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# REFINED BOUNDS FOR THE EIGENVALUES OF THE STOKES OPERATOR

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**We analyze bounds for the sums of eigenvalues of the Stokes operator restricted to a bounded domain $\Omega \subset \mathbb{R}^d$ with $d \geq 2$. We improve upon existing lower bound estimates due to Ilyin as well as those by the second author and S. Yıldırım Yolcu, while preserving sharpness in the sense of Weyl asymptotics.**

## 1. Introduction

Let $\Omega$ be an open bounded set in $\mathbb{R}^d$, $d \geq 2$. We derive sharper estimates for the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ of Stokes problem defined by

$$
\begin{cases} -\Delta \boldsymbol{u}_k + \nabla p_k = \lambda_k \boldsymbol{u}_k & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}_k = 0 & \text{in } \Omega, \\ \boldsymbol{u}_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}
$$

The eigenvalues of the Stokes problem in (1) are important due to their numerous applications in fluid mechanics. They can be interpreted as the eigenvalues associated with linear (small or infinitesimal) self-oscillations of the fluid within the domain $\Omega$ [4]. When the eigenfunctions are required to meet specific conditions that are crucial from a physical perspective in fluid dynamics, exact eigenvalues for the Stokes problem are not obtainable. Consequently, both theoretical and practical aspects necessitate the close identification of the eigenvalues. The literature concerning the eigenvalues of the Stokes problem is vast, and recent studies aimed at deriving estimates for these eigenvalues are documented in [6; 9; 10; 12; 24].

Before presenting the results, we first review some basic facts about the theory of the Stokes operator. Let $\mathcal{U}$ denote the set of smooth, divergence-free vector functions with compact supports. Specifically,

$$
\mathcal{U} = \{\boldsymbol{u} : \Omega \to \mathbb{R}^d,\ \boldsymbol{u} \in \boldsymbol{C}_0^{\infty}(\Omega),\ \operatorname{div} \boldsymbol{u} = 0\}
$$

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and let the closure of $\mathcal{U}$ in $\boldsymbol{L}^2(\Omega)$ and $\boldsymbol{H}_0^1(\Omega)$ be denoted by $L$ and $U$, respectively. In particular,

$$U \subseteq \{\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega),\ \operatorname{div} \boldsymbol{u} = 0\}$$

and if $\Omega$ is open, bounded and locally Lipschitz, we have the equality [3]:

$$U = \{\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega),\ \operatorname{div} \boldsymbol{u} = 0\}.$$

We note that $\boldsymbol{L}^2(\Omega)$ can be written as $\boldsymbol{L}^2(\Omega) = L \oplus L^{\perp}$, where

$$L^{\perp} = \{\boldsymbol{u} \in \boldsymbol{L}^2(\Omega),\ \boldsymbol{u} = \nabla p,\ p \in L^2_{\mathrm{loc}}(\Omega)\}$$

(see [3; 4; 19]). The Stokes operator is formally expressed as

$$A\boldsymbol{u} = -P_L(\Delta \boldsymbol{u}),$$

where the linear operator $P_L : \boldsymbol{L}^2(\Omega) \to L$, defined by

$$P_L(\boldsymbol{v}) = \boldsymbol{v} - \nabla \Delta^{-1}(\operatorname{div} \boldsymbol{v}),$$

is called the Leray projection [4], i.e., $P_L(P_L(\boldsymbol{v})) = P_L(\boldsymbol{v})$. Notice that $P_L$ becomes the identity operator for the divergence-free vector fields because $P_L(\boldsymbol{v}) = \boldsymbol{v}$ for $\operatorname{div} \boldsymbol{v} = 0$. Furthermore, $P_L$ can be understood as the projection onto divergence-free vector fields. This projection is particularly employed to remove some terms and components in the Stokes and Navier–Stokes equations. Going back to the Stokes problem in (1), one can also see that

$$A\boldsymbol{u} = -\Delta \boldsymbol{u} + \nabla p, \quad p = \Delta^{-1}(\operatorname{div} \Delta \boldsymbol{u}),$$

and that for all $\boldsymbol{u}, \boldsymbol{w}$ in $U$, $A$ is defined by

$$(A\boldsymbol{u}, \boldsymbol{w}) = (\nabla \boldsymbol{u}, \nabla \boldsymbol{w}),$$

i.e.,

$$\int_\Omega A\boldsymbol{u}(\boldsymbol{x}) \cdot \boldsymbol{w}(\boldsymbol{x})\, d\boldsymbol{x} = \int_\Omega \sum_{i=1}^{d} \frac{\partial \boldsymbol{u}}{\partial x_i} \frac{\partial \boldsymbol{w}}{\partial x_i}\, d\boldsymbol{x}.$$

The Stokes operator $A$ is an unbounded, linear, self-adjoint, positive definite operator in $L$. In addition to these nice properties, its inverse $A^{-1}$ is a compact, self-adjoint operator [3; 4]. Thus, there exists an orthonormal basis $\{\boldsymbol{u}_k\}_{k=1}^{\infty}$ in $L$ and a set of positive eigenvalues $\{\delta_k\}_{k=1}^{\infty}$ accumulating at zero such that

$$A^{-1}\boldsymbol{u}_k = \delta_k \boldsymbol{u}_k,$$

for $k = 1, 2, 3, \ldots$. Therefore, $\{\boldsymbol{u}_k\}_{k=1}^{\infty} \in U$ with corresponding eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ with $\lambda_k = 1/\delta_k$ are such that

$$A\boldsymbol{u}_k = \lambda_k \boldsymbol{u}_k. \tag{2}$$

The eigenvalues (including multiplicities) satisfy

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \to \infty \quad \text{as } k \to \infty. \tag{3}$$

Observe that by taking the scalar product with orthonormal $\boldsymbol{u}_k$ we have

$$\lambda_k = \|\nabla \boldsymbol{u}_k\|^2 = \int_\Omega \sum_{i=1}^{d} \frac{\partial \boldsymbol{u}_k}{\partial x_i} \cdot \frac{\partial \boldsymbol{u}_k}{\partial x_i}\, d\boldsymbol{x}. \tag{4}$$

The eigenvalues also satisfy the Weyl asymptotic formulas [4; 8; 17] for $d \ge 2$:

$$\lambda_k \sim 4\pi \left( \frac{\Gamma\left(1 + \frac{d}{2}\right)}{(d-1)|\Omega|} \right)^{\frac{2}{d}} k^{\frac{2}{d}}, \tag{5}$$

where $\Gamma(x)$ denotes the gamma function $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$ for $x > 0$ and $|\Omega|$ denotes the Lebesgue measure of $\Omega$.

The eigenvalue bounds that will be established in this paper closely resemble those associated with the Dirichlet Laplacian. A substantial body of literature exists regarding inequalities for the eigenvalues of the Dirichlet Laplacian; notable references include the survey articles cited in [1; 2; 7]. In his renowned paper [18], G. Pólya demonstrated that for small values of $k$, the ratio $4\pi k/|\Omega|$ serves as a lower bound for the eigenvalues $\mu_k$ of the Dirichlet Laplacian in tiling domains in $\mathbb{R}^2$. Pólya conjectured that this result could be extended to any bounded domain in $\mathbb{R}^d$. This conjecture remains unresolved. A significant advance in establishing a lower bound was achieved by P. Li and S.-T. Yau [14], who demonstrated the following inequality related to the sums of the eigenvalues $\mu_j$ of the Dirichlet Laplacian on the domain $\Omega$:

$$\sum_{j=1}^{k} \mu_j \ge 4\pi \frac{d}{d+2} \left( \frac{\Gamma\left(1 + \frac{d}{2}\right)}{|\Omega|} \right)^{\frac{2}{d}} k^{1+\frac{2}{d}}. \tag{6}$$

Recent research has focused significantly on these types of bounds, along with their extensions and enhancements, particularly in relation to other operators. For instance, A. Melas [16] provided an improvement on (6) that incorporates moments of inertia. In two dimensions, H. Kovařík, S. Vugalter, and T. Weidl [13] further advanced Melas's findings by introducing a positive correction term that involves the size of the boundary. T. Weidl [21] achieved an enhancement of the sharp Berezin-type bounds concerning the Riesz means $\sum_k (z - \mu_k)_+^\sigma$ of the eigenvalues associated with the Dirichlet Laplacian operator within a specified domain for $\sigma \ge 3/2$. Regarding other operators, Harrell and Yıldırım Yolcu [5] along with Yolcu [29] proved inequalities of Berezin–Li–Yau type applicable to the eigenvalues of Klein–Gordon operators. Yıldırım Yolcu and Yolcu [25; 28; 27] established

Melas-type improvements for the eigenvalues of fractional Laplacian operators $(-\Delta)^{\alpha/2}, \alpha \in (0, 2]$.

It is noteworthy that the coefficient of $k^{1+\frac{2}{d}}$ in (6) is slightly larger for the tiling domains [18]. See [23] for more extensive research and a generalized conjecture.

Lower bounds on the sums of eigenvalues are important for the theoretical framework of attractors associated with the Navier–Stokes equations [9; 10]. The eigenvalue bounds of the Stokes operator have garnered attention following the work by A. A. Ilyin [9], who proved an inequality of Berezin–Li–Yau type for the Stokes operator:

$$\sum_{j=1}^{k} \lambda_j \geq 4\pi \frac{d}{d+2}\left(\frac{\Gamma\left(1+\frac{d}{2}\right)}{(d-1)\,|\Omega|}\right)^{\frac{2}{d}} k^{1+\frac{2}{d}}, \tag{7}$$

where $k \geq 1$. In this article, we first build upon (7) by establishing upper bounds for the sums of negative powers and lower bounds for the sums of positive powers of the eigenvalues of the Stokes operator as follows:

**Theorem 1.** *For $k \geq 1$, $0 < b < d/2$, and $d \geq 2$, the sums of negative powers of eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ of the Stokes operator on $\Omega$ in (1) satisfy*

$$\sum_{j=1}^{k} \lambda_j^{-b} \leq (4\pi)^{-b} \frac{d}{d-2b}\left(\frac{(d-1)|\Omega|}{\Gamma\left(1+\frac{d}{2}\right)}\right)^{\frac{2b}{d}} k^{1-\frac{2b}{d}}. \tag{8}$$

**Theorem 2.** *For $k \geq 1$, $0 < a \leq 1$, and $d \geq 2$, the sums of positive powers of eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ of the Stokes operator on $\Omega$ in (1) satisfy*

$$\sum_{j=1}^{k} \lambda_j^{a} \geq (4\pi)^{a} \frac{d}{d+2a}\left(\frac{\Gamma\left(1+\frac{d}{2}\right)}{(d-1)|\Omega|}\right)^{\frac{2a}{d}} k^{1+\frac{2a}{d}}. \tag{9}$$

By appropriately translating the open set $\Omega$ if needed, we can assume that the second moment of inertia $I(\Omega)$ is defined as

$$I(\Omega) = \int_{\Omega} |\boldsymbol{x}|^2 \, d\boldsymbol{x}.$$

Ilyin [10] was able to improve the Berezin–Li–Yau inequality in (7) for the Stokes operator by adding a lower-order term involving the moment of inertia $I(\Omega)$ in dimensions 2, 3, and 4 as follows:

$$\sum_{j=1}^{k} \lambda_j \geq 4\pi \frac{d}{d+2}\left(\frac{\Gamma\left(1+\frac{d}{2}\right)}{(d-1)\,|\Omega|}\right)^{\frac{2}{d}} k^{1+\frac{2}{d}} + \frac{d-1}{48}\beta_d^{\mathrm{S}} \frac{|\Omega|}{I(\Omega)}\, k, \tag{10}$$

where in the two-dimensional case $\beta_2^{\mathrm{S}} = \frac{239}{240}$, while for $d = 3, 4$ it suffices to take $\beta_3^{\mathrm{S}} = 0.986$ and $\beta_4^{\mathrm{S}} = 0.978$.

Let

$$B_R(\boldsymbol{x}) := \{\boldsymbol{y} \in \mathbb{R}^d : |\boldsymbol{y} - \boldsymbol{x}| \leq R\}$$

be the ball of radius $R$ centered at $\boldsymbol{x}$ in $\mathbb{R}^d$. Let

$$\omega_d = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(1 + \frac{d}{2}\right)} \tag{11}$$

be the volume of the unit ball $B_1(\boldsymbol{x})$ in $\mathbb{R}^d$.

Building on the improvements in (10), Yıldırım Yolcu and Yolcu [24] introduced additional correction terms beyond those dependent on $k^{1+\frac{2}{d}}$ and $k \geq 1$, along with their explicit coefficients, applicable for any $d \geq 2$:

$$\sum_{j=1}^{k} \lambda_j \geq 4\pi \, \frac{d}{d+2} \left( \frac{\Gamma\left(1 + \frac{d}{2}\right)}{(d-1)\,|\Omega|} \right)^{\frac{2}{d}} k^{1+\frac{2}{d}} + \frac{d-1}{24(d^2+2d)} \frac{|\Omega|}{I(\Omega)} \, k + \frac{((d-1)\,|\Omega|)^{\frac{3d+2}{2d}}}{144\sqrt{\pi}\, d^{\frac{3}{2}}\, (d+2)\, I(\Omega)^{\frac{3}{2}}\, \Gamma\left(1 + \frac{d}{2}\right)^{\frac{1}{d}}} \, k^{1-\frac{1}{d}}. \tag{12}$$

In this paper we establish several effective lower bounds for the sums of the eigenvalues of the Stokes problem, including the following improved estimates:

**Theorem 3** (refinement of the Berezin–Li–Yau inequality for the Stokes operator). *Let $\Omega \subset \mathbb{R}^d$. For any $d \geq m + 1 \geq 2$, the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ of the Stokes operator in (1) satisfy*

$$\sum_{i=1}^{k} \lambda_i \geq 4\pi \left( \frac{\Gamma\left(1 + \frac{d}{2}\right)}{(d-1)\,|\Omega|} \right)^{\frac{2}{d}} k^{1+\frac{2}{d}} - \frac{2\omega_d^{\frac{m-1}{d}} A_{m+2}\varrho^{\frac{(m+1)d+m-1}{d}}}{(d+2)M^{m+1}} k^{1-\frac{m-1}{d}} + c\, \frac{2\omega_d^{\frac{m}{d}} (m+1) A_{m+3}\varrho^{\frac{(m+2)d+m}{d}}}{(d+2)(m+3)M^{m+2}} k^{1-\frac{m}{d}},$$

*where $k \geq 1$, $a$ is defined by the equality in (39),*

$$\varrho = \frac{d-1}{(2\pi)^d}|\Omega|, \quad m_s = \frac{2\sqrt{(d^2-d)|\Omega| I(\Omega)}}{(2\pi)^d}, \quad M = \frac{\sqrt{d^2-d}}{(2\pi)^d}|\Omega|^{\frac{d+1}{d}}\omega_d^{-\frac{1}{d}}, \tag{13}$$

*and the remaining notation is defined by*

$$A_r = (a+1)^r - a^r, \qquad c = \min\{1, \max\{a_1, a_2\}\},$$

$$a_1 = \frac{(d+2)(m+3)}{(m+1)\left((m+2)d+m\right)} \frac{(2\pi)^{m+2}}{A_{m+3}} d^{\frac{m+2}{2}} (d-1)^{-\frac{m(d+2)+2m}{2d}},$$

*and*

$$a_2 = \frac{\sqrt{2}d^{\frac{1}{d}}}{2(d-1)^{\frac{d+2}{2d}}} \frac{[(m+1)d+m-1](m+3)}{[(m+2)d+m](m+1)} \frac{A_{m+2}}{A_{m+3}}.$$

Theorem 3 yields no improvement when $d = 2$, but it quickly yields the following enhanced lower bound for any dimension $d \geq 3$:

**Corollary 1.** *For any bounded domain $\Omega \subseteq \mathbb{R}^d$, with $d \geq 3$, and any $k \geq 1$, the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ of the Stokes operator in (1) satisfy*

$$\sum_{i=1}^{k} \lambda_i \geq \frac{4\pi d}{d+2}\left(\frac{\Gamma\left(1+\frac{d}{2}\right)}{(d-1)\,|\Omega|}\right)^{\frac{2}{d}} k^{1+\frac{2}{d}} + \frac{\varrho^2}{6(d+2)m_s^2}k + \frac{\omega_d^{-\frac{1}{d}}\varrho^{\frac{3d+1}{d}}}{9(d+2)m_s^3}k^{\frac{d-1}{d}} + \frac{3\omega_d^{-\frac{1}{d}}\varrho^{\frac{4d+2}{d}}}{80(n+2)m_s^4}k^{\frac{d-2}{d}},$$

*where $\varrho$ and $m_s$ are as in (13).*

When $d = 3$, we have an improvement:

**Theorem 4.** *Let $\Omega \subset \mathbb{R}^d$ with $d = 3$. Then the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ of the Stokes operator in (1) satisfy*

$$\sum_{i=1}^{k} \lambda_i \geq \frac{2\pi\varrho^6}{15m_s^5}\left((t+1)^6 - t^6\right), \tag{14}$$

*where $m_s$ and $\varrho$ are defined in (13) and*

$$t = \frac{1}{2}\left(\left(T+\sqrt{T^2+\tfrac{1}{27}}\right)^{\frac{1}{3}} - \left(-T+\sqrt{T^2+\tfrac{1}{27}}\right)^{\frac{1}{3}} - 1\right), \quad T = \frac{(d+1)m_s^d}{\omega_d\varrho^{d+1}}k.$$

*Furthermore,*

$$\sum_{i=1}^{k} \lambda_i \geq \frac{3}{5}\left(\frac{2\pi^3}{2\omega_3|\Omega|}\right)^{\frac{2}{3}} k^{\frac{5}{3}} + \frac{1}{24}\frac{|\Omega|}{I(\Omega)}k - \frac{11\pi\omega_3^{-\frac{1}{3}}}{180}\frac{\varrho^{\frac{14}{3}}}{m_s^4}k^{\frac{1}{3}} + \frac{659\pi\omega_3^{\frac{1}{3}}}{75000\cdot 4^{\frac{1}{3}}}\frac{\varrho^{\frac{22}{3}}}{m_s^6}k^{-\frac{1}{3}}.$$

Inspired by Melas's research [16], we adopted a similar methodology using fundamental techniques from prior studies [10; 22; 24; 27; 28; 29; 26]. While preserving the core strategy of these works, we introduced significant modifications to achieve stronger lower bounds.

***Outline.*** Section 2 presents results needed for proving the theorems discussed in this work. The main content is found in Sections 3–6, which respectively establish Theorems 1, 2, 3 and 4, proving intermediate results as needed.

## 2. Preliminaries

The set of eigenfunctions $\{\boldsymbol{u}_j\}_{j=1}^{\infty}$ of the Stokes operator is orthonormal in $L^2(\Omega)$, and so the set of Fourier transforms $\{\hat{\boldsymbol{u}}_j\}_{j=1}^{\infty}$ also forms an orthonormal set in $L^2(\mathbb{R}^d)$, by Plancherel's theorem. Let us set

$$U_k(\xi) = \sum_{j=1}^{k} |\hat{\boldsymbol{u}}_j(\xi)|^2 = \sum_{j=1}^{k} \left| \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\Omega} e^{-i\boldsymbol{x}\cdot\xi} \boldsymbol{u}_j(\boldsymbol{x})\, d\boldsymbol{x} \right|^2 \geq 0. \tag{15}$$

The integral is taken over $\Omega$ instead of $\mathbb{R}^d$ because the support of $\boldsymbol{u}_j$ is $\Omega$.

The following result, proven by A. A. Ilyin [10, Section 2], is essential to our calculations.

**Lemma 1** [10]. *The functions $U_k$ defined by (15) satisfy*

$$\int_{\mathbb{R}^d} U_k(\xi)\, d\xi = k, \tag{16}$$

$$U_k(\xi) \leq \frac{d-1}{(2\pi)^d} |\Omega|, \quad \xi \in \mathbb{R}^d, \tag{17}$$

$$\int_{\mathbb{R}^d} |\xi|^2\, U_k(\xi)\, d\xi = \sum_{j=1}^{k} \lambda_j. \tag{18}$$

Note that equations (16)–(18) bear a striking resemblance to their Laplacian counterparts. However, the proof of Lemma 1 employs suborthonormal functions [10].

We highlight some important properties of the function $U$. Let $U^*(\xi)$ designate the radial decreasing rearrangement of $U(\xi)$. By approximating $U$ if needed, we may assume that there exists a real-valued absolutely continuous function

$$\zeta : [0,\infty) \to \left[0, \frac{d-1}{(2\pi)^d} |\Omega|\right]$$

such that $U^*(\xi) = \zeta(|\xi|)$. Then

$$0 \leq -\zeta' \leq m_s, \tag{19}$$

where $m_s$ depends only on $d$ and $\Omega$ and is explicitly defined in (13). To see (19), let $\mu$ be the distribution function defined by

$$\mu(s) = |\{U(\xi) > s\}| = |\{U^*(\xi) > s\}|.$$

Then, we observe that

$$\mu(\zeta(t)) = \left|\{U^*(\xi) > \zeta(t)\}\right| = \left|\{\xi : |\xi| < t\}\right| = |B_t(0)| = w_d t^d.$$

Invoking the coarea formula in view of (17), we have

$$\mu(s)=\int_s^\infty\int_{\{U^{-1}(t)\}}\frac{1}{|\nabla U|}\,d\mathcal{H}\,dt=\int_s^{\frac{d-1}{(2\pi)^d}|\Omega|}\int_{\{U=t\}}\frac{1}{|\nabla U|}\,d\mathcal{H}\,dt,$$

where $\mathcal{H}$ is the $(d-1)$-dimensional Hausdorff measure. Let us consider $t>0$ values such that $\zeta'(t)<0$.

Let $\overline{K}$ denote the closure of $K\subset\mathbb{R}^d$ and $\partial K$ the boundary of $K$. The isoperimetric inequality,

$$\mathcal{H}(\partial K)\ge d w_d^{\frac{1}{d}}|\overline{K}|^{\frac{d-1}{d}},$$

enables us to deduce

$$\begin{aligned}\frac{dw_d t^{d-1}}{\zeta'(t)}&=\mu'(\zeta(t))=-\int_{\{U=\zeta(t)\}}\frac{1}{|\nabla U|}\,d\mathcal{H}\le-\frac{1}{m_S}\mathcal{H}(\{U=\zeta(t)\})\\&\le-\frac{1}{m_S}dw_d^{\frac{1}{d}}\mu(\zeta(t))^{\frac{d-1}{d}}=-\frac{1}{m_S}dw_d t^{d-1}.\end{aligned}$$

This inequality together with $\zeta'\le 0$ easily yields (19). Note that (19) essentially states that if the gradient vector of the original function is bounded, then the gradient of the rearrangement retains the same bound. (An alternative derivation is possible, using the Pólya–Szegő inequality.)

Now, let $r$ represent the real number such that $|\Omega|=w_d r^d$. We can get a lower bound for $I(\Omega)$:

$$I(\Omega)\ge\int_{B_r(0)}|\boldsymbol{x}|^2\,d\boldsymbol{x}=\frac{dw_d}{d+2}r^{d+2}=\frac{d}{d+2}w_d^{-\frac{2}{d}}|\Omega|^{\frac{d+2}{d}},$$

which quickly results in a lower bound for $m_s$ as follows:

$$m_s=\frac{2\sqrt{(d^2-d)|\Omega|I(\Omega)}}{(2\pi)^d}\ge\frac{\sqrt{d^2-d}}{(2\pi)^d}|\Omega|^{\frac{d+1}{d}}\omega_d^{-\frac{1}{d}}=:M. \tag{20}$$

## 3. Proof of Theorem 1

The following proof is inspired from the proof of the Berezin–Li–Yau inequality in [5; 14; 28; 29]. An analogous proof is also given in [20] by means of the bathtub principle [15].

*Proof of Theorem 1.* Assume the properties (16)–(18). Since $|\hat{\boldsymbol{u}}_j(\xi)|^2\,d\xi$ is a probability measure on $\mathbb{R}^d$ and $t\mapsto t^{-b}$ is convex for $t>0$ and $b>0$, employing Jensen's inequality and (15), we obtain

$$\sum_{j=1}^k\lambda_j^{-b}=\sum_{j=1}^k\left(\int_{\mathbb{R}^d}|\xi|^2\,|\hat{\boldsymbol{u}}_j(\xi)|^2\,d\xi\right)^{-b}\le\int_{\mathbb{R}^d}\frac{U_k(\xi)}{|\xi|^{2b}}\,d\xi. \tag{21}$$

Let $\mathbf{1}_\Omega(t)$ denote the characteristic function of the set $\Omega$. Define

$$\begin{aligned}\Psi_k(\xi) &= \frac{d-1}{(2\pi)^d}|\Omega|\mathbf{1}_{B_{S_k}(0)}(\xi) \quad \text{and}\\ S_k &= \left(\frac{(d-2b)(2\pi)^d\left(\sum_{j=1}^k \lambda_j^{-b}\right)}{d\,(d-1)\,\omega_d\,|\Omega|}\right)^{\frac{1}{d-2b}},\end{aligned} \tag{22}$$

so that

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{\Psi_k(\xi)}{|\xi|^{2b}}\,d\xi &= \frac{d-1}{(2\pi)^d}|\Omega|\,d\,\omega_d \int_0^{S_k} \frac{1}{t^{2b}}\,t^{d-1}\,dt\\ &= \frac{d-1}{(2\pi)^d}|\Omega|\,\frac{d\,\omega_d}{d-2b}S_k^{d-2b} = \sum_{j=1}^k \lambda_j^{-b}.\end{aligned} \tag{23}$$

Notice that

$$\left(\frac{1}{|\xi|^{2b}} - \frac{1}{S_k^{2b}}\right)(U_k(\xi) - \Psi_k(\xi)) \le 0. \tag{24}$$

Integrating (24) on $\mathbb{R}^d$ and using (23) we arrive at

$$\frac{1}{S_k^{2b}} \int_{\mathbb{R}^d} (U_k(\xi) - \Psi_k(\xi))\,d\xi \ge \int_{\mathbb{R}^d} \frac{U_k(\xi) - \Psi_k(\xi)}{|\xi|^{2b}}\,d\xi \ge 0 \tag{25}$$

from which it follows that

$$\int_{\mathbb{R}^d} U_k(\mu)\,dmu \ge \int_{\mathbb{R}^d} \Psi_k(\mu)\,d\mu.$$

Thus, by (16), we obtain

$$k \ge \int_{\mathbb{R}^d} \Psi_k(\mu)\,d\mu = \frac{d-1}{(2\pi)^d}|\Omega|\,d\omega_d\left(\frac{S_k^d}{d}\right). \tag{26}$$

Substituting $\omega_d$ given by (11) and $S_k$ given by (22) into (26) and rearranging the terms, we obtain (8). □

## 4. Proof of Theorem 2

We sketch a direct proof of (9) for the sake of completeness.

*Proof of Theorem 2.* Recall that $U_k$ satisfies (16)–(18). Since $|\hat{\boldsymbol{u}}_j(\xi)|^2\,d\xi$ is a probability measure on $\mathbb{R}^d$ and $r \mapsto r^a$ is concave for $r > 0$ and $0 < a \le 1$, we can use Jensen's inequality to derive that

$$\sum_{j=1}^k \lambda_j^a = \sum_{j=1}^k \left(\int_{\mathbb{R}^d} |\xi|^2\,|\hat{\boldsymbol{u}}_j(\xi)|^2\,d\xi\right)^a \ge \int_{\mathbb{R}^d} |\xi|^{2a}\,U_k(\xi)\,d\xi. \tag{27}$$

Setting

$$\Theta_k(\xi) = \frac{d-1}{(2\pi)^d}|\Omega|\,\mathbf{1}_{B_{T_k}(0)}(\xi), \quad T_k = \left(\frac{(d+2a)(2\pi)^d\left(\sum_{j=1}^k \lambda_j^a\right)}{d\,(d-1)\,\omega_d\,|\Omega|}\right)^{\frac{1}{d+2a}}, \tag{28}$$

we have

$$\begin{aligned}\int_{\mathbb{R}^d} |\xi|^{2a}\,\Theta_k(\xi)\,d\xi &= \frac{d-1}{(2\pi)^d}|\Omega|\,d\,\omega_d \int_0^{T_k} r^{2a} r^{d-1}\,dr \\ &= \frac{d-1}{(2\pi)^d}|\Omega|\,\frac{d\,\omega_d}{d+2a} T_k^{d+2a} = \sum_{j=1}^k \lambda_j^a.\end{aligned} \tag{29}$$

Now observe that

$$(|\xi|^{2a} - T_k^{2a})(U_k(\xi) - \Theta_k(\xi)) \ge 0. \tag{30}$$

Integrating (30) on $\mathbb{R}^d$ and using (27) and (29) we conclude that

$$T_k^{2a}\int_{\mathbb{R}^d}(U_k(\xi)-\Theta_k(\xi))\,d\xi \le \int_{\mathbb{R}^d} |\xi|^{2a}\,(U_k(\xi)-\Theta_k(\xi))\,d\xi \le 0, \tag{31}$$

which yields

$$\int_{\mathbb{R}^d} U_k(\xi)\,d\xi \le \int_{\mathbb{R}^d} \Theta_k(\xi)\,d\xi.$$

Now we use (16) to obtain

$$k \le \int_{\mathbb{R}^d} \Theta_k(\xi)\,d\xi = \frac{d-1}{(2\pi)^d}|\Omega|\,d\,\omega_d\left(\frac{T_k^d}{d}\right). \tag{32}$$

Substituting the values of $\omega_d$ and $T_k$ given in (11) and (28) into (32) and simplifying, we deduce (9). □

## 5. Proof of Theorem 3

Before the proof of Theorem 3, we give some elementary results from [11; 24]. To connect two key integrals in our proof, we introduce an important quantity, $a$, using the idea in [24]:

**Lemma 2.** *For any function* $\alpha : [0,\infty) \to [0,1]$ *and integer* $d \ge 1$ *with*

$$\int_0^\infty \phi(t)\,dt = 1, \quad \int_0^\infty t^d\phi(t)\,dt < \infty, \quad \int_0^\infty t^{d+2}\phi(t)\,dt \le \infty,$$

*there exists an* $a \ge 0$ *such that*

$$\int_a^{a+1} t^d\,dt = \int_0^\infty t^d\phi(t)\,dt \quad \text{and} \quad \int_a^{a+1} t^{d+2}dt \le \int_0^\infty t^{d+2}\phi(t)\,dt.$$

The following sharp inequality is key in estimating the lower bounds for $\sum_{i=1}^{k} \lambda_i$.

**Lemma 3** [11]. *For an integer $d \geq m+1 \geq 2$ and positive real numbers $t$ and $\tau$ we have*

$$dt^{d+2} - (d+2)\tau^2 t^d + 2\tau^{d+2} - \sum_{k=1}^{m+1} 2k t^{k-1} \tau^{n-k+1} (\tau - t)^2 \geq 0. \tag{33}$$

Considering all of these findings, we arrive at the following intermediate estimate.

**Proposition 1.** *If $d \geq m+1 \geq 2$, we have*

$$\sum_{i=1}^{k} \lambda_i \geq \omega_d \, (dA)^{\frac{d+2}{d}} (\zeta(0))^{-\frac{2}{d}} - \frac{2\omega_d A_{m+2} (dA)^{\frac{d-m+1}{d}} (\zeta(0))^{\frac{(m+1)d+m-1}{d}}}{(d+2) m_s^{m+1}} + \frac{2\omega_d (m+1) A_{m+3} (dA)^{\frac{d-m}{n}} (\zeta(0))^{\frac{(m+2)d+m}{d}}}{(d+2)(m+3) m_s^{m+2}}, \tag{34}$$

*where $m_s$ is defined in (13) and*

$$A_r := (a+1)^r - a^r \geq 1, \quad A := \frac{1}{d} \int_a^{a+1} t^d \, dt, \quad 0 \leq \zeta(0) \leq \frac{d-1}{(2\pi)^d} |\Omega|,$$

*the quantity $a$ being defined by*

$$\int_a^{a+1} t^d \, dt = \frac{k}{\omega_d}.$$

*Proof.* We use a strategy similar to that of [11; 24]. Since the map $\xi \mapsto |\xi|^2$ is radial and increasing, Lemma 1 implies that

$$k = \int_{\mathbb{R}^d} U(\xi) \, d\xi = \int_{\mathbb{R}^d} U^*(\xi) \, d\xi = d\omega_d \int_0^\infty t^{d-1} \zeta(t) \, dt, \tag{35}$$

$$\sum_{i=1}^{k} \lambda_i = \int_{\mathbb{R}^d} |\xi|^2 U(\xi) \, d\xi \geq \int_{\mathbb{R}^d} |\xi|^2 U^*(\xi) \, d\xi = d\omega_d \int_0^\infty t^{d+1} \zeta(t) \, dt. \tag{36}$$

For convenience, we rescale $\zeta$ as

$$S(t) = \frac{1}{\zeta(0)} \zeta \left( \frac{\zeta(0)}{m_s} t \right).$$

Obviously, $S(t)$ is a positive function with $S(0) = 1$ and $0 \leq -S'(t) \leq 1$. Let $\phi(t) := -S'(t)$ and define

$$A = \int_0^\infty t^{d-1} S(t) \, dt, \quad B = \int_0^\infty t^{d+1} S(t) \, dt. \tag{37}$$

Integrating by parts, we get

$$\int_0^\infty t^d \phi(t)\,dt = dA, \quad \int_0^\infty t^{d+2}\phi(t)\,dt = (d+2)B. \tag{38}$$

According to (38) and Lemma 2, there exists $a \geq 0$ such that

$$\int_a^{a+1} t^d dt = dA \quad \text{and} \quad \int_a^{a+1} t^{d+2} dt \leq (d+2)B. \tag{39}$$

For $i \geq 0$, $\tau \geq \frac{1}{2}$ and $a \geq 0$, we have

$$\int_a^{a+1} t^i(\tau - t)^2 dt = \frac{A_{i+3}}{i+3} - \frac{2A_{i+2}}{i+2}\tau + \frac{A_{i+1}}{i+1}\tau^2, \tag{40}$$

where

$$A_r = (a+1)^r - a^r \geq 1.$$

Taking into account Lemma 3 and (40), we may integrate (33) in $t$ from $a$ to $a+1$ to get

$$\begin{aligned} d(d+2)B \geq{}& d(d+2)A\tau^2 - 2\tau^{d+2} \\ &+ \sum_{i=1}^{m+1} 2i\tau^{d-i+1}\left(\frac{A_i}{i}\tau^2 - \frac{2A_{i+1}}{i+1}\tau + \frac{A_{i+2}}{i+2}\right). \end{aligned} \tag{41}$$

The summation on the second row of (41) can be rewritten as

$$\begin{aligned} &= 2\tau^{d+2} + 2\sum_{i=1}^{m} A_{i+1}\tau^{d-i+2} - 2\sum_{i=1}^{m+1} \frac{2iA_{i+1}}{i+1}\tau^{d-i+2} + 2\sum_{i=1}^{m+1} \frac{iA_{i+2}}{i+2}\tau^{d-i+1} \\ &= 2\tau^{d+2} + 2A_2\tau^{d+1} + \frac{2mA_{m+2}}{m+2}\tau^{d-m+1} + \frac{2(m+1)A_{m+3}}{m+3}\tau^{d-m} \\ &\quad -2A_2\tau^{d+1} - \frac{4(m+1)A_{m+2}}{m+2}\tau^{d-m+1} + 2\sum_{i=2}^{m}\left(1 + \frac{i-1}{i+1} - \frac{2i}{i+1}\right)A_{i+1}\tau^{d-i+2}. \end{aligned}$$

The parenthetical factor in this last summation vanishes, so the right-hand side of the preceding display boils down to

$$\frac{2(m+1)A_{m+3}}{m+3}\tau^{d-m} + 2\tau^{d+2} - 2A_{m+2}\tau^{d-m+1},$$

and we deduce that

$$d(d+2)B - (d+2)\tau^2 dA \geq \frac{2(m+1)A_{m+3}}{m+3}\tau^{d-m} - 2A_{m+2}\tau^{d-m+1}. \tag{42}$$

Using Jensen's inequality, we obtain $(dA)^{\frac{1}{d}} \geq \int_\delta^{\delta+1} t\,dt \geq \frac{1}{2}$ for any $\delta \geq 0$. Hence, putting $\tau = (dA)^{\frac{1}{d}}$ in (42), we arrive at

$$B \geq \frac{(dA)^{\frac{d+2}{d}}}{d} - \frac{2A_{m+2}(dA)^{\frac{d-m+1}{n}}}{d(d+2)} + \frac{2(m+1)A_{m+3}(nA)^{\frac{d-m}{d}}}{d(d+2)(m+3)}. \tag{43}$$

Taking the rescaling of $\zeta(t)$ into consideration, we obtain from (43) the integral inequality

$$\int_0^\infty t^{s+1}\zeta(t)\,dt \geq \frac{(dA)^{\frac{d+2}{n}}(\zeta(0))^{-\frac{2}{d}}}{d} - \frac{2S_{m+2}(dA)^{\frac{d-m+1}{d}}(\zeta(0))^{\frac{(m+1)d+m-1}{d}}}{d(d+2)m_s^{m+1}} + \frac{2(m+1)S_{m+3}(dA)^{\frac{d-m}{d}}(\zeta(0))^{\frac{(m+2)n+m}{d}}}{d(d+2)(m+3)m_s^{m+2}}.$$

Notice that $\inf_t \zeta(t) \leq \zeta(0) \leq \sup_t \zeta(t)$. Due to (36), the estimate of $\int_0^\infty t^{s+1}\zeta(t)\,dt$ provides a lower bound

$$\sum_{i=1}^k \lambda_i \geq \omega_d\,(dA)^{\frac{d+2}{d}}(\zeta(0))^{-\frac{2}{d}} - \frac{2\omega_d A_{m+2}(dA)^{\frac{d-m+1}{d}}(\zeta(0))^{\frac{(m+1)d+m-1}{d}}}{(d+2)m_s^{m+1}} + \frac{2\omega_d(m+1)A_{m+3}(dA)^{\frac{d-m}{n}}(\zeta(0))^{\frac{(m+2)d+m}{d}}}{(d+2)(m+3)m_s^{m+2}}.$$

This is what we required in Proposition 1. □

The right-hand side of the intermediate lower bound (34) depends on $\zeta(0)$. Our goal is to derive a final eigenvalue inequality that depends solely on $d$ and $k$. To achieve this, we adopt a strategy of minimizing (34) with respect to $\zeta(0)$ over the interval $[0, \varrho]$, where $\varrho = \frac{d-1}{(2\pi)^d}|\Omega|$ as defined in (13).

*Proof of Theorem 3.* Using (35) and (38) in combination with integration by parts, we obtain

$$\frac{k}{\omega_n} = d\int_0^\infty t^{d-1}\zeta(t)\,dt = \int_0^\infty -t^d\zeta'(t)\,dt = \int_0^\infty t^d\phi(t)\,dt = dA.$$

Using the fact that $dA = k/\omega_d$, we define $G_1(x)$, $G_2(x)$ and $G(x)$ for $x \in [0, \varrho]$ by

$$G_1(x) = \omega_d^{-\frac{2}{d}}x^{-\frac{2}{d}}k^{\frac{d+2}{d}} + c_2\frac{2\omega_d^{\frac{m}{d}}(m+1)A_{m+3}x^{\frac{(m+2)d+m}{d}}}{(d+2)(m+3)m_s^{m+2}}k^{\frac{d-m}{d}},$$

$$G_2(x) = -\frac{2\omega_d^{\frac{m-1}{d}}A_{m+2}x^{\frac{(m+1)d+m-1}{d}}}{(d+2)m_s^{m+1}}k^{\frac{d-m+1}{d}},$$

$$G(x) = G_1(x) - G_2(x).$$

Here $0 < c_2 \le 1$ is a positive number to be determined. Since $m \ge 1$, we conclude that $G_2(x)$ is decreasing on $[0, \varrho]$. By direct calculation, we have

$$\frac{d}{2}\frac{\omega_d^{\frac{2}{d}} G_1'(x)}{k^{\frac{d-m}{d}}} = -k^{\frac{2+d}{d}} x^{-\frac{d+2}{d}} + c_2 \omega_d^{\frac{m+2}{d}} \frac{(m+1)\big((m+2)d+m\big)}{(d+2)(m+3)} \frac{A_{m+3}}{m_s^{m+2}} x^{\frac{(m+1)d+m}{d}}.$$

Therefore, when

$$c_2 \le \frac{k^{\frac{d+2}{d}}}{\omega_d^{\frac{m+2}{d}}} \frac{(d+2)(m+3)}{(m+1)[(m+2)d+m]} \frac{m_s^{m+2}}{A_{m+3}} \varrho^{-\frac{(m+2)(d+1)}{d}},$$

we conclude that $G_1'(x) \le 0$ on $[0, \varrho]$. Since

$$m_s \ge M = \frac{\sqrt{d^2-d}}{(2\pi)^d \omega_d^{\frac{1}{d}}} |\Omega|^{\frac{d+1}{d}}, \quad \varrho = \frac{d-1}{(2\pi)^d}|\Omega|, \tag{44}$$

we can choose $c_1 = \min\{1, a_1\}$, where $a_1$ is defined by

$$a_1 = \frac{(d+2)(m+3)}{(m+1)\big[(m+2)d+m\big]} \frac{(2\pi)^{m+2}}{A_{m+3}} \frac{d^{\frac{m+2}{2}}}{(d-1)^{\frac{m(d+2)+2m}{2d}}}, \tag{45}$$

to obtain

$$\sum_{i=1}^{k} \lambda_i \ge \omega_d^{-\frac{2}{d}} \varrho^{-\frac{2}{d}} k^{\frac{d+2}{d}} - \frac{2\omega_d^{\frac{m-1}{d}} A_{m+2} \varrho^{\frac{(m+1)d+m-1}{d}}}{(d+2)M^{m+1}} k^{\frac{d-m+1}{d}} + c_1 \frac{2\omega_d^{\frac{m}{d}}(m+1)A_{m+3}\varrho^{\frac{(m+2)d+m}{d}}}{(d+2)(m+3)M^{m+2}} k^{\frac{d-m}{d}}, \tag{46}$$

where $\varrho$ and $M$ are defined in (13). Similarly, we can split $G(x)$ as a sum of two functions:

$$G(x) = g_1(x) + g_2(x), \quad x \in [0, \varrho],$$

where $g_1$ and $g_2$ are defined by $g_1(x) = \omega_d^{-\frac{2}{d}} x^{-\frac{2}{d}} k^{\frac{d+2}{d}}$ and

$$g_2(x) = c_2 \frac{2\omega_d^{\frac{m}{d}}(m+1)A_{m+3} x^{\frac{(m+2)d+m}{d}}}{(d+2)(m+3)m_s^{m+2}} k^{\frac{d-m}{d}} - \frac{2\omega_d^{\frac{m-1}{d}} A_{m+2} x^{\frac{(m+1)d+m-1}{d}}}{(d+2)m_s^{m+1}} k^{\frac{d-m+1}{d}}.$$

Obviously, $g_1(x)$ is decreasing on $[0, \varrho]$. Rewriting $g_2(x)$ as

$$\frac{d(d+2)}{2} \frac{m_s}{\omega_d^{\frac{m}{d}} k^{\frac{d-m}{d}}} g_2(x) = c_2 \frac{[(m+2)d+m](m+1)A_{m+3}}{(m+3)m_s} x^{\frac{(m+1)d+m}{d}} - ((m+1)d+m-1)\omega_d^{-\frac{1}{d}} A_{m+2} k^{\frac{1}{d}} x^{\frac{(d+1)m-1}{d}}$$

allows us to observe that if

$$c_2 \le \frac{[(m+1)d+m-1](m+3)}{[(m+2)d+m](m+1)}\,\frac{A_{m+2}}{A_{m+3}}\,\frac{m_s}{\omega_d^{1/d}}\,\varrho^{-\frac{d+1}{d}},$$

then $g_2'(x) \le 0$ on $[0,\varrho]$. Using (44) again, we choose $c_3 = \min\{1, a_2\}$, where $a_2$ is defined by

$$a_2 = \frac{\sqrt{2}d^{\frac{1}{d}}}{2(d-1)^{\frac{d+2}{2d}}}\,\frac{[(m+1)d+m-1](m+3)}{[(m+2)d+m](m+1)}\,\frac{A_{m+2}}{A_{m+3}}, \tag{47}$$

to derive that

$$\sum_{i=1}^{k}\lambda_i \ge \omega_d^{-\frac{2}{d}}\varrho^{-\frac{2}{d}}k^{\frac{d+2}{d}} - \frac{2\omega_d^{\frac{m-1}{d}}A_{m+2}\varrho^{\frac{(m+1)d+m-1}{d}}}{(d+2)M^{m+1}}k^{\frac{d-m+1}{d}} + c_3\frac{2\omega_d^{\frac{m}{d}}(m+1)A_{m+3}\varrho^{\frac{(m+2)d+m}{d}}}{(d+2)(m+3)M^{m+2}}k^{\frac{d-m}{d}}. \tag{48}$$

In light of equations (46) and (48), we can use $c = \min\big\{1, \max\{a_1, a_2\}\big\}$, where $a_1$ and $a_2$ are defined in (45) and (47), to get

$$\sum_{i=1}^{k}\lambda_i \ge \omega_d^{-\frac{2}{d}}\varrho^{-\frac{2}{d}}k^{\frac{d+2}{d}} - \frac{2\omega_d^{\frac{m-1}{d}}A_{m+2}\varrho^{\frac{(m+1)d+m-1}{d}}}{(d+2)M^{m+1}}k^{\frac{d-m+1}{d}} + c\,\frac{2\omega_d^{\frac{m}{d}}(m+1)A_{m+3}\varrho^{\frac{(m+2)d+m}{d}}}{(d+2)(m+3)M^{m+2}}k^{\frac{d-m}{d}}. \tag{49}$$

Finally, we use

$$\omega_d^{-\frac{2}{d}}\varrho^{-\frac{2}{d}}k^{\frac{d+2}{d}} = 4\pi\left(\frac{\Gamma\big(1+\frac{d}{2}\big)}{(d-1)\,|\Omega|}\right)^{\frac{2}{d}}k^{1+\frac{2}{d}} \tag{50}$$

to complete the proof of Theorem 3. □

*Proof of Corollary 1.* An argument similar to that of [11] gives

$$d(d+2)B - d(d+2)A\tau^2 + 2\tau^{d+2} \ge \tfrac{1}{6}(\tau^d) + \tfrac{1}{9}(\tau^{d-1}) + \tfrac{3}{80}\tau^{d-2}.$$

Hence, we get

$$\sum_{i=1}^{k}\lambda_i \ge \frac{d\omega_d^{-\frac{2}{d}}\zeta(0)^{-\frac{2}{d}}}{d+2}k^{\frac{d+2}{d}} + \frac{\zeta(0)^2}{6(d+2)m_s^2}k + \frac{c_4\omega_d^{-\frac{1}{d}}\zeta(0)^{\frac{3d+1}{d}}}{(d+2)m_s^3}k^{\frac{d-1}{d}} + \frac{c_5\omega_d^{-\frac{1}{d}}\zeta(0)^{\frac{4d+2}{d}}}{(n+2)m_s^4}k^{\frac{d-2}{d}},$$

where $0 < c_4 \leq \frac{1}{9}$ and $0 < c_5 \leq \frac{3}{80}$ are two constants to be chosen. Following an approach analogous to the discussion in the proof of Theorem 3, we can choose $c_4 = \frac{1}{9}$ and $c_5 = \frac{3}{80}$. Using the equality in (50) completes the proof. □

## 6. Proof of Theorem 4

The following lemma from [10] plays a pivotal role in establishing our eigenvalue inequality and makes use of the piecewise continuous function

$$\Phi_s(r) = \begin{cases} m_s & \text{if } 0 \leq r \leq s, \\ m_s - \varrho(r-s) & \text{if } s \leq r \leq s + m_s/\varrho, \\ 0 & \text{if } s + m_s/\varrho \leq r, \end{cases}$$

where $m_s$ and $\varrho$ are defined in (13).

**Lemma 4** [10]. *Let $\alpha > 0$ be a real number and $\Psi(r)$ be a decreasing and absolutely continuous function such that*

$$\int_0^\infty r^\alpha \, \Psi(r)\, dr = \int_0^\infty r^\alpha \, \Phi_s(r)\, dr, \quad 0 \leq \Psi \leq m_s, \quad -L < \Psi' < 0.$$

*Then, for any $\beta \geq \alpha$, the following integral inequality holds*:

$$\int_0^\infty r^\beta \, \Psi(r)\, dr \geq \int_0^\infty r^\beta \, \Phi_s(r)\, dr.$$

*Moreover, for any $\gamma \geq 0$, we have*

$$\int_0^\infty r^\gamma \, \Phi_s(r)\, dr = \frac{\varrho^{\gamma+2}}{(\gamma+1)(\gamma+2)m_s^{\gamma+1}}\big((t+1)^{\gamma+2} - t^{\gamma+2}\big), \quad s = \frac{t m_s}{\varrho}.$$

*Proof of Theorem 4*. Let $m_s$ be as in (13). Applying Lemma 4 to $\Psi(x) = \zeta(x)$ and setting $\alpha = d-1$, we obtain

$$\int_0^\infty t^{d-1}\zeta(t)\, dt = \frac{k}{d\omega_d} = \frac{\varrho^{d+1}}{d(d+1)m_s^d}\big((t+1)^{d+1} - t^{d+1}\big),$$

which in turn implies that $t$ is the unique root of the equation

$$(t+1)^{d+1} - t^{d+1} = T := \frac{(d+1)m_s^d}{\omega_d \varrho^{d+1}} k.$$

Then Lemma 4 implies

$$\int_0^\infty t^{d+1}\zeta(t)\, dt \geq \frac{\varrho^{d+3}}{(d+2)(d+3)m_s^{d+2}}\big((t+1)^{d+3} - t^{d+3}\big). \tag{51}$$

One can easily check that $T$ satisfies

$$T \geq 1, \quad T \geq \frac{d+1}{d-1} \frac{(4\pi)^d}{\omega_d^2} \left( \frac{d^2}{(d-1)(d+2)} \right)^{\frac{d}{2}}.$$

In particular, when $d = 3$, we consider the equation

$$(t+1)^4 - t^4 = T = \frac{4\varrho^3}{\omega_3 m_s^4} k,$$

whose positive root is

$$t = \frac{1}{2} \left( \left( T + \sqrt{T^2 + \tfrac{1}{27}} \right)^{\frac{1}{3}} - \left( -T + \sqrt{T^2 + \tfrac{1}{27}} \right)^{\frac{1}{3}} - 1 \right).$$

From (36), we obtain our desired result as

$$\sum_{i=1}^{k} \lambda_i \geq \frac{2\pi\varrho^6}{15 m_s^5} \left( (t+1)^6 - t^6 \right). \tag{52}$$

Considering the Taylor series of $t$, we arrive at

$$t \geq -\tfrac{1}{2} + 4^{-\frac{1}{3}} T^{\frac{1}{3}} - \tfrac{1}{6} 2^{-\frac{1}{3}} T^{-\frac{1}{3}} + \tfrac{1}{324} 4^{-\frac{1}{3}} T^{-\frac{5}{3}} + \tfrac{1}{1944} 2^{-\frac{1}{3}} T^{-\frac{7}{3}} - \tfrac{1}{26244} 4^{-\frac{1}{3}} T^{-\frac{11}{3}} - \tfrac{1}{629856} 2^{-\frac{1}{3}} T^{-\frac{11}{3}},$$

where we used the fact that $T \geq 1$. Hence, we obtain

$$(t+1)^6 - t^6 \geq \tfrac{3}{4} 2^{-\frac{1}{3}} T^{\frac{5}{3}} + \tfrac{5}{8} T - \tfrac{11}{24} 2^{\frac{2}{3}} T^{\frac{1}{3}} + \tfrac{659}{10000} T^{-\frac{1}{3}}.$$

Putting the lower bound of $(t+1)^6 - t^6$ into (52), we get

$$\sum_{i=1}^{k} \lambda_i \geq \frac{3}{5} \left( \frac{2\pi^3}{2\omega_3 |\Omega|} \right)^{\frac{2}{3}} k^{\frac{5}{3}} + \frac{1}{24} \frac{|\Omega|}{I(\Omega)} k - \frac{11\pi}{180} \frac{\varrho^{\frac{14}{3}}}{\omega_3^{\frac{1}{3}} m_s^4} k^{\frac{1}{3}} + \frac{659\pi\omega_3^{\frac{1}{3}}}{75000 \cdot 4^{\frac{1}{3}}} \frac{\varrho^{\frac{22}{3}}}{m_s^6} k^{-\frac{1}{3}}.$$

This concludes the proof of Theorem 4. □

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