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DATA FOR SHIMURA VARIETIES INTERSECTING THE TORELLI LOCUS

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For infinitely many Hurwitz spaces parametrizing cyclic covers of the projective line, we provide a method to determine the integral PEL datum of the Shimura variety that contains the image of the Hurwitz space under the Torelli morphism.

1. Introduction

1.1. ***Overview.*** In [15], Shimura studied unitary groups associated with Hermitian spaces over algebraic number fields and their maximal lattices. In [14], he developed this theory to study isomorphism classes of polarized abelian varieties and Riemann forms. Using this, in [16], Shimura determined the lattice and Hermitian matrix associated with each of six families of cyclic covers of the projective line $\mathbb{P}^1$. The lattice and Hermitian matrix determine *the integral PEL datum* of the family, as defined in Section 2.5.

For m an odd prime such that $\mathbb{Q}(\zeta_m)$ has class number one, we provide in Theorem 5.5 a method to determine the lattice and Hermitian matrix, and thus the integral PEL datum, for all positive-dimensional families of degree m cyclic covers of $\mathbb{P}^1$.

Our results compute the integral PEL datum of certain unitary Shimura varieties that contain Hurwitz spaces of cyclic covers of the projective line.

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1.2. ***Results.*** Consider a family H_γ of cyclic covers of the projective line $\mathbb{P}^1$, indexed by the monodromy datum $\gamma = (m, N, a)$, where m is the degree, N is the number of branch points, and a is the inertia type (Section 2.1). Let $g = g_\gamma$ be the genus of the curves in this family; see (2-1).

For $g \geq 1$, let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g. Let Z_γ be the image of the family H_γ in $\mathcal{A}_g$ under the Torelli morphism. In [12, Section 3], Moonen describes the smallest PEL type Shimura substack S_γ of $\mathcal{A}_g$ containing Z_γ. Our goal is to compute the Shimura datum of S_γ or, more precisely, the *integral PEL datum* of S_γ as defined in Definition 2.2. We use [17, Proposition 4.1], which shows that the integral PEL datum of S_γ can be computed at any point P of Z_γ.

Suppose m is an odd prime such that $\mathbb{Q}(\zeta_m)$ has class number 1. In other words, $m \in \{3, 5, 7, 11, 13, 17, 19\}$. In Theorem 5.5, for every monodromy datum γ for degree m covers, we provide a method that determines the integral PEL datum of S_γ. Our approach is to find a point $P \in Z_\gamma$ for which we can analyze the Jacobian J_P represented by P and the lattice $\Lambda = H^1(J_P, \mathbb{Z})$. From this, we can

(1) express $V = \mathbb{Q}^{2g}$ as a vector space over $F = \mathbb{Q}(\zeta_m)$, find the $\mathcal{O}_F$-lattice $\Lambda \subset V$, and

(2) explicitly find the Hermitian form $\langle \cdot , \cdot \rangle$ on V, which takes integral values on Λ.

Our technique also applies to infinitely many monodromy data γ when m is a composite number such that $\mathbb{Q}(\zeta_m)$ has class number 1; see Section 7.

Specifically, we choose P to be a *distinguished point* of Z_γ, meaning that the Jacobian J_P represented by P is a product of principally polarized abelian varieties each having complex multiplication, see Definition 5.1. For every monodromy datum γ with m an odd prime, we prove in Proposition 5.2 that Z_γ has a distinguished point P such that each of the abelian varieties in the product has complex multiplication by $\mathbb{Z}[\zeta_m]$. In fact, J_P is the Jacobian of a singular curve in the boundary of the Hurwitz family.

1.3. ***Comparison of methods.*** Our approach to this problem is different from Shimura's and may be more accessible to people with background in the areas of algebraic number theory and moduli spaces of curves (Hurwitz spaces).

One advantage of our approach is that it provides a straightforward method to determine the lattice and the Hermitian form explicitly. We provide many examples of this in Section 6 (for m prime) and Section 7 (for m composite). The reason is that the decomposition of J_P provides a basis for the lattice as a free $\mathbb{Z}[\zeta_m]$-module. The Hermitian form is diagonal with respect to that basis and is determined by the CM-types of the factors of J_P, which can be explicitly computed from γ. In Remark 6.6 we give more careful attention to the subtleties caused by the choice of primitive m-th root of unity.

With Shimura's approach, it is necessary to find a Witt decomposition of the signature and lattice; this is straightforward for Shimura's six examples, five of which are of the form $y^m = f(x)$ for m an odd prime and $f(x)$ a separable polynomial, but could cause subtleties for more complicated families.

One drawback to our approach is that it only works when the CM-types produced in the method are simple. The simple property guarantees that the principal polarization on each of the abelian varieties is unique up to isomorphism; this guarantees that the Hermitian form we compute is correct. We use several methods to address this issue.

For composite m, there are several complications with both approaches.

1.4. *Outline.* Section 2 contains information about families of cyclic covers of $\mathbb{P}^1$ and information about Shimura varieties and Shimura data.

Section 3 contains some results in algebraic number theory about the narrow class group and units of independent signs that we need for this paper and later papers.

In Section 4, we prove a result about uniqueness of principal polarizations on abelian varieties with complex multiplication (Propositions 4.4 and 4.5).

Section 5 contains the main result of the paper, Theorem 5.5, which determines the integral PEL data of the unitary Shimura varieties S_γ for all families S_γ when m is an odd prime such that $\mathbb{Q}(\zeta_m)$ has class number 1.

Section 6 contains examples of Theorem 5.5: for all γ when $m = 3$; and for all γ with $N = 4$ branch points when $m = 5$ and $m = 7$. In Section 6.4, we apply the technique of Theorem 5.5 to determine the integral PEL datum using a different kind of distinguished point that represents the Jacobian of a curve with extra automorphisms.

In Section 7, we provide some examples when $m = 4, 6, 10$ to illustrate that the technique of Theorem 5.5 sometimes works when m is composite.

Remark 1.1. In [12, Theorem 3.6], Moonen proved that there are exactly 20 equivalence classes of γ for which Z_γ is open and dense in S_γ; in this situation, the family is called *special*. The six families in Shimura's paper are $M[n]$ when $n = 6, 10, 8, 11, 16, 17$, where $M[n]$ denotes the n-th row in [12, Table 1]. As an application of Theorem 5.5, we determine the integral PEL datum for 12 of these 20 families, including the 6 families from [16]. We emphasize that the special property is not necessary for either approach.

We would like to thank the referee for this observation: when $\dim(Z_\gamma) < \dim(S_\gamma)$, the Torelli locus gives a cycle on S_γ which is not obviously associated to a Shimura subvariety.

2. Hurwitz families and Shimura varieties

2.1. ***Families of cyclic covers of the projective line.*** This section is a shortened version of [8, Section 2.2]. Consider the universal family of μ_m-covers $\psi : \mathcal{C} \to \mathbb{P}^1$, branched at N points, with *inertia type* $a = (a(1), \ldots, a(N))$; the data $\gamma = (m, N, a)$ is called a *monodromy datum*. Over $\mathbb{Q}(\zeta_m)$, such a cover ψ has an equation of the form $y^m = \prod_{i=1}^{N} (x - t(i))^{a(i)}$, and a chosen automorphism h of order m given by $(x, y) \mapsto (x, \zeta_m y)$. By the Riemann–Hurwitz formula, the genus of the fibers of $\mathcal{C}$ is

$$g = g_\gamma = 1 + \frac{1}{2}\Big((N-2)m - \sum_{i=1}^{N} \gcd(a(i), m)\Big). \tag{2-1}$$

For $0 \le n < m$, let $\tau_n : \mathbb{Q}[\mu_m] \to \mathbb{C}$ be given by $\tau_n(\zeta_m) = e^{2\pi I n/m}$. We identify $\mathbb{Z}/m\mathbb{Z} = \mathrm{Hom}(\mathbb{Q}[\mu_m], \mathbb{C})$ by $n \mapsto \tau_n$. The *signature type* of γ is a function $\mathfrak{f} : \mathrm{Hom}(\mathbb{Q}[\mu_m], \mathbb{C}) \to \mathbb{Z}_{\ge 0}$ which we denote by $\mathfrak{f} = (\mathfrak{f}(\tau_1), \ldots, \mathfrak{f}(\tau_{m-1}))$, where $\mathfrak{f}(\tau_n)$ is the dimension of the eigenspace of $H^0(\mathcal{C}(\mathbb{C}), \Omega^1)$ on which h acts by multiplication by ζ_m^n.

For any $x \in \mathbb{Q}$, let $\langle x \rangle$ denote the fractional part of x. By [12, Lemma 2.7, §3.2],

$$\mathfrak{f}(\tau_n) = \begin{cases} -1 + \sum_{i=1}^{N} \Big\langle \dfrac{-na(i)}{m} \Big\rangle & \text{if } n \not\equiv 0 \bmod m, \\ 0 & \text{if } n \equiv 0 \bmod m. \end{cases} \tag{2-2}$$

We describe how the inertia type a and signature $\mathfrak{f}$ change under the action of $\mathrm{Aut}(\mu_m) \simeq (\mathbb{Z}/m\mathbb{Z})^*$. Let $\sigma_i \in \mathrm{Aut}(\mu_m)$ denote the automorphism such that $\sigma_i(\zeta_m) = \zeta_m^i$.

Lemma 2.1. *The automorphism $\sigma_i \in \mathrm{Aut}(\mu_m)$ takes the inertia type*

$$a = (a_1, a_2, \ldots, a_N) \quad \textit{to} \quad a' = i^{-1} \cdot a = (i^{-1} \cdot a_1, i^{-1} \cdot a_2, \ldots, i^{-1} \cdot a_N)$$

and the signature

$$\mathfrak{f} \quad \textit{to} \quad \mathfrak{f}', \qquad \textit{where } \mathfrak{f}'(\tau_n) = \mathfrak{f}(\tau_{ni^{-1}}).$$

Proof. The j-th entry a_j of the inertia type signifies that the canonical generator of inertia above the j-th branch point is the a_j-th power of the generator ζ_m of μ_m. The canonical generator of inertia is the $i^{-1}a_j$-th power of the new generator ζ_m^i of μ_m, so the j-th entry of the inertia type a' is $i^{-1}a_j$.

The automorphism σ_i permutes the eigenspaces for the action of h on $H^0(\mathcal{C}(\mathbb{C}), \Omega^1)$, taking the eigenspace indexed by n to the one indexed by ni^{-1}. This yields the formula $\mathfrak{f}'(\tau_n) = \mathfrak{f}(\tau_{ni^{-1}})$, which can also be deduced from the formula for a' and (2-2). □

Consider the Hurwitz family of μ_m-covers of $\mathbb{P}^1$ with monodromy datum $\gamma = (m, N, a)$. As in [8, Definition 2.1], let $Z^0 = Z^0_\gamma$ denote the image of this family in $\mathcal{A}_g$. Let $Z = Z_\gamma$ denote the closure of Z^0_γ in $\mathcal{A}_g$.

2.2. ***Complex abelian varieties.*** Let $g \geq 1$ and let $V = \mathbb{R}^{2g}$. Suppose V has a complex structure and $\Lambda \subset V$ is a lattice of rank $2g$. A Riemann form on a pair (V, Λ) is an alternating pairing $\Psi : \Lambda \times \Lambda \to \mathbb{Z}$ such that the pairing $\Psi_\mathbb{R}(\,\cdot\,, \sqrt{-1}\,\cdot\,) : V \times V \to \mathbb{R}$ is symmetric and positive definite. Every complex abelian variety of dimension g is isomorphic to V/Λ for some pair (V, Λ) that admits a Riemann form.

By Riemann's theorem, the category of polarized abelian varieties of dimension g over $\mathbb{C}$ is equivalent to the category of pairs (V, Λ), where V is a nontrivial $\mathbb{C}$-vector space of dimension g and Λ is a lattice in V together with a Riemann form integral on Λ.

2.3. ***Siegel modular varieties.*** For $g \geq 1$, let $V = \mathbb{Q}^{2g}$ together with the standard lattice $\Lambda = \mathbb{Z}^{2g}$. Let $\Psi : V \times V \to \mathbb{Q}$ be the standard symplectic form, which is integral on Λ. The points of the Siegel upper half space $\mathcal{H}_g$ are complex structures J on $V_\mathbb{R}$ such that $\Psi_J = \Psi_\mathbb{R}(\,\cdot\,, J\,\cdot\,)$ is a Riemann form. These points parametrize principally polarized complex abelian varieties A of dimension g, equipped with a trivialization $\Lambda \simeq H^1(A, \mathbb{Z})$. The Shimura datum for $\mathcal{A}_g$ arises from the symplectic $\mathbb{Q}$-vector space (V, Ψ); its integral datum given by the self-dual $\mathbb{Z}$-lattice Λ. The associated algebraic group over $\mathbb{Q}$ is $G_g := \mathrm{GSp}(V, \Psi)$, the algebraic group of symplectic similitudes; see [11, Example 11.12].

2.4. ***Shimura data of PEL type.*** A *Shimura datum of PEL type* is a Shimura datum arising from a *PEL datum* $(B, *, V, \langle\,,\rangle, h)$, where (see [5, Section 4])

- B is a semisimple finite dimensional $\mathbb{Q}$-algebra;
- $*$ is a positive involution on B;
- V is a finitely generated B-module;
- $\langle\,\cdot\,,\cdot\,\rangle$ is a nondegenerate skew-symmetric $*$-Hermitian form on V; and
- $h : \mathbb{C} \to \mathrm{End}_{B\otimes\mathbb{R}}(V_\mathbb{R})$ is an $\mathbb{R}$-algebra homomorphism satisfying $\langle h(z)\,\cdot\,,\cdot\,\rangle = \langle\,\cdot\,, h(\bar{z})\,\cdot\,\rangle$, for all $z \in \mathbb{C}$; and such that the symmetric bilinear form $\langle\,\cdot\,, h(i)\cdot\,\rangle$ on V is positive definite.

The associated algebraic group H over $\mathbb{Q}$ is $H = \mathrm{GL}_B(V) \cap \mathrm{GSp}(V, \langle\,,\rangle)$. By definition, $H \subset G_g$ for $g = (\dim_\mathbb{Q} V)/2$.

Given a PEL datum $(B, *, V, \langle\,,\rangle, h)$, there is an associated PEL moduli space $\mathrm{Sh} = \mathrm{Sh}(B, *, V, \langle\,,\rangle, h)$, which is a subspace of $\mathcal{A}_g$. The space Sh is defined over a finite extension of $\mathbb{Q}$, called the *reflex field* [5, Section 5]. The manifold $\mathrm{Sh}(\mathbb{C})^{\mathrm{an}}$ is the disjoint union of finitely many connected Shimura varieties [5, Sections 7–8].

2.5. *Integral PEL data.* Let $(B, *, V, \langle\, ,\rangle, h)$ be a PEL datum.

Definition 2.2. An *integral PEL datum* of the PEL datum $(B, *, V, \langle\, ,\rangle, h)$ consists of a tuple $(\mathcal{O}_B, *, \Lambda, \langle\, ,\rangle, h)$ where

- $\mathcal{O}_B$ is an order of B that is $*$-stable, and
- Λ is a lattice of V that is $\mathcal{O}_B$-stable, such that the $*$-Hermitian form $\langle\cdot,\cdot\rangle$ is integral on Λ.

A rational prime p is *good* for an integral PEL datum if $\mathcal{O}_B$ is maximal at p and Λ is self-dual at p. All but finitely many primes are good, [6, Definition 1.4.1.1].

To an integral PEL datum $(\mathcal{O}_B, *, \Lambda, \langle\, ,\rangle, h)$, there is an associated (canonical) integral model $\mathrm{Sh}(\mathcal{O}_B, *, \Lambda, \langle\, ,\rangle, h)$ of $\mathrm{Sh}(B, *, V, \langle\, ,\rangle, h)$, defined over the ring of integers of the reflex field localized at good primes, [6, Theorem 1.4.1.11]. If p is a good prime for the integral PEL datum, then $\mathrm{Sh}(\mathcal{O}_B, *, \Lambda, \langle\, ,\rangle, h)$ has good reduction at p.

2.6. *Shimura varieties given by the action of the covering group.* Let $\gamma = (m, N, a)$ be a monodromy datum, and let Z_γ be as in Section 2.1. Then Z_γ is contained in a subspace of PEL type S_γ of $\mathcal{A}_g$ [12, §3]. The PEL datum $(B, *, V, \langle\, ,\rangle, h)$ of S_γ (defined as in Section 2.4) satisfies the following conditions:

- $B = \mathbb{Q}[\mu_m]$ is the group algebra of μ_m over $\mathbb{Q}$.
- $*$ is the involution on $\mathbb{Q}[\mu_m]$ induced by the inverse map on μ_m.
- $V = \mathbb{Q}^{2g}$ for $g = g_\gamma$ as given in (2-1).

The structure of V as a B-module, and the complex structure on $V_\mathbb{R}$ (namely, h) are uniquely determined by the signature type $\mathfrak{f} = \mathfrak{f}_\gamma$ given in (2-2).

Our goal is to compute the integral PEL datum $(\mathcal{O}_B, *, \Lambda, \langle\, ,\rangle, h)$ of S_γ.

More precisely, we set $\mathcal{O}_B = \mathbb{Z}[\mu_m]$, which is a $*$-stable order of B, and compute an $\mathcal{O}_B$-stable lattice Λ of V and a skew Hermitian form $\langle\cdot,\cdot\rangle$ on V which is integral on Λ. By construction, the order $\mathcal{O}_B$ is maximal away from m and the lattice Λ is self-dual away from m; hence S_γ has good reduction at every prime p not dividing m.

2.7. *Complex multiplication.* Let L be a CM-field and let L_0 be the maximal totally real subfield of L. Let $n = [L_0 : \mathbb{Q}]$; hence, $[L : \mathbb{Q}] = 2n$. A CM-type of L is an ordered set Φ of distinct embeddings $\phi_i : L \hookrightarrow \mathbb{C}$, for $1 \le i \le n$, no two of which are complex conjugate. A CM-type of L is called *simple* if it is not induced from the CM-type of a proper CM-subfield of L.

Let A be a complex torus such that $L \subset \mathrm{End}(A) \otimes \mathbb{Q}$. We say that A is of CM-type (L, Φ) if $\dim(A) = n$ and the complex representation of L on $\mathrm{Lie}(A)$ is isomorphic to $\sum_{\phi\in\Phi} \phi$. If, in addition, $\mathrm{End}(A) \simeq \mathcal{O}_L$, we say A has type $(\mathcal{O}_L, \Phi)$.

The following lemma is well-known. See [9, Lemma 3.1] for some explanation.

Lemma 2.3. *Suppose $\psi : \mathcal{C} \to \mathbb{P}^1$ is a μ_m-cover, branched at $N = 3$ points, with inertia type $a = (a_1, a_2, a_3)$ and signature type $\mathfrak{f}$. Then the Jacobian of $\mathcal{C}$ has complex multiplication by $\prod_d \mathbb{Q}(\zeta_d)$, where d satisfies $1 < d \mid m$, and $d \nmid a(i)$ for $1 \le i \le 3$. Its CM-type Φ satisfies*

$$\phi \in \Phi \iff \mathfrak{f}(\phi) > 0. \tag{2-3}$$

The maximal order in $\prod_d \mathbb{Q}(\zeta_d)$ is $\prod_d \mathbb{Z}[\zeta_d]$. The image of the group ring $\mathbb{Z}[\mu_m]$ in $\prod_d \mathbb{Q}(\zeta_d)$ has finite index inside the maximal order. This index is 1 if and only if there is a unique d satisfying $1 < d \mid m$, and $d \nmid a(i)$ for $1 \le i \le 3$. This is true if m is prime.

The next result is immediate from Lemma 2.1 and (2-3).

Lemma 2.4. *If $N = 3$, then $\sigma_i \in \mathrm{Aut}(\mu_m)$ takes the CM-type Φ to $\Phi' = i \cdot \Phi = \{\tau_{ni} \mid \tau_n \in \Phi\}$.*

For any CM-type (L, Φ), there exists a unique 0-dimensional PEL type moduli space $\mathrm{Sh} = \mathrm{Sh}(L, \Phi)$ whose points represent abelian varieties with complex multiplication of type $(\mathcal{O}_L, \Phi)$. (When $L = \mathbb{Q}(\zeta_m)$, the points of Sh represent the Jacobians of curves for which there is a μ_m-cover of $\mathbb{P}^1$ branched at $N = 3$ points with CM-type Φ.) The signature type $\mathfrak{f}$ of Sh is given by

$$\text{for each } \phi : L \to \mathbb{C}, \quad \mathfrak{f}(\phi) = \begin{cases} 1 & \text{if } \phi \in \Phi, \\ 0 & \text{if } \phi \notin \Phi. \end{cases} \tag{2-4}$$

For any signature type $\mathfrak{f}$ on L taking values in $\{0, 1\}$, there is a unique CM-type Φ of L compatible with $\mathfrak{f}$.

3. Some background from algebraic number theory

3.1. ***Class group and narrow class group.*** For a number field L: let $\mathcal{O}_L$ denote its ring of integers; let $\mathcal{U}_L$ denote the units in $\mathcal{O}_L$; and let $D_{L/\mathbb{Q}}$ denote the different of L over $\mathbb{Q}$.

Let cl_L be the class group of L. Recall that $\mathrm{cl}_L = I_L/P_L$, where I_L is the group of nonzero fractional ideals and P_L is the subgroup of nonzero principal fractional ideals. Let $h_L = |\mathrm{cl}_L|$ be the class number of L.

An element $\alpha \in L$ is totally positive if it is positive under every real embedding of L. Let $\mathcal{U}_L^+ \subset \mathcal{U}_L$ be the subgroup of totally positive units. Let $P_L^+ \subset P_L$ be the subgroup of principal ideals generated by a totally positive element. The narrow class group of L is $\mathrm{cl}_L^+ = I_L/P_L^+$ and the narrow class number is $h_L^+ = |\mathrm{cl}_L^+|$. There is a surjection

$$\nu_L : \mathrm{cl}_L^+ \to \mathrm{cl}_L.$$

3.2. ***Units of independent signs.*** Let L be a CM-field and L_0 its maximal totally real subfield. Let $n = [L_0 : \mathbb{Q}]$; hence, $[L : \mathbb{Q}] = 2n$. By [19, Theorem 4.10], h_{L_0} divides h_L.

We fix an ordering $\tau_1, \dots, \tau_n$ of the n real embeddings $L_0 \hookrightarrow \mathbb{R}$. If $L_0/\mathbb{Q}$ is Galois, we identify these with the elements of $\mathrm{Gal}(L_0/\mathbb{Q})$. Consider the group homomorphism

$$\rho_{L_0} : \mathcal{U}_{L_0} \to \{\pm 1\}^n, \quad \rho_{L_0}(u) = (\tau_i(u)/|\tau_i(u)|)_{1 \le i \le n} \text{ for } u \in \mathcal{U}_{L_0}. \tag{3-1}$$

We say that L_0 has *units of independent signs* if, for each real embedding τ, there is a unit which is negative under τ but positive under all other real embeddings [2, Definition 12.1]. This is equivalent to saying that ρ_{L_0} is surjective or that $\nu_{L_0} : \mathrm{cl}^+_{L_0} \to \mathrm{cl}_{L_0}$ is an isomorphism [2, Lemma 11.2].

We say that L_0 has *units of almost independent signs* if every unit in $\mathcal{U}_{L_0}$ is negative under an even number of real embeddings and, for every pair of real embeddings, there is a unit which is negative under exactly the two embeddings in that pair [2, Definition 12.13]. This condition is equivalent to $|\ker(\nu_{L_0})| = 2$ or $|\mathrm{coker}(\rho_{L_0})| = 2$, by [2, Lemma 11.2].

3.3. ***Norms and the Hasse unit index.*** Consider the norm map $N : \mathcal{U}_L \to \mathcal{U}_{L_0}$ given by $N(y) = N_{L/L_0}(y)$ for $y \in \mathcal{U}_L$. By Dirichlet's unit theorem, $[\mathcal{U}_{L_0} : \mathcal{U}^2_{L_0}] = 2^n$.

Lemma 3.1. *Suppose L_0 is a totally real field, and L/L_0 is a CM-extension. Then*

$$\mathcal{U}^2_{L_0} \subseteq N(\mathcal{U}_L) \subseteq \mathcal{U}^+_{L_0} \subseteq \mathcal{U}_{L_0}. \tag{3-2}$$

Proof. By definition, $\mathcal{U}^+_{L_0} \subseteq \mathcal{U}_{L_0}$. All elements in $N(\mathcal{U}_L)$ are totally positive units because L is a CM-field, quadratic over its totally real subfield L_0. So $N(\mathcal{U}_L) \subseteq \mathcal{U}^+_{L_0}$. Also, $\mathcal{U}^2_{L_0} = N(\mathcal{U}_{L_0})$, hence $\mathcal{U}^2_{L_0} \subset N(\mathcal{U}_L)$. □

Let μ_L be the group of roots of unity of L.

Definition 3.2. The *Hasse unit index* of a CM-extension L/L_0 is

$$Q(L) = [\mathcal{U}_L : \mu_L \mathcal{U}_{L_0}].$$

By [19, Theorem 4.12], $Q(L) = 1$ or $Q(L) = 2$. Since $\mathrm{Ker}(N) = \mu_L$, it follows that $\mathcal{U}_L = \mu_L \mathcal{U}_{L_0}$ if and only if $N(\mathcal{U}_L) = N(\mathcal{U}_{L_0})$. Also $N(\mathcal{U}_{L_0}) = \mathcal{U}^2_{L_0}$. Thus

$$Q(L) = [N(\mathcal{U}_L) : \mathcal{U}^2_{L_0}]. \tag{3-3}$$

Let t be the number of finite primes ramified in L/L_0.

Lemma 3.3. *Suppose L has odd class number. Let L_0 be a totally real field, and L/L_0 a CM-extension. Then $Q(L) = 1$ if and only if $t = 1$; and $Q(L) = 2$ if and only if $t = 0$.*

Proof. The material in [2, Chapter 13] is stated in terms of the type of the CM-extension L/L_0. By a theorem of Kummer [2, Theorem 13.4] L/L_0 has type I (resp. II) if and only if $Q(L) = 1$ (resp. $Q(L) = 2$). The result is then immediate from [2, page 73]. □

We summarize the result from this section that we need in this and later papers.

Lemma 3.4. *Suppose L is a CM-field with maximal totally real subfield L_0.*

(1) *L_0 has units of independent signs if and only if $Q(L)[\mathcal{U}^+_{L_0} : N(\mathcal{U}_L)] = 1$.*

(2) *L_0 has units of almost independent signs if and only if $Q(L)[\mathcal{U}^+_{L_0} : N(\mathcal{U}_L)] = 2$.*

Proof. (1) By [2, Lemma 12.2], L_0 has units of independent signs if and only if every element of $\mathcal{U}^+_{L_0}$ is a square. The result then follows from Lemma 3.1.

(2) It follows from the definitions that L_0 has units of almost independent signs if and only if $[\mathcal{U}_{L_0} : \mathcal{U}^+_{L_0}] = 2^{n-1}$. Since $[\mathcal{U}_{L_0} : \mathcal{U}^2_{L_0}] = 2^n$, this is equivalent to $[\mathcal{U}^+_{L_0} : \mathcal{U}^2_{L_0}] = 2$. □

3.4. *Cyclotomic fields.* Let m be a positive integer and let $\zeta_m = e^{2\pi I/m} \in \mathbb{C}$. Let $\mu_m \subset \mathbb{C}$ be the group of m-th roots of unity. Let $\mathbb{Q}[\mu_m]$ be the group algebra of μ_m over $\mathbb{Q}$.

The cyclotomic field $F = \mathbb{Q}(\zeta_m)$ is a CM-field over $\mathbb{Q}$ with maximal totally real subfield $F_0 = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$. The degree of F_0 over $\mathbb{Q}$ is $n = \phi(m)/2$.

Notation 3.5. The choice of ζ_m fixes an embedding $\sigma_1 : F \hookrightarrow \mathbb{C}$. For $1 \le i < m$, with $\gcd(i, m) = 1$, let σ_i be the embedding $F \hookrightarrow \mathbb{C}$ (or automorphism in $\mathrm{Gal}(F/\mathbb{Q})$) determined by $\sigma_i(\zeta_m) = \zeta_m^i$. For $x \in F$, let $\bar{x} = \sigma_{m-1}(x)$ denote its complex conjugate.

If $m = p^r$ is a prime power, then F/F_0 is ramified at the unique prime of F_0 above p and at the n infinite primes of F_0 and is unramified at all other primes. See [19, Proposition 2.15].

Lemma 3.6. *Let $F = \mathbb{Q}(\zeta_m)$ and $F_0 = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$. Suppose F has class number* 1.

(1) *If m is a prime power* (*or twice a prime power*), *then F_0 has narrow class number* 1 *and thus has units of independent signs. The complete list of these m is*

$m = 1, 2, 4, 8, 16, 32;$
$m = 3, 5, 7, 9, 11, 13, 17, 19, 25, 27;$ *or twice a value of m on this row.*

(2) *If m is not a prime power* (*or twice a prime power*), *then F_0 has narrow class number* 2 *and thus has units of almost independent signs. The complete list of these m is*

$m = 12, 16, 20, 24, 28, 36, 40, 44, 48, 60, 84;$
$m = 15, 21, 33, 35, 45;$ *or twice a value of m on this row.*

Proof. The complete list of m such that $h_F = 1$ is well-known; it is the union of the lists in parts (1) and (2). For these m, since $h_F = 1$, also $h_{F_0} = 1$.

(1) By a result of Hasse (see [2, Corollary 3.9]), since h_F and h_{F_0} are odd, F_0 has units of independent signs if and only if F/F_0 is ramified at exactly one finite prime. This happens if and only if $m = p^r$ or $m = 2 \cdot p^r$, for some prime p.

(2) Since h_F and h_{F_0} are odd, then F_0 has units of almost independent signs if and only if F/F_0 is not ramified at any finite prime [2, Corollary 13.10]. This happens if m is not a prime power (or twice a prime power). □

3.5. *A generator for the different of cyclotomic fields.*

Lemma 3.7. *Let $F = \mathbb{Q}(\zeta_m)$. The element β_0 below generates $D_{F/\mathbb{Q}}$ and $\beta_0 = -\bar{\beta}_0$.*

(1) *If m is an odd prime, then*

$$\beta_0 = \frac{m}{\zeta_m^{(m+1)/2} - \zeta_m^{(m-1)/2}}. \tag{3-4}$$

(2) *If $m = 2^k$ with $k \geq 2$, then $\beta_0 = -2^{k-1} i$.*

(3) *If $m = 3^k$, then $\beta_0 = -3^{k-1}\sqrt{3}i$.*

(4) *If $m = pq$ with p, q distinct odd primes, then*

$$\beta_0 = m \frac{\zeta_m^{(m+1)/2} - \zeta_m^{(m-1)/2}}{\left(\zeta_m^{q(p+1)/2} - \zeta_m^{q(p-1)/2}\right)\left(\zeta_m^{p(q+1)/2} - \zeta_m^{p(q-1)/2}\right)}. \tag{3-5}$$

Proof. In each case, β_0 is on the imaginary axis, so $\beta_0 = -\bar{\beta}_0$.

Let $c_m(x)$ denote the m-th cyclotomic polynomial. The different $D_{F/\mathbb{Q}}$ is generated by $\langle c'_m(\zeta_m)\rangle$ by [13, III, Proposition 2.4]. To show that β_0 generates $D_{F/\mathbb{Q}}$, it suffices to show that it is an associate of $\langle c'_m(\zeta_m)\rangle$ in $\mathcal{O}_F$.

(1) When m is an odd prime, then $c_m(x) = (x^m - 1)/(x - 1)$. Then

$$c'_m(x) = \frac{(x-1)mx^{m-1} - (x^m - 1)}{(x-1)^2}.$$

Hence $c'_m(\zeta_m) = m\zeta_m^{-1}/(\zeta_m - 1)$, and β_0 from (3-4) is an associate of $c'_m(\zeta_m)$ in $\mathcal{O}_F$.

(2) If $m = 2^k$, then $c_m(x) = x^{2^{k-1}} + 1$. Thus $c'_m(\zeta_m) = 2^{k-1}\zeta_m^{2^{k-1}-1} = -2^{k-1}/\zeta_m$, and the corresponding β_0 is an associate of $c'_m(\zeta_m)$ in $\mathcal{O}_F$.

(3) If $m = 3^k$, then $c_m(x) = x^{2\cdot 3^{k-1}} + x^{3^{k-1}} + 1$. Thus $c'_m(\zeta_m) = 3^{k-1}\zeta_m^{3^{k-1}-1}\sqrt{-3}$. and this β_0 is an associate of $c'_m(\zeta_m)$ in $\mathcal{O}_F$.

(4) In this case, $c_m(x) = \dfrac{(x^m-1)(x-1)}{(x^p-1)(x^q-1)}$, and β_0 in (3-5) is an associate of $c'_m(\zeta_m) = \dfrac{m(\zeta_m - 1)}{\zeta_m(\zeta_m^p - 1)(\zeta_m^q - 1)}$. □

3.6. ***Simple types.*** Let $F = \mathbb{Q}(\zeta_m)$ and recall that $\mathrm{Gal}(F/\mathbb{Q}) \simeq (\mathbb{Z}/m\mathbb{Z})^*$. A CM-type Φ of F is simple, if it is not induced from any CM-field $F' \subset F$.

Let $H \subset \mathrm{Gal}(F/\mathbb{Q})$ and let F^H be the fixed field of F under H. If F^H is a CM-field, then Φ is induced from F^H if and only if it is a union of cosets of H.

Lemma 3.8. *Suppose $m = 4$ or m is a Fermat prime or twice a Fermat prime. Then Φ is simple.*

We will use Lemma 3.8 for $m = 4$ and $m = 3, 5, 17$ and $m = 6, 10$.

Proof. For these values of m, the field F contains no proper nontrivial CM-fields. □

Lemma 3.9. *Let $m > 3$ be prime. Let $\gamma = (m, 3, a)$ be a monodromy datum with inertia type $a = (1, a_2, a_3)$. Let Φ be the CM-type of F corresponding to a. Then Φ is simple if and only if $a \neq (1, x, x^2)$ for some $x \in (\mathbb{Z}/m\mathbb{Z})^*$ with order* 3.

Proof. This follows from [4, Theorem 2]. See also [7, Theorem 6.2]. □

For example, when $m = 7$ and $a = (1, a_2, a_3)$, Φ is simple unless $a = (1, 2, 4)$.

By a direct computation, one can check that Φ is always simple when $m = 25$ or $m = 27$.

4. Principal polarizations of CM abelian varieties

Let L be a CM-field and L_0 its maximal totally real subfield. We will now study principal polarizations on abelian varieties having complex multiplication by L.

Section 4.1 contains background about complex tori with complex multiplication. In Section 4.2, we study principal polarizations on CM-abelian varieties, following work of van Wamelen. In Section 4.3, we prove a result about existence and uniqueness of principal polarizations for abelian varieties with simple CM-type when L has class number 1. In Section 4.4, we specialize to the case that L is a cyclotomic field with class number 1.

4.1. ***Complex tori.*** We review some complex multiplication theory, following Lang in [7, Chapter 1]. Let A be a complex torus such that $L \subset \mathrm{End}(A) \otimes \mathbb{Q}$. We say that A is of type (L, Φ) if the complex representation of $\mathrm{End}(A) \otimes \mathbb{Q}$ is isomorphic to $\sum_{\phi\in\Phi} \phi$. If, in addition, $\mathrm{End}(A) \simeq \mathcal{O}_L$, we say A has type $(\mathcal{O}_L, \Phi)$.

Theorem 4.1 [7, Chapter 1: Theorems 3.3, 3.5, 4.1, 4.2].

(1) *If $\mathfrak{a}$ is a lattice in L and Φ is a CM-type of L, then $\mathbb{C}^n/\Phi(\mathfrak{a})$ is a complex torus of type (L, Φ).*

(2) *If A is a complex torus of type (L, Φ), then there exists a lattice a of L such that $A \simeq \mathbb{C}^n/\Phi(\mathfrak{a})$.*

(3) *If Φ is a simple type and $\mathfrak{a}$ is a fractional ideal of L, then* $\mathrm{End}(\mathbb{C}^n/\Phi(\mathfrak{a})) \simeq \mathcal{O}_L$.

(4) *If* Φ *is a simple type and* $\mathfrak{a}, \mathfrak{b}$ *are fractional ideals of* L, *then* $\mathbb{C}^n/\Phi(\mathfrak{a}) \simeq \mathbb{C}^n/\Phi(\mathfrak{b})$ *if and only if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are in the same ideal class.*

In particular, if (L, Φ) is a simple CM-type, then the set of isomorphism classes of complex tori of type $(\mathcal{O}_L, \Phi)$ is in bijection with the class group of L.

Furthermore, by [7, Chapter 1, Theorem 4.5], every (admissible, nondegenerate) Riemann form on $\mathbb{C}^n/\Phi(\mathfrak{a})$ is given by

$$\mathbb{E}(\Phi(x), \Phi(y)) = \mathrm{tr}_{L/\mathbb{Q}}(\xi x \bar{y}) \quad \text{for } x, y \in L, \tag{4-1}$$

for some ξ such that $L = L_0(\xi)$, $\xi^2 \in L_0$ is totally negative, and $\mathrm{Im}(\phi(\xi)) > 0$ for $\phi \in \Phi$.

4.2. *Principal polarizations on CM abelian varieties.* In [18, p. 310], van Wamelen developed an algorithm to produce isomorphism classes of principally polarized abelian varieties of type $(\mathcal{O}_L, \Phi)$ based on the following result.

Theorem 4.2 (van Wamelen [18]). *Let* (L, Φ) *be a CM-type.*

(1) (Theorem 4)[1] *Writing* $L = L_0(\sqrt{-c})$ *for some* $c \in \mathcal{O}_{L_0}$, *there exist a fractional ideal* $\mathfrak{a} \subset L$ *and an element* $b \in \mathcal{O}_{L_0}$ *such that* $D_{L/\mathbb{Q}} \cdot \mathfrak{a}\bar{\mathfrak{a}} = \langle b\sqrt{-c}\rangle$.

(2) (Theorem 3) *Let* $\xi \in L$ *be such that* $L = L_0(\xi)$, $\xi^2 \in L_0$, *and* $D_{L/\mathbb{Q}} \cdot \mathfrak{a}\bar{\mathfrak{a}} = \langle \xi^{-1}\rangle$, *for some fractional ideal* $\mathfrak{a}$ *of* L (*for example*, *with the notation of part* (1), $\xi = (b\sqrt{-c})^{-1}$). *Define a Riemann form* $\mathbb{E} : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ *by*

$$\mathbb{E}(z, w) = \sum_{i=1}^{n} \phi_i(\xi)(\bar{z}_i w_i - z_i \bar{w}_i),$$

for $z, w \in \mathbb{C}^n$. *If* $\mathrm{Im}(\phi(\xi)) > 0$ *for* $\phi \in \Phi$, *then* $\mathbb{E}$ *defines a principal polarization on* $\mathbb{C}^n/\Phi(\mathfrak{a})$ *of type* $(\mathcal{O}_L, \Phi)$. *Furthermore*, *if* (L, Φ) *is a simple CM-type*, *then all principal polarizations on* $\mathbb{C}^n/\Phi(\mathfrak{a})$ *of type* $(\mathcal{O}_L, \Phi)$ *are given by such a* ξ.[2]

(3) (Corollary 1) *Two principal polarizations of the same CM-type on* $\mathbb{C}^n/\Phi(\mathfrak{a})$ *arising from* ξ_1 *and* ξ_2 *give isomorphic principally polarized abelian varieties if and only if there exists a unit* $u \in \mathcal{O}_L^*$ *such that* $\xi_1 = u\bar{u}\xi_2$.

Corollary 4.3. *Let* Φ *be a CM-type of* L. *An element* $\xi = \beta^{-1}$ *for some* $\beta \in \mathcal{O}_L$ *defines a principal polarization on* $A_\Phi = \mathbb{C}^n/\Phi(\mathcal{O}_L)$ *of CM-type* $(\mathcal{O}_L, \Phi)$ *if and only if*

(1) β *generates the different* $D_{L/\mathbb{Q}}$,

(2) $\beta = -\overline{\beta}$, *and*

(3) $\mathrm{Im}(\phi(\beta)) < 0$, *for each* $\phi \in \Phi$.

[1] We have made a small adjustment to the notation to be consistent with other parts of this theorem.

[2] Note that type (L, Φ) and type $(\mathcal{O}_L, \Phi)$ are equivalent in the last two statements.

Two elements β, β' satisfying the above conditions yield isomorphic principally polarized abelian varieties if and only if there exists a unit $u \in \mathcal{U}_L$ such that $\beta = u\bar{u}\beta'$. If the CM-type Φ is simple, then all principal polarizations of A_Φ of CM-type $(\mathcal{O}_L, \Phi)$ arise this way.

Proof. This follows from Theorem 4.2, replacing ξ, ξ_1, ξ_2 with β^{-1}, β^{-1}, β'^{-1}. □

In Lemma 3.7, we determined an element $\beta_0 \in \mathcal{O}_F$ satisfying conditions (1) and (2) in Corollary 4.3 when $F = \mathbb{Q}(\zeta_m)$ for many values of m.

4.3. *Existence and uniqueness of principal polarizations.* In this section, we study principal polarizations on CM-abelian varieties. Under conditions on the class group and unit group of the field, we show such principal polarizations exist and are uniquely determined. Recall that L is a CM-field with maximal totally real subfield L_0 and $n = [L_0 : \mathbb{Q}]$.

Proposition 4.4. *Suppose L_0 has units of independent signs and (L, Φ) is a CM-type.*

(1) *There exists a principally polarized CM-abelian variety of type $(\mathcal{O}_L, \Phi)$.*

(2) *If (L, Φ) is simple, then any CM-abelian variety of type $(\mathcal{O}_L, \Phi)$ has at most one principal polarization, up to isomorphism.*

(3) *If (L, Φ) is simple and L has class number* 1, *then there exists a unique principally polarized CM-abelian variety of type $(\mathcal{O}_L, \Phi)$, up to isomorphism.*

Proof. By Theorem 4.2(1), there exists a fractional ideal $\mathfrak{a}$ in L and an element $\beta_0 \in \mathcal{O}_L$ such that (1) β_0 generates $D_{L/\mathbb{Q}}\mathfrak{a}\bar{\mathfrak{a}}$ and (2) $\beta_0 = -\bar{\beta}_0$. An element $\beta' \in \mathcal{O}_L$ satisfies conditions (1) and (2) if and only if $\beta' = u_0\beta_0$ for a totally real unit $u_0 \in \mathcal{U}_{L_0}$.

By Theorem 4.2(2), $\beta \in \mathcal{O}_L$ defines a principal polarization of type $(\mathcal{O}_L, \Phi)$ on $\mathbb{C}^n/\Phi(\mathfrak{a})$ if and only if it satisfies conditions (1) and (2), and also (3) $\mathrm{Im}(\phi(\beta)) < 0$, for each $\phi \in \Phi$.

Hence, to finish the first claim, it suffices to check that there exists $u_0 \in \mathcal{U}_{L_0}$ such that $\beta = u_0\beta_0$ satisfies condition (3), for any CM-type Φ of L. If L_0 has units of independent signs, then ρ_{L_0} is surjective; this shows that the unit u_0 described above exists.

By Theorem 4.2(2), if (L, Φ) is simple, all principal polarizations of type Φ on $\mathbb{C}^n/\Phi(\mathfrak{a})$ arise from some $\beta \in \mathcal{O}_L$ satisfying conditions (1)–(3). By Theorem 4.2(3), $\beta, \beta' \in \mathcal{O}_L$ satisfying conditions (1)–(3) define isomorphic principally polarized abelian varieties if and only if $\beta' = N(u)\beta$ for some unit $u \in \mathcal{U}_L$.

By definition, given a fractional ideal $\mathfrak{a}$ of L, if $\beta \in \mathcal{O}_L$ satisfying conditions (1)–(3) exists, then $\beta' \in \mathcal{O}_L$ also satisfies conditions (1)–(3) if and only if $\beta' = u^+\beta$ for some totally positive unit $u^+ \in \mathcal{U}_{L_0}^+$.

Hence, to finish the second claim, it suffices to check that for any $u^+ \in \mathcal{U}_{L_0}^+$ there is $u \in \mathcal{U}_L$ such that $u^+ = N(u)$. Since L_0 has units of independent signs, this follows from Lemma 3.4. Finally, if L has class number 1, then by Theorem 4.2 and [18, Theorem 5], any CM-abelian variety of type $(\mathcal{O}_L, \Phi)$ is isomorphic to $A_\Phi = \mathbb{C}^n/\Phi(\mathcal{O}_L)$. □

Proposition 4.5. *Suppose L_0 has units of almost independent signs and $Q(L) = 2$, where $Q(L)$ is defined in Definition 3.2. Suppose (L, Φ) is a simple CM-type.*

(1) *The number of isomorphism classes of principal polarizations on a CM-abelian variety of type $(\mathcal{O}_L, \Phi)$ is at most one.*

(2) *Suppose in addition that L has class number* 1. *If there is a principally polarized CM-abelian variety of type $(\mathcal{O}_L, \Phi)$, then it is unique up to isomorphism.*

Proof. The arguments in the proof of Proposition 4.4 still apply, after observing that $U_{L_0}^+ = N(\mathcal{U}_L)$ when L_0 has units of almost independent signs and $Q(L) = 2$ by Lemma 3.4. □

4.4. ***CM-types for cyclotomic fields.*** We consider the case when L is the cyclotomic field $F = \mathbb{Q}(\zeta_m)$. By Section 3.2, the next result is a special case of Propositions 4.4 and 4.5.

Corollary 4.6. *Let m such that $F = \mathbb{Q}(\zeta_m)$ has class number* 1. *Let Φ be a CM-type of F.*

(1) *If F_0 has narrow class number* 1, *then there exists a principally polarized CM-abelian variety of type $(\mathcal{O}_F, \Phi)$. Furthermore, if (F, Φ) is simple, then it is unique up to isomorphism.*

(2) *Suppose F_0 has narrow class number* 2 *and Φ is simple. If there exists a principally polarized CM-abelian variety of type $(\mathcal{O}_F, \Phi)$, then it is unique up to isomorphism.*

We will now explicitly describe the data of the principal polarization.

Corollary 4.7. *Let m, $F = \mathbb{Q}(\zeta_m)$, and β_0 be as in Lemma 3.7. Let Φ be a CM-type of F. Consider conditions* (1)–(3) *in Corollary 4.3.*

(1) *If $\Phi' = i\Phi$ and β satisfies conditions* (1)–(3) *for Φ, then $\beta' = \sigma_{i^{-1}}(\beta)$ satisfies conditions* (1)–(3) *for Φ'.*

(2) *Suppose F_0 has units of independent signs. Then, for any CM-type Φ of F, there exists a unit $u_0 \in \mathcal{U}_{F_0}$ such that $\beta = u_0\beta_0$ satisfies conditions* (1)–(3) *for Φ. Furthermore, β is the unique element satisfying conditions* (1)–(3) *for Φ, up to multiplication by an element of $N_{F/F_0}(\mathcal{U}_F)$.*

The condition that F_0 has units of independent signs is satisfied if the narrow class number of F_0 is odd [2, bottom of page 58].

Proof. (1) This follows from Lemma 2.1 and Lemma 2.4.

(2) By Lemma 3.7, β_0 satisfies conditions (1) and (2) of Corollary 4.3. Since F_0 has units of independent signs, there exists $u_0 \in \mathcal{U}_{F_0}$ such that $\beta = u_0\beta_0$ satisfies condition (3). Since $u_0 \in \mathcal{U}_{F_0}$ is a real unit, β also satisfies conditions (1) and (2) of Corollary 4.3. Note that β is unique up to multiplication by a totally positive unit of F_0. By Lemma 3.4, $\mathcal{U}_{F_0}^+ = N(\mathcal{U}_F)$, proving the uniqueness statement. □

See Examples 6.3 and 6.7 for illustrations of Corollary 4.7 when $m = 5$ and $m = 7$, respectively. As another example, if $m = 8$, then $\beta_0 = -4i$ from Lemma 3.7: if $\mathfrak{f} = (0, 1, 0, 1)$, set $u_0 = -1$ and $\beta = 4i$; if $\mathfrak{f} = (1, 1, 0, 0)$, set $u_0 = \sqrt{2} - 1$ and $\beta = 4(1 - \sqrt{2})i$.

5. Shimura data

Consider a monodromy datum $\gamma = (m, N, a)$ as in Section 2.1. Recall that H_γ is the Hurwitz space of μ_m-cyclic covers of $\mathbb{P}^1$ with monodromy datum γ. Recall also that Z_γ is the closure in $\mathcal{A}_g$ of the image of H_γ under the Torelli morphism and S_γ is the smallest PEL type Shimura substack of $\mathcal{A}_g$ containing Z_γ. In this section, under certain assumptions on m, we provide a method to determine the integral PEL datum of S_γ.

A point P in Z_γ represents the Jacobian of a stable curve C_P of compact type, where compact type means that the dual graph of C_P is a tree. Furthermore, the curve C_P is a μ_m-cover of a tree of projective lines. The main idea of the proof is to find a point P in Z_γ such that the Jacobian of every irreducible component of C_P has complex multiplication by a CM-field.

5.1. ***Admissible covers.*** We briefly review some information about the boundary of H_γ, see [20, Chapter 4] or [3, Section 1]. The generic point of Z_γ represents the Jacobian of a smooth curve. More generally, a point P of Z_γ represents the Jacobian J_P of a stable curve C_P of compact type. There is a μ_m-cover $\psi : C_P \to T$ where T is a tree of projective lines. The nodes of C_P and T are ordinary double points.

Since P is in Z_γ, the closure of Z_γ^0, the cover ψ is *admissible* as defined below. If η is a node of C_P, consider the restrictions ϕ_1 and ϕ_2 of ϕ to the two irreducible components of C_P that intersect at η. When m is prime, the compact type condition implies that ϕ_1 and ϕ_2 are each totally ramified at η. For $i = 1, 2$, let α_i be the canonical generator of inertia of ϕ_i at η. The admissible condition is that α_1 and α_2 are inverses in μ_m.

5.2. ***Distinguished points.*** We prove that there exists a distinguished point P in Z_γ such that we can compute the lattice and Hermitian form for J_P.

Definition 5.1. A point P in Z_γ is a *distinguished point* if the Jacobian J_P is a principally polarized abelian variety with complex multiplication by a maximal order in a CM-field, or the direct sum of such together with the product polarization.

Proposition 5.2. *Let m be an odd prime. Let $\gamma = (m, N, a)$ be a monodromy datum. Then Z_γ has a distinguished point P. More specifically, for $r = N - 2$:*

(1) *In the family H_γ of μ_m-covers of a genus 0 curve with monodromy datum γ, there is a point which represents an admissible μ_m-cover $\psi : C_P \to T$, where T is a tree of r projective lines and C_P is a curve of compact type, with r irreducible components $\mathcal{C}_1, \dots, \mathcal{C}_r$, each of which is a curve of genus $(m-1)/2$ admitting a μ_m-cover of $\mathbb{P}^1$ branched at 3 points.*

(2) *The Jacobian J_P of C_P is of the form $A \simeq \bigoplus_{j=1}^r A_j$, where each A_j is a principally polarized abelian variety of dimension $(m-1)/2$ having complex multiplication by $\mathcal{O}_F = \mathbb{Z}[\zeta_m]$, together with the product polarization.*

Proof. The fact that Z_γ has a distinguished point P is immediate from part (2), which we will show follows from part (1).

(1) Let $a = (a(1), \dots, a(N))$ be the inertia type. If $N \geq 4$ and m is an odd prime, then a has the following property: there is a pair (i, j) with $1 \leq i < j \leq N$, such that $a(i) + a(j) \not\equiv 0 \bmod m$. Without loss of generality, we can suppose $i = 1$ and $j = 2$. Let $\alpha_2 = a(1) + a(2)$ and $\alpha_1 = -\alpha_2$.

When $N \geq 4$, then Z_γ^0 is affine. Consider a family of μ_m-covers with monodromy datum γ, where the first branch point b_1 approaches the second branch point b_2. When $b_1 = b_2$, the curve is singular and its normalization is a μ_m-cover ϕ of a tree of two projective lines. Let ϕ_1 (resp. ϕ_2) be the restriction of ϕ over the first (resp. second) of these.

The cover ϕ_1 (resp. ϕ_2) is ramified at the specializations of 2 (resp. $N-2$) ramification points. For each of these, the canonical generator of inertia remains the same at the specialization. The values in the inertia type of ϕ_1 (or ϕ_2) sum to 0 modulo m. Also, ϕ is admissible and ramified at the ordinary double point. Thus ϕ_1 is a μ_m-cover of $\mathbb{P}^1$ branched at 3 points with inertia type $(a(1), a(2), \alpha_1)$, and ϕ_2 is a μ_m-cover of $\mathbb{P}^1$ branched at $N-1$ points with inertia type $(\alpha_2, a(3), \dots, a(N))$. The cover ϕ is called *a degeneration of compact type* in [8, Remark 5.2].

By induction on N, we see that H_γ contains a family of μ_m-covers which degenerates completely to an admissible μ_m-cover $\psi : C_P \to T$, where T is a tree of r projective lines and the restriction of ψ above each component of T is branched at 3 points. By (2-1), each irreducible component of C_P has genus $(m-1)/2$.

(2) Choose a labeling $C_1, \dots, C_r$ of the irreducible components of C_P. For $1 \leq j \leq r$, let $A_j = \mathrm{Jac}(C_j)$. Note that A_j is a principally polarized abelian variety of

dimension $(m-1)/2$. By [1, Section 9.2, Example 8, page 246], $J_P \simeq \bigoplus_{j=1}^{r} A_j$, and the principal polarization on J_P decomposes as the product polarization.

Consider the CM-field $F = \mathbb{Q}(\zeta_m)$. Then $\mathcal{O}_F = \mathbb{Z}[\zeta_m] \subset \mathrm{End}(A_j)$ since μ_m acts on C_j for $1 \le j \le r$. Since $\deg(F/\mathbb{Q}) = 2 \cdot \dim(A_j)$, the abelian variety A_j has complex multiplication by $\mathcal{O}_F$ for $1 \le j \le r$. □

5.3. *Shimura datum for Hurwitz spaces.* We turn to the main result of the paper. Recall that $\gamma = (m, N, a)$ and $r = N - 2$.

Definition 5.3. Let $P \in Z_\gamma$ be a distinguished point as described in Proposition 5.2. Suppose $\mathbb{Q}(\zeta_m + \zeta_m^{-1})$ has units of independent signs. (This is guaranteed by Lemma 3.6 when m is an odd prime such that $F = \mathbb{Q}(\zeta_m)$ has class number 1.)

For $1 \le j \le r$, define β_j and $\xi_j = \beta_j^{-1}$ as follows. Recall that C_j is an irreducible component of C_P. The inertia type of the μ_m-cover $C_j \to \mathbb{P}^1$ determines its signature $\mathfrak{f}_j$ by (2-2), which determines its CM-type Φ_j as in (2-4). Let β_0 be as in Lemma 3.7. By Corollary 4.7, there exists $u_j \in \mathcal{U}_{F_0}$ such that $\beta_j := u_j \beta_0$ satisfies conditions (1)–(3) of Corollary 4.3 for Φ_j. Furthermore, β_j is the unique element satisfying conditions (1)–(3) for Φ_j, up to multiplication by an element of $N_{F/F_0}(\mathcal{U}_F)$.

Remark 5.4. The CM-type (F, Φ_j) of A_j may not be simple. For example, for the special family $M[17]$ with $m = 7$, we show in Section 6.3 that (F, Φ_2) is not simple.

Let $F = \mathbb{Q}(\zeta_m)$ and denote by $\mathcal{O}_F = \mathbb{Z}[\zeta_m]$ the ring of integers of F.

Theorem 5.5. *Let m be an odd prime. Let $\gamma = (m, N, a)$ be a monodromy datum with $N \ge 4$. Suppose $F = \mathbb{Q}(\zeta_m)$ has class number 1. Let $r = N - 2$.*

Let $P \in Z_\gamma$ be a distinguished point as described in Proposition 5.2. For $1 \le j \le r$, this determines an abelian variety A_j with CM-type $(\mathcal{O}_F, \Phi_j)$; let $\xi_j = \beta_j^{-1}$ be as in Definition 5.3.

Then the integral PEL datum for S_γ is given by

- *the F-vector space $V = F^r$, together with the standard $\mathcal{O}_F$-lattice $\Lambda = (\mathcal{O}_F)^r \subseteq V$;*
- *the Hermitian form $H_B = \langle \cdot , \cdot \rangle$ on V taking integral values on Λ and defined by*

$$\langle x, y \rangle = \mathrm{tr}_{F/\mathbb{Q}}(x B \bar{y}^T) \quad \text{for } B = \mathrm{diag}[\xi_1, \dots \xi_r] \in \mathrm{GL}_r(F) = \mathrm{GL}(V). \tag{5-1}$$

It is straightforward to compute B from γ; we give many examples in Section 6.

Proof. The variety Z_γ is an irreducible algebraic subvariety of $\mathcal{A}_g$. By [17, Proposition 4.1], the lattice and Hermitian form for the integral PEL datum of S_γ can be computed at any point of Z_γ. Since m is an odd prime, there is a distinguished point $P \in Z_\gamma$ as described in Proposition 5.2. We compute the lattice and Hermitian form at P.

Let $V = H^1(C_P(\mathbb{C}), \mathbb{Q})$. Note that $r = N - 2 = 2g/(m-1)$. From (2-2), we deduce that there is an isomorphism of $\mathbb{Q}[\mu_m]$-modules $V \simeq F^r$, where multiplication of $\mathbb{Q}[\mu_m]$ on F^r is given by the natural homomorphism $\mathbb{Q}[\mu_m] \to F$. The complex structure on $V_{\mathbb{R}}$ is given as $V_{\mathbb{R}} \simeq (F \otimes_{\mathbb{Q}} \mathbb{R})^r \simeq \oplus_{n=1}^{m} \mathbb{C}^{\mathfrak{f}(\tau_n)}$. By (2-2), $\mathfrak{f}(\tau_n) + \mathfrak{f}(\tau_{-n}) = r$ for all $n \not\equiv 0 \bmod m$.

Let $\Lambda = H^1(C_P(\mathbb{C}), \mathbb{Z})$ be the first Betti cohomology of the curve C_P. When F has class number 1, we prove that there is an isomorphism of $\mathbb{Q}[\mu_m]$-modules $V \simeq F^r$ such that the $\mathbb{Z}[\mu_m]$-lattice $\Lambda \subset V$ maps isomorphically onto $(\mathcal{O}_F)^r \subset F^r$ and the Hermitian form ψ on V maps to the diagonal Hermitian form H_B on F^r in (5-1). Note $(\mathcal{O}_F)^r \subset F^r$ is a $\mathbb{Z}[\mu_m]$-lattice, and H_B is integral on $(\mathcal{O}_F)^r$.

By Lemma 3.6, if m is prime and F has class number 1, then $F_0 = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$ has narrow class number 1 and has units of independent signs. For $1 \le j \le r$, by hypothesis, A_j has dimension $(m-1)/2$; it has complex multiplication by $\mathcal{O}_F$ and has CM-type $(\mathcal{O}_F, \Phi_j)$. Furthermore, β_j defines a principal polarization on A_j; the corresponding Hermitian form is given by $\mathbb{E}(\Phi(x), \Phi(y)) = \mathrm{tr}_{F/\mathbb{Q}}(\xi_j x \bar{y})$ for $x, y \in L$ by (4-1).

Since F has class number 1, there is a unique complex torus of CM-type $(\mathcal{O}_F, \Phi_j)$ up to isomorphism by Theorem 4.1; namely, $A_j \simeq \mathbb{C}^g/\Phi_j(\mathcal{O}_F)$. Since $J_P \simeq \oplus_{j=1}^{r} A_j$, we deduce that $\Lambda \simeq (\mathcal{O}_F)^r$. The product polarization on J_P defines an integral Hermitian form of the prescribed signature on Λ. By [16, Appendix, Proposition 8], there is a unique Hermitian form of given signature on Λ. Thus the Hermitian form on Λ is the one in (5-1). □

Remark 5.6. The values of m for which Theorem 5.5 applies are $3, 5, 7, 11, 13, 17, 19$.

For $m = 3$, 5, 11, 17, it is not necessary to refer to [16, Appendix, Proposition 8]; the reason is that (F, Φ_j) is automatically simple for $1 \le j \le r$ by Lemmas 3.8 and 3.9. Thus by Corollary 4.6(1), there is a unique principally polarized CM-abelian variety of type $(\mathcal{O}_F, \Phi_j)$ up to isomorphism. In Section 6.4, we give an alternative approach for $m = 7$ that relies on Corollary 4.6(2) and does not refer to [16, Appendix, Proposition 8].

The strategy of Theorem 5.5 may apply even when m is composite; see Section 7.

6. Applications for m prime

Theorem 5.5 gives a method to determine the integral PEL datum for S_γ for all monodromy data $\gamma = (m, N, a)$ when $m = 3$, 5, 7, 11, 13, 17, 19, so that $F = \mathbb{Q}(\zeta_m)$ has class number 1. In this section, we give examples of this. Specifically:

- In Section 6.1, when $m = 3$, we explicitly determine the Hermitian form for all γ.
- In Sections 6.2 and 6.3, when $m = 5, 7$, we give explicit examples for all equivalence classes of γ when $N = 4$.

- In Section 6.4, we follow an alternative approach for $m = 7$ which avoids the possible occurrence of nonsimple CM-types at the distinguished point.

In particular, we determine the integral PEL datum for the 6 special families for which the degree m is an odd prime; these are denoted by $M[n]$ as in [12, Table 1].

Given a signature $\mathfrak{f}$, let Φ be the CM-type of F determined by $\mathfrak{f}$, as in (2-4). Given the F-vector space $V = F^r$, together with the standard $\mathcal{O}_F$-lattice $\Lambda = (\mathcal{O}_F)^r \subseteq V$, let $H_B = \langle \cdot, \cdot \rangle$ denote the Hermitian form on V defined in (5-1).

6.1. *Integral PEL datum for all families when m = 3.*

Example 6.1. When $m = 3$, then $\beta_0 = 3/(\zeta_3^2 - \zeta_3) = \sqrt{-3}$ by Lemma 3.7.

For $\mathfrak{f}_1 = (1, 0)$, set $u_1 = -1$ and $b_1 = u_1\beta_0$ so that $b_1 = -\sqrt{-3}$. Then $\mathrm{Im}(\sigma_1(b_1)) < 0$.

For $\mathfrak{f}_2 = (0, 1)$, set $u_2 = 1$ and $b_2 = u_2\beta_0$ so that $b_2 = \sqrt{-3}$. Then $\mathrm{Im}(\sigma_2(b_2)) < 0$.

The CM-type Φ_i is simple by Lemma 3.8 and b_i satisfies Corollary 4.7 with respect to Φ_i.

For $m = 3$, there are simple formulas relating the signature $\mathfrak{f}$ and the inertia type a. If a has d_1 entries of 1 and d_2 entries of 2, then $\mathfrak{f} = (f_1, f_2)$ with $f_1 = (2d_1 + d_2 - 3)/3$ and $f_2 = (d_1 + 2d_2 - 3)/3$. Then $d_1 = 2f_1 - f_2 + 1$ and $d_2 = 2f_2 - f_1 + 1$. One can check that $0 \le \max(f_1, f_2) \le 2\min(f_1, f_2) + 1$.

Corollary 6.2. *Let $m = 3$ and $N \ge 4$. Let $\gamma = (3, N, a)$ be a monodromy datum with signature (f_1, f_2). Let $g = f_1 + f_2$. Let S_γ be the component of the Shimura variety containing Z_γ.*

Let $F = \mathbb{Q}(\zeta_3)$ and $\xi = -1/\sqrt{-3}$. Then the integral PEL datum of S_γ has lattice $(\mathcal{O}_F)^g$ with Hermitian form H_B where $B \in \mathrm{GL}_g(\mathcal{O}_F)$ is diagonal with f_1 entries of ξ and f_2 entries of $-\xi$.

Proof. Consider the admissible μ_3-cover $\varphi : \mathcal{C}_P \to T$ represented by the distinguished point from Proposition 5.2. Here T is a tree of $r = f_1 + f_2$ projective lines. Above f_1 (resp. f_2) components of T, the restriction of φ is a μ_3-cover branched at 3 points with signature $\mathfrak{f}_1 = (1, 0)$ (resp. $\mathfrak{f}_2 = (0, 1)$). The result then follows from Theorem 5.5. □

In particular, Corollary 6.2 includes the three special families $M[3]$, $M[6]$, $M[10]$:

M	(m, N, a)	B
$M[3]$	(3, 4, (1, 1, 2, 2))	$\mathrm{diag}[\xi, -\xi]$
$M[6]$	(3, 5, (1, 1, 1, 1, 2))	$\mathrm{diag}[\xi, \xi, -\xi]$
$M[10]$	(3, 6, (1, 1, 1, 1, 1, 1))	$\mathrm{diag}[\xi, \xi, \xi, -\xi]$

The cases $M[6] : \xi[1, 1, -1]$ and $M[10] : \xi[1, 1, 1, -1]$ match the table on [16, p. 1].

6.2. ***Integral PEL datum when m = 5.*** We illustrate Theorem 5.5 for several well-chosen examples when $m = 5$, including all families branched at $N = 4$ points up to equivalence and one family branched at $N = 5$ points. This includes the two special families when $m = 5$, namely $M[11]$ and $M[16]$. A similar result can be obtained for any monodromy datum γ with $N > 4$ by an inductive process.

Example 6.3. The following table summarizes the data for CM-types when $m = 5$.

a	$\mathfrak{f}$	Φ	β
(4,3,3)	(0,1,0,1)	{2,4}	$\beta_1 = 5/(\zeta_5^3 - \zeta_5^2)$
(3,1,1)	(1,1,0,0)	{1,2}	$\beta_2 = 5/(\zeta_5 - \zeta_5^4)$
(1,2,2)	(1,0,1,0)	{1,3}	$\beta_3 = -\beta_1$
(2,4,4)	(0,0,1,1)	{3,4}	$\beta_4 = -\beta_2$

In the i-th line of the table, the CM-type Φ_i is simple by Lemma 3.8 and β_i satisfies Corollary 4.7 with respect to Φ_i. The automorphism σ_3 permutes the rows via the cycle (1, 2, 3, 4) and its inverse σ_2 permutes the fourth column via $\beta_1 \to \beta_2 \to -\beta_1 \to -\beta_2 \to \beta_1$.

Proof. When $m = 5$, then $\beta_0 = 5/(\zeta_5^3 - \zeta_5^2)$ by Lemma 3.7. For $\mathfrak{f}_1 = (0, 1, 0, 1)$, set $u_1 = 1$ and $\beta_1 = u_1\beta_0$. We compute that $\mathrm{Im}(\sigma_j(\beta_1)) < 0$ for $j = 2, 4$. For $\mathfrak{f}_2 = (1, 1, 0, 0)$, set $u_2 = (\zeta_5^3 - \zeta_5^2)/(\zeta_5 - \zeta_5^4)$ and $\beta_2 = u_2\beta_0$. We compute that $\mathrm{Im}(\sigma_j(\beta_2)) < 0$ for $j = 1, 2$. The signature $\mathfrak{f}_3 = (1, 0, 1, 0)$ (resp. $\mathfrak{f}_4 = (0, 0, 1, 1)$) is the complex conjugate of $\mathfrak{f}_1$ (resp. $\mathfrak{f}_2$), which negates the value of β. □

Corollary 6.4. *Let $m = 5$ and $F = \mathbb{Q}(\zeta_5)$. Every family of μ_5-covers of $\mathbb{P}^1$ with $N = 4$ is equivalent to either* (i), (ii) *or* $M[11]$ *in the table below. Recall $\beta_1, \beta_2 \in \mathcal{O}_F$ from Example 6.3 and let $\xi_i = \beta_i^{-1}$. For the monodromy data $\gamma = (5, N, a)$ and $r = N - 2$ as below, the integral PEL datum of S_γ has lattice $(\mathcal{O}_F)^r$ with Hermitian form H_B where $B \in \mathrm{GL}_r(\mathcal{O}_F)$ is as below.*

M	$(5, N, a)$	B
(i)	(5,4,(1,1,4,4))	$\mathrm{diag}[\xi_2, -\xi_2]$
(ii)	(5,4,(1,2,3,4))	$\mathrm{diag}[-\xi_1, \xi_1]$
$M[11]$	(5,4,(1,3,3,3))	$\mathrm{diag}[\xi_1, \xi_2]$
$M[16]$	(5,5,(2,2,2,2,2))	$\mathrm{diag}[-\xi_1, -\xi_2, -\xi_1]$

Proof. Suppose $m = 5$ and $N = 4$ and let a be the inertia type of γ. If three of the values of a are the same, the family is equivalent to the one with $a = (1, 3, 3, 3)$, which is $M[11]$. If two of the values of a are the same, the family is equivalent to (i). If all values of a are distinct, the family is equivalent to (ii).

By Theorem 5.5, it suffices to find the CM-type (F, Φ_i) for the abelian varieties A_i in the decomposition of the Jacobian of $\mathcal{C}_P$. We refer to [8, Remark 5.2, Lemma 6.4] for information about the admissible degeneration, given in shorthand by: (i) $(1,1,3)+(2,4,4)$; (ii) $(1,2,2)+(3,3,4)$; $M[11]$ $(1,3,1)+(4,3,3)$; and $M[16]$ $(2,2,1)+(4,2,4)+(1,2,2)$. Using the table in Example 6.3, we find the entries of the diagonal of B. □

Remark 6.5. The family (ii), with monodromy datum $\gamma=(5,4,(1,2,3,4))$, has a second degeneration of the form $(1,3,1)+(4,2,4)$, whose Hermitian form has matrix $B'=\operatorname{diag}[\xi_2,-\xi_2]$. We give two reasons why the Hermitian forms determined by B' and $B=\operatorname{diag}[-\xi_1,\xi_1]$ are isomorphic.

First, by Lemma 2.1, the automorphism σ_2 takes the inertia types in the first degeneration to those in the second by multiplying the entries by 3. By Corollary 4.7(1), the action on the entries of B is via $\sigma_{2^{-1}}=\sigma_3$ and

$$\sigma_3(B)=\operatorname{diag}[\sigma_3(-\xi_1),\sigma_3(\xi_1)]=\operatorname{diag}[\xi_2,-\xi_2]=B'.$$

Second, for family (ii), S_γ has signature type $(1,1,1,1)$; hence its reflex field is $\mathbb{Q}$ (which is smaller than $F_0\subset\mathbb{R}$). The matrices B and B' are conjugate under the action of $\sigma_3\in\operatorname{Gal}(F_0/\mathbb{Q})$ and correspond to the two choices of a $\mathbb{Q}$-linear embedding $F_0\hookrightarrow\mathbb{R}$.

Remark 6.6. To compare with Shimura's work, write $w=\zeta_5+\zeta_5^4$. Then we have $w^2+w-1=0$. So $w=(-1+\sqrt{5})/2$. Then $\xi_2/\xi_1=-w-1=-(1+\sqrt{5})/2$ so $\xi_1/\xi_2=(1-\sqrt{5})/2$.

Consider the family $\gamma'=(5,4,(1,1,1,1,1))$. A careful look at [16, Section 5] shows that Shimura replaced ζ_5 by ζ_5^3 in his computation for this family. This has the effect of switching to the family $M[16]$ with $\gamma=(5,4,(2,2,2,2,2))$; indeed, Shimura computes that the signature is $(2,0,3,1)$. By line 4 of the table in Corollary 6.4, the family γ has $B=-\xi_1[1,1,\xi_2/\xi_1]=-\xi_1[1,1,-(1+\sqrt{5})/2]$. This does not exactly match what is written in line 5 of the table on [16, page 1], namely $[1,1,\xi_1/\xi_2]=[1,1,(1-\sqrt{5})/2]$, but it has the same sign signature and thus yields an isomorphic Hermitian form.

Consider the family $\gamma'=(5,4,(2,1,1,1))$. The details for this family are not included in [16] but it appears that Shimura replaced ζ_5 by ζ_5^3 in his computation for this family also. This has the effect of switching to the family $M[11]$ with $\gamma=(5,4,(1,3,3,3))$. By line 3 of the table in Corollary 6.4, the family γ has $B=\xi_2[1,\xi_1/\xi_2]=\xi_2[1,(1-\sqrt{5})/2]$. This matches what is written in line 4 of the table on [16, page 1].

6.3. ***Integral PEL datum when $m=7$.*** We illustrate Theorem 5.5 for several well-chosen examples when $m=7$, including all families branched at $N=4$ points up to equivalence. This includes the special family $M[17]$. A similar result can be obtained for any monodromy datum γ with $N>4$ by an inductive process.

Example 6.7. The following table summarizes the cases when $m = 7$.

a	$\mathfrak{f}$	Φ	β
$(1,1,5)$	$(1,1,1,0,0,0)$	$\{1,2,3\}$	$\beta_1 = 7/(\zeta_7 - \zeta_7^6)$
$(3,3,1)$	$(1,0,1,0,1,0)$	$\{1,3,5\}$	$\beta_2 = 7/(\zeta_7^3 - \zeta_7^4)$
$(2,2,3)$	$(1,0,0,1,1,0)$	$\{1,4,5\}$	$\beta_3 = 7/(\zeta_7^2 - \zeta_7^5)$
$(6,6,2)$	$(0,0,0,1,1,1)$	$\{4,5,6\}$	$\beta_4 = -\beta_1$
$(4,4,6)$	$(0,1,0,1,0,1)$	$\{2,4,6\}$	$\beta_5 = -\beta_2$
$(5,5,4)$	$(0,1,1,0,0,1)$	$\{2,3,6\}$	$\beta_6 = -\beta_3$
$(1,2,4)$	$(1,1,0,1,0,0)$	$\{1,2,4\}$	$\beta = -\dfrac{7(\zeta_7^3-\zeta_7^4)}{(\zeta_7-\zeta_7^6)(\zeta_7^2-\zeta_7^5)}$
$(3,1,5)$	$(0,0,1,0,1,1)$	$\{3,5,6\}$	$\beta' = -\beta$

Lemma 6.8. *In the i-th line of the table in Example 6.7, for $1 \le i \le 6$, the CM-type Φ_i is simple, and the element β_i satisfies Corollary 4.7 with respect to Φ_i. The generator σ_3 of the Galois group* $\mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})$ *permutes these via $\sigma_3^{i-1}(\beta_1) = \beta_i$, for $i = 1, \ldots, 6$.*

In the last two lines of the table, the element β (resp. β') satisfies Corollary 4.7 with respect to the CM-type $\Phi = \{1, 2, 4\}$ (resp. $\Phi' = \{3, 5, 6\}$). The CM types Φ, Φ' are not simple.

Proof. When $m = 7$, then $\beta_0 = 7/(\zeta_7^4 - \zeta_7^3)$ by Lemma 3.7.

For $a_1 = (1, 1, 5)$ and $\mathfrak{f}_1 = (1, 1, 1, 0, 0, 0)$, set $u_1 = (\zeta_7^4 - \zeta_7^3)/(\zeta_7 - \zeta_7^6)$ and $\beta_1 = u_1\beta_0$. We compute that $\mathrm{Im}(\sigma_j(\beta_1)) < 0$ for $j = 1, 2, 3$.

Let $\sigma = \sigma_5$, which is a generator of $\mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q})$ and has inverse σ_3. Then, by Lemmas 2.1 and 2.4, for $1 \le i \le 6$, the action of σ_5^{i-1} changes the inertia type to $a_i := (3^{i-1}) \cdot a_1$ and the CM-type to $\Phi_i := 5^{i-1} \cdot \Phi$, which also determines the signature $\mathfrak{f}_i$. Also, the element $\beta_i = \sigma_3^{i-1}(\beta_1)$ satisfies Corollary 4.7 with respect to Φ_i.

For $a = (1, 2, 4)$ and $\mathfrak{f} = (1, 1, 0, 1, 0, 0)$, set

$$u = -\frac{(\zeta_7^4 - \zeta_7^3)^2}{(\zeta_7 - \zeta_7^6)(\zeta_7^2 - \zeta_7^5)} \quad \text{and} \quad \beta = u\beta_0 = -\frac{7(\zeta_7^3 - \zeta_7^4)}{(\zeta_7 - \zeta_7^6)(\zeta_7^2 - \zeta_7^5)}.$$

Then $\mathrm{Im}(\sigma_j(\beta_2)) < 0$ for $j = 1, 2, 4$. The last line follows from Lemma 2.1 by applying complex conjugation to the previous line.

By Lemma 3.9, Φ_i is simple unless it is $\{1, 2, 4\}$ or $\{3, 5, 6\}$. □

For $m = 7$ and $N = 4$, every family is equivalent to one in the next result.

Corollary 6.9. *Let $m = 7$ and $F = \mathbb{Q}(\zeta_7)$. Recall $\beta_1, \beta_2, \beta_3, \beta \in \mathcal{O}_F$ from Example 6.7. Let $\xi_i = \beta_i^{-1}$ for $i = 1, 2, 3$ and $\xi = \beta^{-1}$. For the monodromy* *data $\gamma = (7, 4, a)$ as below, the integral PEL datum of S_γ has lattice $(\mathcal{O}_F)^2$ with Hermitian form H_B where $B \in \mathrm{GL}_2(\mathcal{O}_F)$ is as below.*

M	a	B
(i)	$(1,1,2,3)$	$\mathrm{diag}[\xi_1, \xi_3]$
(ii)	$(1,1,6,6)$	$\mathrm{diag}[\xi_1, -\xi_1]$
(iii)	$(1,2,5,6)$	$\mathrm{diag}[\xi_1, -\xi_1]$
$M[17]$	$(2,4,4,4)$	$\mathrm{diag}[-\xi_2, \xi]$

Proof. We refer to [8, Remark 5.2, Lemma 6.4] for information about the admissible degeneration of each family, given in shorthand by (i) $(1, 1, 5) + (2, 2, 3)$; (ii) $(1, 1, 5) + (2, 6, 6)$; (iii) $(1, 5, 1) + (6, 2, 6)$; and $M[17]$ $(4, 4, 6) + (1, 4, 2)$. For each family, consider the CM-types (F, Φ_1) and (F, Φ_2) for the abelian varieties A_i in the decomposition of the Jacobian at the distinguished point arising from the degeneration. We compute the associated integral PEL data by Theorem 5.5, using the table in Example 6.7. □

Remark 6.10. In cases (i), (ii), and (iii), both (F, Φ_1) and (F, Φ_2) are simple. For $M[17]$, the CM-type (F, Φ_2) is not simple, but this is not a concern by [16, Appendix, Proposition 8] or Section 6.4. For $M[17]$, we note that $B = \xi[1, -\xi_2/\xi]$ where $-\xi_2/\xi = \zeta_7 + \zeta_7^6 - 1$. In line 6 of the table on [16, page 1], for the related family $\gamma = (7, 4, (4, 1, 1, 1))$, Shimura writes the Hermitian form as H_B with $B = \xi[1, -\sin(3\pi/7)/\sin(2\pi/7)]$.

6.4. ***Another approach for M[17].*** In this section, we compute the integral PEL datum for the family $M[17]$ using another approach. This approach utilizes a different kind of distinguished point Q in the family, namely one that represents a curve C_Q with extra automorphisms. In this section, we see that the Jacobian J_Q has complex multiplication by a larger field and that its CM-type is simple.

In fact, we proved that every positive-dimensional family of μ_7-covers of $\mathbb{P}^1$ has a distinguished point representing a product of principally polarized abelian varieties, each of which has complex multiplication with a CM-type that is simple. In this way, one can avoid the use of [16, Appendix, Proposition 8] when $m = 7$. We omit the details of this.

We start by finding another distinguished point in the family Z_γ, under certain restrictive conditions on γ. Similarly to Notation 3.5, for $0 \leq n \leq 3m$ with $\gcd(n, 3m) = 1$, let σ_n be the embedding $\mathbb{Q}(\zeta_{3m}) \hookrightarrow \mathbb{C}$ determined by $\sigma_n(\zeta_{3m}) = \zeta_{3m}^n$.

Proposition 6.11. *Let $\gamma = (m, 4, a)$ where $m > 3$ is prime and $a = (1, 1, 1, m - 3)$. Then Z_γ has a distinguished point Q. More specifically: in the family of μ_m-covers with monodromy datum γ, there is a point which represents a μ_m-cover*

$\psi : C_Q \to \mathbb{P}^1$, where C_Q is a curve of genus $m-1$ having an automorphism of order 3.

The Jacobian of C_Q has complex multiplication by $(\mathbb{Z}[\zeta_{3m}], \Phi_Q)$, where, for any $0 < n < 3m$ with $(n, 3m) = 1$, the embedding σ_n is in Φ_Q if and only if

$$n \in [0, 2m/3] \cup [m, 5m/3] \cup [2m, 8m/3] \quad \textit{if } n \equiv 1 \bmod 3, \quad \textit{or}$$

$$n \in [0, m/3] \cup [m, 4m/3] \cup [2m, 7m/3] \quad \textit{if } n \equiv 2 \bmod 3.$$

It is possible to generalize Proposition 6.11, by replacing 3 by an odd prime ℓ relatively prime to m and letting $N = \ell + 1$ and $a = (1, \dots, 1, m - \ell)$. We omit this generalization.

Proof of Proposition 6.11. Let C_Q be the smooth projective curve with equation $y^m = x^3 - 1$. It admits a μ_m-cover to $\mathbb{P}^1$ branched at $1, \zeta_3, \zeta_3^2, \infty$ with inertia type $a = (1, 1, 1, m-3)$. The genus of C_Q is $m-1$ by (2-1). Also C_Q has an automorphism $(x, y) \mapsto (\zeta_3 x, y)$ of order 3.

The Jacobian $J_Q = \mathrm{Jac}(C_Q)$ is a principally polarized abelian variety having dimension $m-1$. Let $F = \mathbb{Q}(\zeta_m)$, and $L = \mathbb{Q}(\zeta_{3m})$. The field L is a CM-field of degree $2 \cdot \dim(J_Q)$. Then, the inclusion $\mathcal{O}_F \subset \mathrm{End}(J_Q)$ extends to an inclusion $\mathcal{O}_L = \mathcal{O}_F[\zeta_3] \subset \mathrm{End}(J_Q)$. Thus J_Q has complex multiplication by $\mathcal{O}_L = \mathbb{Z}[\zeta_{3m}]$.

Consider the morphism $\phi : C_Q \to \mathbb{P}^1$, taking $(x, y) \mapsto x^3$. Then ϕ is a μ_{3m}-cover of $\mathbb{P}^1$, branched at $0, 1, \infty$. We compute the CM-type of J_Q by finding the inertia type and signature type of ϕ. Let b_0 (resp. b_1, resp. b_∞) denote the element of $\mathbb{Z}/3m\mathbb{Z}$ that determines the canonical generator of inertia of ϕ above 0, (resp. 1, resp. ∞).

Note that $b_1 = 3$. This is because ϕ is branched at the 3 points $x = 1, \zeta_3, \zeta_3^2$ that lie above $x^3 = 1$; the canonical generator of inertia of ϕ at the points of C_Q above these is, by definition, the automorphism identified with $\zeta_7^1 = \zeta_{21}^3$.

Without loss of generality, $b_0 = m$. To see this, note there are m points of C_Q above $x^3 = 0$, so $\gcd(b_0, 3m) = m$; this implies that $b_0 = 2m$ or $b_0 = m$; possibly after replacing the order 3 automorphism with its inverse, we can suppose that $b_0 = m$. Third, $b_\infty = 3m - b_0 - b_1 = 2m - 3$, so $(b_0, b_1, b_\infty) = (m, 3, 2m-3)$.

We compute the signature $\mathfrak{f}$ of ϕ. Let $0 < n < 3m$ and $(n, 3m) = 1$. By (2-2):

$$\mathfrak{f}(\sigma_n) = -1 + \langle -n/3 \rangle + \langle -n/m \rangle + \langle (-2n)/3 + (n/m) \rangle;$$

that is,

$$\mathfrak{f}(\sigma_n) = \begin{cases} -1 + \frac{2}{3} + \langle -n/m \rangle + \langle -\frac{2}{3} + n/m \rangle & \text{if } n \equiv 1 \bmod 3, \\ -1 + \frac{1}{3} + \langle -n/m \rangle + \langle -\frac{1}{3} + n/m \rangle & \text{if } n \equiv 2 \bmod 3, \end{cases}$$

We deduce that $\mathfrak{f}(\sigma_n) = 1$ if $n \equiv 1 \bmod 3$ and $n \in [0, 2m/3] \cup [m, 5m/3] \cup [2m, 8m/3]$ or if $n \equiv 2 \bmod 3$ and $n \in [0, m/3] \cup [m, 4m/3] \cup [2m, 7m/3]$; otherwise $\mathfrak{f}(\sigma_n) = 0$.

Finally, by (2-3), the CM-type Φ_Q of J_Q is as given in the statement. □

For $m = 7$, we verify that the CM-type Φ_Q of the Jacobian of C_Q is simple.

Lemma 6.12. *When $m = 7$, then the CM-type Φ_Q in Proposition 6.11 is simple.*

Proof. Let C_Q be the smooth projective curve with equation $y^7 = x^3 - 1$. From Proposition 6.11, its CM-type is $\Phi = \{\tau_n \mid n = 1, 2, 4, 8, 10, 16\}$. We check that Φ is simple by showing that it is not induced from any proper CM-field. Let $\sigma_i : \mathbb{Q}(\zeta_{3m}) \to \mathbb{C}$ be the embedding given by $\sigma_i(\zeta_{3m}) = \zeta_{3m}^i$, for $1 \leq i \leq 3m$ with $\gcd(i, 3m) = 1$. Every subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_{3m})/\mathbb{Q})$ not containing complex conjugation contains either σ_8 (order 2), σ_{13} (order 2) or σ_4 (order 3). It suffices to show that the subgroup generated by each of these has a coset not contained in Φ but having nontrivial intersection with Φ, so that Φ is not a union of cosets. For $\langle\sigma_8\rangle$, this is true because of the coset $\{\sigma_4, \sigma_{11}\}$. For $\langle\sigma_{13}\rangle$, this is true because of the coset $\{\sigma_1, \sigma_{13}\}$. For $\langle\sigma_4\rangle$, this is true because of the coset $\{\sigma_2, \sigma_8, \sigma_{11}\}$. □

The following lemma will be helpful.

Lemma 6.13. *In the field $L = \mathbb{Q}(\zeta_{21})$, write $\zeta = \zeta_{21}$. Let $\beta_3 = 7/(\zeta^6 - \zeta^{15})$. Let*

$$\alpha = (\zeta^7 - \zeta^{14})(\zeta^2 - \zeta^{19}) = (\zeta^9 + \zeta^{12}) - (\zeta^5 + \zeta^{16}). \tag{6-1}$$

With respect to the CM-type $\Phi = \{1, 2, 4, 8, 10, 16\}$ of L, the following element z satisfies conditions (1)–(3) *in Corollary 4.3*:

$$z = \beta_3\alpha = 21(\zeta^2 - \zeta^{19})/((\zeta^{14} - \zeta^7)(\zeta^6 - \zeta^{15})). \tag{6-2}$$

Proof. We verify the first claim by computation using that $\zeta^{14} - \zeta^7 = -\sqrt{-3}$. We verify the second claim from Lemma 3.7, part (4). Note that $z = -\sigma_4(\beta_0)$ for β_0 as in loc. cit. □

We now compute the integral PEL datum of $M[17]$ using Proposition 6.11.

Proposition 6.14. *Let $m = 7$ and $F = \mathbb{Q}(\zeta_7)$. Let $\xi_1 = \beta_1^{-1}$ where $\beta_1 \in \mathcal{O}_F$ is defined in Example 6.7. Consider the units $u_1 = \zeta_7^2 + \zeta_7^5$ and $v = (1 + \zeta_7^3 + \zeta_7^4)^{-1}$ of $\mathbb{Z}[\zeta_7]$. For the family $M[17]$, with monodromy datum $\gamma = (7, 4, (2, 4, 4, 4))$, the integral PEL datum of S_γ has lattice $\Lambda = (\mathcal{O}_F)^2$ with Hermitian form H_B where*

$$B = \xi_1 v \begin{bmatrix} u_1 & -\zeta_7^2 \\ -\zeta_7^5 & u_1 \end{bmatrix}.$$

Proof. Consider the family $Z = Z_\gamma$ with monodromy datum $\gamma = (7, 4, (1, 1, 1, 4))$. Because $(2, 4, 4, 4) = 4 \cdot (4, 1, 1, 1)$ and $4 \equiv 2^{-1} \bmod 7$, the action of σ_2 takes Z to the family $M[17]$.

Let

$$A = \frac{\zeta_7^2 - \zeta_7^5}{7(1 + \zeta_7 + \zeta_7^6)} \begin{bmatrix} \zeta_7^3 + \zeta_7^4 & -\zeta_7^4 \\ -\zeta_7^3 & \zeta_7^3 + \zeta_7^4 \end{bmatrix}. \tag{6-3}$$

One can check that A is integral away from 7. We claim that the integral PEL datum of Z is given by the lattice $\Lambda = (\mathcal{O}_F)^2$ and the Hermitian form for the matrix A. This is sufficient to prove the proposition because, by the last statement of Corollary 4.3, the integral PEL datum of $M[17]$ is then given by the lattice $\Lambda = (\mathcal{O}_F)^2$ and the Hermitian form for the matrix $B = \sigma_2^{-1}(A)$.

We turn to computing the integral PEL datum of Z. Let $L = \mathbb{Q}(\zeta_{21})$. Consider the CM-type $\Phi = \{1, 2, 4, 8, 10, 16\}$ for L. Consider the curve C_Q of genus 6 given by the equation $y^7 = x^3 - 1$ and the associated μ_7-cover $C_Q \to \mathbb{P}^1$. By Proposition 6.11: this cover is represented by a point Q of Z; the Jacobian $\mathrm{Jac}(C_Q)$ is a principally polarized abelian variety with complex multiplication by $(\mathcal{O}_L, \Phi_Q)$; and this CM-type is simple by Lemma 6.12. In other words, Q is a simple distinguished point of Z.

Note that L has class number 1 and L_0 has narrow class number 2. By Corollary 4.6, there exists a unique principally polarized CM-abelian variety of type $(\mathcal{O}_L, \Phi)$ up to isomorphism. By Lemma 6.13, the element z from (6-2) satisfies conditions (1)–(3) in Corollary 4.3 with respect to Φ. Thus $\mathrm{Jac}(C_Q)$ is isomorphic to the torus $\mathbb{C}^6/\Phi(\mathcal{O}_L)$, with principal polarization given by $\xi = z^{-1}$. Furthermore, the integral PEL datum of Z is given by the lattice $\Lambda = \mathcal{O}_L \subset V = L$ and the Hermitian form $\langle x, y\rangle = \mathrm{tr}_{L/Q}(x\xi\bar{y}^T)$. Note that $\mathcal{O}_L = \mathbb{Z}[\zeta_{21}] = \mathcal{O}_F[\zeta_3]$. With respect to the ordered basis $1, \zeta_3$ for $\mathcal{O}_L$ over $\mathcal{O}_F$, the Hermitian form is given by a matrix in $\mathrm{GL}_2(\mathcal{O}_F)$. By Lemma 6.15, this matrix is A. □

Lemma 6.15. *With respect to the ordered basis* $1, \zeta_3$ *for* $\mathcal{O}_L$ *over* $\mathcal{O}_F$*, the Hermitian form* $\langle x, y\rangle = \mathrm{tr}_{L/Q}(x\xi\bar{y}^T)$ *is given by the matrix* $A \in \mathrm{GL}_2(\mathcal{O}_F)$*, where* A *is given in* (6-3).

Proof. Let $x, y \in \Lambda$. By Lemma 6.13, since $\beta_3 = 7/(\zeta_7^2 - \zeta_7^5)$ is in $\mathcal{O}_{F_0}$, we have

$$\langle x, y\rangle = \mathrm{tr}_{L/\mathbb{Q}}(xz\bar{y}) = \mathrm{tr}_{F/\mathbb{Q}}(\beta_3^{-1}\,\mathrm{tr}_{L/F}(x\alpha^{-1}\bar{y})),$$

where α is as in (6-1).

Let $\tau = \sigma_8 \in \mathrm{Gal}(L/\mathbb{Q})$. Then τ is the generator of $\mathrm{Gal}(L/F)$, and

$$\mathrm{tr}_{L/\mathbb{Q}}(x\alpha^{-1}\bar{y}) = \mathrm{tr}_{F/\mathbb{Q}}(x\alpha^{-1}\bar{y} + \tau(x)\tau(\alpha^{-1})\tau(\bar{y})).$$

Write $x = x_1 + x_2\zeta_3$ and $y = y_1 + y_2\zeta_3$, for $x_1, x_2, y_1, y_2 \in \mathcal{O}_F$. We compute

$$\mathrm{tr}_{L/\mathbb{Q}}(x\alpha^{-1}\bar{y}) = \mathrm{tr}_{F/\mathbb{Q}}\big((x_1+x_2\zeta_3)\alpha^{-1}(\bar{y}_1+\bar{y}_2\zeta_3^2)+(x_1+x_2\zeta_3^2)\tau(\alpha^{-1})(\bar{y}_1+\bar{y}_2\zeta_3)\big).$$

Write $a_{1,1} = \alpha^{-1}+\tau(\alpha^{-1})$, $a_{1,2} = \zeta_3^2\alpha^{-1}+\zeta_3\tau(\alpha^{-1})$, and $a_{2,1} = \zeta_3\alpha^{-1}+\zeta_3^2\tau(\alpha^{-1})$. Then $\mathrm{tr}_{L/\mathbb{Q}}(x\alpha^{-1}\bar{y}) = \mathrm{tr}_{F/\mathbb{Q}}(x_1\bar{y}_1a_{1,1}+x_1\bar{y}_2a_{1,2}+x_2\bar{y}_1a_{2,1}+x_2\bar{y}_2a_{1,1})$. We compute that $a_{1,1} = \alpha^{-1}+\tau(\alpha^{-1}) = (\zeta_7^4+\zeta_7^3)/(1+\zeta_7+\zeta_7^6)$, $a_{1,2} = -\zeta_7^4/(1+\zeta_7+\zeta_7^6)$, and $a_{2,1} = -\zeta_7^3/(1+\zeta_7+\zeta_7^6)$. Thus, $\langle x, y\rangle = \mathrm{tr}_{F/\mathbb{Q}}((x_1, x_2)A(\bar{y}_1, \bar{y}_2)^T)$. □

7. Applications for composite *m*

We illustrate how to extend Theorem 5.5 by computing the integral PEL datum in some cases when m is composite. In particular, we determine the integral PEL datum for 6 of the Moonen special families where $m = 4, 6, 10$.

Remark 7.1. Our techniques do not apply well for symplectic Shimura varieties, so we exclude the modular curve $M[1]$ with $m = 2$, the Picard surface $M[2]$ with $m = 2$, the family $M[7]$ with $m = 4$, and $M[12]$ with $m = 6$. We exclude $M[15]$ with $m = 8$ and $M[20]$ with $m = 12$, since the CM-types for biquadratic CM-fields are never simple. Our techniques are not sufficient to handle $M[13]$ with $m = 6$ or $M[19]$ with $m = 9$.

7.1. ***Difficulties when m is composite.*** The key to Theorem 5.5 is the existence of a distinguished point P in Z_γ satisfying the properties in Proposition 5.2. When m is composite, this is not always possible for the following reasons.

Remark 7.2. One issue when m is composite is that the curve C_P might not have compact type. For example, the family $M[12]$, with $\gamma = (6, 4, (1, 1, 1, 3))$, has no admissible degenerations of compact type. The reason is that the two covers with inertia types $a_1 = (1, 1, 4)$ and $a_2 = (2, 1, 3)$ would need to be joined at two points, leading to a cycle in the dual graph of C_P.[3] The same is true for $M[7]$, with $\gamma = (4, 4, (1, 1, 1, 1))$.

Remark 7.3. Another issue when m is composite is that the integral group algebra $\mathbb{Z}[\mu_m]$ has nontrivial index in $\prod_{1 \le d | m} \mathbb{Z}[\zeta_d]$. Hence, the Jacobian of a μ_m-cover of $\mathbb{P}^1$ branched at 3 points might not have complex multiplication by a maximal order. See Section 7.5.

7.2. ***Integral PEL data for two families with m = 4.*** We find the integral PEL datum for two of the three special families with $m = 4$: $M[4]$ and $M[8]$; we exclude $M[7]$ as it is not unitary. When $m = 4$, the hypotheses of Proposition 5.2 are not satisfied but we can find a distinguished point in the family by direct computation.

Example 7.4. If $m = 4$, then $\beta_0 = -2i$ from Lemma 3.7. Set $u_1 = -1$, so $\beta_1 = -\beta_0 = 2i$. If $\mathfrak{f}_1 = (0, 1)$, then β_1 satisfies the 3 conditions of Corollary 4.3 and all CM-types of $F = \mathbb{Q}(i)$ are simple.

Corollary 7.5. *Let $m = 4$ and $F = \mathbb{Q}(i)$. Let $\xi = \beta_1^{-1} = 1/2i$. For the family M with monodromy datum γ as in the next table* (top of page 206), *the integral PEL datum for S_γ has lattice $(\mathcal{O}_F)^r$ with Hermitian form H_B, where r and $B \in \mathrm{GL}_r(\mathcal{O}_F)$*

[3]There is a typo in this case in [8, Lemma 6.4].

are as follows:

M	$\gamma = (m, N, a)$	r	$B \in \mathrm{GL}_r(\mathcal{O}_F)$
$M[4]$	$(4, 4, (1, 2, 2, 3))$	2	$\mathrm{diag}[\xi, -\xi]$
$M[8]$	$(4, 5, (1, 1, 2, 2, 2))$	3	$\mathrm{diag}[\xi, -\xi, \xi]$

Proof. When $m = 4$, then $a = (3, 2, 3)$ has signature $\mathfrak{f} = (0, 1)$ and $\sigma_2(\beta_1) < 0$. Also $a = (1, 2, 1)$ has signature $\mathfrak{f} = (1, 0)$ and $\sigma_1(-\beta_1) < 0$. By [8, Lemma 6.4], there is a distinguished point in the family $M[4]$ and the family $M[8]$. The family $M[4]$ has an admissible degeneration, expressed in short as $(1, 2, 1) + (3, 2, 3)$. The family $M[8]$ has an admissible degeneration, expressed as $(1, 2, 1) + (3, 2, 3) + (1, 2, 1)$. The result then follows from Theorem 5.5. □

The case $M[8] : \xi[1, -1, 1]$ matches line 3 of the table on [16, page 1].

7.3. ***Remark when m = 9.*** The techniques in this paper are not sufficient to find the integral PEL datum for the special family $M[19]$ when $m = 9$. This family has inertia type $a = (3, 5, 5, 5)$ and its only admissible degeneration can be expressed in short hand as $(3, 5, 1) + (8, 5, 5)$. For $a_2 = (8, 5, 5)$, the Jacobian has complex multiplication by $\mathbb{Q}[\zeta_3] \times \mathbb{Q}[\zeta_9]$. As the action of $\mathbb{Z}[\mu_9]$ does not extend to the maximal order $\mathbb{Z}[\zeta_3] \times \mathbb{Z}[\zeta_9]$, we do not know how to compute its lattice.

We expect that our technique is sufficient to determine the integral PEL datum when $m = 9$, $N = 4$ and $a = (3, 1, 8, 6)$. We leave the details to the reader to check.

7.4. ***Generalities for twice a prime.***

Notation 7.6. Let $m = 2m'$ where m' is an odd prime. Let $F = \mathbb{Q}(\zeta_m)$ and $F' = \mathbb{Q}(\zeta_{m'})$. We identify $F = F'$ and $\mathbb{Z}[\zeta_m] = \mathbb{Z}[\zeta_{m'}]$, by $\zeta_m = -\zeta_{m'}^{(m'+1)/2}$. There is a bijection between CM types Φ for F and CM types Φ' for F' where $\sigma_j \in \Phi$ for an odd integer $1 \le j \le m$ if and only if $\sigma_{j'} \in \Phi'$, where j' is the reduction of j modulo m'.

If β satisfies Corollary 4.7 with respect to the CM type $(\mathbb{Z}[\zeta_{m'}], \Phi')$ then it also satisfies Corollary 4.7 with respect to the CM type $(\mathbb{Z}[\zeta_m], \Phi)$. This can be verified in general but we only need a few cases from Example 6.1 when $m' = 3$ and Example 6.3 when $m' = 5$.

Example 7.7.

m	β	$(\mathbb{Z}[\zeta_{m'}], \Phi')$	$(\mathbb{Z}[\zeta_m], \Phi)$
6	$b_2 = \sqrt{-3}$	$(\mathbb{Z}[\zeta_3], \{2\})$	$(\mathbb{Z}[\zeta_6], \{5\})$
10	$\beta_1 = 5/(\zeta_5^3 - \zeta_5^2)$	$(\mathbb{Z}[\zeta_5], \{2, 4\})$	$(\mathbb{Z}[\zeta_{10}], \{7, 9\})$
10	$\beta_2 = 5/(\zeta_5 - \zeta_5^4)$	$(\mathbb{Z}[\zeta_5], \{1, 2\})$	$(\mathbb{Z}[\zeta_{10}], \{1, 7\})$

7.5. ***Integral PEL data for 4 families with m = 6, 10.*** We find the integral PEL datum for three special families with $m=6$, namely $M[5]$, $M[9]$ and $M[14]$, and for the special family with $m=10$, namely $M[18]$. For $m=6$, we exclude $M[12]$ as it is not unitary and $M[13]$ (Remark 7.10). For m composite, the hypotheses of Proposition 5.2 are not satisfied. In each case, we find a distinguished point P in the family for which the issues in Remarks 7.2 and 7.3 do not occur.

Corollary 7.8. (Recall Notation 7.6.) *For the special family* $M[18]$ *with monodromy datum* $\gamma=(10,4,(3,5,6,6))$, *the integral PEL datum of* S_γ *has lattice*

$$\Lambda=(\mathcal{O}_{F'})\oplus(\mathcal{O}_F)^2$$

with Hermitian form H_B, *where* $B\in \mathrm{GL}_1(\mathcal{O}_{F'})\times\mathrm{GL}_2(\mathcal{O}_F)$ *is* $\mathrm{diag}[\xi_2]\oplus\mathrm{diag}[\xi_1,\xi_2]$ *with* $\xi_i=\beta_i^{-1}$.

Proof. Let $m=10$ and $m'=5$. Let $M=M[18]$ with $\gamma=(10,4,(3,5,6,6))$ and signature type $\mathfrak{f}=(1,1,0,1,0,0,2,0,1)$. The proof is similar to that of Theorem 5.5. We produce a simple distinguished point P in the family similar to that in Proposition 5.2(1); it represents an admissible μ_m-cover $\psi: C_P\to T$, where T is a tree of projective lines and C_P is a curve of compact type such that each of its irreducible components is a curve admitting a μ_m-cover of $\mathbb{P}^1$ branched at 3 points. We verify by direct computation that each irreducible component of C_P has complex multiplication by either $\mathbb{Z}[\zeta_m]$ or $\mathbb{Z}[\zeta_{m'}]$.

Consider the μ_5-cover $\psi_2: C_2\to\mathbb{P}^1$ branched at three points, with inertia type $a_2=(4,3,3)$, and signature $\mathfrak{f}_2=(0,1,0,1)$. Then $A_2=\mathrm{Jac}(C_2)$ has complex multiplication by $(\mathbb{Z}(\zeta_5),\{2,4\})$. Consider the induced curve $\tilde{C}_2=\mathrm{Ind}_5^{10}(C_2)$, which is the disconnected curve consisting of two copies of C_2, and the induced μ_{10}-cover $\Psi_2:\tilde{C}_2\to\mathbb{P}^1$. Then Ψ_2 is branched at three points and, somewhat imprecisely, we can say that it has inertia type $(8,6,6)=\mathrm{Ind}_5^{10}(4,3,3)$. Above the first branch point, there are two points η_2 and η_2' on $\tilde{C}_2$, and they are labeled by the two cosets of $\mu_5\subset\mu_{10}$. Let $\mathcal{A}_2=\mathrm{Jac}(\tilde{C}_2)\simeq A_2^2$.

As explained in [10, Section 3.1], the signature type of $\mathcal{A}_2$ is

$$\mathfrak{f}_2=(0,1,0,1,0,0,1,0,1)=(0,1,0,1,0,0,0,0,0)+(0,0,0,0,0,0,1,0,1).$$

The action of $\mathbb{Z}[\mu_{10}]$ on $\mathcal{A}_2$ is the diagonal action of $\mathbb{Z}[\zeta_5]\times\mathbb{Z}[\zeta_{10}]$ on A^2. The first (resp. second) copy of A_2 has CM-type $(\mathbb{Z}[\zeta_5],\{2,4\})$ (resp. $(\mathbb{Z}[\zeta_{10}],\{7,9\})$); for this CM-type, the element β_1 (resp. β_1) satisfies Corollary 4.7, as seen in Example 7.7.

Consider the μ_{10}-cover $\psi_1: C_1\to\mathbb{P}^1$ branched at three points, with inertia type $a_1=(3,5,2)$. Above the 3rd branch point, there are two points η_1 and η_1' on C_1 and they are labeled by the two cosets of $\mu_5\subset\mu_{10}$. Let $A_1=\mathrm{Jac}(C_1)$; it has signature type $\mathfrak{f}_1=(1,0,0,0,0,0,1,0,0)$. By Lemma 2.3, A_1 has complex multiplication by $(\mathbb{Z}[\zeta_{10}], \{1, 7\})$. Since $\{1, 7\} = \{3 \cdot 7, 3 \cdot 9\}$ mod 10, the element β that satisfies Corollary 4.7 is $\sigma_7(\beta_1) = \beta_2$. All CM types of $\mathbb{Q}(\zeta_5) = \mathbb{Q}(\zeta_{10})$ are simple.

Let $\mathcal{C}$ denote the singular curve, whose components are C_1 and $\tilde{C}_2$, formed by identifying $\eta_1\ \eta_1'$ with η_2 and η_2', so that the cosets are matched correctly, in two ordinary double points. The curve $\mathcal{C}$ admits an admissible μ_{10}-cover Ψ to a chain of two projective lines. By construction, $\mathcal{C}$ has an action by $\mathcal{O}_{F'} \oplus \mathcal{O}_F^2$, with CM-type given by $\mathrm{diag}[\xi_2] \oplus \mathrm{diag}[\xi_1, \xi_2]$.

The last thing to check is that Ψ is represented by a point $P \in Z_\gamma$. Since Ψ is admissible, it can be deformed to a μ_{10}-cover of $\mathbb{P}^1$ with inertia type $(3, 5, 6, 6)$. This has signature type $\mathfrak{f} = (1, 1, 0, 1, 0, 0, 2, 0, 1)$. Hence, Ψ is represented by a point $P \in Z_\gamma$ and by the preceding paragraph P is a simple distinguished point. □

Corollary 7.9. (Recall Notation 7.6.) *Let* $\xi = 1/\sqrt{-3}$. *For the special families* M *with monodromy datum* $\gamma = (6, N, a)$ *as below, the integral PEL datum of* S_γ *has lattice* $\Lambda = (\mathcal{O}_{F'})^{r'} \oplus (\mathcal{O}_F)^r$ *with Hermitian form* H_B *where* r', r, *and* B *are as follows*:

M	$\gamma = (m, N, a)$	(r', r)	$B \in \mathrm{GL}_{r'}(\mathcal{O}_{F'}) \times \mathrm{GL}_r(\mathcal{O}_F)$
$M[5]$	$(6, 4, (2, 3, 3, 4))$	$(0, 2)$	$\mathrm{diag}[\xi, -\xi]$
$M[9]$	$(6, 4, (1, 3, 4, 4))$	$(1, 2)$	$\mathrm{diag}[-\xi] \oplus \mathrm{diag}[\xi, -\xi]$
$M[14]$	$(6, 5, (2, 2, 2, 3, 3))$	$(1, 3)$	$\mathrm{diag}[\xi] \oplus \mathrm{diag}[\xi, -\xi, \xi]$

Proof. The proof is very similar to that of Corollary 7.8 so we provide only a sketch.

(1) Let $M = M[5]$ with $\gamma = (6, 4, (2, 3, 3, 4))$ and $\mathfrak{f} = (1, 0, 0, 0, 1)$. Then C_P is the join of two μ_6-covers with inertia types $a_1 = (1, 2, 3)$ and $a_2 = (3, 4, 5)$. These have signatures $\mathfrak{f}_1 = (1, 0, 0, 0, 0)$ and $\mathfrak{f}_2 = (0, 0, 0, 0, 1)$. By Lemma 2.3, A_1 has CM by $\mathcal{O}_F$ of type $\Phi_1 = \{1\}$, and A_2 has CM by $\mathcal{O}_F$ of type $\Phi_2 = \{5\}$. Let $b_1 = -b_2 = -\sqrt{-3}$. For $i = 1, 2$, the CM-type Φ_i is simple, and the element $b_i \in \mathcal{O}_F$ satisfies Corollary 4.7 with respect to Φ_i.

(2) Let $M = M[9]$ with $\gamma = (6, 4, (1, 3, 4, 4))$ and $\mathfrak{f} = (1, 1, 0, 0, 1)$. Then C_P is the join of two covers with inertia types $a_1 = (1, 2, 3)$ and $a_2 = \mathrm{Ind}_3^6(2, 2, 2)$. These have signatures $\mathfrak{f}_1 = (1, 0, 0, 0, 0)$ and $\mathfrak{f}_2 = (0, 1, 0, 0, 1)$. By Lemma 2.3, $A_1 = \mathrm{Jac}(C_1)$ has CM-type $(\mathcal{O}_F, \{1\})$, which has $\beta = b_1 = -b_2$. Also $\mathcal{A}_2 \simeq A^2$ and the action of $\mathbb{Z}[\mu_6]$ on $\mathcal{A}_2$ is given by the diagonal action of $\mathbb{Z}[\zeta_3] \times \mathbb{Z}[\zeta_6]$ on A^2. The first copy of A has CM-type $(\mathbb{Z}[\zeta_3], \{2\})$ which has $\beta = b_2$ and the second copy of A has CM-type $(\mathbb{Z}[\zeta_6], \{5\})$ which has $\beta = b_2$.

(3) Let $M = M[14]$ with $\gamma = (6, 5, (2, 2, 2, 3, 3))$ and $\mathfrak{f} = (2, 0, 0, 1, 1)$. Then C_P is the join of three covers with inertia types $a_1 = (1, 2, 3)$, $a_2 = (3, 4, 5)$ and $a_3 = \mathrm{Ind}_3^6(1, 1, 1)$. These have signatures $\mathfrak{f}_1 = (1, 0, 0, 0, 0)$, $\mathfrak{f}_2 = (0, 0, 0, 0, 1)$, and $\mathfrak{f}_3 = (1, 0, 0, 1, 0)$. By Lemma 2.3, A_1 has CM-type $(\mathcal{O}_F, \{1\})$ which has $\beta = b_1$

and A_2 has CM-type $(\mathcal{O}_F, \{5\})$ which has $\beta = b_2$. Also $\mathcal{A}_3 \simeq A^2$ and the action of $\mathbb{Z}[\mu_6]$ on $\mathcal{A}_3$ is given by the diagonal action of $\mathbb{Z}[\zeta_3] \times \mathbb{Z}[\zeta_6]$ on A^2. The first copy of A has CM-type $(\mathbb{Z}[\zeta_3], \{1\})$ which has $\beta = b_1$ and the second copy of A has CM-type $(\mathbb{Z}[\zeta_6], \{4\})$ which has $\beta = b_1$. □

We checked Corollaries 7.8 and 7.9 independently, using the fact that the special families for $m = 6, 10$ are subspaces of those for $m' = 3, 5$, up to a Galois twist.

Remark 7.10. Let $M = M[13]$ be the special family with $\gamma = (6, 4, (1, 1, 2, 2))$ and $\mathfrak{f} = (2, 1, 0, 1, 0)$. This has an admissible degeneration to the join of two covers with inertia types $a_1 = (1, 1, 4)$ and $a_2 = \mathrm{Ind}_3^6(1, 1, 1)$. The first one has signature $\mathfrak{f}_1 = (1, 1, 0, 0, 0)$ and its Jacobian A_1 has CM by an order of finite index in $\mathcal{O}_{F'} \times \mathcal{O}_F$. The index is nontrivial since $\langle x^3 - 1, x^3 + 1\rangle = \langle 2\rangle$ in $\mathbb{Z}[x]/\langle x^6 - 1\rangle$, so our technique does not apply. The other degeneration to $a_1 = (1, 2, 3)$ and $a_2 = (3, 1, 2)$ does not have compact type.

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WANLIN LI
DEPARTMENT OF MATHEMATICS
VANDERBILT UNIVERSITY
NASHVILLE, TN
UNITED STATES
wanlin.li@vanderbilt.edu

ELENA MANTOVAN
DEPARTMENT OF MATHEMATICS
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CA
UNITED STATES
mantovan@caltech.edu

RACHEL PRIES
DEPARTMENT OF MATHEMATICS
COLORADO STATE UNIVERSITY
FORT COLLINS, CO
UNITED STATES
pries@colostate.edu

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