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We consider a new class Q of distribution functions F that have the property of rational-infinite divisibility: there exist some infinitely divisible distribution functions F_1 and F_2 such that $F_1 = F * F_2$. A distribution function of class Q is quasi-infinitely divisible in the sense that its characteristic function admits the Lévy-type representation with a "signed spectral measure". This class is a wide natural extension of the fundamental class of infinitely divisible distribution functions and it is being actively studied now. We are interested in conditions for a distribution function F to belong to class Q for the unexplored case, where F may have a continuous singular part. We propose a criterion under the assumption that the continuous singular part of F is dominated by the discrete part in a certain sense. The criterion generalizes the previous results by Alexeev and Khartov for discrete probability laws and the results by Berger and Kutlu for the mixtures of discrete and absolutely continuous laws. In addition, we describe the characteristic triplet of the corresponding Lévy-type representation, which may contain a continuous singular part. We also show that the assumption of the dominated continuous singular part cannot be omitted or even slightly extended (without some special assumptions). We apply the general criterion to some interesting particular examples. We also positively solve the decomposition problem stated by Lindner, Pan and Sato within the case being considered.

1. Introduction

This paper is devoted to the study of a new class of probability laws that naturally extends the fundamental class of infinitely divisible distributions.

Let F be a distribution function on the real line $\mathbb{R}$. Recall that F and the corresponding probability law are called *infinitely divisible* if for every positive integer n there exists a distribution function $F_{1/n}$ such that $F = (F_{1/n})^{*n}$, where

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"$*$" denotes the convolution operation, i.e., F is the n-fold convolution power of $F_{1/n}$. Let f be the characteristic function of F, i.e.,

$$f(t) := \int_{\mathbb{R}} e^{itx}\, dF(x), \quad t \in \mathbb{R}.$$

It is well-known (see [12; 31; 37]) that F is infinitely divisible if and only if f admits the *Lévy representation*

$$f(t) = \exp\left\{it\gamma - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \bigl(e^{itx} - 1 - it\sin(x)\bigr)\, dL(x)\right\}, \quad t \in \mathbb{R}, \tag{1}$$

with a *shift parameter* $\gamma \in \mathbb{R}$, a *the Gaussian variance* $\sigma^2 \geqslant 0$, and a *Lévy spectral function* $L : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, which is nondecreasing on each of the intervals $(-\infty, 0)$ and $(0, +\infty)$, and satisfies

$$\lim_{x\to-\infty} L(x) = \lim_{x\to+\infty} L(x) = 0, \tag{2}$$

and also

$$\int_{O_\delta} x^2\, dL(x) < \infty \quad \text{for any} \quad \delta > 0,$$

where $O_\delta := (-\delta, 0) \cup (0, \delta)$. The function L is assumed to be right-continuous at every point of the real line. Importantly, the *characteristic triplet* (γ, σ^2, L) is uniquely determined by f and hence by F. Due to this representation, the class of infinitely divisible probability laws has found a lot of applications through Lévy processes (see [31]), the stochastic calculus (see [4]), teletraffic models (see [25]), and actuarial mathematics (see [32]).

Let $\boldsymbol{I}$ denote the class of all infinitely divisible distribution functions on the real line. This class is naturally extended in the following way. We call a distribution function F (and the corresponding probability law) *rational-infinitely divisible* if there exist some infinitely divisible distribution functions F_1 and F_2 such that $F_1 = F * F_2$. In terms of characteristic functions, this definition is equivalent to the formula $f(t) = f_1(t)/f_2(t)$, $t \in \mathbb{R}$, for the characteristic function f of F, where f_1 and f_2 are the characteristic functions of some infinitely divisible distribution functions F_1 and F_2. We denote by $\boldsymbol{Q}$ the class of all rational-infinitely divisible distribution functions. Since F_2 may be chosen as degenerate at some point a (i.e., $f_2(t) = e^{ita}$, $t \in \mathbb{R}$), it is clear that, indeed, $\boldsymbol{I} \subset \boldsymbol{Q}$. Moreover, from the definition, it is seen that the characteristic function f of any $F \in \boldsymbol{Q}$ admits a *Lévy-type representation*. Namely, if F_1 and F_2 have characteristic triplets $(\gamma_1, \sigma_1^2, L_1)$ and $(\gamma_2, \sigma_2^2, L_2)$, then formula (1) holds with *the shift parameter* $\gamma = \gamma_1 - \gamma_2 \in \mathbb{R}$, *the Gaussian variance* $\sigma^2 = \sigma_1^2 - \sigma_2^2$, and *the spectral function* $L = L_1 - L_2$. In that case, L has a bounded total variation on $\mathbb{R} \setminus O_\delta$ for every $\delta > 0$, and, in general, it is nonmonotonic on the intervals $(-\infty, 0)$ and $(0, +\infty)$. The function L also inherits from L_1 and L_2 right-continuity on $\mathbb{R}$ and property (2). Moreover,

$$\int_{O_\delta} x^2\, d|L|(x) < \infty \quad \text{for any} \quad \delta > 0,$$

where we integrate over the variation of L. We now suppose that, conversely, f admits representation (1) with some $\gamma \in \mathbb{R}$, $\sigma^2 \geqslant 0$, and L satisfying the above conditions. Following Lindner and Sato [26], the corresponding distribution function F and probability law are called *quasi-infinitely divisible*. Let us fix any real γ_1 and γ_2 such that $\gamma = \gamma_1 - \gamma_2$. Let us fix any nonnegative σ_1^2 and σ_2^2 such that $\sigma_1^2 \geqslant \sigma_2^2$ and $\sigma^2 = \sigma_1^2 - \sigma_2^2$. Due to the Hahn–Jordan decomposition, it is not difficult to show that there exist some canonical Lévy spectral functions L_1 and L_2 satisfying the usual conditions with monotonicity such that $L = L_1 - L_2$. Then $f(t) = f_1(t)/f_2(t)$, $t \in \mathbb{R}$, where f_1 and f_2 are represented by the canonical Lévy formula (1) with characteristic triplets $(\gamma_1, \sigma_1^2, L_1)$ and $(\gamma_2, \sigma_2^2, L_2)$. So f_1 and f_2 are the characteristic functions for some infinitely divisible distribution functions and hence the distribution function F corresponding to such an f is rational-infinitely divisible. Thus $F \in \boldsymbol{Q}$ if and only if f admits representation (1) with some (γ, σ^2, L) satisfying the above conditions. Additionally, *the characteristic triplet* (γ, σ^2, L) is uniquely determined by f and hence by F as for infinitely divisible laws (this can be concluded from the assertion in [12, p. 80]). It is also clear that for any rational-infinitely divisible F its characteristic function f has no zeroes on the real line, i.e., $f(t) \neq 0$, $t \in \mathbb{R}$.

The class $\boldsymbol{Q}$ and its multivariate analog are objects of active study (see [2; 8; 26]) and they find interesting applications in probability limit and compactness theorems (see Sections 4 and 8 in [26], Section 3 in [3], the paper [19], and also [1; 17]), and in other areas (see, for instance, [10; 29; 30]). But, actually, nondegenerate representatives of $\boldsymbol{Q} \setminus \boldsymbol{I}$ appeared even earlier in the theory of decompositions of probability laws as components of certain infinitely divisible distribution functions (see [12, pp. 81–83; 27, p. 165]).

The class $\boldsymbol{Q}$ is seen to be rather wide. For instance, it contains the distribution function of every probability law that has a mass $> 1/2$ at some point. Hence the class $\boldsymbol{Q}$ contains nondegenerate distribution functions of some probability laws with bounded supports (see examples in [26]), which are “far” from the infinite divisibility property in a known sense (see [5]). So it is interesting and important to obtain criteria for belonging to the class $\boldsymbol{Q}$. The existing results in this direction usually have simple and nice formulations in terms of characteristic functions. The first quite general result of such type was obtained by Lindner, Pan, and Sato in [26] (see Theorem 8.1, p. 30). It states that a lattice distribution function F belongs to the class $\boldsymbol{Q}$ if and only if its characteristic function f does not have zeroes on the real line, i.e., $f(t) \neq 0$ for any $t \in \mathbb{R}$.

In [3] and [18], this result was generalized to the class of arbitrary discrete probability laws. Namely, a discrete distribution function F belongs to $\boldsymbol{Q}$ if and only if its characteristic function f is separated from zero, i.e., $|f(t)| \geqslant \mu$ for any $t \in \mathbb{R}$ and for some constant $\mu > 0$. Moreover, in that case, the components of the characteristic triplet are fully described. This result is a generalization of the previous one for discrete lattice distributions, because the absolute value $|f(\cdot)|$ of the characteristic function f of a discrete lattice distribution is a periodic continuous function on $\mathbb{R}$. Therefore such f is zero-free on the period segment (and hence on $\mathbb{R}$) if any only if it is separated from zero.

In [6], Berger considered mixtures of a degenerate law (with a nonzero coefficient) and absolutely continuous distributions. According to his result, a distribution function F of such type belongs to $\boldsymbol{Q}$ if and only if $f(t) \neq 0$, $t \in \mathbb{R}$. Moreover, the result describes the structure of the components of the characteristic triplet in that case. The author also formulated more general criterion for the case, when the degenerate law from the previous one is replaced by a discrete lattice distribution with characteristic function, which has no zeroes on the real line. At present, however, the most general criterion (among those that use assumptions about the type of distribution) is the following result for the mixtures of discrete and absolutely continuous probability laws, which was obtained by Berger and Kutlu in the paper [7]. Let us formulate it with more details here. Namely, assume that $F(x) = c_d F_d(x) + c_a F_a(x)$, $x \in \mathbb{R}$, where F_d is a discrete distribution function, F_a is an absolutely continuous distribution function, $c_d > 0$, $c_a \geqslant 0$, and $c_d + c_a = 1$. We write the characteristic function f in the corresponding form: $f(t) = c_d f_d(t) + c_a f_a(t)$, $t \in \mathbb{R}$. Then $F \in \boldsymbol{Q}$ if and only if $f(t) \neq 0$ and $|f_d(t)| \geqslant \mu$ for any $t \in \mathbb{R}$ with some constant $\mu > 0$. It is equivalent to the condition $|f(t)| \geqslant \mu'$ for any $t \in \mathbb{R}$ and for some constant $\mu' > 0$. Moreover, Berger and Kutlu showed the existence of some discrete part in the spectral function and they fully described its absolutely continuous part for this case. It should be noted that we are not aware of any similar results for purely absolutely continuous distribution functions F. However, for some cases the problem of membership in class $\boldsymbol{Q}$ for a given distribution function of such type is not difficult to solve by the general criteria proposed in [20] with some additional analysis.

This article is devoted to generalizing and complementing all the mentioned results (except [20]) for the case, when F may have a continuous singular part. Namely, we propose a criterion for a distribution function F to belong to class $\boldsymbol{Q}$ under the assumption that its continuous singular part is dominated by its discrete part in a certain sense. In fact, we show that the conditions on f from the results [6] and [7] are carried over to this case. In addition, we describe the characteristic triplet of the corresponding Lévy-type representation, which may contain some continuous singular part.

We next show that the assumption of a dominated continuous singular part cannot be omitted or even slightly extended without some additional assumptions. In addition, for any $F \in \boldsymbol{Q}$ we solve the decomposition problem, which was stated by Lindner, Pan, Sato in [26] (see Open Question 8.4), within the considered case. Here we obtain a positive solution generalizing similar results from [6] and [7].

The article has the following structure. Section 2 contains necessary preliminaries, more detailed statements of some preexisting results mentioned above and the formulations of the new results of the paper. In Section 3, we formulate some important known theorems and useful lemmata, which will be auxiliary tools needed for the proofs of our results. In Section 4, we first prove a key auxiliary lemma and we next propose the proofs of the main results of the article.

Throughout the paper, we use the following notation. We denote by $\mathbb{N}$ the set of positive integers and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The symbols $\mathbb{Z}$ and $\mathbb{Q}$ denote as usual the sets of all integers and rational numbers, respectively. Next, $\mathbb{C}$ is the set of all complex numbers. For any $z \in \mathbb{C}$ we denote by $\mathrm{Im}\{z\}$ and $\arg(z)$ its imaginary part and the principal value of the argument of z, respectively. If ψ is a complex-valued continuous function on $\mathbb{R}$ satisfying $\psi(0) = c \in \mathbb{C} \setminus \{0\}$ and $\psi(t) \neq 0$ for any $t \in \mathbb{R}$, then *the distinguished logarithm* $\mathrm{Ln}\,\psi$ is defined by the formula $\mathrm{Ln}\,\psi(t) := \ln|\psi(t)| + i\,\mathrm{Arg}\,\psi(t)$, $t \in \mathbb{R}$, where $\mathrm{Arg}\,\psi(t)$ is the argument of $\psi(t)$ uniquely defined on $\mathbb{R}$ by the continuity with the condition $\mathrm{Arg}\,\psi(0) = \arg(c) \in (-\pi, \pi]$. The symbol $\mathbb{1}_a$ with fixed $a \in \mathbb{R}$ denotes the distribution function of the degenerate law concentrated at the point a, i.e., $\mathbb{1}_a(x) = 1$ for $x \geqslant a$ and $\mathbb{1}_a(x) = 0$ for $x < a$. For any set A we denote by $\mathbb{I}_A$ the indicator function of A, i.e., $\mathbb{I}_A(x) = 1$ for any $x \in A$ and $\mathbb{I}_A(x) = 0$ for any $x \notin A$. The signum function is denoted by $\mathrm{sgn}(\,\cdot\,)$, i.e., $\mathrm{sgn}(x) = +1$ for $x > 0$, $\mathrm{sgn}(x) = -1$ for $x < 0$, and $\mathrm{sgn}(0) = 0$. For any finite set A the symbol $|A|$ denotes the number of elements of A. We always set $\sum_{k\in A} a_k = 0$ and $\prod_{k\in A} a_k = 1$ in the case $A = \varnothing$. For any two vectors x and y from $\mathbb{R}^n$ the standard scalar product is denoted by $\langle x, y\rangle$.

For any function G defined on $\mathbb{R}$ the limits $\lim_{x\to\pm\infty} G(x)$ are denoted by $G(\pm\infty)$, respectively, if these limits exist. The class of all functions $G : \mathbb{R} \to \mathbb{R}$ of bounded total variation on $\mathbb{R}$ (nonmonotonic in general), which are right-continuous at every point and $G(-\infty) = 0$, is denoted by $\boldsymbol{V}$. We denote by $\boldsymbol{V}_{\mathbb{C}}$ the set of all functions $G : \mathbb{R} \to \mathbb{C}$ of the form $G(x) = G_1(x) + iG_2(x)$, $x \in \mathbb{R}$, where G_1 and G_2 are from $\boldsymbol{V}$. For every $G \in \boldsymbol{V}$ (or $\boldsymbol{V}_{\mathbb{C}}$) its total variation on $\mathbb{R}$ will be denoted by $\|G\|$ and the total variation on $(-\infty, x]$ by $|G|(x)$, $x \in \mathbb{R}$. So we have $|G(x)| \leqslant |G|(x) \leqslant \|G\|$, $x \in \mathbb{R}$, and $|G|(+\infty) = \|G\|$. Next, we adopt the following convention. Let G be a function from $\boldsymbol{V}$ with the Fourier–Stieltjes transform g, i.e., $g(t) = \int_{\mathbb{R}} e^{itx}\, dG(x)$, $t \in \mathbb{R}$. In view of the uniqueness theorem for functions from $\boldsymbol{V}$, we set $\|g\| := \|G\|$. So $\|g\| = 0$ if and only if $g(t) = 0$, $t \in \mathbb{R}$, and $\|c \cdot g\| = |c| \cdot \|g\|$ for any $c \in \mathbb{R}$. Let G_1 and G_2 be functions from $\boldsymbol{V}$ with the

Fourier–Stieltjes transforms g_1 and g_2, correspondingly. The known inequalities $\|G_1 + G_2\| \leqslant \|G_1\| + \|G_2\|$ and $\|G_1 * G_2\| \leqslant \|G_1\| \cdot \|G_2\|$ are correspondingly written as $\|g_1 + g_2\| \leqslant \|g_1\| + \|g_2\|$ and $\|g_1 \cdot g_2\| \leqslant \|g_1\| \cdot \|g_2\|$. We recall that both $\boldsymbol{V}$ and the corresponding space of functions g with norm $\|\cdot\|$ will be complete normed spaces (see [11, p. 165]).

2. Criteria for belonging to class $\boldsymbol{Q}$

Let F be an arbitrary distribution function on the real line. According to the Lebesgue decomposition theorem, F admits the representation

$$F(x) = c_d F_d(x) + c_a F_a(x) + c_s F_s(x), \quad x \in \mathbb{R}, \tag{3}$$

where F_d, F_a, and F_s are discrete, absolutely continuous and continuous singular distribution functions, respectively. Here the coefficients c_d, c_a, and c_s are nonnegative constants such that $c_d + c_a + c_s = 1$. Let f be the characteristic function of F. It is represented in a similar way:

$$f(t) = c_d f_d(t) + c_a f_a(t) + c_s f_s(t), \quad t \in \mathbb{R}, \tag{4}$$

where f_d, f_a, and f_s are the characteristic functions corresponding to F_d, F_a, and F_s, respectively. It is well known that the summands in (3) or (4) are uniquely determined. So if any of the terms is not identically zero, then the corresponding coefficient, distribution function, and characteristic function are uniquely determined.

We will consider only the case that F has nonzero discrete part, i.e., $c_d > 0$ in (3). We write the distribution function F_d in the form

$$F_d(x) = \sum_{\substack{k \in \mathbb{N}_0 \\ x_k \leqslant x}} p_k, \quad x \in \mathbb{R}, \tag{5}$$

where x_k are distinct reals associated with weights $p_k \geqslant 0$, $k \in \mathbb{N}_0$, $\sum_{k=0}^{\infty} p_k = 1$. Hence f_d has the form

$$f_d(t) = \sum_{k \in \mathbb{N}_0} p_k e^{itx_k}, \quad t \in \mathbb{R}. \tag{6}$$

We define the carrier of the distribution corresponding to F_d:

$$\mathcal{X} := \{x_k : p_k > 0, k \in \mathbb{N}_0\}.$$

Obviously, $\mathcal{X} \neq \varnothing$. We also need the set of all finite $\mathbb{Z}$-linear combinations of elements from the set $\mathcal{X}$:

$$\langle \mathcal{X} \rangle := \left\{ \sum_{k=1}^{m} a_k z_k : a_k \in \mathbb{Z},\ z_k \in \mathcal{X},\ m \in \mathbb{N} \right\}.$$

So $\langle\mathcal{X}\rangle$ is the module over the ring $\mathbb{Z}$ with the generating set $\mathcal{X}$. It easily seen that, in particular, $\mathcal{X}\subset\langle\mathcal{X}\rangle$ and $0\in\langle\mathcal{X}\rangle$. If $\mathcal{X}\neq\{0\}$, then $\langle\mathcal{X}\rangle$ is an infinite countable set.

Now we are ready to formulate in detail the most general existing results on criteria for belonging to class $\boldsymbol{Q}$. We start with the result obtained in [3] and [18] by Alexeev and Khartov for the case of discrete F.

Theorem 1. *Suppose that F is a discrete distribution function, $c_d=1$ and $c_a=c_s=0$ in (3) (F and F_d are identical and hence f and f_d are too; F has the form (5), f is represented by (6)). Then $F\in\boldsymbol{Q}$ if and only if $\inf_{t\in\mathbb{R}}|f(t)|>0$. In that case, f admits the representation*

$$f(t)=\exp\Big\{it\gamma_0+\sum_{u\in\langle\mathcal{X}\rangle\setminus\{0\}}\lambda_u\big(e^{itu}-1\big)\Big\},\quad t\in\mathbb{R}, \tag{7}$$

with some $\gamma_0\in\langle\mathcal{X}\rangle$ and $\lambda_u\in\mathbb{R}$ for all $u\in\langle\mathcal{X}\rangle\setminus\{0\}$, and $\sum_{u\in\langle\mathcal{X}\rangle\setminus\{0\}}|\lambda_u|<\infty$.

It is not difficult to rewrite the representation (7) in integral form (1) with some $\gamma\in\mathbb{R}$, $\sigma^2=0$, and some discrete L, which will satisfy all the conditions for a spectral function for the quasi-infinitely divisibility. We will do this below for a more general case. Hence, if the characteristic function f is represented by (7), then $F\in\boldsymbol{Q}$.

It is clear that if $F\in\boldsymbol{Q}$ and $\lambda_u\geqslant 0$ for all $u\in\langle\mathcal{X}\rangle\setminus\{0\}$ in representation (7), then F is infinitely divisible, i.e., $F\in\boldsymbol{I}$. If there exists $\lambda_v<0$ with some $v\in\langle\mathcal{X}\rangle\setminus\{0\}$, then $F\in\boldsymbol{Q}\setminus\boldsymbol{I}$, because the (uniquely defined) function L will be decreasing in the neighborhood of v. For examples of the latter case see [26, p. 10] and [27, p. 165].

We now formulate Berger and Kutlu's result in [7] for the important case $c_a\geqslant 0$. As mentioned, at present, this is the most general criterion using information about the type of the distribution function F.

Theorem 2. *Suppose that F has the decomposition (3) with some $c_d>0$, $c_a\geqslant 0$, and $c_s=0$. Then the following statements are equivalent:*

(i) $F\in\boldsymbol{Q}$.

(ii) $\inf_{t\in\mathbb{R}}|f(t)|>0$.

(iii) $f(t)\neq 0$ *for any* $t\in\mathbb{R}$, *and* $\inf_{t\in\mathbb{R}}|f_d(t)|>0$.

If one (hence all) of the conditions is satisfied, then f admits the representation

$$f(t)=\exp\Big\{it\gamma_0+\sum_{u\in\mathcal{U}}\lambda_u\big(e^{itu}-1\big)+\int_{\mathbb{R}}(e^{itx}-1)\Big(v(x)+\operatorname{sgn}(x)\,\frac{\mathfrak{m}\cdot e^{-|x|}}{|x|}\Big)\,dx\Big\},\quad t\in\mathbb{R}, \tag{8}$$

where $\gamma_0\in\mathbb{R}$, $\mathfrak{m}\in\mathbb{Z}$, $\mathcal{U}$ is a discrete set, $\lambda_u\in\mathbb{R}$ for all $u\in\mathcal{U}$, with $\sum_{u\in\mathcal{U}}|\lambda_u|<\infty$, and v is a real-valued function from $L_1(\mathbb{R})$.

In addition to this theorem, the paper [7] contains a number of interesting assertions concerning decomposition of F and characterization of the existence of the H-moment of F for certain positive functions H. Additionally, there is a rather general theorem similar to Theorem 2 for the function f, but without the assumption that it is the characteristic function of some probability law (Theorem 2.1 in [7]).

Theorem 2 generalizes the results by Berger ([6, Theorems 4.5 and 4.12]) mentioned in the introduction, where F_d was assumed to be a discrete lattice. It is also seen that, in fact, Theorem 2 is a strengthening of Theorem 1. However, we note that γ_0 and the discrete part in (7) are described in greater detail than in formula (8).

We now propose a generalization of the previous criteria for the case when F may have a continuous singular part, i.e., when $c_s \geqslant 0$ in (3). The following theorem is the main result of this article.

For convenience, we preliminarily select the following property of distributions. Let us define $\mu_d := \inf_{t\in\mathbb{R}} |f_d(t)|$. We say that a distribution function F has a *dominated continuous singular part* if $c_s < c_d\mu_d$ for the case $\mu_d > 0$ and if $c_s = 0$ for the case $\mu_d = 0$.

Theorem 3. *Suppose that F has a decomposition* (3) *with $c_d > 0$, $c_a \geqslant 0$, $c_s \geqslant 0$, and that F has a dominated continuous singular part. Then the following statements are equivalent*:

(i) $F \in \boldsymbol{Q}$.

(ii) $\inf_{t\in\mathbb{R}} |f(t)| > 0$.

(iii) $f(t) \neq 0$ *for any* $t \in \mathbb{R}$, *and* $\inf_{t\in\mathbb{R}} |f_d(t)| > 0$.

If one (*hence all*) *of the conditions is satisfied*, f *admits the representation*

$$f(t) = \exp\bigg\{it\gamma_0 + \sum_{u\in\langle\mathcal{X}\rangle\setminus\{0\}} \lambda_u\big(e^{itu}-1\big) + \int_{\mathbb{R}\setminus\{0\}} (e^{itx}-1)\bigg(v_a(x) + \operatorname{sgn}(x)\,\frac{\mathfrak{m}_a\cdot e^{-|x|}}{|x|}\bigg)\,dx + \int_{\mathbb{R}\setminus\{0\}} (e^{itx}-1)\;dW(x)\bigg\}, \quad t\in\mathbb{R}. \tag{9}$$

Here $\gamma_0 \in \langle\mathcal{X}\rangle$, $\lambda_u \in \mathbb{R}$ for all $u \in \langle\mathcal{X}\rangle \setminus \{0\}$, and $\sum_{u\in\langle\mathcal{X}\rangle\setminus\{0\}} |\lambda_u| < \infty$. Next, the function $v_a : \mathbb{R} \mapsto \mathbb{R}$ satisfies $\int_{\mathbb{R}} |v_a(x)|\,dx < \infty$, and, in the case $c_a = 0$, v_a is identically 0; *the constant $\mathfrak{m}_a \in \mathbb{Z}$ is defined by the formula*

$$\mathfrak{m}_a := \frac{1}{2\pi}\Big(\lim_{t\to\infty} \operatorname{Arg} R_a(t) - \lim_{t\to-\infty} \operatorname{Arg} R_a(t)\Big), \tag{10}$$

with

$$R_a(t) := 1 + \frac{c_a f_a(t)}{c_d f_d(t) + c_s f_s(t)}, \quad t\in\mathbb{R},$$

where, in particular, $\mathfrak{m}_a = 0$ *for the case* $c_a = 0$. *Next, the function* $W : \mathbb{R} \to \mathbb{R}$ *belongs to* $\boldsymbol{V}$ *and it is always continuous on* $\mathbb{R}$. *If* $c_s = 0$ *then* W *is identically* 0. *If* $c_s \neq 0$ *then* W *is not absolutely continuous on* $\mathbb{R}$, *i.e., it always contains some continuous singular part. In addition, if all the functions* F_s^{*k}, $k \in \mathbb{N}$, *are continuous singular, then the function* W *is (purely) continuous singular.*

Note that the discrete part in the exponent in (9) depends only on discrete part of f, i.e., on f_d. More precisely, we have the following remark, which will be seen from the proof of Theorem 3.

Remark 1. In the representation (9),

$$\operatorname{Ln} f_d(t) = it\gamma_0 + \sum_{u \in \langle \mathcal{X} \rangle \setminus \{0\}} \lambda_u (e^{itu} - 1), \quad t \in \mathbb{R}.$$

It is easily seen that if we set $c_a := 0$ and $c_s := 0$ in the assumptions of this theorem, then we come to Theorem 1. If we set only $c_s := 0$, then we conclude the statement of Theorem 2, but with a full description of the discrete part in (8). In the unexplored case $c_s > 0$, the new function W appears and it always has some continuous singular part.

Using Theorem 3, it is easy to construct a lot of particular examples of $F \in \boldsymbol{Q}$ with nonzero continuous singular parts.

Example 1. Suppose that

$$F(x) := c_d \mathbb{1}_0(x) + c_a F_a(x) + c_s F_s(x), \quad x \in \mathbb{R},$$

with $c_d > c_s > 0$, $c_a \geqslant 0$, and $c_d + c_a + c_s = 1$. Let F_s be an arbitrary continuous singular function, but F_a be an absolutely continuous distribution function, whose characteristic function f_a is real and nonnegative (for instance, f_a is a Pólya-type characteristic function, or f_a corresponds to a symmetric continuous stable distribution). Then $F \in \boldsymbol{Q}$. Indeed, F has dominated continuous singular part, because f_d is identically 1 here, i.e., $\mu_d = 1$, and hence $c_d \mu_d = c_d > c_s$. Let us check (ii) from Theorem 3. For this, we consider the characteristic function of F,

$$f(t) = c_d + c_a f_a(t) + c_s f_s(t), \quad t \in \mathbb{R}.$$

Under the assumptions, we observe that

$$|f(t)| \geqslant |c_d + c_a f_a(t)| - c_s |f_s(t)| = c_d + c_a f_a(t) - c_s |f_s(t)|, \quad t \in \mathbb{R}.$$

Since $f_a(t) \geqslant 0$ and $|f_s(t)| \leqslant 1$ for any $t \in \mathbb{R}$, we have $|f(t)| \geqslant c_d - c_s > 0$ for any $t \in \mathbb{R}$, i.e., (ii) holds. Thus $F \in \boldsymbol{Q}$ by Theorem 3.

In the general case, of course, checking (ii) or (iii) of Theorem 3 may require more subtle analysis. We illustrate this with the following special example, which is interesting also because there is a (purely) continuous singular function W in (9).

Example 2. Let us consider the mixture of the degenerate law, the uniform distribution on [0, 1] and the classical Cantor distribution. Namely, we set

$$F(x) := c_d \mathbb{1}_0(x) + c_a U(x) + c_s S(x), \quad x \in \mathbb{R},$$

where U is an absolutely continuous distribution function with density $\mathbb{I}_{[0,1]}$, and S is the cumulative function of the classical Cantor distribution supported on the Cantor set $\mathcal{C} \subset [0, 1]$; $c_d > c_s > 0$, $c_a \geqslant 0$, and $c_d + c_a + c_s = 1$.

We first observe that f_d is identically 1 here and, as in Example 1, F has dominated continuous singular part. We next check condition (iii) of Theorem 3. Since $\inf_{t\in\mathbb{R}} |f_d(t)| = 1$, it remains to show that $f(t) \neq 0$ for any $t \in \mathbb{R}$. The function f is expressed by the formula

$$f(t) = c_d + c_a \frac{e^{it}-1}{it} + c_s e^{it/2} \prod_{k=1}^{\infty} \cos(t/3^k), \quad t \in \mathbb{R}.$$

We set

$$\hat{f}(t) := f(t)e^{-it/2} = c_d e^{-it/2} + c_a \frac{\sin(t/2)}{t/2} + c_s \prod_{k=1}^{\infty} \cos(t/3^k), \quad t \in \mathbb{R}.$$

The functions f and $\hat{f}$ have the same set of zeroes on $\mathbb{R}$, a subset of the set of zeroes of the function $\operatorname{Im}\{\hat{f}(t)\} = -c_d \sin(t/2)$, $t \in \mathbb{R}$. The last set is exactly $\{2\pi m : m \in \mathbb{Z}\}$. Therefore it is sufficient to show that $f(2\pi m) \neq 0$ for any $m \in \mathbb{Z}$. Obviously, we can exclude $m = 0$. For any $m \in \mathbb{Z} \setminus \{0\}$ we write

$$\begin{aligned} |f(2\pi m)| = |\hat{f}(2\pi m)| &= \left| c_d \cos(\pi m) + c_a \frac{\sin(\pi m)}{\pi m} + c_s \prod_{k=1}^{\infty} \cos(2\pi m/3^k) \right| \\ &= \left| c_d (-1)^m + c_s \prod_{k=1}^{\infty} \cos(2\pi m/3^k) \right|. \end{aligned}$$

It follows that

$$|f(2\pi m)| \geqslant c_d - c_s \prod_{k=1}^{\infty} \left|\cos(2\pi m/3^k)\right| \geqslant c_d - c_s > 0.$$

Thus $f(t) \neq 0$ for any $t \in \mathbb{R}$ and condition (iii) of Theorem 3 is satisfied. So, by the theorem, $F \in \boldsymbol{Q}$.

Let us consider the representation (9) for f. Since $f_d(t) = 1$ for any $t \in \mathbb{R}$, the sum

$$it\gamma_0 + \sum_u \lambda_u (e^{itu} - 1)$$

is identically 0 according to Remark 1. There are the function $v_a \in L_1(\mathbb{R})$ and the constant $\mathfrak{m}_a \in \mathbb{Z}$, but we will not determine these here. Next, we turn to the function

W. It is known that all convolution powers S^{*n}, $n \in \mathbb{N}$, of the Cantor distribution function S are continuous singular; this follows from the famous Jessen–Wintner theorem (see [16], Theorem 35) and the representation of $S\big(x+\frac{1}{2}\big)$, $x \in \mathbb{R}$, as the infinite symmetric Bernoulli convolution, and it was also explicitly shown in [36] (see p. 520). According to Theorem 3, this implies that W is purely continuous singular. Moreover, it will be seen from the proof of Theorem 3 that W is the limit of the sums

$$W_n(x) := \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d}\right)^k S^{*k}(x), \quad x \in \mathbb{R}, \quad n \in \mathbb{N},$$

as $n \to \infty$ with respect to convergence in total variation on $\mathbb{R}$.

So Theorem 3 yields a sufficient condition for W to be (purely) continuous singular. Is it true that W is always a (purely) continuous singular function in the case $c_s \neq 0$?

Proposition 1. *Suppose that $F \in \boldsymbol{Q}$ satisfies the assumptions of Theorem 3 with $c_s > 0$ and the characteristic function f is represented by (9) with some W. Suppose there is an integer $n_a \geqslant 2$ such that the function $F_s^{*(n_a-1)}$ is (purely) continuous singular, but the function $F_s^{*n_a}$ is not, i.e., $F_s^{*n_a}(x) = \alpha H_a(x) + (1-\alpha) H_s(x)$, $x \in \mathbb{R}$, where α is a number in $(0, 1]$, H_a is an absolutely continuous distribution function, and H_s is a continuous singular distribution function. If*

$$\alpha \geqslant \frac{n_a}{n_a+1} \cdot \frac{c_s}{c_d}$$

then the function W is not (purely) continuous singular. In particular, this is always true for the case $\alpha = 1$.

It should be recalled here that there exist examples of continuous singular distribution functions (say of F_s), whose convolution squares (say F_s^{*2}) are absolutely continuous (see [14; 15; 36]). Therefore we answer the question asked before Proposition 1 in the negative.

The following remark yields the characteristic triplet for $F \in \boldsymbol{Q}$ in the explicit form under the assumptions of Theorem 3.

Remark 2. The representation (9) can be written in the form

$$\begin{aligned} f(t) &= \exp\left\{it\gamma_0 + \int_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1)\, dL(x)\right\} \\ &= \exp\left\{it\gamma + \int_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1 - it\sin(x))\, dL(x)\right\}, \quad t \in \mathbb{R}, \end{aligned}$$

where

$$\gamma := \gamma_0 + \int_{\mathbb{R}\setminus\{0\}} \sin(x)\, dL(x)$$

and $L(x) := L_d(x) + L_a(x) + L_s(x)$ for any $x \in \mathbb{R} \setminus \{0\}$ with

$$L_d(x) := \begin{cases} \sum\limits_{\substack{u \in \langle \mathcal{X} \rangle \setminus \{0\} \\ u \leqslant x}} \lambda_u & \text{if } x < 0, \\ -\sum\limits_{\substack{u \in \langle \mathcal{X} \rangle \setminus \{0\} \\ u > x}} \lambda_u & \text{if } x > 0, \end{cases} \tag{11}$$

$$L_a(x) := \begin{cases} \int\limits_{u \leqslant x} \Big(v_a(u) + \operatorname{sgn}(u) \, \frac{\mathfrak{m}_a \cdot e^{-|u|}}{|u|} \Big) \, du & \text{if } x < 0, \\ -\int\limits_{u > x} \Big(v_a(u) + \operatorname{sgn}(u) \, \frac{\mathfrak{m}_a \cdot e^{-|u|}}{|u|} \Big) \, du & \text{if } x > 0, \end{cases} \tag{12}$$

$$L_s(x) := \begin{cases} W(x) & \text{if } x < 0, \\ W(x) - W(+\infty) & \text{if } x > 0. \end{cases} \tag{13}$$

It is seen that L satisfies all admissible conditions for a spectral function in the Lévy type representation. Thus, under the assumptions of Theorem 3, $(\gamma, 0, L)$ is the characteristic triplet for F satisfying one of the conditions (i)–(iii). On the other hand, if we know that (9) represents the characteristic function of some probability law, then its distribution function F is quasi-infinitely divisible by the definition and hence $F \in \boldsymbol{Q}$.

The following remark shows that the property of dominated singular part is not a necessary condition for belonging to class $\boldsymbol{Q}$.

Remark 3. Let F satisfy the assumptions of Theorem 3 with $c_s > 0$. Suppose that $F \in \boldsymbol{Q}$. Then $F^{*n} \in \boldsymbol{Q}$ for any $n \in \mathbb{N}$, but F^{*n} does not have dominated singular part for all sufficiently large n.

Indeed, it is seen from the definition of the rational-infinite divisibility that the convolution of two any distribution functions from $\boldsymbol{Q}$ belongs to $\boldsymbol{Q}$. Therefore $F^{*n} \in \boldsymbol{Q}$ for any $n \in \mathbb{N}$. Next, we consider the characteristic function f of F. It admits the decomposition (4) with $c_d > 0$, $c_a \geqslant 0$, $c_s > 0$, and $c_s < c_d \mu_d$, where $\mu_d = \inf_{t \in \mathbb{R}} |f_d(t)|$ as before. Then every F^{*n} has the characteristic function

$$f(t)^n = \big(c_d f_d(t) + c_a f_a(t) + c_s f_s(t)\big)^n, \quad t \in \mathbb{R}.$$

From this it is easily seen that $t \mapsto c_d^n f_d(t)^n$, $t \in \mathbb{R}$, is the Fourier–Stieltjes transform of the discrete part of F^{*n}. Hence c_d^n is the weight of this part and $\inf_{t \in \mathbb{R}} |f_d(t)^n| = \mu_d^n$. Next, there are n terms $c_s c_d^{n-1} f_s(t) f_d(t)^{n-1}$ that appear in the Fourier–Stieltjes transform of the continuous singular part of F^{*n}. This is because a convolution of continuous singular function with a discrete function is continuous singular (see [28, p. 190] or [34, p. 319]). So the weight of the continuous singular part of F^{*n} is not less than $n c_s c_d^{n-1}$. For any integer $n \geqslant c_d / c_s$ we have $n c_s c_d^{n-1} \geqslant c_d^n \geqslant c_d^n \mu_d^n$. Therefore the exact weight of the continuous singular part of F^{*n} is not less than

$c_d^n \mu_d^n$ too; i.e., the condition of dominated singular part doesn't hold for F^{*n} for $n \geqslant c_d/c_s$.

The following interesting example shows that the condition of a dominated singular part cannot be simply omitted, or even extended to the case $c_s = c_d \mu_d$ with $\mu_d > 0$ without certain additional assumptions.

Example 3. Let B denote the distribution function of the Bernoulli law on the points ± 1 with equal probabilities, i.e., $B(x) = \frac{1}{2}\,\mathbb{1}_{-1}(x) + \frac{1}{2}\,\mathbb{1}_1(x)$, $x \in \mathbb{R}$. We set

$$F_*(x) := \tfrac{1}{2}\,\mathbb{1}_0(x) + \tfrac{1}{2}\,F_s(x), \qquad x \in \mathbb{R}, \tag{14}$$

where F_s is the following infinite symmetric Bernoulli convolution

$$F_s = B_1 * B_2 * \cdots * B_n * \cdots$$

with $B_k(x) := B(k!\,x)$, $x \in \mathbb{R}$, $k \in \mathbb{N}$. Let f_* and f_s denote the characteristic functions of F_* and F_s. Then $f_*(t) = \frac{1}{2} + \frac{1}{2}\,f_s(t)$, $t \in \mathbb{R}$, and

$$f_s(t) = \prod_{k=1}^{\infty} \cos(t/k!), \qquad t \in \mathbb{R}. \tag{15}$$

It is known (see [28, pp. 20, 67]) that the function F_s is continuous singular. Let us consider Lebesgue's decomposition (4) for $f = f_*$. We have $c_d = \frac{1}{2}$, $c_a = 0$, and $c_s = \frac{1}{2}$. Observe that the component f_d is identically 1 for f_* and hence $\mu_d = \inf_{t \in \mathbb{R}} |f_d(t)| = 1$. So $c_d \mu_d = c_s$, i.e., the condition of dominated continuous singular part, doesn't hold. Next, $f_*(t) \neq 0$ for any $t \in \mathbb{R}$, since otherwise $f_s(t_0) = -1$ for some $t_0 \neq 0$ and hence $|f_s(t_0)| = 1$, which would mean that F_s is a discrete lattice distribution function (see [28, Theorem 2.1.4]), a contradiction. Thus F_* satisfies condition (iii) of Theorem 3 and the assumption $c_d > 0$, but F_* doesn't have dominated continuous singular part.

Proposition 2. *The function F_* doesn't belong to $\boldsymbol{Q}$.*

Thus, in general, conditions (i) and (iii) from Theorem 3 are not equivalent even if $c_s = c_d \mu_d > 0$.

By the way, we recall that distribution functions of discrete laws with a point mass $\frac{1}{2}$, which have characteristic functions without zeroes on the real line, don't always belong to class $\boldsymbol{Q}$ (see [22] for more details). However, if the distribution is an atom of mass $\frac{1}{2}$ plus a continuous part, which is not purely singular, then the answer is definite.

Example 4. Suppose that

$$F(x) := \tfrac{1}{2}\,\mathbb{1}_{\gamma_0}(x) + c_a F_a(x) + c_s F_s(x), \qquad x \in \mathbb{R},$$

where F_a is an absolutely continuous distribution function, F_s is a continuous singular distribution function, $\gamma_0 \in \mathbb{R}$, $c_a > 0$, $c_s \geqslant 0$, and $c_a + c_s = \frac{1}{2}$. Such a function F always belongs to class $\boldsymbol{Q}$. Let us check it. We consider the characteristic function of F:

$$f(t) = \tfrac{1}{2}\, e^{it\gamma_0} + c_a f_a(t) + c_s f_s(t), \quad t \in \mathbb{R}.$$

Here $c_d = \frac{1}{2}$ and $f_d(t) = e^{it\gamma_0}$, $t \in \mathbb{R}$. Hence $\mu_d = \inf_{t\in\mathbb{R}} |f_d(t)| = 1$ and $c_s = \frac{1}{2} - c_a < \frac{1}{2} = c_d\mu_d$, i.e., F has dominated continuous singular part. Next, $f(t) \neq 0$ for any $t \in \mathbb{R}$. For otherwise $2c_a f_a(t_0) + 2c_s f_s(t_0) = -e^{it_0\gamma_0}$ for some $t_0 \neq 0$ and so $|2c_a f_a(t_0) + 2c_s f_s(t_0)| = 1$. This equality implies that the distribution function with the characteristic function $2c_a f_a(\,\cdot\,) + 2c_s f_s(\,\cdot\,)$ is discrete lattice, but this is actually continuous by the assumptions, a contradiction.

So F satisfies condition (iii) and the other assumptions of Theorems 3. It follows that $F \in \boldsymbol{Q}$.

The next assertion solves the decomposition problem for any distribution function $F \in \boldsymbol{Q}$ satisfying the assumptions of Theorem 3.

Proposition 3. *Let F be a distribution function with decomposition (3) with some $c_d > 0$, $c_a \geqslant 0$, $c_s \geqslant 0$, and a dominated continuous singular part. Suppose that $F \in \boldsymbol{Q}$ and $F = F_1 * F_2$ with some distribution functions F_1 and F_2. Then F_1 and F_2 belong to $\boldsymbol{Q}$.*

This proposition generalizes a similar result from [7] (see Corollary 2.3 to Theorem 2.2), which solves the problem for $F \in \boldsymbol{Q}$ satisfying the assumptions of Theorem 2. We note that there is a general decomposition problem for arbitrary $F \in \boldsymbol{Q}$, which was stated by Lindner, Pan and Sato in [26] (see Open Question 8.4): *Is it true that if $F \in \boldsymbol{Q}$ and $F = F_1 * F_2$ (F_1 and F_2 being distribution functions on $\mathbb{R}$), then $F_1 \in \boldsymbol{Q}$ and $F_2 \in \boldsymbol{Q}$?* So Proposition 3 answers this question in the affirmative for any $F \in \boldsymbol{Q}$ satisfying the assumptions of Theorem 3. However, there is a result in [21] that asserts that the general answer is negative.

3. Auxiliary theorems and lemmata

The proof of the main result (Theorem 3) of the article essentially uses the following Wiener–Pitt theorem [35] (see also [33; 11, p. 191]).

Theorem 4. *Let H be a function in $\boldsymbol{V}_{\mathbb{C}}$ with Lebesgue decomposition*

$$H(x) = H_d(x) + H_a(x) + H_s(x), \quad x \in \mathbb{R},$$

where H_d, H_a, H_s are the discrete, absolutely continuous and singular parts of H, each of which belongs to $\boldsymbol{V}_{\mathbb{C}}$. Let

$$h(t) := \int_{\mathbb{R}} e^{itx}\, dH(x), \quad \text{and} \quad h_d(t) := \int_{\mathbb{R}} e^{itx}\, dH_d(x), \quad t \in \mathbb{R}.$$

Suppose that $\inf_{t\in\mathbb{R}} |h(t)| > 0$ *and* $\|H_s\| < \inf_{t\in\mathbb{R}} |h_d(t)|$. *Then there exists a function* K *in* $\boldsymbol{V}_{\mathbb{C}}$ *such that*

$$\frac{1}{h(t)} = \int_{\mathbb{R}} e^{itx}\, dK(x), \quad t \in \mathbb{R}.$$

The following theorem was proposed by Berger [6]. In fact, it is a modification of one of Krein's results (see [23, Theorem L, p. 15]).

Theorem 5. *Suppose that a function* $h : \mathbb{R} \to \mathbb{C}$ *is defined by the formula*

$$h(t) := c + \int_{\mathbb{R}} e^{itx} u(x)\, dx, \quad t \in \mathbb{R}, \tag{16}$$

where $c \in \mathbb{C} \setminus \{0\}$ *is a constant*, $u : \mathbb{R} \to \mathbb{C}$ *is a function satisfying the condition* $\int_{\mathbb{R}} |u(x)|\, dx < \infty$. *Assume that* $h(0) = 1$ *and* $h(t) \neq 0$ *for any* $t \in \mathbb{R}$. *Then*

$$h(t) = \exp\left\{\int_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1)\left(v(x) + \operatorname{sgn}(x)\,\frac{\mathfrak{m}\cdot e^{-|x|}}{|x|}\right) dx\right\}, \quad t \in \mathbb{R},$$

where $\mathfrak{m}$ *is a constant defined by the formula*

$$\mathfrak{m} := \frac{1}{2\pi}\Big(\lim_{t\to\infty} \operatorname{Arg} h(t) - \lim_{t\to-\infty} \operatorname{Arg} h(t)\Big),$$

and the function $v : \mathbb{R} \to \mathbb{C}$ *satisfies the condition* $\int_{\mathbb{R}} |v(x)|\, dx < \infty$.

Let us consider the quantity $\mathfrak{m}$ from the theorem. It is the *index* of h (see [23]). By the Riemann–Lebesgue lemma, for the function h defined by (16) we have $h(t) \to c$ as $t \to \pm\infty$. Therefore it is not difficult to conclude the following fact.

Remark 4. The quantity $\mathfrak{m}$ from Theorem 5 is well-defined and it is an integer.

We next formulate a very useful result obtained by Berger [6].

Theorem 6. *Let* F *be a distribution function on* $\mathbb{R}$ *with the characteristic function* f. *Suppose that* f *admits the representation*

$$f(t) = \exp\left\{it\gamma_1 - \frac{\sigma^2 t^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \big(e^{itx} - 1 - it\, \mathbb{1}_{[-1,1]}(x)\big)\, dL(x)\right\}, \quad t \in \mathbb{R},$$

with some $\gamma_1 \in \mathbb{C}$, $\sigma^2 \in \mathbb{C}$, *and the function* $L(x) = L_1(x) + iL_2(x)$, $x \in \mathbb{R} \setminus \{0\}$. *Here for every* $j \in \{1, 2\}$ *the function* L_j *has bounded variation on* $\mathbb{R} \setminus O_\delta$ *for any* $\delta > 0$ (*in general*, L_j *may be nonmonotonic on the intervals* $(-\infty, 0)$ *and* $(0, +\infty)$), L_j *is right-continuous at every point of* $\mathbb{R}$, $L_j(-\infty) = L_j(+\infty) = 0$, *and*

$$\int_{O_\delta} x^2\, d|L_j|(x) < \infty \quad \text{for any} \quad \delta > 0,$$

where $O_\delta := (-\delta, 0) \cup (0, \delta)$. *Then* $\gamma_1 \in \mathbb{R}$, $\sigma^2 \geqslant 0$, *and* $\operatorname{Im}\{L(x)\} = L_2(x) = 0$ *for any* $x \in \mathbb{R} \setminus \{0\}$. *In addition*, $F \in \boldsymbol{Q}$.

The next theorem yields one special property of characteristic functions of rational-infinitely divisible laws. Its proof can be found in [18, p. 3] or [19, p. 360].

Theorem 7. *Let F be a distribution function on $\mathbb{R}$ with the characteristic function f. If $F \in \boldsymbol{Q}$ then for any $\tau > 0$ there exists $C_\tau > 0$ such that*

$$\sup_{t\in\mathbb{R}} \left| \frac{f(t-\tau)f(t+\tau)}{f(t)^2} \right| \leqslant C_\tau .$$

Lemma 1 (Bochner [9]). *Let $(W_n)_{n\in\mathbb{N}}$ be a sequence in $\boldsymbol{V}$ such that $\sup_{n\in\mathbb{N}} \|W_n\| = B < \infty$. Let*

$$w_n(t) := \int_{\mathbb{R}} e^{itx}\, dW_n(x), \quad t \in \mathbb{R}, \quad n \in \mathbb{N}.$$

Suppose that $w_n(t) \to w(t)$ as $n \to \infty$ uniformly on every bounded interval. Then there exists $W \in \boldsymbol{V}$ with $\|W\| \leqslant B$ such that

$$w(t) = \int_{\mathbb{R}} e^{itx}\, dW(x), \quad t \in \mathbb{R}.$$

We will need the following multivariate version of the Leibniz rule for partial derivatives of a product of two functions (see [13, p. 10]).

Lemma 2. *Suppose that the functions $f : \mathbb{R}^d \to \mathbb{C}$ and $g : \mathbb{R}^d \to \mathbb{C}$ have continuous mixed partial derivatives up to and including order d in some open set $T \subset \mathbb{R}^d$. Then, for any $t = (t_1, \dots, t_d) \in T$,*

$$\frac{\partial^d (f(t)\cdot g(t))}{\partial t_1 \cdots \partial t_d} = \sum_{\mathfrak{u}\subset D} \left(\frac{\partial^{|\mathfrak{u}|} f(t)}{\prod_{j\in\mathfrak{u}} \partial t_j} \cdot \frac{\partial^{d-|\mathfrak{u}|} g(t)}{\prod_{j\in D\setminus\mathfrak{u}} \partial t_j} \right),$$

where $D = \{1, \dots, d\}$, and the parameter $\mathfrak{u}$ of the sum ranges over all the subsets of D.

We will also need the multivariate form of Faà di Bruno's formula for the partial derivatives of a composition of two functions (see [13, p. 4]).

Lemma 3. *Suppose that a function $g : \mathbb{R}^d \to \mathbb{C}$ has continuous mixed partial derivatives up to and including order d in some open set $T \subset \mathbb{R}^d$. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function in some domain in $\mathbb{C}$ that which include the values of $g(t)$, $t \in T$. Suppose that $D := \{1, \dots, d\}$, $\mathfrak{u}$ is nonempty subset of D, $\mathcal{P}(\mathfrak{u})$ is the set of all partitions of the set $\mathfrak{u}$. Then, for any $t = (t_1, \dots, t_d) \in T$,*

$$\frac{\partial^{|\mathfrak{u}|} f(g(t))}{\prod_{j\in\mathfrak{u}} \partial t_j} = \sum_{P\in\mathcal{P}(\mathfrak{u})} \left(\frac{d^{|P|} f(z)}{dz^{|P|}}\bigg|_{z=g(t)} \cdot \prod_{s\in P} \frac{\partial^{|s|} g(t)}{\prod_{j\in s} \partial t_j} \right).$$

We now formulate a lemma that will be a key tool in proving of Theorem 3. Its proof is given at the start of the next section.

Lemma 4. *Suppose a function $\varphi : \mathbb{R}^d \to \mathbb{C}$, where $d \in \mathbb{N}$, admits the representation*

$$\varphi(t) = \sum_{m=1}^{N} q_m e^{i\langle t, c_m\rangle}, \quad t = (t_1, \ldots, t_d) \in \mathbb{R}^d,$$

with $N \in \mathbb{N}$, $q_m \in \mathbb{C}$, and $c_m = (c_{m,1}, \ldots, c_{m,d}) \in \mathbb{Z}^d$, $m = 1, \ldots, N$. Then, for any $k \in \mathbb{N}$,

$$\|\varphi^k\| \leqslant \left(\frac{\pi}{\sqrt{6}}\right)^d \sum_{\mathfrak{u}\subset D}\Bigg(\prod_{j\in D\setminus\mathfrak{u}} |k\alpha_j - 1| \cdot \sum_{\substack{P\in\mathcal{P}(\mathfrak{u})\\ |P|\leqslant k}} \Bigg(\prod_{l=1}^{|P|}(k-l+1)\cdot S_\varphi^{k-|P|}\cdot R_\varphi(P)\Bigg)\Bigg), \tag{17}$$

where $D = \{1, \ldots, d\}$,

$$\begin{aligned} &\alpha_j := \min\nolimits_{m=1,\ldots,N} c_{m,j}, \quad j = 1, \ldots, d,\\ &S_\varphi := \sup_{t\in[-\pi,\pi]^d} |\varphi(t)|, \quad R_\varphi(P) := \sup_{t\in[-\pi,\pi]^d} \prod_{s\in P} \left|\frac{\partial^{|s|}\varphi(t)}{\prod_{j\in s}\partial t_j}\right|. \end{aligned} \tag{18}$$

In particular,

$$\|\varphi^k\| \leqslant A_\varphi k^d S_\varphi^k \quad \text{for any } k \in \mathbb{N}, \tag{19}$$

with

$$A_\varphi := \left(\frac{\pi}{\sqrt{6}}\right)^d \sum_{\mathfrak{u}\subset D}\Bigg(\prod_{j\in D\setminus\mathfrak{u}} \big(|\alpha_j|+1\big)\cdot \sum_{P\in\mathcal{P}(\mathfrak{u})} S_\varphi^{-|P|} R_\varphi(P)\Bigg), \tag{20}$$

which is independent of k.

4. Proofs

Proof of Lemma 4. We will first prove inequality (17) for $k = 1$. We define a vector $a := (a_1, \ldots, a_d)$ with $a_j := 1 - \alpha_j$, $j = 1, \ldots, d$, and we introduce the function

$$\hat{\varphi}_a(t) := \varphi(t)e^{i\langle t, a\rangle} = \sum_{m=1}^{N} q_m e^{i\langle t, c_m + a\rangle}, \quad t \in \mathbb{R}^d.$$

Note that $c_m + a \in \mathbb{N}^d$ for any $m = 1, \ldots, N$. Without loss of generality we can assume that c_m are distinct if $N > 1$. Then we have

$$\begin{aligned} \|\varphi\| &= \sum_{m=1}^{N} |q_m| = \sum_{m=1}^{N}\Bigg(|q_m|\cdot\prod_{j=1}^{d}(c_{m,j}+a_j)\cdot\prod_{j=1}^{d}\frac{1}{c_{m,j}+a_j}\Bigg)\\ &\leqslant \Bigg(\sum_{m=1}^{N}\Bigg(|q_m|^2\cdot\prod_{j=1}^{d}(c_{m,j}+a_j)^2\Bigg)\Bigg)^{1/2}\cdot\Bigg(\sum_{m=1}^{N}\prod_{j=1}^{d}\frac{1}{(c_{m,j}+a_j)^2}\Bigg)^{1/2}. \end{aligned} \tag{21}$$

Since $c_m + a = (c_{m,1} + a_1, \ldots, c_{m,d} + a_d) \in \mathbb{N}^d$ are distinct vectors for different m, the following estimate holds:

$$\sum_{m=1}^{N} \prod_{j=1}^{d} \frac{1}{(c_{m,j} + a_j)^2} \leqslant \prod_{j=1}^{d} \Bigl(1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2} + \cdots \Bigr) = \Bigl(\frac{\pi^2}{6}\Bigr)^d.$$

We next write the function $\hat{\varphi}_a$ in expanded form as

$$\hat{\varphi}_a(t) = \sum_{m=1}^{N} q_m \exp\Bigl\{ i \sum_{j=1}^{d} t_j (c_{m,j} + a_j) \Bigr\}, \quad t = (t_1, \ldots, t_d) \in \mathbb{R}^d.$$

So it is easily seen that

$$\frac{\partial^d \hat{\varphi}_a(t)}{\partial t_1 \cdots \partial t_d} = \sum_{m=1}^{N} \Bigl(q_m \cdot i^d \cdot \prod_{j=1}^{d} (c_{m,j} + a_j) \cdot \exp\Bigl\{ i \sum_{j=1}^{d} t_j (c_{m,j} + a_j) \Bigr\} \Bigr), \qquad t = (t_1, \ldots, t_d) \in \mathbb{R}^d.$$

We consider the written exponential functions (additionally normed by $(2\pi)^d$) as a rank-N orthonormal system of $L_2\bigl([-\pi, \pi]^d\bigr)$ and, by Parseval's identity, we get

$$\sum_{m=1}^{N} \Bigl(|q_m|^2 \cdot \prod_{j=1}^{d} (c_{m,j} + a_j)^2 \Bigr) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \Bigl| \frac{\partial^d \hat{\varphi}_a(t)}{\partial t_1 \cdots \partial t_d} \Bigr|^2 \, dt_1 \cdots dt_d.$$

Hence

$$\sum_{m=1}^{N} \Bigl(|q_m|^2 \cdot \prod_{j=1}^{d} (c_{m,j} + a_j)^2 \Bigr) \leqslant \sup_{t \in [-\pi,\pi]^d} \Bigl| \frac{\partial^d \hat{\varphi}_a(t)}{\partial t_1 \cdots \partial t_d} \Bigr|^2.$$

Using these estimates in (21), we come to the inequality

$$\|\varphi\| \leqslant \Bigl(\frac{\pi}{\sqrt{6}} \Bigr)^d \sup_{t \in [-\pi,\pi]^d} \Bigl| \frac{\partial^d \hat{\varphi}_a(t)}{\partial t_1 \cdots \partial t_d} \Bigr|.$$

We next find an upper estimate for the latter supremum in terms of φ and a. We apply Lemma 2 for the function $\hat{\varphi}_a(t) = \varphi(t) e^{i\langle t, a\rangle}$ with $f(t) := \varphi(t)$ and $g(t) := e^{i\langle t, a\rangle}$, $t \in \mathbb{R}^d$:

$$\frac{\partial^d \hat{\varphi}_a(t)}{\partial t_1 \cdots \partial t_d} = \sum_{\mathfrak{u} \subset D} \Bigl(\frac{\partial^{|\mathfrak{u}|} \varphi(t)}{\prod_{j \in \mathfrak{u}} \partial t_j} \cdot \frac{\partial^{d-|\mathfrak{u}|} e^{i\langle t, a\rangle}}{\prod_{j \in D \setminus \mathfrak{u}} \partial t_j} \Bigr) = \sum_{\mathfrak{u} \subset D} \Bigl(\frac{\partial^{|\mathfrak{u}|} \varphi(t)}{\prod_{j \in \mathfrak{u}} \partial t_j} \, e^{i\langle t, a\rangle} \, i^{d-|\mathfrak{u}|} \prod_{j \in D \setminus \mathfrak{u}} a_j \Bigr), \qquad t \in \mathbb{R}^d.$$

Here $D := \{1, \ldots, d\}$ and the index $\mathfrak{u}$ of the sum ranges over all subsets of D.

Hence we get the estimate

$$\sup_{t\in[-\pi,\pi]^d}\left|\frac{\partial^d\hat{\varphi}_a(t)}{\partial t_1\cdots\partial t_d}\right|\leqslant\sum_{\mathfrak{u}\subset D}\left(\prod_{j\in D\setminus\mathfrak{u}}|a_j|\cdot\sup_{t\in[-\pi,\pi]^d}\left|\frac{\partial^{|\mathfrak{u}|}\varphi(t)}{\prod_{j\in\mathfrak{u}}\partial t_j}\right|\right).$$

Thus (on account of the equalities $a_j=1-\alpha_j$, $j=1,\dots,d$) we have

$$\|\varphi\|\leqslant\left(\frac{\pi}{\sqrt{6}}\right)^d\sum_{\mathfrak{u}\subset D}\left(\prod_{j\in D\setminus\mathfrak{u}}|\alpha_j-1|\cdot\sup_{t\in[-\pi,\pi]^d}\left|\frac{\partial^{|\mathfrak{u}|}\varphi(t)}{\prod_{j\in\mathfrak{u}}\partial t_j}\right|\right).\tag{22}$$

The right-hand side of this inequality coincides with that of (17) for $k=1$. Indeed, in the inner sum in (17), the index P can be equal to $\{\mathfrak{u}\}$, i.e., $|P|=1$, and, in the product in $R_\varphi(P)$, the index s assumes only the value $\mathfrak{u}$. Therefore $\prod_{l=1}^{|P|}(k-l+1)=1$, $S_\varphi^{k-|P|}=S_\varphi^0=1$, and

$$R_\varphi(P)=\sup_{t\in[-\pi,\pi]^d}\left|\frac{\partial^{|\mathfrak{u}|}\varphi(t)}{\prod_{j\in\mathfrak{u}}\partial t_j}\right|.$$

Substituting these values in (17), we see the match with (22).

We now prove that (17) is true for any $k\in\mathbb{N}$. Let us fix an arbitrary $k\in\mathbb{N}$. We consider the function φ^k given by

$$\varphi^k(t)=\left(\sum_{m=1}^{N}q_me^{i\langle t,c_m\rangle}\right)^k,\quad t\in\mathbb{R}^d.$$

Clearly, it can be written in the form

$$\varphi^k(t)=\sum_{m=1}^{M}Q_me^{i\langle t,C_m\rangle},\quad t\in\mathbb{R}^d,$$

with some $M\in\mathbb{N}$, $Q_m\in\mathbb{C}$, and distinct $C_m=(C_{m,1},\dots,C_{m,d})\in\mathbb{Z}^d$, $m=1,\dots,M$. Observe that

$$C_m\in\left\{\sum_{j=1}^{k}c_{m_j}:m_1,\dots,m_k\in\{1,\dots,N\}\right\},\quad m=1,\dots,M.$$

Therefore

$$\min_{m=1,\dots,M}C_{m,j}=k\min_{m=1,\dots,N}c_{m,j}=k\alpha_j,\quad j=1,\dots,d.$$

Taking this into account, we use inequality (22) for the function φ^k:

$$\|\varphi^k\|\leqslant\left(\frac{\pi}{\sqrt{6}}\right)^d\sum_{\mathfrak{u}\subset D}\left(\prod_{j\in D\setminus\mathfrak{u}}|k\alpha_j-1|\cdot\sup_{t\in[-\pi,\pi]^d}\left|\frac{\partial^{|\mathfrak{u}|}\varphi^k(t)}{\prod_{j\in\mathfrak{u}}\partial t_j}\right|\right).$$

We now write expressions for the mixed partial derivatives from the right-hand side. We apply Lemma 3 with $g(t) := \varphi(t)$, $t \in \mathbb{R}^d$, and $f(z) := z^k$, $z \in \mathbb{C}$:

$$\frac{\partial^{|\mathfrak{u}|}\varphi^k(t)}{\prod_{j\in\mathfrak{u}}\partial t_j} = \sum_{P\in\mathcal{P}(\mathfrak{u})}\left(\frac{d^{|P|}z^k}{dz^{|P|}}\bigg|_{z=\varphi(t)} \cdot \prod_{s\in P}\frac{\partial^{|s|}\varphi(t)}{\prod_{j\in s}\partial t_j}\right)$$
$$= \sum_{\substack{P\in\mathcal{P}(\mathfrak{u})\\ |P|\leqslant k}}\left(\prod_{l=1}^{|P|}(k-l+1)\cdot\varphi(t)^{k-|P|}\cdot\prod_{s\in P}\frac{\partial^{|s|}\varphi(t)}{\prod_{j\in s}\partial t_j}\right).$$

This formula holds even in the case $\mathfrak{u} = \varnothing$, where P must be $\varnothing$, i.e., $|P| = 0$ and $\mathcal{P}(\mathfrak{u}) = \{\varnothing\}$. Indeed, there is only one summand in the latter sum, in which the written products are equal to 1 (because the indexes formally belong to the empty set). Hence the right-hand side equals $\varphi(t)^k$, as does the left-hand side, since $|\mathfrak{u}| = 0$.

We have the estimate

$$\sup_{t\in[-\pi,\pi]^d}\left|\frac{\partial^{|\mathfrak{u}|}\varphi^k(t)}{\prod_{j\in\mathfrak{u}}\partial t_j}\right| \leqslant \sum_{\substack{P\in\mathcal{P}(\mathfrak{u})\\ |P|\leqslant k}}\left(\prod_{l=1}^{|P|}(k-l+1)\cdot S_\varphi^{k-|P|}\cdot R_\varphi(P)\right),$$

where S_φ and $R_\varphi(P)$ are defined by (18). Thus we obtain the required inequality (17).

We next show that (19) holds. Let us obtain an upper estimate for the right-hand side of inequality (17). We fix $k \geqslant 1$. Observe that

$$\prod_{j\in D\setminus\mathfrak{u}}|k\alpha_j - 1| \leqslant \prod_{j\in D\setminus\mathfrak{u}}\big(k|\alpha_j|+1\big) \leqslant \prod_{j\in D\setminus\mathfrak{u}}\big(k|\alpha_j|+k\big) \leqslant k^{d-|\mathfrak{u}|}\cdot\prod_{j\in D\setminus\mathfrak{u}}\big(|\alpha_j|+1\big).$$

Next, since $|P| \leqslant |\mathfrak{u}|$ for any $P \in \mathcal{P}(\mathfrak{u})$, we have

$$\prod_{l=1}^{|P|}(k-l+1) \leqslant \prod_{l=1}^{|P|}(k-1+1) = k^{|P|} \leqslant k^{|\mathfrak{u}|}.$$

We apply these inequalities to (17), obtaining

$$\|\varphi^k\| \leqslant \left(\frac{\pi}{\sqrt{6}}\right)^d \sum_{\mathfrak{u}\subset D}\left(k^{d-|\mathfrak{u}|}\prod_{j\in D\setminus\mathfrak{u}}(|\alpha_j|+1)\cdot\sum_{\substack{P\in\mathcal{P}(\mathfrak{u})\\ |P|\leqslant k}}k^{|\mathfrak{u}|}\cdot S_\varphi^{k-|P|}\cdot R_\varphi(P)\right)$$
$$= k^d S_\varphi^k\cdot\left(\frac{\pi}{\sqrt{6}}\right)^d\sum_{\mathfrak{u}\subset D}\left(\prod_{j\in D\setminus\mathfrak{u}}\big(|\alpha_j|+1\big)\cdot\sum_{\substack{P\in\mathcal{P}(\mathfrak{u})\\ |P|\leqslant k}}S_\varphi^{-|P|}R_\varphi(P)\right).$$

Since S_φ and $R_\varphi(P)$ are always nonnegative, we come to the needed inequality

$$\|\varphi^k\| \leqslant k^d S_\varphi^k\left(\frac{\pi}{\sqrt{6}}\right)^d\sum_{\mathfrak{u}\subset D}\left(\prod_{j\in D\setminus\mathfrak{u}}\big(|\alpha_j|+1\big)\cdot\sum_{P\in\mathcal{P}(\mathfrak{u})}S_\varphi^{-|P|}R_\varphi(P)\right) = A_\varphi k^d S_\varphi^k. \quad \square$$

Proof of Theorem 3. We assume that F admits the decomposition (3) with some $c_d \in (0, 1]$, and that F has a dominated singular part. We set $\mu_d := \inf_{t\in\mathbb{R}} |f_d(t)|$.

(i) $\Rightarrow$ (ii). Suppose that $F \in \boldsymbol{Q}$. If $\mu_d = 0$ then $c_s = 0$ by the assumption of a dominated singular part, and hence we can apply Theorem 2. According to that theorem, (i) implies (ii). Next, if $\mu_d > 0$ then we know that $c_s < c_d\mu_d$, i.e., $c_d\mu_d - c_s > 0$. Observe that, for any $t \in \mathbb{R}$,

$$\begin{aligned} |f(t)| &= \big|c_d f_d(t) + c_s f_s(t) + c_a f_a(t)\big| \geqslant c_d|f_d(t)| - c_s\big|f_s(t)| - c_a|f_a(t)\big| \\ &\geqslant c_d\mu_d - c_s - |f_a(t)|. \end{aligned}$$

Since $f_a(t) \to 0$ as $t \to \pm\infty$, there exists $T > 0$ such that $|f_a(t)| < \frac{1}{2}(c_d\mu_d - c_s)$ as $|t| > T$. Hence $|f(t)| > \frac{1}{2}(c_d\mu_d - c_s) > 0$ as $|t| > T$. Next, let us consider the function $t \mapsto |f(t)|$ on the segment $[-T, T]$. Since it is continuous, there exists $t_{\min} \in [-T, T]$ such that $|f(t)| \geqslant |f(t_{\min})|$ for any $t \in [-T, T]$. Since $F \in \boldsymbol{Q}$, we know that $f(t) \neq 0$ for any $t \in \mathbb{R}$ (see the introduction) and thus $C_T := |f(t_{\min})| > 0$. So we get $|f(t)| \geqslant C_T > 0$ for any $t \in [-T, T]$. Thus, for any $t \in \mathbb{R}$,

$$|f(t)| \geqslant \min\Big\{\tfrac{1}{2}(c_d\mu_d - c_s), C_T\Big\} > 0,$$

i.e., $\inf_{t\in\mathbb{R}} |f(t)| > 0$. So we come to (ii).

(ii) $\Rightarrow$ (iii). Obviously, (ii) yields that $f(t) \neq 0$ for any $t \in \mathbb{R}$. To obtain a contradiction, suppose that $\mu_d = \inf_{t\in\mathbb{R}} |f_d(t)| = 0$. Then $c_s = 0$ by the condition of dominated singular part. According to Theorem 2, (iii) follows from (ii), i.e., in particular, we have $\mu_d > 0$, a contradiction. Thus we conclude that $\inf_{t\in\mathbb{R}} |f_d(t)| > 0$ and (iii) holds.

(iii) $\Rightarrow$ (i). Let us consider f represented by formula (4) with $c_d > 0$. Here we assume that $f(t) \neq 0$ for any $t \in \mathbb{R}$, that

$$\mu_d = \inf_{t\in\mathbb{R}} |f_d(t)| > 0, \tag{23}$$

and that $c_s < c_d\mu_d$. Due to these assumptions, for any $t \in \mathbb{R}$ we have

$$\big|c_d f_d(t) + c_s f_s(t)\big| \geqslant c_d|f_d(t)| - c_s|f_s(t)| \geqslant c_d\mu_d - c_s > 0,$$

i.e.,

$$\inf_{t\in\mathbb{R}} |c_d f_d(t) + c_s f_s(t)| > 0. \tag{24}$$

Then for any $t \in \mathbb{R}$ we can write

$$\begin{aligned} f(t) &= c_d f_d(t) + c_s f_s(t) + c_a f_a(t) \\ &= f_d(t) \cdot \frac{c_d f_d(t) + c_s f_s(t) + c_a f_a(t)}{c_d f_d(t) + c_s f_s(t)} \cdot \frac{c_d f_d(t) + c_s f_s(t)}{f_d(t)} \\ &= f_d(t) \cdot \left(1 + \frac{c_a f_a(t)}{c_d f_d(t) + c_s f_s(t)}\right) \cdot \frac{c_d f_d(t) + c_s f_s(t)}{f_d(t)}. \end{aligned}$$

So it is convenient to represent f as

$$f(t) = f_d(t) f_{a,ds}(t) f_{s,d}(t), \quad t \in \mathbb{R}, \tag{25}$$

where

$$f_{a,ds}(t) := c_d + c_s + \frac{c_a(c_d+c_s) f_a(t)}{c_d f_d(t) + c_s f_s(t)}, \quad f_{s,d}(t) := \frac{c_d f_d(t) + c_s f_s(t)}{(c_d+c_s) f_d(t)}, \quad t \in \mathbb{R}.$$

Let us consider f_d represented by (6). Due to (23), by Theorem 1, it admits the representation

$$f_d(t) = \exp\left\{ it\gamma_0 + \sum_{u \in \langle \mathcal{X} \rangle \setminus \{0\}} \lambda_u \bigl(e^{itu} - 1\bigr) \right\}, \quad t \in \mathbb{R}, \tag{26}$$

with some $\gamma_0 \in \langle \mathcal{X} \rangle$ and $\lambda_u \in \mathbb{R}$ for all $u \in \langle \mathcal{X} \rangle \setminus \{0\}$, and $\sum_{u \in \langle \mathcal{X} \rangle \setminus \{0\}} |\lambda_u| < \infty$.

We now consider the function $f_{a,ds}$. If $c_a = 0$ then $f_{a,ds}(t) = c_d + c_s = 1 - c_a = 1$. We next suppose that $c_a > 0$. The function $t \mapsto c_d f_d(t) + c_s f_s(t)$, $t \in \mathbb{R}$, is separated from 0 according to (24) and for its continuous singular part we have

$$\|c_s f_s\| = c_s < c_d \mu_d = c_d \inf_{t \in \mathbb{R}} |f_d(t)|.$$

By Theorem 4, there exists a function $I_{ds} : \mathbb{R} \to \mathbb{C}$ in $\boldsymbol{V}_{\mathbb{C}}$ such that

$$\frac{1}{c_d f_d(t) + c_s f_s(t)} = \int_{\mathbb{R}} e^{itx}\, dI_{ds}(x), \quad t \in \mathbb{R}.$$

We next observe that

$$\frac{f_a(t)}{c_d f_d(t) + c_s f_s(t)} = \int_{\mathbb{R}} e^{itx}\, dF_a(x) \cdot \int_{\mathbb{R}} e^{itx}\, dI_{ds}(x) = \int_{\mathbb{R}} e^{itx}\, d(F_a * I_{ds})(x), \quad t \in \mathbb{R},$$

and we write

$$\frac{c_a(c_d+c_s) f_a(t)}{c_d f_d(t) + c_s f_s(t)} = \int_{\mathbb{R}} e^{itx}\, d\widetilde{F}_a(x), \quad t \in \mathbb{R},$$

where $\widetilde{F}_a(x) := c_a(c_d+c_s)(F_a * I_{ds})(x)$, $x \in \mathbb{R}$. The function $\widetilde{F}_a \in \boldsymbol{V}_{\mathbb{C}}$ inherits absolute continuity from F_a, i.e., there exists a function $\tilde{p}_a : \mathbb{R} \to \mathbb{C}$ such that $\int_{\mathbb{R}} |\tilde{p}_a(x)|\, dx < \infty$ and $\widetilde{F}_a(x) = \int_{y \leqslant x} \tilde{p}_a(y)\, dy$, $x \in \mathbb{R}$. Hence

$$\int_{\mathbb{R}} e^{itx}\, d\widetilde{F}_a(x) = \int_{\mathbb{R}} e^{itx}\, \tilde{p}_a(x)\, dx, \quad t \in \mathbb{R},$$

and we have

$$f_{a,ds}(t) = c_d + c_s + \frac{c_a(c_d+c_s) f_a(t)}{c_d f_d(t) + c_s f_s(t)} = c_d + c_s + \int_{\mathbb{R}} e^{itx}\, \tilde{p}_a(x)\, dx, \quad t \in \mathbb{R},$$

where $c_d + c_s \geqslant c_d > 0$. Note that $f_{a,ds}(0) = 1$. Since $f(t) \neq 0$ for any $t \in \mathbb{R}$ and the function $t \mapsto c_d f_d(t) + c_s f_s(t)$, $t \in \mathbb{R}$, is bounded, we conclude that $f_{a,ds}(t) \neq 0$

for any $t \in \mathbb{R}$. By Theorem 5, $f_{a,ds}$ admits the representation

$$(27)\quad f_{a,ds}(t) = \exp\left\{ \int_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1)\left(v_a(x) + \operatorname{sgn}(x)\,\frac{\mathfrak{m}_a \cdot e^{-|x|}}{|x|} \right) dx \right\}, \qquad t \in \mathbb{R},$$

where $v_a : \mathbb{R} \to \mathbb{C}$ is a function satisfying the condition $\int_{\mathbb{R}} |v_a(x)|\,dx < \infty$, $\mathfrak{m}_a$ is a constant defined by the formula

$$\mathfrak{m}_a := \frac{1}{2\pi}\big(\lim_{t\to\infty} \operatorname{Arg} f_{a,ds}(t) - \lim_{t\to-\infty} \operatorname{Arg} f_{a,ds}(t) \big).$$

Since $c_d + c_s$ is positive, formula (10) gives an equivalent definition of $\mathfrak{m}_a$. By the way, it is easily seen that $\mathfrak{m}_a = 0$ if $c_a = 0$. In general, $\mathfrak{m}_a \in \mathbb{Z}$ (see Remark 4). Therefore, in the case $c_a = 0$, we set $v_a(x) := 0$ for every $x \in \mathbb{R}$ and the formula (27) remains valid.

We now turn to the function

$$f_{s,d}(t) = \frac{c_d f_d(t) + c_s f_s(t)}{(c_d + c_s) f_d(t)} = \left(1 + \frac{c_s f_s(t)}{c_d f_d(t)} \right) \cdot \frac{c_d}{c_d + c_s}, \qquad t \in \mathbb{R}.$$

From (24) we know that $f_{s,d}(t) \neq 0$ for any $t \in \mathbb{R}$ and we see that $f_{s,d}(0) = 1$. Hence the distinguished logarithm $\operatorname{Ln} f_{s,d}$ is uniquely defined on $\mathbb{R}$ with condition $\operatorname{Ln} f_{s,d}(0) = 0$. Due to the assumptions,

$$(28)\qquad \left| \frac{c_s f_s(t)}{c_d f_d(t)} \right| \leqslant \frac{c_s}{c_d \mu_d} < 1 \quad \text{for any} \quad t \in \mathbb{R},$$

and we can write

$$(29)\qquad \operatorname{Ln} f_{s,d}(t) = \ln\left(1 + \frac{c_s f_s(t)}{c_d f_d(t)} \right) - \ln\left(1 + \frac{c_s}{c_d} \right), \qquad t \in \mathbb{R},$$

where $\ln(\,\cdot\,)$ returns the principal value of the logarithm. Let us consider the first logarithm in the right-hand side. Due to (28), we have the expansion

$$(30)\quad \ln\left(1 + \frac{c_s f_s(t)}{c_d f_d(t)} \right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{c_s f_s(t)}{c_d f_d(t)} \right)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d} \right)^k f_s(t)^k g_d(t)^k, \qquad t \in \mathbb{R},$$

where we have set $g_d(t) := 1/f_d(t)$, $t \in \mathbb{R}$. The series converges uniformly in $\mathbb{R}$ by the Weierstrass M-test, because, due to (28), for any $t \in \mathbb{R}$ the absolute values of its terms are majorized by the terms of the convergent numerical series

$$\sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{c_s}{c_d \mu_d} \right)^k.$$

Let us consider the function g_d and show that it is actually a Fourier–Stieltjes transform of some function from class $\boldsymbol{V}$. Indeed, according to (26),

$$g_d(t) = e^{-it\gamma_0} \cdot e^{-\lambda_0} \cdot \exp\Big\{-\sum_{u\in\langle\mathcal{X}\rangle\setminus\{0\}} \lambda_u e^{itu}\Big\}, \quad t \in \mathbb{R},$$

where we have set

$$\lambda_0 := -\sum_{u\in\langle\mathcal{X}\rangle\setminus\{0\}} \lambda_u.$$

There is an absolutely convergent Fourier series with real coefficients in the last exponential function. Hence this function itself expands into an absolutely convergent Fourier series with coefficients in $\mathbb{R}$ and exponents in $\langle\mathcal{X}\rangle$. This remains true after multiplying this series by $e^{-\lambda_0}$ and $e^{-it\gamma_0}$ (which leads to g_d), because $\lambda_0 \in \mathbb{R}$ and $\gamma_0 \in \langle\mathcal{X}\rangle$. Thus we have

$$g_d(t) = \sum_{y\in\langle\mathcal{X}\rangle} q_y e^{ity}, \quad t \in \mathbb{R}, \tag{31}$$

where $q_y \in \mathbb{R}$ for every $y \in \langle\mathcal{X}\rangle$ and $\sum_{y\in\langle\mathcal{X}\rangle} |q_y| < \infty$. Obviously, the series (31) can be written as a Fourier–Stieltjes transform,

$$g_d(t) = \int_{\mathbb{R}} e^{itx}\, dI_d(x), \quad t \in \mathbb{R},$$

with the discrete function $I_d \in \boldsymbol{V}$ given by

$$I_d(x) := \sum_{\substack{y\in\langle\mathcal{X}\rangle\\ y\leqslant x}} q_y, \quad x \in \mathbb{R}. \tag{32}$$

Let us return to the formula (30). We observe that for any $k \in \mathbb{N}$ the functions f_s^k, g_d^k, $f_s^k \cdot g^k$ are the Fourier–Stieltjes transforms of functions from $\boldsymbol{V}$:

$$\begin{gathered} f_s(t)^k = \int_{\mathbb{R}} e^{itx}\, dF_s^{*k}(x), \quad g_d(t)^k = \int_{\mathbb{R}} e^{itx}\, dI_d^{*k}(x), \\ f_s(t)^k g_d(t)^k = \int_{\mathbb{R}} e^{itx}\, d(F_s^{*k} * I_d^{*k})(x), \quad t \in \mathbb{R}. \end{gathered} \tag{33}$$

Consequently, the partial sums of the series (30) are the Fourier–Stieltjes transforms of the functions

$$W_n(x) := \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d}\right)^k \big(F_s^{*k} * I_d^{*k}\big)(x), \quad x \in \mathbb{R},\ n \in \mathbb{N}. \tag{34}$$

Let us show that the sum of the whole series (30) admits the representation

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d}\right)^k f_s(t)^k g_d(t)^k = \int_{\mathbb{R}} e^{itx}\, dW(x), \quad t \in \mathbb{R}, \tag{35}$$

with some function $W \in \boldsymbol{V}$. According to Lemma 1, for this it is sufficient that (1) the partial sums of (30) uniformly converge to the infinite sum on every bounded interval and (2) $\sup_{n\in\mathbb{N}} \|W_n\| < \infty$. The uniform convergence of the partial sums (even on whole real line) was shown above, i.e., (1) holds. Let us prove (2). For any $n \in \mathbb{N}$ we have the estimate

$$\|W_n\| \leqslant \sum_{k=1}^{n} \frac{1}{k}\left(\frac{c_s}{c_d}\right)^k \|F_s^{*k} * I_d^{*k}\| \leqslant \sum_{k=1}^{n} \left\{\frac{1}{k}\left(\frac{c_s}{c_d}\right)^k \|F_s^{*k}\| \cdot \|I_d^{*k}\|\right\}.$$

Here $\left\|F_s^{*k}\right\| = 1$ for every $k \in \mathbb{N}$, because all the F_s^{*k} are distribution functions. Also, it is convenient for us to set $\|g_d^k\| = \|I_d^{*k}\|$, $k \in \mathbb{N}$. Thus we come to the inequalities

$$\|W_n\| \leqslant \sum_{k=1}^{n} \frac{1}{k}\left(\frac{c_s}{c_d}\right)^k \|g_d^k\|, \quad n \in \mathbb{N}. \tag{36}$$

We now estimate $\|g_d^k\|$ for every $k \in \mathbb{N}$. Let us return to decomposition (31) for the function g_d. The set $\langle \mathcal{X} \rangle$ is countable, and $\sum_{y\in\langle\mathcal{X}\rangle} |q_y| < \infty$. So we fix an arbitrary $\varepsilon \in (0, 1)$ and choose $N_\varepsilon \in \mathbb{N}$ and distinct points $y_1, \ldots, y_{N_\varepsilon} \in \langle \mathcal{X} \rangle$ such that

$$\sum_{y\in\langle\mathcal{X}\rangle\setminus\{y_1,\ldots,y_{N_\varepsilon}\}} |q_y| < \varepsilon.$$

We introduce the polynomial

$$\tilde{g}_\varepsilon(t) := \sum_{m=1}^{N_\varepsilon} q_{y_m} e^{ity_m}, \quad t \in \mathbb{R}. \tag{37}$$

So we have $\|g_d - \tilde{g}_\varepsilon\| < \varepsilon$. We also observe that

$$\left\|\frac{g_d - \tilde{g}_\varepsilon}{g_d}\right\| \leqslant \|g_d - \tilde{g}_\varepsilon\| \cdot \left\|\frac{1}{g_d}\right\| = \|g_d - \tilde{g}_\varepsilon\| \cdot \|f_d\| = \|g_d - \tilde{g}_\varepsilon\| < \varepsilon. \tag{38}$$

Let us fix $k \in \mathbb{N}$ and write

$$\|g_d^k\| = \left\|\tilde{g}_\varepsilon^k \cdot \left(\frac{g_d}{\tilde{g}_\varepsilon}\right)^k\right\| \leqslant \|\tilde{g}_\varepsilon^k\| \cdot \left\|\frac{g_d}{\tilde{g}_\varepsilon}\right\|^k.$$

In order to get a convenient representation for $g_d/\tilde{g}_\varepsilon$, we observe that for any $n \in \mathbb{N}$

$$\begin{aligned} 1 - \left(\frac{g_d(t) - \tilde{g}_\varepsilon(t)}{g_d(t)}\right)^n &= \left(1 - \frac{g_d(t) - \tilde{g}_\varepsilon(t)}{g_d(t)}\right) \cdot \sum_{j=0}^{n-1} \left(\frac{g_d(t) - \tilde{g}_\varepsilon(t)}{g_d(t)}\right)^j \\ &= \frac{\tilde{g}_\varepsilon(t)}{g_d(t)} \cdot \sum_{j=0}^{n-1} \left(\frac{g_d(t) - \tilde{g}_\varepsilon(t)}{g_d(t)}\right)^j, \quad t \in \mathbb{R}. \end{aligned}$$

So we have

$$\frac{g_d(t)}{\tilde{g}_\varepsilon(t)} = \sum_{j=0}^{n-1}\left(\frac{g_d(t)-\tilde{g}_\varepsilon(t)}{g(t)}\right)^j + \left(\frac{g_d(t)-\tilde{g}_\varepsilon(t)}{g_d(t)}\right)^n \cdot \frac{g_d(t)}{\tilde{g}_\varepsilon(t)}, \quad t \in \mathbb{R}, \quad n \in \mathbb{N}.$$

Hence, due to (38), we get

$$\left\|\frac{g_d}{\tilde{g}_\varepsilon}\right\| \leqslant \sum_{j=0}^{n-1}\left\|\frac{g_d-\tilde{g}_\varepsilon}{g_d}\right\|^j + \left\|\frac{g_d-\tilde{g}_\varepsilon}{g_d}\right\|^n \cdot \left\|\frac{g_d}{\tilde{g}_\varepsilon}\right\| < \sum_{j=0}^{n-1}\varepsilon^j + \varepsilon^n \cdot \left\|\frac{g_d}{\tilde{g}_\varepsilon}\right\|, \quad n \in \mathbb{N}.$$

Since $\varepsilon \in (0, 1)$, letting $n \to \infty$ yields

$$\left\|\frac{g_d}{\tilde{g}_\varepsilon}\right\| \leqslant \sum_{j=0}^{\infty}\varepsilon^j = \frac{1}{1-\varepsilon}.$$

Thus we have

$$\|g_d^k\| \leqslant \frac{\|\tilde{g}_\varepsilon^k\|}{(1-\varepsilon)^k}. \tag{39}$$

So the estimation of $\|I_d^{*k}\| = \|g_d^k\|$ is reduced to finding an upper bound for $\|\tilde{g}_\varepsilon^k\|$.

Let us consider $\tilde{g}_\varepsilon$ defined by formula (37). Let $Y_\varepsilon := \{y_1, y_2, \ldots, y_{N_\varepsilon}\}$. Suppose that $Y_\varepsilon = \{0\}$, i.e., $\sum_{y\in\langle\mathcal{X}\rangle\setminus\{0\}} |q_y| < \varepsilon$ and $\tilde{g}_\varepsilon(t) = q_0$ for any $t \in \mathbb{R}$. Then $\|\tilde{g}_\varepsilon^k\| = |q_0|^k$ for any $k \in \mathbb{N}$. Observe that

$$|q_0| = |\tilde{g}_\varepsilon(0)| \leqslant |g_d(0)| + |\tilde{g}_\varepsilon(0) - g_d(0)|,$$

where $|g_d(0)| = |1/f_d(0)| = 1$ and

$$|\tilde{g}_\varepsilon(0) - g_d(0)| = \left|\sum_{y\in\langle\mathcal{X}\rangle\setminus\{0\}} q_y\right| \leqslant \sum_{y\in\langle\mathcal{X}\rangle\setminus\{0\}} |q_y| < \varepsilon.$$

Thus $|q_0| < 1+\varepsilon$ and hence $\|\tilde{g}_\varepsilon^k\| < (1+\varepsilon)^k$.

We next assume that $Y_\varepsilon \neq \{0\}$, i.e., Y_ε contains nonzero elements. We select a basis over $\mathbb{Q}$ in the set Y_ε (see [24, p. 67–68]), i.e., we choose nonzero elements $\beta_1, \ldots, \beta_d \in Y_\varepsilon$ with some $d \in \{1, \ldots, N_\varepsilon\}$, which are linearly independent over $\mathbb{Q}$ and for any $m \in \{1, \ldots, N_\varepsilon\}$ there exist some $r_{m,1}, \ldots, r_{m,d} \in \mathbb{Q}$ such that $y_m = \sum_{l=1}^d r_{m,l}\beta_l$. Linear independence over $\mathbb{Q}$ means that with $r_1, \ldots, r_d \in \mathbb{Q}$, vanishes only in the case $r_1 = r_2 = \cdots = r_d = 0$. It is clear that the coefficients $r_{m,l}$ of the decomposition of y_m are uniquely determined. Let $\varkappa$ be the minimal positive integer such that $\bar{r}_{m,l} := \varkappa \cdot r_{m,l} \in \mathbb{Z}$ for any admissible m and l. We set $\bar{\beta}_l := \beta_l/\varkappa$ for every $l \in \{1, \ldots, d\}$. Then we have $y_m = \sum_{l=1}^d \bar{r}_{m,l}\bar{\beta}_l$ for any $m \in \{1, \ldots, N_\varepsilon\}$. Here the coefficients $\bar{r}_{m,l}$ are uniquely determined by y_m too. We define the vectors $\bar{r}_m := (\bar{r}_{m,1}, \ldots, \bar{r}_{m,d}) \in \mathbb{Z}^d$, $m = 1, \ldots, N_\varepsilon$. Since $y_1, \ldots, y_{N_\varepsilon}$ are assumed to be distinct, all $\bar{r}_m$ are distinct too. Let us introduce the function

$$\varphi_\varepsilon(t_1, \ldots, t_d) := \sum_{m=1}^{N_\varepsilon} q_{y_m} \exp\left\{i\sum_{l=1}^{d} \bar{r}_{m,l}t_l\right\}, \quad t_1, \ldots, t_d \in \mathbb{R}.$$

It is easily seen that this function is continuous and 2π-periodic over every variable. We will also use a shorthand for it:

$$\varphi_\varepsilon(t) = \sum_{m=1}^{N_\varepsilon} q_{y_m} e^{i\langle t,\bar{r}_m\rangle}, \quad t = (t_1, \dots, t_d) \in \mathbb{R}^d.$$

The functions φ_ε and $\tilde{g}_\varepsilon$ are related:

$$\varphi_\varepsilon\big(\bar{\beta}_1 t, \dots, \bar{\beta}_d t\big) = \sum_{m=1}^{N_\varepsilon} q_{y_m} \exp\Big\{it\sum_{l=1}^{d} \bar{r}_{m,l}\bar{\beta}_l\Big\} = \sum_{m=1}^{N_\varepsilon} q_{y_m} e^{ity_m} = \tilde{g}_\varepsilon(t), \quad t \in \mathbb{R},$$

i.e., $\tilde{g}_\varepsilon$ is diagonal for the function $(t_1, \dots, t_d) \mapsto \varphi_\varepsilon(\bar{\beta}_1 t_1, \dots, \bar{\beta}_d t_d)$. Hence the image of the first function is dense in the image of the second one (see [24, p. 116, Theorem 2.4.1]) and, consequently, in the image of φ_ε. This yields

$$\sup_{t\in\mathbb{R}} |\tilde{g}_\varepsilon(t)| = \sup_{t\in[-\pi,\pi]^d} |\varphi_\varepsilon(t)|. \tag{40}$$

Let us show that

$$\|\tilde{g}_\varepsilon^k\| = \|\varphi_\varepsilon^k\| \quad \text{for any } k \in \mathbb{N}. \tag{41}$$

We fix $k \in \mathbb{N}$ and write

$$\tilde{g}_\varepsilon^k(t) = \Big(\sum_{m=1}^{N_\varepsilon} q_{y_m} e^{ity_m}\Big)^k = \sum_{m_1=1}^{N_\varepsilon} \cdots \sum_{m_k=1}^{N_\varepsilon} \big(q_{y_{m_1}} \cdot \dots \cdot q_{y_{m_k}} e^{it(y_{m_1}+\dots+y_{m_k})}\big), \quad t \in \mathbb{R},$$

$$\varphi_\varepsilon^k(t) = \Big(\sum_{m=1}^{N_\varepsilon} q_{y_m} e^{i\langle t,\bar{r}_m\rangle}\Big)^k = \sum_{m_1=1}^{N_\varepsilon} \cdots \sum_{m_k=1}^{N_\varepsilon} \big(q_{y_{m_1}} \cdot \dots \cdot q_{y_{m_k}} e^{i\langle t,\bar{r}_{m_1}+\dots+\bar{r}_{m_k}\rangle}\big), \ t \in \mathbb{R}^d.$$

We next have the representation

$$\tilde{g}_\varepsilon^k(t) = \sum_{z\in\mathcal{Y}_\varepsilon^{(k)}} \Bigg(\sum_{\substack{1\leqslant m_1,\dots,m_k\leqslant N_e \\ y_{m_1}+\dots+y_{m_k}=z}} q_{y_{m_1}} \cdot \dots \cdot q_{y_{m_k}}\Bigg) e^{itz}, \quad t \in \mathbb{R},$$

$$\varphi_\varepsilon^k(t) = \sum_{s\in\mathcal{R}_\varepsilon^{(k)}} \Bigg(\sum_{\substack{1\leqslant m_1,\dots,m_k\leqslant N_e \\ \bar{r}_{m_1}+\dots+\bar{r}_{m_k}=s}} q_{\bar{r}_{m_1}} \cdot \dots \cdot q_{\bar{r}_{m_k}}\Bigg) e^{i\langle t,s\rangle}, \quad t \in \mathbb{R}^d,$$

where

$$\mathcal{Y}_\varepsilon^{(k)} := \big\{y_{m_1} + \dots + y_{m_k} : m_1, \dots, m_k \in \{1, \dots, N_\varepsilon\}\big\},$$
$$\mathcal{R}_\varepsilon^{(k)} := \big\{\bar{r}_{m_1} + \dots + \bar{r}_{m_k} : m_1, \dots, m_k \in \{1, \dots, N_\varepsilon\}\big\}.$$

Thus we have

$$\|\tilde{g}_\varepsilon^k\| = \sum_{z\in\mathcal{Y}_\varepsilon^{(k)}} \Bigg| \sum_{\substack{m_1=1,\dots,N_\varepsilon,\\ \dots\\ m_k=1,\dots,N_\varepsilon\\ y_{m_1}+\dots+y_{m_k}=z}} q_{y_{m_1}}\cdot\dots\cdot q_{y_{m_k}} \Bigg|, \tag{42}$$

$$\|\varphi_\varepsilon^k\| = \sum_{s\in\mathcal{R}_\varepsilon^{(k)}} \Bigg| \sum_{\substack{m_1=1,\dots,N_\varepsilon,\\ \dots\\ m_k=1,\dots,N_\varepsilon\\ \bar{r}_{m_1}+\dots+\bar{r}_{m_k}=s}} q_{y_{m_1}}\cdot\dots\cdot q_{y_{m_k}} \Bigg|. \tag{43}$$

There is a natural map between the two finite sets $\mathcal{Y}_\varepsilon^{(k)}$ and $\mathcal{R}_\varepsilon^{(k)}$: every number $z=\sum_{j=1}^k y_{m_j}$ from the first set is paired with a vector $s=\sum_{j=1}^k \bar{r}_{m_j}$ from the second set. Here s is the vector of coefficients of the decomposition for z with the basis $\bar{\beta}_1,\dots,\bar{\beta}_d$. This map is actually a bijection. Indeed, it is injective, because distinct numbers from $\mathcal{Y}_\varepsilon^{(k)}$ have the distinct decompositions, i.e., they are paired with the distinct vectors from the set $\mathcal{R}_\varepsilon^{(k)}$. The map is surjective, because any vector $v\in\mathcal{R}_\varepsilon^{(k)}$ is a sum $\bar{r}_{m'_1}+\dots+\bar{r}_{m'_k}$ with some indices $m'_1,\dots,m'_d\in\{1,\dots,N_\varepsilon\}$, and this sum corresponds to the number $z=y_{m'_1}+\dots+y_{m'_k}$ from the set $\mathcal{Y}_\varepsilon^{(k)}$ by construction. Next, it is clear from the basis decompositions that for any fixed pair of corresponding elements $z\in\mathcal{Y}_\varepsilon^{(k)}$ and $s\in\mathcal{R}_\varepsilon^{(k)}$ the equalities $y_{m_1}+\dots+y_{m_k}=z$ and $\bar{r}_{m_1}+\dots+\bar{r}_{m_k}=s$ are equivalent for varying index vectors $(m_1,\dots,m_k)$. Since the inner sums in (42) and (43) add the same weight $q_{y_{m_1}}\cdot\dots\cdot q_{y_{m_k}}$ for every index vector $(m_1,\dots,m_k)$, we conclude that these inner sums are equal. Thus we come to the equality $\|\tilde{g}_\varepsilon^k\|=\|\varphi_\varepsilon^k\|$.

We now apply Lemma 4 to estimate $\|\varphi_\varepsilon^k\|$ for any $k\in\mathbb{N}$ (we set $\varphi:=\varphi_\varepsilon$, $N:=N_\varepsilon$, $q_m:=q_{y_m}$ and $c_m:=\bar{r}_m$ for every $m=1,\dots,N_\varepsilon$). Using inequality (19), we get

$$\|\varphi_\varepsilon^k\| \leqslant A_{\varphi_\varepsilon} k^d \sup_{t\in[-\pi,\pi]^d} |\varphi_\varepsilon(t)|^k, \quad k\in\mathbb{N},$$

where A_{φ_ε} is a constant defined by (20), which doesn't depend on k. Applying (40) and (41), we come to the inequality

$$\|\tilde{g}_\varepsilon^k\| \leqslant A_{\varphi_\varepsilon} k^d \sup_{t\in\mathbb{R}} |\tilde{g}_\varepsilon(t)|^k, \quad k\in\mathbb{N}.$$

Observe that

$$\sup_{t\in\mathbb{R}} |\tilde{g}_\varepsilon(t)| \leqslant \sup_{t\in\mathbb{R}} |g_d(t)| + \sup_{t\in\mathbb{R}} |g_d(t)-\tilde{g}_\varepsilon(t)|.$$

On account of (38), we have

$$\frac{\sup_{t\in\mathbb{R}} |g_d(t)-\tilde{g}_\varepsilon(t)|}{\sup_{t\in\mathbb{R}} |g_d(t)|} \leqslant \sup_{t\in\mathbb{R}} \left| \frac{g_d(t)-\tilde{g}_\varepsilon(t)}{g_d(t)} \right| \leqslant \left\| \frac{g_d-\tilde{g}_\varepsilon}{g_d} \right\| < \varepsilon,$$

i.e., $\sup_{t\in\mathbb{R}} |g_d(t)-\tilde{g}_\varepsilon(t)| < \varepsilon \sup_{t\in\mathbb{R}} |g_d(t)|$. Therefore

$$\sup_{t\in\mathbb{R}} |\tilde{g}_\varepsilon(t)| \leqslant \sup_{t\in\mathbb{R}} |g_d(t)| + \varepsilon \sup_{t\in\mathbb{R}} |g_d(t)| = (1+\varepsilon) \sup_{t\in\mathbb{R}} |g_d(t)|,$$

where

$$\sup_{t\in\mathbb{R}} |g_d(t)| = \sup_{t\in\mathbb{R}} \left|\frac{1}{f_d(t)}\right| = \frac{1}{\inf_{t\in\mathbb{R}} |f_d(t)|} = \frac{1}{\mu_d}.$$

So we obtain the estimate

$$\|\tilde{g}_\varepsilon^k\| \leqslant A_{\varphi_\varepsilon} k^d \cdot \frac{(1+\varepsilon)^k}{\mu_d^k}, \quad k \in \mathbb{N}.$$

We note that here $k^d \geqslant 1$ and $1/\mu_d^k \geqslant 1$ for any $k \in \mathbb{N}$. Therefore the estimates of $\|\tilde{g}_\varepsilon^k\|$ in the cases $Y_\varepsilon = \{0\}$ and $Y_\varepsilon \neq \{0\}$ can be unified as

$$\|\tilde{g}_\varepsilon^k\| \leqslant C_\varepsilon k^d \cdot \frac{(1+\varepsilon)^k}{\mu_d^k} \quad \text{for any } k \in \mathbb{N},$$

with some constant $C_\varepsilon > 0$. We use this in (39), obtaining

$$\|g_d^k\| \leqslant \frac{\|\tilde{g}_\varepsilon^k\|}{(1-\varepsilon)^k} \leqslant C_\varepsilon k^d \cdot \frac{1}{\mu_d^k} \cdot \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^k, \quad k \in \mathbb{N}.$$

Let us return to (36). Due to the last estimate, for any $\varepsilon \in (0, 1)$ and $n \in \mathbb{N}$ we have

$$\|W_n\| \leqslant \sum_{k=1}^n \left(\frac{1}{k}\left(\frac{c_s}{c_d}\right)^k C_\varepsilon k^d \cdot \frac{1}{\mu_d^k} \cdot \left(\frac{1+\varepsilon}{1-\varepsilon}\right)^k\right) = C_\varepsilon \sum_{k=1}^n k^{d-1} \left(\frac{c_s}{c_d \mu_d} \cdot \frac{1+\varepsilon}{1-\varepsilon}\right)^k. \tag{44}$$

Since $c_s < c_d \mu_d$ by assumption, the fixed number $\varepsilon \in (0, 1)$ can be specified by

$$0 < \frac{c_s}{c_d \mu_d} \cdot \frac{1+\varepsilon}{1-\varepsilon} < 1.$$

Then we obtain

$$\sup_{n\in\mathbb{N}} \|W_n\| \leqslant C_\varepsilon \sum_{k=1}^\infty k^{d-1} \left(\frac{c_s}{c_d \mu_d} \cdot \frac{1+\varepsilon}{1-\varepsilon}\right)^k < \infty,$$

i.e., we showed that the sufficient condition (2) for (35) is satisfied (condition (1) was proved above). So, according to (30) and (35), we have

$$\ln\left(1 + \frac{c_s f_s(t)}{c_d f_d(t)}\right) = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d}\right)^k f_s(t)^k g_d(t)^k = \int_{\mathbb{R}} e^{itx}\, dW(x), \quad t \in \mathbb{R}, \tag{45}$$

with some function $W \in \boldsymbol{V}$. Hence (29) takes the form

$$\operatorname{Ln} f_{s,d}(t) = \int_{\mathbb{R}} e^{itx}\, dW(x) - \ln\left(1 + \frac{c_s}{c_d}\right), \quad t \in \mathbb{R}.$$

The equality $\operatorname{Ln} f_{s,d}(0) = 0$, which was mentioned above, implies

$$\ln\left(1 + \frac{c_s}{c_d}\right) = \int_{\mathbb{R}} dW(x).$$

Then

$$\mathrm{Ln}\, f_{s,d}(t) = \int_{\mathbb{R}} (e^{itx} - 1)\; dW(x), \quad t \in \mathbb{R},$$

i.e., we come to the representation

$$f_{s,d}(t) = \exp\left\{\int_{\mathbb{R}} (e^{itx} - 1)\; dW(x)\right\} \\ = \exp\left\{\int_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1)\; dW(x)\right\}, \quad t \in \mathbb{R}. \tag{46}$$

Let us investigate the function W. If $c_s = 0$, then W is identically 0, which is seen from (45). We next focus on the case $c_s \neq 0$. We will prove that W is continuous on $\mathbb{R}$. For this we first observe that the W_n, $n \in \mathbb{N}$, are continuous functions on $\mathbb{R}$. Indeed, according to (34), W_n is a finite linear combination of the functions $F_s^{*k} * I_d^{*k}$, $k \in \mathbb{N}$, which are continuous on $\mathbb{R}$, being convolutions with continuous F_s. Next, we observe that, similarly to (44), we have the estimate

$$\|W_{n_2} - W_{n_1}\| \leqslant C_\varepsilon \sum_{k=n_1+1}^{n_2} k^{d-1}\left(\frac{c_s}{c_d \mu_d} \cdot \frac{1+\varepsilon}{1-\varepsilon}\right)^k,$$

with the same ε, C_ε and for any positive integers n_1 and n_2 such that $n_1 \leqslant n_2$. The sum on the right can be made arbitrarily small for all sufficiently large n_1 and n_2, because its terms (for $k \in \mathbb{N}$) form a convergent series. This means that $(W_n)_{n\in\mathbb{N}}$ is a fundamental sequence in the space $\boldsymbol{V}$. Since $\boldsymbol{V}$ is a complete norm space (see the comments at the end of the introduction), there exists $W_* \in \boldsymbol{V}$ such that $\|W_n - W_*\| \to 0$ as $n \to \infty$. Hence

$$\lim_{n\to\infty} \int_{\mathbb{R}} e^{itx}\, dW_n(x) = \int_{\mathbb{R}} e^{itx}\, dW_*(x), \quad t \in \mathbb{R}.$$

On the other hand, we know that

$$\lim_{n\to\infty} \int_{\mathbb{R}} e^{itx}\, dW_n(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}\left(\frac{c_s}{c_d}\right)^k f_s(t)^k g_d(t)^k \\ = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\left(\frac{c_s}{c_d}\right)^k f_s(t)^k g_d(t)^k = \int_{\mathbb{R}} e^{itx}\, dW(x), \quad t \in \mathbb{R}.$$

Then $\int_{\mathbb{R}} e^{itx}\, dW_*(x) = \int_{\mathbb{R}} e^{itx}\, dW(x)$ for any $t \in \mathbb{N}$, and we conclude that $W_* = W$. So we have proved that

$$\|W_n - W\| \to 0 \quad \text{as } n \to \infty. \tag{47}$$

In general, from the well-known relation $\sup_{x\in\mathbb{R}} |U(x)| \leqslant \|U\|$ for any $U \in \boldsymbol{V}$, we have uniform convergence:

$$\sup_{x\in\mathbb{R}} \bigl|W_n(x) - W(x)\bigr| \to 0 \quad \text{as } n \to \infty.$$

Due to the already proved continuity of W_n, $n \in \mathbb{N}$, we conclude that W is continuous on $\mathbb{R}$ too.

We now show that W cannot be purely absolutely continuous (in the case $c_s \neq 0$). For suppose that W is absolutely continuous. Let

$$w(t) := \int_{\mathbb{R}} e^{itx}\, dW(x), \quad t \in \mathbb{R}.$$

Then, according to (45), we have

$$1 + \frac{c_s f_s(t)}{c_d f_d(t)} = \exp\left\{\int_{\mathbb{R}} e^{itx}\, dW(x)\right\} = \exp\{w(t)\} = 1 + \sum_{k=1}^{\infty} \frac{w(t)^k}{k!}, \quad t \in \mathbb{R}.$$

Hence, on the one hand,

$$\begin{aligned} \frac{c_s f_s(t)}{c_d f_d(t)} &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{w(t)^k}{k!} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k!} \int_{\mathbb{R}} e^{itx}\, dW^{*k}(x) \\ &= \lim_{n\to\infty} \int_{\mathbb{R}} e^{itx}\, d\left(\sum_{k=1}^{n} \frac{1}{k!}\, W^{*k}(x)\right), \quad t \in \mathbb{R}. \end{aligned}$$

On the other hand, on account of (33), we have

$$\frac{c_s f_s(t)}{c_d f_d(t)} = \frac{c_s}{c_d} \cdot f_s(t) \cdot g_d(t) = \frac{c_s}{c_d} \cdot \int_{\mathbb{R}} e^{itx}\, d(F_s * I_d)(x) = \int_{\mathbb{R}} e^{itx}\, d\left(\frac{c_s}{c_d} \cdot (F_s * I_d)(x)\right),$$

for $t \in \mathbb{R}$. Since the variance

$$\left\| \sum_{k=n_1+1}^{n_2} \frac{1}{k!}\, W^{*k} \right\| \leqslant \sum_{k=n_1+1}^{n_2} \frac{\|W^{*k}\|}{k!} \leqslant \sum_{k=n_1+1}^{n_2} \frac{\|W\|^k}{k!}$$

can be made arbitrarily small for all sufficiently large n_1 and n_2, the sequence of sums $\sum_{k=1}^{n} W^{*k}/k!$, $n \in \mathbb{N}$, is fundamental in the space $\boldsymbol{V}$. Due to the completeness of $\boldsymbol{V}$, these sums converge in variation to some function from $\boldsymbol{V}$, namely, to $(c_s/c_d) \cdot (F_s * I_d)$ by the uniqueness of the Fourier–Stieltjes transform:

$$\left\| \sum_{k=1}^{n} \frac{1}{k!}\, W^{*k} - \frac{c_s}{c_d} \cdot (F_s * I_d) \right\| \to 0 \quad \text{as } n \to \infty.$$

But this is impossible, because $\sum_{k=1}^{n} W^{*k}/k!$, $n \in \mathbb{N}$, are absolutely continuous as linear combinations of convolution powers of absolutely continuous W by the assumption, but $(c_s/c_d) \cdot (F_s * I_d)$ is continuous singular as the convolution of the continuous singular function F_s and the discrete function I_d (see the comments after Remark 3). Thus W is not a (purely) absolutely continuous function from $\boldsymbol{V}$, i.e., it always has some continuous singular part.

Let us return to (47) and prove the sufficient condition (from the statement of the theorem) for W to be (purely) continuous singular. Suppose that all the functions F_s^{*k}, $k \in \mathbb{N}$, are continuous singular. Hence all W_n, $n \in \mathbb{N}$ are too. Let W_a and W_s

be the absolutely continuous part and the continuous singular part of the Lebesgue decomposition for W: $W = W_a + W_s$. Then

$$\|W_n - W\| = \|W_n - W_a - W_s\| = \|W_n - W_s\| + \|W_a\| \geqslant \|W_a\| \geqslant 0.$$

Due to (47), we conclude that $\|W_a\| = 0$, i.e., $W_a(x) = 0$ for any $x \in \mathbb{R}$. Thus $W = W_s$.

We now combine the representations (26), (27) and (46) with formula (25):

$$\begin{aligned} f(t) = \exp\Bigg\{ & it\gamma_0 + \sum_{u \in \langle \mathcal{X} \rangle \setminus \{0\}} \lambda_u \big(e^{itu} - 1\big) \\ & + \int_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1)\bigg(v_a(x) + \operatorname{sgn}(x)\, \frac{\mathfrak{m}_a e^{-|x|}}{|x|}\bigg)\, dx + \int_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1)\, dW(x)\Bigg\} \\ = \exp\bigg\{ & it\gamma_0 + \int_{\mathbb{R}\setminus\{0\}} (e^{itx} - 1)\, dL(x)\bigg\}, \quad t \in \mathbb{R}, \end{aligned}$$

with $L(x) := L_d(x) + L_a(x) + L_s(x)$, $x \in \mathbb{R} \setminus \{0\}$, where L_d, L_a, and L_s are defined by formulas (11), (12), and (13). It is not difficult to check that L_d, L_a, L_s, and, consequently, L satisfy all conditions the for a spectral function of the Lévy type representation (see introduction or Theorem 6). However, we recall that the function v_a is potentially complex-valued and hence L is too. Next, it is seen that

$$\int_{S_1} |x|\, d|L_*|(x) < \infty$$

with $S_1 := [-1, 1] \setminus \{0\}$ and for any $L_* \in \{L_d, L_a, L_s\}$. Hence it is also true for $L_* = L$, and we can write

$$f(t) = \exp\bigg\{ it\gamma_1 + \int_{\mathbb{R}\setminus\{0\}} \big(e^{itx} - 1 - itx\, \mathbb{I}_{[-1,1]}(x)\big)\, dL(x)\bigg\}, \quad t \in \mathbb{R},$$

with $\gamma_1 := \gamma_0 + \int_{\mathbb{R}\setminus\{0\}} x\, \mathbb{I}_{[-1,1]}(x)\, dL(x)$, which is potentially complex.

We now apply Theorem 6 to derive that L is actually real-valued, which means that v_a is a real-valued function. Moreover, we have $F \in \boldsymbol{Q}$, i.e., (i) is proved. □

Proof of Proposition 1. To obtain a contradiction, we assume that W is continuous singular. For convenience, we introduce the operator $[\,\cdot\,]_s$, which acts from $\boldsymbol{V}$ to $\boldsymbol{V}$, and for any $U \in \boldsymbol{V}$ it returns the continuous singular part of U as $[U]_s$ (which can be identically zero). We start with (47) from the proof of Theorem 3, which holds under the assumptions of the proposition. For every $n \in \mathbb{N}$ we write

$$\|W_n - W\| = \big\|(W_n - [W_n]_s) + ([W_n]_s - W)\big\| = \big\|W_n - [W_n]_s\big\| + \big\|[W_n]_s - W\big\|,$$

where the latter equality is valid because $W_n - [W_n]_s$ is absolutely continuous (or identically zero) and $[W_n]_s - W$ is continuous singular by assumption. Therefore $\|W_n - W\| \geqslant \big\|[W_n]_s - W\big\| \geqslant 0$ and, due to (47), we conclude that

$$\big\|[W_n]_s - W\big\| \to 0 \quad \text{as } n \to \infty. \tag{48}$$

Next, according to (34), we have

$$[W_n]_s = \left(\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d} \right)^k F_s^{*k} * I_d^{*k} \right)_s = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d} \right)^k \left[F_s^{*k} * I_d^{*k} \right]_s$$

for $n \in \mathbb{N}$. Since I_d is discrete (see (32)), the functions I_d^{*k} are discrete and hence $\left[F_s^{*k} * I_d^{*k}\right]_s = \left[F_s^{*k}\right]_s * I_d^{*k}$ for any $k \in \mathbb{N}$. Thus we come to the equalities

$$[W_n]_s = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d} \right)^k \left[F_s^{*k} \right]_s * I_d^{*k}, \quad n \in \mathbb{N}. \tag{49}$$

Let us consider the functions $\left[F_s^{*k}\right]_s$ and numbers $\left[F_s^{*k}\right]_s(\infty)$. By the definition of n_a, for any $k < n_a$ we have $\left[F_s^{*k}\right]_s = F_s^{*k}$ and, in particular, $\left[F_s^{*k}\right]_s(\infty) = 1$. For the case $k = n_a$, by the assumption, we have $\left[F_s^{*n_a}\right]_s(x) = (1-\alpha)H_s(x)$, $x \in \mathbb{R}$, and hence $\left[F_s^{*n_a}\right]_s(\infty) = 1-\alpha$. Next, for any integer $k \geqslant n_a$ we observe that

$$\left[F_s^{*(k+1)}\right]_s(x) = \left[F_s^{*k} * F_s\right]_s(x) = \left[[F_s^{*k}]_s * F_s\right]_s(x) \leqslant \left([F_s^{*k}]_s * F_s\right)(x), \quad x \in \mathbb{R}.$$

By the Lebesgue dominated convergence theorem, we conclude that

$$\left([F_s^{*k}]_s * F_s\right)(\infty) = \lim_{z\to\infty} \int_{\mathbb{R}} F_s(z-x)\, d[F_s^{*k}]_s(x) = \int_{\mathbb{R}} d[F_s^{*k}]_s(x) = \left[F_s^{*k}\right]_s(\infty).$$

Thus $\left[F_s^{*(k+1)}\right]_s(\infty) \leqslant \left[F_s^{*k}\right]_s(\infty)$ for any integer $k \geqslant n_a$. Let us introduce the sequence $A_k := \left[F_s^{*k}\right]_s(\infty)$, $k \in \mathbb{N}$. So we have

$$\begin{gathered} A_1 = \cdots = A_{n_a-1} = 1, \qquad A_{n_a} = 1-\alpha, \\ A_k \geqslant A_{k+1} \geqslant 0 \quad \text{for any } \; k \geqslant n_a. \end{gathered} \tag{50}$$

Due to (48), we next conclude that

$$\int_{\mathbb{R}} e^{itx}\, dW(x) = \lim_{n\to\infty} \int_{\mathbb{R}} e^{itx}\, d[W_n]_s(x), \quad t \in \mathbb{R}.$$

Then, according to (45) and (49), we have

$$\ln\left(1 + \frac{c_s f_s(t)}{c_d f_d(t)}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d}\right)^k f_{k,s}(t) g_d(t)^k \quad \text{for any} \quad t \in \mathbb{R},$$

where

$$f_{k,s}(t) := \int_{\mathbb{R}} e^{itx}\, d\left[F_s^{*k}\right]_s(x), \quad t \in \mathbb{R}.$$

In particular, we write

$$\ln\left(1 + \frac{c_s f_s(0)}{c_d f_d(0)}\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d}\right)^k f_{k,s}(0) g_d(0)^k.$$

Recall that $f_s(0) = f_d(0) = g_d(0) = 1$ and $f_{k,s}(0) = [F_s^{*k}]_s(\infty) = A_k$ for any $k \in \mathbb{N}$. Then, on the one hand,

$$\ln\left(1 + \frac{c_s}{c_d}\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d}\right)^k A_k.$$

On the other hand, since $c_s < c_d \mu_d \leqslant c_d$, we have the expansion

$$\ln\left(1 + \frac{c_s}{c_d}\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d}\right)^k.$$

Thus we come to the equality

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d}\right)^k A_k = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \left(\frac{c_s}{c_d}\right)^k.$$

Due to (50), it is reduced to

$$\sum_{k=n_a}^{\infty} \frac{(-1)^{k-n_a}}{k} \left(\frac{c_s}{c_d}\right)^k A_k = \sum_{k=n_a}^{\infty} \frac{(-1)^{k-n_a}}{k} \left(\frac{c_s}{c_d}\right)^k. \tag{51}$$

The series on the left is alternating with nonincreasing absolute values of the terms. Therefore

$$\sum_{k=n_a}^{\infty} \frac{(-1)^{k-n_a}}{k} \left(\frac{c_s}{c_d}\right)^k A_k \leqslant \frac{1}{n_a} \left(\frac{c_s}{c_d}\right)^{n_a} A_{n_a} = \frac{1}{n_a} \left(\frac{c_s}{c_d}\right)^{n_a} (1 - \alpha).$$

Since the series on the right in (51) is alternating with strictly decreasing absolute values, the following inequality holds:

$$\sum_{k=n_a}^{\infty} \frac{(-1)^{k-n_a}}{k} \left(\frac{c_s}{c_d}\right)^k > \frac{1}{n_a} \left(\frac{c_s}{c_d}\right)^{n_a} - \frac{1}{n_a+1} \left(\frac{c_s}{c_d}\right)^{n_a+1} = \frac{1}{n_a} \left(\frac{c_s}{c_d}\right)^{n_a} \left(1 - \frac{n_a}{n_a+1} \cdot \frac{c_s}{c_d}\right).$$

We see that the assumption $\alpha \geqslant \frac{n_a}{n_a+1} \cdot \frac{c_s}{c_d}$ implies the inequality

$$\sum_{k=n_a}^{\infty} \frac{(-1)^{k-n_a}}{k} \left(\frac{c_s}{c_d}\right)^k A_k < \sum_{k=n_a}^{\infty} \frac{(-1)^{k-n_a}}{k} \left(\frac{c_s}{c_d}\right)^k.$$

Thus we come to a contradiction with (51). So the assumption that W is continuous singular is false.

Due to the condition of dominated continuous singular part, we have $c_s < c_d$. Since $\frac{n_a}{n_a+1} < 1$, the inequality $\alpha \geqslant \frac{n_a}{n_a+1} \cdot \frac{c_s}{c_d}$ always holds in the case $\alpha = 1$. □

Proof of Proposition 2. Let us consider the function f_s defined by formula (15). We first find the limit of the sequence $f_s(t_n)$, $n \in \mathbb{N}$, with $t_n := \pi(2n)!$ as $n \to \infty$. We write $f_s(t_n) = M_n \cdot R_n$, $n \in \mathbb{N}$, where

$$M_n := \prod_{k=1}^{2n} \cos(t_n/k!) \quad \text{and} \quad R_n := \prod_{k=2n+1}^{\infty} \cos(t_n/k!), \quad n \in \mathbb{N}.$$

For any $n \in \mathbb{N}$

$$\begin{aligned} M_n &= \cos\left(\frac{\pi(2n)!}{(2n)!}\right) \cdot \cos\left(\frac{\pi(2n)!}{(2n-1)!}\right) \cdot \cos\left(\frac{\pi(2n)!}{(2n-2)!}\right) \cdot \cdots \cdot \cos\left(\frac{\pi(2n)!}{1!}\right) \\ &= \cos(\pi) \cdot \cos(\pi \cdot 2n) \cdot \cos\big(\pi \cdot 2n(2n-1)\big) \cdot \cdots \cdot \cos\big(\pi \cdot 2n(2n-1) \cdot \cdots \cdot 1\big) \\ &= (-1) \cdot 1 \cdot 1 \cdot \cdots \cdot 1 = -1. \end{aligned}$$

Next, for every $n \in \mathbb{N}$ and for every integer $k \geqslant 2n+1$ we have

$$0 < \frac{t_n}{k!} = \frac{\pi(2n)!}{k \cdot (k-1)!} \leqslant \frac{\pi}{k} \leqslant \frac{\pi}{3},$$

and, by the well-known inequality $\cos(x) \geqslant 1 - x^2/2$, $x \in \mathbb{R}$, we get

$$\cos(t_n/k!) \geqslant 1 - \frac{(t_n/k!)^2}{2} \geqslant 1 - \frac{\pi^2}{2k^2} \geqslant 1 - \frac{\pi^2}{18} > 0.$$

Therefore, on the one hand, it is clear that for any $n \in \mathbb{N}$ we have $\cos(t_n/k!) < 1$ for any $k \geqslant 2n+1$ and hence $R_n < 1$. On the other hand,

$$R_n \geqslant \prod_{k=2n+1}^{\infty} \left(1 - \frac{\pi^2}{2k^2}\right) = \exp\left\{\sum_{k=2n+1}^{\infty} \ln\left(1 - \frac{\pi^2}{2k^2}\right)\right\}, \quad n \in \mathbb{N},$$

where the sum in the exponent tends to 0 as $n \to \infty$. So we conclude that $R_n \to 1$ as $n \to \infty$. Thus

$$f_s(t_n) = M_n \cdot R_n = -R_n \to -1 \quad \text{as} \quad n \to \infty,$$

and we know that $f_s(t_n) > -1$ for any $n \in \mathbb{N}$.

We next observe that

$$f_s(t_n \pm \pi) = \prod_{k=1}^{\infty} \cos\left(\frac{t_n \pm \pi}{k!}\right) = 0.$$

Indeed,

$$\cos\left(\frac{t_n \pm \pi}{k!}\right)\bigg|_{k=2} = \cos\left(\frac{\pi(2n)! \pm \pi}{2!}\right) = \cos\left(\pi \cdot 3 \cdot \cdots \cdot (2n) \pm \frac{\pi}{2}\right) = 0.$$

We now return to the characteristic function $f_*(t) = \frac{1}{2} + \frac{1}{2} f_s(t)$, $t \in \mathbb{R}$, and we consider the quantities

$$\frac{f_*(t_n - \pi) f_*(t_n + \pi)}{f_*(t_n)^2} = \frac{\left(\frac{1}{2} + \frac{1}{2} f_s(t_n - \pi)\right)\left(\frac{1}{2} + \frac{1}{2} f_s(t_n + \pi)\right)}{\left(\frac{1}{2} + \frac{1}{2} f_s(t_n)\right)^2}, \quad n \in \mathbb{N}.$$

By the above,

$$\frac{f_*(t_n - \pi) f_*(t_n + \pi)}{f_*(t_n)^2} = \frac{1}{(1 + f_s(t_n))^2} \to \infty \quad \text{as} \quad n \to \infty.$$

Therefore, by Theorem 7, f_* cannot be the characteristic function of a distribution function from class $\boldsymbol{Q}$. Thus $F_* \notin \boldsymbol{Q}$. □

Proof of Proposition 3. We have decompositions (3) and (4) for the distribution function F and its characteristic function f, respectively. Since $F \in \boldsymbol{Q}$, we know that $f(t) \neq 0$ for any $t \in \mathbb{R}$ and $\mu_d = \inf_{t\in\mathbb{R}} |f_d(t)| > 0$ according to equation (9). Due to the assumption of dominated singular part and $c_d > 0$, we have $c_s < c_d \mu_d$.

Let f_1 and f_2 denote the characteristic functions of F_1 and F_2. The assumed decomposition $F = F_1 * F_2$ means that $f(t) = f_1(t) f_2(t)$, $t \in \mathbb{R}$. This means that $f_1(t) \neq 0$ and $f_2(t) \neq 0$ for all $t \in \mathbb{R}$.

Let us write the Lebesgue decompositions for F_1 and F_2:

$$F_j(x) = c_{j,d} F_{j,d}(x) + c_{j,a} F_{j,a}(x) + c_{j,s} F_{j,s}(x), \quad x \in \mathbb{R}, \quad j \in \{1, 2\}.$$

We also write the corresponding decompositions for f_1 and f_2:

$$f_j(t) = c_{j,d} f_{j,d}(t) + c_{j,a} f_{j,a}(t) + c_{j,s} f_{j,s}(t), \quad t \in \mathbb{R}, \quad j \in \{1, 2\}.$$

Here $c_{j,d}$, $c_{j,a}$, $c_{j,s}$ are nonnegative and $c_{j,d} + c_{j,a} + c_{j,s} = 1$ for $j = 1$ and $j = 2$. For clarity, we write the equality $F = F_1 * F_2$ in the expanded form:

$$(52) \quad c_d F_d + c_a F_a + c_s F_s = \left(c_{1,d} F_{1,d} + c_{1,a} F_{1,a} + c_{1,s} F_{1,s}\right) * \left(c_{2,d} F_{2,d} + c_{2,a} F_{2,a} + c_{2,s} F_{2,s}\right).$$

Since F has nonzero discrete part ($c_d > 0$), the functions F_1 and F_2 have nonzero discrete parts too, i.e., $c_{1,d} > 0$ and $c_{2,d} > 0$. Since a convolution of any two distribution functions is discrete if and only if these functions are discrete, we conclude that $c_d F_d(x) = c_{1,d} c_{2,d} (F_{1,d} * F_{2,d})(x)$, $x \in \mathbb{R}$, i.e., $c_d = c_{1,d} c_{2,d}$ and $F_d = F_{1,d} * F_{2,d}$. Thus we have $f_d(t) = f_{1,d}(t) f_{2,d}(t)$, $t \in \mathbb{R}$. Since $|f_{1,d}(t)| \leqslant 1$ and $|f_{2,d}(t)| \leqslant 1$ for any $t \in \mathbb{R}$, we conclude that

$$\mu_{1,d} := \inf_{t\in\mathbb{R}} |f_{1,d}(t)| \geqslant \inf_{t\in\mathbb{R}} |f_d(t)| = \mu_d > 0,$$

and, analogously,

$$\mu_{2,d} := \inf_{t\in\mathbb{R}} |f_{2,d}(t)| \geqslant \inf_{t\in\mathbb{R}} |f_d(t)| = \mu_d > 0.$$

We next observe that $F_{1,s} * F_{2,d}$ and $F_{2,s} * F_{1,d}$ are continuous singular. Therefore the corresponding summands from (52) are included in the continuous part of F, i.e., $c_s F_s(x) \geqslant c_{1,s}c_{2,d}(F_{1,s} * F_{2,d})(x) + c_{2,s}c_{1,d}(F_{2,s} * F_{1,d})(x)$, $x \in \mathbb{R}$. Consequently,

$$c_s = c_s F_s(\infty) \geqslant c_{1,s}c_{2,d}(F_{1,s} * F_{2,d})(\infty) + c_{2,s}c_{1,d}(F_{2,s} * F_{1,d})(\infty) \\ = c_{1,s}c_{2,d} + c_{2,s}c_{1,d}.$$

Then we get

$$c_{1,s}c_{2,d} \leqslant c_s < c_d \mu_d = c_{1,d}c_{2,d}\mu_d \leqslant c_{1,d}c_{2,d}\mu_{1,d},$$

i.e., $c_{1,s} < c_{1,d}\mu_{1,d}$. Analogously,

$$c_{2,s}c_{1,d} \leqslant c_s < c_d \mu_d = c_{1,d}c_{2,d}\mu_d \leqslant c_{1,d}c_{2,d}\mu_{2,d},$$

i.e., $c_{2,s} < c_{2,d}\mu_{2,d}$. Thus F_1 and F_2 have dominated continuous singular parts.

We have shown that F_1 and F_2 satisfy the assumptions of Theorem 3 and condition (iii) from it. Thus, according to that theorem, F_1 and F_2 belong to class $\boldsymbol{Q}$. □

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