \frac{a_\varepsilon}{\varepsilon\sqrt{e}} + b_\varepsilon \right) \mathrm{Vol}(B^n) \\
&=: c_\varepsilon \mathrm{Vol}(B^n).
\end{aligned} \tag{14}$$

Since (13) is valid when $s > \frac{1}{2\varepsilon}$, we have to divide the integral into two parts. The integral from 0 to $\frac{1}{2\varepsilon}$ is standard, since $0 \leqslant \varphi_s \leqslant \frac{1}{2\varepsilon}$; the integral from $\frac{1}{2\varepsilon}$ to infinity is estimated via (13), and it leads to the second inequality. The rest is routine.

Proof of Theorem 1.3. Now we are ready to prove the openness for the higher-rank case. Let E be a holomorphic vector bundle of rank r over Δ_R^n, where Δ_R^n is a polydisc with radius R slightly larger than 1. Let $(e^1, \dots, e^{r-1})$ be a fiber coordinate system of $\mathbb{P}(E^*)$, and let $de \wedge d\bar{e}$ be short for

$$de^1 \wedge d\bar{e}^1 \wedge \cdots \wedge de^{r-1} \wedge d\bar{e}^{r-1}.$$

Take a finite coordinate chart $\{B_j^{n+r-1}\}$ of $\mathbb{P}(E^*)$. For an arbitrary singular Hermitian metric h on E, denote by φ the induced metric on $\mathcal{O}_E(1)$, and

$$\int_{B^{n+r-1}} e^{-(r+1)\varphi} d\lambda := \max_j \{ \int_{B_j^{n+r-1}} e^{-(r+1)\varphi} d\lambda \}$$

with the convention that

$$\int_{B^{n+r-1}} e^{-(r+1)\varphi} d\lambda = +\infty \quad \text{if some} \quad \int_{B_j^{n+r-1}} e^{-(r+1)\varphi} d\lambda = +\infty.$$

We restate and prove Theorem 1.3.

Theorem 4.2. *Assume that* (E, h) *is Griffiths-semipositive, and*

$$\mathscr{I}((r+1)\varphi) = \mathcal{O}_{\mathbb{P}(E^*)}.$$

Then there exist positive numbers $\hat{R}$ *and* ε *such that*

$$\int_{\Delta^n_{\hat{R}}} |F|_h^{2(1+\varepsilon)} d\lambda < \infty$$

for any holomorphic section F *of* E. *Moreover,* ε *can be taken so that*

$$\varepsilon \geqslant \frac{\delta_{n+r-1}}{\int_{B^{n+r-1}} e^{-(r+1)\varphi} d\lambda},$$

where δ_{n+r-1} *depends only on the dimension.*

Proof. Pick an $R' \in (1, R)$. By Proposition 2.1(1), there exists a sequence of smooth, Griffiths positive metrics $\{h_\nu\}$ over $\Delta^n_{R'}$ increasing pointwise to h. Let φ_ν be the metric on $\mathcal{O}_E(1)$ induced by h_ν, and let $\omega_{x,\nu} := (i\partial\bar{\partial}\varphi_\nu)|_{\mathbb{P}(E^*)_x}$. Consequently, φ_ν deceases to φ. Since both φ and φ_ν are plurisubharmonic, we can assume $\varphi, \varphi_\nu \leqslant 0$ without loss of generality.

Now by assumption we have

$$\int_{B^{n+r-1}} e^{-(r+1)\varphi_\nu} d\lambda \leqslant \int_{B^{n+r-1}} e^{-(r+1)\varphi} d\lambda < \infty.$$

Then apply Theorem 4.1 ((14), more precisely), we obtain

$$\int_{B^{n+r-1}/2} e^{-(1+\varepsilon)(r+1)\varphi_\nu} d\lambda \leqslant c_\varepsilon \mathrm{Vol}(B^{n+r-1}), \tag{15}$$

where ε can be taken so that

$$\varepsilon \geqslant \frac{\delta_{n+r-1}}{\int_{B^{n+r-1}} e^{-(r+1)\varphi_\nu} d\lambda}.$$

Since φ_ν decreases to φ,

$$\frac{\delta_{n+r-1}}{\int_{B^{n+r-1}} e^{-(r+1)\varphi_\nu} d\lambda} \searrow \frac{\delta_{n+r-1}}{\int_{B^{n+r-1}} e^{-(r+1)\varphi} d\lambda}.$$

Let

$$\varepsilon = \frac{\delta_{n+r-1}}{\int_{B^{n+r-1}} e^{-(r+1)\varphi} d\lambda},$$

and take the limit of (15) with respect to ν. We finally obtain

$$\int_{B^{n+r-1}/2} e^{-(1+\varepsilon)(r+1)\varphi} d\lambda \leqslant c_\varepsilon \mathrm{Vol}(B^{n+r-1}) < \infty.$$

In the end, observe that we can take a positive number $\hat{R} < R'$ such that the total space of $\mathbb{P}(E^*)|_{\Delta^n_{\hat{R}}}$ is finitely covered by $B_j^{n+r-1}/2$. It exactly implies that

$$\begin{aligned}
\int_{\Delta^n_{\hat{R}}} |F|^{2(1+\varepsilon)}_{h_\nu} d\lambda &= \int_{\Delta^n_{\hat{R}}} \Big(\int_{\mathbb{P}(E^*)_x} |F|^2 e^{-\varphi_\nu} \omega^{r-1}_{x,\nu} \Big)^{1+\varepsilon} d\lambda \\
&\leqslant \int_{\Delta^n_{\hat{R}}} \Big(\int_{\mathbb{P}(E^*)_x} (|F|^2 e^{-\varphi_\nu})^{1+\varepsilon} \omega^{r-1}_{x,\nu} \Big) \Big(\int_{\mathbb{P}(E^*)_x} \omega^{r-1}_{x,\nu} \Big)^{\frac{\varepsilon}{1+\varepsilon}} d\lambda \\
&= \int_{\Delta^n_{\hat{R}}} \Big(\int_{\mathbb{P}(E^*)_x} (|F|^2 e^{-(\frac{r}{1+\varepsilon}+1)\varphi_\nu})^{1+\varepsilon} \cdot \pi^* \det h^*_\nu \cdot i^{r-1} de \wedge d\bar{e} \Big) d\lambda \\
&\leqslant \sum_j \int_{B_j^{n+r-1}/2} |F|^{2(1+\varepsilon)} e^{-(r+1+\varepsilon)\varphi_\nu} \cdot \pi^* \det h^*_\nu d\lambda \\
&\leqslant C \sum_j \int_{B_j^{n+r-1}/2} e^{-(r+1+\varepsilon)\varphi_\nu} d\lambda \\
&\leqslant C \sum_j \int_{B_j^{n+r-1}/2} e^{-(r+1+\varepsilon)\varphi} d\lambda < \infty
\end{aligned}$$

for any F. In the forth inequality we abuse the notation that $d\lambda$ refers to the Lebesgue measure on both $\Delta^n_{\hat{R}}$ and $B_j^{n+r-1}/2$. The fifth inequality is due to the fact that $\{\det h^*_\nu\}$ is a decreasing sequence of smooth functions. So $|F|^{2(1+\varepsilon)}\pi^* \det h^*_\nu$ is bounded by some positive constant C only depends on F. As ν tends to zero, we obtain

$$\int_{\Delta^n_{\hat{R}}} |F|^{2(1+\varepsilon)}_{h} d\lambda < \infty$$

for any F. The proof is complete. □

Let $\{U_j\}$ be a finite coordinate covering of $\mathbb{P}(E^*)$, and let $\{\rho_j\}$ be the associated partition of unity. When $\mathscr{I}((r+1)\varphi) = \mathcal{O}_{\mathbb{P}(E^*)}$, for any holomorphic section F we have

$$\begin{aligned}
\int_{\Delta^n} |F|^2_{h_\nu} d\lambda &= \int_{\Delta^n} \Big(\int_{\mathbb{P}(E^*)_x} |F|^2 e^{-\varphi_\nu} \omega^{r-1}_{x,\nu} \Big) d\lambda \\
&= \int_{\Delta^n} \Big(\int_{\mathbb{P}(E^*)_x} |F|^2 e^{-(r+1)\varphi_\nu} \cdot \pi^* \det h^*_\nu \cdot i^{r-1} de \wedge d\bar{e} \Big) d\lambda \\
&= \sum_j \int_{U_j} \rho_j |F|^2 e^{-(r+1)\varphi_\nu} \cdot \pi^* \det h^*_\nu d\lambda \\
&\leqslant C \sum_j \int_{U_j} e^{-(r+1)\varphi_\nu} d\lambda \\
&\leqslant C \sum_j \int_{U_j} e^{-(r+1)\varphi} d\lambda < \infty.
\end{aligned}$$

As ν tends to zero, we obtain

$$\int_{\Delta^n} |F|^2_h d\lambda < \infty$$

for any F. Hence

$$\mathcal{E}(h) = E. \tag{16}$$

However, if one attempts to deduce $\mathscr{I}((r+1)\varphi) = \mathcal{O}_{\mathbb{P}(E^*)}$ from (16), the influence of the zero locus of $\det h^*$ cannot be ignored. Hence the converse is not clear.

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